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NON-SEMIBOUNDED CLOSED SYMMETRIC FORMS ASSOCIATED WITH A GENERALIZED FRIEDRICHS EXTENSION

ANDREAS FLEIGE, SEPPO HASSI, HENK DE SNOO, AND HENRIK WINKLER

Abstract. The theory of closed sesquilinear forms in the non-semibounded situation exhibits some new features, as opposed to the semibounded situation. In particular, there can be more than one closed form associated with the generalized Friedrichs extension S_F of a non-semibounded symmetric operator S (if S_F exists). However, there is one unique form $\mathfrak{t}_F[\cdot,\cdot]$ satisfying Kato's second representation theorem and, in particular, $\operatorname{dom}\mathfrak{t}_F = \operatorname{dom}|S_F|^{1/2}$. In the present paper another closed form $\mathfrak{t}^F[\cdot,\cdot]$ is constructed which is also uniquely associated with S_F. The relation between these two forms is analyzed and it is shown that these two non-semibounded forms can indeed differ from each other. Some general criteria for their equality are established. The results induce solutions to some open problems concerning generalized Friedrichs extensions and complete some earlier results about them in the literature. The study is connected to the spectral functions of definitizable operators in Kreĭn spaces.

1. Introduction

The Friedrichs extension plays an essential role in the representation of closed semibounded sesquilinear forms. An analog of the Friedrichs extension for nonsemibounded forms has been proposed by A.G.R. McIntosh [19, 20, 21]. He introduced a notion of closed nonsemibounded sesquilinear forms, established analogs of Kato's first and second representation theorems, and formulated some open problems; see also [10, 11] for a more explicit framework. Another operator theoretic approach (via associated Q-functions) to such generalized Friedrichs extensions was developed in [12, 13, 14], where a connection with extension theory was established, solving some of McIntosh's open problems. This operator theoretic approach was augmented by a systematic study of associated sesquilinear forms via Kreĭn space methods in [7]. The present paper completes the last two approaches with solutions to some open problems going back to [12]; cf. [19].

For motivation first recall the classical semibounded setting. In this case the concepts of selfadjoint operators and of closed symmetric sesquilinear forms are equivalent. More precisely, the following identity in Kato's first representation theorem

$$\mathfrak{t}[u,v] = (Tu,v), \quad u \in \operatorname{dom} T, \quad v \in \operatorname{dom}\mathfrak{t}, \tag{1.1}$$

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establishes a one-to-one correspondence between all closed semibounded symmetric sesquilinear forms $\mathfrak{t}[\cdot,\cdot]$ and all semibounded selfadjoint operators T acting on a Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$; cf. [17, Theorem VI-2.7]. Furthermore, Kato's second representation theorem shows that the domain of a closed semibounded form $\mathfrak{t}[\cdot,\cdot]$ is given by

$$\operatorname{dom}\mathfrak{t} = \operatorname{dom}|T|^{\frac{1}{2}}, \tag{1.2}$$

where T is the selfadjoint operator associated with $\mathfrak{t}[\cdot,\cdot]$ by (1.1); cf. [17, Theorem VI-2.23]. If S is a closed densely defined symmetric and semibounded operator, then the classical Friedrichs extension is given by the selfadjoint operator $S_F = T$ associated with the closure $\mathfrak{t}[\cdot,\cdot]$ of the semibounded form

$$\mathfrak{s}[u,v] := (Su,\, v), \quad u,v \in \operatorname{dom}\mathfrak{s} := \operatorname{dom} S, \tag{1.3}$$

[17, VI-2]. The form domain $\operatorname{dom}\mathfrak{t}$ is sometimes called the "energy space" generated by S. It yields the characterization

$$S_F = \{\, \{f,g\} \in S^* : \, f \in \operatorname{dom}\mathfrak{t}\,\} \tag{1.4}$$

(using the notation of relations). Recall that a symmetric sesquilinear form $\mathfrak{t}[\cdot,\cdot]$, semibounded from below, is closed if and only if for some $\lambda \in \mathbb{R}$ the form

$$\mathfrak{t}[u,v]_\lambda := \mathfrak{t}[u,v] - \lambda(u,v), \quad u,v \in \operatorname{dom}\mathfrak{t}, \tag{1.5}$$

defines a Hilbert space $(\operatorname{dom}\mathfrak{t},\, \mathfrak{t}[\cdot,\cdot]_\lambda)$, which is continuously embedded in the underlying Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$; cf. [17, Theorem VI-1.11].

In the non-semibounded situation there is a general approach to representation theorems based on Kreĭn space theory, cf. [7]. However, the connection between forms and operators becomes more involved and requires a more delicate analysis. A form $\mathfrak{t}[\cdot,\cdot]$ is now said to be *closed* if the inner product space $(\operatorname{dom}\mathfrak{t},\, \mathfrak{t}[\cdot,\cdot]_\lambda)$ (see (1.5)) is a Kreĭn space. Whereas the first representation theorem remains true, the second representation theorem is not true in the new setting; cf. [5], [7]. There may be closed non-semibounded forms associated with a non-semibounded selfadjoint operator T by (1.1) which do not satisfy (1.2); cf. [7], [9]. A closed form $\mathfrak{t}[\cdot,\cdot]$ for which the identity (1.2) is satisfied is said to be *regular*. The identity (1.1) now defines a one-to-one correspondence between all regular closed forms $\mathfrak{t}[\cdot,\cdot]$ and all selfadjoint operators T with spectrum $\sigma(T) \neq \mathbb{R}$; cf. [7]. The present paper completes this theory and studies further analogies and differences with the classical semibounded theory.

Let S be a closed densely defined symmetric, in general nonsemibounded, operator with defect numbers $(1,1)$. Furthermore, assume that the form $\mathfrak{s}[\cdot,\cdot]$ in (1.3) is closable (now in the Kreĭn space setting) and that S has a generalized Friedrichs extension S_F. Then, in analogy with the classical situation $\mathfrak{s}[\cdot,\cdot]$ has a regular closure $\mathfrak{t}_F[\cdot,\cdot]$ (again in the Kreĭn space setting) which is uniquely defined and S_F is the associated operator by the first repesentation theorem. In fact, similar to (1.4), one has the characterization

$$S_F = \{\, \{f,g\} \in S^* : \, f \in \operatorname{dom}|S_F|^{\frac{1}{2}}\ (= \operatorname{dom}\mathfrak{t}_F)\,\}, \tag{1.6}$$

cf. [7, Theorem 7.2] (see Theorem 2.7). This paper presents a new construction of a closed form, denoted by $\mathfrak{t}^F[\cdot,\cdot]$, which is also uniquely defined and for which S_F is also the associated operator by the first representation theorem. Moreover, one has the characterization

$$S_F = \{ \{f, g\} \in S^* : f \in \operatorname{dom} \mathfrak{t}^F \}. \tag{1.7}$$

The construction of the form $\mathfrak{t}^F[\cdot,\cdot]$ is based on the operator theoretic approach to the generalized Friedrichs extension in [12], where $\operatorname{dom} S$ is completed with respect to a topology generated by a selfadjoint extension different from S_F; cf. [12].

The main open problem which arises from [12] is to describe the relation between the "energy space" $\operatorname{dom} \mathfrak{t}^F$ and the domain $\operatorname{dom} |S_F|^{\frac{1}{2}}$ appearing in (1.6) and (1.7); in particular, the question going back to [12] is when these spaces are equal. Using the present approach of closed nonsemibounded forms this problem is reduced to the regularity of $t^F[\cdot,\cdot]$. The regularity of a closed form can be characterized in terms of the regularity of the critical point ∞ of a certain definitizable operator; cf. [7]. Hence the present theory is connected to the study of a number of formally different problems appearing in the spectral theory of definitizable operators in a Kreĭn space, such as the similarity problem of a nonnegative operator in a Kreĭn space (cf. [7, 2]) or the Riesz basis property of eigenfunctions of S_F (cf. [5, Proposition 5], [9, Theorem 2.6], [6, Theorem 2.8]). The present theory has applications in indefinite Sturm-Liouville problems (cf. [1, 2, 3, 4, 5, 6, 7, 9, 18]); in particular, using the approach from [9], the above closed forms $\mathfrak{t}_F[\cdot,\cdot]$ and $\mathfrak{t}^F[\cdot,\cdot]$ may then be described more explicitly and an example shows that $\mathfrak{t}^F[\cdot,\cdot]$ need not be regular. The present paper shows that the situation, described in [9] for the indefinite Sturm-Liouville setting, also appears in general.

2. Basic facts on closed forms and generalized Friedrichs extensions

The general theory of closed non-semibounded sesquilinear forms can be found in [5], [7], and [9]. Here some basic facts from this theory are recalled for the construction of the regular closed form associated with the generalized Friedrichs extension (if it exists).

2.1. Closed symmetric sesquilinear forms and representation theorems.

Let $\mathfrak{t}[\cdot,\cdot]$ be a densely defined symmetric sesquilinear form in a Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$. Assume for a moment that $\mathfrak{t}[\cdot,\cdot]$ is semibounded from below, i.e., the inner product

$$\mathfrak{t}[u,v]_\lambda := \mathfrak{t}[u,v] - \lambda(u,v), \quad u,v \in \operatorname{dom} \mathfrak{t}, \tag{2.1}$$

is nonnegative for some $\lambda \in \mathbb{R}$. Then the form $\mathfrak{t}[\cdot,\cdot]$ is closed in the classical sense (cf. [17]) if and only if for some $\lambda \in \mathbb{R}$ the form domain $\operatorname{dom} \mathfrak{t}$ provided with the inner product $\mathfrak{t}[\cdot,\cdot]_\lambda$ in (2.1) is a Hilbert space which is continuously embedded in $(\mathfrak{H}, (\cdot,\cdot))$; cf. [17, Theorem VI-1.11]. In the following the assumption of semiboundedness is dropped. Then, according to [7] the form $\mathfrak{t}[\cdot,\cdot]$ is said to be *closed* if there exists a so-called *gap point* $\lambda \in \mathbb{R}$ such that $\operatorname{dom} \mathfrak{t}$ provided with the inner product $\mathfrak{t}[\cdot,\cdot]_\lambda$ in (2.1) is a Kreĭn space which is continuously embedded in $(\mathfrak{H}, (\cdot,\cdot))$. The topology of the Kreĭn space does not depend on the choice of the gap point; see [7, Lemma 3.1]. A densely defined symmetric sesquilinear form $\mathfrak{s}[\cdot,\cdot]$ in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ is said to be *closable* if it has a closed extension $\mathfrak{t}[\cdot,\cdot]$, such that $\operatorname{dom} \mathfrak{s}$ is dense in the Kreĭn space $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ for some (and hence for all) gap points $\lambda \in \mathbb{R}$ of $\mathfrak{t}[\cdot,\cdot]$. In this case $\mathfrak{t}[\cdot,\cdot]$ is called a *closure* of $\mathfrak{s}[\cdot,\cdot]$.

The following two theorems from [7] generalize Kato's Representation Theorems [17, Theorem VI-2.1, Theorem VI-2.23] to the non-semibounded situation.

Theorem 2.1 (First representation theorem)**.** *Let $\mathfrak{t}[\cdot,\cdot]$ be a densely defined closed symmetric sesquilinear form in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ with gap point $\lambda \in \mathbb{R}$. Then the following statements are true:*

(i) *There exists a unique selfadjoint operator $T_\mathfrak{t}$ in $(\mathfrak{H}, (\cdot,\cdot))$ such that* $\operatorname{dom} T_\mathfrak{t} \subset \operatorname{dom} \mathfrak{t}$ *and*

$$\mathfrak{t}[u,v] = (T_\mathfrak{t} u, v), \quad u \in \operatorname{dom} T_\mathfrak{t}, \quad v \in \operatorname{dom} \mathfrak{t}.$$

(ii) $\operatorname{dom} T_\mathfrak{t}$ *is dense in the Kreĭn space* $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$.

(iii) *If $u \in \operatorname{dom}\mathfrak{t}$, $w \in \mathfrak{H}$ and $\mathfrak{t}[u,v] = (w,v)$ for all v in a dense linear subspace of the Kreĭn space* $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ *then $u \in \operatorname{dom} T_\mathfrak{t}$ and $T_\mathfrak{t} u = w$.*

(iv) *The range restriction*

$$A_\mathfrak{t} = \{\, \{u, T_\mathfrak{t} u\} : u \in \operatorname{dom} T_\mathfrak{t},\ T_\mathfrak{t} u \in \operatorname{dom}\mathfrak{t} \,\}$$

of $T_\mathfrak{t}$ is selfadjoint and definitizable in the Kreĭn space $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$.

(v) *All gap points of $\mathfrak{t}[\cdot,\cdot]$ belong to the resolvent set of $T_\mathfrak{t}$.*

The theory of definitizable operators in Kreĭn spaces can be found in [18]. Observe that the critical points of definitizable operators may be regular or singular.

Theorem 2.2 (Second representation theorem)**.** *Let $\mathfrak{t}[\cdot,\cdot]$ be a densely defined closed symmetric sesquilinear form in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ with gap point $\lambda \in \mathbb{R}$ and let $T_\mathfrak{t}$ and $A_\mathfrak{t}$ be the associated operators. Then*

$$\operatorname{dom}\mathfrak{t} = \operatorname{dom} |T_\mathfrak{t}|^{\frac{1}{2}} \tag{2.2}$$

if and only if ∞ is not a singular critical point of $A_\mathfrak{t}$. In this case the topology of the Kreĭn space $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ *is induced by the graph inner product*

$$(|T_\mathfrak{t}|^{1/2} u, |T_\mathfrak{t}|^{1/2} v) + (u,v), \quad u, v \in \operatorname{dom} |T_\mathfrak{t}|^{\frac{1}{2}},$$

or, equivalently, by the inner product $(|T_\mathfrak{t} - \lambda|^{\frac{1}{2}} u, |T_\mathfrak{t} - \lambda|^{\frac{1}{2}} v)$.

A closed symmetric form $\mathfrak{t}[\cdot,\cdot]$ is said to be *regular* if (2.2) is satisfied. The following result can be found in [7, Theorem 5.2].

Theorem 2.3. *The mapping $\mathfrak{t}[\cdot,\cdot] \to T_\mathfrak{t}$ defines a one-to-one correspondence between all regular densely defined closed symmetric forms in $(\mathfrak{H}, (\cdot,\cdot))$ and all selfadjoint operators in $(\mathfrak{H}, (\cdot,\cdot))$ with spectrum different from the whole real axis $\mathbb{R}$.*

In [9, Proposition 2.5] it was shown that in (2.2) domain inclusion instead of equality is enough for regularity.

Proposition 2.4. *Let $\mathfrak{t}[\cdot,\cdot]$ be a densely defined closed symmetric sesquilinear form in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ and let $T_\mathfrak{t}$ be the associated operator. Then the following statements are equivalent:*

(i) $\operatorname{dom}\mathfrak{t} \subset \operatorname{dom} |T_\mathfrak{t}|^{\frac{1}{2}}$;

(ii) $\operatorname{dom}\mathfrak{t} \supset \operatorname{dom} |T_\mathfrak{t}|^{\frac{1}{2}}$;

(iii) $\operatorname{dom}\mathfrak{t} = \operatorname{dom} |T_\mathfrak{t}|^{\frac{1}{2}}$.

It should be noted that another type of criterion for regularity already appears in [19, 20], see also [10]. Furthermore in [10] there is a simple example of a selfadjoint operator (being an infinite complex matrix) with an associated form which is not regular.

According to [7, Proposition 5.1] the statement of Theorem 2.1 (v) can be sharpened for regular closed forms:

Proposition 2.5. *The set of gap points of a regular closed form* $\mathfrak{t}[\cdot,\cdot]$ *coincides with the real part of the resolvent set of its representing operator, i.e. with* $\mathbb{R}\cap\rho(T_{\mathfrak{t}})$.

2.2. **The generalized Friedrichs extension of a closed symmetric operator.** Let S be a closed densely defined symmetric operator in the Hilbert space $(\mathfrak{H},\,(\cdot,\cdot))$ with defect $(1,1)$. Associated with S is the following densely defined symmetric sesquilinear form in the Hilbert space $(\mathfrak{H},\,(\cdot,\cdot))$:

$$\mathfrak{s}[u,v] := (Su,\, v), \quad u,v\in \operatorname{dom}\mathfrak{s} := \operatorname{dom} S. \tag{2.3}$$

If the operator S is semibounded then $\mathfrak{s}[\cdot,\cdot]$ is closable and among all selfadjoint extensions of S the selfadjoint operator associated with the closure of $\mathfrak{s}[\cdot,\cdot]$ is called the Friedrichs extension, cf. [17, Theorem VI-2.11]. As in [7, Theorem 7.1] the following alternative from [13, Theorem 2.1] is used to introduce the *generalized Friedrichs extension* of S in the general non-semibounded case:

Theorem 2.6. *Let S be a densely defined closed symmetric operator with defect numbers* $(1,1)$ *in the Hilbert space* $(\mathfrak{H},(\cdot,\cdot))$. *Then, either for all selfadjoint extensions T of S the domain* $\operatorname{dom} S$ *is dense in the Hilbert space* $\operatorname{dom}|T|^{\frac12}$ *equipped with the graph inner product*

$$(|T|^{1/2}u, |T|^{1/2}v) + (u,v), \quad u,v\in \operatorname{dom}|T|^{\frac12},$$

or this is true for precisely one selfadjoint extension T of S, the so-called generalized Friedrichs extension of S.

The following result rephrases the above alternative in terms of forms, see [7, Theorem 7.2]. Recall the definition of the *essential spectrum* of S:

$$\sigma_e(S) := \{\,\lambda\in\mathbb{R}:\ \operatorname{ran}(S-\lambda) \text{ is not closed or } \dim\ker\,(S-\lambda)=\infty\,\}.$$

Theorem 2.7. *Let S be a densely defined closed symmetric operator with defect numbers* $(1,1)$ *in the Hilbert space* $(\mathfrak{H},\,(\cdot,\cdot))$ *and let the form* $\mathfrak{s}[\cdot,\cdot]$ *be defined by* (2.3). *Then the following statements are equivalent:*

(i) $\mathfrak{s}[\cdot,\cdot]$ *is closable;*
(ii) $\mathfrak{s}[\cdot,\cdot]$ *has a regular closure;*
(iii) *there is a non-empty open interval* $I\subset\mathbb{R}$ *such that* $\sigma_e(S)\cap I=\emptyset$.

If a regular closure $\mathfrak{t}_F[\cdot,\cdot]$ *of* $\mathfrak{s}[\cdot,\cdot]$ *exists and is unique, then S has a generalized Friedrichs extension* S_F *which is given by the operator* $T_{\mathfrak{t}_F}$ *associated with* $\mathfrak{t}_F[\cdot,\cdot]$:

$$T_{\mathfrak{t}_F} = S_F = \{\,\{f,g\}\in S^*:\ f\in\operatorname{dom}\mathfrak{t}_F\,\}.$$

If a regular closure $\mathfrak{t}[\cdot,\cdot]$ *of* $\mathfrak{s}[\cdot,\cdot]$ *exists but is not unique, then S does not have a generalized Friedrichs extension and the mapping* $\mathfrak{t}[\cdot,\cdot]\to T_{\mathfrak{t}}$ *defines a one-to-one correspondence between all regular closures of* $\mathfrak{s}[\cdot,\cdot]$ *and all selfadjoint extensions of S.*

Remark 2.8. Note that condition (iii) of Theorem 2.7 implies $\rho(T)\cap\mathbb{R}\neq\emptyset$ for all selfadjoint extensions T of S and by Proposition 2.5 $\rho(T)\cap\mathbb{R}$ is the set of gap points of the regular closed form $\mathfrak{t}[\cdot,\cdot]$ associated with T as in Theorem 2.3.

3. Some useful extensions of the general theory

This section contains some new results as additions to the facts explained in the previous section.

3.1. **Uniqueness of closed forms.** By Theorem 2.3 there is only one regular closed form associated with a selfadjoint operator T with $\sigma(T) \neq \mathbb{R}$. It was already shown in [7, Example 6.2] that there may be other (non-regular) closed forms also associated with T by Theorem 2.1. It will now be shown that a closed form associated with T is at least uniquely determined by its form domain. However, first note the following useful fact.

Lemma 3.1. *Let* $\mathfrak{t}_1[\cdot,\cdot]$ *and* $\mathfrak{t}_2[\cdot,\cdot]$ *be two densely defined closed symmetric sesquilinear forms in the Hilbert space* $(\mathfrak{H}, (\cdot,\cdot))$ *with gap points* λ_1 *and* λ_2*, respectively. Then the following statements hold true:*

(i) *If* $\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2$ *then the Kreĭn space* $(\operatorname{dom}\mathfrak{t}_1,\ \mathfrak{t}_1[\cdot,\cdot]_{\lambda_1})$ *is continuously embedded in the Kreĭn space* $(\operatorname{dom}\mathfrak{t}_2,\ \mathfrak{t}_2[\cdot,\cdot]_{\lambda_2})$.

(ii) *If* $\operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$ *then the topologies of the Kreĭn spaces* $(\operatorname{dom}\mathfrak{t}_1,\ \mathfrak{t}_1[\cdot,\cdot]_{\lambda_1})$ *and* $(\operatorname{dom}\mathfrak{t}_2,\ \mathfrak{t}_2[\cdot,\cdot]_{\lambda_2})$ *coincide.*

Proof. It is enough to prove (i) since (ii) is a direct consequence of (i). Assume $\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2$. Then the embedding operator id from $(\operatorname{dom}\mathfrak{t}_1,\ \mathfrak{t}_1[\cdot,\cdot]_{\lambda_1})$ to $(\operatorname{dom}\mathfrak{t}_2,\ \mathfrak{t}_2[\cdot,\cdot]_{\lambda_2})$ with $id(u) = u$ is closed. To see this, let $u_n, u \in \operatorname{dom}\mathfrak{t}_1$, $n \in \mathbb{N}$ and $\tilde{u} \in \operatorname{dom}\mathfrak{t}_2$ such that $u_n \to u$ as $n \to \infty$ in $(\operatorname{dom}\mathfrak{t}_1,\ \mathfrak{t}_1[\cdot,\cdot]_{\lambda_1})$ and $id(u_n) = u_n \to \tilde{u}$ as $n \to \infty$ in $(\operatorname{dom}\mathfrak{t}_2,\ \mathfrak{t}_2[\cdot,\cdot]_{\lambda_2})$. Then there is also convergence in $(\mathfrak{H}, (\cdot,\cdot))$ since both spaces are continuously embedded in $(\mathfrak{H}, (\cdot,\cdot))$; consequently $\tilde{u} = u (= id(u))$. Hence id is closed, and hence also continuous by the closed graph theorem. □

Proposition 3.2. *Let* $\mathfrak{t}_1[\cdot,\cdot]$ *and* $\mathfrak{t}_2[\cdot,\cdot]$ *be two densely defined closed symmetric sesquilinear forms in the Hilbert space* $(\mathfrak{H}, (\cdot,\cdot))$ *with the same associated selfadjoint operator, i.e.* $T_{\mathfrak{t}_1} = T_{\mathfrak{t}_2}$. *Then* $\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2$ *implies* $\mathfrak{t}_1[\cdot,\cdot] = \mathfrak{t}_2[\cdot,\cdot]$.

Proof. According to [9, Lemma 2.4] the inclusion $\operatorname{dom}\mathfrak{t}_1 \subset \operatorname{dom}\mathfrak{t}_2$ is equivalent to the equality $\operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$. Then for $u \in \operatorname{dom} T_{\mathfrak{t}_1}$ $(= \operatorname{dom} T_{\mathfrak{t}_2})$ and $v \in \operatorname{dom}\mathfrak{t}_1$ $(= \operatorname{dom}\mathfrak{t}_2)$ one has

$$\mathfrak{t}_1[u,v] = (T_{\mathfrak{t}_1}u, v) = (T_{\mathfrak{t}_2}u, v) = \mathfrak{t}_2[u,v]. \tag{3.1}$$

By Theorem 2.1 $\operatorname{dom} T_{\mathfrak{t}_1}$ is dense in the Kreĭn space $(\operatorname{dom}\mathfrak{t}_1,\ \mathfrak{t}_1[\cdot,\cdot]_{\lambda_1})$ and in the Kreĭn space $(\operatorname{dom}\mathfrak{t}_2,\ \mathfrak{t}_2[\cdot,\cdot]_{\lambda_2})$, where λ_1 and λ_2 are gap points. By Lemma 3.1 both Kreĭn spaces have the same topology and hence, equation (3.1) remains true for all $u, v \in \operatorname{dom}\mathfrak{t}_1$ by continuity. □

3.2. **A characterization of the generalized Friedrichs extension via defect spaces.** Let S be a closed densely defined symmetric operator in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ with defect numbers $(1,1)$. If the operator S is semibounded then the Friedrichs extension $T = S_F$ of S satisfies

$$\ker\,(S^* - \lambda) \cap \operatorname{dom}|T|^{1/2} = \{0\}, \quad \lambda \in \rho(T). \tag{3.2}$$

On the other hand, for all other selfadjoint extensions T of S one has

$$\ker\,(S^* - \lambda) \subset \operatorname{dom}|T|^{1/2}, \quad \lambda \in \rho(T). \tag{3.3}$$

If S is not semibounded, then the generalized Friedrichs extension can be characterized by means of the properties (3.2) and (3.3). For this the following result is useful; it connects the property (3.3) with the denseness of $\operatorname{dom} S$ in the Kreĭn space associated with the selfadjoint extension T of S in Theorem 2.3.

Proposition 3.3. *Let S be a densely defined closed symmetric operator with defect numbers* $(1,1)$ *in the Hilbert space* $(\mathfrak{H},(\cdot,\cdot))$ *and assume that the form* $\mathfrak{s}[\cdot,\cdot]$ *defined in* (2.3) *is closable. Let T be a selfadjoint extension of S and let* $\mathfrak{t}[\cdot,\cdot]$ *be the regular closed form associated with T as in Theorem 2.3 (cf. Remark 2.8). Then for all gap points* λ $(\in \rho(T)\cap\mathbb{R})$ *the following statements are equivalent:*

(i) $\operatorname{dom} S$ *is not dense in the Kreĭn space* $(\operatorname{dom}\mathfrak{t},\mathfrak{t}[\cdot,\cdot]_\lambda)$*;*
(ii) $\ker\,(S^*-\lambda)\subset \operatorname{dom}\mathfrak{t}$.

If these conditions hold, then the subspaces $\ker\,(S^*-\lambda)$ *and* $\operatorname{dom} S$ *are orthogonal in the Kreĭn space* $(\operatorname{dom}\mathfrak{t},\mathfrak{t}[\cdot,\cdot]_\lambda)$.

Proof. Recall that a linear subspace $\mathfrak{L}\subset\operatorname{dom}\mathfrak{t}$ is not dense in the Kreĭn space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}[\cdot,\cdot]_\lambda)$ if and only if there exists an element $v_0\neq 0$, such that $\mathfrak{t}[u,v_0]_\lambda=0$ for all $u\in\mathfrak{L}$.

(i) $\Rightarrow$ (ii) Assume that $\operatorname{dom} S$ is not dense in the Kreĭn space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}[\cdot,\cdot]_\lambda)$ and let $v_0\in\operatorname{dom}\mathfrak{t}$ be a nontrivial element such that $\mathfrak{t}[u,v_0]_\lambda=0$ for all $u\in\operatorname{dom} S$. Then

$$((S-\lambda)u,v_0)=((T-\lambda)u,v_0)=\mathfrak{t}[u,v_0]_\lambda, \tag{3.4}$$

which shows that $v_0\perp\operatorname{ran}(S-\lambda)$ in the Hilbert space $(\mathfrak{H},(\cdot,\cdot))$. Therefore, $v_0\in\ker\,(S^*-\lambda)$ and, consequently, $\ker\,(S^*-\lambda)=\operatorname{span}\{v_0\}\subset\operatorname{dom}\mathfrak{t}$.

(ii) $\Rightarrow$ (i) Assume that $\ker\,(S^*-\lambda)\subset\operatorname{dom}\mathfrak{t}$ and let $0\neq v_0\in\ker\,(S^*-\lambda)$. Then $((S-\lambda)u,v_0)=0$ for all $u\in\operatorname{dom} S$. Using (3.4) again this means that $\operatorname{dom} S$ is not dense in the Kreĭn space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}[\cdot,\cdot]_\lambda)$. □

Using Proposition 3.3 and the result on the graph topology from Theorem 2.2 the alternative from Theorem 2.6 can be formulated as follows.

Theorem 3.4. *Let S be a densely defined closed symmetric operator with defect numbers* $(1,1)$ *in the Hilbert space* $(\mathfrak{H},(\cdot,\cdot))$ *and assume that the form* $\mathfrak{s}[\cdot,\cdot]$ *defined in* (2.3) *is closable. Then, either for all selfadjoint extensions T of S and their associated regular closed forms* $\mathfrak{t}[\cdot,\cdot]$ *(according to Theorem 2.3) one has*

$$\ker\,(S^*-\lambda)\cap\operatorname{dom}\mathfrak{t}=\{0\},\quad \lambda\in\rho(T)\cap\mathbb{R},$$

or for all but one selfadjoint extensions T of S one has

$$\ker\,(S^*-\lambda)\subset\operatorname{dom}\mathfrak{t},\quad \lambda\in\rho(T)\cap\mathbb{R}. \tag{3.5}$$

Precisely in the last case S has a generalized Friedrichs extension S_F and it is given by the exceptional extension not satisfying (3.5).

Remark 3.5. Recall that if the condition (3.5) holds for some selfadjoint extension T of S, then it holds for all but one selfadjoint extensions T of S. In Theorem 3.4 the form $\mathfrak{s}[\cdot,\cdot]$ is assumed to be closable in order to obtain $\rho(T)\cap\mathbb{R}\neq\emptyset$ which then allows to associate with T the regular closed form $\mathfrak{t}[\cdot,\cdot]$ with gap points $\lambda\in\rho(T)\cap\mathbb{R}$ (see Remark 2.8). In (3.5) one can equivalently use an arbitrary, not necessarily real, point $\lambda\in\rho(T)$. For details see [12, Proposition 2.1], [16].

Lemma 3.1 allows the following slight extension of [9, Theorem 2.8] (see also [12, Theorem 4.1]) on the invariance of the regular closed forms $\mathfrak{t}[\cdot,\cdot]$ associated by Theorem 2.3 with the selfadjoint extensions T of S satisfying $T\neq S_F$.

Proposition 3.6. *Assume that* $\mathfrak{s}[\cdot,\cdot]$ *has a unique regular closure and let* T_1 *and* T_2 *be selfadjoint extensions of* S *such that* $T_j \neq S_F$; $j = 1, 2$. *Let* $\mathfrak{t}_j[\cdot,\cdot]$ *be the regular closed form with a gap point* λ_j *associated with* T_j *by Theorem 2.3. Then*

$$\operatorname{dom}\mathfrak{t}_1 = \operatorname{dom}\mathfrak{t}_2$$

and, moreover, the topologies of the Kreĭn spaces $(\operatorname{dom}\mathfrak{t}_j,\ \mathfrak{t}_j[\cdot,\cdot]_{\lambda_j})$ *coincide.*

Proof. The equality of the domains $\operatorname{dom}\mathfrak{t}_1$ and $\operatorname{dom}\mathfrak{t}_2$ was proved in [9, Theorem 2.8]. Now apply Lemma 3.1 to conclude that the topologies of the Kreĭn spaces $(\operatorname{dom}\mathfrak{t}_j,\ \mathfrak{t}_j[\cdot,\cdot]_\lambda)$, $j = 1, 2$, are the same. □

4. A closed form associated with the energy space

Let again S be a densely defined closed symmetric operator in the Hilbert space $(\mathfrak{H},\ (\cdot,\cdot))$ with defect numbers $(1,1)$. Assume that $\mathfrak{s}[\cdot,\cdot]$ has a unique regular closure $\mathfrak{t}_F[\cdot,\cdot]$ and hence S has a generalized Friedrichs extension S_F. Now consider a fixed selfadjoint extension $T \neq S_F$ of S and let $\mathfrak{t}[\cdot,\cdot]$ be the associated regular closed form as in Theorem 2.3 (cf. Remark 2.8). Since the defect numbers of S are finite, it follows from Theorem 2.7 (iii) that $\rho(S_F) \cap \rho(T) \cap \mathbb{R} \neq \emptyset$. Furthermore, each $\lambda \in \rho(S_F) \cap \rho(T) \cap \mathbb{R}$ is a gap point of the closed forms $\mathfrak{t}_F[\cdot,\cdot]$ and $\mathfrak{t}[\cdot,\cdot]$; cf. Proposition 2.5. The form domain $\operatorname{dom}\mathfrak{t}$ $(= \operatorname{dom}|T|^{1/2})$ is a Kreĭn space with the inner product $\mathfrak{t}[u,v]_\lambda$ in (2.1). Let $\operatorname{dom}\mathfrak{t}^F$ be the closure of $\operatorname{dom} S$ in the Kreĭn space $(\operatorname{dom}\mathfrak{t},\ \mathfrak{t}[\cdot,\cdot]_\lambda)$ and define the form $\mathfrak{t}^F[\cdot,\cdot]$ as the restriction of $\mathfrak{t}[\cdot,\cdot]$ to $\operatorname{dom}\mathfrak{t}^F$:

$$\mathfrak{t}^F[f,g] = \mathfrak{t}[f,g], \quad f,g \in \operatorname{dom}\mathfrak{t}^F. \tag{4.1}$$

Generalizing the classical terminology, $\operatorname{dom}\mathfrak{t}^F$ is called the "energy space".

Lemma 4.1. *The energy space* $\operatorname{dom}\mathfrak{t}^F$ *does not depend on the choice of the selfadjoint extension* $T \neq S_F$ *of* S.

Proof. According to Proposition 3.6 the topology of the Kreĭn space $(\operatorname{dom}\mathfrak{t},\ \mathfrak{t}[\cdot,\cdot]_\lambda)$ is the same for all selfadjoint extensions $T \neq S_F$ of S. Therefore, the closure of $\operatorname{dom} S$ does not depend on the selfadjoint extension $T \neq S_F$ of S. □

The following statement gives an analog in the non-semibounded case of a decomposition result in the nonnegative case; see [15, Proposition 2.3] and the references therein. Recall that a linear subspace $\mathfrak{L}$ of a Kreĭn space is called *degenerate* if there is an element $0 \neq u \in \mathfrak{L}$ such that u is orthogonal to the whole subspace $\mathfrak{L}$ with respect to the inner product of the Kreĭn space. Of course, a one-dimensional subspace $\mathfrak{L}$ is degenerate if and only if it is neutral, i.e. the inner product vanishes on $\mathfrak{L}$.

Theorem 4.2. *Let* S *be a densely defined closed symmetric operator in the Hilbert space* $(\mathfrak{H},\ (\cdot,\cdot))$ *with defect numbers* $(1,1)$ *and assume that* $\mathfrak{s}[\cdot,\cdot]$ *has a unique regular closure* $\mathfrak{t}_F[\cdot,\cdot]$. *Let* $T \neq S_F$ *be a selfadjoint extension of* S, *let* $\mathfrak{t}[\cdot,\cdot]$ *be the regular closed form associated with* T *according to Theorem 2.3 with a gap point* $\lambda \in \rho(S_F) \cap \rho(T) \cap \mathbb{R}$, *and let* $\mathfrak{t}^F[\cdot,\cdot]$ *be as in (4.1). Then the defect space* $\ker(S^* - \lambda)$ *is a one-dimensional non-degenerate subspace of the Kreĭn space* $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ *and the following decomposition*

$$(\operatorname{dom}\mathfrak{t} =)\operatorname{dom}|T|^{1/2} = \operatorname{dom}\mathfrak{t}^F + \ker(S^* - \lambda), \tag{4.2}$$

is a direct orthogonal sum in this Kreĭn space.

Proof. By assumption S is a densely defined symmetric operator with defect numbers $(1,1)$. Hence, for some $h \in \operatorname{dom} T \setminus \operatorname{dom} S$ there is a direct sum decomposition $\operatorname{dom} T = \operatorname{dom} S + \operatorname{span}\{h\}$. Since $\operatorname{dom} T$ is dense in the Kreĭn space $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ by Theorem 2.1 (ii), the co-dimension of the closure of $\operatorname{dom} S$, i.e. of the subspace $\operatorname{dom} \mathfrak{t}^F$, is at most one. Since $T \neq S_F$, Theorem 3.4 shows that $\ker\,(S^*-\lambda) \subset \operatorname{dom} \mathfrak{t}$. Now it follows from Proposition 3.3 that the co-dimension of $\operatorname{dom} \mathfrak{t}^F$ in the Kreĭn space $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ is at least one and hence equal to one.

Next it is shown that the sum in the right-hand side of (4.2) is direct. Assume that $v_0 \in \ker\,(S^*-\lambda)$ belongs to $\operatorname{dom} \mathfrak{t}^F$. Thus $v_0 \in \operatorname{dom} S^* \cap \operatorname{dom} \mathfrak{t}^F$ and it follows from (1.7) that $v_0 \in \operatorname{dom} S_F$ (cf. [12, Proposition 3.5]). Since $\lambda \in \rho(S_F)$, one concludes that $\ker\,(S^*-\lambda) \cap \operatorname{dom} S_F = \{0\}$ and, thus, $v_0 = 0$. Therefore, the sum in (4.2) is direct. Using the co-dimension argument from above this proves the decomposition of $\operatorname{dom} \mathfrak{t}$ in (4.2). By Proposition 3.3 the subspaces $\ker\,(S^*-\lambda)$ and $\operatorname{dom} S$ (and hence $\operatorname{dom} \mathfrak{t}^F$) are orthogonal in the Kreĭn space $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$. Therefore, the defect space $\ker\,(S^*-\lambda)$ cannot be degenerate, since otherwise the whole space $\operatorname{dom} \mathfrak{t}$ is degenerate. □

Corollary 4.3. *There is an element $v_0 \in \operatorname{dom} \mathfrak{t}$, $\mathfrak{t}[v_0, v_0]_\lambda \neq 0$, such that*

$$\operatorname{dom} \mathfrak{t}^F = \{u \in \operatorname{dom} \mathfrak{t} \;:\; \mathfrak{t}[u, v_0]_\lambda = 0\}.$$

Theorem 4.4. *Under the same assumptions as in Theorem 4.2 the form $\mathfrak{t}^F[\cdot,\cdot]$ on $\operatorname{dom} \mathfrak{t}^F$ is closed with gap point λ and the associated operator is S_F, i.e. $S_F = T_{\mathfrak{t}^F}$.*

Proof. By Corollary 4.3 $\operatorname{dom} \mathfrak{t}^F$ is the orthogonal complement of v_0 in the Kreĭn space $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$. Since $\operatorname{span}\{v_0\}$ is non-degenerate, $\operatorname{dom} \mathfrak{t}^F$ remains a Kreĭn space with $\mathfrak{t}[\cdot,\cdot]_\lambda$ and its topology is the restriction of the topology of $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$. This Kreĭn space can be written as $(\operatorname{dom} \mathfrak{t}^F, \mathfrak{t}^F[\cdot,\cdot]_\lambda)$ with $\mathfrak{t}^F[\cdot,\cdot]_\lambda = \mathfrak{t}^F[\cdot,\cdot] - \lambda(\cdot,\cdot)$ and it is also continuously embedded in $(\mathfrak{H}, (\cdot,\cdot))$; hence, $\mathfrak{t}^F[\cdot,\cdot]$ is closed with gap point λ. It follows from (1.7) that $\operatorname{dom} S_F \subset \operatorname{dom} \mathfrak{t}^F$; cf. [12, Proposition 3.5]. Now, let $u \in \operatorname{dom} S_F$. Then for all $v \in \operatorname{dom} S$ one has

$$(S_F u, v) = (S^* u, v) = (u, Sv) = (u, Tv) = \mathfrak{t}[u,v] = \mathfrak{t}^F[u,v].$$

Since $\operatorname{dom} S$ is dense in $(\operatorname{dom} \mathfrak{t}^F, \mathfrak{t}^F[\cdot,\cdot]_\lambda)$ by definition of $\operatorname{dom} \mathfrak{t}^F$, Theorem 2.1 (iii) implies that $u \in \operatorname{dom} T_{\mathfrak{t}^F}$ and $T_{\mathfrak{t}^F} u = S_F u$. Therefore, $S_F \subset T_{\mathfrak{t}^F}$ and, since both are selfadjoint, the equality $T_{\mathfrak{t}^F} = S_F$ follows. □

Now Theorem 4.4 allows to extend the uniqueness of the form domain $\operatorname{dom} \mathfrak{t}^F$ according to Lemma 4.1 to the form $\mathfrak{t}^F[\cdot,\cdot]$ itself by Proposition 3.2.

Corollary 4.5. *The form $\mathfrak{t}^F[\cdot,\cdot]$ does not depend on the choice of the selfadjoint extension T of S if $T \neq S_F$.*

Since both closed forms $\mathfrak{t}_F[\cdot,\cdot]$ and $\mathfrak{t}^F[\cdot,\cdot]$ are associated with the generalized Friedrichs extension S_F, the following result is an immediate consequence of Theorem 2.3.

Corollary 4.6. *The equality $\mathfrak{t}_F[\cdot,\cdot] = \mathfrak{t}^F[\cdot,\cdot]$ holds if and only if the form $\mathfrak{t}^F[\cdot,\cdot]$ is regular.*

5. Inclusions of square root domains

Assume the same situation as in the previous section. The next theorem gives a new regularity criterion; it can be seen as an improvement of Proposition 2.4 with regard to $\mathfrak{t}^F[\cdot,\cdot]$.

Theorem 5.1. *Under the same assumptions as in Theorem 4.2 the closed form* $\mathfrak{t}^F[\cdot,\cdot]$ *is regular if and only if*

$$(\operatorname{dom}\mathfrak{t}_F =) \operatorname{dom}|S_F|^{1/2} \subset \operatorname{dom}|T|^{1/2} \ (= \operatorname{dom}\mathfrak{t}). \tag{5.1}$$

Proof. ($\Rightarrow$) If $\mathfrak{t}^F[\cdot,\cdot]$ is regular then by Corollary 4.6 $\operatorname{dom}\mathfrak{t}_F = \operatorname{dom}\mathfrak{t}^F \subset \operatorname{dom}\mathfrak{t}$.

($\Leftarrow$) Let $\operatorname{dom}\mathfrak{t}_F \subset \operatorname{dom}\mathfrak{t}$. Then, by Lemma 3.1 the embedding of the Kreĭn space $(\operatorname{dom}\mathfrak{t}_F, \mathfrak{t}_F[\cdot,\cdot]_\lambda)$ in $(\operatorname{dom}\mathfrak{t}, \mathfrak{t}[\cdot,\cdot]_\lambda)$ is continuous where $\mathfrak{t}_F[\cdot,\cdot]_\lambda = \mathfrak{t}_F[\cdot,\cdot] - \lambda(\cdot,\cdot)$. Now consider the element $v_0 \in \operatorname{dom}\mathfrak{t}$ according to Corollary 4.3. It satisfies $\mathfrak{t}[u,v_0] - \lambda(u,v_0) = 0$ for all $u \in \operatorname{dom}S_F$ ($\subset \operatorname{dom}\mathfrak{t}^F$). However, by Theorem 2.1 $\operatorname{dom}S_F$ is dense in $(\operatorname{dom}\mathfrak{t}_F, \mathfrak{t}_F[\cdot,\cdot]_\lambda)$ since S_F is the selfadjoint operator associated with $\mathfrak{t}_F[\cdot,\cdot]$. Therefore one has also $\mathfrak{t}[u,v_0] - \lambda(u,v_0) = 0$ for all $u \in \operatorname{dom}\mathfrak{t}_F$ by continuity. Then $(\operatorname{dom}|S_F|^{1/2} =) \operatorname{dom}\mathfrak{t}_F \subset \operatorname{dom}\mathfrak{t}^F$ follows from Corollary 4.3. This is the regularity of $\mathfrak{t}^F[\cdot,\cdot]$ by Proposition 2.4. □

The regularity criterion in Theorem 5.1 can be reformulated in a stronger form by means of the decomposition result established in Theorem 4.2.

Corollary 5.2. *The form* $\mathfrak{t}^F[\cdot,\cdot]$ *is regular if and only if the following decomposition holds true:*

$$(\operatorname{dom}\mathfrak{t} =) \operatorname{dom}|T|^{1/2} = \operatorname{dom}|S_F|^{1/2} \dot{+} \ker\,(S^* - \lambda) \ (= \operatorname{dom}\mathfrak{t}_F \dot{+} \ker\,(S^* - \lambda)). \tag{5.2}$$

Proof. ($\Rightarrow$) If the form $\mathfrak{t}^F[\cdot,\cdot]$ is regular, then $\operatorname{dom}\mathfrak{t}^F = \operatorname{dom}\mathfrak{t}_F = \operatorname{dom}|S_F|^{1/2}$ and hence the decomposition (4.2) in Theorem 4.2 can be rewritten as in (5.2).

($\Leftarrow$) The decomposition (5.2) implies that $\operatorname{dom}|S_F|^{1/2} \subset \operatorname{dom}|T|^{1/2}$ and, therefore, the form $\mathfrak{t}^F[\cdot,\cdot]$ is regular by Theorem 5.1. □

Proposition 2.4 and Theorem 5.1 together lead to the following result.

Corollary 5.3. *The following statements are equivalent (each being equivalent to the regularity of* $\mathfrak{t}^F[\cdot,\cdot]$*):*

(i) $\operatorname{dom}|S_F|^{1/2} = \operatorname{dom}\mathfrak{t}^F$;
(ii) $\operatorname{dom}|S_F|^{1/2} \subset \operatorname{dom}\mathfrak{t}^F$;
(iii) $\operatorname{dom}|S_F|^{1/2} \supset \operatorname{dom}\mathfrak{t}^F$;
(iv) $\operatorname{dom}|S_F|^{1/2} \subset \operatorname{dom}\mathfrak{t}$.

Observe that (4.2) holds independent of $\mathfrak{t}^F[\cdot,\cdot]$ being regular. In general $\mathfrak{t}^F[\cdot,\cdot]$ is not regular, so that (5.2) and the inclusion $\operatorname{dom}|S_F|^{1/2} \subset \operatorname{dom}|T|^{1/2}$ may fail to hold. For an example of this situation see Section 6. The next proposition collects all relations between square root domains of selfadjoint extensions of S from the present paper combined with results from [12, Section 4].

Proposition 5.4. *Let S be a closed densely defined symmetric operator in $\mathfrak{H}$ with defect numbers $(1,1)$ and let T_1 and T_2 be selfadjoint extensions of S. Assume that the form $\mathfrak{s}[\cdot,\cdot]$ is closable. Then the following statements are true:*

(i) *If S has a generalized Friedrichs extension S_F, then*

$$\operatorname{dom}|T_1|^{1/2} = \operatorname{dom}|T_2|^{1/2} \quad \textit{for all} \quad T_1, T_2 \neq S_F.$$

(ii) *If the inclusion* $\operatorname{dom}|T_1|^{1/2} \subset \operatorname{dom}|T_2|^{1/2}$ *holds for some* $T_1 \neq T_2$ *then* S *has a generalized Friedrichs extension* S_F.

(iii) *The inclusion* $\operatorname{dom}|T_1|^{1/2} \subset \operatorname{dom}|T_2|^{1/2}$ *or* $\operatorname{dom}|T_2|^{1/2} \subset \operatorname{dom}|T_1|^{1/2}$ *holds for all selfadjoint extensions* T_1 *and* T_2 *of* S *if and only if* S_F *exists and the closed form* $\mathfrak{t}^F[\cdot,\cdot]$ *associated with* S_F *is regular.*

(iv) *If* S_F *exists then* $\operatorname{dom}|S_F|^{1/2} \not\subset \operatorname{dom}|T_1|^{1/2}$ *for some (equivalently for all)* $T_1 \neq S_F$ *if and only if the closed form* $\mathfrak{t}^F[\cdot,\cdot]$ *associated with* S_F *is not regular.*

Proof. The statement (ii) follows immediately from [12, Theorems 4.1, 4.2]. The rest is obtained from Proposition 3.6, (ii), and Theorem 5.1. □

Remark 5.5. The validity of the domain inclusion in (5.1) and some general criteria for (5.1) to hold have been open problems on generalized Friedrichs extensions (cf. [12, 13]). In particular, it was not clear how the sufficient condition (ii) for the existence of S_F (going back to [12]) could be modified into a necessary and sufficient condition like (iii). Theorem 5.1 and Proposition 5.4 together with the example in Section 6 below therefore provide a complete answer to these problems.

6. Closed forms associated with indefinite Sturm-Liouville operators

The present theory can be illustrated by some Sturm-Liouville operators and associated forms which were studied in detail in [8, 9]. Let $-DpD$ be a Sturm-Liouville expression on the compact interval $[-b, b]$, whose real coefficient p satisfies $tp(t) > 0$ almost everywhere and $1/p$ in $L^1[-b, b]$. In the Hilbert space $L^2[-b, b]$ this differential expression induces the densely defined closed symmetric minimal differential operator $T_{\min}$ by Dirichlet boundary conditions $u(-b) = u(b) = 0$ and the additional interface conditions

$$u(0+) = u(0-), \quad (pu')(0+) = (pu')(0-) = 0. \tag{6.1}$$

The operator $T_{\min}$ has defect numbers $(1, 1)$ and a selfadjoint extension of $T_{\min}$ is given by the operator T_∞, determined by the Dirichlet boundary conditions and no interface condition. Using partial integration this extension induces the form $\mathfrak{t}_\infty[\cdot,\cdot]$ given by

$$\mathfrak{t}_\infty[u, v] = \int_{-b}^{b} u'(t)\overline{v'(s)}p(s)\, ds,$$

again subject to Dirichlet boundary conditions. Another selfadjoint extension T_0 of $T_{\min}$ is given by Dirichlet boundary conditions and by the interface conditions

$$(pu')(0+) = (pu')(0-) = 0 \tag{6.2}$$

allowing functions with a jump at 0 (i.e. $u(0+) \neq u(0-)$). Now, partial integration leads to the form $\mathfrak{t}_0[\cdot,\cdot]$ defined similarly to $\mathfrak{t}_\infty[\cdot,\cdot]$ but allowing also functions with a jump at 0. By [9, Theorem 5.5, Proposition 6.1] the form $\mathfrak{t}_0[\cdot,\cdot]$ is closed and regular and T_0 is the associated operator, i.e. $T_0 = T_{\mathfrak{t}_0}$.

As in the definite situation (i.e. $p(t) > 0$) also here, T_∞ is the generalized Friedrichs extension of the non-semibounded minimal operator $T_{\min}$; cf. [8, Proposition 4.3] and [9, Theorem 6.3].

Theorem 6.1. *Assume that the function* p *satisfies* $tp(t) > 0$ *almost everywhere and* $1/p \in L^1[-b,b]$. *Let* $\lambda \in \rho(T_0) \cap \rho(T_\infty) \cap \mathbb{R}$. *Then for* $S = T_{\min}$ *the form* $\mathfrak{t}^F[\cdot,\cdot]$ *defined in (4.1) (using* $T := T_0$*) coincides with* $\mathfrak{t}_\infty[\cdot,\cdot]$. *In particular,* $\mathfrak{t}_\infty[\cdot,\cdot]$ *is closed with gap point* λ *and* T_∞ $(= S_F)$ *is the associated selfadjoint operator, i.e.* $T_\infty = T_{\mathfrak{t}_\infty}$.

Proof. Note that the domains $\operatorname{dom} T_{\min}$, $\operatorname{dom} \mathfrak{t}_\infty$, and $\operatorname{dom} \mathfrak{t}_0$ remain unchanged if the function p is replaced by $|p|$. In this case $(\operatorname{dom} \mathfrak{t}_0,\ \mathfrak{t}_0[\cdot,\cdot])$ is a Hilbert space and by [9, Proposition 4.6] the closure of $\operatorname{dom} T_{\min}$ $(= \operatorname{dom} S)$ in this space is given by the form domain $\operatorname{dom} \mathfrak{t}_\infty$. Now, returning to the original function p, note that by [9, Lemma 5.2] this is also the closure of $\operatorname{dom} T_{\min}$ in the Kreĭn space $(\operatorname{dom} \mathfrak{t}_0,\ \mathfrak{t}_0[\cdot,\cdot]_\lambda)$ for $\lambda = 0\,(\in \rho(T_0))$ and hence also for $\lambda \in \rho(T_0) \cap \rho(T_\infty) \cap \mathbb{R}$. Since each of the forms $\mathfrak{t}^F[\cdot,\cdot]$ and $\mathfrak{t}_\infty[\cdot,\cdot]$ is a restriction of the form $\mathfrak{t}_0[\cdot,\cdot]$, they coincide. By Theorem 4.4 the form $\mathfrak{t}^F[\cdot,\cdot]$ is closed with gap point $\lambda \in \rho(T_0) \cap \rho(T_\infty) \cap \mathbb{R}$, and the associated operator is S_F. Therefore the form $\mathfrak{t}_\infty[\cdot,\cdot]$ is closed. □

Remark 6.2. Define the function v_0 on $[-b,b]$ by

$$v_0(t) = \int_{-b}^{t} \frac{ds}{p(s)}, \quad t \in [-b,0); \quad v_0(t) = -\int_{t}^{b} \frac{ds}{p(s)}, \quad t \in (0,b]. \tag{6.3}$$

Then $v_0 \in \operatorname{dom} \mathfrak{t}_0$ $(= \operatorname{dom} |T_0|^{1/2})$ and v_0 spans the kernel of $T_{\min}^*$ $(= S^*)$, and moreover

$$\mathfrak{t}_0[v_0, v_0] = \int_{-b}^{b} |v_0'(s)|^2\, p(s)\, ds = \int_{-b}^{b} \frac{ds}{p(s)}.$$

Hence ker S^* $(= \operatorname{span}\{v_0\})$ is non-degenerate in the Kreĭn space $(\operatorname{dom} \mathfrak{t}_0,\ \mathfrak{t}_0[\cdot,\cdot])$ if and only if

$$\int_{-b}^{b} \frac{ds}{p(s)} \neq 0. \tag{6.4}$$

However, condition (6.4) is equivalent to $0 \in \rho(T_\infty)$ $(= \rho(S_F))$; cf. [9, Lemma 3.2]. Therefore, under the additional condition (6.4) $\lambda \in \rho(T_0) \cap \rho(T_\infty) \cap \mathbb{R}$ can be chosen as $\lambda := 0$ and the (non-degenerate) element v_0 in Corollary 4.3 can be chosen as the function in (6.3). Then this allows the characterization

$$\operatorname{dom} \mathfrak{t}_\infty = \{u \in \operatorname{dom} \mathfrak{t}_0 \ :\ \mathfrak{t}_0[u, v_0] = 0\}.$$

In [9, Theorem 5.5] the function v_0 was used to prove the closedness of the form $\mathfrak{t}_\infty[\cdot,\cdot]$ under the additional condition (6.4); this condition has been relaxed in Theorem 6.1.

Sufficient conditions on the function p for the regularity of the closed form $\mathfrak{t}_\infty[\cdot,\cdot]$ can be found in [5, Corollary 11] [1] and in [6].

Explicit functions p which lead to non-regular forms $\mathfrak{t}^F[\cdot,\cdot]$ were presented in [9, Section 6.3] and [6], e.g.

$$p(t) = \begin{cases} t\log^2|t|, & t \in [-\frac{3}{4}, \frac{1}{4}], \\ \frac{1}{4}\log^2 4, & t \in (\frac{1}{4}, 1], \\ -\frac{3}{4}(\log 3 - \log 4)^2, & t \in [-1, -\frac{3}{4}), \end{cases}$$

cf. [6, Example 3.7]. Note that in [6] also the set difference between $\operatorname{dom} \mathfrak{t}_\infty$ and $\operatorname{dom} |T_\infty|^{\frac{1}{2}}$ was discussed.

[1] The factor γ in $h(x)$ of [5, Corollary 11] should be in the denominator.

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