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The effect of finite rank perturbations on Jordan chains of linear operators

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Abstract

A general result on the structure and dimension of the root subspaces of a matrix or a linear operator under finite rank perturbations is proved: The increase of dimension from the n-th power of the kernel of the perturbed operator to the $(n+1)$-th power differs from the increase of dimension of the corresponding powers of the kernels of the unperturbed operator by at most the rank of the perturbation and this bound is sharp.

Keywords: Finite rank perturbation, Jordan chain, root subspace

MSC 2010: 47A55, 47A10, 15A18

1 Introduction

Perturbation theory for linear operators and their spectra is one of the main objectives in operator theory and functional analysis, with numerous applications in mathematics, physics and engineering sciences, here we mention only [4]. In many approaches compact perturbations and perturbations small in size are investigated, e.g. when stability properties of the index, nullity and deficiency of Fredholm and semi-Fredholm operators are analysed. A widely used and well-known fact on the effect of compact perturbations on eigenvalues is the following: If S and T are bounded operators in a Banach space, $K = S - T$ is compact and $\lambda \in \mathbb{C}$ is an isolated eigenvalue with finite algebraic multiplicity of S or $\lambda \notin \sigma(S)$ then λ is an eigenvalue with finite algebraic multiplicity of T or $\lambda \notin \sigma(T)$. It is clear that for an arbitrary compact perturbation K there exists no bound in λ on the dimensions of $\ker(T-\lambda)$ or $\ker(T-\lambda)^{n+1}/\ker(T-\lambda)^n$. The situation is different when the perturbation is not only compact but of finite rank. This is the case which is considered in the present note. It follows easily that the dimensions of $\ker(S-\lambda)$ and $\ker(T-\lambda)$ differ at most by k if the perturbation $K = S-T$ is an operator with $\operatorname{rank}(K) = k$ (see e.g. [5, Theorem 2.2] for the case of matrices).

Our main objective is to explore such connections between kernels of consecutive higher powers of $S-\lambda$ and $T-\lambda$, and to prove the following general result on the structure and dimensions of the root subspaces under finite rank perturbations: Given a linear operator S acting on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$), consider the space $\ker(S-\lambda)^{n+1}/\ker(S-\lambda)^n$, where $\lambda \in \sigma(S)$. Its dimension coincides with the number of linearly independent Jordan chains of S at λ of length at least $n+1$. It then turns out that the change of the number of these Jordan chains of S at λ under a rank k perturbation can be bounded by k,

$$\left|\dim\left(\frac{\ker(S-\lambda)^{n+1}}{\ker(S-\lambda)^n}\right) - \dim\left(\frac{\ker(T-\lambda)^{n+1}}{\ker(T-\lambda)^n}\right)\right| \le k, \tag{1.1}$$

where $K = S - T$ is an operator with $\operatorname{rank}(K) = k$ and this bound is sharp.

The most interesting case is when S has a rich structure of Jordan chains in the sense that the dimensions of $\ker(S-\lambda)$ and of $\ker(S-\lambda)^{n+1}/\ker(S-\lambda)^n$ are large compared with the rank k of the perturbation. Moreover (1.1) is valid not only for bounded operators/matrices but also for unbounded operators and a slightly more general variant of finite rank perturbations, see Hypothesis 2.1 below.

We were not able to find this general fact in the mathematical literature. Even for matrices the statements in Theorem 2.2 are only known for the special case of so-called generic perturbations; cf. [1, 2, 3, 5, 6, 7, 8, 9].

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2 Main result

Let X be a vector space over $\mathbb{K}$, where $\mathbb{K}$ stands either for $\mathbb{R}$ or $\mathbb{C}$. Let S and T be linear operators in X defined on some linear subspaces $\operatorname{dom} S$ and $\operatorname{dom} T$ of X, respectively. We shall assume that the following holds:

Hypothesis 2.1. *There exists a linear subspace M contained in* $\operatorname{dom} S \cap \operatorname{dom} T$ *such that the restrictions $S \upharpoonright M$ and $T \upharpoonright M$ coincide on M and*

$$\max\bigl\{\dim(\operatorname{dom} S/M), \dim(\operatorname{dom} T/M)\bigr\} = k < \infty.$$

Three typical situations where the above hypothesis is satisfied are the following:

(i) X is a finite dimensional space, S and T are defined on X and the rank of $S-T$ is k. In this case, for a fixed basis of X, S and T are represented by matrices.

(ii) S and T are defined on the same subspace M of X and we have

$$\dim(\operatorname{ran}(S-T))=k.$$

(iii) The operators $S-\mu_0$ and $T-\mu_0$ are bijective for some $\mu_0 \in \mathbb{K}$ and

$$\dim\bigl(\operatorname{ran}\left((S-\mu_0)^{-1}-(T-\mu_0)^{-1}\right)\bigr)=k.$$

A *Jordan chain of S at $\lambda \in \mathbb{K}$ of length n* is a finite ordered set of non-zero vectors $\{x_0,\dots,x_{n-1}\}$ of X such that $(S-\lambda)x_0=0$ and $(S-\lambda)x_i=x_{i-1}$, $i=1,\dots,n-1$. The elements of a Jordan chain are linearly independent. The first $n-1$ elements of a Jordan chain of length n form a Jordan chain of length $n-1$. Two Jordan chains $\{x_0,\dots,x_n\}$ and $\{y_0,\dots,y_m\}$ are called *linearly independent* if the vectors $x_0,\dots,x_n,y_0,\dots,y_m$ are linearly independent. Furthermore, we say that S has k Jordan chains at λ of length n if there exist k linearly independent Jordan chains of length n. The *root subspace $\mathcal{L}_\lambda(S)$ of S at λ* is the collection of all Jordan chains of S at λ, $\mathcal{L}_\lambda(S)=\cup_{j=1}^{\infty}\ker(S-\lambda)^j$.

The following theorem is the main result of this note. We postpone its proof to Section 4.

Theorem 2.2. *Let S and T be linear operators in X satisfying Hypothesis 2.1. Then, the following holds for every $\lambda \in \mathbb{K}$:*

(i) *If $\ker(S-\lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(T-\lambda)^n$ and*

$$|\dim\ker(S-\lambda)^n-\dim\ker(T-\lambda)^n|\leq k\,n. \tag{2.1}$$

(ii) *If $\ker(S-\lambda)^{n+1}/\ker(S-\lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(T-\lambda)^{n+1}/\ker(T-\lambda)^n$ and*

$$\left|\dim\left(\frac{\ker(S-\lambda)^{n+1}}{\ker(S-\lambda)^n}\right)-\dim\left(\frac{\ker(T-\lambda)^{n+1}}{\ker(T-\lambda)^n}\right)\right|\leq k. \tag{2.2}$$

In the following corollary the bounds in Theorem 2.2 are considered in the context of the dimensions of the root subspaces.

Corollary 2.3. *Let S and T be linear operators in X which satisfy Hypothesis 2.1. Assume that the root subspace $\mathcal{L}_\lambda(S)$ of S at $\lambda \in \mathbb{K}$ is finite dimensional. Then, the following holds:*

(i) *If the maximal length of Jordan chains of S at λ is bounded by p then*

$$|\dim\mathcal{L}_\lambda(S)-\dim\ker(T-\lambda)^p|\leq k\,p.$$

(ii) *If the maximal lengths of Jordan chains of S at λ and Jordan chains of T at λ are bounded by p and q, respectively, then*

$$\bigl|\dim\mathcal{L}_\lambda(S)-\dim\mathcal{L}_\lambda(T)\bigr|\leq k\,\max\{p,q\}.$$

Proof. In (i) we have $\mathcal{L}_\lambda(S) = \ker(S-\lambda)^p$. In (ii) we have, in addition, $\mathcal{L}_\lambda(T) = \ker(T-\lambda)^q$. Then (i) and (ii) follow from (2.1). $\square$

The estimates in Theorem 2.2 are sharp in the following sense.

Example 2.4. *In $\mathbb{K}^m$ consider a fixed basis $\{e_1, \ldots, e_m\}$ and let with respect to this basis the linear operators A_1 and B_1 be given via their $m \times m$ matrix-representation*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \textit{and} \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then A_1 and B_1 satisfy Hypothesis 2.1 with $k = 1$ and we have for $j \le m$

$$\ker A_1^j = \operatorname{span}\{e_1, \ldots, e_j\} \quad \textit{and} \quad \ker B_1^j = \{0\}.$$

Hence the assertions in Theorem 2.2 are sharp for the case $\lambda = 0$ and $k = 1$. In order to obtain sharpness for general $k \in \mathbb{N}$ consider the $(mk \times mk)$-matrices

$$A = A_1 \oplus \cdots \oplus A_1 \quad \textit{and} \quad B = B_1 \oplus \cdots \oplus B_1.$$

3 Auxiliary statements

In the following we collect some observations which will be used in the proofs.

Observation 3.1. *Let S and T be linear operators in X satisfying Hypothesis 2.1. If $\{x_0, \ldots, x_n\}$ is a Jordan chain of S at λ such that $x_k \in M$ for every $k = 0, \ldots, n$, then $\{x_0, \ldots, x_n\}$ is also a Jordan chain of T at λ. Indeed, notice that if $x_k \in M$ then $Sx_k = Tx_k$. Therefore, $(T-\lambda)x_0 = (S-\lambda)x_0 = 0$ and $(T-\lambda)x_k = (S-\lambda)x_k = x_{k-1}$ for every $k = 1, \ldots, n$.*

Observation 3.2. *Given an operator A acting on a vector space Y, the family of cosets $\{y_1 + \ker A, \ldots, y_m + \ker A\}$ is linearly independent in $Y/\ker A$ if and only if the family $\{Ay_1, \ldots, Ay_m\}$ is linearly independent in Y. This follows from the fact that*

$$A' : Y/\ker A \to Y, \qquad y + \ker A \mapsto Ay,$$

is a linear isomorphism between the vector spaces $Y/\ker A$ and $\operatorname{ran} A$. Considering the subspaces $\operatorname{span}\{y_1 + \ker A, \ldots, y_m + \ker A\}$ and $\operatorname{span}\{Ay_1, \ldots, Ay_m\}$, the assertion above follows immediately.

Notice that it suffices to prove Theorem 2.2 for $\lambda = 0$, otherwise replace S and T by $S - \lambda$ and $T - \lambda$. In the following lemma we discuss this situation in the case $k = 1$.

Lemma 3.3. *Let S and T be linear operators in X satisfying Hypothesis 2.1 with $k = 1$. Then the following holds:*

(i) *If $\ker S^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker T^n$ and*

$$|\dim \ker S^n - \dim \ker T^n| \leq n. \tag{3.1}$$

(ii) *If $\ker S^{n+1} / \ker S^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker T^{n+1} / \ker T^n$ and*

$$\left|\dim\left(\ker S^{n+1} / \ker S^n\right) - \dim\left(\ker T^{n+1} / \ker T^n\right)\right| \leq 1. \tag{3.2}$$

Proof. We show (i) for the case $n = 1$, i.e.

$$|\dim \ker S - \dim \ker T| \leq 1. \tag{3.3}$$

Assume that $\ker S$ is finite dimensional and $\dim \ker T > \dim \ker S + 1$. Then there exist $m := \dim \ker S + 2$ linearly independent vectors $\{x_1, \ldots, x_m\}$ in $\ker T$. If $x_j \in M$ then $Sx_j = Tx_j$. So, if $x_j \in M$ for all $j = 1, \ldots, m$ then $\{x_1, \ldots, x_m\} \subseteq \ker S$, a contradiction.

Hence, there exists $1 \leq k_0 \leq m$ such that $x_{k_0} \in \ker T \setminus M$. After reordering we can assume that $k_0 = m$. As $\dim(\operatorname{dom} T/M) \leq 1$ it is easy to see that there exist $\alpha_k \in \mathbb{K}$ such that

$$z_k := x_k - \alpha_k x_m \in M, \qquad k = 1, \ldots, m-1.$$

Thus $Sz_k = Tz_k = 0$ for $k = 1, \ldots, m-1$, and we conclude that $\{z_1, \ldots, z_{m-1}\}$ is a linearly independent set in $\ker S$; a contradiction. Therefore, $\dim \ker T \leq \dim \ker S + 1$ and, in particular, $\ker T$ is finite dimensional. By interchanging S and T we also obtain $\dim \ker S - 1 \leq \dim \ker T$ and hence (3.3) follows.

In the following we prove (ii). Let $n \in \mathbb{N}$, $n \geq 1$, such that $\ker S^{n+1} / \ker S^n$ is finite dimensional and set

$$m := \dim(\ker S^{n+1} / \ker S^n) + 2. \tag{3.4}$$

Assume that the set $\{x_{1,n} + \ker T^n, \ldots, x_{m,n} + \ker T^n\}$ is linearly independent in $\ker T^{n+1} / \ker T^n$. For $k = 1, \ldots, m$ construct the following Jordan chains of T at 0:

$$x_{k,0} := T^n x_{k,n}, \quad x_{k,1} := T^{n-1} x_{k,n}, \quad \ldots, \quad x_{k,n-1} := T x_{k,n}.$$

Then, $x_{k,0} \in \ker T$ for $k = 1, \ldots, m$ and, applying Observation 3.2 to T^n it follows that

$$\{x_{1,0}, \ldots, x_{m,0}\} \text{ is a linearly independent set in } \ker T. \tag{3.5}$$

Define the index set $\mathfrak{I}$ by

$$\mathfrak{I} := \big\{(k,j): \; x_{k,j} \notin M, 1 \le k \le m, 1 \le j \le n\big\}.$$

The set $\mathfrak{I}$ is non-empty. Otherwise $\{x_{k,0}, \dots x_{k,n}\} \subset M$ for every $1 \le k \le m$ and, by Observation 3.1, these m (linearly independent) Jordan chains of T at 0 of length $n+1$ are as well (linearly independent) Jordan chains of S at 0 of length $n+1$, a contradiction to (3.4). Set

$$h := \min\big\{j: \; (k,j) \in \mathfrak{I} \text{ for some } k \text{ with } 1 \le k \le m\big\}.$$

Without loss of generality, after a reordering of the indices, assume that $(m,h) \in \mathfrak{I}$, i.e. $x_{m,h} \notin M$. Then,

$$j < h \text{ implies } x_{k,j} \in M \text{ for all } k = 1, \dots, m. \tag{3.6}$$

In what follows we construct $m-1$ elements $z_1, \dots, z_{m-1}$ in $\ker S^{n+1}$ such that $\{z_1 + \ker S^n, \dots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$, which is a contradiction to (3.4). We consider three different cases.

Case I: $h = n$**.** Since $x_{m,n} \notin M$, there exist $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := x_{k,n} - \alpha_{k,n} x_{m,n} \in M \cap \ker T^{n+1} \quad \text{for } k = 1, \dots, m-1.$$

From (3.6) it follows that, for every $k = 1, \dots, m-1$, the Jordan chain $\{x_{k,0} - \alpha_{k,n} x_{m,0}, \dots, x_{k,n-1} - \alpha_{k,n} x_{m,n-1}, z_k\}$ of T at 0 is contained in M. Then, by Observation 3.1 these are also $m-1$ (linearly independent) Jordan chains of S at 0 of length n. In particular, the set $\{z_1 + \ker S^n, \dots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$.

Case II: $h = n-1$**.** Since $x_{m,n-1} \notin M$, there exist $\alpha_{k,n-1} \in \mathbb{K}$ such that

$$v_{k,n-1} := x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} \in M \cap \ker T^n \quad \text{for } k = 1, \dots, m-1.$$

Let $w_{k,n} := x_{k,n} - \alpha_{k,n-1} x_{m,n} \in \ker T^{n+1}$ for $k = 1, \dots, m-1$ and choose $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,n-1} \in M \cap \ker T^{n+1} \quad \text{for} \;\; k = 1, \dots, m-1.$$

Since $z_k \in M$ and $v_{k,n-1} \in M$, $k = 1, \dots, m-1$, we conclude from $T w_{k,n} = v_{k,n-1}$ together with (3.6) that

$$\begin{aligned}
S^{n+1} z_k &= S^n S z_k = S^n T z_k \\
&= S^n T (w_{k,n} - \alpha_{k,n} x_{m,n-1}) = S^n (v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1} T (v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1} T (x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= S^{n-1} (x_{k,n-2} - \alpha_{k,n-1} x_{m,n-2} - \alpha_{k,n} x_{m,n-3}) \\
&\;\;\vdots \\
&= S^2 (x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= S T (x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= S (x_{k,0} - \alpha_{k,n-1} x_{m,0}) = T (x_{k,0} - \alpha_{k,n-1} x_{m,0}) = 0,
\end{aligned}$$

and $S^n z_k = x_{k,0} - \alpha_{k,n-1} x_{m,0} \neq 0$ for all $k = 1, \ldots, m-1$. By (3.5) the set $\{x_{1,0} - \alpha_{1,n-1} x_{m,0}, \ldots, x_{m-1,0} - \alpha_{m-1,n-1} x_{m,0}\}$ is linearly independent. Then, by Observation 3.2 applied to S^n, $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is also linearly independent in $\ker S^{n+1} / \ker S^n$.

Case III: $0 \le h \le n-2$**.** In this case we construct, as in Case II, two sets of vectors

$$\big\{v_{k,j} \in M \cap \ker T^{j+1} : k = 1, \ldots, m-1, j = h, \ldots, n-1\big\}, \tag{3.7}$$

and

$$\big\{w_{k,j+1} \in \ker T^{j+2} : k = 1, \ldots, m-1, j = h, \ldots, n-1\big\}. \tag{3.8}$$

By assumption, $x_{m,h} \notin M$. We start the construction with $j = h$, that is, with the definition of the vectors $v_{k,h}$ and $w_{k,h+1}$ for $k = 1, \ldots, m-1$: There exist $\alpha_{k,h} \in \mathbb{K}$ such that

$$v_{k,h} := x_{k,h} - \alpha_{k,h} x_{m,h} \in M \cap \ker T^{h+1} \quad \text{for } k = 1, \ldots, m-1.$$

Using the same coefficients $\alpha_{k,h} \in \mathbb{K}$, let

$$w_{k,h+1} := x_{k,h+1} - \alpha_{k,h} x_{m,h+1} \in \ker T^{h+2} \quad \text{for} \quad k = 1, \ldots, m-1.$$

Notice that $T w_{k,h+1} = v_{k,h}$ for $k = 1, \ldots, m-1$. The vectors $v_{k,j}$ and $w_{k,j+1}$ for $k = 1, \ldots, m-1$ are defined inductively for $j = h+1, \ldots, n-1$, in the following way: Fix $j = h+1, \ldots, n-1$ and assume that we have constructed $v_{k,j-1} \in M \cap \ker T^j$ and $w_{k,j} \in \ker T^{j+1}$for $k = 1, \ldots, m-1$. Then there exist $\alpha_{k,j} \in \mathbb{K}$ such that

$$v_{k,j} := w_{k,j} - \alpha_{k,j} x_{m,h} \in M \cap \ker T^{j+1} \quad \text{for} \quad k = 1, \ldots, m-1.$$

Also, define

$$w_{k,j+1} := x_{k,j+1} - \sum_{i=0}^{j-h} \alpha_{k,h+i} x_{m,j-i+1} \in \ker T^{j+2} \quad \text{for} \quad k = 1, \ldots, m-1.$$

A straightforward computation shows $T w_{k,j+1} = v_{k,j}$ for $k = 1, \ldots, m-1$. We have constructed the sets in (3.7) and (3.8).

Finally, observe that there also exist $\alpha_{k,n} \in \mathbb{K}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,h} \in M \cap \ker T^{n+1} \quad \text{for} \quad k = 1, \ldots, m-1.$$

Hence,

$$\begin{aligned}
S z_k &= T z_k = T(w_{k,n} - \alpha_{k,n} x_{m,h}) = v_{k,n-1} - \alpha_{k,n} x_{m,h-1}, \\
S^2 z_k &= S(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= T(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= T(w_{k,n-1} - \alpha_{k,n-1} x_{m,h} - \alpha_{k,n} x_{m,h-1}) \\
&= v_{k,n-2} - \alpha_{k,n-1} x_{m,h-1} - \alpha_{k,n} x_{m,h-2},
\end{aligned}$$

and, in the same way, we show that

$$S^{n-h} z_k = \ v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i},$$

where $x_{m,l} = 0$ if $l < 0$. Also, observe that

$$\begin{aligned}
S^{n-h+1} z_k &= S(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= T(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= T(x_{k,h} - \alpha_{k,h} x_{m,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}, \\
S^{n-h+2} z_k &= S(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}) \\
&= T(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}) \\
&= x_{k,h-2} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-2}, \\
&\vdots \\
S^n z_k &= x_{k,0} - \alpha_{k,h} x_{m,0}, \qquad \text{and} \\
S^{n+1} z_k &= 0.
\end{aligned}$$

Furthermore, the set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$. In fact, by (3.5), the set $\{x_{1,0} - \alpha_{1,h} x_{m,0}, \ldots, x_{m-1,0} - \alpha_{m-1,h} x_{m,0}\}$ is linearly independent in $\ker S$. Then, applying Observation 3.2 to S^n, it follows that $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ is linearly independent in $\ker S^{n+1}/\ker S^n$.

Summing up, we have shown in Cases I-III above that there exists a linearly independent set $\{z_1 + \ker S^n, \ldots, z_{m-1} + \ker S^n\}$ in $\ker S^{n+1}/\ker S^n$, which contradicts (3.4). Therefore,

$$\dim(\ker T^{n+1}/\ker T^n) \leq \dim(\ker S^{n+1}/\ker S^n) + 1,$$

and, in particular, $\ker T^{n+1}/\ker T^n$ is finite dimensional. By interchanging S and T we obtain

$$\dim(\ker S^{n+1}/\ker S^n) - 1 \leq \dim(\ker T^{n+1}/\ker T^n),$$

and (3.2) follows. Finally, (3.1) is a consequence of (3.3) and repeated applications of (3.2). $\square$

Before proving Theorem 2.2 we will improve the upper bound in (ii) of Lemma 3.3 for a particular class of rank-one perturbations.

Assume that S is a linear operator in X and M is a linear subspace in $\operatorname{dom} S$ such that $\dim\bigl(\operatorname{dom} S/M\bigr) = k$. Then, there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} S) \setminus M$ such that

$$\operatorname{dom} S = M \dotplus \operatorname{span}\{x_1, \ldots, x_k\}$$

We define the restrictions

$$S_p := S \upharpoonright \left(M \dotplus \operatorname{span}\{x_1, \ldots, x_p\}\right), \qquad 1 \le p \le k.$$

Lemma 3.4. *Given $2 \le p \le k$, if $\ker S_p^{n+1}/\ker S_p^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker S_{p-1}^{n+1}/\ker S_{p-1}^n$ and*

$$\dim\left(\frac{\ker S_p^{n+1}}{\ker S_p^n}\right) - 1 \le \dim\left(\frac{\ker S_{p-1}^{n+1}}{\ker S_{p-1}^n}\right) \le \dim\left(\frac{\ker S_p^{n+1}}{\ker S_p^n}\right).$$

Proof. By Lemma 3.3 only the second inequality needs to be proved. Assume that $\dim\left(\ker S_p^{n+1}/\ker S_p^n\right) = i < \infty$ and that the set $\{z_1 + \ker S_{p-1}^n, \ldots, z_{i+1} + \ker S_{p-1}^n\}$ is linearly independent in $\ker S_{p-1}^{n+1}/\ker S_{p-1}^n$. Then, since $\ker S_{p-1}^{n+1} \subset \ker S_p^{n+1}$, there exist $\alpha_1, \ldots, \alpha_{i+1} \in \mathbb{K}$ (not all equal to zero) such that

$$z := \alpha_1 z_1 + \cdots + \alpha_{i+1} z_{i+1} \in \ker S_p^n.$$

Together with $z \in \operatorname{dom} S_{p-1}^{n+1} \subset \operatorname{dom} S_{p-1}^n$ we conclude $z \in \ker S_{p-1}^n$, a contradiction, and Lemma 3.4 is shown. $\square$

4 Proof of Theorem 2.2

We start the proof with some preparations. By assumption S and T satisfy Hypothesis 2.1. We discuss the case

$$\dim\bigl(\operatorname{dom} S/M\bigr) = k \quad \text{and} \quad \dim\bigl(\operatorname{dom} T/M\bigr) = l \le k.$$

Then there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} S) \setminus M$ and $y_1, \ldots, y_l \in (\operatorname{dom} T) \setminus M$ such that

$$\operatorname{dom} S = M \dotplus \operatorname{span}\{x_1, \ldots, x_k\} \quad \text{and} \quad \operatorname{dom} T = M \dotplus \operatorname{span}\{y_1, \ldots, y_l\}.$$

Also, we can assume that $\operatorname{span}\{x_1, \ldots, x_k\} \cap \operatorname{span}\{y_1, \ldots, y_l\} = \{0\}$ (otherwise M can be enlarged). Next, consider the restrictions

$$S_p := S \upharpoonright \left(M \dotplus \operatorname{span}\{x_1, \ldots, x_p\}\right), \qquad 1 \le p \le k,$$

and

$$T_q := T \upharpoonright \left(M \dotplus \operatorname{span}\{y_1, \ldots, y_q\}\right), \qquad 1 \le q \le l.$$

Clearly $S = S_k$ and $T = T_l$. As mentioned before, it is sufficient to prove Theorem 2.2 for $\lambda = 0$. Let $\ker S^{n+1}/\ker S^n$ be finite dimensional for some $n \in \mathbb{N}$, $n \ge 1$. Applying repeatedly Lemma 3.4 to $S = S_k, S_{k-1}, \ldots, S_2$, we see that $\ker S_1^{n+1}/\ker S_1^n$ is finite dimensional and

$$\dim\left(\frac{\ker S^{n+1}}{\ker S^n}\right) - (k-1) \le \dim\left(\frac{\ker S_1^{n+1}}{\ker S_1^n}\right) \le \dim\left(\frac{\ker S^{n+1}}{\ker S^n}\right). \tag{4.1}$$

The operators S_1 and T_1 satisfy Hypothesis 2.1 with $k = 1$. Hence, by Lemma 3.3, $\ker T_1^{n+1}/\ker T_1^n$ is finite dimensional and

$$\left|\dim\left(\ker S_1^{n+1}/\ker S_1^n\right) - \dim\left(\ker T_1^{n+1}/\ker T_1^n\right)\right| \le 1. \tag{4.2}$$

Similarly, repeated application of Lemma 3.4 to $T_2, T_3, \ldots, T_l = T$ shows that $\ker T^{n+1}/\ker T^n$ is finite dimensional and

$$\dim\left(\frac{\ker T^{n+1}}{\ker T^n}\right) - (l-1) \le \dim\left(\frac{\ker T_1^{n+1}}{\ker T_1^n}\right) \le \dim\left(\frac{\ker T^{n+1}}{\ker T^n}\right). \tag{4.3}$$

Since $l \le k$, notice that $-(k-1) \le -(l-1)$. Therefore with (4.1), (4.2) and (4.3)

$$\begin{aligned}
\dim&\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right)\\
&\ge \dim\left(\ker S_1^{n+1}/\ker S_1^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right)\\
&\ge \dim\left(\ker T_1^{n+1}/\ker T_1^n\right) - 1 - \dim\left(\ker T^{n+1}/\ker T^n\right)\\
&\ge -(l-1) - 1\\
&\ge -(k-1) - 1 = -k.
\end{aligned}$$

An analogous calculation for the upper bound shows

$$\dim\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right) \le k,$$

which yields

$$\left|\dim\left(\ker S^{n+1}/\ker S^n\right) - \dim\left(\ker T^{n+1}/\ker T^n\right)\right| \le k,$$

and assertion (ii) in Theorem 2.2 holds. Finally, assertion (i) in Theorem 2.2 follows from

$$|\dim\ker S - \dim\ker T| \le k,$$

which is shown in a similar way as in the proof of Lemma 3.3, and a repeated application of (2.2).

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