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Ekeland's variational principle for vector optimization with variable ordering structure

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Abstract

There are many generalizations of Ekeland's variational principle for vector optimization problems with fixed ordering structures, i.e., ordering cones. These variational principles are useful for deriving optimality conditions, ε-Kolmogorov conditions in approximation theory, and ε-maximum principles in optimal control. Here, we present several generalizations of Ekeland's variational principle for vector optimization problems with respect to variable ordering structures. For deriving these variational principles we use nonlinear scalarization techniques. Furthermore, we derive necessary conditions for approximate solutions of vector optimization problems with respect to variable ordering structures using these variational principles and the subdifferential calculus by Mordukhovich.

Key Words: Nonconvex vector optimization, Variable ordering structure, Ekeland's variational principle, Separation theorem, Limiting (Mordukhovich) subdifferential.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C26,

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1 Introduction

Ekeland's variational principle is a deep assertion concerning the existence of an exact solution of a perturbed optimization problem in a neighborhood of an approximate solution of the original optimization problem under the assumption that the objective function of the original problem is bounded from below and lower semicontinuous (l.s.c.). Applications of Ekeland's variational principles can be seen in economics, control theory, game theory, nonsmooth analysis and many others. Here we establish several generalizations of Ekeland's variational principle for vector optimization problems with respect to variable ordering structures. For an introduction to variable ordering structures and for some recent results in this area we refer to [2, 3, 4, 8, 13, 19, 20, 22, 24, 25, 29, 35]. We use a concept of approximate solutions of vector optimization problems with respect to variable ordering structures which can be considered as a generalization of ε-efficiency by Loridan [23].

In this paper we impose two standing assumptions.

(A1) X is a Banach space, Ω is a closed set in X, Y is a real topological linear space, $f : X \to Y$ is a vector-valued function with $\operatorname{dom} f \neq \emptyset$, and $\varepsilon \geq 0$.

(A2) The set-valued mapping $C : Y \rightrightarrows Y$ satisfies $\mathbf{0} \in \operatorname{bd}(C(y))$ and $C(y)$ is closed for all $y \in Y$. The nonzero vector $k^0 \in Y \setminus \{\mathbf{0}\}$ satisfies $C(y) + [0, +\infty)k^0 \subset C(y)$ for all $y \in Y$.

Under assumptions (A1) and (A2) we consider the following vector optimization problem with respect to a variable ordering structure:

$$\varepsilon k^0\text{–Min} \quad f(x) \text{ subject to } \; x \in \Omega \text{ with respect to } \; C, \tag{VVOP}$$

where εk^0-minimality stands for two different kinds of optimal solution concepts: εk^0-minimal solutions in Definition 3.1 and εk^0-nondominated solutions in Definition 4.1. The set-valued mapping C is called a variable ordering structure (or ordering map).

The aim of this paper is to establish new variational principles of Ekeland-type for these two kinds of solutions by using a nonlinear scalarization technique and derive from them necessary conditions for approximate solutions of (VVOP).

With our new variants of Ekeland's type variational principles we improve and extend results recently shown in the literature (see [3, 30]) in several directions: We derive the results for εk^0-minimal solutions as well as for εk^0-nondominated solutions of (VVOP). Furthermore, in the variational principle in [30] the existence of an element belonging to the set of weakly εk^0-minimal solutions of the original problem that is a weakly minimal solution of a perturbed optimization problem is shown. We show a sharper result, namely that there exists an εk^0-minimal solution of the original problem that is a minimal solution of a perturbed vector optimization problem with variable ordering structure. Moreover, we study the solid case and additionally, the important nonsolid case.

In contrast to vector optimization with fixed ordering structure, i.e. ordering cone, where both minimality and nondomination are identical, an εk^0-nondominated solution of (VVOP) might not be εk^0-minimal to (VVOP) in the case of vector optimization problems with variable ordering structure; see, e.g. [31, 32]. Generalizations of Ekeland's variational principle for vector optimization with fixed ordering structure have been extensively studied by many authors in the literature, see, e.g. [34] and references therein. Our technique is based on the nonlinear scalarization technique used in [33] for vector optimization problems with fixed ordering structure. In Section 3, we recall the definition of εk^0-minimal solutions of (VVOP) and derive corresponding variational principles for both solid and nonsolid cases. Section 4 is devoted to results related to εk^0-nondominated solutions of (VVOP). In the last Section 5, we derive necessary conditions for approximate solutions of (VVOP) in terms of subdifferential of functions and normal cones to sets by Mordukhovich [26].

2 Preliminaries

Let Y be a real linear topological space and C be a nonempty set in Y. The notations $\mathrm{int}(C)$, $\mathrm{cl}(C)$, and $\mathrm{bd}(C)$ stand for the topological interior, the topological closure, and the topological boundary of the set C, respectively. For a nonconvex set C, the convex hull of C is denoted by $\mathrm{conv}(C)$. The set C is said to be *solid* iff $\mathrm{int}(C) \neq \emptyset$, *proper* iff $C \neq \emptyset$ and $C \neq Y$, *pointed* iff $C \cap (-C) \subset \{\mathbf{0}\}$, and *a cone* iff $\lambda c \in C$ for all $c \in C$ and $\lambda \geq 0$. See [13, 17, 18, 21] for basic definitions and concepts of vector optimization, and [16, 28, 33] for some scalarization methods to convert (convex and nonconvex vector) optimization problems with fixed ordering structure/ordering cone and important properties of these methods.

Let us recall a powerful nonlinear scalarization tool from [16] by Tammer and Weidner; cf. [17] which is used in the sequel.

Let A be a nonempty subset of Y and $k \neq \mathbf{0}$ be an element of Y. The functional $\varphi_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - A\} \tag{1}$$

is called a *nonlinear (separating) scalarization* function (with respect to the set A and the direction k). The following lemmas provide several important properties of $\varphi_{A,k}$.

Lemma 2.1. *([17, Theorem 2.3.1]) Let Y be a real topological linear space, A be a closed proper set in Y, and $k \in Y \setminus \{\mathbf{0}\}$ be a nonzero vector; namely, a direction of Y. Assume that the pair (A, k) satisfies the following condition*

$$A + [0, +\infty) \cdot k \subset A. \tag{2}$$

Then the following hold:

(a) *The functional* $\varphi_{A,k}$ *is l.s.c. over its domain* $\operatorname{dom}\varphi_{A,k} = \mathbb{R}k - A$. *Moreover, its* t*-level set is given by*

$$\{y \in Y \mid \varphi_{A,k}(y) \leq t\} = tk - A,\ \forall\, t \in \mathbb{R} \tag{3}$$

and the transformation of $\varphi_{A,k}$ *along the direction* k *is calculated by*

$$\varphi_{A,k}(y + tk) = \varphi_{A,k}(y) + t,\ \forall\, y \in Y,\ \forall\, t \in \mathbb{R}. \tag{4}$$

(b) $\varphi_{A,k}$ *is convex if and only if the set* A *is convex, and* $\varphi_{A,k}$ *is positively homogeneous, i.e.* $\varphi_{A,k}(ty) = t\varphi_{A,k}(y)$ *for all* $t \geq 0$ *and* $y \in Y$, *if and only if* A *is a cone.*

(c) $\varphi_{A,k}$ *is proper if and only if* A *does not contain lines parallel to* k, *i.e.*

$$\forall\, y \in Y,\ \exists\, t \in \mathbb{R}\ :\ y + tk \notin A. \tag{5}$$

(d) $\varphi_{A,k}$ *is finite-valued, i.e.* $\operatorname{dom}\varphi_{A,k} = Y$, *if and only if*

$$\mathbb{R}k - A = Y. \tag{6}$$

(e) *Given* $B \subset Y$. $\varphi_{A,k}$ *is* B*-monotone, i.e.* $\big[a \in b - B \implies \varphi_{A,k}(a) \leq \varphi_{A,k}(b)\big]$ *if and only if* $A + B \subset A$.

(f) $\varphi_{A,k}$ *is subadditive if and only if* $A + A \subset A$.

In many common situations, we need stronger properties of the functional $\varphi_{A,k}$ such as continuity or even Lipschitz continuity.

Lemma 2.2. *([17, Theorem 2.3.1]) Let* Y, A, B, k, *and* $\varphi_{A,k}$ *be as in Lemma 2.1. Suppose additionally* $\operatorname{int}(A) \neq \emptyset$ *and*

$$A + (0, +\infty) \cdot k \subset \operatorname{int}(A). \tag{7}$$

Then, one has:

(g) $\varphi_{A,k}$ *is continuous and*

$$\begin{aligned} \{y \in Y \mid \varphi_{A,k}(y) < t\} &= tk - \operatorname{int}(A), \quad \forall\ t \in \mathbb{R}, & (8)\\ \{y \in Y \mid \varphi_{A,k}(y) = t\} &= tk - \operatorname{bd}(A), \quad \forall\ t \in \mathbb{R}. & (9) \end{aligned}$$

(h) *If* $\varphi_{A,k}$ *is proper, then* $\varphi_{A,k}$ *is* B*-monotone* $\Leftrightarrow A + B \subset A \Leftrightarrow \operatorname{bd} A + B \subset A$. *Moreover, if* $\varphi_{A,k}$ *is finite-valued, then* $\varphi_{A,k}$ *is strictly* B*-monotone, i.e.* $\big[a \in b - B\ \wedge\ a \neq b \implies \varphi_{A,k}(a) < \varphi_{A,k}(b)\big] \Leftrightarrow A + (B \setminus \{\mathbf{0}\}) \subset \operatorname{int}(A) \Leftrightarrow \operatorname{bd}(A) + (B \setminus \{\mathbf{0}\}) \subset \operatorname{int}(A)$.

(i) *If* $\varphi_{A,k}$ *is proper, then* $\varphi_{A,k}$ *is subadditive* $\Leftrightarrow A + A \subset A \Leftrightarrow \operatorname{bd}(A) + \operatorname{bd}(A) \subset A$.

Remark 2.3. *Assume that $A \subset Y$ is a closed proper set and $k \in Y$ and there exists a cone $D \subset Y$ such that $k \in \operatorname{int}(D)$ and*

$$A + \operatorname{int}(D) \subset A. \tag{10}$$

Then (5), (6) and (7) hold (see [17, Proposition 2.3.4 (i)]).

Note that the existence of such a cone D is only sufficient; indeed, the set

$$A := \{(x, y) \in I\!R^2 \mid \ x \geq |y| \ \ and \ \ |y| \leq 1\}$$

and the element $k = (1, 0) \in \operatorname{int} A$ satisfy condition (2)*, but not conditions* (6) *and* (10)*. By Lemma 2.1 (d), the functional $\varphi_{A,k}$ is not finite-valued everywhere; we have $\varphi_{A,k}(1, 2) = +\infty$.*

Note also that the set

$$A := \{(x, y) \in I\!R^2 \mid \ x \geq \sqrt{|y|}\}$$

and the element $k = (1, 0)$ satisfy conditions (7) and (6), but not condition (10). By Lemma 2.1 (d), the functional $\varphi_{A,k}$ is finite-valued.

The next lemma provides, in addition to some properties in the previous two lemmas, several important ones broadly used in vector optimization.

Lemma 2.4. *([17, Corollary 2.3.5 and Theorem 2.3.6] and [9, Lemma 2.1]) Let Y be a real topological linear space, $C \subset Y$ be a proper, closed, convex and solid cone as an ordering cone of Y, and $k^0 \in \operatorname{int}(C)$ be a (positive) direction of Y. The functional φ_{C,k^0} defined in* (1) *is a finite-valued continuous, sublinear, and strictly-*int(C)*-monotone function. Moreover, if $\overline{y}$ is a weakly minimal element of a nonempty set S of Y with respect to C, i.e., $S \cap (\overline{y} - \operatorname{int}(C)) = \emptyset$, then one has*

$$\forall y \in S : \ \varphi_{C,k^0}(y - \overline{y}) \geq 0.$$

Finally, in this section let us recall the variational principle initiated by Ekeland in 1972. It is one of the most important results in nonlinear analysis. It ensures the existence of an exact solution of a perturbed problem in a neighborhood of an approximate solution of the original problem without convexity and compactness assumptions. It has been become a very useful tool to solve problems in optimization, optimal control theory, game theory, nonlinear equations and dynamical systems.

Theorem 2.5. *(cf. [15, Theorem 1]) Let X be a real Banach space, and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, not identical to $+\infty$, and bounded from below on a closed set Ω in X. Let $\varepsilon > 0$ be given, and an element $\overline{x} \in \Omega$ such that $\varphi(\overline{x}) \leq \inf_{x \in \Omega} \varphi(x) + \varepsilon$. Then there exists an element $x_\varepsilon \in \operatorname{dom} \varphi \cap \Omega$ such that*

(i) $\quad \varphi(x_\varepsilon) \leq \varphi(\overline{x}) \leq \inf_{x \in \Omega} \varphi(x) + \varepsilon,$

(ii) $\quad \|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon},$

(iii) $\varphi(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| > \varphi(x_\varepsilon),\ \forall x \in \Omega \setminus \{x_\varepsilon\}$.

Remark 2.6. (Strong form of Ekeland's variational principle). *Theorem 2.5 is known as the weak version of Ekeland's variational principle since we can find an element $x_\varepsilon \in \operatorname{dom}\varphi \cap \Omega$ which satisfies, in addition to* (i)–(iii)*, the following condition (see [15])*

(i′) $\varphi(x_\varepsilon) + \sqrt{\varepsilon}\,\|\overline{x} - x_\varepsilon\| \le \varphi(\overline{x})$.

Obviously, (i′) *implies* (i) *and* (ii)*.*

3 Variational principles for εk^0-minimal solutions of (VVOP)

In this section, we first recall the concept of εk^0-minimal solutions of (VVOP) (see [32]), then present a revised version of Ekeland's variational principle for vector-valued functions obtained in [30, Theorem 5.1] and [3, Corollary 3.1].

Consider problem (VVOP) with assumptions (A1) and (A2). We define a binary relation on Y with respect to the variable ordering structure C, denoted by $\le_1$, by

$$y^1 \le_1 y^2 \ \text{ iff } \ y^1 \in y^2 - C(y^2). \tag{11}$$

We can also define the weak ordering relation of $\le_1$, denoted by $<_1$, by replacing the set $C(y^2)$ in (11) by its interior, i.e.

$$y^1 <_1 y^2 \ \text{ iff } \ y^1 \in y^2 - \operatorname{int}(C(y^2)).$$

We are interested in the following concepts of solutions of (VVOP).

Definition 3.1. *Consider problem* (VVOP) *and $\varepsilon \ge 0$. Then:*

(a) *An element x_ε is said to be an εk^0-minimal solution of* (VVOP) *with respect to the variable ordering structure $C(\cdot)$ iff there is no element $y \in f(\Omega) := \cup_{x\in\Omega}\{f(x)\}$ such that $y + \varepsilon k^0 \le_1 f(x_\varepsilon)$, i.e.*

$$\big(f(x_\varepsilon) - \varepsilon k^0 - (C(f(x_\varepsilon)) \setminus \{\mathbf{0}\})\big) \cap f(\Omega) = \emptyset.$$

(b) *Suppose that $\operatorname{int} C(f(x_\varepsilon)) \ne \emptyset$. An element x_ε is said to be a weakly εk^0-minimal solution of* (VVOP) *with respect to $C(\cdot)$ iff*

$$\big(f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int}(C(f(x_\varepsilon)))\big) \cap f(\Omega) = \emptyset.$$

When $\varepsilon = 0$, it coincides with the usual definition of (weakly) minimal solutions; see, e.g. [10, 20]. We denote the sets of εk^0-minimal and weakly εk^0-minimal solutions by εk^0-Min(Ω, f, C) and εk^0-WMin(Ω, f, C), respectively. For $\varepsilon = 0$, we also write Min(Ω, f, C) and WMin(Ω, f, C).

In this section, we present several variants of Ekeland's variational principle for εk^0-minimal solutions of (VVOP) for both solid and nonsolid cases. In Theorems 3.4 and 3.8 a scalarization by means of a nonlinear function and Ekeland's variational principle for scalar optimization problems (see Theorem 2.5) are applied. So it is important to formulate the boundedness as well as lower semicontinuity assumptions in such a way that the scalarized function has corresponding properties as supposed in Theorem 2.5.

We will introduce different boundedness (see Definitions 3.2 and 3.9) as well as different lower semicontinuity (see Definitions 3.3 and 3.7) properties.

In the first result, we suppose boundedness and lower semicontinuity in the sense of the following definitions.

Definition 3.2. *Consider problem (VVOP). We say that the function f is bounded from below over Ω with respect to $\underline{y}$ and $\Theta \subset Y$ iff $f(\Omega) \subseteq \underline{y} + \Theta$.*

Definition 3.3. *Consider problem (VVOP). We say that the function f is (k^0, C)-lower semicontinuous over Ω iff all the sets*

$$M(y, t) := \big\{u \in \Omega \mid f(u) \in tk^0 - C(y)\big\}$$

are closed in X for all $y \in f(\Omega)$ and $t \in \mathbb{R}$.

The following theorem (cf. [30, Theorem 5.1]) gives the first generalization of the Ekeland's variational principle (Theorem 2.5) for εk^0-minimal solutions of (VVOP) provided that $f : X \to Y$ is bounded from below and that f is (k^0, C)-lower semicontinuous over Ω.

Theorem 3.4. *([30, Theorem 5.1]) Consider problem* (VVOP), let $\overline{x} \in \varepsilon k^0 -$ Min$\,(\Omega, f, C)$ and set $\overline{y} := f(\overline{x})$. *Impose in addition to (A1) and (A2) the following assumptions:*

(A3) *The images $C(y)$ are proper, closed, pointed, and solid sets satisfying $C(y) + (0, +\infty)k^0 \subset \operatorname{int}(C(y))$ and $C(y) + C(y) \subset C(y)$ for all $y \in f(\Omega)$.*

(A4) *There exists a cone-valued mapping $B : Y \rightrightarrows Y$ satisfying $k^0 \in \operatorname{int}(B(y))$ and $C(y) + B(y) \setminus \{\mathbf{0}\} \subset C(y)$ for all $y \in f(\Omega)$.*

(A5) f is bounded from below over Ω with respect to an element $\underline{y}$ and the set $\Theta := C(\overline{y})$ in the sense of Definition 3.2.

(A6) f is (k^0, C)-lower semicontinuous over Ω in the sense of Definition 3.3.

(A7) $C(y) \subset C(\overline{y})$ for all $y \in f(\Omega)$.

Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that

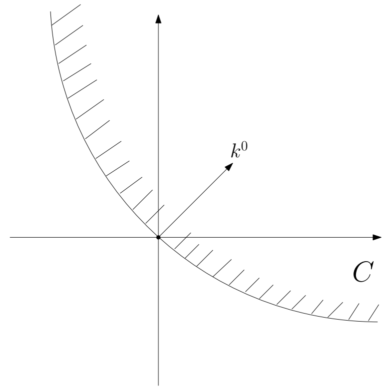

Figure 1: A convex set C satisfies assumptions (A1) and (A3).

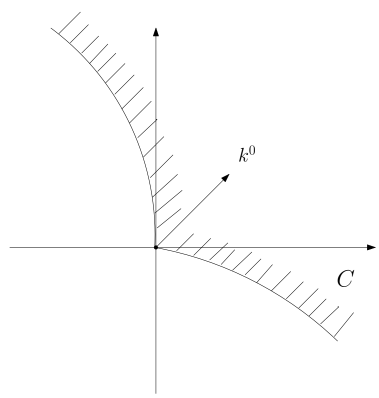

Figure 2: A nonconvex set C satisfies assumptions (A1) and (A3).

(i) $x_\varepsilon \in \varepsilon k^0 - \mathrm{WMin}(\Omega, f, B)$, *i.e.* $\big(f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(f(x_\varepsilon))\big) \cap f(\Omega) = \emptyset$.

(ii) $\|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}$.

(iii) $x_\varepsilon \in \mathrm{WMin}(\Omega, f_{\varepsilon k^0}, B)$, *where* $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0$.

Remark 3.5. *Figures 1 and 2 give examples for sets C where assumptions* (A1) *and* (A3) *in Theorem 3.4 are fulfilled (for $C(y) \equiv C$).*

Remark 3.6. *Observe that Theorem 3.4 can be improved much by using the nonlinear scalarization*

$$\varphi(y) := \varphi_{C(\overline{y}),k^0}(y) = \inf\left\{t \in \mathbb{R} \mid y \in tk^0 - C(\overline{y})\right\} \text{ with } \overline{y} := f(\overline{x})$$

and the properties of this scalarization functional established in Lemmas 2.1, 2.2, and 2.4. In this case, we suppose instead of assumption (A3), (A3*) *and instead of* (A4), (A4*) *as following:*

(A3*) *$C(\overline{y})$ is a proper, closed, pointed, and solid set satisfying* $\mathbb{R}k^0 - C(\overline{y}) = Y$.

(A4*) *There exists a cone-valued mapping* $B : Y \rightrightarrows Y$ *satisfying* $k^0 \in \operatorname{int}(B(\overline{y}))$, $C(\overline{y}) + B(\overline{y}) \setminus \{\mathbf{0}\} \subset C(\overline{y})$ *and* $B(f(x)) \subset B(\overline{y})$ *for all* $x \in \Omega$ *with* $\|x - \overline{x}\| \leq \sqrt{\varepsilon}$.

In addition, since $C(y)$ *for* $y \in f(\Omega) \setminus \{\overline{y}\}$ *do not play any role in the definition of the scalarization functional* φ*, we need to assume the closedness of all the sets*

$$M(\overline{y}, t) := \{u \in \Omega \mid f(u) \in t \cdot k^0 - C(\overline{y})\}$$

for all $t \in \mathbb{R}$*; in other words, we do not need the closedness of the sets* $M(y, t)$ *in Definition 3.3 when* $y \neq \overline{y}$ *and* $y \in f(\Omega)$.

This leads us to a refined variational principle for εk^0-minimal solutions of (VVOP) under a weaker assumption on lower semicontinuity.

Definition 3.7. *Consider problem (VVOP),* $\overline{x} \in \Omega \cap \operatorname{dom} f$*, and* $\overline{C} := C(\overline{y})$*. The function* f *is* $(k^0, \overline{C})$*-lower semicontinuous over* Ω *iff the sets*

$$M(t) := \{u \in \Omega \mid f(u) \in t \cdot k^0 - \overline{C}\}$$

are closed in X *for all* $t \in \mathbb{R}$.

Theorem 3.8. (Variational principle for εk^0-minimal solutions, solid case). *Consider problem* (VVOP)*, let* $\overline{x} \in \varepsilon k^0 - \operatorname{Min}(\Omega, f, C)$ *and set* $\overline{y} := f(\overline{x})$*. Assume that in addition to (A1) and (A2) the following conditions hold:*

(A3′) *The image* $\overline{C} := C(\overline{y})$ *is a proper, closed, pointed, and solid set satisfying* $\mathbb{R}k^0 - \overline{C} = Y$.

(A4′) *There exists a cone-valued mapping* $B : Y \rightrightarrows Y$ *such that* $k^0 \in \operatorname{int}(\overline{B})$ *with* $\overline{B} := B(\overline{y})$*,* $\overline{C} + \overline{B} \setminus \{\mathbf{0}\} \subset \operatorname{int}(\overline{C})$*, and* $B(f(x)) \subset \overline{B}$ *for all* $x \in \Omega$ *with* $\|x - \overline{x}\| \leq \sqrt{\varepsilon}$.

(A5) f *is bounded from below over* Ω *with respect to an element* $\underline{y} \in Y$ *and the cone* $\overline{C}$ *in the sense of Definition 3.2.*

(A6′) f *is* $(k^0, \overline{C})$*-lower semicontinuous over* Ω *in the sense of Definition 3.7.*

Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ *such that*

(i′) $x_\varepsilon \in \varepsilon k^0 - \operatorname{Min}(\Omega, f, B)$*, i.e.* $\big(f(x_\varepsilon) - \varepsilon k^0 - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\}\big) \cap f(\Omega) = \emptyset$,

(ii) $\|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}$,

(iii′) $x_\varepsilon \in \operatorname{Min}(\Omega, f_{\varepsilon k^0}, B)$*, where* $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0$.

Proof. Consider the nonlinear scalarization functional

$$\varphi(y) := \varphi_{\overline{C},k^0}(y) = \inf\left\{t \in \mathbb{R} \mid y \in tk^0 - \overline{C}\right\}.$$

By Lemmas 2.1 and 2.2, φ has the following properties under the assumption made in the theorem:

— $\varphi(y+tk^0) = \varphi(y)+t$ for all $y \in Y$ and for all $t \in \mathbb{R}$ due to (A3′) by Lemma 2.1 (a).

— φ is continuous due to (A3′) by Lemma 2.2 (g).

— φ is strictly $\overline{B}$-monotone (and thus $\overline{B}$-monotone) in the sense that

$$y_2 - y_1 \in \overline{B} \setminus \{\mathbf{0}\} \implies \varphi(y_1) < \varphi(y_2)$$

due to (A3′) and (A4′) by Lemma 2.1 (d) and by Lemma 2.2 (h).

Similar to Lemma 2.4, we prove that $\overline{x}$ is an ε-minimal solution of some scalar optimization problem. To proceed, set $g(x) := f(x) - f(\overline{x})$ with $\operatorname{dom} g = \operatorname{dom} f$. Obviously, $g(\overline{x}) = \mathbf{0}$. We get from the εk^0-minimality of $\overline{x}$ to the function f with respect to the variable ordering structure $C(\cdot)$ in Definition 3.1 that

$$\begin{aligned}
& (f(x) - f(\overline{x}) + \varepsilon k^0) \notin -C(f(\overline{x})),\ \forall\, x \in \Omega \cap \operatorname{dom} f \text{ with } \ f(x) \neq f(\overline{x}) \\
\iff\ & (g(x) + \varepsilon k^0) \notin \mathbf{0} - \overline{C},\ \forall\, x \in \Omega \cap \operatorname{dom} g \text{ with } \ g(x) \neq 0 \\
\implies\ & \varphi(g(x) + \varepsilon k^0) = \varphi(g(x)) + \varepsilon \geq 0,\ \forall\, x \in \Omega \cap \operatorname{dom} g,
\end{aligned}$$

where the implication holds due to the strict $\overline{B}$-monotonicity of φ and $\varphi(\mathbf{0}) = 0$ which only holds because of (A2) and the pointedness of $\overline{C}$ in (A3′). This together with $\varphi(g(\overline{x})) = \varphi(\mathbf{0}) = 0$ yields

$$\inf_{x \in \Omega} \varphi(g(x)) + \varepsilon \geq \varphi(g(\overline{x})), \tag{12}$$

i.e. $\overline{x}$ is an ε-minimal solution of the composition function $\varphi \circ g : X \to \mathbb{R} \cup \{+\infty\}$ over Ω.

Observe that the validity of (A5) and (A6′) ensures the boundedness from below and the lower semicontinuity of the composition $\varphi \circ g$, respectively. Employing now the classical Ekeland's variational principle in Theorem 2.5 to the composition function $\varphi \circ g$ and its ε-minimal solution $\overline{x}$, we can find some $x_\varepsilon \in \Omega \cap \operatorname{dom} g = \Omega \cap \operatorname{dom} f$ such that

(a) $\varphi(g(x_\varepsilon)) \leq \varphi(g(\overline{x})) = 0$;

(b) $\|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}$;

(c) $\varphi(g(x)) + \sqrt{\varepsilon}\|x - \overline{x}\| > \varphi(g(x_\varepsilon)), \forall x \in \operatorname{dom} f \cap \Omega$ and $x \neq x_\varepsilon$.

Obviously, (ii) holds. Next, we will show that x_ε satisfies also the two major relations (i′) and (iii′) in the theorem. Arguing by contradiction, we assume that (i′) does not hold, i.e. x_ε is not an εk^0-minimal solution of (VVOP) with respect to the ordering structure $B(\cdot)$. By Definition 3.1, we get from $x_\varepsilon \notin \varepsilon k^0 - \mathrm{Min}\,(\Omega, f, B)$ the existence of $x \in \Omega$ such that

$$\begin{aligned} & f(x) \in f(x_\varepsilon) - \varepsilon k^0 - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \\ \iff\ & f(x) - f(\overline{x}) + \varepsilon k^0 \in (f(x_\varepsilon) - f(\overline{x})) - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \\ \iff\ & g(x) + \varepsilon k^0 \in g(x_\varepsilon) - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\} \overset{(\mathrm{A4}')}{\subset} g(x_\varepsilon) - \overline{B} \setminus \{\mathbf{0}\} \end{aligned}$$

By the strict $\overline{B}$-monotonicity of φ we get from the last inclusion that

$$\varphi(g(x_\varepsilon)) > \varphi(g(x) + \varepsilon k^0) = \varphi(g(x)) + \varepsilon \geq \inf_{u \in \Omega} \varphi(g(u)) + \varepsilon \geq \varphi(g(\overline{x}))$$

where the last estimate ($\geq 0 = \varphi(g(\overline{x}))$) holds due to (12). The latter contradicts (a). This contradiction ensures the validity of (i′) in the theorem.

To complete the proof, it remains to show the fulfillment of condition (iii′). Arguing by contradiction, we assume that x_ε is not a minimal solution of the perturbed function $f_{\varepsilon k^0} = f + \sqrt{\varepsilon}\| \cdot - x_\varepsilon\| k^0$ with respect to $B(\cdot)$, i.e. there is some $x \in \Omega \cap \mathrm{dom}\, f = \Omega \cap \mathrm{dom}\, g$ and $x \neq x_\varepsilon$ such that

$$\begin{aligned} & f(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0 \in f(x_\varepsilon) - B(f(x_\varepsilon)) \\ \iff\ & f(x) - f(\overline{x}) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0 \in f(x_\varepsilon) - f(\overline{x}) - B(f(x_\varepsilon)) \\ \iff\ & g(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0 \in g(x_\varepsilon) - B(f(x_\varepsilon)) \overset{(\mathrm{A4}')}{\subset} g(x_\varepsilon) - \overline{B} \\ \implies\ & \varphi\Big(g(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0\Big) = \varphi(g(x)) + \sqrt{\varepsilon}\|x - x_\varepsilon\| \leq \varphi(g(x_\varepsilon)) \end{aligned}$$

where the implication holds due to the (strict) $\overline{B}$-monotonicity of φ. The latter inequality contradicts (c). The contradiction justifies (iii′) and thus completes the proof of the theorem. □

In the proof of the next variational principle, we will use Theorem 3.4 by Bao and Mordukhovich [1] such that we adapt our assumptions concerning boundedness as well as lower semicontinuity to this theorem. Furthermore, we suppose in the next results that Y is a Banach space.

Definition 3.9. *Consider problem (VVOP). We say that $f : X \to Y$ is bounded from below over Ω with respect to $\Theta \subset Y$ iff there is a bounded set $M \subset Y$ such that $f(\Omega) \subseteq M + \Theta$.*

Remark 3.10. *Of course, in the case of Banach spaces X and Y, the boundedness in the sense of Definition 3.9 is weaker than the boundedness in the sense of Definition*

3.2. However, in Definition 3.9 the boundedness of the set M is supposed and we are dealing with Banach spaces. The boundedness in the sense of Definition 3.9 is used in [1] and called quasiboundedness there.

Note that in many Ekeland-type results in the literature; see, e.g. [1, 2, 3] and the references therein, the function f is assumed to be C-level-closed, known also as C-lower semicontinuous, where C is a fixed ordering cone of the ordered image space.

Definition 3.11. *Consider problem (VVOP), $\overline{x} \in \Omega \cap \operatorname{dom} f$, $\overline{y} := f(\overline{x})$ and $\overline{C} := C(\overline{y})$. The function f is said to be $\overline{C}$-lower semicontinuous over Ω iff the sets*

$$\operatorname{lev}(y; f) := \big\{x \in \Omega \mid f(x) \in y - \overline{C}\big\}$$

are closed in X for all $y \in Y$.

Obviously, if f is $\overline{C}$-lower semicontinuous over Ω in the sense of Definition 3.11, then it is $(k^0, \overline{C})$-lower semicontinuous over Ω in the sense of Definition 3.7.

The next result is another improved version of [30, Theorem 5.1] for the nonsolid case.

Theorem 3.12. (Variational principle for εk^0-minimal solutions, nonsolid case). *Consider problem* (VVOP)*, where both X and Y are Banach spaces. Let $\overline{x} \in \varepsilon k^0\text{-}\mathrm{Min}\,(\Omega, f, C)$. Set $\overline{y} := f(\overline{x})$ and $\overline{C} := C(\overline{y})$. Assume, in addition to the standing assumptions* (A1) *and* (A2)*, the following conditions:*

(A3″) *$\overline{C}$ is a proper, closed, convex and pointed cone.*

(A4″) *$C(f(x)) \subset \overline{C}$ for all $x \in \Omega$ with $\|x - \overline{x}\| \leq \sqrt{\varepsilon}$.*

(A5″) *f is bounded from below over Ω with respect to the cone $\overline{C}$ in the sense of Definition 3.9.*

(A6″) *f is $\overline{C}$-lower semicontinuous over Ω in the sense of Definition 3.11.*

Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that

(i) *$f(x_\varepsilon) \in f(\overline{x}) - C(f(\overline{x}))$, and thus $x_\varepsilon \in \varepsilon k^0\text{-}\mathrm{Min}\,(\Omega, f, C)$,*

(ii) *$\|\overline{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$,*

(iii) *$x_\varepsilon \in \mathrm{Min}(\Omega, f_{\varepsilon k^0}, C)$, where $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}\|x_\varepsilon - x\| k^0$.*

Proof. By the $\overline{C}$-lower semicontinuity of f over Ω in (A6″) and the continuity of the norm, the function $f(\cdot) + \sqrt{\varepsilon}\|\overline{x} - \cdot\| k^0$ is $\overline{C}$-lower semicontinuous over Ω and thus the $\overline{y}$-level-set of $f(\cdot) + \sqrt{\varepsilon}\|\overline{x} - \cdot\| k^0$ with respect to the ordering cone $\overline{C}$ denoted by

$$\Xi := \operatorname{lev}(\overline{y}; \overline{C}) = \big\{x \in \Omega \mid f(x) + \sqrt{\varepsilon}\|\overline{x} - x\| k^0 \in \overline{y} - \overline{C}\big\}$$

is a closed set in X. Observe that the restriction f_Ξ of f on Ξ with $\operatorname{dom} f_\Xi = \Xi$ satisfies all the assumptions of the vector version of Ekeland's variational principle in vector optimization with ordering cone/fixed ordering structure; see, e.g., [1, Theorem 3.4]. Observe also that $\overline{x}$ is an εk^0-minimal solution of f_Ξ with respect to the closed, convex and pointed cone $\overline{C}$, i.e.

$$f_\Xi(x) \notin f_\Xi(\overline{x}) - \varepsilon k^0 - \overline{C} \setminus \{\mathbf{0}\}, \ \forall\, x \in \Xi. \tag{13}$$

Employing [1, Theorem 3.4] to the function f_Ξ, its εk^0-minimal solution $\overline{x}$, the cone $\overline{C}$, k^0, ε, and $\lambda = \sqrt{\varepsilon}$, we can find some $x_\varepsilon \in \Xi$ with $\|\overline{x} - x_\varepsilon\| \le \sqrt{\varepsilon}$ such that

$$f(x) + \sqrt{\varepsilon}\|x_\varepsilon - x\|k^0 \notin f(x_\varepsilon) - \overline{C}, \ \forall\, x \in \Xi \setminus \{x_\varepsilon\}. \tag{14}$$

Obviously, (ii) is satisfied. (i) follows directly from $x_\varepsilon \in \Xi$ as follows:

$$\begin{aligned} x_\varepsilon \in \Xi \ &\Longleftrightarrow \ f(x_\varepsilon) + \sqrt{\varepsilon}\|\overline{x} - x_\varepsilon\|k^0 \in \overline{y} - \overline{C} &\quad (15)\\ &\Longleftrightarrow \ f(x_\varepsilon) \in f(\overline{x}) - \big(\sqrt{\varepsilon}\|\overline{x} - x_\varepsilon\|k^0 + \overline{C}\big) \\ &\overset{(A2)}{\Longrightarrow} \ f(x_\varepsilon) \in f(\overline{x}) - \overline{C} = f(\overline{x}) - C(f(\overline{x})). &\quad (16)\end{aligned}$$

Obviously, (16) verifies the first part of (i). To justify the second part of (i), we use (16), the εk^0-minimality of $\overline{x}$, the inclusion $C(f(x_\varepsilon)) \subset \overline{C}$ by assumption (A4″), and the convexity of the cone $\overline{C}$ in (A3″) ensuring that $\overline{C} + \overline{C} \setminus \{\mathbf{0}\} \subset \overline{C} \setminus \{\mathbf{0}\}$. Details below.

$$\begin{aligned} \overline{x} \in \varepsilon k^0 - \operatorname{Min}(\Omega, f, C) \ &\Longleftrightarrow \ \big(f(\overline{x}) - \varepsilon k^0 - (\overline{C} \setminus \{\mathbf{0}\})\big) \cap f(\Omega) = \emptyset \\ &\Longleftrightarrow \ \big(f(\overline{x}) - \overline{C} - \varepsilon k^0 - \overline{C} \setminus \{\mathbf{0}\}\big) \cap f(\Omega) = \emptyset \\ &\overset{(16)}{\Longrightarrow} \ \big(f(x_\varepsilon) - \varepsilon k^0 - C(f(x_\varepsilon)) \setminus \{\mathbf{0}\}\big) \cap f(\Omega) = \emptyset \\ &\Longleftrightarrow \ x_\varepsilon \in \varepsilon k^0 - \operatorname{Min}(\Omega, f, C). \end{aligned}$$

Finally, we will justify (iii) by contradiction. Assume that it does not hold, and then find some $x \in \Omega \cap \operatorname{dom} f$ with $x \neq x_\varepsilon$ such that $f(x) + \sqrt{\varepsilon}\|x_\varepsilon - x\|k^0 \in f(x_\varepsilon) - C(f(x_\varepsilon))$. By (A4″), we get

$$f(x) + \sqrt{\varepsilon}\|x_\varepsilon - x\|k^0 \in f(x_\varepsilon) - C(f(\overline{x})) = f(x_\varepsilon) - \overline{C}. \tag{17}$$

Using (14) this implies $x \notin \Xi$. Summing up the inclusion (17) and the one in (15) gives

$$f(x) + \sqrt{\varepsilon}\big(\|x_\varepsilon - x\| + \|\overline{x} - x_\varepsilon\|\big)k^0 \in f(\overline{x}) - \overline{C} - \overline{C} = f(\overline{x}) - \overline{C}, \tag{18}$$

where $\overline{C} + \overline{C} = \overline{C}$ holds due to the convexity of the cone $\overline{C}$ in (A3″). Since $\|x_\varepsilon - x\| + \|\overline{x} - x_\varepsilon\| - \|\overline{x} - x\| \ge 0$ by the triangle inequality of the norm, we will further manipulate (18) as follows:

$$\begin{aligned} & f(x)+\sqrt{\varepsilon}\|\overline{x}-x\|k^0 \\ \in\ & f(\overline{x})-\sqrt{\varepsilon}(\|x_\varepsilon-x\|+\|\overline{x}-x_\varepsilon\|-\|\overline{x}-x\|)k^0-\overline{C} \\ \subset\ & f(\overline{x})-\overline{C}, \end{aligned}$$

where the inclusion holds due to (A2). By the construction of Ξ, we have $x \in \Xi$ and arrive at a contradiction. This contradiction verifies the validity of (iii) and thus completes the proof of the theorem. □

Remark 3.13. *1. Theorem 3.8 improves much the earlier result in Theorem 3.4 due to the fact that* (A3)–(A7) *imply* (A3′)–(A6′).

2. In Theorem 3.12 we do not impose any nonempty interiority condition on the variable ordering structure under consideration in comparison to Condition (A3) in Theorem 3.4 and Condition (A4′) *in Theorem 3.8.*

3. In contrast to our approach, Bao at al. obtained in [3] a version of Ekeland's variational principle for vector-valued/set-valued mappings with variable ordering structures in which condition (A4″) is exchanged by the following condition

$$\text{(A}'''\text{)}\ \forall\, x_1, x_2 \in \Omega,\ f(x_1) \leq_1 f(x_2) \quad \Longrightarrow \quad C(f(x_1)) \leq_1 C(f(x_2)).$$

It is important to emphasize that the validity of (A‴) implies that the binary relation $\leq_1$ defined in (11) is transitive, and thus it is a preorder. In our present paper, it is not necessarily a preorder.

4 Variational principles for εk^0-nondominated solutions

In this section, we give an extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problem with variable ordering structure, where the εk^0-nondominatedness for solutions of (VVOP) is defined in Definition 4.1. It is important to emphasize that there is no difference between εk^0-nondominated and εk^0-minimal solutions in the case of fixed ordering structure. The reader can find many examples illustrating that this statement is, in general, not true in the case of variable ordering structure in [2, 10, 13, 32]. Let us begin this section with several definitions.

As usual, consider problem (VVOP) and assume that assumptions (A1) and (A2) are satisfied. In contrast to the binary relation defined in (11), we define another one, namely, the domination relation, on the space Y by

$$y^1 \leq_2 y^2 \ \text{ iff } \ \ y^2 \in y^1 + C(y^1). \tag{19}$$

When $C(y) \equiv \overline{C}$ for some fixed set $\overline{C}$ in Y, the two binary relations defined in (11) and (19) are identical.

We now define approximate nondominated solutions of vector optimization problems with respect to a variable ordering structure $C(\cdot)$; see [31, 32] for more details and properties of approximate optimal solutions to problem (VVOP). By using the domination binary relation $\leq_2$ in (19) instead of the ordering relation $\leq_1$ in (11) in Definition 3.1 we have the corresponding εk^0-nondominated solutions of (VVOP).

Definition 4.1. *Consider problem* (VVOP) *and* $\varepsilon \geq 0$. *Then:*

(a) *An element* $x_\varepsilon \in \Omega$ *is said to be an* εk^0*-nondominated solution of* (VVOP) *with respect to the variable ordering structure* $C(\cdot)$ *iff there is no element* $y \in f(\Omega)$ *such that* $y \leq_2 f(x_\varepsilon) - \varepsilon k^0$, *i.e.*

$$\forall x \in \Omega, \quad f(x_\varepsilon) - \varepsilon k^0 \notin f(x) + C(f(x)) \setminus \{\mathbf{0}\}.$$

(b) *Suppose that* $\mathrm{int}(C(f(x))) \neq \emptyset$ *for all* $x \in \Omega$. *An element* $x_\varepsilon \in \Omega$ *is said to be a weakly* εk^0*-nondominated solution of* (VVOP) *with respect to* $C(\cdot)$ *iff*

$$\forall x \in \Omega, \quad f(x_\varepsilon) - \varepsilon k^0 \notin f(x) - \mathrm{int}(C(f(x))).$$

We denote the sets of all the εk^0-nondominated and weakly εk^0-nondominated solutions of (VVOP) by εk^0-ND(Ω, f, C) and εk^0-WND(Ω, f, C), respectively. If $\varepsilon = 0$, these nondominated solution concepts coincide with the usual definitions of nondominated solutions in [2, 10, 14, 35] and they are denoted by ND(Ω, f, C) and WND(Ω, f, C), respectively.

In order to prove the main theorem, we use the functional $\varphi_{\overline{y},C,k^0} : Y \mapsto \mathbb{R} \cup \{\pm\infty\}$ to some $\overline{y} \in Y$ and some $k^0 \in Y$ defined by

$$\varphi_{\overline{y},C,k^0}(y) = \inf\{t \in \mathbb{R} \mid \overline{y} + tk^0 - y \in C(y)\} \text{ for all } \; y \in Y. \tag{20}$$

This functional was studied in [11]; see also [13] for characterizing nondominated elements with respect to a variable cone-valued ordering structure. A slight modification of the scalarization was studied already by Chen and Yang [6] and later also by Chen and colleagues [5, 7].

The following lemma characterizes (approximate) solutions with the help of the above functional. Part (c) was already proven under slightly stronger assumptions on C and for a similar but different scalarization functional in [28, Theorem 4.3 and 4.4]. For cone-valued maps, part (a) and (b) of the following lemma was already proven in [13, Theorem 5.11]. Of course, (b) also follows from (a) and (c). Note that in [30] functions θ_z are considered, so for each z we have a different functional. Then a result of type (c) is proven which compares values of the scalarization functional for each element $z \in \Omega$ but for each z an individual functional θ_z is taken. Here, we have one functional φ for all $z \in \Omega$.

Lemma 4.2. *Let* (A1) *hold and let* $C : Y \rightrightarrows Y$ *be a set-valued map where* $C(y)$ *is closed for each* $y \in Y$ *satisfying the following two conditions for some* $k^0 \in Y \setminus \{\mathbf{0}\}$*:*

(C1) $(-\infty, 0)k^0 \cap C(y) = \emptyset$ *and* $\mathbf{0} \in \mathrm{bd}(C(y))$ *for all* $y \in f(\Omega)$*.*

(C2) $C(y) + (0, +\infty)k^0 \subset \mathrm{int}(C(y))$ *for all* $y \in f(\Omega)$*.*

We consider the functional $\varphi_{\overline{y},C,k^0} : Y \to \mathbb{R}$ *defined in (20) for some* $\overline{y} \in f(\Omega)$*. Then the following hold:*

(a) *One, under condition* (C1) *only, has*

$$\varphi_{\overline{y},C,k^0}(\overline{y}) = 0. \tag{21}$$

(b) *Let* $\overline{x} \in \Omega$ *and* $\overline{y} = f(\overline{x})$*. Then* $\overline{x} \in \mathrm{WND}(\Omega, f, C)$ *if and only if*

$$\inf_{y \in f(\Omega)} \varphi_{\overline{y},C,k^0}(y) = 0. \tag{22}$$

(c) *Let* $\varepsilon \geq 0$*,* $\overline{x} \in \Omega$*, and* $\overline{y} = f(\overline{x})$*. Then* $\overline{x} \in \varepsilon k^0$*-*$\mathrm{WND}(\Omega, f, C)$ *if and only if*

$$\inf_{y \in f(\Omega)} \varphi_{\overline{y},C,k^0}(y) \geq -\varepsilon. \tag{23}$$

Proof. We set $\varphi(y) := \varphi_{\overline{y},C,k^0}(y)$ for all $y \in f(\Omega)$. As (b) follows from (c) for $\varepsilon = 0$, we prove only (a) and (c).

(a) We have $\varphi(\overline{y}) = \inf\{t \in \mathbb{R} \mid tk^0 \in C(\overline{y})\}$. As $\mathbf{0} \in \mathrm{bd}(C(y))$ for all $y \in f(\Omega)$ and $(-\infty, 0)k^0 \cap C(y) = \emptyset$ we get $\varphi(\overline{y}) = 0$.

(c) Assume $\varphi(y) \geq -\varepsilon$ for all $y \in f(\Omega)$ but $\overline{x} \notin \varepsilon k^0$-$\mathrm{WND}(\Omega, f, C)$. Then there exists $y \in f(\Omega)$ with $\overline{y} - \varepsilon k^0 - y \in \mathrm{int}(C(y))$. Thus there is a scalar $t < 0$ such that
$$(\overline{y} - y) + (t - \varepsilon)\, k^0 \in C(y),$$
i.e. $\overline{y} + (t - \varepsilon)\, k^0 - y \in C(y)$ and hence $\varphi(y) \leq t - \varepsilon < -\varepsilon$, which is a contradiction.
Next, let $\overline{x} \in \varepsilon k^0$-$\mathrm{WND}(\Omega, f, C)$ but assume the existence of $t \in \mathbb{R}$, $t < -\varepsilon$ and $y \in f(\Omega)$ such that
$$\overline{y} + t\, k^0 - y \in C(y).$$
As $C(y) + (-t - \varepsilon)\, k^0 \in \mathrm{int}(C(y))$ by (C2), we have
$$\overline{y} - \varepsilon k^0 \in y + C(y) + (-t - \varepsilon)\, k^0 \subset y + \mathrm{int}(C(y))$$
in contradiction to the weak εk^0-nondominatedness of $\overline{x}$ to problem (VVOP).

□

Lemma 4.3. *Assume that* (A1) *holds and let* $C : Y \rightrightarrows Y$ *be a set-valued map where* $C(y)$ *is closed for each* $y \in Y$ *satisfying the following condition for some* $k^0 \in Y \setminus \{\mathbf{0}\}$*:*

$$\forall y \in f(\Omega) : \quad C(y) + (0, +\infty)k^0 \subset C(y) \setminus \{0\}. \tag{24}$$

Let $\varepsilon \geq 0$*,* $\overline{x} \in \Omega$*, and* $\overline{y} = f(\overline{x})$*. Then* $\overline{x} \in \varepsilon k^0\text{-ND}(\Omega, f, C)$ *implies*

$$\inf_{y \in f(\Omega)} \varphi_{\overline{y},C,k^0}(y) \geq -\varepsilon. \tag{25}$$

Proof. The assertion follows analogously to the proof of Lemma 4.2 (c) taking into account (24) instead of (C2). □

The next lemma provides a condition under which the composition function $\varphi_{\overline{y},C,k^0} \circ f$ is l.s.c. over Ω.

Lemma 4.4. *Consider problem* (VVOP)*,* $\overline{x} \in \Omega$*,* $\overline{y} = f(\overline{x})$*, and the scalarization functional* $\varphi_{\overline{y},C,k^0}$ *defined by (20). Assume that the ordering structure* $C : Y \rightrightarrows Y$ *satisfies condition* (C3)*:*

(C3) *C has a closed graph over $f(\Omega)$ in the sense that for every sequence of pairs $\{(y_n, v_n)\}$, if $y_n \in f(\Omega)$ and $v_n \in C(y_n)$ for all $n \in \mathbb{N}$ and $(y_n, v_n) \to (y_*, v_*)$ as $n \to +\infty$, then $y_* \in f(\Omega)$ and $v_* \in C(y_*)$.*

Then if f is a continuous function over Ω, the composition $\varphi_{\overline{y},C,k^0} \circ f$ is a lower semicontinuous function over Ω.

Proof. Assume that f is a continuous function over Ω. To prove the lower semicontinuity of $\varphi_{\overline{y},C,k^0} \circ f$ over Ω, it is sufficient to show that the set

$$A := \operatorname{lev}(t; \varphi_{\overline{y},C,k^0} \circ f) = \left\{x \in \Omega \mid \varphi_{\overline{y},C,k^0}(f(x)) \leq t\right\}$$

is closed in X for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ arbitrarily and take any sequence $\{x_n\}$ in A and thus in Ω such that $x_n \to x_*$ as $n \to +\infty$. By the description of A, we have $\varphi_{\overline{y},C,k^0}(f(x_n)) \leq t$ and thus

$$\overline{y} + tk^0 - f(x_n) \in C(f(x_n)).$$

Since f is continuous over Ω, the sequence of pairs $(y_n, v_n) \in \operatorname{gph} C$ with $y_n := f(x_n)$ and $v_n := \overline{y} + tk^0 - f(x_n)$ converges to $(f(x_*), \overline{y} + tk^0 - f(x_*))$. By (C3), we have

$$\overline{y} + tk^0 - f(x_*) \in C(f(x_*))$$

and thus $\varphi_{\overline{y},C,k^0}(f(x_*)) \leq t$ by the definition of $\varphi_{\overline{y},C,k^0}$ in (20). The last inequality justifies $x_* \in A$ and thus the closedness of the set A. The proof is complete. □

Remark 4.5. *In Lemma 4.4 is it required that C has a closed graph over $f(\Omega)$ which is related to the upper semicontinuity of C. In [12] is was shown that any cone-valued map C mapping in a reflexive Banach space with $C(y^0)$ a closed convex cone for $y^0 \in Y$ is upper semicontinuous at y^0 if and only if there is a neighborhood U of y^0 such that $C(y) \subset C(y^0)$ for all $y \in U$. Upper semicontinuous maps with closed images have a closed graph. The converse holds true if the images are compact.*

Lemma 4.6. *Consider problem* (VVOP) *and the scalarization functional $\varphi_{\overline{y},C,k^0}$ defined by (20). Assume that the assumptions* (A1) *and* (A2) *hold.*

Suppose that the function f enjoys the following boundedness condition:

(A5) *f is bounded from below over Ω with respect to $\underline{y} \in Y$ and the set $\Theta = C(\underline{y})$ in the sense of Definition 3.2.*

Furthermore, suppose that the ordering structure $C : Y \rightrightarrows Y$ satisfies for $\underline{y}$ from assumption (A5):

(C4) *$C(y) + C(\underline{y}) \subset C(\underline{y})$ for all $y \in f(\Omega)$; which holds provided that $C(y) \subset C(\underline{y})$ for all $y \in f(\Omega)$ and $C(\underline{y})$ is a convex cone. Furthermore, suppose that there exists a cone D with $k^0 \in \operatorname{int}(D)$ and $C(\underline{y}) + \operatorname{int}(D) \subset C(\underline{y})$.*

Then the functional $\varphi_{\overline{y},C,k^0} \circ f$ is bounded from below over Ω.

Proof. Consider the element $\underline{y}$ given by assumption (A5). Taking into account assumption (C4) and Remark 2.3 with $A = C(\underline{y}) - \overline{y}$, there exists $\underline{t} \in \mathbb{R}$ such that

$$\overline{y} + \underline{t}k^0 - \underline{y} \notin C(\underline{y}). \tag{26}$$

Assume now that f is bounded from below over Ω by $\underline{y}$ with respect to $C(\underline{y})$, but $\varphi_{\overline{y},C,k^0} \circ f$ is not bounded from below over Ω. The former ensures that $-\underline{y} \in -f(x) + C(\underline{y})$. The latter allows us to find some $x \in \Omega$ such that $\varphi_{\overline{y},C,k^0}(f(x)) < \underline{t}$. By (20) and (A2), we have

$$\overline{y} + \underline{t}k^0 - f(x) \in C(f(x)).$$

Combining the last two inclusions while taking into account (C4), we have

$$\overline{y} + \underline{t}k^0 - \underline{y} \in C(f(x)) + C(\underline{y}) \subset C(\underline{y})$$

which contradicts (26). The contradiction clearly verifies the lower boundedness of $\varphi_{\overline{y},C,k^0}$ over Ω and completes the proof. □

We now are ready to present an extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structure.

Theorem 4.7. *Consider problem* (VVOP)*, let $\overline{x} \in \varepsilon k^0\text{-ND}(\Omega, f, C)$, and set $\overline{y} := f(\overline{x})$. Assume, in addition to the standing assumptions* (A1) *and* (A2) *that the following conditions hold:*

(A5) *f is bounded from below over Ω with respect to the element $\underline{y} \in Y$ and the set $\Theta = C(\underline{y})$ in the sense of Definition 3.2.*

(A6‴) *f is continuous over Ω.*

Furthermore, suppose that the conditions (C1)*, (24),* (C3) *and* (C4) *from Lemmata 4.2, 4.3, 4.4 and 4.6 hold.*

Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that

(i') $\varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\overline{x} - x_\varepsilon\| \leq \varphi_{\overline{y},C,k^0}(f(\overline{x}))$,

(ii) $\|\overline{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$,

(iii) *$x_\varepsilon \in \Omega$ is an exact solution of the scalar problem*

$$\min_{x\in\Omega} \quad \varphi_{\overline{y},C,k^0}(f(x)) + \sqrt{\varepsilon}\|x - x_\varepsilon\|.$$

Proof. Consider $\overline{x} \in \varepsilon k^0\text{-ND}(\Omega, f, C)$, $\overline{y} := f(\overline{x})$, and the functional $\varphi_{\overline{y},C,k^0}$ defined in (20). By Lemmata 4.2 and 4.3 we get under the imposed conditions (C1) and (24) that

$$0 = \varphi_{\overline{y},C,k^0}(\overline{y}) \leq \inf_{y\in f(\Omega)\varphi_{\overline{y},C,k^0}(y)+\varepsilon},$$

i.e., $\overline{y}$ is an ε-minimal solution of $\varphi_{\overline{y},C,k^0} \circ f$ over Ω. Under the assumptions made in the theorem, the functional $\varphi_{\overline{y},C,k^0} \circ f$ is lower semicontinuous and bounded from below on Ω because of Lemmas 4.4 and 4.6. This means that all the assumptions of Theorem 2.5 are fulfilled. Therefore, we get from Theorem 2.5 the existence of $x_\varepsilon \in \Omega$ such that

(i) $\varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) \leq \varphi_{\overline{y},C,k^0}(f(\overline{x})) \leq \inf_{x\in\Omega} \ \varphi_{\overline{y},C,k^0}(f(x)) + \varepsilon$.

(ii) $\|\overline{x} - x_\varepsilon\| \leq \sqrt{\varepsilon}$.

(iii) $\varphi_{\overline{y},C,k^0}(f(x)) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| > \varphi_{\overline{y},C,k^0}(f(x_\varepsilon))$ for all $x \in \Omega$ and $x \neq x_\varepsilon$.

(i′) follows from Remark 2.6. The proof is complete. □

5 Applications to necessary optimality conditions

In this section, we use the variational principles presented in the previous sections in order to derive necessary conditions for approximate solutions of problem (VVOP) based on subdifferential calculus by Mordukhovich [26].

Definition 5.1. *A Banach space is* Asplund *if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.*

The class of Asplund spaces is quite broad including every reflexive Banach space and every Banach space with a separable dual; in particular, c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < +\infty$ are Asplund spaces, but ℓ_1 and ℓ_∞ are not Asplund spaces.

Definition 5.2. *Let Ω be a subset of a Banach space X and let $\overline{x} \in \Omega$.*

(a) *The* Fréchet normal cone *of Ω at $\overline{x} \in \Omega$ is defined by*

$$\widehat{N}(\overline{x};\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq 0 \right\}, \tag{27}$$

where $x \xrightarrow{\Omega} \overline{x}$ means $x \to \overline{x}$ with $x \in \Omega$.

(b) *Assume that X is an Asplund space and Ω is locally closed around $\overline{x} \in \Omega$, i.e., there is a neighborhood U of $\overline{x}$ such that $\Omega \cap \operatorname{cl} U$ is a closed set. The* (basic, limiting, Mordukhovich) normal cone *of Ω at $\overline{x}$ is defined by*

$$\begin{aligned} N(\overline{x};\Omega) &:= \operatorname*{Lim\,sup}_{x \to \overline{x}} \widehat{N}(x;\Omega) \\ &= \left\{ x^* \in X^* \;\middle|\; \exists\, x_k \to \overline{x},\ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\Omega) \right\}, \end{aligned} \tag{28}$$

where Lim sup *stands for the sequential Painlevé-Kuratowski outer limit of Fréchet normal cones to Ω at x as x tends to $\overline{x}$.*

Note that, in contrast to (27), the basic normal cone (28) is often *nonconvex* enjoying nevertheless *full calculus*, and that both the cones (28) and (27) reduce to the normal cone of convex analysis when Ω is convex.

Definition 5.3. *Let X be an Asplund space and consider a functional $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ and a point $\overline{x} \in \operatorname{dom} \varphi$.*

(a) *The set*

$$\partial_M \varphi(\overline{x}) := \left\{ x^* \in X^* \mid (x^*, -1) \in N((\overline{x}, \varphi(\overline{x})); \operatorname{epi} \varphi) \right\}$$

is the (basic, limiting) subdifferential *of φ at $\overline{x}$, and its elements are basic subgradients of φ at this point.*

(b) *The set*

$$\partial^\infty \varphi(\overline{x}) := \left\{ x^* \in X^* \mid (x^*, 0) \in N((\overline{x}, \varphi(\overline{x})); \operatorname{epi} \varphi) \right\}$$

is the singular subdifferential of φ at $\overline{x}$, and its elements are singular subgradients of φ at this point.

If φ is locally Lipschitz at $\overline{x}$, then $\partial^\infty \varphi(\overline{x}) = \{\mathbf{0}\}$. If φ is strictly Lipschitz continuous at $\overline{x}$; in particular, it is $C^{1,1}$, then $\partial_M \varphi(\overline{x}) = \{\nabla \varphi(\overline{x})\}$.

Lemma 5.4. ([26, Theorem 3.36 and Corollary 3.43]) *Assume that X and Y are Asplund spaces.*

(a) *If $\varphi_1, \varphi_2 : X \to \mathbb{R}$ are proper functionals and there exists a neighborhood U of $\overline{x} \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$ such that φ_1 is Lipschitz and φ_2 is lower semicontinuous on U then*

$$\partial_M(\varphi_1 + \varphi_2)(\overline{x}) \subset \partial_M\varphi_1(\overline{x}) + \partial_M\varphi_2(\overline{x}).$$

(b) *If a function $f : X \to Y$ is strictly Lipschitz at $\overline{x}$ and a functional $\varphi : Y \to \mathbb{R}$ is finite and lower semicontinuous on some neighborhood of $\overline{y} := f(\overline{x})$, then*

$$\partial_M(\varphi \circ f)(\overline{x}) \subset \bigcup_{y^* \in \partial_M\varphi(f(\overline{x}))} \partial_M(y^* \circ f)(\overline{x})$$

provided that the pair of functions (φ, f) satisfies the qualification condition

$$\partial^\infty\varphi(\overline{y}) \cap \ker\partial_M\langle\cdot, f\rangle(\overline{x}) = \{\mathbf{0}\}, \tag{29}$$

where $\ker\partial_M\langle\cdot, f\rangle(\overline{x}) = \{y^* \in Y^* \mid \mathbf{0} \in \partial_M\langle y^*, f\rangle(\overline{x})\}$.

Theorem 5.5. *Consider problem* (VVOP), *let $\overline{x} \in \varepsilon k^0\text{-ND}(\Omega, f, C)$ and $\overline{y} := f(\overline{x})$. Assume that X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitz, Ω is a closed subset of X, $C : Y \rightrightarrows Y$ is a set-valued mapping. Suppose in addition to the standing assumptions* (A1)–(A2) *that all assumptions* (A5), (A6‴), (C1), *(24),* (C3) *and* (C4) *from Theorem 4.7 are fulfilled. Assume also that the pair of functions $\{\varphi_{\overline{y},C,k^0}, f\}$ satisfies the qualification condition (29) for all $x \in \Omega$ such that $\|x - \overline{x}\| \le \sqrt{\varepsilon}$. Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that*

(i′) $\varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) + \sqrt{\varepsilon}\,\|\overline{x} - x_\varepsilon\| \le \varphi_{\overline{y},C,k^0}(f(\overline{x}))$,

(ii) $\|\overline{x} - x_\varepsilon\| \le \sqrt{\varepsilon}$,

(iii) $v^* \in \partial_M\varphi_{\overline{y},C,k^0}(f(x_\varepsilon))$ *such that* $\mathbf{0} \in \partial_M(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \Omega) + \sqrt{\varepsilon}B_{X^*}$.

Proof. By Theorem 4.7, there exists $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that it satisfies (i′), (ii) and it is an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over Ω with

$$h(x) := (\varphi_{\overline{y},C,k^0} \circ f)(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\| \quad \text{for all } x \in X.$$

By [27, Proposition 5.1] we have

$$\mathbf{0} \in \partial_M h(x_\varepsilon) + N(x_\varepsilon; \Omega).$$

By Lemma 4.4, the composition $\varphi_{\overline{y},C,k^0} \circ f$ is lower-semicontinuous on a neighborhood of x_ε. Employing Lemma 5.4 (a) to the lower semicontinuous functional $\varphi_{\overline{y},C,k^0} \circ f$ and the Lipschitz continuous function $\|.\|$, we have

$$\partial_M h(x_\varepsilon) \;\; \subset \;\; \partial_M\big(\varphi_{\overline{y},C,k^0} \circ f\big)(x_\varepsilon) + \partial_M(\sqrt{\varepsilon}\,\|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

By Lemma 5.4 (b), we have

$$\partial_M(\varphi_{\overline{y},C,k^0} \circ f)(x_\varepsilon) \subset \bigcup \left\{ \partial_M(v^* \circ f)(x_\varepsilon) \mid v^* \in \partial_M \varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) \right\}.$$

Combining these three inclusions together while taking into account the subdifferential of the norm $\partial_M \|\cdot - x_\varepsilon\|(x_\varepsilon) = B_{X^*}$, we can find $v^* \in \partial_M \varphi_{\overline{y},C,k^0}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial_M(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \Omega) + \sqrt{\varepsilon} B_{X^*}.$$

The proof is complete. □

Corollary 5.6. *Consider problem* (VVOP)*,* $\overline{x} \in \mathrm{ND}(\Omega, f, C)$ *be a nondominated solution of problem* (VVOP)*, and* $\overline{y} := f(\overline{x})$*. Assume that* X *and* Y *are Asplund spaces and* $f : X \to Y$ *is strictly Lipschitz. Suppose in addition to the standing assumptions* (A1)–(A2) *that all assumptions* (A5)*,* (A6‴)*,* (C1)*, (24),* (C3) *and* (C4) *from Theorem 4.7 are fulfilled for some* k^0*. Assume also that the pair of functions* $\{\varphi_{\overline{y},C,k^0}, f\}$ *satisfies the qualification condition (29) at* $\overline{x}$*. Then, for any* $\lambda > 0$*, there is an element* $v^* \in \partial_M(\varphi_{\overline{y},C,k^0}(f(\overline{x}))$ *such that*

$$\mathbf{0} \in \partial_M(v^* \circ f)(\overline{x}) + N(\overline{x}; \Omega) + \lambda B_{X^*}. \tag{30}$$

Proof. Since $\overline{x} \in \mathrm{ND}(\Omega, f, C)$, i.e. $\overline{x}$ is a $0k^0$-nondominated solution of (VVOP), it is also εk^0-nondominated to (VVOP) with $\varepsilon = \lambda^2 > 0$ for all $\lambda > 0$ and a weak nondominated solution of (VVOP). By Theorem 5.5 the only point $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ which satisfies condition (i′)

$$\varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) + \lambda \|\overline{x} - x_\varepsilon\| \le \varphi_{\overline{y},C,k^0}(\overline{y}) \;\text{ with }\; \overline{y} = f(\overline{x})$$

is $\overline{x}$ since Lemma 4.2 (a)–(b) says $\varphi_{\overline{y},C,k^0}(\overline{y}) = 0$ and $\varphi_{\overline{y},C,k^0}(f(x_\varepsilon)) \ge 0$. Since $\overline{x}$ satisfies (iii), we can find $v^* \in \partial_M \varphi_{\overline{y},C,k^0}(f(\overline{x}))$ such that

$$\mathbf{0} \in \partial_M(v^* \circ f)(\overline{x}) + N(\overline{x}; \Omega) + \lambda B_{X^*}$$

clearly verifying (30). The proof is complete. □

Note that the necessary conditions for nondominated solutions of problem (VVOP) obtained in this section are different from those in [2, 14].

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