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Achim Ilchmann, Tilman Selig, Carsten Trunk

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Hrsg.: Leiter des Instituts für Mathematik

Weimarer Straße 25

98693 Ilmenau

Tel.: +49 3677 69-3621

Fax: +49 3677 69-3270

<http://www.tu-ilmenau.de/math/>

# The Byrnes-Isidori form for infinite-dimensional systems

Achim Ilchmann\*      Tilman Selig\*      Carsten Trunk\*

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## Abstract

We define a Byrnes-Isidori form for a class of infinite-dimensional systems with relative degree  $r$  and show that any system belonging to this class can be transformed into this form. We also analyze the concept of (stable) zero dynamics and show that it is, together with the Byrnes-Isidori form, instrumental for static proportional high-gain output feedback stabilization. Moreover, we show that funnel control is feasible for any system with relative degree one and with exponentially stable zero dynamics; a funnel controller is a time-varying proportional output feedback controller which ensures, for a large class of reference signals, that the error between the output and the reference signal evolves within a prespecified funnel. Therefore transient behavior of the error is obeyed.

**Key words.** Byrnes-Isidori form, relative degree, infinite-dimensional systems, high-gain stabilizability, funnel control

**AMS subject classification** 93B10, 93C20, 93D21

## 1 Introduction

We consider the class of linear infinite-dimensional systems

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \tag{1.1a}$$

$$y(t) = (x(t), c), \tag{1.1b}$$

where  $(A, b, c)$  satisfy, for some  $r \in \mathbb{N}$ , the assumptions

**(A1)**  $A : \text{dom } A \subset H \rightarrow H$  is the generator of a strongly continuous semigroup  $(T(t))_{t \geq 0}$  on a real Hilbert space  $(H, (\cdot, \cdot))$ ,

**(A2)**  $b \in \text{dom } A^r$  and  $c \in \text{dom } A^{*r}$ ,

**(A3)** system  $(A, b, c)$  has *relative degree*  $r$ :  $(A^{r-1}b, c) \neq 0$  and  $(A^j b, c) = 0 \forall j = 0, 1, \dots, r-2$ ;

this class of systems is denoted by  $\Sigma_r$  and we write  $(A, b, c) \in \Sigma_r$ .

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\*Institut für Mathematik, Technische Universität Ilmenau, Weimarer Straße 25, D-98693 Ilmenau, Germany, achim.ilchmann, tilman.selig, carsten.trunk@tu-ilmenau.de

The mild solution of (1.1a) is given for any  $u \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R})$  and any  $x^0 \in H$  by

$$x(t) = T(t)x^0 + \int_0^t T(t-s) b u(s) ds, \quad t \geq 0. \quad (1.2)$$

It is well-known that  $x$  and  $y$  are continuous functions; see [5, Lemma 3.1.5]. Therefore, we may define the (*mild*) *behavior* of (1.1) by

$$\mathfrak{B}_{(A,b,c)} := \left\{ (x, u, y) \in \mathcal{C}(\mathbb{R}_{\geq 0}; H) \times L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}) \times \mathcal{C}(\mathbb{R}_{\geq 0}; H) \left| \begin{array}{l} x(t) = T(t)x(0) + \int_0^t T(t-s) b u(s) ds \\ \text{and } y(t) = (x(t), c), \quad t \geq 0 \end{array} \right. \right\}.$$

For finite-dimensional systems, Assumptions (A1) and (A2) are superfluous, and Assumption (A3) means that (A,b,c) has relative degree  $r$  in frequency domain terms, see e.g. [11, p. 137]. If the system (1.1) is finite-dimensional and satisfies (A3), then it is easy to see that the relative degree  $r$  is the minimal number so that  $u$  appears explicitly for the first time in the  $r$ -th derivative of  $y$  or, more formally,

$$y^{(r)}(t) = cA^r x + cA^{r-1} b u \quad \text{and} \quad y^{(j)}(t) = cA^j x(t) \quad \forall j = 0, 1, \dots, r-1. \quad (1.3)$$

For infinite-dimensional systems, Assumption (A1) is standard in systems theory, see e.g. [5]; Assumption (A2) is clearly restrictive; Assumption (A2) together with (A3) means that the system has relative degree  $r$ , see [14, Definition 1.3 and Lemma 2.9].

The guiding research idea of the present paper is whether it is possible to extend the following well known results from finite-dimensional systems to the class  $\Sigma_r$  of infinite-dimensional systems. For linear, finite-dimensional systems the Byrnes-Isidori form is well understood; see [1, 10, 11]. This form together with the concept of stable zero dynamics is instrumental for various high-gain stabilization and tracking results; see again [11]. If the nominal system has relative degree one and stable zero dynamics (often called minimum phase), then it is high-gain stabilizable by proportional a output feedback  $u(t) = -ky(t)$ , provided the gain  $k$  is sufficiently large. The drawback is that it is unknown how large the gain has to be chosen. It can be resolved by an adaptive controller of the form  $u(t) = -k(t)y(t)$ ,  $\dot{k}(t) = y(t)^2$ ; see for example [4]. However, the drawback of the adaptive controller is that the gain  $k(\cdot)$  is, although bounded, monotone and may become too large, whence additive noise corrupting the output may lead to instability and, more important, transient behavior is not obeyed at all. This drawback was resolved by the funnel controller introduced by [10].

A generalization of these finite-dimensional results to infinite-dimensional systems cannot be expected in full generality. However, we show that the class  $\Sigma_r$  is an appropriate class to allow for a Byrnes-Isidori form and subsequently for control theoretic consequences such as funnel control.

We describe the literature on infinite-dimensional systems related to our results. The basic idea to identify the zero dynamics is the splitting the state space into the two subspaces

$$H = c^\perp \dot{+} \text{ls}\{b\}, \quad (1.4)$$

this is sketched in [2, 3] for the class  $\Sigma_1$  of systems with relative degree one. For higher relative degree, this decomposition generalizes to

$$H = \underbrace{c^\perp \cap (A^*c)^\perp \cap \dots \cap (A^{*r-1}c)^\perp}_{=H_{c,A}^\perp} \dot{+} \underbrace{\text{ls}\{b\} \dot{+} \text{ls}\{Ab\} \dot{+} \dots \dot{+} \text{ls}\{A^{r-1}b\}}_{=H_{A,b}^\perp} \quad (1.5)$$

and has been studied in [14]; it is also the basis considered in the finite dimensional case when deriving the Byrnes-Isidori form, see for example [10]. We consider instead the decomposition

$$H = \text{ls}\{c\} \dot{+} b^\perp \quad (1.6)$$

which generalizes to

$$H = \underbrace{\text{ls}\{c\} \dot{+} \text{ls}\{A^*c\} \dot{+} \dots \dot{+} \text{ls}\{A^{*r-1}c\}}_{=:H_{c,A}} \dot{+} \underbrace{\{b\}^\perp \cap \{Ab\}^\perp \cap \dots \cap \{A^{r-1}b\}^\perp}_{=:H_{A,b}}. \quad (1.7)$$

The difference of the two decompositions (1.5) and (1.7) is essentially taking orthogonal complements; (1.7) is instrumental for our approach. In [14], it is shown that the space  $H_{c,A}^\perp$  is the largest feedback invariant subspace of  $c^\perp$ ; Furthermore, a more general definition of relative degree in the frequency domain is given, and it is shown that (A2)-(A3) imply that  $(A, b, c)$  fulfills this general frequency domain definition of relative degree.

Similarly, in [13] a multi-input multi-output system is called of generalized degree one if, and only if, its transfer-function matrix  $G(\cdot)$  is meromorphic on  $\mathbb{C}_0 := \{z \in \mathbb{C} : \text{Re } z > 0\}$  and admits a representation as

$$G^{-1}(s) = sD^{-1} + H(s),$$

where  $D$  is an invertible matrix and  $H(\cdot)$  a bounded analytic function defined on  $\mathbb{C}_0$ . If  $\sigma(D) \subset \mathbb{C}_0$ , then [12] show that the plant described by  $G(\cdot)$  can be stabilized by static output feedback of the form  $u(t) = -ky(t)$ , provided the feedback gain  $k$  is sufficiently large.

Clearly, the results presented here give an alternative proof of the finite dimensional results on the Byrnes-Isidori form and the funnel controller as presented in [10, 11].

The present paper is organized as follows. In Section 2 we define a Byrnes-Isidori form for systems  $\Sigma_r$  of relative degree  $r$  and show that any system belonging to  $\Sigma_r$  can be transformed into Byrnes-Isidori form. Furthermore, an internal loop form is derived; this form is instrumental for proving regulation results later.

In Section 3, we define the concept of (stable) zero dynamics and characterize it in terms of the Byrnes-Isidori form.

From Section 4 onwards we restrict our attention to relative degree one systems belonging to  $\Sigma_1$ . The previous results allow to show in Section 4 that any system of class  $\Sigma_1$  with exponentially stable zero dynamics satisfies the high-gain property, that means it is high-gain stabilizable by a static proportional output feedback if the gain is sufficiently large.

Finally, in Section 5 we show that the well known funnel controller from finite-dimensional systems is also feasible for systems belonging to  $\Sigma_1$  and having exponentially stable zero dynamics. This means, tracking of a large class of reference signals within a prespecified funnel is possible with a time-varying non-monotone gain.

The above theoretical results are verified for the one-dimensional heat equation in Section 6.

To make the presentation more readable, we have delegated some of the lemmata and proofs to Appendices A and B. Appendix A also contains some basic definitions from the theory of evolution equations in Hilbert spaces.

## Nomenclature

$\mathbb{R}_{\geq 0}$	$[0, \infty)$
$0_{\mathbb{R}^r}$	$(0, \dots, 0)^\top \in \mathbb{R}^r$
$H$ (or $H, (\cdot, \cdot)$ )	Hilbert space $H$ with inner product $(\cdot, \cdot)$
$\mathcal{B}(H, X), \mathcal{B}(H)$	the set of bounded linear operators from $H$ to a Hilbert space $X$ or to $H$ , resp.
$L^\infty(\mathbb{R}_{\geq 0}; H)$	the set of equivalence classes of essentially bounded, strongly measurable functions $f : \mathbb{R}_{\geq 0} \rightarrow H$
$L^1(\mathbb{R}_{\geq 0}; H)$	the set of equivalence classes of integrable functions $f : \mathbb{R}_{\geq 0} \rightarrow H$
$L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, H)$	the set of locally integrable functions $f : \mathbb{R}_{\geq 0} \rightarrow H$
$\mathcal{C}^\ell(\mathbb{R}_{\geq 0}; H)$	the set of $\ell$ -times continuously differentiable functions $f : \mathbb{R}_{\geq 0} \rightarrow H$
$\mathcal{C}(\mathbb{R}_{\geq 0}; H)$	the set of continuous functions $f : \mathbb{R}_{\geq 0} \rightarrow H$
$W^{1,\infty}(\mathbb{R}_{\geq 0}, H)$	the set of functions which together with their first (distributional) derivative belong to $L^\infty(\mathbb{R}_{\geq 0}; H)$
$\text{dom } A$	the domain of the linear operator $A$
$A^{-*} := (A^{-1})^*$	the adjoint of the inverse of a closed, bijective operator $A$ in $H$ , note that $A^{-*} = (A^*)^{-1}$
$M \dot{+} N$	the direct sum of two (linear) subspaces $M, N$ . Here, a subspace is always closed.
$\text{ls } \{x\}$	the linear span of $x \in H$
$\pi_{X_i}$	the projector onto the $i$ -th component of a vector $x \in X_1 \times X_2$ , $i = 1, 2$

## 2 The Byrnes-Isidori form

We start with the definition of the Byrnes-Isidori form.

**Definition 2.1.** A system  $(A, b, c) \in \Sigma_r$  is said to be in *Byrnes-Isidori form* if, and only if,  $H = \mathbb{R}^r \times V$  for some Hilbert space  $V$  and  $(A, b, c)$  satisfies the following:

- (i) There exists an operator  $Q : \text{dom } Q \subset V \rightarrow V$  which generates a strongly continuous semigroup in  $V$ .
- (ii) The operator  $A$  has the domain

$$\text{dom } A = \mathbb{R}^r \times \text{dom } Q$$

and a representation with respect to of the form

$$A \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-2} & P_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \quad \forall \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \in \text{dom } A, \quad (2.1)$$

where

(iii) the operators  $P_0, \dots, P_{r-1} : \mathbb{R} \rightarrow \mathbb{R}$ ,  $S : V \rightarrow \mathbb{R}$ ,  $R : \mathbb{R} \rightarrow V$  are bounded;

(iv) The vectors  $b$  and  $c$  have the form

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_r \\ 0 \end{pmatrix} \in \mathbb{R}^r \times V, \quad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^r \times V.$$

◇

The main result of the present section is to show that every system  $(A, b, c) \in \Sigma_r$  can be transformed into Byrnes-Isidori form. The appropriate transformation is constructed with the help of the following three lemmata.

A simple, but for our analysis instrumental, observation is that the Hilbert space  $H$  may be decomposed, for any  $(A, b, c) \in \Sigma_r$ , into the direct sum

$$H = \underbrace{\text{ls}\{c\} \dot{+} \text{ls}\{A^*c\} \dot{+} \dots \dot{+} \text{ls}\{A^{*r-1}c\}}_{=:H_{c,A}} \dot{+} \underbrace{\{b\}^\perp \cap \{Ab\}^\perp \cap \dots \cap \{A^{r-1}b\}^\perp}_{=:H_{A,b}}. \quad (2.2)$$

This follows immediately from Assumptions (A2) and (A3): first,  $c, A^*c, \dots, A^{*r-1}c$  are linearly independent, and secondly

$$\forall i \in \{0, \dots, r-1\} : H_{A,b} \cap \text{ls}\{A^{*i}c\} = \{0\};$$

hence the sum  $H_{c,A} \dot{+} H_{A,b}$  is direct and since  $H_{A,b}$  has by definition at most codimension  $r$ , equation (2.2) follows.

**Lemma 2.2.** *Let  $(A, b, c) \in \Sigma_r$  and define the operators*

$$P^m : H \rightarrow \mathbb{R}, \quad x \mapsto P^m x := P_{m+1}^m x - \sum_{j=m+2}^r P_j^m x, \quad m = 0, \dots, r-1, \quad (2.3)$$

where

$$P_{m+1}^m : H \rightarrow \mathbb{R}, \quad x \mapsto P_{m+1}^m x := \frac{(x, A^{r-(m+1)}b)}{(c, A^{r-1}b)}, \quad m = 0, 1, \dots, r-1,$$

$$P_j^m : H \rightarrow \mathbb{R}, \quad x \mapsto P_j^m x := \left( P_{m+1}^m A^{*j-1}c - \sum_{k=m+2}^{j-1} P_k^m A^{*j-1}c \right) \frac{(x, A^{r-j}b)}{(c, A^{r-1}b)}, \quad j = m+2, \dots, r.$$

Then for any  $\ell, m \in \{0, \dots, r-1\}$  we have

$$P^m H_{A,b} = \{0\} \quad (2.4)$$

$$P^m A^{*\ell}c = \begin{cases} 1, & \text{if } \ell = m, \\ 0, & \text{if } \ell \neq m. \end{cases} \quad (2.5)$$

*Proof.* Assertion (2.4) follows from Assumption (A3) and the definition of  $P^m$ . Similarly, Assertion (2.5) follows for  $\ell = m$  and  $\ell = 0, \dots, m-1$ . It remains to show (2.5) for  $\ell \in \{m+1, \dots, r-1\}$ . By definition of  $P_j^m$  and Assumption (A3) we have

$$P_j^m A^{*\ell} c = 0 \quad \text{for all } j = \ell + 2, \dots, r, \quad (2.6)$$

and therefore

$$\begin{aligned} P^m A^{*\ell} c &= P_{m+1}^m A^{*\ell} c - \sum_{j=m+2}^r P_j^m A^{*\ell} c \\ &= P_{m+1}^m A^{*\ell} c - P_{\ell+1}^m A^{*\ell} c - \sum_{j=m+2}^{\ell} P_j^m A^{*\ell} c \\ &= P_{m+1}^m A^{*\ell} c - \sum_{j=m+2}^{\ell} P_j^m A^{*\ell} c - \underbrace{\left( P_{m+1}^m A^{*\ell} c - \sum_{k=m+2}^{\ell} P_k^m A^{*\ell} c \right)}_{=1} \frac{(A^{*\ell} c, A^{r-1-\ell} b)}{(c, A^{r-1} b)} \\ &= 0. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 2.3.** *Let  $(A, b, c) \in \Sigma_r$  and use the notation as in Lemma 2.2. Then the operator*

$$P_{A,b} : H \rightarrow H, \quad x \mapsto P_{A,b} x := \left( I - \sum_{j=0}^{r-1} A^{*j} c P^j \right) x \quad (2.7)$$

is a projection onto  $H_{A,b}$ , and every  $x \in H$  has a unique decomposition with respect to (2.2) of the form

$$x = (P^0 x) c + (P^1 x) A^* c + \dots + (P^{r-1} x) A^{*r-1} c + P_{A,b} x. \quad (2.8)$$

*Proof.* By definition of  $P_{A,b}$  and (2.4) we have

$$P_{A,b} x = x \quad \text{for all } x \in H_{A,b}$$

and, by (2.5), we have  $\text{ls}\{c\} \dot{+} \text{ls}\{A^*c\} \dot{+} \dots \dot{+} \text{ls}\{A^{*r-1}c\} = H_{c,A} \subset \ker P_{A,b}$ . Hence, in view of (2.2),  $P_{A,b}$  is a projection. Finally, (2.8) is a direct consequence of the definition of  $P_{A,b}$ .  $\square$

**Lemma 2.4.** *The operator*

$$U : H \rightarrow \mathbb{R}^r \times H_{A,b}, \quad x \mapsto Ux := \begin{pmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1} x \\ P_{A,b} x \end{pmatrix} \quad (2.9)$$

is bounded and bijective with inverse

$$U^{-1} : \mathbb{R}^r \times H_{A,b} \rightarrow H, \quad \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \mapsto \sum_{j=0}^{r-1} \alpha_j A^{*j} c + \eta. \quad (2.10)$$

Furthermore, with the orthogonal projector  $P^\perp : H \rightarrow H$  onto  $H_{A,b}$ , we have

$$U^{-*} : H \rightarrow \mathbb{R}^r \times H_{A,b}, \quad x \mapsto \begin{pmatrix} (x, c) \\ (x, A^*c) \\ \vdots \\ (x, A^{*(r-1)}c) \\ P^\perp x \end{pmatrix}. \quad (2.11)$$

*Proof.* The assertions about  $U$  and its inverse are a direct consequence of Lemma 2.3. The formula for  $U^{-*}$  follows since for all  $(\alpha_0, \dots, \alpha_{r-1}, \eta)^\top \in \mathbb{R}^r \times H_{A,b}$  and all  $x \in H$  we have

$$\left( x, U^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \right)_H = \left( x, \sum_{j=0}^{r-1} \alpha_j A^{*j} c + \eta \right)_H = \left( \begin{pmatrix} (x, c) \\ (x, A^*c) \\ \vdots \\ (x, A^{*(r-1)}c) \\ P^\perp x \end{pmatrix}, \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \right)_{\mathbb{R}^r \times H_{A,b}}.$$

□

We are now in a position to state the main result of this section and show that any system  $(A, b, c) \in \Sigma_r$  may be transformed into Byrnes-Isidori form.

**Theorem 2.5** (Byrnes-Isidori form). *Let  $(A, b, c) \in \Sigma_r$ . Then the bijective and bounded operator  $U : H \rightarrow \mathbb{R}^r \times H_{A,b}$  defined in (2.9) converts the system  $(A, b, c)$  into the system*

$$(\widehat{A}, \widehat{b}, \widehat{c}) := (U^{-*}AU^*, U^{-*}b, Uc), \quad \text{with} \quad \text{dom } \widehat{A} := U^{-*} \text{dom } A, \quad (2.12)$$

which is in Byrnes-Isidori form. More precisely, we have

$$(\widehat{A}, \widehat{b}, \widehat{c}) = \left( \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-2} & P_{r-1} & S \\ R & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \begin{pmatrix} 0 \\ \vdots \\ (A^{r-1}b, c) \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right), \quad (2.13)$$

where  $P^\perp : H \rightarrow H$  denotes the orthogonal projector onto  $H_{A,b}$  and

$$P_i = P^i A^{*r} c \quad \forall i \in \{0, \dots, r-1\} \quad (2.14)$$

$$R : \mathbb{R} \rightarrow H_{A,b}, \quad \alpha \mapsto R\alpha = \frac{\alpha}{(c, A^{r-1}b)} P^\perp A^r b, \quad (2.15)$$

$$S : H_{A,b} \rightarrow \mathbb{R}, \quad \eta \mapsto S\eta = (P_{A,b} A^{*r} c, \eta) \quad (2.16)$$

$$Q\eta = P^\perp A\eta - R(c, \eta) \quad \forall \eta \in \text{dom } Q = H_{A,b} \cap \text{dom } A, \quad (2.17)$$

$$\text{dom } \widehat{A} = \mathbb{R}^r \times (H_{A,b} \cap \text{dom } A) = \mathbb{R}^r \times \text{dom } Q, \quad (2.18)$$

and  $Q$  generates a strongly continuous semigroup  $(T_Q(t))_{t \geq 0}$  in  $H_{A,b}$ .



The proof is in Appendix B.

**Remark 2.6.**

- (i) Let  $(A, b, c) \in \Sigma_r$  be in Byrnes-Isidori form as in Definition 2.1. Then  $A$  has a block operator structure of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & Q \end{bmatrix}, \quad (2.19)$$

where  $A_{11} : \mathbb{R}^r \rightarrow \mathbb{R}^r$ ,  $A_{12} : H_{A,b} \rightarrow \mathbb{R}^r$  and  $A_{21} : \mathbb{R}^r \rightarrow H_{A,b}$  are bounded operators, and only  $Q$  is (possibly) unbounded.

- (ii) Let  $(A, b, c) \in \Sigma_r$  be transformed into Byrnes-Isidori form  $(\hat{A}, \hat{b}, \hat{c})$  as in (2.12). Then  $U^{-*}$  applied to the mild solution (1.2) yields

$$U^{-*}x(t) = U^{-*}x^0 + \int_0^t U^{-*}T(t-s)U^*\hat{b}u(s) ds, \quad t \geq 0, \quad (2.20)$$

and since the semigroup  $(\hat{T}(t))_{t \geq 0}$  generated by  $\hat{A}$  is  $\hat{T}(t) = U^{-*}T(t)U^*$ , we conclude

$$(x, u, y) \in \mathfrak{B}_{(A,b,c)} \iff (U^{-*}x, u, y) \in \mathfrak{B}_{(\hat{A}, \hat{b}, \hat{c})}.$$

Next we rewrite the mild solution of a system in Byrnes-Isidori form as a functional differential equation. This gives an equation in the output variable  $y(\cdot)$  only and shows a simpler input/output structure.

**Proposition 2.7** (Internal loop form). *Let  $(A, b, c) \in \Sigma_r$ ,  $x^0 \in H$ ,  $u \in L_{\text{loc}}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ , and consider the system (1.1). Then, with the notation as in Theorem 2.5, the following are equivalent.*

- (i)  $\exists (x, u, y) \in \mathfrak{B}_{(A,b,c)}$  with  $x(0) = x^0$ .

- (ii) The function  $y$  is  $r - 1$ -times continuously differentiable and satisfies

$$\begin{pmatrix} y(t) \\ \vdots \\ y^{(r-2)}(t) \\ y^{(r-1)}(t) \end{pmatrix} = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, A^{*r-2}c) \\ (x^0, A^{*r-1}c) \end{pmatrix} + \begin{pmatrix} \int_0^t y^{(1)}(s) ds \\ \vdots \\ \int_0^t y^{(r-1)}(s) ds \\ \int_0^t \sum_{i=0}^{r-1} P_i y^{(i)}(s) + S\eta(s) + (A^{r-1}b, c)u(s) ds \end{pmatrix}, \quad (2.21)$$

$$\eta(t) = T_Q(t)P^\perp x^0 + \int_0^t T_Q(t-s)[R, 0, \dots, 0](y(s), \dots, y^{(r-1)}(s))^\top ds \quad \forall t \geq 0. \quad (2.22)$$

- (iii) The function  $y$  is  $r - 1$ -times continuously differentiable and its  $r$ th derivative satisfies

$$y^{(r)}(t) = \sum_{i=0}^{r-1} P_i y^{(i)}(t) + (\mathbf{T}_{\eta^0} y)(t) + (A^{r-1}b, c)u(t) \quad a.e., \quad (2.23)$$

and, for the orthogonal projector  $P^\perp : H \rightarrow H$  onto  $H_{A,b}$ ,

$$\begin{pmatrix} y(0) \\ \vdots \\ y^{(r-1)}(0) \\ \eta(0) \end{pmatrix} = U^{-*}x^0 = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, A^{*r-1}c) \\ P^\perp x^0 \end{pmatrix}, \quad (2.24)$$

where for  $\eta^0 = \eta(0)$  the causal linear operator  $\mathbf{T}_{\eta^0}$  is defined by

$$\begin{aligned} \mathbf{T}_{\eta^0} : L_{\text{loc}}^1(\mathbb{R}_{\geq 0}, \mathbb{R}) &\rightarrow \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}), \\ y &\mapsto \left( t \mapsto ST_Q(t)\eta^0 + S \int_0^t T_Q(t-s) Ry(s) ds \right). \end{aligned}$$

The functions  $x$  and  $\eta$  in (i) and (ii) are related by  $x(t) = U^*(y(t), y^{(1)}(t), \dots, y^{(r-1)}(t), \eta(t))^\top$ .

*Proof.* (i)  $\Rightarrow$  (ii): Let  $(x, u, y) \in \mathfrak{B}_{(A, b, c)}$  with  $x(0) = x^0$ . Then  $x$  is a mild solution of (1.1a). By Lemma 2.4 and Remark 2.6 (ii), the boundedly invertible transformation  $U^{-*}$  maps  $x$  onto the mild solution  $(\alpha_0, \dots, \alpha_{r-1}, \eta)^\top := U^{-*}x$  of

$$\frac{d}{dt} \begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-1}(t) \\ \eta(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & 1 & 0 \\ P_0 & P_1 & \cdots & P_{r-1} & S \\ R & 0 & \cdots & 0 & Q \end{bmatrix} \begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-1}(t) \\ \eta(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b, c) \\ 0 \end{pmatrix} u(t). \quad (2.25)$$

Since  $\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & 1 \\ P_0 & P_1 & \cdots & P_{r-1} \end{bmatrix}$  is bounded, we may apply Lemma A.4 to (2.25) to conclude from (A.13) that

$$\begin{pmatrix} \alpha_0(t) \\ \vdots \\ \alpha_{r-2}(t) \\ \alpha_{r-1}(t) \end{pmatrix} = \begin{pmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-2}(0) \\ \alpha_{r-1}(0) \end{pmatrix} + \begin{pmatrix} \int_0^t \alpha_1(s) ds \\ \vdots \\ \int_0^t \alpha_{r-1}(s) ds \\ \int_0^t \sum_{i=0}^{r-1} P_i \alpha_i(s) + S\eta(s) + (A^{r-1}b, c)u(s) ds \end{pmatrix}, \quad (2.26)$$

$$\eta(t) = T_Q(t)\eta(0) + \int_0^t T_Q(t-s) [R, 0 \dots, 0] (\alpha_0(s), \dots, \alpha_{r-1}(s))^\top ds \quad \forall t \geq 0, \quad (2.27)$$

and

$$\begin{pmatrix} \alpha_0(0) \\ \vdots \\ \alpha_{r-1}(0) \\ \eta(0) \end{pmatrix} = U^{-*}x(0) = \begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, A^{*r-1}c) \\ P^\perp x^0 \end{pmatrix}.$$

Since, by (2.12) and (2.13),  $Uc = [1, 0, \dots, 0]^\top$ , we have

$$y(t) = (x(t), c) = (U^{-*}x, Uc) = \alpha_0(t) \quad \forall t \geq 0, \quad (2.28)$$

and we conclude from (2.26) that  $y^{(i)} = \alpha_0^{(i)} = \alpha_i$  for all  $i = 0, \dots, r-1$ . Hence, (ii) is shown.

(ii)  $\Rightarrow$  (iii): If (ii) holds, then the lower line of (2.21) shows that the function  $y^{(r-1)} = \alpha_{r-1}$  is absolutely continuous. Therefore, it is almost everywhere differentiable and its derivative satisfies (2.23).

(iii)  $\Rightarrow$  (i): Assume  $y$  satisfies (iii). Define  $(\alpha_0, \dots, \alpha_{r-1}) := (y, \dots, y^{(r-1)})$  and  $\eta$  by (2.22). Then (2.26) and (2.27) are fulfilled. Therefore, by Remark 2.6 (ii) and Lemma A.4, the function  $x(\cdot) := U^*(\alpha_0(\cdot), \dots, \alpha_{(r-1)}(\cdot), \eta(\cdot))^\top$  is a mild solution of (1.1a) with initial value  $x^0$ . Finally, (2.28) yields (i). This completes the proof of the proposition.  $\square$

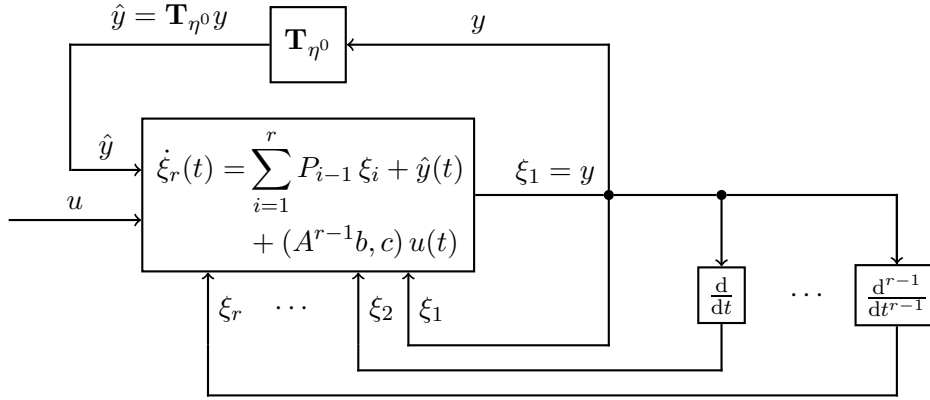


Figure 1: The internal loop form

Note that the right hand side of equation (2.23) may be interpreted as a ordinary differential term  $\sum_{i=0}^{r-1} P_i y^{(i)}(t) + (A^{r-1}b, c)u(t)$  which is perturbed by a functional term  $(\mathbf{T}_{\eta^0}y)(t)$ ; see Figure 1. This structure will be exploited to control the system in Section 5.

We close this section with a result on the uniqueness of the Byrnes-Isidori form. It shows in particular that all  $P_i$  in the entries of the representation of  $A$  in (2.1) are uniquely defined. In terms of the internal loop form in Figure 1, this means that the main block, i.e. the ordinary differential equation, is uniquely given. Moreover, the input-output behavior of the perturbation block  $\mathbf{T}_{\eta^0}$  is unique, although the triple  $(Q, R, S)$  which determines the mapping  $\mathbf{T}_{\eta^0}$  is internally only unique up to a bounded bijective invertible transformation. This is made precise in the following proposition.

**Proposition 2.8** (Uniqueness of the Byrnes-Isidori form). *If  $(A, b, c) \in \Sigma_r$  is transformed by a bijective and bounded operator  $W : H \rightarrow \mathbb{R}^r \times \tilde{V}$  into the Byrnes-Isidori form*

$$(W^{-*}AW^*, W^{-*}b, Wc) = \left( \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \tilde{P}_0 & \tilde{P}_1 & \cdots & \tilde{P}_{r-2} & \tilde{P}_{r-1} & \tilde{S} \\ \tilde{R} & 0 & \cdots & 0 & 0 & \tilde{Q} \end{bmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{b}_r \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right), \quad (2.29)$$

then the entries of (2.13) and (2.29) are related as follows:

- (i)  $P_i = \tilde{P}_i$  for all  $i = 0, \dots, r-1$ , and they are uniquely defined by  $(A, b, c)$ ,
- (ii)  $(\tilde{Q}, \tilde{R}, \tilde{S}) = (\tilde{Y}Q\tilde{Y}^{-1}, \tilde{Y}R, S\tilde{Y}^{-1})$ , with  $\text{dom } \tilde{Q} = \tilde{Y} \text{ dom } \hat{Q}$  for some bounded and bijective operator  $\tilde{Y} : \tilde{V} \rightarrow H_{A,b}$ .
- (iii)  $\tilde{b}_r = b_r = (A^{r-1}b, c)$ .

The proof is in Appendix B.

### 3 Zero dynamics

In this section the zero dynamics of a system  $(A, b, c) \in \Sigma_r$  are investigated. Roughly speaking, the zero dynamics are those dynamics of the system which are not visible at the output.

**Definition 3.1.** The *zero dynamics* of a system  $(A, b, c) \in \Sigma_r$  is the real subvector space of the mild behavior defined by

$$\mathcal{ZD}_{(A,b,c)} := \{(x, u, y) \in \mathfrak{B}_{(A,b,c)} \mid y \equiv 0\}.$$

The Byrnes-Isidori form is instrumental to simplify the presentation of the zero dynamics as will be shown in the following proposition.

**Proposition 3.2.** *Let  $(A, b, c) \in \Sigma_r$ . Then, with the notation as in Theorem 2.5, the zero dynamics is given by*

$$\mathcal{ZD}_{(A,b,c)} = \left\{ \left( U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot) P^\perp x^0 \end{pmatrix}, -\frac{ST_Q(\cdot) P^\perp x^0}{(A^{r-1}b, c)}, 0 \right) \middle| x^0 \in H \right\}, \quad (3.1)$$

where  $(T_Q(t))_{t \geq 0}$  denotes the semigroup generated by  $Q$ .

*Proof.* Let  $(x, u, y) \in \mathcal{ZD}_{(A,b,c)}$  with  $x(0) = x^0$ . By Proposition 2.7,  $(y, \dots, y^{(r-1)})^\top$  and  $\eta(\cdot) := P^\perp x(\cdot)$  satisfy (2.21) and (2.22). Since  $y = 0$ , we have  $y^{(i)} = 0$  for all  $i = 0, \dots, r-1$ , and solving equations (2.21) and (2.22) for  $u$  and  $\eta$  yields  $\eta(t) = T_Q(t) P^\perp x^0$  and  $u(t) = -(A^{r-1}b, c)^{-1} S \eta(t)$ . Since Proposition 2.7 also states that  $x(t) = U^*(y(t), \dots, y^{(r-1)}(t), \eta(t))^\top$ , the triple  $(x, u, y)$  belongs to the right hand side of (3.1).

Conversely, let some  $\tilde{x}^0 \in H$  be given and define  $x^0 := U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ P^\perp \tilde{x}^0 \end{pmatrix}$ . Then (2.11) shows

$$\begin{pmatrix} (x^0, c) \\ \vdots \\ (x^0, A^{*r-1}c) \\ P^\perp x^0 \end{pmatrix} = U^{-*} x^0 = \begin{pmatrix} 0_{\mathbb{R}^r} \\ P^\perp \tilde{x}^0 \end{pmatrix}.$$

Using this equation, it can be seen that the functions

$$y(t) := 0, \quad u(t) := -\frac{ST_Q(t) P^\perp \tilde{x}^0}{(A^{r-1}b, c)}, \quad \eta(t) = T_Q(t) P^\perp \tilde{x}^0 \quad \forall t \geq 0.$$

satisfy (ii) of Proposition 2.7 with  $x^0$ . Hence, this proposition implies that

$$\left( U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot) P^\perp \tilde{x}^0 \end{pmatrix}, -\frac{ST_Q(\cdot) P^\perp \tilde{x}^0}{(A^{r-1}b, c)}, 0 \right) = \left( U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ \eta(\cdot) \end{pmatrix}, u(\cdot), y(\cdot) \right) \in \mathfrak{B}_{(A,b,c)}.$$

Since  $y \equiv 0$ , the left hand side is an element of the zero dynamics. □

Exponential stability of the zero dynamics and of a semigroup are defined as follows.

**Definition 3.3.** A system  $(A, b, c) \in \Sigma_r$  is said to have *exponentially stable zero dynamics* if, and only if,

$$\exists M, \mu > 0 \forall (x, u, 0) \in \mathcal{ZD}_{(A,b,c)} \forall t \geq 0 : \|(x(t), u(t))\| \leq M \|x(0)\| e^{-\mu t}. \quad (3.2)$$

A semigroup  $(T(t))_{t \geq 0}$  is called *exponentially stable* if, and only if, there exist  $M, \mu > 0$  such that

$$\|T(t)\| \leq M e^{-\mu t} \quad \forall t \geq 0. \quad (3.3)$$

◇

Now Proposition 3.2 leads to a characterization of exponential stability of the zero dynamics in terms of the Byrnes-Isidori form:

**Proposition 3.4.** *A system  $(A, b, c) \in \Sigma_r$  has exponentially stable zero dynamics if, and only if, the operator  $Q$  in Theorem 2.5 (i) generates an exponentially stable semigroup.*

*Proof.* First note that Theorem 2.5 implies that  $Q$  generates a strongly continuous semigroup  $T_Q$ . If this semigroup is exponentially stable, then the assertion is an immediate consequence of Proposition 3.2. Assume on the other hand that  $(A, b, c)$  has exponentially stable zero dynamics and let  $x^0 \in H_{A,b}$  be arbitrary. Then  $x^0 = P^{-1}x^0$  and equation (3.1) shows that

$$\left( U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(\cdot)x^0 \end{pmatrix}, -\frac{ST_Q(\cdot)x^0}{(A^{r-1}b, c)}, 0 \right) \in \mathcal{ZD}_{(A,b,c)}.$$

Thus, the stability assumption (3.2) implies that

$$\forall t \geq 0 : \left\| U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ T_Q(t)x^0 \end{pmatrix} \right\| \leq M \left\| U^* \begin{pmatrix} 0_{\mathbb{R}^r} \\ x^0 \end{pmatrix} \right\| e^{-\mu t}.$$

Since  $U^*$  is boundedly invertible, we conclude  $\|T_Q(t)x^0\| \leq \|U^*\| \|U^{-*}\| M e^{-\mu t} \|x^0\|$ . This shows the exponential stability of the semigroup, because  $M$  and  $\mu$  are by assumption independent of  $x^0$ .  $\square$

In view of the internal loop form in Figure 1 we may observe: If  $(A, b, c) \in \Sigma_r$  has exponentially stable zero dynamics, then Proposition 3.4 and (3.3) show that  $\mathbf{T}_{\eta^0}$  maps bounded functions to bounded functions. This property is crucial for the high-gain stabilizability results that we derive in the next two sections.

## 4 High-gain stabilizability

In this section we concentrate on systems with relative degree  $r = 1$  and show *high-gain stabilizability*: if  $(A, b, c) \in \Sigma_1$  has exponentially stable zero dynamics, then it is stabilizable by proportional output feedback  $u(t) = -k \operatorname{sgn}(b, c) y(t)$  provided the gain  $k > 0$  is sufficiently large. This feedback applied to (1.1) yields a closed-loop system

$$\dot{x}(t) = Ax(t) - \operatorname{sgn}(b, c) k b(x(t), c),$$

which is by the following proposition exponentially stable.

**Proposition 4.1** (High-gain stabilizability). *Let  $(A, b, c) \in \Sigma_1$  have relative degree  $r = 1$  and exponentially stable zero dynamics. Then there exists a  $k^* > 0$  such that for all  $k \geq k^*$  the operator*

$$A_k : \operatorname{dom} A \subset H \rightarrow H, \quad x \mapsto Ax - \operatorname{sgn}(b, c) k b(x, c)$$

*generates an exponentially stable semigroup.*

*Proof.* Note that according to [6, Section III.1.3],  $A_k$  generates a semigroup since  $A - A_k$  is a bounded operator. Furthermore, by Theorem 2.5 we have for  $x \in H$

$$b(x, c) = U^* \widehat{b}(U^{-*}x, \widehat{c}) = U^* \begin{pmatrix} (b, c) \\ 0 \end{pmatrix} \left( U^{-*}x, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = U^* \begin{bmatrix} -(b, c) & 0 \\ 0 & 0 \end{bmatrix} U^{-*}x$$

and hence

$$A_k = U^* \left( \widehat{A} - \begin{bmatrix} k|(b, c)| & 0 \\ 0 & 0 \end{bmatrix} \right) U^{-*} = U^* \begin{bmatrix} P_0 - k|(b, c)| & S \\ R & Q \end{bmatrix} U^{-*}$$

with the boundedly invertible transformation  $U^*$ . So it suffices to show that the semigroup generated by

$$\widehat{A}_k : \text{dom } \widehat{A} \subset \mathbb{R}^r \times H_{A,b} \rightarrow \mathbb{R}^r \times H_{A,b}, \quad \widehat{A}_k := \begin{bmatrix} P_0 - k|(b,c)| & S \\ R & Q \end{bmatrix}$$

is exponentially stable for large  $k$ . By [5, Theorem 5.1.5] this is the case if, and only if,  $\sigma(\widehat{A}_k) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda < 0\}$  and

$$\sup_{\text{Re } \lambda > 0} \|(\lambda - \widehat{A}_k)^{-1}\| < \infty. \quad (4.1)$$

By Proposition 3.4,  $Q$  generates an exponentially stable semigroup and thus, there exists a  $k_Q > 0$  with

$$\sup_{\text{Re } \lambda > 0} \|(\lambda - Q)^{-1}\| \leq k_Q. \quad (4.2)$$

Now choose  $k^*$  such that

$$k^*|(b,c)| - P_0 > k_Q \|R\| \|S\|.$$

Then for all  $k \geq k^*$  and all  $\lambda \in \mathbb{C}$  such that  $\text{Re } \lambda > 0$ , the number  $h_\lambda := \lambda - P_0 + k|(b,c)| - S(\lambda - Q)^{-1}R \neq 0$  and

$$\left| \frac{1}{h_\lambda} \right| \leq \left| \frac{1}{|\lambda - P_0 + k|(b,c)| - \|S\| \|R\| k_Q} \right| < \frac{1}{k^*|(b,c)| - P_0 - \|S\| \|R\| k_Q}. \quad (4.3)$$

It is well known, see e.g. [15, Lemma A.4.2(iii)], that

$$\lambda - \widehat{A}_k = \begin{bmatrix} \lambda - P_0 + k|(b,c)| & -S \\ -R & \lambda - Q \end{bmatrix}$$

is invertible if the operators  $h_\lambda$  and  $\lambda - Q$  are invertible and that in this case

$$(\lambda - \widehat{A}_k)^{-1} = \begin{bmatrix} \frac{1}{h_\lambda} & \frac{1}{h_\lambda} S(\lambda - Q)^{-1} \\ -(\lambda - Q)^{-1} R \frac{1}{h_\lambda} & (\lambda - Q)^{-1} + (\lambda - Q)^{-1} R \frac{1}{h_\lambda} S(\lambda - Q)^{-1} \end{bmatrix}.$$

The uniform bounds (4.2) and (4.3) imply that (4.1) holds and hence, the claim follows from [5, Theorem 5.1.5].  $\square$

The high-gain result in Proposition 4.1 has some drawbacks: First, the size of the gain depends on the system data, and this data may not be given explicitly. In fact, the assumptions that  $(A, b, c)$  is in  $\Sigma_1$  and has exponential stability are only structural. So if we did not have to determine the size of  $k$  in Proposition 4.1 a priori, we would need very little information to stabilize the system. Secondly, if a feasible size of the gain is chosen, it may be too large so that corruption of the output is amplified. A different feedback to resolve these drawbacks is introduced in the next section.

## 5 Funnel control

In this section, we assume that the system  $(A, b, c) \in \Sigma_r$  has exponentially stable zero dynamics and relative degree  $r = 1$ . Note that these assumptions on the system are only structural, that means no system data are required. We are going to design a special time-varying proportional feedback-gain such that the closed-loop system has a global solution that is bounded with respect to the state space norm. Besides this stability, we desire to achieve two further control objectives: The first one is approximate tracking, by the output  $y$ , of reference signals  $y_{\text{ref}}$  of class  $W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ . In particular, for arbitrary  $\lambda > 0$ , we seek an output feedback strategy which ensures that, for every  $y_{\text{ref}} \in$

$W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ , the closed-loop system has bounded solution and the tracking error  $e(t) = y(t) - y_{\text{ref}}(t)$  is ultimately bounded by  $\lambda$  (that is,  $\|e(t)\| \leq \lambda$  for all  $t$  sufficiently large). The second control objective is prescribed transient behavior of the tracking error signal. We capture both objectives in the concept of a performance funnel

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid e \in \left(-\frac{1}{\varphi(t)}, \frac{1}{\varphi(t)}\right) \right\},$$

the boundary of which is determined by the reciprocal of a function  $\varphi$  belonging to

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \mid \begin{array}{l} \varphi(0) = 0, \varphi(s) > 0 \forall s > 0, \liminf_{s \rightarrow \infty} \varphi(s) > 0 \\ \forall \delta > 0 \exists \text{ global Lipschitz bound of } \varphi^{-1} \text{ on } [\delta, \infty) \end{array} \right\}.$$

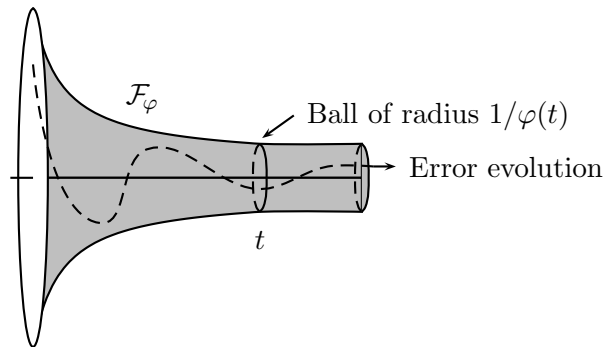


Figure 2: Prescribed performance funnel  $\mathcal{F}_\varphi$ .

The aim is an output feedback strategy ensuring that, for every reference signal  $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$ , the tracking error  $e = y - y_{\text{ref}}$  evolves within the funnel  $\mathcal{F}_\varphi$ ; see Fig. 2. For example, if  $\liminf_{t \rightarrow \infty} \varphi(t) > 1/\lambda$ , then evolution within the funnel ensures that the first control objective is achieved. If  $\varphi$  is chosen as the function  $t \mapsto \min\{t/T, 1\}/\lambda$ , then evolution within the funnel ensures that the prescribed tracking accuracy  $\lambda > 0$  is achieved within the prescribed time  $T > 0$ .

Loosely speaking, funnel control exploits an inherent benign high-gain property of the system by designing – with appropriate choice of  $\varphi \in \Phi$  – a proportional error feedback  $u(t) = -k(t)e(t)$  in such a way that  $k(t)$  becomes large if  $|e(t)|$  approaches the performance funnel boundary (equivalently, if  $\varphi(t)|e(t)|$  approaches the value 1), thereby precluding contact with the funnel boundary. We emphasize that the gain is non-monotone and decreases as the error recedes from the funnel boundary. The essence of the proof of the main result lies in showing that the closed-loop system is well-posed in the sense that  $u$  and  $k$  are bounded functions and the error evolves strictly within the performance funnel.

For  $\varphi \in \Phi$ , the *funnel controller* can be expressed in its simplest form as

$$\boxed{u(t) = -k(t) \operatorname{sgn}(b, c) e(t), \quad k(t) = \frac{1}{1 - \varphi(t)|e(t)|}, \quad e(t) = y(t) - y_{\text{ref}}(t).} \quad (5.1)$$

If (5.1) is applied to any system  $(A, b, c) \in \Sigma_1$  with exponentially stable zero dynamics, then the following theorem shows that the tracking error  $e(t)$  evolves within the performance funnel  $\mathcal{F}_\varphi$  and moreover, the error evolution is strictly bounded away from the funnel boundary, thereby ensuring that the gain function  $k(\cdot)$  and the control function  $u(\cdot)$  are bounded.

**Theorem 5.1.** Consider a system  $(A, b, c) \in \Sigma_1$  with relative degree  $r = 1$  and exponentially stable zero dynamics. Let  $\varphi \in \Phi$  specify the performance funnel  $\mathcal{F}_\varphi$ . Then for an arbitrary reference signal  $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$  the control (5.1) applied to (1.1) yields the closed-loop system

$$\begin{aligned} \dot{x}(t) &= Ax(t) - b \frac{1}{1 - \varphi(t)|y(t) - y_{\text{ref}}(t)|} \text{sgn}(b, c)(y(t) - y_{\text{ref}}(t)), & x(0) &= x^0, \\ y(t) &= (x(t), c). \end{aligned} \quad (5.2)$$

The system (5.2) has a solution  $x \in \mathcal{C}(\mathbb{R}_{\geq 0}, H)$  in the sense that the following equations hold for all  $t \geq 0$ :

$$\begin{aligned} x(t) &= T(t)x^0 + \int_0^t T(t-s)bu(s)ds, & \text{with} \\ u(s) &= \frac{-\text{sgn}(b, c)}{1 - \varphi(s)|x(s), c - y_{\text{ref}}(s)|} ((x(s), c) - y_{\text{ref}}(s)). \end{aligned} \quad (5.3)$$

Every function that satisfies (5.3) on an interval can be extended to a solution on  $\mathbb{R}_{\geq 0}$  in the sense of (5.3) and satisfies:

- (a) The functions  $u$ ,  $k$ , and  $e$  defined in (5.1) are bounded;
- (b)  $\exists \varepsilon \in (0, 1) \forall t > 0 : |e(t)| \leq (1 - \varepsilon)\varphi(t)^{-1}$ .

*Proof.* We use the equivalence of (i) and (iii) in Proposition 2.7. Since  $r = 1$ , Proposition 2.7 states that (5.3) is, for any fixed initial-value  $x^0 \in H$ , equivalent to the functional differential equation

$$\begin{aligned} \dot{y}(t) &= P_0y(t) + (\mathbf{T}_{\eta^0}y)(t) + (b, c)u(t) & \text{a.e.}, \\ u(t) &= \frac{-\text{sgn}(b, c)}{1 - \varphi(t)|y(t) - y_{\text{ref}}(t)|} [y(t) - y_{\text{ref}}(t)], \\ y(0) &= \pi_{\mathbb{R}}U^{-*}x^0 = (x^0, c), \end{aligned} \quad (5.4)$$

with the operator  $\mathbf{T}_{\eta^0}$  parametrized by  $\eta^0 := P^\perp x^0$ . More precisely, for any solution  $x$  of (5.3), the function  $y(t) := (x(t), c)$  satisfies (5.4), and, conversely, if (5.4) has a solution  $y \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$ , then the function  $x(\cdot) := U^*(y(\cdot), \eta(\cdot))^\top$ , with  $\eta(t) := T_Q(t)\eta^0 + \int_0^t T_Q(t-s)Ry(s)ds$ , fulfils  $(x, u, y) \in \mathfrak{B}_{(A, b, c)}$ , which by definition means that (5.3) holds.

In order to apply results from [9], we write (5.4) in the equivalent form

$$\dot{y}(t) = f(p(t), (Ty)(t), \tilde{u}(t)) = P_0y(t) + \mathbf{T}_{\eta^0}(y)(t) + (b, c)u(t) \quad (5.5)$$

$$\tilde{u}(t) := \text{sgn}((b, c))u(t). \quad (5.6)$$

where

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R}, & (x_1, x_2, x_3) &\mapsto f(x_1, x_2, x_3) := x_2 + \text{sgn}(b, c)(b, c)x_3, \\ T : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}) &\rightarrow L_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}), & y &\mapsto P_0y(\cdot) + \mathbf{T}_{\eta^0}(y). \end{aligned}$$

The triple  $(0, f, T)$  satisfies [9, Definition 3]; note that the first entry of  $f$  in [9] is a perturbation which in our case is non-existent. Furthermore,  $\tilde{u}$  is a feedback of the form as in [9, Theorem 7]. By Proposition 3.4, the semigroup  $T_Q$  is exponentially stable and it is already mentioned in [9, Section 4.2] that operators of the form  $T$  with exponentially stable semigroups belong to the class considered in [9]. So all the prerequisites of [9, Theorem 7] are fulfilled and therefore there is a solution  $y \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$  of (5.5), all solutions of (5.5) can be extended to solutions on  $\mathbb{R}_{\geq 0}$  and all solutions on  $\mathbb{R}_{\geq 0}$  satisfy (a) and (b). Since (5.5)-(5.6) is equivalent to (5.4) and (5.3) is equivalent to (5.4), it follows that all claims hold for solutions of (5.3).  $\square$



## 6 Example heat equation

We consider a metal bar of length one that can be heated on every point simultaneously according to

$$\begin{aligned}
 \partial_t x(\xi, t) &= \partial_\xi^2 x(\xi, t) + u(\xi, t), & \xi \in [0, 1], \quad t \geq 0, \\
 x(\xi, 0) &= x^0(\xi), & \xi \in [0, 1], \\
 \partial_\xi x(0, t) &= \partial_\xi x(1, t) = 0, & t \geq 0, \\
 y(t) &= \int_0^1 \cos^2(\pi\xi)x(\xi, t) \, d\xi.
 \end{aligned} \tag{6.1}$$

The evaluation of the function  $x(\xi, t)$  represents the temperature at position  $\xi$  and time  $t$ ; the initial temperature profile is  $x^0(\xi)$ , and  $u(\xi, t)$  denotes the input function for the heat. It is well-known (see [5, Example 2.1.1]) that the partial differential equation (6.1) can be modelled in the form (1.1) by choosing  $H = L^2(0, 1)$  and

$$A : \text{dom } A \subset H \rightarrow H, \quad Af := f'', \quad \text{dom } A = \{f \in W^{1,2}(0, 1) \mid f'(0) = f'(1) = 0\},$$

$b(\xi) := 1$  and  $c(\xi) := \cos^2(\pi\xi)$ . Clearly,  $A$  is a self-adjoint operator in  $H$ , the vectors  $b$  and  $c$  are in  $\text{dom } A = \text{dom } A^*$ , and since

$$(b, c) = \int_0^1 \cos^2(\pi\xi) \, d\xi = \frac{1}{2},$$

it follows that  $(A, b, c) \in \Sigma_1$ . According to Theorem 2.5 we may transform  $(A, b, c)$  into Byrnes-Isidori form. We calculate the entries of this form:

$$\begin{aligned}
 P^0 x &= P_1^0 x = \frac{(x, b)}{(b, c)} = 2 \int_0^1 x(\xi) \, d\xi, \\
 P_{A,b} x &= x - cP^0 x = x - 2 \cos^2(\pi \cdot) \int_0^1 x(\xi) \, d\xi \quad \forall x \in H,
 \end{aligned}$$

and by (2.2) we have  $H_{A,b} = \{b\}^\perp$ . Since

$$(A^*c)(\xi) = \frac{d^2}{d\xi^2} \cos^2(\pi\xi) = 2\pi^2 (\sin^2(\pi\xi) - \cos^2(\pi\xi)) \quad \text{and} \quad Ab = 0 \quad \text{for all } \xi \in [0, 1],$$

it follows that

$$\begin{aligned}
 P_0 &= P^0 A^* c = 2 \int_0^1 2\pi^2 (\sin^2(\pi\xi) - \cos^2(\pi\xi)) \, d\xi = 0, \\
 R &= \frac{1}{(b, c)} Ab = 0, \\
 S\eta &= (P_{A,b} A^* c, \eta) = 2\pi^2 \int_0^1 (\sin^2(\pi\xi) - \cos^2(\pi\xi)) \eta(\xi) \, d\xi \quad \forall \eta \in \{b\}^\perp, \\
 Q\eta &= P^\perp A\eta - R(c, \eta) = P^\perp A\eta, \quad \forall \eta \in \{b\}^\perp \cap \text{dom } A.
 \end{aligned}$$

It is well-known that the eigenvalues of  $A$  are  $\{-n^2\pi^2 \mid n = 0, 1, 2, \dots\}$  with corresponding eigenvectors  $\phi_n(\xi) := \sqrt{2} \cos(n\pi\xi)$ ; see for example [5, Example 2.3.7]. Since  $\phi_0 = b$  and  $\{\phi_n(\xi) \mid n = 0, 1, \dots\}$  is an orthonormal basis of  $L^2(0, 1)$ , we have

$$H_{A,b} = \overline{\text{ls}\{\phi_n \mid n = 1, 2, \dots\}},$$

which shows that  $H_{A,b}$  is invariant under  $A$  and hence

$$Q = P^\perp A|_{H_{A,b} \cap \text{dom } A} = A|_{H_{A,b} \cap \text{dom } A}, \quad \text{dom } Q = \{b\}^\perp \cap \text{dom } A.$$

So the Byrnes-Isidori form reads

$$\widehat{A} = \begin{bmatrix} P_0 & S \\ R & Q \end{bmatrix} = \begin{bmatrix} 0 & S \\ 0 & A|_{\{b\}^\perp \cap \text{dom } A} \end{bmatrix}, \quad \widehat{b} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad \widehat{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

By Proposition 3.4, the system  $(A, b, c)$  has exponentially stable zero dynamics if, and only if,  $Q$  generates an exponentially stable semigroup in  $H_{A,b}$ . The latter is true because the operator  $A|_{H_{A,b} \cap \text{dom } A}$  has the eigenvalues  $\{-n^2\pi^2 \mid n = 1, 2, \dots\}$  with corresponding eigenvectors  $\{\phi_n \mid n = 1, 2, \dots\}$  and hence, by [15, Example 3.3.3 and 3.3.5], it generates a semigroup with growth bound  $-\pi^2$ .

Therefore, the system (6.1) fulfills the prerequisites of Theorem 5.1 and the funnel control (5.1) is feasible.

We illustrate this by a numerical simulation. The class  $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$  of reference signals consists of bounded signals with essentially bounded derivative, or in other words, bounded functions that are uniformly Lipschitz. For our simulation, we have chosen a rather vivid signal: the first component of the chaotic Lorenz system

$$\begin{aligned} \dot{\zeta}_1 &= 10(\zeta_2 - \zeta_1), & \zeta_1(0) &= -8 \\ \dot{\zeta}_2 &= 28\zeta_1 - \zeta_2 - \zeta_1\zeta_3, & \zeta_2(0) &= 8 \\ \dot{\zeta}_3 &= \zeta_1\zeta_2 - \frac{8}{3}\zeta_3, & \zeta_3(0) &= 27, \end{aligned} \tag{6.2}$$

see [8, Ex. 3.1.27]. The signal  $y_{\text{ref}}(\cdot) := \zeta_1$  is indeed of class  $W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$  since the unique solution of (6.2) is bounded and has a bounded derivative as by [8, Ex. 3.2.33] solutions are attracted to a compact set. The function determining the funnel boundary was chosen to be  $\varphi(t) = \min\{2t, 6\}$ . In Figure 3 the numerical solution of the partial differential equation (6.1) controlled by the funnel controller (5.1) and initial value

$$x^0(\xi) = \begin{cases} 5, & \xi \in [0, 1/2], \\ -5, & \xi \in (1/2, 1]. \end{cases}$$

is depicted. It can be seen in Figure 3 that the output trajectory  $y$  evolves strictly within the funnel, indicated by solid red lines, around the reference trajectory  $y_{\text{ref}}$ , and that the input is large only if the output is close to the reference signal.

## Appendix A Mild solutions

In this section we collect some basic facts on mild solutions needed for the proofs of our main results.

**Definition A.1.** Let  $A : \text{dom } A \subset H \rightarrow H$  generate a semigroup on  $H$ . Then we define for some  $\lambda \in \rho(A)$  the norm

$$\|x\|_{H_{A,-1}} := \|(\lambda - A)^{-1}x\|_H \quad \forall x \in H,$$

and denote by  $H_{A,-1}$  the completion of  $H$  with respect to  $\|\cdot\|_{H_{A,-1}}$ . ◇

It is well known (see e.g. [15, Section 3.6]) that  $\|\cdot\|_{H_{A,-1}}$  is independent of the choice of  $\lambda \in \rho(A)$  in the sense that different choices of  $\lambda$  yield equivalent norms. Furthermore,  $A$  admits an extension to a continuous operator  $A|_H : H \rightarrow H_{A,-1}$ .

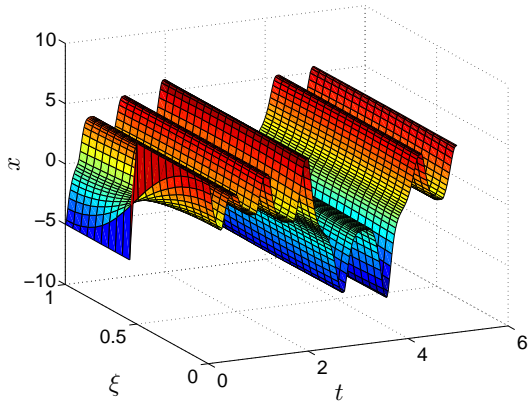


Fig. a: Solution of (6.1) with feedback (5.1).

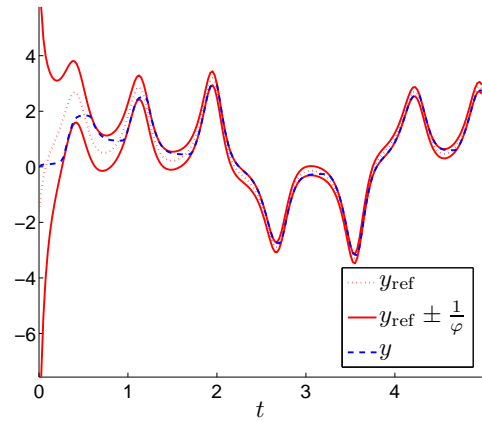


Fig. b: The output-trajectory (dashed) and the funnel (red) around  $y_{\text{ref}}$  (dotted).

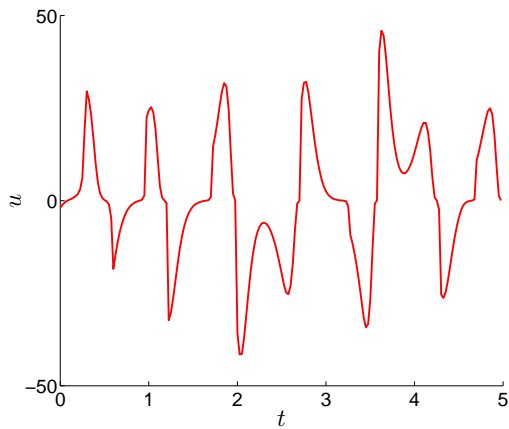


Fig. c: The input signal  $u$  produced by the controller (5.1).

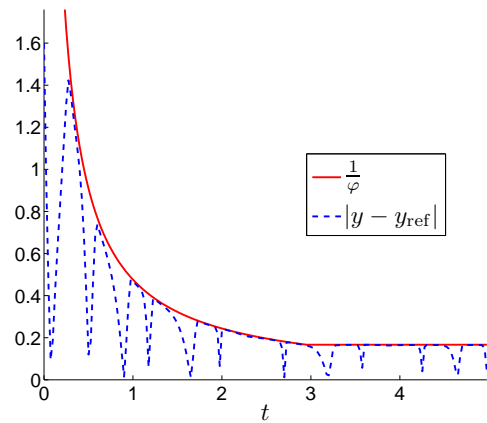


Fig. d: The error  $e = |y - y_{\text{ref}}|$ .

Figure 3: Funnel controller (5.1) applied to the heat equation (6.1) to track a chaotic reference signal  $y_{\text{ref}}(\cdot) := \zeta_1$  from the Lorenz equation (6.2) within the funnel specified by  $\varphi(t) = \min\{2t, 6\}$ .

**Definition A.2** (Mild and strong solution). Let  $(T(t))_{t \geq 0}$  denote the semigroup generated by  $A : \text{dom } A \subset H \rightarrow H$ ,  $x^0 \in H$  and  $f \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, H)$ . Then the *mild solution* of the Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t), \quad x(0) = x^0 \quad (\text{A.1})$$

is the function

$$x(t) = T(t)x^0 + \int_0^t T(t-s)f(s) \, ds \quad t \geq 0. \quad (\text{A.2})$$

A function  $x$  is called *strong solution* of (A.1) in  $H$  if  $x \in \mathcal{C}(\mathbb{R}_{\geq 0}, H)$ ,  $A|_H x(\cdot) \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, H_{A,-1})$  and

$$x(t) = x^0 + \int_0^t A|_H x(\tau) + f(\tau) \, d\tau \quad \forall t \geq 0, \quad (\text{A.3})$$

where this Bochner-integral is defined with respect to the norm of  $H_{A,-1}$ .  $\diamond$

In view of [15, Definition 3.2.2 (i)], our definition of strong solutions is just a reformulation of [15, Definition 3.8.1], (put  $n = 0$  there).

**Lemma A.3.** *Assume that  $A : \text{dom } A \subset H \rightarrow H$  generates the semigroup  $(S(t))_{t \geq 0}$ , the operator  $D \in \mathcal{B}(H)$  is bounded and  $f \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, H)$ . Denote the semigroup generated by  $A + D$  by  $S_D$ . Then  $x \in \mathcal{C}(\mathbb{R}_{\geq 0}, H)$  is a mild solution of*

$$\dot{x}(t) = (A + D)x(t) + f(t), \quad x(0) = x^0, \quad (\text{A.4})$$

if, and only if, it satisfies

$$x(t) = S(t)x^0 + \int_0^t S(t-s)(Dx(s) + f(s)) \, ds \quad \forall t \geq 0. \quad (\text{A.5})$$

*Proof.* The mild solution  $x$  of (A.4) satisfies, by definition,

$$x(t) = S_D(t)x^0 + \int_0^t S_D(t-s)f(s) \, ds \quad \forall t \geq 0. \quad (\text{A.6})$$

This equation coincides with [15, (3.8.2)] and [15, Theorem 3.8.2 (iv)] shows that  $x$  is a strong solution of (A.4) in  $H$ .

This means, by Definition A.2, that  $x$  satisfies the equation

$$x(t) = x^0 + \int_0^t \dot{x}(s) \, ds = x^0 + \int_0^t (A + D)|_H x(s) + f(s) \, ds \quad \forall t \geq 0 \quad (\text{A.7})$$

in terms of the integrals defined with respect to the norm  $\|\cdot\|_{H_{A+D,-1}}$  of the rigged space  $H_{A+D,-1}$  belonging to  $A + D$ .

As  $D$  is bounded, the norms of  $H_{A,-1}$  and  $H_{A+D,-1}$  are equivalent and these two spaces coincide. With similar arguments as above, it follows that any  $\tilde{x} \in \mathcal{C}(\mathbb{R}_{\geq 0}, H)$  satisfying (A.5) is a strong solution in  $H$  of

$$\tilde{x}(t) = A\tilde{x}(t) + \tilde{f}(t), \quad \tilde{x}(0) = x^0,$$

with  $\tilde{f}(t) := D\tilde{x}(t) + f(t)$ . This means, that  $\tilde{x}$  fulfills, in  $H_{A,-1}$ , the equation

$$\tilde{x}(t) = x^0 + \int_0^t \dot{\tilde{x}}(s) \, ds = x^0 + \int_0^t A|_H \tilde{x}(s) + D\tilde{x}(s) + f(s) \, ds \quad \forall t \geq 0, \quad (\text{A.8})$$

where the integrals are now defined with respect to the  $H_{A,-1}$ -norm, which is equivalent to the  $H_{A+D,-1}$ -norm. Due to the boundedness of  $D$ , the extension  $(A + D)|_H$  satisfies

$$(A + D)|_H x = A|_H x + Dx \quad \forall x \in H.$$

Thus, the equation (A.8) is equivalent to

$$\tilde{x}(t) = x^0 + \int_0^t (A + D)|_H \tilde{x}(s) + f(s) ds \quad \forall t \geq 0.$$

Therefore, any  $\tilde{x}$  which satisfies (A.8) also fulfills (A.7), and vice versa. Since by [15, Theorem 3.8.2 (ii)] the solutions of (A.7) and (A.8) in  $H$  are unique, the claim follows.  $\square$

**Lemma A.4.** *Let  $U$ ,  $X_1$  and  $X_2$  be Hilbert spaces and let  $A_{ii}$  be an operator which generates the semigroup  $(S_{ii}(t))_{t \geq 0}$  in  $X_i$  for  $i = 1, 2$ . Let  $A_{12} \in \mathcal{B}(X_2, X_1)$ ,  $A_{21} \in \mathcal{B}(X_1, X_2)$  and  $B_1 \in \mathcal{B}(U, X_1)$ ,  $B_2 \in \mathcal{B}(U, X_2)$  be bounded operators,  $x^0 \in H$  and  $u \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0}, U)$ . Then  $x$  is a mild solution of*

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t), \quad x(0) = x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}, \quad (\text{A.9})$$

if, and only if, the projected functions  $x_1(\cdot) = \pi_{X_1} x(\cdot)$  and  $x_2(\cdot) = \pi_{X_2} x(\cdot)$  satisfy

$$x_1(t) = S_{11}(t)x_1^0 + \int_0^t S_{11}(t-s) (A_{12}x_2(s) + B_1u(s)) ds, \quad (\text{A.10})$$

$$x_2(t) = S_{22}(t)x_2^0 + \int_0^t S_{22}(t-s) (A_{21}x_1(s) + B_2u(s)) ds. \quad (\text{A.11})$$

If (A.10) holds and  $A_{11}$  is bounded, then  $x_1$  satisfies

$$x_1(t) = x_1^0 + \int_0^t A_{11}x_1(s) + A_{12}x_2(s) + B_1u(s) ds \quad \forall t \geq 0, \quad (\text{A.12})$$

where the integral is defined with respect to the norm of  $X_1$ ; and furthermore,  $x_1$  is differentiable with respect to the norm of  $X_1$  almost everywhere with

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \quad \text{for a.a. } t \geq 0. \quad (\text{A.13})$$

*Proof.* We apply Lemma A.3 to the operators

$$A := \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad \text{and} \quad D := \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}.$$

It states that  $x$  is a mild solution of (A.9) if, and only if,

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \begin{bmatrix} S_{11}(t) & 0 \\ 0 & S_{22}(t) \end{bmatrix} \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} + \int_0^t \begin{bmatrix} S_{11}(t-s) & 0 \\ 0 & S_{22}(t-s) \end{bmatrix} \left( \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(s) \right) ds \\ &= \begin{pmatrix} S_{11}(t)x_1^0 \\ S_{22}(t)x_2^0 \end{pmatrix} + \int_0^t \begin{bmatrix} 0 & S_{11}(t-s)A_{12} \\ S_{22}(t-s)A_{21} & 0 \end{bmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} + \begin{pmatrix} S_{11}(t-s)B_1u(s) \\ S_{22}(t-s)B_2u(s) \end{pmatrix} ds \\ &= \begin{pmatrix} S_{11}(t)x_1^0 \\ S_{22}(t)x_2^0 \end{pmatrix} + \int_0^t \begin{pmatrix} S_{11}(t-s)A_{12}x_2(s) + S_{11}(t-s)B_1u(s) \\ S_{22}(t-s)A_{21}x_1(s) + S_{22}(t-s)B_2u(s) \end{pmatrix} ds. \end{aligned}$$

The additional claim follows from [15, Theorem 3.8.2]: If the operator  $A_{11}$  is bounded, the rigged space  $X_{1A_{11},-1}$  introduced in Definition A.1 coincides with  $X_1$ , and therefore  $x_1$  satisfies

$$x_1(t) = x_1^0 + \int_0^t (A_{11}x_1(s) + A_{12}x_2(s) + B_1u(s)) ds \quad \forall t \geq 0, \quad (\text{A.14})$$

where the integration is carried out in  $X_1$ .

To prove (A.13), note that equation (A.12) and Corollary 2 of [7, Theorem 3.8.5] imply that, for almost all  $t$ , the limit

$$\lim_{h \rightarrow 0} \frac{x_1(t+h) - x_1(t)}{h} = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$$

with respect to  $\|\cdot\|_{X_1}$  exists. This shows (A.13) and completes the proof of the lemma.  $\square$

## Appendix B Proofs of Theorem 2.5 and Proposition 2.8

We first collect three technical lemmata which are essential for the proof of Theorem 2.5.

**Lemma B.1.** *Let  $(A, b, c) \in \Sigma_r$ . Then*

$$A^*\eta = (P^0A^*\eta)c + P_{A,b}A^*\eta \in \text{ls}\{c\} + H_{A,b} \quad \forall \eta \in H_{A,b} \cap \text{dom } A^*. \quad (\text{B.1})$$

*Proof.* If  $r = 1$ , then (B.1) follows immediately from (2.2) and (2.7). Assume  $r > 1$  and let  $\eta \in H_{A,b} \cap \text{dom } A^*$ . Then (2.2) and (2.7) yield

$$A^*\eta = \alpha_0c + \alpha_1A^*c + \cdots + \alpha_{r-1}A^{*r-1}c + P_{A,b}A^*\eta \quad \text{with } \alpha_i = P^iA^*\eta \in \mathbb{R} \text{ and } P_{A,b}A^*\eta \in H_{A,b},$$

and thus

$$0 \stackrel{(2.2)}{=} (\eta, Ab) = (A^*\eta, b) \stackrel{(A3)}{=} \alpha_{r-1}(A^{*r-1}c, b)$$

and (A3) yields  $\alpha_{r-1} = 0$ . Next,

$$0 \stackrel{(2.2)}{=} (\eta, A^2b) = (A^*\eta, Ab) \stackrel{(A3)}{=} \alpha_{r-2}(A^{*r-2}c, Ab) = \alpha_{r-2}(A^{*r-1}c, b)$$

and (A3) yields  $\alpha_{r-2} = 0$ . Proceeding in this way, we conclude

$$0 \stackrel{(2.2)}{=} (\eta, A^{r-1}b) = (A^*\eta, A^{r-2}b) \stackrel{(A3)}{=} \alpha_1(A^*c, A^{r-2}b) = \alpha_1(A^{*r-1}c, b)$$

and arrive at  $0 = \alpha_{r-1} = \cdots = \alpha_1$ . This proves the lemma.  $\square$

**Lemma B.2.** *Let  $(A, b, c) \in \Sigma_r$ . Then, for any  $m = 0, \dots, r-1$ , the operator  $P^m A^*$  is closable and densely defined, and its closure is the bounded operator*

$$\overline{P^m A^*} : H \rightarrow \mathbb{R}, \quad x \mapsto \frac{(x, A^{r-m}b)}{(c, A^{r-1}b)} - \sum_{j=m+2}^r \left( P_{m+1}^m A^{*j-1}c - \sum_{k=m+2}^{j-1} P_k^m A^{*j-1}c \right) \frac{(x, A^{r+1-j}b)}{(c, A^{r-1}b)}. \quad (\text{B.2})$$

*The operator  $P_{A,b}A^*|_{H_{A,b}}$  with domain  $H_{A,b} \cap \text{dom } A^*$  is a closed and densely defined operator in  $H_{A,b}$  and satisfies*

$$P_{A,b}A^*\eta = A^*\eta - \left( \overline{P^0 A^*} \eta \right) c = A^*\eta - (P^0 A^*\eta) c \quad \forall \eta \in H_{A,b} \cap \text{dom } A^*. \quad (\text{B.3})$$

*Proof.* Obviously, for  $x \in \text{dom } A^*$  the mapping defined in (B.2) coincides with  $P^m A^* x$ , see Lemma 2.2. Since  $b$  belongs to  $\text{dom } A^r$ , the right hand side of (B.2) is also defined for arbitrary  $x \in H$ , hence  $P^m A^*$  is closable and its closure  $\overline{P^m A^*}$  is given by (B.2).

Statement (B.3) follows from (B.1) and the fact that  $P^0 A^* \eta = \overline{P^0 A^*} \eta$  for  $\eta \in \text{dom } A^*$ . Since  $A^*$  is closed and densely defined in  $H$  and  $\overline{P^0 A^*}$  is a bounded operator by (B.2),  $P_{A,b} A^*|_{H_{A,b}}$  is by (B.3) a closed and densely defined operator in  $H_{A,b}$ . This completes the proof of the lemma.  $\square$

**Lemma B.3.** *Let  $(A, b, c) \in \Sigma_r$  and let  $P^m$  be as in Lemma 2.2. Assume  $P_i$ ,  $R$  and  $S$  are given by (2.14)–(2.16). Then*

$$R = (P^0 A^*)^* = (\overline{P^0 A^*})^*, \quad R^* = \overline{P^0 A^*}, \quad (\text{B.4})$$

$$S^* \alpha = \alpha P_{A,b} A^{*r} c \quad \forall \alpha \in \mathbb{R}. \quad (\text{B.5})$$

Let  $Q$  be defined by (2.17), then  $Q$  is a densely defined closed operator in  $H_{A,b}$  and satisfies

$$Q = (P_{A,b} A^*|_{H_{A,b}})^*, \quad Q^* = P_{A,b} A^*|_{H_{A,b}} \quad \text{with} \quad \text{dom } Q^* = H_{A,b} \cap \text{dom } A^*. \quad (\text{B.6})$$

*Proof.* The operator  $\overline{P^0 A^*}$  is bounded, see Lemma B.2. Hence  $(P^0 A^*)^* = (\overline{P^0 A^*})^*$ . If  $P^\perp$  denotes the orthogonal projector onto  $H_{A,b}$  in  $H$ , we have, for all  $\eta \in H_{A,b}$  and all  $\alpha \in \mathbb{R}$ ,

$$\left( \overline{P^0 A^*} \eta, \alpha \right)_{\mathbb{R}} \stackrel{(\text{B.2}), (2.2)}{=} \left( \frac{(\eta, A^r b)_H}{(c, A^{r-1} b)}, \alpha \right)_{\mathbb{R}} = \left( \eta, \alpha \frac{P^\perp A^r b}{(c, A^{r-1} b)} \right)_{H_{A,b}} \stackrel{(2.15)}{=} (\eta, R\alpha)_{H_{A,b}}.$$

This proves (B.4). Equation (B.5) follows immediately from the definition (2.16).

It remains to show (B.6). We have, for arbitrary  $\eta \in H_{A,b} \cap \text{dom } A^*$  and  $\xi \in H_{A,b} \cap \text{dom } A$ ,

$$\begin{aligned} (P_{A,b} A^* \eta, \xi)_{H_{A,b}} &\stackrel{(\text{B.3})}{=} (A^* \eta, \xi)_H - \left( (\overline{P^0 A^*} \eta) c, \xi \right)_H = (\eta, A \xi)_H - \left( \overline{P^0 A^*} \eta, 1 \right)_{\mathbb{R}} (c, \xi)_H \\ &\stackrel{(\text{B.4})}{=} \left( \eta, P^\perp A \xi - R(c, \xi)_H \right)_{H_{A,b}} \stackrel{(2.17)}{=} (\eta, Q \xi)_{H_{A,b}}. \end{aligned}$$

Hence,  $\text{dom } Q \subset \text{dom } (P_{A,b} A^*|_{H_{A,b}})^*$  and  $Qx = (P_{A,b} A^*|_{H_{A,b}})^* x$  for all  $x \in \text{dom } Q$ . Thus it remains to show that  $\text{dom } (P_{A,b} A^*|_{H_{A,b}})^* \subset \text{dom } Q$ . Since  $\text{dom } Q = H_{A,b} \cap \text{dom } A$  it is sufficient to show  $\text{dom } (P_{A,b} A^*|_{H_{A,b}})^* \subset \text{dom } A$ . Let  $\xi \in \text{dom } (P_{A,b} A^*|_{H_{A,b}})^*$  and  $x \in \text{dom } A^*$ . We write  $x$  in the form (2.8) and observe that, by Assumption (A2),  $\eta := P_{A,b} x \in \text{dom } A^*$ . We compute

$$\begin{aligned} (A^* x, \xi) &\stackrel{(2.8)}{=} (A^* \eta, \xi) + \sum_{j=0}^{r-1} (P^j x) \left( A^{*j+1} c, \xi \right) \\ &\stackrel{(\text{B.1})}{=} \left( (P^0 A^* \eta) c, \xi \right) + (P_{A,b} A^* \eta, \xi) + \sum_{j=0}^{r-1} (P^j x) \left( A^{*j+1} c, \xi \right) \\ &\stackrel{(\text{B.4})}{=} (\eta, R(c, \xi)_H)_{H_{A,b}} + (\eta, (P_{A,b} A^*|_{H_{A,b}})^* \xi) + \sum_{j=0}^{r-1} (P^j x) \left( A^{*j+1} c, \xi \right). \quad (\text{B.7}) \end{aligned}$$

The mapping which maps  $x \in \text{dom } A^*$  to the right hand side of (B.7) is continuous with respect to the norm of  $H$ , and thus  $\xi \in \text{dom } (A^*)^* = \text{dom } A$ . So the first equation in (B.6) holds and it implies that  $Q$  is closed. The second equation in (B.6) follows immediately from the first one and therefore the proof of the lemma is complete.  $\square$

**Proof of Theorem 2.5:** We claim that  $(\widehat{A}, \widehat{b}, \widehat{c})$  fulfills Definition 2.1 with  $V = H_{A,b}$ . For the proof we proceed in several steps.

*Step 1:* We show that  $\widehat{b}$  and  $\widehat{c}$  satisfy Definition 2.1 (iv):

Applying  $U^{-*}$  to  $b \in H_{A,b}^\perp$  yields, with the representation (2.11) and (A3), the equality  $\widehat{b} = U^{-*}b = (0, \dots, 0, (A^{r-1}b, c), 0)^\top$ . The equations (2.5), (2.7), and (2.9) show that  $\widehat{c} = Uc = (1, 0, \dots, 0)^\top$ .

*Step 2:* For  $\widehat{A}$  we show that if  $\widehat{A}$  is defined by (2.13), (2.18) with entries (2.14)–(2.17), then it satisfies

$$\text{dom } \widehat{A} = U^{-*} \text{dom } A \quad \text{and} \quad \widehat{A} = U^{-*}AU^*. \quad (\text{B.8})$$

Since the entries of  $\widehat{A}$  defined by (2.14)–(2.16) are bounded, the domain of the adjoint of  $\widehat{A}$  is given by  $\mathbb{R}^r \times \text{dom } Q^*$  and Lemma B.3 yields

$$\text{dom } \widehat{A}^* = \mathbb{R}^r \times (H_{A,b} \cap \text{dom } A^*). \quad (\text{B.9})$$

Moreover, by Lemma B.3, the operator  $\widehat{A}^*$  is given, with respect to the decomposition  $\mathbb{R}^r \times H_{A,b}$ , by

$$\widehat{A}^* \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0 A^{*r} c & P^0 A^* \\ 1 & 0 & \cdots & 0 & P^1 A^{*r} c & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & P^{r-2} A^{*r} c & 0 \\ 0 & 0 & & 1 & P^{r-1} A^{*r} c & 0 \\ 0 & 0 & \cdots & 0 & P_{A,b} A^{*r} c & P_{A,b} A^* \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \eta \end{pmatrix} \quad \begin{array}{l} \forall \alpha_0, \dots, \alpha_{r-1} \in \mathbb{R}, \\ \forall \eta \in H_{A,b} \cap \text{dom } A^*. \end{array} \quad (\text{B.10})$$

Next, we show

$$\text{dom } A^* = U^{-1} \text{dom } \widehat{A}^* \quad \text{and} \quad UA^*x = \widehat{A}^*Ux \quad \forall x \in \text{dom } A^*. \quad (\text{B.11})$$

Let  $(\alpha_0, \dots, \alpha_{r-1}, \eta)^\top \in \text{dom } \widehat{A}^*$ . Then  $\eta \in H_{A,b} \cap \text{dom } A^*$ , which, together with (A2), shows

$$U^{-1}(\alpha_0, \dots, \alpha_{r-1}, \eta)^\top = \sum_{i=0}^{r-1} \alpha_i A^{*i} c + \eta \in \text{dom } A^*.$$

Conversely, fix  $x \in \text{dom } A^*$ . Then

$$x \stackrel{(2.8)}{=} (P^0 x)c + (P^1 x)A^*c + \cdots + (P^{r-1} x)A^{*r-1}c + P_{A,b}x,$$

and (A2) yields

$$\eta := P_{A,b}x \stackrel{(2.8)}{=} x - \sum_{j=0}^{r-1} (P^j x)A^{*j}c \in H_{A,b} \cap \text{dom } A^*,$$

which implies, in view of the definition of  $U$  and (B.9), that  $Ux \in \text{dom } \widehat{A}^*$ . Hence the first equality in (B.11) follows.

The decomposition (2.8) applied to the vector  $A^{*r}c$  gives

$$P^{r-1}xA^{*r}c = P^{r-1}x \left( \sum_{j=0}^{r-1} (P^j A^{*r}c)A^{*j}c + P_{A,b}A^{*r}c \right) \quad (\text{B.12})$$



and we conclude

$$\begin{aligned}
UA^*x &\stackrel{(2.8)}{=} U \left( A^*\eta + \sum_{j=0}^{r-1} (P^j x) A^{*j+1} c \right) \\
&\stackrel{(B.1)}{=} U \left( (P^0 A^* \eta) c + P_{A,b} A^* \eta + \sum_{j=0}^{r-2} (P^j x) A^{*j+1} c + (P^{r-1} x) A^{*r} c \right) \\
&\stackrel{(B.12)}{=} U \left( (P^0 A^* \eta) c + P_{A,b} A^* \eta + \sum_{j=0}^{r-2} (P^j x) A^{*j+1} c + P^{r-1} x \left( \sum_{j=0}^{r-1} (P^j A^{*r} c) A^{*j} c + P_{A,b} A^{*r} c \right) \right) \\
&\stackrel{(2.4),(2.5)}{=} \begin{pmatrix} P^0 A^{*r} c(P^{r-1} x) + P^0 A^* \eta \\ P^0 x + P^1 A^{*r} c(P^{r-1} x) \\ P^1 x + P^2 A^{*r} c(P^{r-1} x) \\ \vdots \\ P^{r-2} x + P^{r-1} A^{*r} c(P^{r-1} x) \\ P_{A,b} A^{*r} c(P^{r-1} x) + P_{A,b} A^* \eta \end{pmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & P^0 A^{*r} c & P^0 A^* \\ 1 & 0 & \cdots & 0 & P^1 A^{*r} c & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & & \ddots & 0 & P^{r-2} A^{*r} c & 0 \\ 0 & 0 & & 1 & P^{r-1} A^{*r} c & 0 \\ 0 & 0 & \cdots & 0 & P_{A,b} A^{*r} c & P_{A,b} A^* \end{bmatrix} \begin{pmatrix} P^0 x \\ P^1 x \\ \vdots \\ P^{r-1} x \\ \eta \end{pmatrix} \\
&= \widehat{A}^* U x.
\end{aligned}$$

This proves (B.10) and the remaining part of (B.11).

Recall that  $(XT)^* = T^* X^*$  for any densely defined operator  $T$  and bounded operator  $X$  and, if in addition  $X$  is boundedly invertible, we have  $(TX)^* = X^* T^*$ ; see e.g. [16, Section 4.4]. Hence, (B.8) follows from (B.11). This completes the proof of (B.8). Since  $U^*$  is a boundedly invertible transformation and  $(A, b, c) \in \Sigma_r$  we have  $(\widehat{A}, \widehat{b}, \widehat{c}) \in \Sigma_r$ .

*Step 3:* It remains to show that  $Q$  generates a semigroup, i.e. it fulfills Definition 2.1 (i).

It is clear that  $\widehat{A}$  generates a semigroup on  $\mathbb{R}^r \times H_{A,b}$ . In the block operator notation (2.19) for  $\widehat{A}$  the operators  $\widehat{A}_{11}$ ,  $\widehat{A}_{12}$ , and  $\widehat{A}_{21}$  are bounded. So

$$\text{diag}(0, Q) := \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} = \widehat{A} - \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & 0 \end{bmatrix},$$

is a bounded perturbation of  $\widehat{A}$ , and, in view of [6, Section III.1.3], it is a semigroup generator whose domain equals  $\text{dom } \widehat{A}$ . Obviously  $\{0\} \times H_{A,b}$  is a closed,  $\text{diag}(0, Q)$ -invariant subspace of  $\mathbb{R}^r \times H$ . Since the spectrum of  $Q$  is equal to the spectrum of  $\text{diag}(0, Q)$  up to the value 0, the condition (iv) of [15, Theorem 3.14.4] is satisfied. This theorem implies that  $\text{diag}(0, Q)|_{\{0\} \times H_{A,b}}$  with domain

$$\text{dom } \widehat{A} \cap (\{0\} \times H_{A,b}) = \{0\} \times (H_{A,b} \cap \text{dom } A),$$

generates a strongly continuous semigroup on  $\{0\} \times H_{A,b}$ . Now the identification of  $H_{A,b}$  with  $\{0\} \times H_{A,b}$  and  $Q$  with  $\text{diag}(0, Q)|_{\{0\} \times H_{A,b}}$  completes the proof.  $\square$

**Proof of Proposition 2.8:** We mimic the proof for finite-dimensional time-varying linear systems

in [1] and proceed in several steps.

*Step 1:* Consider the bounded bijective linear operator

$$Y : \mathbb{R}^r \times H_{A,b} \rightarrow \mathbb{R}^r \times \tilde{V}, \quad Y := W^{-*}U^* \quad (\text{B.13})$$

which admits a representation with respect to  $\mathbb{R}^r \times H_{A,b}$  and  $\mathbb{R}^r \times \tilde{V}$  of the form

$$Y = \begin{bmatrix} Y_{00} & Y_{01} & \cdots & Y_{0r} \\ Y_{10} & Y_{11} & \cdots & Y_{1r} \\ \vdots & \vdots & & \vdots \\ Y_{r0} & Y_{r1} & \cdots & Y_{rr} \end{bmatrix} \quad \text{with bounded} \quad \begin{array}{l} Y_{ij} : \mathbb{R} \rightarrow \mathbb{R}, \quad i, j \in \{0, \dots, r-1\} \\ Y_{ir} : H_{A,b} \rightarrow \mathbb{R}, \quad i \in \{0, \dots, r-1\} \\ Y_{rj} : \mathbb{R} \rightarrow \tilde{V}, \quad j \in \{0, \dots, r-1\} \\ Y_{rr} : H_{A,b} \rightarrow \tilde{V} \end{array}$$

and write

$$(\tilde{A}, \tilde{b}, \tilde{c}) := (W^{-*}AW^*, W^{-*}b, Wc). \quad (\text{B.14})$$

Then we have, with  $\hat{A}$  from Theorem 2.5,

$$Y\hat{A}Y^{-1} = W^{-*}U^*\hat{A}U^{-*}W^* \stackrel{(\text{B.11})}{=} W^{-*}AW^* \stackrel{(\text{B.14})}{=} \tilde{A}. \quad (\text{B.15})$$

Simply applying  $\tilde{A}$  from (2.29)  $(r-1)$ -times to  $\tilde{b}$  yields

$$(A^{r-1}b, c) = (W^{-*}A^{r-1}W^*W^{-*}b, Wc) = (\tilde{A}^{r-1}\tilde{b}, \tilde{c}) = b_r.$$

Hence, we have shown (iii) and

$$Y \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b, c) \\ 0 \end{pmatrix} \stackrel{(2.13)}{=} W^{-*}b \stackrel{(2.29)}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (A^{r-1}b, c) \\ 0 \end{pmatrix}. \quad (\text{B.16})$$

We calculate

$$\begin{pmatrix} Y_{00}^* \\ \vdots \\ Y_{0,r-1}^* \\ Y_{0r}^* \end{pmatrix} = Y^* \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \stackrel{(2.29)}{=} Y^*Wc \stackrel{(\text{B.13})}{=} Uc \stackrel{(2.13)}{=} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This, together with (B.16), gives

$$Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ Y_{10} & Y_{11} & \cdots & Y_{1,r-2} & 0 & Y_{1r} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ Y_{r-2,0} & Y_{r-2,1} & \cdots & Y_{r-2,r-2} & 0 & Y_{r-2,r} \\ Y_{r-1,0} & Y_{r-1,1} & \cdots & Y_{r-1,r-2} & 1 & Y_{r-1,r} \\ Y_{r0} & Y_{r1} & \cdots & Y_{r,r-2} & 0 & Y_{rr} \end{bmatrix}, \quad (\text{B.17})$$

and

$$\begin{aligned} [0, 1, 0, \dots, 0] &\stackrel{(2.1)}{=} [1, 0, \dots, 0]\hat{A} \stackrel{(\text{B.17})}{=} [1, 0, \dots, 0]Y\hat{A} \\ &\stackrel{(\text{B.15})}{=} [1, 0, \dots, 0]\tilde{A}Y \stackrel{(2.29)}{=} [0, 1, 0, \dots, 0]Y \\ &\stackrel{(\text{B.13})}{=} [Y_{10}, \dots, Y_{1,r-2}, 0, Y_{1,r}] \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned}
[0, 0, 1, 0, \dots, 0] &\stackrel{(2.1)}{=} [0, 1, 0, \dots, 0] \widehat{A} \stackrel{(B.18)}{=} [0, 1, 0, \dots, 0] Y \widehat{A} \\
&\stackrel{(B.15)}{=} [0, 1, 0, \dots, 0] \widetilde{A} Y \stackrel{(2.29)}{=} [0, 0, 1, 0, \dots, 0] Y \\
&\stackrel{(B.13)}{=} [Y_{20}, \dots, Y_{2,r-2}, 0, Y_{2r}].
\end{aligned}$$

We proceed by calculating the first  $r$  rows of  $Y$  in this way until

$$\begin{aligned}
[0, \dots, 0, 1, 0] &\stackrel{(2.1)}{=} [0, \dots, 0, 1, 0, 0] \widehat{A} = [0, \dots, 0, 1, 0, 0] Y \widehat{A} \\
&\stackrel{(B.15)}{=} [0, \dots, 0, 1, 0, 0] \widetilde{A} Y \stackrel{(2.29)}{=} [0, \dots, 0, 0, 1, 0] Y \\
&\stackrel{(B.13)}{=} [Y_{r-1,0}, \dots, Y_{r-1,r-2}, 1, Y_{r-1,r}],
\end{aligned}$$

and arrive at

$$Y = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r,0} & \dots & Y_{r,r-2} & 0 & Y_{rr} \end{bmatrix}. \quad (B.19)$$

The operator  $Y$  is bounded and bijective. Therefore,

$$\widetilde{Y} : H_{A,b} \rightarrow \widetilde{V}, \quad \widetilde{Y} := Y_{rr}$$

is a bounded, bijective operator. This, together with (B.15), already shows that  $\widetilde{S} = \widetilde{S} \widetilde{Y}^{-1}$ . Now the special structure of  $\widehat{A}$ ,  $\widetilde{A}$ ,  $Y$  in (2.1), (2.29), (B.19), resp., yields

$$\begin{aligned}
&[\widetilde{Y}R, Y_{r0}, \dots, Y_{r,r-2}, \widetilde{Y}Q] \\
&= [0, \dots, 0, I] \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r0} & \dots & Y_{r,r-2} & 0 & \widetilde{Y} \end{bmatrix} \begin{bmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & & 0 & 1 & 0 \\ P^0 A^{*r} c & P^1 A^{*r} c & \dots & P^{r-2} A^{*r} c & P^{r-1} A^{*r} c & S \\ R & 0 & \dots & 0 & 0 & Q \end{bmatrix} \\
&= [0, \dots, 0, I] Y \widehat{A} \\
&\stackrel{(B.15)}{=} [0, \dots, 0, I] \widetilde{A} Y \\
&= [0, \dots, 0, I] \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \widetilde{P}_0 & \widetilde{P}_1 & \dots & \widetilde{P}_{r-2} & \widetilde{P}_{r-1} & \widetilde{S} \\ \widetilde{R} & 0 & \dots & 0 & 0 & \widetilde{Q} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ Y_{r0} & \dots & Y_{r,r-2} & 0 & \widetilde{Y} \end{bmatrix} \\
&= [\widetilde{R} + \widetilde{Q}Y_{r0}, \widetilde{Q}Y_{r1}, \dots, \widetilde{Q}Y_{r,r-2}, 0, \widetilde{Q}\widetilde{Y}].
\end{aligned}$$

By successively comparing the blocks in order from  $(r-2)$ th to first and by finally considering the last entry, we see that

$$Y_{r,r-2} = 0 = \cdots = Y_{r0} = 0 \quad \text{and} \quad \tilde{R} = \tilde{Y}R, \quad \tilde{Q} = \tilde{Y}Q\tilde{Y}^{-1}.$$

Finally,

$$Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & \tilde{Y} \end{bmatrix}$$

and (B.15) give (ii). This completes the proof.  $\square$

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