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# Analysis of Dynamic System Properties

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# ANALYSIS OF DYNAMIC SYSTEM PROPERTIES

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## Preface

This monograph is dedicated to the memory of Prof. Ing. Anton Vrban, CSc.

Professor Anton Vrban was a respected scientific researcher and pedagogue in the area of automation and control. He supervised doctoral studies as well as the study program itself, introduced new topics and created a new focus in engineering studies.

He was a well respected professional in his field not only in Slovakia but internationally as well. He participated in both basic and applied research and developed his own method for the identification of systems, which has been named after him and used throughout the world. He published articles and papers in the most respected forums and was a member of several national and international committees and editorial boards.

## Abstract

This publication deals with the sensitivity, tolerance and robustness of dynamic systems. It brings general approach to solve specific issues in the field of presented topic. This approach is unusual, as it does not consist of partial solutions and summarization of knowledge, but it offers new methodology for problem solving, which is targeted to the nature of the problem. The methodology is designed to be well applicable. The textbook is useful for studying topics related to sensitivity, tolerance and robustness in dynamic systems. It can be also used by the designers of various complex dynamic systems and also as the stimulus for further theoretical, algorithmic and software supported development in the field of presented topic.



## Key words

sensitivity, tolerance, robustness, linear dynamic systems

# 1. INTRODUCTION

The development of science, technology, computer science and informatics allows for increasingly sophisticated automated production machines, equipment, control systems, manipulators and robots whose properties are gradually approaching the characteristics of intelligent systems. Although the problems associated with the development of "intelligent systems" (especially in the areas of algorithms and computer programs) involves ***artificial intelligence***, study of the ***immanent properties*** of mechanisms (mechatronic systems) that determine the skills, capabilities and behavior in the desired mode of function, at acceptable changes in external and internal parameters, are needed more than ever. Rapid development of ***computer science***, as well as the existence of powerful computing resources, creates new opportunities for the effective use of methods for examining the properties of dynamic systems (***computer simulations***). The existence of these options, parallel with the trend to develop "***intelligent systems***" also evokes the need for closer inspection of such properties (or characteristics) of dynamical systems, which is an analogous system with natural intelligence, in which the effectiveness of their cooperation culminates. Such characteristics include the ***sensitivity***, ***tolerance*** and ***robustness*** of the system. Although in theory, this problem has been given generous attention, especially in the measurement of physical quantities (and recently also to the sensitivity and robustness of economic and financial system), in the area of examining internal dynamic changes of the systems structure and their impact on external changes, computer simulations are not possible to use, and thus it was not possible to effectively research and implement these relations. Therefore it is desirable to update the problem area and look for methods and procedures for its solution, by which the use of computer simulations provide new theoretical insight and practical results which deepens the knowledge of "more intimate" properties of dynamic systems, reflected in their behavior, especially in the dynamic mode. This publication is the authors own contribution to the solution of this problem, which is directed at the formulation of goals within the area of ***linear dynamic systems***. In the work the achieved results are partially annotated can be useful in analyzing the properties of the system, operation monitoring, design of adaptive systems for the diagnosis of mechatronic systems of known structure, as well as for educational purposes. Therefore, the structure of this publication is characterized as a hybrid between textbook and monograph presented with the results of the authors own research. Articles in which the results of some original solutions are presented, can be found in the list of publications (2, 3, 11, 13, 15, 17, 18, 19, 20).

## 2. GENERALLY

The general system is understood as the purposeful definition (created) of the final and bounded set of definite elements. **$A=\{a_1,a_2,\ldots,a_k,\ldots,a_n\}$** and many (simple and multiple)mutual relations (relationships, constraints) between them **$R=\{r_{i,j}\}$ for I = 1,…,n; j= 1,…,n**. The physical character of the elements, their essential properties, and organization of the sets of mutual constraints between them, together generate the internal ***structure of the system***, which determines its ***immanent properties*** and also the methods (possibilities, abilities) behavior in the interaction with the important existential surroundings. If the system definition is capable, in the presence of time-varying effects of the surroundings (stimulations, actions), to depict the time-varying responses (reactions) also, then the **dynamic system (DS)** is undergoing the so-called ***dynamic process***. It is clear that the DS's waveform of reaction to the input (inputs) will not only depend on the surroundings instantaneous action (with respect to time), but will also depend on the instantaneous state of its structure (elements, their parameters and constraints). If we define the initial state of the DS (initial conditions), state of the surroundings, explicit excitation function (input signal), and assume that the constraints between elements of the system remain unchanged, then the reaction of the system (output) will be dependent only on the properties (parameters) of its elements. Any change in the parameters of the system is more or less reflected in its behavior. Therefore it is necessary and useful to analyze the effects of changing DS element parameters. That is, to predict the behavior of the system when these parameters change, and apply this knowledge in creating and monitoring activities (operational) of the dynamic (mechatronic) system.

The best mathematical described and theoretically analyzed class of systems consist of linear dynamic systems (LDS), in which the dynamic behavior is expressed by a system of linear differential equations (in the continuous waveform of the process), that is a system of linear differential equations ( in the discrete waveform of the process), or their integral transforms. If the properties (parameters) of the system do not change with time, then we consider a system of linear differential/ differential equations with constant coefficients. Such systems are known as stationary (time invariant). The properties of the system obviously change if the value of the constants change, arising in the need to examine, analyze and evaluate the effects of these changes. For this analysis we base our mathematical model on the linear stationary system (LSS) and this model will also be used in examining the properties of the system with changes in element parameters.

For easier physical interpretation, we will base the analysis on the linear stationary system with one input and one output, which will be known as the **linear system (LS).** The **LS** consists of only linear functional elements (members), among which there are fixed (unchanging over time) internal constraints. Linear elements can be understood as elements, in which the dependence of the output variable **y(t)** on the input variable **u(t)** is linear, that is: for input **u(t)** and output **y(t) of the n$^{th}$** element forming the LS the following applies:

$y_n(t) = L_n\{u_i(t)\}$, where $\boldsymbol{L_n}$ **is linear operator**.

**The operator is linear** if the following property is true:

$$\mathbf{L}\left\{\sum_i u_i(t)\right\} = \sum_i L\{u_i(t)\} = \sum_i y_i(t) \qquad [2.1]$$

$$\mathbf{L}\{k\,u_i(t)\} = k\;L\{u_i(t)\} = k\,y_i(t)$$

The first equation expresses what is known as **the principle of superposition**, by which the transformation of the sums of independent input signals $\mathbf{u_1}$**,** $\mathbf{u_2}$**, …,** $\mathbf{u_i}$**, …** is given by the sum of linear transforms for each individual signal. The second condition requires that the first condition for any value of real or complex constants k is met (interval for constant values where the linear system is **an interval (region) of linearity**; **linearization**). In real systems this condition is rarely met, and the linearity of models from real systems is limited only for definite region of the input signals amplitude – region of linearization. Equation [2.1] and its resulting conclusions can be used to confirm the linearity of the examined system.

## 3. LINEAR DYNAMIC SYSTEM

To have continuity in the description of issues from the presented problems, we introduce some well-known themes **in the behavior of linear dynamic systems**. From the properties of the system we describe their characteristics. These characteristics (external) describe the behavior of the dynamic system from the observer's point of view, which monitors the feedback of the system output on the defined input excitation (signal) with known initial state of the system.

### 3.1 Image and frequency transfer

If the input signal **u(t)** of a one-dimensional **LS** is a continuous (that is, continuous in parts) function of argument **t** (time), then the input signal **y(t)** will also be a continuous function of the same argument. The dependence between changes in input variables and output variables for a linear dynamic system, whose structure doesn't change with time **(LSS)** is described by linear differential equations with constant coefficients, which can be written in the following form

$$\sum_{\nu'=0}^{n} a_\nu \frac{d^\nu y}{dt^\nu} = \sum_{\mu=0}^{m} b_\mu \frac{d^\mu u}{dt^\mu}, \qquad [3.1]$$

in which m ≤ n for real systems.

Let us consider then a **LSS**, with its input excited by a regular continuous signal u(t), for which the following is true: If **u(t) = 0** for **t < 0**, and **y(t)** and all of its derivatives up to **(n-1)** degrees for **t < 0** are zero, then we have a passive system in steady state.  We can then apply the unilateral **Laplace transformation** to signals **u(t)** and **y(t)**

$$L\{u(t)\} = \int_0^\infty u(t)e^{-pt}dt = U(p) \qquad [3.2]$$
$$L\{y(t)\} = \int_0^\infty y(t)e^{-pt}dt = Y(p)$$

Where **U(p)** is the Laplace image **(L)** of **u(t), Y(p)** is the **Laplace image of y(t)**, and **p = c+jω** is a complex argument, where *ω* physically defines the angular frequency.

Application of the ***Laplace transformation*** on equation [**3.1**] when considering zero initial conditions and taking into account the **principle of superposition**, we can write:

$$Y(p)\sum_{\nu=0}^{n} a_\nu p^\nu = U(p)\sum_{\mu=0}^{m} b_\mu p^\mu \qquad [3.3]$$

**where (*for a real system*) *m ≤ n*.**

The ratio of **L**-image output signal **y(t)** on the **L**-image input signal **u(t)** with zero initial conditions, is defined as **the image transformation of the LSS**:

$$G(p) = \frac{\mathrm{L}\{y(t)\}}{\mathrm{L}\{u(t)\}} = \frac{Y(p)}{U(p)} = \frac{\sum_{\mu=0}^{m} b_\mu p^\mu}{\sum_{\nu=0}^{n} a_\nu p^\nu} = \frac{M(p)}{N(p)} \qquad [3.4]$$

If we apply the Fourier transformation on signals u(t) and y(t)

$$F\{u(t)\} = \int_0^\infty u(t)\, e^{-j\omega t}\, dt \overset{!}{=} U(j\omega)$$

$$F\{y(t)\} = \int_0^\infty y(t)\, e^{-j\omega t}\, dt \overset{!}{=} Y(j\omega) \qquad [3.5]$$

Then the ratio

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{\sum_{\mu=0}^{m} b_\mu (j\omega)^\mu}{\sum_{\nu=0}^{n} a_\nu (j\omega)^\nu} \overset{!}{=} \frac{M(j\omega)}{N(j\omega)} \qquad [3.6]$$

Gives the so called **frequency transfer of the LSS**.

Comparing equations **[3.4]** and **[3.6]** shows that:

$$G(j\omega) = \{G(p)\}_{p=j\omega} \qquad [3.7]$$

Because coefficient $a_\nu$, are dependent on properties (parameters) of system elements and how they connect, with the **structure** of the system, **expresses the transfer function, or the explicit frequency transfer of the dynamic properties for the given LSS.**

### 3.2 Frequency characteristics

As we can see from relation **[3.6],** the frequency transfer is a complex function of argument **ω** (angular frequency). **The geometric image of the frequency transfer in the gauss complex plane for $0 \le \omega \le \infty$ is the frequency characteristic (fig. 3.1)**, which for every value of the frequency

$\omega=\omega_k$ gives the real $P_k(\omega)$ and imaginary $Q_k(\omega)$ coordinate of the frequency transfer, that is its ***module*** $|G(j\omega_k)| = f(\omega_k)$ and ***phase*** $\Phi(\omega_k)$.

Obviously it applies

$$G(j\omega_k) = \frac{Y(j\omega_k)}{U(j\omega_k)} = \frac{M(j\omega_k)}{N(j\omega_k)} = P(\omega_k) + jQ(\omega_k) = W(\omega_k)e^{j\Phi(\omega_k)} \qquad [3.8]$$

where

$$G(\omega_k) = \sqrt{P^2(\omega_k) + Q^2(\omega_k)}$$

$$\Phi(\omega_k) = arctg\frac{Q(\omega_k)}{P(\omega_k)}$$

***For k = 1,2,3,… give the values of the module and phase of the frequency characteristic.***

***Representing points of the frequency transfer*** (module coordinates and phase, which are the real and imaginary components) in the gauss complex plane for changing argument ω, creates ***the continuous hodograph of parametric curves, which we call*** **the frequency (or sometimes the amplitude-phase frequency, or Nyquist frequency)** ***characteristic. The path of the Nyquist characteristic for a system with the transformation***

$$G_1(p) = \frac{p+6}{p^3 + 6p^2 + 11p + 6} = \frac{M(p)}{N(p)},$$

or the ***frequency transfer***

$$F(j\omega)_1 = \{F_1(p)\}_{p=j\omega}$$

Using the **MATLAB** software the representation can be seen in **figure 3.1**

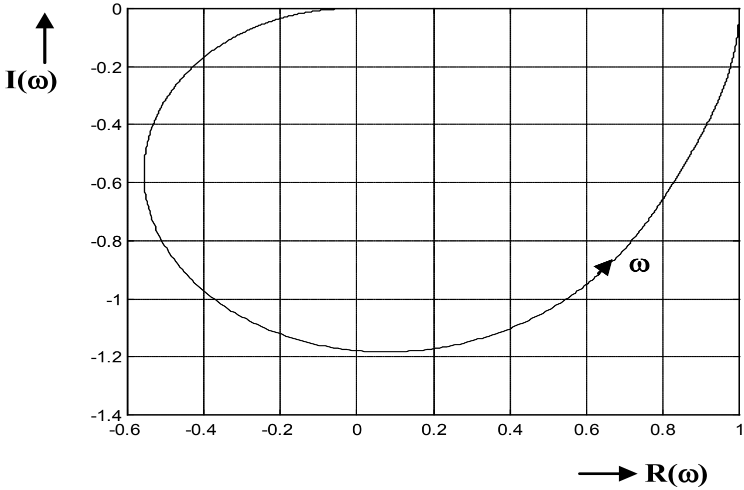



***Fig. 3.1*** *Nyquist characteristic $G_1(p)$*

**<u>Point characteristics</u>**

These characteristics represent the transfer (amplitude and phase) of properties **LDS** in a steady state, with harmonic excitation of the input signal. They are defined with the help of frequency transfer of the system $G_1(j\omega)$ in the logarithmic scale of angular frequency ω. **Amplitude-logarithmic frequency** characteristic is defined by the relation

$$20\log|G_1(j\omega)| = f\{\log(\omega)\} \quad \text{[dB]} \qquad [3.9]$$

And **the phase** by relation

$$\Phi(\omega) = \varphi\{\log\omega\} \qquad [\text{ rad}], \text{ resp. } [\text{deg}] \qquad [3.10]$$

Usually, both characteristics are given together. ***Characteristic of the system transfer***

$$G_2(p) = \frac{5p + 20}{p^3 + 4p^2 + 14p + 20} = \frac{M(p)}{N(p)},$$

and the ***frequency transfer***

$$G_2(j\omega) = G_2(p)_{p=j\omega}$$

Are represented in **fig. 3.2**

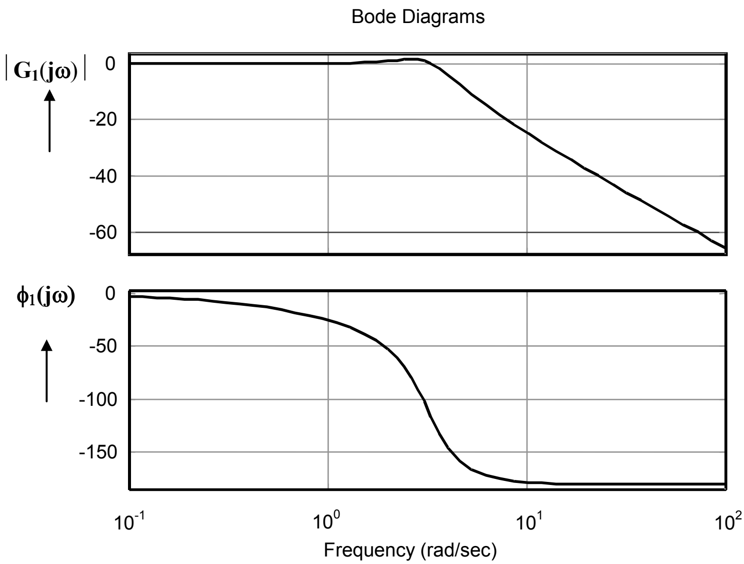



***Fig. 3.2*** *Bode characteristics of a linear system*

## 3.3 Time characteristics

Characterizing the behavior of the dynamic system in time is the definition of two basic functions and their corresponding (graphical representation) characteristics, ***which express the behavior*** of the dynamic systems feedback on the input excitation, and thus their properties, and immediate state, or change in internal states and properties of the analyzed system.

**Transfer function and characteristic**

The transfer function of the **LDS** is the time response on individual step signals applied on the input of the system, which is in steady state (zero initial conditions) and is the graphical image of the transfer function **h(t)**, which is the original **Laplace-Wagner** transfer function **G(p),** that is

$$h(t) = W^{-1}\{G(p)\} \quad \text{for } t \geq 0 \qquad [3.11]$$

**Impulse function and characteristic**

**The impulse function g(t)** is the time response of the system on the Dirac's impulse with zero initial conditions. It is given by the original Laplace transfer function **G(p),** thus

$$g(t) = L^{-1}\{G(p)\} \quad \text{for } t \geq 0 \qquad [3.12]$$

and its graph is the **impulse characteristic**.

***The time characteristic*** of a real, stable, linear, stationary system are continuous curves, that always initially begin at ($t = 0$), and for increasing t steadily (a periodic or periodically damped) change, and for $t \rightarrow \infty$ the transfer function ends at the value of static gain of the system $K_0=\{G(p)\}p=0$, and the impulse ends with a value of zero. ***The impulse function is the derivation of the transfer function with respect to time.***

Transfer and impulse characteristics of the **LDS** given by the transfer function $G_1$ are represented in fig. 3.3, the transfer function $G_2$ can be seen in fig. 3.4.

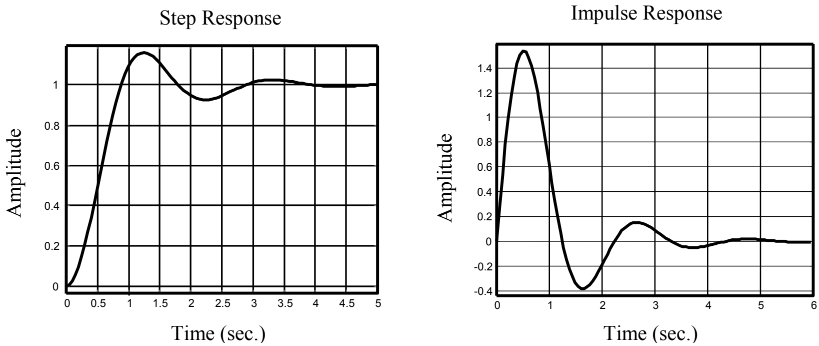



***Fig. 3.3*** *Time characteristic of the system $G_2(p)$*

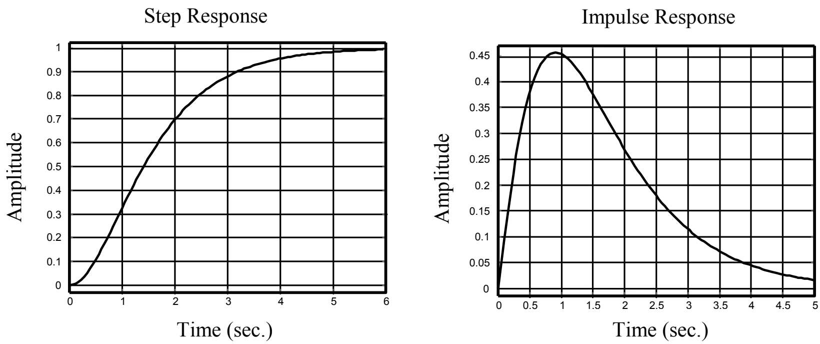



***Fig. 3.4*** *Time characteristics of system $G_2(p)$*

# 4. SENSITIVITY OF A DYNAMIC SYSTEM

Sensitivity of the dynamic system in the broader sense, is understood as the change of behavior under the influence of elementary changes of the surroundings state, as well as the changes in value of variables, which characterize the physical properties (parameters) of its elements, or the properties of its internal constraints. Changes to the systems output variables, which display its reaction to changes in input parameters, characterize the behavior of the system. Changes in state, and the corresponding system behavior when elementary parameters change with respect to its elements, will be known as parametric sensitivity of the dynamic system, this will be kept in mind for future considerations.

## 4.1 Sensitivity functions

Because the waveform of a dynamic process can be described, in general, by a system of ordinary or partial differential equations whose shape reflects the arrangements and constraints of its elements, where their coefficients implicitly or explicitly include the parameters of the elements. Then these differential equations, or their transformation, can serve as a **mathematical model of the system**.

For simplicity of the mathematical description, we consider the dynamic system with input $u(\xi)$ with the reaction $y(\xi)$. The ***input-output image*** will be given by the equation

$$y(\xi) = F\{\xi, \alpha, u_0, u(\xi)\} \quad [4.1]$$

where $F\{\xi,\alpha\}$ represents the display, or that is, the system function of argument $\xi$, whose shape and coefficients $\alpha=\{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n\}$depend on the internal structure of the system.

**Absolute differential sensitivity (*sensitivity function*)** of the given dynamic system for changes in the k^th^ parameter **(*parametric sensitivity*)** is defined by the relation

$$S_k(\xi,\alpha_k)=\frac{\partial}{\partial\alpha_k}\{F(\xi,\alpha)\} \qquad [4.2]$$

Relation **[4.2]** gives the absolute sensitivity function within its definite domain (***sensitivity function***). Its dimension is given as the ratio between the dimension of the function with respect to the dimension of the parameter $\alpha$.

**Absolute sensitivity of the function on the relative change in parameter values** is given by

$$S_a(\xi,\alpha_k)=\lim_{\Delta\alpha\to 0}\frac{\Delta F(\xi,\alpha)}{\frac{\Delta\alpha_k}{\alpha_k}}=\frac{\partial F(\xi,\alpha)}{\partial\alpha_k}\alpha_k$$

It is the ***function of argument*** $\xi$, whose ***dimension is identical with that of the natural function*** $G(\xi,\alpha_k)$.

**Relative differential sensitivity** is defined by the relation

$$S_r(\xi,\alpha_k)=\frac{\partial\,\mathbf{ln}\,F(\xi,\alpha)}{\partial\,\mathbf{ln}\,\alpha_k}=\frac{\partial F(\xi,\alpha)}{\partial\alpha_k}\frac{\alpha_k}{F(\xi,\alpha)}=\frac{\alpha_k}{F(\xi,\alpha)}.S_k(\xi,\alpha_k) \qquad [4.3]$$

**It *describes the change in values of the systems function [%] with respect to the one-percentile change of the parameter. Its value is a dimensionless number.* At the same time, relation [4.3] gives the dependence between the absolute and relative differential sensitivity.**

As can be seen in relations [4.2] and [4.3], the defined ***sensitivity***, with parameters of α, and shape of the system function, are all dependent on the argument ξ (in the real form of time "t", or the frequency ω) and ***determine the effect of changing the parameters*** of individual system elements on the waveform of the dynamic process. For the ***permissible*** (or allowed) ***change in behavior of the system***, we can also determine the tolerable changes – ***tolerance coefficient***. Creating a system function as a mathematical model of its internal properties and resulting external behavior is generally very complex. It is relatively easier with **linear dynamic systems (LDS)** since their structure is described by linear differential equations, and its behavior for any given initial conditions and excitation are expressed by the solution of these equations. Although there exist a number of solutions for partial problems of sensitivity, it is desirable to generalize the knowledge for individual types of systems and systematically organize and create an appropriate method of solution, which would be ***"tailored" to be used in computer simulation program***s and computer simulations in general. This work will describe, comment, and document some results, in logical form, achieved in the previous term, as well as in the solution of research tasks.

## 4.2 Sensitivity on additional parameters

***The systems sensitivity on a parameter of any element is the specific property of the system relative to that specific element***. Understanding sensitivity this way is generally not a variable with the additive properties, even though the figurative meaning often refers to the whole set of property changes of the elements, or the properties of the surroundings (for example; the sensitivity of the mechanism, or organism on the climate conditions, etc…). In terms of the systems behavior, which is characterized only by the change of some of its defined parameters, or characteristics, we can investigate the effect of element sensitivity on the changes of these parameters (for example; gain, frequency oscillators, changes in start-up characteristics of the drive, etc…).

The total differential system function $F(\xi,\alpha)$ from argument $\xi$ (usually time) with coefficients (or more precisely, element parameters) $\alpha=\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n$ which indicate the change of values with elementary changes of the coefficients will be

$$dF(\xi,\alpha) = \frac{\partial F}{\partial \alpha_1} d\alpha_1 + .... + \frac{\partial F}{\partial \alpha_k} d\alpha_k + .... + \frac{\partial F}{\partial \alpha_n} d\alpha_n = \sum_{k=1}^{n} \frac{\partial F(\xi,\alpha)}{\partial \alpha_k} d\alpha_k$$

from which **the overall change in value of the system function** can be written

$$\Delta F(\xi,\alpha) \cong \sum_{k=1}^{n} \alpha_i \frac{\partial F(\xi,\alpha)}{\partial \alpha_k} \cdot \delta_k = \sum_{k=1}^{n} \alpha_k S(\xi,\alpha_k)\delta_k \; ; \quad \text{for } k = 1,2,....,i,....,n$$

where $\alpha_k$ – are values of the coefficients for the model (containing element parameters),

$\delta_k = \frac{\Delta\alpha_k}{\alpha_k}$ – are relatively small changes in coefficient values of the model and

$S(\xi,\alpha_k) = \frac{\partial F(\xi,\alpha)}{\partial\alpha_k}$ ) – is the system functions absolute sensitivity on the coefficient $\alpha_i$.

Because individual members within expression **[4.4]** can have different signs, the value of $\Delta F(\xi,\alpha)$ cannot be a measure of the integral (total) system sensitivity on the changes in element parameters. Considering the changes in parameter values to be independent we can evaluate the **total change of the system function** by the following relation

$$\overline{\Delta F(\xi,\alpha_k)} = \sqrt{\sum_{k=1}^{n}\left[\alpha_k S(\xi,\alpha_k)\delta_k\right]^2} \qquad [4.4]$$

The value (waveform) of **relative changes in system function** against the original (original values of element parameters $\alpha=\alpha_{01}, \alpha_{02},\dots, \alpha_{0i},\dots, \alpha_{0n}$ ) will be then expressed by relation

$$\delta F(\xi,\alpha) = \frac{\overline{\Delta F(\xi,\alpha_k)}}{\left|F(\xi,\alpha\ )\right|} = \frac{\sqrt{\sum_{i=1}^{n}\left[\alpha_k S(\xi,\alpha_k)\delta_k\right]^2}}{\left|F(\xi,\alpha\ \right|} \qquad [4.5]$$

The value of expression [4.5] in relation to values of argument $\xi$ (time, frequency), with possible changes in value of individual elements, can serve to ***identify*** the “***most sensitive intervals***” (***phase***), ***activity*** of the observed dynamic system, as well as the assessment of total sensitivity (or more precisely, ***instability***). It is not however, a guarantee of the correct functioning of the system (stability, process characteristic, etc…). Therefore the possibility of these states, particularly in the areas of increased sensitivity (for example; the critical start-up of a machine, etc…), should be checked. At the same time **δ*F*(ξ,α)** can be a ***measure (criteria) of quality (and robustness) of the overall design*** of the proposed or implemented dynamic system.

For a robust system it is an obvious desire to achieve the lowest possible value for the relative change of the system function $\delta F$ (ξ, α) over the whole mode of operation (parameter ξ).

### 4.3 Sensitivity of the frequency transfer and frequency characteristic

For the sensitivity analysis of a linear one-dimensional system, it is suitable to use as the system function – **transfer function** $G(p)$, defined by the ratio between the feedback of the **Laplace image** $Y(p)$ with the image of excitation $U(p)$, with zero initial conditions. Or more accurately; the **frequency transfer** $G(j\omega)$, defined by

$$G(j\omega,\alpha)=\frac{Y(j\omega,\alpha)}{U(j\omega,\alpha)}=\{G(p,\alpha)\}p=j\omega=\frac{M(j\omega,\alpha)}{N(j\omega,\alpha)} \qquad [4.6]$$

The characteristic simulation, it is suitable to use the frequency transfer **(4.6)** in exponential form, where we indicate the dependence on parameter $\alpha=\alpha_1, \ldots, \alpha_\chi, \ldots, \alpha_n$ as well.

$$G(j\omega,\alpha)=G(\omega,\alpha)\cdot e^{j\Phi(\omega,\alpha)} \qquad [4.7]$$

If relation **[4.3]** is now applied to the ***frequency transfer*** **[4.7]** we can write the **relative parametric sensitivity of the frequency transfer on coefficient** $\alpha_k$ as

$$S_r^k(j\omega)=\frac{\partial \ln G(j\omega,\alpha)}{\partial \ln \alpha_k}=\frac{\partial G(\omega,\alpha)}{\partial \alpha_k}\cdot\frac{\alpha_k}{G(\omega,\alpha)}+j\frac{\partial \Phi(\omega,\alpha)}{\partial \alpha_k}\cdot\alpha_k=$$

$$=Re\{S_r^k(j\omega)\}+j.Im\{S_r^k(j\omega)\} \qquad [4.8]$$

where

$$Re\{S_r^k(j\omega,\alpha_k)\}=\frac{\delta G(\omega,\alpha)}{\delta\alpha_k}.\frac{\alpha_k}{G(\omega,\alpha)}=S_r^k(\omega,\alpha_k) \qquad [4.9]$$

is the ***real part*** of the relative sensitivity of the ***frequency response***, which ***describes the relative change of its model attributable to the elementary relative change in the value of the k^th^ coefficient*** $\alpha_k$, **that is: the relative differential sensitivity** ***of the frequency transfer module***, and

$$Im\{S_r^k(j\omega)\}=\frac{\delta\Phi(\omega,\alpha)}{\delta\alpha_k}.\alpha_k=S_a^{\Phi}(\omega,\alpha_k) \qquad [4.10]$$

Is the imaginary part of the relative sensitivity of the **frequency response**, which ***describes the change in phase with elementary relative change in the value of the k^th^ coefficient*** $\alpha_k$***, that is the absolute phase sensitivity for relative unit change in value of coefficient*** $\alpha_k$**.**

In terms of equation **[4.7]** and **[4.8]** and considering **[4.9]** and **[4.10]** results in the frequency transfer **[4.6]** able to directly derive relations for the relative sensitivity of the module $S_r^k(\omega,\alpha_k)$ and the phase sensitivity

$S_r^{\phi}(\omega, \alpha_k)$ **on the elementary relative change in value of any arbitrary** coefficient $a_\gamma$, $b_\mu$ frequency transfer of a linear system.

Consider the frequency transfer of a **LDS** in the form

$$G(j\omega,\alpha,\beta) = \frac{\sum_0^m b_\mu (j\omega)^\mu}{\sum_0^n a_\nu (j\omega)^\nu} = \frac{M(j\omega,\beta)}{N(j\omega,\alpha)} \qquad [4.11]$$

where $\beta$=$\beta_0$, $\beta_1$, $\beta_2$, …, $\beta_\mu$, …, $\beta_m$ are polynomial coefficients in the numerator of the transfer and

$\alpha$=$\alpha_0$, $\alpha_1$, $\alpha_2$, …, $\alpha_\gamma$, …, $\alpha_n$ are polynomial coefficients in the denominator of the transfer where $m \le n$.

If we apply relation **[4.3]** on the frequency transfer in the form of **[4.11],** with respect to **[4.9]** and **[4.10]** we can derive relations for:

- **relative differential sensitivity of the amplitude function** (also characteristic) on polynomial coefficients in the numerator $\beta_\mu$ and denominator $\alpha_\gamma$ by

$$S_\mu^G(\omega) = Re\left\{\frac{b_\mu (j\omega)^\mu}{\sum_0^m \beta_\mu (j\omega)^\mu}\right\} = \mathrm{Re}\left\{\frac{b_\mu (j\omega)^\mu}{M(j\omega)}\right\}$$

$$S_\nu^G(\omega) = Re\left\{-\frac{a_\nu (j\omega)^\nu}{\sum_0^n a_\nu (j\omega)^\nu}\right\} = \mathrm{Re}\left\{-\frac{a_\nu (j\omega)^\nu}{N(j\omega)}\right\} \qquad [4.12]$$

- and for the **semi-logarithmic phase sensitivity** of the frequency transfer by

$$S_{\mu}^{\Phi}(\omega) = \operatorname{Im}\left\{ \frac{b_{\mu}(j\omega)^{\mu}}{\sum_{0}^{m} b_{\mu}(j\omega)^{\mu}} \right\} = \operatorname{Im}\left\{ \frac{b_{\mu}(j\omega)^{\mu}}{M(j\omega)} \right\}$$

$$S_{\nu}^{\Phi}(\omega) = \operatorname{Im}\left\{ -\frac{a_{\nu}(j\omega)^{\nu}}{\sum_{0}^{n} a_{\nu}(j\omega)^{\nu}} \right\} = \operatorname{Im}\left\{ -\frac{a_{\nu}(j\omega)^{\nu}}{N(j\omega)} \right\} \qquad [4.13]$$

Relations **[4.12]** and **[4.13]** reflect the sensitivity of the module and the system phase frequency transfer on the change in coefficient values of the frequency transfers numerator polynomial $b_{\mu}$, and denominator polynomial $a_{\gamma}$ **[4.11].** It is a routine affaire to graphically model these relations on a computer using suitable software (***Matlab, Mathcad***, and others), the results are informative and very useful, even indispensible, in the design of dynamic (mechatronic) systems. From the waveform of the phase and amplitude sensitivity characteristic of the analyzed (modeled) dynamic system on each individual coefficient by expression **[4.12]** and **[4.13]** we can assess the effect of changes in their values on the transfer properties of the system. This has great ***significance*** not only in audiovisual and telecommunications systems, but also in ***assessing the changes in behavior of the dynamic (mechatronic) systems when changes in values of parameter elements occur***.

In designing a dynamic system, the designer can assess the effects that sensitivity analysis has on individual elements of its structure, and through modification, negative or undesirable effects can be avoided or eliminated altogether. In particular, it is necessary to know the ***sensitivity of dynamic systems with consideration for its stability.*** Characteristics of sensitivity (mainly the module sensitivity) point directly the coefficients (elements) whose change represents the greatest impact for the emergence of instability (weight). From knowledge about how changes in coefficient values (and parameters) effect the waveform of the process, It is possible to assess the internal causes (***diagnostics***). The aforementioned method for determining sensitivity of the frequency characteristic for the LDS was published in literature **[2, 17]**. The relations derived can be graphical illustrated in the example.

Illustrating the waveform of the sensitivity for the amplitude and frequency characteristic with respect to the given relations, the parametric sensitivity of the frequency characteristics on the transfer coefficients, for the third order system with aperiodic step response, can be given by the transfer

$$G(p)=\frac{b_1p+b_0}{a_3p^3+a_2p^2+a_1p+a_0}=\frac{M(p)}{N(p)}=\frac{p+6}{p^3+6p^2+11p+6}$$

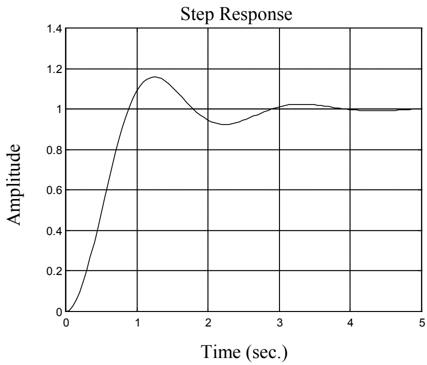



***Fig 4.1*** *Step response of the system*

The waveform of ***the relative differential system sensitivity*** on individual coefficients in the numerator of the transfer (characteristic equation) dependent on frequency ω (in the transfer band of the system) is represented in ***fig. 4.2 and the waveform of the phase sensitivity can be seen in fig. 4.3.***

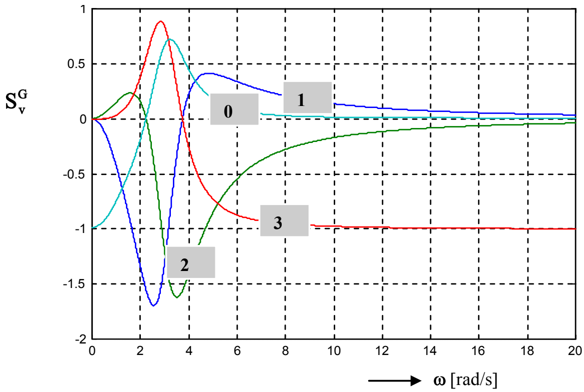
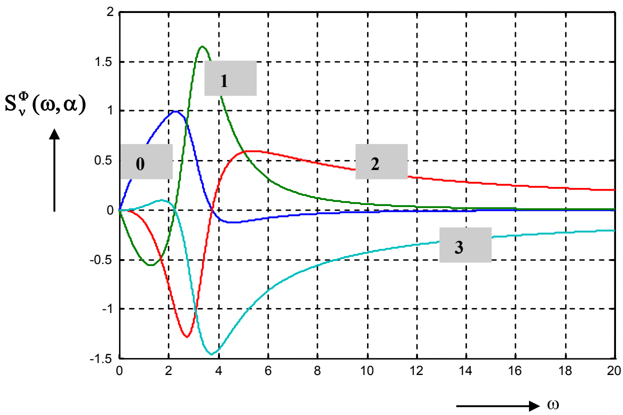



***Fig. 4.2*** *Relative differential sensitivity of the module G(jω) on the coefficients* $a_0$, $a_1$, $a_2$, $a_3$




***4.3*** *Sensitivity of the phase [rad] on the one-percentile change in coefficient N(jω)*

The waveform of the systems sensitivity functions on the change in coefficients is dependent on the waveform of the system characteristics at original (unchanged) values of coefficients (parameters of individual elements). The waveforms of these functions are different with a monotone, or aperiodic waveform of the original transfer characteristic. The described methodology and derived relations for simulating the sensitivity characteristic are generally applicable for one-dimensional linear systems and we can also use them ***for the analysis of sensitivity for multi-dimensional systems with known structure***.

Using the method for the analysis of parametric sensitivity of linear dynamic systems, published in articles and papers (see literature), it is possible to quantitatively determine the sensitivity of the system on the change in coefficients in its mathematical model given by differential equations, or transfer function $G(p)$.

It is important to know the sensitivity of a dynamic system when there is a change in its elements parameters, at the stage of creation (design, construction), in order to be able to choose its appropriate (desirable) structure with respect to the operational properties and requirements of the system. Because the properties (behavior of the system) present themselves in a dynamic regime, it is desirable to know, in detail, the effects of these changes have on the waveform of the transfer events. The differential equation, transfer function (in terms of Laplace transform), transfer function and frequency transfer, give complete information about the dynamic properties of the system. Therefore, all ***relations*** established for the analysis are ***formulated in such*** that they can be directly applied for calculation, or in better words; for the ***computer simulation of the sensitivity characteristics*** in appropriate programs (***such as MATLAB***).

The derived relations allow for the determination of the waveform of parametric sensitivity within the frequency and time domain. Because the coefficients of the transfer function implicitly contain element parameters of the dynamic system, it is apparent that with this methodology the ***sensitivity on any arbitrary parameter*** (parametric sensitivity) can be determined. By the same procedure, it is possible to establish the effects of sensitivity in a closed-loop control system on changes in parameters and controller of the system.

## 4.4 Sensitivity on the parameter element structure

Because parameter $\lambda_k$ of the real systems individual elements are implicitly contained in the coefficients of the dynamic model, that is $\alpha_\nu = f(\lambda_k)$, with known relative sensitivity $S_r^{G(\alpha_\nu)}$ on coefficient $\alpha_\nu$, in which parameter $\lambda_k$ is implicitly contained, the ***relative sensitivity of the module*** on the relative change in parameter $\lambda_k$ can be expressed by

$$S_r^{G(\lambda_k)} = Re\left\{\frac{\lambda_k}{G(j\omega,\alpha)}\frac{\partial G(j\omega,\alpha)}{\partial\alpha_\nu}\frac{\partial\alpha_\nu}{\partial\lambda_k}\right\} = \frac{\lambda_k}{\alpha_\nu}\frac{\partial\alpha_\nu}{\partial\lambda_k}S_r^{G(\alpha_\nu)} \qquad [4.14]$$

And the ***phase sensitivity*** on the relative change in values of parameter element $\lambda_k$

$$S^{\Phi(\lambda_k)} = Im\left\{\frac{\lambda_k}{G(j\omega,\alpha)}\frac{\partial G(j\omega,\alpha)}{\partial\alpha_\nu}\frac{\partial\alpha_\nu}{\lambda_k}\right\} = \frac{\lambda_k}{\alpha_\nu}\frac{\partial\alpha_\nu}{\lambda_k}S^{\Phi(\lambda_k)} \qquad [4.15]$$

where values of sensitivity on coefficient α are determined by relations **[4.12, 4.13].** According to these relations and with the known mathematical model of the system, we can compute and graphically represent the sensitivity of the linear dynamic system on element parameters of its structure, with the use of appropriate software (such as Matlab).

Graphically representing the waveform of the parametric sensitivity of characteristic functions, or in better words, the characteristic of the designed or known dynamic system (control loop on element parameters) can qualitatively assess the effect of changing parameter values, or control parameters, on the behavior of the system. That is to use the control loop, in the waveform of the dynamic process, and obtained information for the design, operation, noninvasive diagnostic of faults of the complex dynamic system (system made-up of sub-systems).

### 4.5 Sensitivity analysis of the structure

**<u>Series structure</u>**

We begin with the simple series structure created by "i" sub-components with transfer $G_i(p)$, for $i$=1,2,3.., $k$, …N. The ***transfer*** of such a ***structure*** is

$$G(p,\alpha) = \prod_{i=1}^{N} G_i(p,\alpha) \qquad [4.16]$$

The **relative sensitivity for the transfer** of the structure on the k$^{th}$ coefficient of the i$^{th}$ component will be

$$S_r^i(p,\alpha_k)=\frac{1}{G(p,\alpha)}\frac{\partial G(p,\alpha)}{\partial G_i(p,\alpha)}\frac{\partial G_i(p,\alpha)}{\partial \alpha_k}\alpha_k=\frac{1}{G_i(p,\alpha)}\frac{\partial G_i(p,\alpha)}{\partial \alpha_k}\alpha_k \qquad [4.17]$$

For the **complex sensitivity of the frequency transfer** (for the structure in series) on the k[th] coefficient of the i[th] component, will then be

$$S_r^i(j\omega,\alpha_k)=\{S_r^i(p,\alpha_k)\}p=j\omega=\frac{1}{G_i(j\omega,\alpha)}\frac{\partial G_i(j\omega,\alpha)}{\partial \alpha_k}\alpha_k \qquad [4.18]$$

The **relative differential sensitivity of the amplitude transfer** (for the structure in series) on the k[th] coefficient of the i[th] component, will then be

$$S_r^{G_i}(\omega,\alpha_k)=Re\left\{\frac{\alpha_k}{G_i(j\omega,\alpha)}\frac{\partial G_i(j\omega,\alpha)}{\partial \alpha_k}\right\} \qquad [4.19]$$

**and the absolute phase sensitivity of series connection on the relative change in the k[th] parameter of the i[th] component** will be

$$S^{\Phi_i}(\omega,\alpha_k)=Im\left\{\frac{\alpha_k}{G_i(j\omega,\alpha)}\frac{\partial G_i(j\omega,\alpha)}{\partial \alpha_k}\right\} \qquad [4.20]$$

From relations **[4.19, 4.20]** results in

- **the Relative sensitivity of the frequency transfer module of the structure in series (system) on the k[th] coefficient of the only i[th] component, is equal to the real part of the relative sensitivity of the i[th] member on its k[th] coefficient. Each displays the percentual change in amplitude of the amplitude-frequency characteristic of**

**the system corresponding to the one-percentile change in parameter value of the kth element, ith component.**

- **Absolute sensitivity of the phase frequency transfer (of a structure in series) on the relative change in value of parameter of the k$^{th}$ element, i$^{th}$ component is given by the imaginary value of the relative sensitivity of the i$^{th}$ component on its k$^{th}$ coefficient. Each displays the phase change percentage of the system on the one-percentile change in parameter value of this element.**

Because the probability that parameters of independent system elements changing at the same time is small, ***for the analysis of the system, it is suitable to determine the sensitivity on individual elements separately***. According to this, we can determine the elements which the system is most sensitive to and pay extra attention to them.

***Knowledge of the sensitivity on individual coefficients (on parameter elements within the domain of the transfer band) allows for the prediction of the systems behavior as a result of changes in individual coefficients (element parameters). It also allows for the compensation in unwanted changes of critical elements, or a more appropriate design of the system structure.***

**Parallel structure**

Let us consider a system consisting of parallel cell combination with the transfer $G_i(p,\alpha)$ **for** $i$ **=1, 2, . ,** $n$**. the resulting structure is**

$$G(p,\alpha)=\sum_{i=1}^{n} G_i(p,\alpha) \qquad [4.21]$$

The **relative sensitivity of the system transfer** on the **k^th** parameter, **i^th** cell can be determined

$$S_r^i(p,\alpha_k)=\frac{1}{G(p,\alpha)}\frac{\partial G(p,\alpha)}{\partial G_i(p,\alpha)}\frac{\partial G_i(p,\alpha)}{\partial \alpha_k}\alpha_k=\alpha_k\frac{1}{G(p,\alpha)}\frac{\partial G_i(p,\alpha)}{\partial \alpha_k}\alpha_k \qquad [4.22]$$

The **relative sensitivity of the frequency transfer module** of the parallel structure on the **k^th** parameter, **i^th** cell will be

$$S_r^{G_i}(\omega,\alpha_k)=Re\left\{\frac{\alpha_k}{G(j\omega,\alpha)}\frac{\partial G_i(j\omega,\alpha)}{\partial \alpha_k}\right\} \qquad [4.23]$$

**For the phase sensitivity** of the system transfer on the **k^th** coefficient of the **i^th** cell, we can then write

$$S_r^{\Phi_i}(\omega,\alpha_k)=Im\left\{\frac{\alpha_k}{G(j\omega,\alpha)}\frac{\partial G_i(j\omega,\alpha)}{\partial \alpha_k}\right\} \qquad [4.24]$$

Comparing relations **[4.17]** with **[4.22]** for relatively complex sensitivity of the structure formed by the same elements for any element "**i**" of coefficient "**k**" for the structure in series and parallel it follows

$$\frac{S_r^i(j\omega,\alpha_k)}{S_r^i(j\omega,\alpha_k)}=\frac{G(j\omega,\alpha)}{G_i(j\omega,\alpha)}=\frac{\sum_{i=1}^{n} G_i(j\omega,\alpha)}{G_i(j\omega,\alpha)} \qquad [4.25]$$

**Relation [4.25] shows that:**

- ***The ratio between the frequency transfer sensitivity of the structure in series on the transfer coefficient of the selected cell, to the***

***sensitivity of the parallel structure on the coefficient of the same cell connected in a parallel structure, is (for frequencies lying within the transfer band) given by the ratio between the frequency transfer of the parallel structure and the transfer of this cell. It then follows, that changes in the parameter of an element connected in a series structure will always have a significantly larger effect on the response of the system when compared to the same change in parameter of the same element connected in the parallel system (creation of stable circuits).***

Graphic representation of relations **[4.19, 2.10]** and **[4.23, 4.24]** illustrate the waveform of the sensitivity on a series or parallel system with respect to either, the transfer coefficients of the sub-cells, or by using **[4.14, 4.15]** on parameters of individual structure elements (if they are known).

For ***the illustration of the preceding results*** we include the solution of a series and parallel connection of two, first order astatic systems to determining the relative sensitivity of both structures on the same parameter of the same element.

Transfer of elements: $G_1(p) = \dfrac{K_1}{p\tau_1 + 1}$; $G_2(p) = \dfrac{K_2}{p\tau_2 + 1}$;

Transfer of series structure: $G_s(p) = G_1(p) \cdot G_2(p)$

Transfer of parallel structure: $G_p(p) = G_1(p) + G_2(p)$

**Relative sensitivity of the amplitude characteristic on the time constant $\tau_1$ for a series structure**

$$S_r^s(j\omega,\tau_1) = Re\left\{\frac{-j\omega\tau_1}{p\tau_1+1}\right\}$$

**Relative sensitivity of the amplitude characteristic on $\tau_1$ for a parallel structure**

$$S_r^p(\omega,\tau_1) = Re\left\{\frac{-j\omega\tau_1 K_1}{K_1(j\omega\tau_2+1)+K_2(j\omega\tau_1+1)}\cdot\frac{j\omega\tau_2+1}{j\omega\tau_1+1}\right\}$$

**Choosing the value of the parameters to be: $K_1$=5; $K_2$=10; $\tau_1$=2; $\tau_2$=1.**

***The sensitivity function for each structure are represented in figure 4.4***

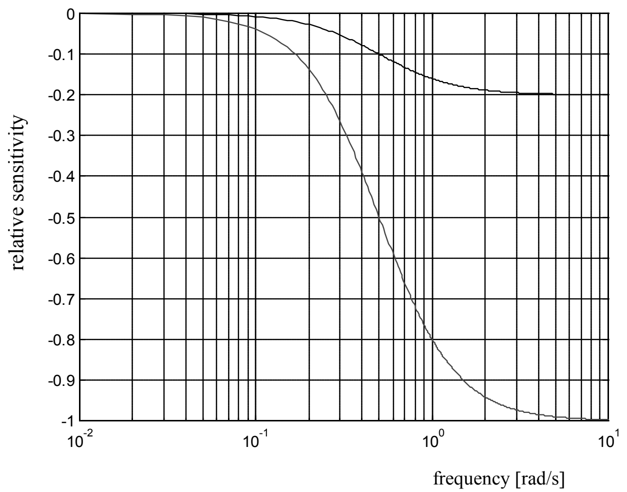



***Fig. 4.4*** *Relative sensitivity of the frequency transfer module of each structure*

The upper curve represents the relative sensitivity for parallel connected elements, while the lower curve represents the relative sensitivity for elements connected in series. From this graphical representation of the sensitivity functions on the same element parameter, once in series and then in parallel connections, we can see that ***in this case, the relative sensitivity within the frequency transfer band are 5 times greater than connections in parallel***.

## 4.6 Sensitivity of a reciprocal system function

Let us consider the general ***complex system function***, which describe the static and dynamic properties of the system ***in the form of frequency transfer for the LDS***

$$G(j\omega,\alpha) = R(\omega,\alpha) + jI(\omega,\alpha) = G(\omega,\alpha)\cdot e^{j\Phi(\omega,\alpha|} \qquad [4.26]$$

where $\omega$ is an argument of the function (usually frequency), and

$\alpha$ is a coefficient of the frequency transfer (element parameter of the system).

The ***relative differential sensitivity*** for the concrete parameter $\alpha_k$ is defined by relation

$$S_r^G(j\omega,\alpha_k) = \frac{\alpha_{\mathbf{k}}}{G(j\omega,\alpha)}\frac{\partial G(j\omega,\alpha)}{\partial\alpha_k} = S_r^G(\omega,\alpha).e^{j\Phi(\omega,\alpha)} =$$

$$= Re\{S_r^G(j\omega,\alpha_k\} + Im\{S_r^G(j\omega,\alpha_k\} \qquad [4.27]$$

If

$$H(j\omega,\alpha_k) = \frac{1}{G(j\omega,\alpha_k)} = \frac{1}{G(\omega,\alpha_k)}e^{-\Phi(\omega,\alpha_k)} = H(\omega,\alpha_k)e^{\Psi(\omega,\alpha_k)}$$

is the ***reciprocal system function of function G(jω, α$_k$)***, then for the sensitivity of the reciprocal system function ***H(jω,α$_k$)*** we can derive the relations:

- ***For the relative sensitivity of module H(ω,α$_k$)***

$$S_r^H(\omega,\alpha_k) = -S_r^G(\omega,\alpha_k) = -Re\left\{\frac{\alpha_k}{G(j\omega,\alpha)}\frac{\partial G(j\omega,\alpha)}{\partial\alpha_k}\right\} \quad [4.28]$$

- ***For the absolute sensitivity of the phase ψ(ω,α$_k$)***

$$S_{a,r}^{\Psi}(\omega,\alpha_k) = -S_{a,r}^{\Phi}(\omega,\alpha_k) = -Im\left\{\frac{\alpha_k}{G(j\omega,\alpha)}\frac{\partial G(j\omega,\alpha)}{\partial\alpha_k}\right\} \quad [4.29]$$

Relations **[4.28 and 4.29]** ***show the direct correlation*** between the relative sensitivity of the system function and its reciprocal function.

- ***The relative sensitivity on the same parameters of the reciprocal function to the system function of the LDS is taken to be negative with respect to the relative sensitivity of the system function on this parameter***

From the sensitivity of the amplitude and phase of the system function according to relations **[4.29** and **4.28]** it is possible to easily determine the sensitivity of the reciprocal system function, which can be important in special cases.

The derived expressions can be applied, ***for example***, to the ***sensitivity analysis of the series resonant R, L, C circuit***.

The reciprocal value for the impedance, that is for the ***admittance G(p)***,from ***Kirchhoff's laws*** we can derive the following relation for such a circuit

$$G(p) = \frac{I(p)}{U(p)} = \frac{p}{Lp^2 + Rp + \frac{1}{C}}$$

According to the known relations **[4.17, 4.22]** we can derive for the ***relative sensitivity of the admittance module*** on parameters ***R***, ***L***, ***C*** by the following relations:

- For ***sensitivity of the admittance module on the resistance R***

$$S_r^G(\omega, R) = Re\left\{\frac{-pR}{Lp^2 + Rp + \frac{1}{C}}\right\} \quad p = j\omega$$

- For ***sensitivity of the admittance module on the inductance L***

$$S_r^G(\omega, L) = Re\left\{\frac{-p^2L}{Lp^2 + Rp + \frac{1}{C}}\right\} p = j\omega$$

- For ***sensitivity of the admittance module on the capacitance C***

$$S_r^G(\omega, C) = Re\left\{-\frac{1}{C}\frac{1}{Lp^2 + Rp + \frac{1}{C}}\right\} p = j\omega$$

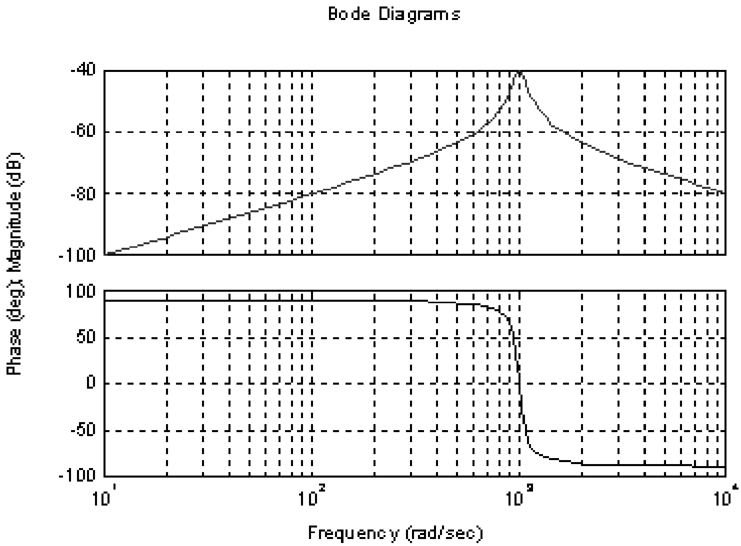



***Fig 4.5*** *Waveform of the module and phase admittance for the series R, L, C circuit*

The ***waveform of the module and phase admittance (conductivity)*** circuit with element values **R=100Ω, L=1H, C=1µF** is represented on figure (4.5).

For the ***relative impedance sensitivity*** on individual elements **R, L, and C** it is possible, either classically [through relations **4.17** and **4.22**], or with respect to relations **[4.29, 4.30]** to derive relations for the calculation of the ***sensitivity*** of ***admittance*** $G(j\omega,\alpha)$ and ***circuit impedance*** $\boldsymbol{Z(j\omega,\alpha)=1/G(j\omega,\alpha)}$ on individual circuit elements which will then be:

$$S_r^Z(\omega, R) = -S_r^G(\omega, R)$$
$$S_r^Z(\omega, L) = -S_r^G(\omega, L)$$
$$S_r^Z(\omega, C) = -S_r^G(\omega, C)$$

***and their waveform is represented*** on fig. 4.6 and 4.7.

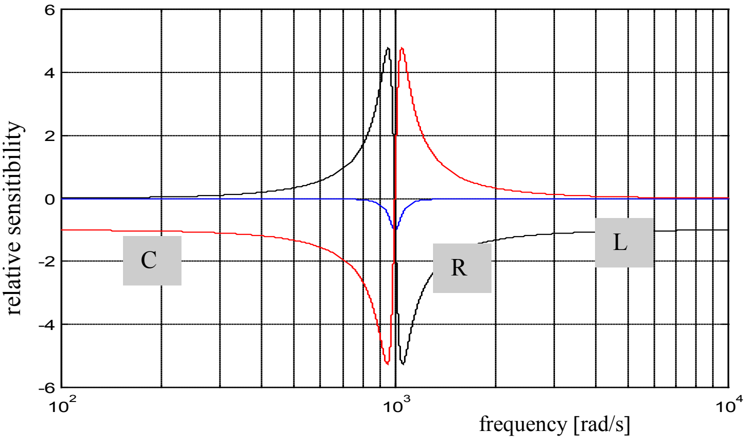



***Fig 4.6*** *Relative sensitivity of circuit admittance on elements R, L, and C*

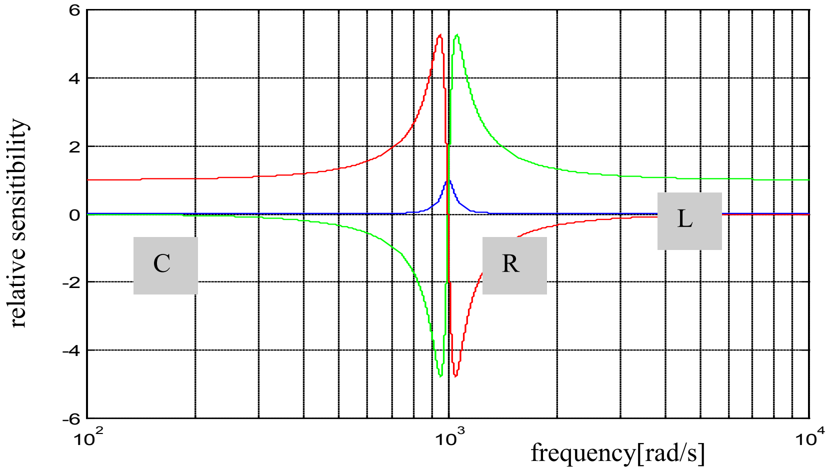



***Fig. 4.7*** *Relative sensitivity of the impedance on elements R, L, and C*

***From figures*** [**4.6**, **4.7**] it is possible to ***determine*** some facts about the series resonant circuit:

- ***The circuit admittance in the the resonant state results in a sharp maximum, which is equal to the reciprocal value of the ohmic resistance*** **R**,
- ***The phase admittance in resonance is zero and the circuit behaves as an ohmic one,***
- ***In the state below resonance, a large portion of energy is concentrated in the electric field of the capacitor and the circuit behaves as in capacitance, in the state above resonance a substantial part of energy is within the magnetic field of the coil, and the circuit behaves inductively,***

- ***Because, at low frequency, the voltage is largely influenced by the capacitance, the sensitivity of the circuit on the resistance and inductance is zero at these frequencies,***
- ***In the lower area close to the resonant frequency, the value of the sensitivity on both reactance elements (L and C) grows, obtains its maximum, then shrinks to zero at the resonant frequency, changes its sign, and obtains its maximum once again in the upper area close to the resonant frequency,***
- ***over the resonant state, the sensitivity of admittance on R and on C approaches zero as the frequency increases, and the sensitivity on L approaches -1,***
- ***It is clear, that the value of the resonant frequency is also sensitive to the change in values of inductance and capacitance.***

Because the ***resonant frequency of the*** **RLC** ***circuit*** (series and parallel) is given by the relation

$$\Omega_0 = \frac{1}{\sqrt{LC}}$$

the ***relative value of the sensitivity*** of this frequency ***on the capacitance C*** is given by

$$S_r^{\Omega_0}(C) = \frac{\partial \Omega_0}{\partial C} \frac{1}{\Omega_0} \cdot C$$

and ***on the inductance*** **L**

$$S_r^{\Omega_0}(L) = \frac{\partial \Omega_0}{\partial L} \frac{1}{\Omega_0} \cdot L$$

After substituting into the expressions we get

$$S_r^{\Omega_0}(L) = S_r^{\Omega_0}(C) = -\frac{1}{2}$$

This relation shows that the relative change of inductance or capacitance by 1% causes a 0.5% relative change in resonant frequency of the circuit $\Omega_0$.

## 4.7 Multi-parametric sensitivity

In the sensitivity analysis of a complex structure consisting of additional elements with more parameters, the formulation of relations for the study of systems in the general context, can become confusing (complex expressions).

System with known structures can usually be composed and decomposed into appropriate subsystems. We can first derive relations for the determination of the systems sensitivity, created by the structure consisting of **i=1, 2, 3, …, N** chosen elements (subsystems) with sensitivity on one parameter of the **i$^{th}$** coefficient on the transfer of the series structure.

### Series structure

***Analysis of the serial structure sensitivity***, we can derive for the ***relative change in amplitude of the system transfer,*** after neglecting the product of the differences, formed by the serial structure of the elements (or subsystems) for relative changes in coefficient values of the transfer function for the elements structure

$$\Delta_{r,r}^{G}(\omega,\alpha)_{s} \cong \mathrm{Re}\left[\sum_{i=1}^{N}\delta_{i,k}\left\{\frac{\alpha_{i,k}}{G_{i}(p,\alpha)}\cdot\frac{\partial G_{i}(p,\alpha)}{\partial\alpha_{i,k}}\right\}p = j\omega\right] \qquad [4.30]$$

where $\alpha_{i,k}$ is the value of the k$^{th}$ transfer function coefficient of the element of the structure $G_i(p,\alpha)$,

$\delta_{i,k}$ is the relative change in value of the k$^{th}$ coefficient of the i$^{th}$ element of the structure (+).

$\Delta_{r,r}^{G}(\omega,\alpha)_{s}$ is the relative amplitude change of the frequency transfer for a serial structure

**Parallel structure**

The sensitivity analysis of the parallel structure can be, by analogues procedures, derived the relation for the relative change in value of the frequency transfer amplitude

$$\Delta_{r,r}^{G}(\omega,\alpha)_{p} = Re\left[\left\{\frac{1}{G(p)}\cdot\sum_{i=1}^{N}\frac{\partial G_{i}(p,\alpha)}{\partial\alpha_{i,k}}\alpha_{i,k}\cdot\delta_{i,k}\right\}p = j\omega\right] \qquad [4.31]$$

where $G(p) = \sum_{i=1}^{N} G_{i}(p,\alpha)$

is the transfer of the parallel structure composed of N elements.

According to expression [4.30 and 4.31] it is possible to determine the waveform of the resulting amplitude deviation of the system transfer in series, or elements of the parallel structure, or subsystems, within the transfer band of the system. If we consider the same relative change in value for all elements, then the expressions indicate the relative change in the system transfer amplitude for the specified (desired, allowable) tolerance, or change in value of their parameters.

Complex structures can then be analyzed for the composition or decomposition of their serial/parallel branches.

**Series – parallel structure**

We will analyze the structure in fig. 4.8.

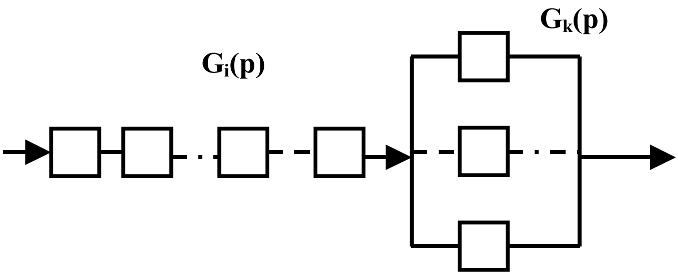



***Fig. 4.8*** *Circuit structure*

Transfer of the series part will be $G_s(p) = \prod_{i=1}^{N} G_i(p)$

Transfer of the parallel part will be $G_p(p) = \sum_{k=1}^{M} G_k(p)$

Analysis of the resulting transfer

$$G(p) + \Delta G(p) = \prod_{i=1}^{N} \left[G_i(p) + \Delta G_i(p)\right] \cdot \sum_{k-1}^{M} \left[G_k(p) + \Delta G_k(p)\right]$$

Neglecting higher order differences the ***relative deviation of the resulting transfer module for a series-parallel structure***, we can derived the useful relationship

$$\delta_{G(\omega)} = \frac{\Delta G(\omega)}{G(\omega)} \cong Re\left\{\sum_{k=1}^{M} \delta_k(j\omega)\right\} + Re\left\{G_s(j\omega).\sum_{i=1}^{N} \frac{\delta_i(j\omega)}{G_i(j\omega)}\right\} \qquad [4.32]$$

where

$\delta_k(j\omega)$ - is a complex function of the relative change in frequency transfer of the $\mathbf{k^{th}}$ element on the nominal transfer of the structures parallel part,

$G_s(j\omega)$ – is the frequency transfer of the series part of the structure

$\delta_i(j\omega)$ – is a complex function of the relative change in frequency transfer of the $\mathbf{i^{th}}$ element on the nominal transfer of the structures series part,

$G_i(j\omega)$ – is the frequency transfer of the series structure's $\mathbf{i^{th}}$ element.

***For illustration*** of a series-parallel connection see **fig. 4.9**

$$G(p) = \frac{K}{pT+1}$$

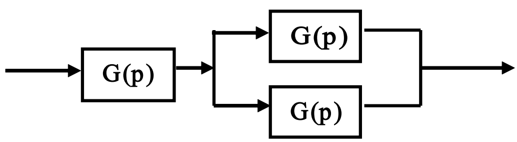



***Fig. 4.9*** *Serial-parallel structure*

For the relative change in amplitude of the frequency transfer structure according to fig. 4.9 we can derive the expression

$$\delta G(\omega) = \frac{\Delta G(\omega)}{G(\omega)} = \delta_T . Re\left[\left\{\frac{-2pT}{pT+1}\right\} p = j\omega\right]$$

**If we substitute K5, $\delta_T$ = 0.05, T=0.1 then**

$$\delta G(\omega) = -Re\left[\left\{\frac{p}{10p+100}\right\} p = j\omega\right]$$

the waveform of the relative change in module of the frequency transfer structure over a 5% change in value of the time constant *T* can be seen in fig. 4.10

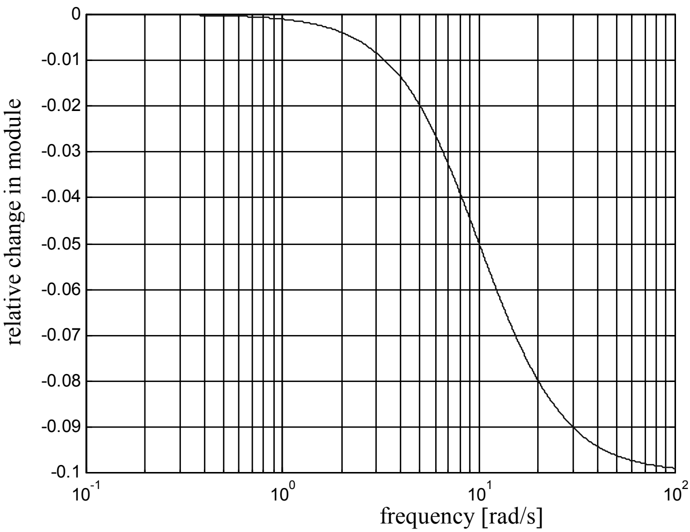



***Fig. 4.10*** *Waveform of the frequency characteristics change in amplitude*

Using the derived relations [4.31] and [4.32] it is possible to analyze the arbitrary structure, consisting of series and parallel connections between elements (blocks).

## 4.8 Sensitivity of the time characteristic

Change in parameter values (and thus coefficients also) of the LDS impacts the time character of the transfer effects within the system. The real waveform of these effects in time represent the transfer and impulse functions (their corresponding characteristics). These characteristics directly depend on the characteristics defined within the complex regions, such as in the transfer function $G(p)$ and frequency transfer $G(j\omega)$. Transfer characteristic h(t) is the Laplace – Wagner original and impulse characteristic g(t) is the Laplace original transfer function of the system $G(p)$. The **relative sensitivity of the transfer function** on the coefficient (parameter) can be defined as the relative change of the function by an elementary change in the differential equations coefficient (transfer function).

$$\xi_r(t,\alpha_k)=\frac{\alpha_k}{h(t)}\frac{\partial h(t,\alpha)}{\partial \alpha_k}=\frac{1}{h(t)}\left\{\alpha_k\frac{\partial h(t,\alpha)}{\partial \alpha_k}\right\}=\frac{\alpha_k}{W^{-1}\{G(p,\alpha)\}}W^{-1}\left\{\frac{\partial G(p,\alpha)}{\partial \alpha_k}\right\} \quad [4.33]$$

where

$$W^{-1}\left\{\frac{\partial G(p,\alpha)}{\partial \alpha_k}\right\}=\xi(t,\alpha_k)$$

is the **absolute sensitivity ξ(*t*,α*k*) of the transfer function on *the change in coefficient value* α*k*** and has the physical dimension defined by the proportion of the transfer functions amplitude on the size of the coefficient.

If we consider that the transfer function has the form

$$G(p)=\frac{\sum_{0}^{m} b_{\mu} p^{\mu}}{\sum_{0}^{n} a_{\nu} p^{\nu}}=\frac{M(p)}{N(p)}$$

then the ***absolute sensitivity of the transfer characteristic on coefficient*** $b_\mu$ will be

$$\xi(t,b_{\mu})=W^{-1}\left\{\frac{p^{\mu}}{N(p)}\right\} \qquad [4.34]$$

And the ***coefficients*** of the polynomial in the denominator $a_\nu$ will be

$$\xi(t,a_{\nu})=W^{-1}\left\{-\frac{p^{\nu}}{N(p)}G(p)\right\} \qquad [4.35]$$

For the growth in the amplitude of the systems transfer characteristic $\Delta h(t,\alpha_k)$ with transfer $G(p,\alpha_k)$on the perceptual change $\frac{\Delta\alpha_k}{\alpha_k}$ **[%]** = $\delta_k$ of coefficient $\alpha_k$, we can be easily derive the expression

$$\Delta h_k=\frac{\alpha_{\nu}.\delta_k[\%]}{100}W^{-1}\left\{\frac{\partial G(p,\alpha)}{\partial\alpha_{\nu}}\right\} \qquad [4.36]$$

For the coefficients of the polynomial in the numerator of the transfer $b_\mu$ and denominator $a_\nu$ will then be

$$\Delta h(t, b_\mu) = \frac{b_\mu . \delta_\mu [\%]}{100} . W^{-1} \left\{ \frac{p^\mu}{N(p)} \right\} \quad [4.37]$$

$$\Delta h(t, a_\nu) = \frac{a_\nu . \delta_\nu [\%]}{100} . W^{-1} \left\{ - \frac{p^\nu}{N(p)} G(p) \right\} \quad [4.38]$$

To illustrate, we introduce the waveform of the transfer characteristics sensitivity of the system (fig. 4.8)

$$G(p) = \frac{b_1 p + b_0}{b_3 p^3 + b_2 p^2 + b_1 p + b_0} = \frac{5p + 20}{p^3 + 4p^2 + 14p + 20} = \frac{M(p)}{N(p)}$$

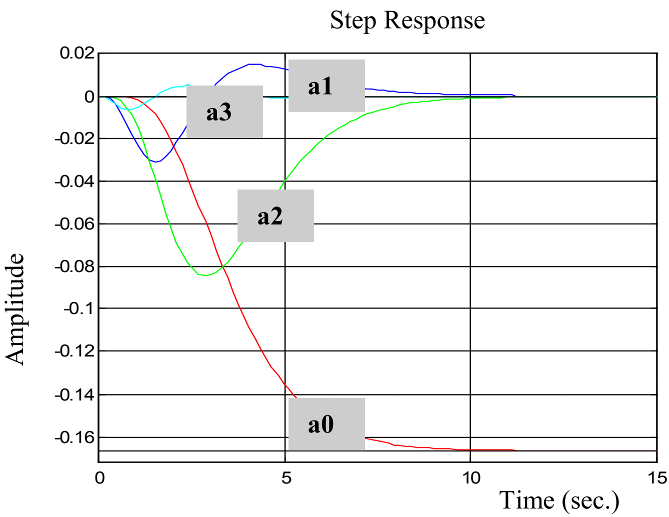



***Fig. 4.11*** *Sensitivity of the transfer characteristics amplitude on the coefficients $a_0$, ..., $a_3$*

The displayed functions give the growth, or voltage loss of the transfer characteristics amplitude within the waveform of the transfer process for a given change in coefficient value.

The change in the absolute value of the transfer characteristics amplitude for the one-percent change in the coefficient values of the denominator can be seen in fig. 4.12.

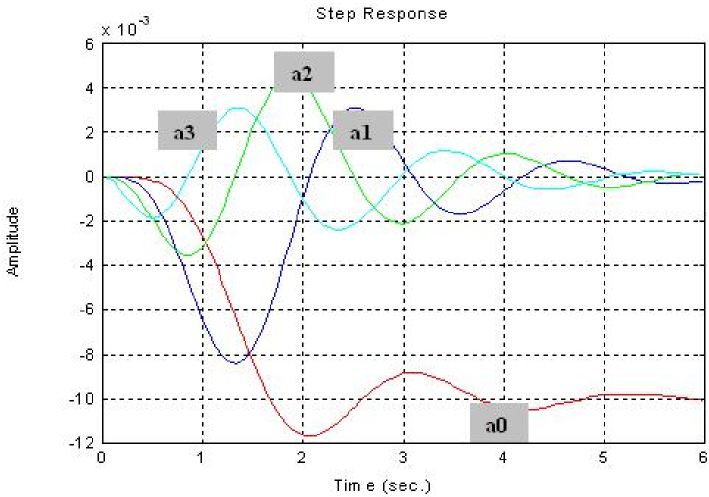



***Fig. 4.12** Difference in the transfer characteristic at a one-percent change in coefficient value (in the transfer denominator)*

**The absolute sensitivity η(*t*, *α*) of the impulse function *g*(*t*)**, which is the derivation of the transfer function with respect to time, for individual coefficients of the transfer function, we can substitute the original ***Wagner*** $\mathbf{W}^{-1}$ with the original ***Laplace*** $L^{-1}$ to obtain

$$\eta(t, b_\mu) = L^{-1}\left\{\frac{p^\mu}{N(p)}\right\} \qquad [4.39]$$

$$\eta(t, a_\nu) = L^{-1}\left\{\frac{-p^\nu}{N(p)} G(p)\right\} \qquad [4.40]$$

***A more detailed analysis of the transfer function sensitivity, which characterizes the effects of changing values in individual coefficients on the waveform of the transfer process, we can sense the change in waveform of the transfer process. On the other hand, from the observed changes of the transfer process (start-up for example) we can judge the change in parameters of some coefficient (element directly) which is possible to use in the diagnostics.***

Using the proposed ***analysis method*** for the parametric sensitivity of a linear dynamic system, partly published in articles (**literature L**) and given in this work, it is possible to quantitatively ***identify and evaluate the sensitivity of the system on the change in coefficients of its mathematical model given in the form of the transfer function G(p), as well as directly on the change in parameter value of its functional elements.***

It is very important to know the sensitivity of a dynamic system on the change in parameters of its elements already ***at the design stage*** of the system in order to choose the appropriate (desired) structure in respect with the operating properties and requirements. Because the properties present themselves in the dynamic regime, it is desirable to understand these effects mainly on the waveform of the transfer effects. The dynamic properties of the system gives complete information about the transfer function and ***transfer (impulse) function/characteristic***. Therefore all relations determined for the analysis are formulated in such a way that they can be used directly for calculation or ***computer simulation***.

***The derived relations allow us to determine the waveform of the parametric sensitivity within the frequency and time domain (argument) on the polynomial coefficients of the numerator and denominator, that is; directly on the element parameters of the transfer for simple and complex linear systems (whether open or with feedback). According to the presented methodology it is possible to analyze relatively complex systems and control circuits as well. The application of the methodology procedures is useful also for the design of adaptive and robust systems and controllers.***

# 5. SENSITIVITY OF THE CONTROL CIRCUIT

Assuming a control circuit with the structure in **fig. 5.1**, in which **S(p,α)** represents the transfer function of the controlled system with coefficients **α=(α$_1$, …, α$_k$, …), R(p,β)** and the transfer of the controller with coefficients **β=(β$_1$, β$_2$, …, β$_k$, …)**. **U(p)** ***is the image of the action (input variables), Y(p) is the image of the output variables (feedback), and W(p) is the image of the desired output variables***.

**Transfer of the desired value on** the ***output value*** will be

$$G(p,\alpha,\beta)=\frac{Y(p)}{W(p)}=\frac{S(p,\alpha)R(p,\beta)}{1+S(p,\alpha)R(p,\beta)} \qquad [5.1]$$

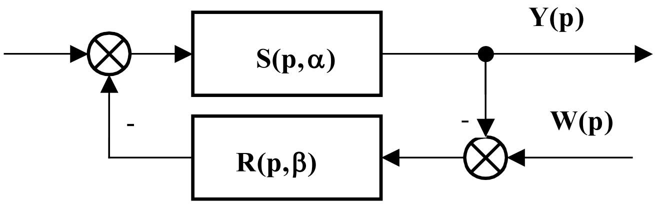



***Fig. 5.1*** *Structure of the control circuit*

## 5.1 Determining the sensitivity

The relative transfer sensitivity of the control circuit *G*(p,α,β) on the systems $\alpha_k^{th}$ coefficient will be defined by relation

$$S_{r,r}^{\alpha_k}(p)=\frac{1}{G(p,\alpha,\beta)}\frac{\partial G(p,\alpha,\beta)}{\partial S(p,\alpha)}\frac{\partial S(p,\alpha)}{\partial \alpha_k}\alpha_k \qquad [5.2]$$

After substituting *G*(p,α,β) we obtain

$$S_{r,r}^{\alpha_k}(p) = \frac{1}{1+S(p,\alpha)R(p,\beta)} \cdot \frac{\alpha_k}{S(p,\alpha)} \frac{\partial S(p,\alpha)}{\partial \alpha_k} \qquad [5.3]$$

For the relative sensitivity of the control circuits frequency transfer it then becomes

$$S_{r,r}^{\alpha_k}(j\omega) = \frac{1}{1+S(j\omega,\alpha)R(j\omega,\beta)} \cdot \frac{\alpha_k}{S(j\omega,\alpha)} \frac{\partial S(j\omega,\alpha)}{\partial \alpha_k} \qquad [5.4]$$

The relative sensitivity of the amplitude of the control circuit's frequency response on the $\alpha_k^{th}$ coefficient of the systems control model will be given by relation

$$S_{r,r}^{\alpha_k}(\omega) = \alpha_k . Re\left\{\frac{1}{1+S(j\omega,\alpha)R(j\omega,\beta)} . \frac{1}{S(j\omega,\alpha)} \frac{\partial S(j\omega,\alpha)}{\partial \alpha_k}\right\} \qquad [5.5]$$

Analogically with the relative sensitivity of the amplitude of the control circuits frequency transfer on the $\beta_k^{th}$ coefficient of the controller it follows

$$S_{r,r}^{\beta_k}(\omega) = \beta_k \cdot \mathrm{Re}\left\{\frac{1}{1+S(j\omega,\alpha)R(j\omega,\beta)} \cdot \frac{1}{R(j\omega,\beta)} \frac{\partial R(j\omega,\beta)}{\partial \beta_k}\right\} \qquad [5.6]$$

***According to the derived relations [5.5 and 5.6], for the regular control process we can determine the relative sensitivity of the frequency transfer module of the closed circuit on any transfer coefficient of the controlled system and controller.***

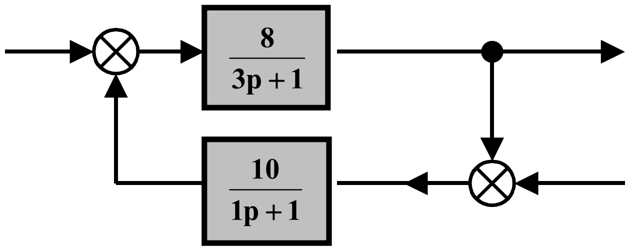
**To illustrate** the derived expressions, the waveform of the amplitude-frequency characteristic's relative sensitivity of the control circuit created by the first order static system ***S(p)*** with time constant ***T*** and controller ***R(p)*** with time constant **τ** can be represented.

***Transfer of the desired values W(p) on the control variable is***

$$G(p)=\frac{S(p)R(p)}{1+S(p)R(p)}, \text{ where } S(p)=\frac{K_s}{pT+1}; \quad R(p)=\frac{K_r}{p\tau+1};$$

and their values $\mathbf{K_s=8; K_r=10; T=3; \tau=1}$.

**Scheme of the analyzed circuit**:




Bode characteristic are on fig. 5.2

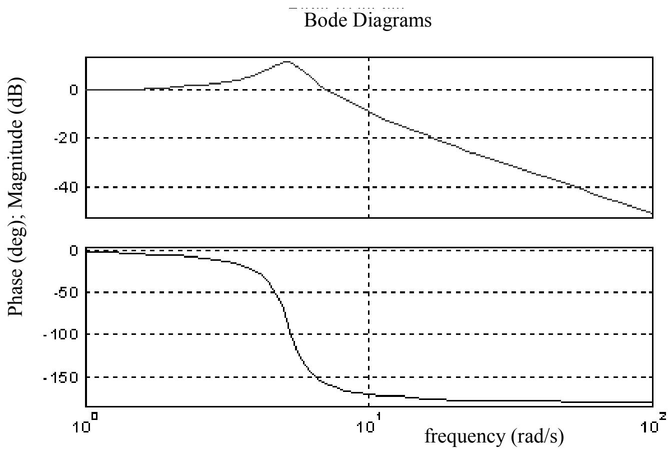



***Fig. 5.2** Logarithmic amplitude and phase characteristic of the control circuit*

From these it can be seen that in the region of angular frequency (approximately **5 rad/s**) the amplitude-frequency characteristic of the circuit achieves its maximum (the circuit has the tendency to vibrate). Also In the region of this frequency, the relative sensitivity of the circuit on the values of the time constants and controller are most significant (**fig. 5.3**).

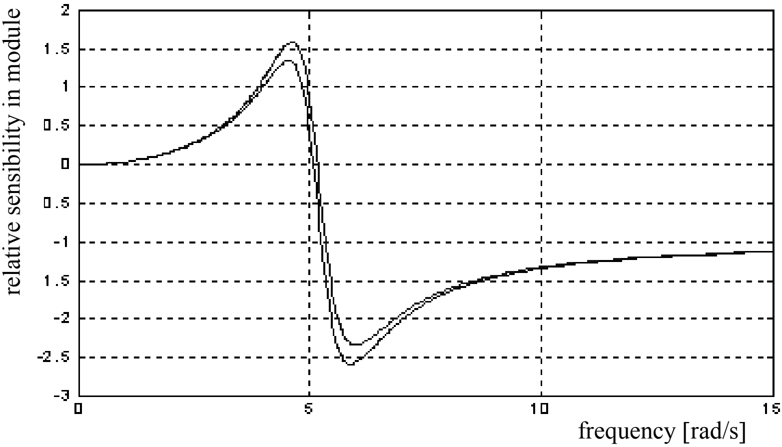



***Fig. 5.3*** *Relative sensitivity of the circuit's frequency transfer module on the system's time constant T and controller τ*

## 5.2 Behavior of the control circuit

To assess the effects in changing time constants of the controller on the transfer properties (ability) of the control circuit we pre-analyze the ***sensitivity of the transfer characteristic on the ratio change between the time constants. For the assessment we will assume the control circuit whose transfer characteristic is <u>aperiodic</u>***.

***Transfer*** of the closed control circuit is

$$G(p)=\frac{Y(p)}{X(p)}=\frac{K_0}{p^2\tau T+p(\tau+T)+K_0+1} \qquad [5.7]$$

where $\boldsymbol{K_0=K_s=K_r}$ **is** the gain of the control circuit, T is the time constant of the system and τ is the time constant of the controller.

If we denote $\frac{T}{\tau}=\xi$, then

$$G(p)=\frac{K_0}{p^2\xi\tau^2+p\tau(1+\xi)+K_0+1}=\frac{K_0}{N(p)}=F(p,\xi,\tau,K_0) \qquad [5.8]$$

***Characteristic equation*** of the control circuit

$$N(p)=p^2\xi\tau+p\tau(1+\xi)+K_0+1=0 \qquad [5.9]$$

has the roots

$$p_{1,2}=\frac{-\tau(1+\xi)\pm\sqrt{\tau^2(1+\xi)^2-4\xi\tau^2(K_0+1)}}{2\xi\tau^2} \qquad [5.10]$$

In terms of (**6**), the ***process*** will be ***stable and aperiodic*** in the circuit if

$$(1+\xi)^2-4\xi(1+K_0)=0 \qquad [5.11]$$

The time constant of the controller (fast or slow) has a significant impact on the waveform of the control process for the given transfer of the system ***S*(*p*).** Therefore we will analyze ***sensitivity of the process on the constants of the controller***. The waveform of the sensitivity depends however, also on the character of the process, therefore the analysis will be performed for the case where the control circuits transfer characteristic will be aperiodic (just below the boundary that the formation of the aperiodic process occurs).

For the aperiodic waveform of the transfer characteristic from relation [5.11] for amplified $K_o$ depending on the ratio of the controller's and system's time constants ξ results in

$$K_0 \leq \frac{1}{4}\frac{(1+\xi)^2}{\xi} - 1 = \frac{1}{4}\eta - 1 \qquad [5.12]$$

Corresponding values of the variables according to relation (5.12) are given in the following table.

VALUES OF THE RATIO BETWEEN TIME CONSTANTS
AND GAIN FOR THE APERIODIC RESPONSE Table 5.1

| ξ | 1 | 2 | 4 | 5 | 10 | 20 | 30 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|---|
| η | 4 | 4,5 | 6,25 | 7,2 | 12,1 | 22,05 | 32,03 | 52,02 | 102 |
| ***Ko*** | **0** | **0,25** | **0,56** | **0,8** | **2,03** | **4,62** | **7,01** | **12** | **24,5** |

In respect to relation [5.12] we can create the graph $K_o = f(\xi)$, (**fig. 5.4**) in which, for the chosen value **ξ ≥ 1** we can subtract the desired gain of the control circuit for the aperiodic response

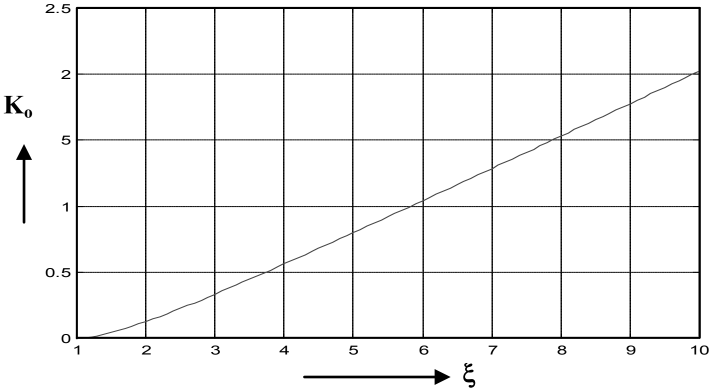



***Fig. 5.4*** *Dependence of the control circuit gain $K_o$ on the ratio of the time constant $\xi$*

## Sensitivity on the time constant of the controller

In the sensitivity analysis we will be based upon the transfer function of a closed loop control circuit $G$(p). The inputs of the desired values of the control variables are in the form [5.7], and through the definition of sensitivity, we will gradually find relations for the determination and graphical representation for the sensitivity of the frequency characteristic, with differing values for the ratio of the controller time constants and values of time constants for the dynamic delay of the control system. This analysis allows us to assess the values of sensitivity for the control circuit and the waveform of the sensitivity for the control variables on the change in time constant within individual phases of the transfer function (21).

We will define the ***relative sensitivity of the frequency transfer*** with the help of the transfer function [5.7]

$$S_\tau(j\omega,\tau) = \lim_{\Delta\tau\to 0}\left\{ \frac{\dfrac{\Delta G(p,\tau)}{G(p,\tau)}}{\dfrac{\Delta\tau}{\tau}} \right\}_{p=j\omega} = \left\{ \frac{\tau}{G(p,\tau)} \frac{\partial G(p,\tau)}{\partial\tau} \right\}_{p=j\omega} \qquad [5.13]$$

The relative sensitivity of the control circuit transfer defined this way is a dimensionless number and gives the relative change in value of the transfer corresponding to the relative value of the elementary change in time constants of the controller τ.

The module **$|S_\tau(j\omega,\tau)|$** represents the ***amplitude sensitivity***, which will describe the sensitivity of the frequency transfer module on the change in time constant of the controller at individual frequencies within the band-pass of the control circuit. This dependence, for different values of the controller's time constants, is assessed with the waveform of the amplitude characteristic.

If we substitute [**5.7**] into relation [**5.13**] for ***G*(*p*,τ)** and we perform the relative operations such that we can assess the effects of changing the time constant of the controller **τ** against the given time constant of the system **T** by means of coefficient ξ, then ***for the absolute value of the frequency transfer sensitivity for the chosen time constant ratio ξ = T/τ we get***

$$S_r(\xi,\omega) = \mathrm{Re}\left\{ \frac{p^2T^2 + pT}{p^2T^2 + pT(1+\xi) + \xi(K_0+1)} \right\} \qquad [5.14]$$

For the graphical illustration let us choose the value of the time constant for the system ***T*=1 s**. ***For differing values of τ*** we will determine the gain of $\boldsymbol{K_o}$ for the aperiodic waveform of the process (Tab. no.1) and will calculated and represent the sensitivity of the frequency characteristic for different ratios of **ξ=T/τ**. For the value of coefficient **ξ=*T*/τ** let us choose: $\boldsymbol{\xi_1=2}$ $\boldsymbol{(\tau_1=0.5)}$; $\boldsymbol{\xi_2=5}$ $\boldsymbol{(\tau_2=0.2)}$; $\boldsymbol{\xi_3=10}$ $\boldsymbol{(\tau_3=0.1)}$. The waveform of the sensitivity of the amplitude-frequency characteristic for the control circuit is represented on **fig. 5.5**.

***In terms of the waveform of the function, it can be seen that the frequency characteristic amplitude of the control circuit for the chosen value of the system's time constant becomes sensitive early (at lower frequencies) for the circuit with smaller values of ξ (greater values of τ). In other words, control with relatively greater values of the controller time constant is more sensitive to the change in the system's time constants already at lower frequencies of the signal's transfer spectrum.***

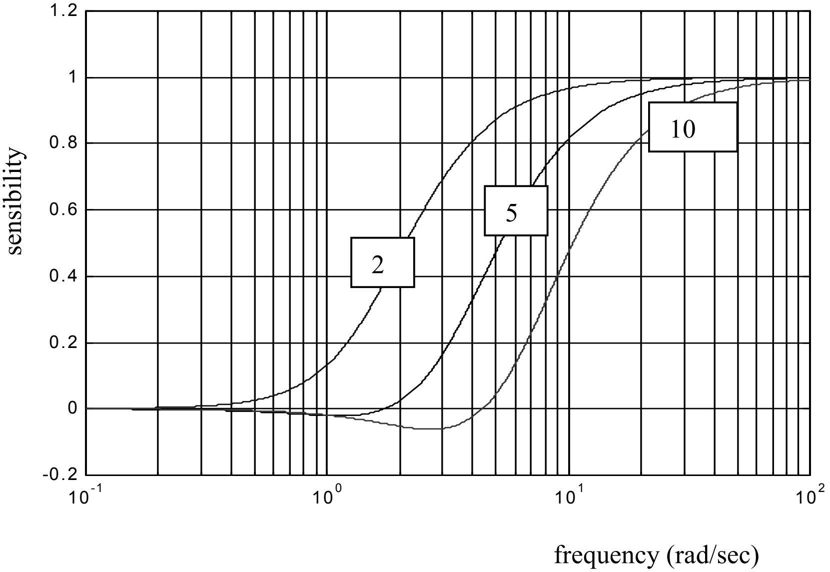



***Fig. 5.5*** *Relative sensitivity of the control circuit's amplitude characteristic for different values of ξ*

***The methodology for analyzing the sensitivity of linear dynamic system, given in this part of the work, is based on the basic description of the system structure in the form of differential equations (transfer function) whose coefficients also implicitly inherit the values of the element parameters of the system***. Because the properties of the systems behavior (external character) are sensitive on the change in parameters, it is logical to study the sensitivity of these characteristics on the element parameters which form the structure of the analyzed system. ***The advantages of the given methodology is that expression for the calculation or simulation of the sensitivity on individual parameters can be relatively easily derived***

***from the description of the system's structure in the form of the transfer function or the transfer of its subsystems as well. The description of the dynamic structure is mainly transformed into the field where most operations with the transfer function or the frequency transfer are performed by algebraic procedures and the functional dependence in time, is obtained by the inverse transformation (Laplace, Wagner or Fourier) which are part of mathematical software in PC's***. With the known structure of the dynamic system it is not needed to solve for differential equations, but the description in the form of the transfer function can be (with an understanding in the transfer of elements and the laws of the transfer composition) created directly ***"in terms" of the structure of the object***.

Sensitivity of the complex dynamic system depends on the properties of its elements and their mutual physical, and also informational, constraints (automatically controlled system). ***This property is not only significant for mechanical systems (mechanisms), but also for living (biological) systems (organisms).*** The dynamic behavior is dependent on the sensitivity of the system to changes in internal parameters (caused also by external effects), that is to say the auto-controlled processes and the external manifestation of their incorrect function, such as: malfunction, instability, change in performance, incorrect response to external stimulations, as well as the overall manifestation of the systems behavior. The effect of parametric sensitivity on the behavior of dynamic systems is also addressed in further parts of this work.

## 6. TOLERANCE OF THE DYNAMIC SYSTEM

The ability of a system to meet the desired (defined) function even with the occurrence of relatively small changes in element values of its parameters is known as the ***system tolerance***. In the following chapter we will have in mind the tolerance with small changes in element parameters or parameter of the medium (for example in hydraulic systems). Requirements on the system behavior can be generally varied, and thus can be formulated in different ways. We will consider "a priori" that the analyzed system with given (original) structure is capable of implementing the demands on its behavior.

Originating from the ***input-output representation of the DS described by the system function G(ξ,α)***, where **ξ** is an argument of the function and $\boldsymbol{\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots, \alpha_n)}$ are coefficients of the system function (element parameter of its structure). The change in response on the elementary change of the $\mathbf{k^{th}}$ parameter, at the defined initial state of the system and waveform of excitation, will be expressed in the form

$$\Delta y(\xi)_k = \frac{\partial G(\xi,\alpha)}{\partial \alpha_k} \cdot \Delta\alpha_k \qquad [6.1]$$

If the permitted (allowable, desired) change in response on the change of the $\mathbf{k^{th}}$ parameter is written as $\mathbf{\Delta y(\xi)_k}$, then from relation (**6.1**) the ***tolerance of parameter*** $\boldsymbol{\alpha_k}$ becomes

$$\Delta\alpha_k = \frac{1}{\dfrac{\partial G(\xi,\alpha)}{\partial \alpha_k}} \cdot \Delta y(\xi)_K = \frac{1}{S_k(\xi)} \cdot \Delta y(\xi)_k \qquad [6.2]$$

Where $S_k(\xi)$ is the absolute differential sensitivity of the system on the change of the $\mathbf{k}^{\mathbf{th}}$ parameter.

***At the same time, relation (6.2) also results in: the tolerance of the kth parameter for the allowable change in reaction (response) of the system, is directly proportional to the reciprocal value of the differential sensitivity on the change of this parameter***.

With the same methodology (procedure) and application of basic relations for the sensitivity and tolerance of the frequency transfer of the linear system, the newly formulated method which, with the use of computer simulations allows us to determine, with relative accuracy, the tolerance for the coefficient values of the original's transfer function of the stable system in the region of its stability (14). This is true also within the band where the transfer characteristic changes from aperiodic to the optimal waveform (in terms of the criteria of the optimal module) (15).

## 6.1 Tolerance of coefficients in the region of stability

Unwanted changes in the system's element parameter values and with it, the coefficients of its mathematical model as well, can cause instability of the system. Therefore it is useful (mainly for the designer) ***to recognize the range of allowable changes in coefficient values, in which the originally stable system remains stable***. This interval will dictate the ***tolerance of coefficients in the region of stability***.

Stability of the linear system depends on the roots of the characteristic equation, which for changes in the v[th] coefficient of a value $\mathbf{\Delta\alpha_v}$ will have the form

$$a_n p^n + a_n p^{n-1} + .... + (a_v + \Delta a_v)p^v + ....a_1 p + a_0 = N(p) + \Delta a_v p^v = 0 \quad [6.3]$$

For the roots laying in the left half of the gauss complex plane (GCP) (in limit cases, on the imaginary axis) and must fulfill the condition

$$N(j\omega) + \Delta a_v (j\omega)^v = \{N(p) + \Delta a_v p^v\}_{p=j\omega} \geq 0$$

In which for $\mathbf{\Delta\alpha_v}$ it follows

$$\Delta a_v \geq \mathrm{Re}\left\{-\frac{N(j\omega)}{(j\omega)^v}\right\} \quad [6.4]$$

In order to represented the waveforms defined by curves ***in the GCP*** for $\omega \in (0, \infty)$ to have a finite dimension, we express the reciprocal value of the relative coefficient tolerance, that is

$$\frac{a_v}{\Delta a_v} \leq \mathrm{Re}\left\{-\frac{a_v (j\omega)^v}{N(j\omega)}\right\} = \frac{1}{\delta a_v} = \mathrm{Re}\{K_v(j\omega)\} \quad [6.5]$$

***The geometric image of K(jω) within the GCP*** *for the continuous change in value* **ω ∈ <0, ∞ )** ***are continuously oriented*** *(similar in shape)* ***curves, which for ω=0*** *start either at the beginning (for* ***v=1…, 2…, …, n****), or at* ***-1*** *(for* ***v=0****), with growing* ***ω*** *continuing in the clockwise direction and for* ***ω*** *→ ∞ end (for* ***v=0, …, 1, n-1****) in the early stages, or at* ***-1*** *(for* ***v=n****), in which* ***their intersections with the real axis defines the limit of allowable***

***changes in the relative reciprocal values of the corresponding coefficient*** *with respect to relation (6.5). Fig6.1 illustrates the situation*

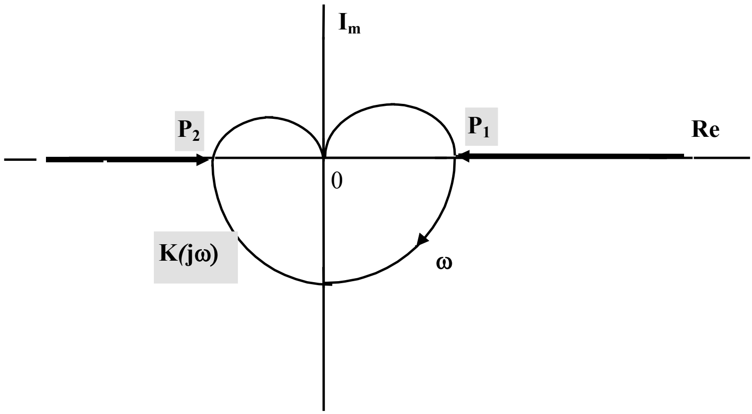



***Fig. 6.1*** *Typical waveform of the curve $K_v(j\omega)$ for v=n-1=3*

With the coordinates of the intersections $\mathbf{P_1}$ and $\mathbf{P_2}$ of the curve $\boldsymbol{K_v(j\omega)}$ with real positive GCP axis, with respect to relation [**6.5**] we can define the ***range of allowable relative changes*** in values of individual ***coefficients*** $\boldsymbol{\Delta a_v}$ in which the originally stable system will remain stable.

$$\frac{a_v}{P_2} \leq \Delta a_v \leq \frac{a_v}{P_1} \qquad [6.6]$$

Application of this method on the determination of allowable coefficient tolerance, as well as system and controller parameters for the second order system controlled by the proportional controller with first order dynamic delay in the general analytical form, can be seen in literature (15). A program for the calculation of tolerance and at the same time, its graphical representation throughout the waveform of the transfer function, is created to automate the solution procedure on a PC in the **MATLAB** program (3).

As such, for a stable, non-static third order system with an aperiodic waveform of the transfer function given by the transfer

$$G(p) = \frac{6}{p^3 + 6p^2 + 11p + 6} = \frac{b_0}{a_3 p^3 + a_2 p^2 + a_1 p + a_0}$$

we can calculate the interval of allowable changes in coefficient values and suitably represent the waveform of the transfer characteristic for their allowable change. In **fig. 6.2** and **6.3** are the **2D** and **3D** representations for the waveform of the transfer characteristic at allowable changes in coefficient $\boldsymbol{a_3}$ respectively, and for the change in coefficient $\boldsymbol{a_0}$ in **fig. 6.4**.

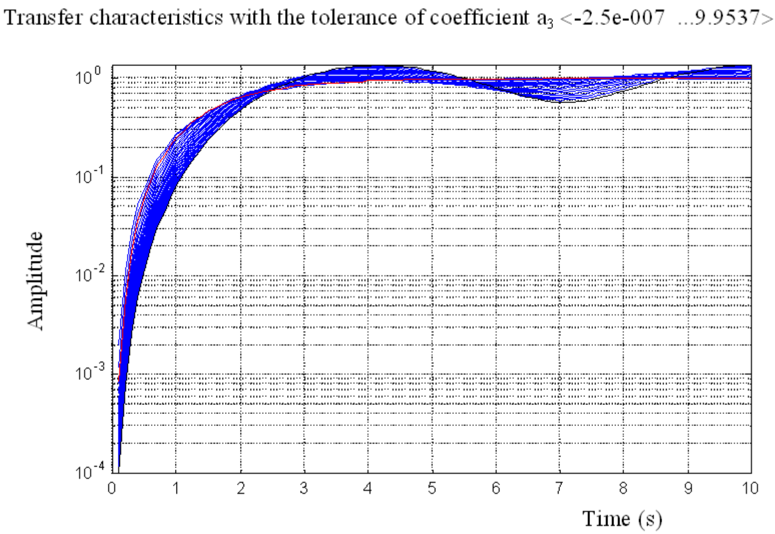



***Fig. 6.2** The band which represents the change in transfer characteristic for the tolerance of coefficient $a_3$*

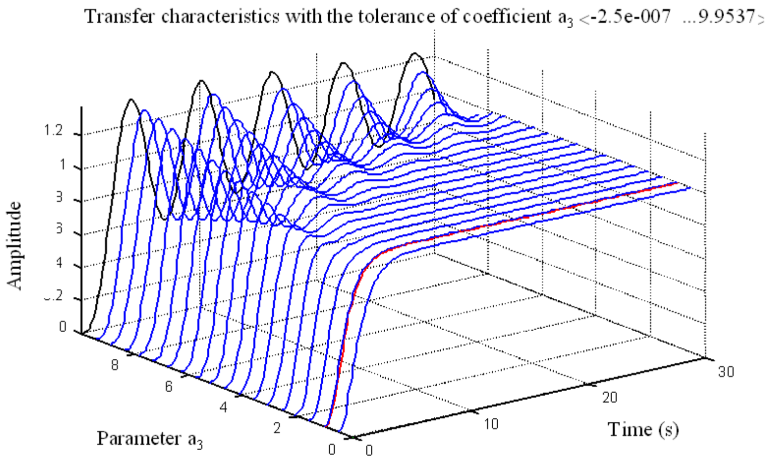



***Fig. 6.3*** *Waveform of the characteristic within the tolerance band of coefficient $a_3$*

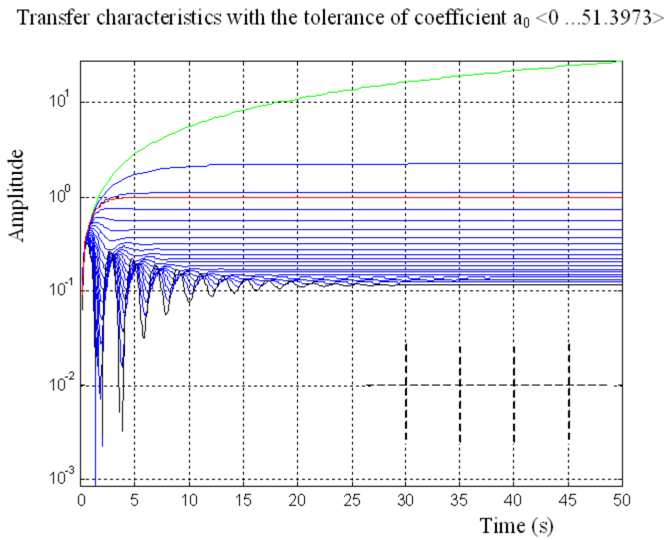



***Fig. 6.4*** *Waveform of the characteristic within the tolerance band of coefficient $a_0$*

## 6.2 Tolerance of the coefficients for optimal response

In the proposal (design) of dynamic systems, it is usually required that their transfer characteristic is aperiodic, or that (mainly for control) it meets the ***requirements of an optimal module***. These requirements resulted in the pursuit to create a method for the definition of allowable changes (tolerance) of individual coefficients of the system's transfer function, in which the original ***aperiodic and stable transfer characteristic turns into the optimal characteristic***. The result of these efforts is the ***original method***, ***which allows us*** (through the use of computer simulation) ***to determine the tolerance or boundary for coefficient values in which the transfer characteristic will stall have the optimal waveform***. The method is based upon the requirements for the optimal model imposed on the waveform of the amplitude-frequency characteristic. In the detailed analysis on the dependence of the waveform of these characteristics on the change in coefficient values of the characteristic polynomial of the frequency transfer, as well as the use of information from simulations of the defined curves within the GCP, a relation can be found for the determination of allowable changes in original coefficient values, in which the transfer characteristic of the system moves within the band defined from the aperiodic to the optimal waveform. The method has been presented on the scientific conference (14, 17) and published in the journal (15).

We will suggest the ***procedure for deriving this method***, based upon the changes in transfer function, or frequency transfer, with a change of any coefficient $a_v$ of the characteristic equation, or characteristic polynomial of the transfer function.

Frequency transfer of the LDS will be described in component form

$$G(j\omega)=\frac{M(j\omega)}{N(j\omega)}=R(\omega)+jI(\omega) \qquad [6.7]$$

The module (absolute value) of the frequency transfer $G(j\omega)$ will be

$$|G(j\omega)|=\sqrt{R(\omega)^2+I(\omega)^2}\equiv G(\omega) \qquad [6.8]$$

The ***optimal module criterion*** for the optimal waveform of the transfer characteristic of the system requires that the following conditions be met:

$$\left.\begin{aligned}&\frac{\partial}{\partial\omega}|G(j\omega)|=\frac{\partial}{\partial\omega}\{G(\omega)\}\leq 0\\ &\qquad\text{and}\\ &\frac{\partial}{\partial\omega}|G(j\omega)|^2=\frac{\partial}{\partial\omega}\left\{G(\omega)^2\right\}\leq 0\end{aligned}\right| \qquad [6.9]$$

After application of the second condition in [**6.9]** on ***G*(ω)** within expression [**6.8**] we obtain

$$\frac{\partial}{\partial\omega}\left\{G(\omega)^2\right\}=2R(\omega)\cdot\frac{\partial R(\omega)}{\partial\omega}+2I(\omega)\cdot\frac{\partial I(\omega)}{\partial\omega}\leq 0 \qquad [6.10]$$

where, ***for the real part of the system's frequency transfer*** *with optimal response (i.e. at the transfer of the frequency characteristic of the real axis)* ***meets the following condition****:*

$$\frac{\partial}{\partial\omega}\left[\mathrm{Re}\{G(j\omega)\}\right]=Re\left\{\frac{\partial G(j\omega)}{\partial\omega}\right\}\leq 0 \qquad [6.11]$$

If we describe the transfer of the system $G(j\omega)$ in the form [6.7], we must consider the change in coefficient of the characteristic polynomial with a value of $\Delta a_v$, and we apply the condition [6.11] on the changed frequency transfer

$$\overline{G}(j\omega) = \frac{M(j\omega)}{N(j\omega) - \Delta a_{\nu}(j\omega)^{\nu}} \qquad [6.12]$$

will be $$Re\left\{\frac{\partial}{\partial\omega}\left[\frac{M(j\omega)}{N(j\omega) - \Delta a_{\nu}(j\omega)^{\nu}}\right]\right\} \leq 0 \qquad [6.13]$$

After carrying out the operations in relation [6.13] and ***adjusting*** the derived expressions, we can derive for the allowable change in the $v^{th}$ coefficient of the characteristic polynomial in the direction of its lower boundary

$$|\Delta a_{\nu}| = \frac{|P_2 - P_1|}{|2P_2P_3 - P_1P_3 + P_4|} \qquad [6.14]$$

where values for **P1, P2, P3**, and **P4** give the coordinates of the intersections defined by the curves with real positive axis in the **GCP**, specifically:

$$\left.\begin{aligned}
P_1 &= Re\{K_1(j\omega)\} = Re\left\{\frac{M'(p)}{M(p)}\right\} p = j\omega; I(\omega) = 0 \\
P_2 &= Re\{K_2(j\omega)\} = Re\left\{\frac{N'(p)}{N(p)}\right\} p = j\omega; I(\omega) = 0 \\
P_3 &= Re\{K_3(j\omega)\} = Re\left\{\frac{p^{\nu}}{N(p)}\right\} p = j\omega; I(\omega) = 0 \\
P_4 &= Re\{K_4(j\omega)\} = Re\left\{\frac{\nu p^{\nu.1}}{N(p)}\right\} p = j\omega; I(\omega) = 0
\end{aligned}\right| \qquad [6.14]$$

The curves $K_1(j\omega)$, and $K_2(j\omega)$ at $\omega=0$ begin at the relative positive axis of the **GCP** at points $P_1$ and $P_2$ (as long as the transfer polynomial are at least of the first degree), with rising frequency they continue in the clockwise direction and for $\omega \rightarrow \infty$ end at the beginning of the coordinates or at the real positive axis of the **GCP** (curve $K_3$ for $v=n$). Coordinates of the intersection of these curves with real positive axis determine the values of coefficients $P_3$ and $P_4$.

For determining the allowable value of growth for coefficient $a_v$ we use the second of the conditions in [6.10] in view of relation [6.11] we can write

$$\left|\operatorname{Re}\left\{N(j\omega)+\Delta a_{v}(j\omega)^{v}\right\}\right| \geq 0$$

from which the allowable deviation (growth) in value of the coefficient is

$$\left|\Delta a_{v}\right| \leq \frac{1}{\left|\operatorname{Re}\left\{\dfrac{(j\omega)^{v}}{N(j\omega)}\right\}\right|}=\frac{1}{\left|P_{3}\right|}$$

The allowable range of the vth coefficient of the LDS's characteristic polynomial, according to the performed analysis using [**4.15**] and [**4.16**], can then be determine by the following relation

$$\boxed{a_{v}-\frac{\left|P_{2}-P_{1}\right|}{\left|2 P_{2} P_{3}-P_{1} P_{3}+P_{4}\right|} \leq \bar{a}_{v} \leq a_{v}+\frac{1}{\left|P_{3}\right|}} \qquad [6.17]$$

where $\overline{a_v}$ are values in which the coefficient can grow from their original values ***so that the system's transfer characteristic, with changes in***

***coefficient values, will be aperiodic, or in better words, have the optimal waveform (in terms of the criteria for the optimal module)***.

Along with the conditions arising from the requirements of the optimal module, the conditions for stability were also used in deriving the relations for determining the tolerance of the coefficients. Therefore the method for a stable system with aperiodic transfer characteristic with no overshot, determines the tolerance of the transfer coefficients for the waveform of the characteristic within the monotone to the optimal band.

Because the **method** analyzes the one-dimensional **LDS** regardless of its structure, ***it is useable for open systems as well as control circuits and in the analysis of multidimensional systems***. Procedures for realization of the method through computer programs such as MATLAB were performed by Ing. R. Halenarom [2] and allow for the elegant computation for the solution of tolerance as well as the graphical representation of the transfer characteristic in 2D and 3D subject to changes in coefficient values within the calculated tolerance interval. Examples of the applied method, together with the simulation of the transfer characteristic's waveform, for systems with aperiodic transfer characteristic without overshot and the transfer

$$G_0(p) = \frac{6}{p^3 + 6p^2 + 11p + 6}$$

are seen in fig. 6.4 to 6.7.

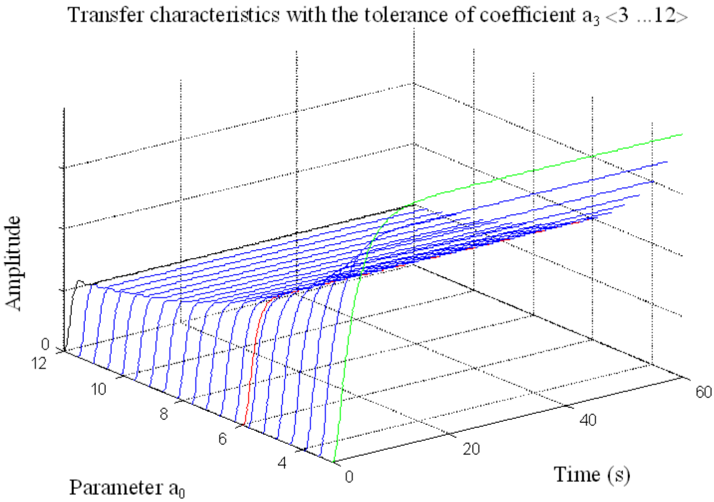



***Fig. 6.5*** *Transfer characteristic for changes in coefficient* $a_0$

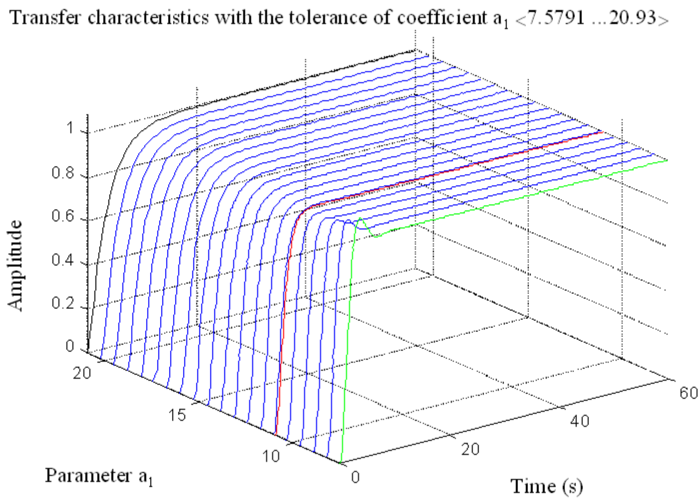



***Fig. 6.6*** *Transfer characteristic within the interval of change for coefficient* $a_1$

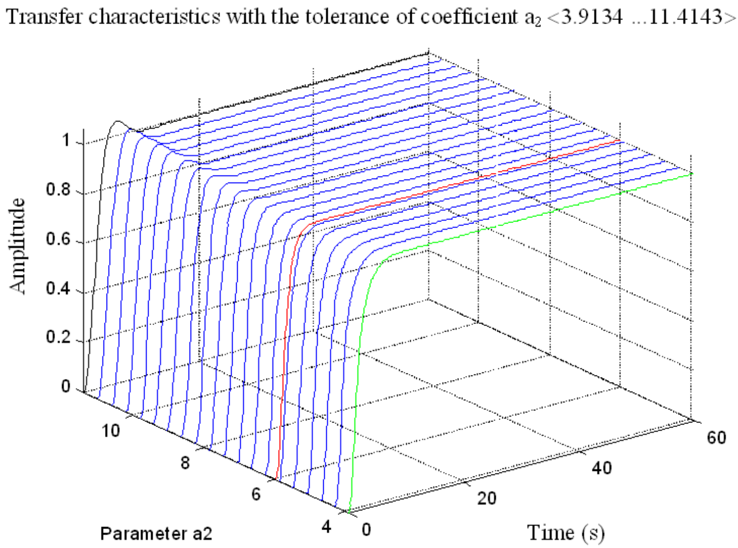



***Fig. 6.7*** *Transfer characteristic within the interval of change for coefficient $a_2$*

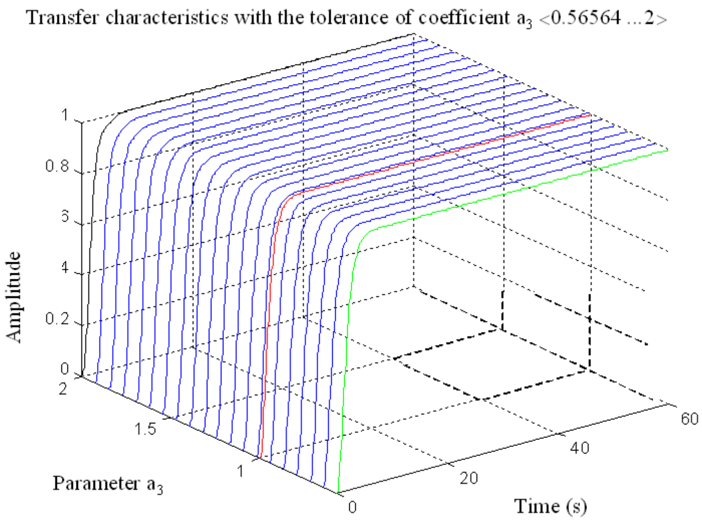



***Fig 6.8*** *Transfer characteristic within the interval for change $a_3$*

General solution of the inverse problem, that is to define equations for the tolerance of coefficient values on the desired output of the waveform, or response is not a simple task, and whether it be an algebraic or differential system it is practically unsolvable. The proposed procedure for determining the tolerance of coefficients therefore is based on the originally defined (given) structure of the dynamic system and allows for the determination of tolerance for individual coefficients (element parameters) of the differential equations, with the assumption that original values of the other coefficients are maintained for the defined change in system response (behavior from the stable to unstable and from the aperiodic to the optimal). It was necessary to modify the mathematical procedure in the theoretical determination and modification of relations, regarding the determination of the tolerance, such that the calculation procedure could be realizable for the determination of the result. Thus the solution methodology has its benefits: in the area of theoretical analysis as well as in the area of synthesis of systems and their subsequent implementation in to software such as **MATLAB**.

***The use of the results is interesting (if not necessary) for the designer (creator of the dynamic system) for the analysis of the dynamic behavior of the system subject to changes in some dominant parameter (robustness), which for known system structures allows us to judge, for the changes in behavior, the unwanted change in a specific functional element (non-mounted diagnostics) and also the deeper understanding of a dynamic system's properties (region of knowledge).***

## 6.3 Tolerance of element parameters of the control circuit

Examining tolerance intervals of individual coefficients of the dynamic system is necessary especially in control where, for known parameter values of the system, it is desirable to know the allowable tolerance of the controller parameters. A simple illustration of a possible procedure can be shown through the example for identifying the tolerance of control parameters for the control circuit of a 1st order dynamic system, with proportional controller and first order delay, proposed for the aperiodic waveform of the controller transfer characteristic (***see chapter* 5.1**). For a given time constant of the system $\boldsymbol{T}$, chosen ratio $\boldsymbol{T/\tau = \xi \geq 1}$ and circuit gain $\boldsymbol{K_0}$ for aperiodic waveform of the control circuit's transfer characteristic, we pre-analyze instances for the time constant of the controlled system $\boldsymbol{T = 1}$, and values $\boldsymbol{\xi = 2}$ and $\boldsymbol{\xi = 10}$. Transfer of the circuit has the form:

$$G(p,\xi,T) = \frac{K_0\xi}{pT^2 + pT(1+\xi) + \xi(K_0+1)} = \frac{b_0}{a_2 p^2 + a_1 p + a_0}$$

***To illustrate the properties*** of the given ***control circuit***, we can derive tolerance coefficients from the results of its analysis from the mathematical model (transfer function) for the allowable change in the ***control characteristic's*** waveform.

If we use the method described in chapter 6.1 created by Vrban including the program module created by Halenar (3, 15) we can then determine the tolerance intervals of individual coefficients (as well as values of parameter elements), in which the transfer characteristic of the control circuit remains within the aperiodic to optimal limits of the waveform, and illustrate the waveform of the control characteristic.

The ***graphical illustrations of the characteristic are*** shown **in fig. 6.9, 6.10, 6.11, 6.12, 6.13, 6.14.**

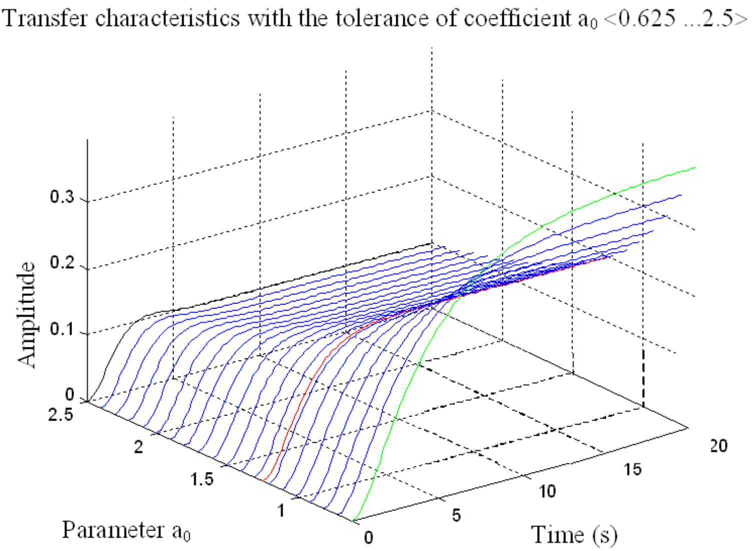



***Fig. 6.9*** *Characteristic for the change in coefficient $a_0$; for $\xi = 2$, $K_0 = 0.25$*

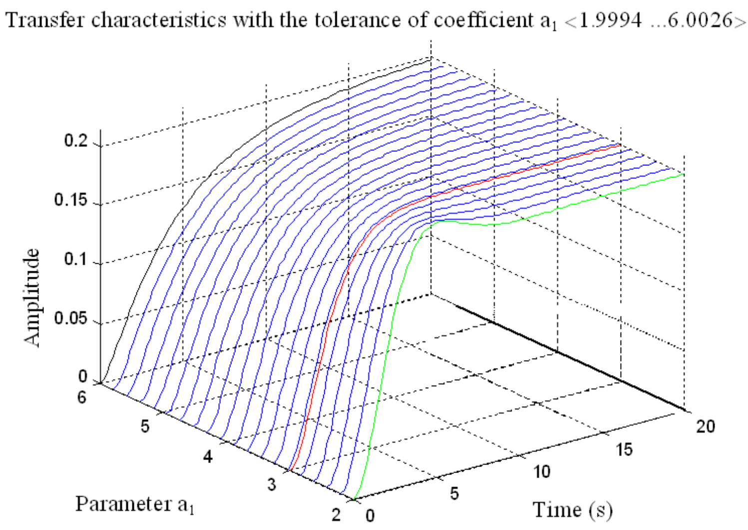



***Fig. 6.10*** *Characteristic for the change in coefficient $a_1$; for $\xi = 2$, $K_0 = 0.2$*

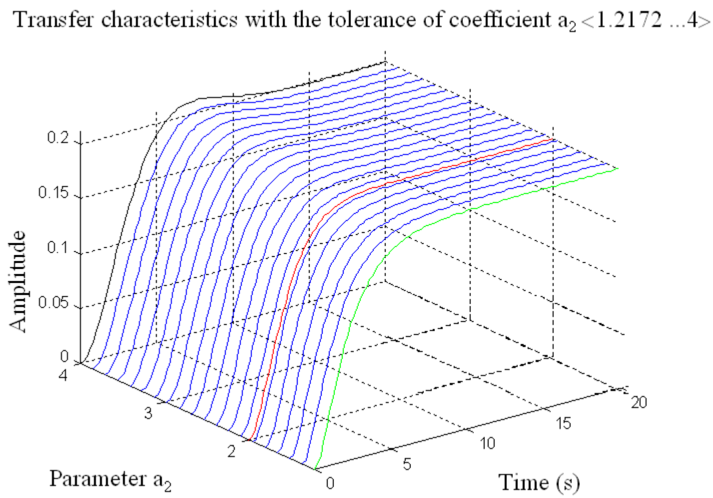



***Fig. 6.11*** *Characteristic for the change in coefficient $a_2$; for $\xi = 2$, $K_0 = 0.25$*

In comparing the interval of allowable change in value of the coefficients, then after calculation of the allowable change in time constant for the ***allowable interval change in time constant of the controller τ*** will show

$$\mathbf{0.5 \leq \tau \leq 2}$$

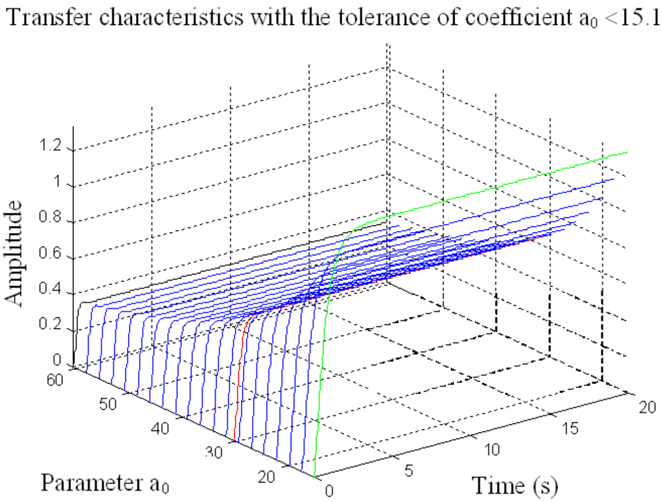



***Fig. 6.12*** *Characteristic for the change in coefficient $a_0$; for $\xi = 10$, $K_0 = 2.03$*

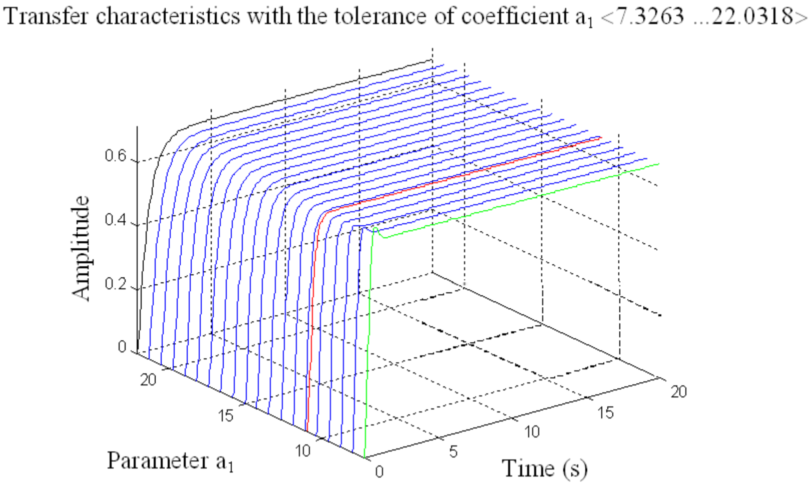



***Fig. 6.13*** *Characteristic for the change in coefficient $a_1$; for $\xi$ = 10, $K_0$ = 2.03*

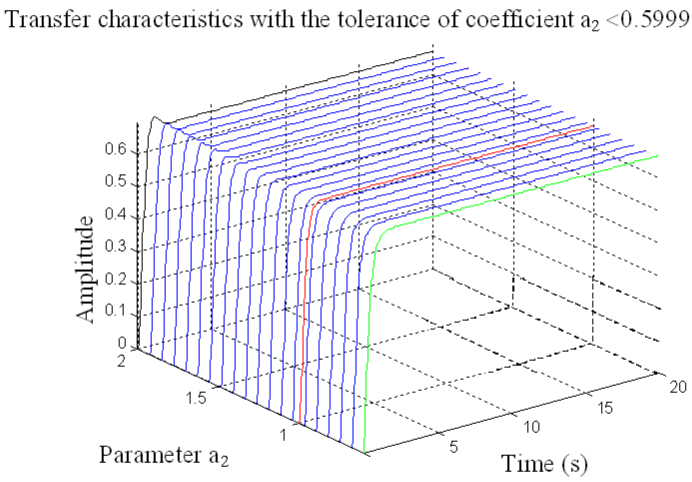



***Fig. 6.14*** *Characteristic for the change in coefficient $a_2$; for $\xi$ = 10, $K_0$ = 2.03*

For the tolerance of the controller's original time constant value **$\tau$=0.1**, change in the transfer characteristic from aperiodic to optimal (in terms of the optimal module) we obtain

$$\mathbf{0{,}09 \leq \tau \leq 0{,}16}$$

From the calculated tolerance coefficient $\Delta a_v$ it is possible to determine the ***tolerance band for individual element parameters and we can consider the real change*** and at the same time, from the waveform of shown characteristic, ***assess the waveform of the control process. This allows us to more deeply pre-analyze the properties and increase the quality of complex control designs.***

# 7. ROBUSTNESS OF LINEAR DYNAMIC SYSTEMS

The term "***robust***" or rather, ***robustness*** within the theory of systems, are used mainly in relation with automatic control (***Robust Control System***). Robustness can also be understood as one of the immanent properties of the system existing within a given real environment. With that said, particular importance of the concept itself is obviously dependent on the internal properties of the system, which have (in terms of its desired determination and behavior) relevant importance

We understand ***robustness of the system in a broader sense*** as an internal property of the system to perform the required functions (behaving in a desirable way) also for the relatively significant changes in parameters of its structure as well as the negative effects of its surroundings. It is clear that ***intervals of allowable changes in parameter values of the structure and its surroundings to maintain the systems functions are limited***. Understanding the above we can then also discuss the ***internal (immanent) robustness of the system*** that is manifested by its ability to meet function requirements even with significant changes in the element parameters and constraints (for example: aging, wear, damage etc...) . We can also discuss ***external robustness***, which presents itself as the resistance of system behavior to the surroundings (temperature, humidity, etc…). The system's capacity to maintain its functional abilities when subject to negative physical effects of the surroundings will be known as the ***resistance***. The ability to respond in an intentional manner (meet its function) even with undesirable changes in parameter values of its own elements and structure is known as '***robust behavior***' or just '***robustness of the system'***. Because the robust behavior of a complex, automatically controlled dynamic system is usually achievable through the appropriate choice of controller, that is to say the use of ***compensation through feedback control*** (in various modifications), we are talking about ***robust control of systems***, or in other words the '***Robust Control Problem'***. Robust control, in the field of automatic control theory, has gained increased attention [**1, 4, 6, 7, 9, 10, 22**] especially in the fields of space, aviation, military, information technology and control of energy resources (mainly nuclear).

## 7.1 System Robustness

Let us now focus our attention on the ***robustness as an internal property of the dynamic system, which reflects its external behavior***. Even in viewing the structure qualitatively the properties and behavior of the dynamic system result in the ***clear*** and ***very close relationship between its sensitivity, tolerance and robustness***. Even if all of these properties characterize the behavior of the system (mechanisms, machines, automated machine) for its use, there are a number of specific operational properties and parameters which greatly affect its usefulness (quality, reliability, security, etc…). This problem, however, belongs to another category, therefore it is not further discussed.

***In the designer's perspective, it is essential to define the sensitivity, tolerance and resistance (otherwise known as the robustness of the designed system), at the initial stages of the equipments development and with respect to its intended use. In the following stages it is then necessary to continuously analyze, review and correct the design such, that the***

***system truly meets the requirements***. When designing open systems without feedback, it is possible to determine the degree of the system's robustness behavior by using the ***internal compensation structure***, which is used quite often in electronic circuits and networks. However, in mechanical (machine) systems this procedure is usually more difficult to implement (more expensive). Also, it is ***practically impossible to achieve such compensation which would completely counter the effects of aging and degradation***.

***The robustness of the system is generally not a constant variable***. It obviously depends on the duration of use (even if this effect can, within the normal life span, be neglected), but also significantly on the properties and parameters of the environment in which the system is operating (system working on earth may not be as robust in space). ***Therefore, in order to characterize the robustness of some quantity (quantitative characteristic), it is necessary to specify the time interval that represents the normal life span of the system as well as the interval of values for parameters of the environment and conditions in which the system will operate. Clearly, it is necessary to define the system's situation where its behavior must be robust and properties (characteristic) which in respect, must be robust*** (robust system in one situation in the perspective of certain external properties, they may not be robust in other situations and in terms of other properties or characteristic).

***Robustness*** of the system's behavior can be, in terms of mathematical or graphical representation, ***described*** and mathematically characterized (***for example: in the phase area***) in which the image of the trajectory indicate the system's dynamic properties and thus also their change with varying parameters. Let us describe the robustness of the system's behavior in more detail.

***A Dynamic system (in terms of its behavior) will be robust if for small enough changes in its parameter retain the topological structure of phase trajectories, for which the coordinates of each points trajectory changes by an arbitrarily small value***. In this description the area of parameters (coefficients) of the dynamic system created by regions where each point in the system, in terms of the defined requirements, is robust. The boundaries between these regions create the ***bifurcation surfaces*** on which the system ceases to be robust.

To illustrate the above defined robustness, we characterize the ***portrait of the phase trajectories of the linear dynamic system for its robustness on its stability. The linear system will be robust on the stability*** in the range in which for such a change in parameters (coefficients) that the topological structure (portrait) of the phase trajectories will correspond to the stability of the system. This means that the phase trajectories, which start from the initial state point, proceed through the state space to its starting point in time. They meet at this (singular) point in the time $t \rightarrow \infty$ without ever really intersecting.

***Quantitative evaluation*** and expression of the ***robustness***, which ***includes the synergy effects of different elements in the general structure***, which could serve as a parameter for comparison between different (in terms of complexity and function) dynamic systems, is ***generally difficult to express***

***clearly***. As a ***comparison*** figure ***for information as to the robustness*** of the system with similar physical base elements, comparable complexity, structurally related and with the same intention (for example: automobiles) we relate to the same ***characteristic properties*** of the system. Thus, it appears that we can define the "***robustness parameter***" using the ***sensitivity and tolerance of essential (dominant) system elements***. The ***dominant elements***, in terms of robustness, we consider as those whose small change in parameters causes a significantly adverse effect in the behavior of the system.

Let us ***consider a general*** dynamic, purpose (to fulfill the required functions) created by a functional system consisting of ***N dominant elements***, which have a ***significant effect*** on its desired ***behavior*** (function), located in the normal environment and with normal operational conditions (***normal situation***). The effects of changing parameter values of relevant (dominant) elements on the change in the desired system behavior (properties) are expressible by the sensitivity of the system on these elements and the tolerance of the change in their parameters. With this it is obvious that ***robust systems are required to have relatively low sensitivity and sufficient tolerance to undesirable changes in parameters of the structure***. On the basis of these dependencies it is possible, as comparative information for specific '***related***' types of systems with typical dynamic characteristics, to define the ***robustness parameter*** whose value characterizes the robustness of the system in terms of the given properties (for example: brake system). This parameter can represent an important indicator for assessing the quality of the dynamic (mechatronic) system. We will attempt to define such a parameter in terms of the ***robustness of linear systems***.

It is obvious that the dynamic system will be, in a specific regime of function, more robust when its behavior will not deviate from its predefined limits (desired tolerance), while the ***deviation of the system from these limits will be effected by the most sensitive and least tolerant (critical) parameters of the system (its mathematical model)***.

To assess the robustness of the system operating in a given (desired) regime, defining also the allowable (acceptable) changes in properties can be made to consider all of the parameters, which will quantify the robustness of the system. This parameter will be defined as the minimum in the ratio between the relative value of the tolerance parameter of actual, dominant elements (coefficients) of the corresponding mathematical model, to the maximum value of the relative sensitivity of the system's characteristic function on the change in these parameters (coefficients). It is clear that this value will be minimum for a critical (in terms of robustness, crucial) element which at the same time can be found by this method. In accordance to [3.1] we consider a mathematical model of the system in the form $G(\xi,\alpha)$, where $\xi$ represents the argument of the system function and $\boldsymbol{\alpha} = [\alpha_1,\ldots, \alpha_k,\ldots, \alpha_n]$ is the vector from values of the coefficients (parameters), then the ***robustness coefficient of the system*** (in terms of the defined properties) ***can be defined by relation***

$$\Omega_s = \min\left\{ \frac{\left|\frac{\Delta\alpha_k}{\alpha_k}\right|}{\max\left|R_k(\xi)\right|} \right\} \quad \text{for } k=1, 2, ..., N \qquad [7.1]$$

Where **max|*$R_k(\xi)$*|** describes the maximum value of the ***relative sensitivity of the system's characteristics on the $k^{th}$ parameter of the selected dominant element.***

Expression $\Delta\alpha_k/\alpha_k$ gives the r***elative value of the tolerance parameter (coefficient) for the allowable change in the system's defined properties*** (behavior). This value can be found by implementing a suitable method (for example [3, 15]) if such a method is available. The value of expression **max|*$R_k(\xi)$*|** represents the maximum value of the relative sensitivity of the $k^{th}$ coefficient in the waveform of the transfer process, that is to say within the transmission frequency band of the analyzed dynamic system. ***For the robustness coefficient of the system we consider the number $\Omega_s$, which for some dominant elements, results in the minimum value*** (***robustness of the system determines the least robust element***).

In terms of relation [7.1] it is seen that Ωs is a non-dimensional real, positive number which can result in a value from zero (for infinite sensitivity, zero tolerance) to an infinite value (for zero sensitivity). Physically it gives the relative value of the allowable change in magnitude of element parameters attributable to the relative value of the change in amplitude of the system's observed characteristic with unit change in value of the corresponding coefficient (parameter). The value of the parameter for the robustness of the system for varying device structures intended for the same function can be useful comparative indicator of their robust behavior in terms of the preferred operational properties. Similarly, the structural analysis of the designed (modeled) system on the robustness in terms of its desired behavior (properties), can also significantly affect the workload on the designer and thus ensure achievement of the robustness for the desired behavior.

To illustrate this methodology on determining the robustness in relation to previous chapters (4 and 6) where, on the basis of simulation results and calculation of the sensitivity and tolerance, the effect of individual coefficients given in the LDS on the robustness of its behavior in terms of the optimal waveform of the transfer response, we determine the value of such a defined robustness coefficient of the system with the transfer

$$G(p)=\frac{p+6}{p^3+6p^2+11p+6}=\frac{b_1p+b_0}{a_3p^3+a_2p^2+a_1p+a_0}$$

The robustness parameter of the system is determined by using the following relation

$$\Omega_s=\min\left\{\frac{\left|\dfrac{\Delta a_\nu}{a_\nu}\right|}{\max\left|R_\nu(t)\right|}\right\}\quad \text{for } \nu=1,\ 2,\ 3,\ ..$$

From the results of the simulation **(15)** and relations defined within, it is possible to determine required information, create a table and determine robustness parameters of the system for allowable change in the transfer response (characteristics).

| ***Coefficient*** | $\mathbf{a_v}$ | **8vd** | **0vh** | **A $\mathbf{a_v/a_v}$** | **max$\mathbf{R_v}$(i;)** | **Q** |
|---|---|---|---|---|---|---|
| **ao** | 6 | 3,14 | 12 | 1,50 | 0,16 | 9,2 |
| **ai** | 11 | 7,77 | 21 | 1,21 | 0,06 | 21,2 |
| **B2** | 6 | 5,4 | 11,45 | 1,00 | 0,07 | 14,3 |
| $\mathbf{a_3}$ | 1 | 0,58 | 2,00 | 1,42 | 0,24 | 5,9 |

Values of the tolerance coefficients given in the third and fourth column of the table above are specified for the given system, as the ratio of their tolerance to the original value and the value of $R_v(\tau)$ represents the maximum value of the absolute sensitivity for its corresponding coefficient. Because the system's robustness parameter, with respect to our definition, depends on the sensitivity of its characteristics properties on the change in value of the corresponding parameter and also on the allowable tolerance value of the coefficient, in order to maintain the ***allowable change*** in characteristic properties (behavior), then as it is defined from the robustness parameter; the "robustness" of the system depends on the robustness of parameters of its individual, dominant elements (only dominant parameters). In terms of such a definition for the robustness of a system, it is only obvious that its robustness is determined by the robustness of critical elements, for which the coefficient value is minimum. ***Determining In such a way the parameter from the group (family) of related systems shows that a greater value expresses a greater robustness of a particular system.*** In the case of structure analysis, we are talking about coefficient $\mathbf{a_3}$ *which obtains the minimum value which we assume to be the robustness value of its parameter.*

Analogically, as in an open system, we can also understand ***robustness of a closed control circuit***, where we relate the robustness of the behavior to the waveform of the controller when changes in parameters of the control system occur.

## 7.2 Control sensitivity

We will understand ***control sensitivity*** as the change in response of the control circuit as a result of the control signal for elementary changes in transfer of the controlled system with allowable change in parameters (perturbation of the system) described by its transfer function ***S*(*p*)**. we can then express the ***absolute sensitivity***.

Transfer of the classic closed control circuit on the control is

$$G(p) = \frac{S(p)R(p)}{1 + S(p)R(p)}$$

***Absolute sensitivity*** **Cs(p)** will be

$$C_s(p) = \frac{\partial G(p)}{\partial S(p)} = \frac{R(p)}{\left[1 + S(p)R(p)\right]^2}$$

***Absolute sensitivity within the frequency band ω***

$$C_s(j\omega) = \left\{ \frac{R(p)}{\left[1 + S(p)R(p)\right]^2} \right\} \mapsto p = j\omega$$

***Relative sensitivity*** of the control circuit transfer on the change in system transfer will be

$$C_s^r(p) = \frac{C_s(p)}{G(p)} = \frac{1}{S(p)\left[1 + S(p)R(p)\right]}$$ [7.2]

from which results; ***for greater values of R(p) (also for amplified controller) the sensitivity of the controller is lower and thus the control circuit is more robust. It applies, of course, only in the limits of stability.***

## 7.3 Control Robustness

Assuming now that the transfer function of the control circuit (control transfer) has the form:

$$G(p) = \frac{S(p).R(p)}{1 + S(p).R(p)}$$

Where ***G*(*p*)** is the ***transfer of the controlled system*** and ***R*(*p*)** is the ***control transfer***.

If the control circuit, with the control system ***G*(*p*,α)** and controller ***R*(*p*,β)**, have such properties that with ***small changes in the system's parameters Δα as well as small changes in parameters of the controller R(p,β) correspond to the allowable (negligible) change of the transfer function of the control G(p), then such a circuit can be said to be robust (in terms of control). In other words it is to say that such a control will be known as*** **robust control**.

For the change in ***transfer function G*(*p*)**, which is a function of change in parameters of the control system and controller, we can write:

$$\Delta G(p)=\frac{\partial G(p)}{\partial R(p)}\Delta R(p)+\frac{\partial G(p)}{\partial G_{TS}(p)}\Delta S(p) \qquad [7.3]$$

In which the ***transfer G(p)***

$$\frac{\partial G(p)}{\partial R(p)}=\frac{S(p)}{[1+S(p).R(p)]^2}$$

$$\frac{\partial G(p)}{\partial S(p)}=\frac{R(p)}{[1+S(p).R(p)]^2} \qquad [7.4]$$

Substituting into [**7.3**]we get:

$$\Delta G(p)=\frac{1}{[1+S(p).R(p)]^2}.[S(p).\Delta R(p)+R(p).\Delta S(p)] \qquad [7.5]$$

We require that **Δ*G*(*p*)** for small changes in **Δ*R*(*p*)** and **Δ*S*(*p*)** are small (ideally zero). Then from relation [**7.5**] we obtain:

$$\{S(p)\Delta R(p)+R(p)\Delta S(p)\}\rightarrow 0 \qquad [7.6]$$

From which $S(p)\Delta R(p)\Rightarrow -R(p)\Delta S(p)$

A more appropriate relation is to modify [**7.6**] into the form:

$$\frac{\Delta R(p)}{R(p)}\rightarrow -\frac{\Delta S(p)}{S(p)}$$

And for the ***frequency transfer***:

$$\frac{\Delta R(j\omega)}{R(j\omega)}\rightarrow -\frac{\Delta S(j\omega)}{S(j\omega)} \qquad [7.7]$$

Because ***R*(*j*ω)** and ***S*(*j*ω)** are complex expressions, we can write:

$$\frac{\Delta R(\omega)}{R(\omega)}.e^{j[\Delta\varphi(\omega)-\Delta\varphi(\omega)]} \rightarrow -\frac{\Delta S(\omega)}{S(\omega)}.e^{j[\Delta\psi(\omega)-\psi(\omega)]}$$

***We therefore consider that***

$$\frac{\Delta R(\omega)}{R(\omega)} \rightarrow -\frac{\Delta S(\omega)}{G(\omega)}$$

$$\Delta\psi(\omega) \rightarrow -\Delta\psi(\omega) \qquad [7.8]$$

***From relation* [7.8] *we can deduce***:

**If the control circuit is to become a robust control system (small dependence of the control process on changes in element parameters which are members of the control circuit) then, theoretically and at the same time they should also somewhat satisfy the following conditions:**

- *Relative change of the controllers frequency transfer module must be proportionally or equally as large as (but contrary to) the relative change of the transfer module of the controlled system within the whole range of its transfer band.*
- *Change in phase of the controller's frequency transfer must be approximately opposite as the change in phase of the system's frequency transfer*

As an interpretation of this assumption we presume the ***Nyquist stability criterion***, which allows for a certain margin of stability **δ**:

$$S(j\omega)R(j\omega) = |1-\delta|\ e^{j\pi};\ 0<\delta<1$$

Lets say that ***S*(*j*ω)** is the frequency transfer of the system and ***R*(*j*ω)** is the frequency transfer of the controller.

We ***require*** that; for changes in values of the system's transfer and controller, the stability reserve remains relatively unchanged:

$$[S(\omega) + \Delta S(\omega)][R(\omega) + \Delta R(\omega)]e^{j[\varphi(\omega)+\Delta\varphi(\omega)+\psi(\omega)+\Delta\psi(\omega)]} = |1 - \delta| e^{j\pi}$$

For the transfer module of open control circuit with the transition into the negative, real axis, the following should apply:

$$[S(\omega) + \Delta S(\omega)] \cdot [R(\omega) + \Delta R(\omega)] \cong R(\omega) S(\omega) = |1 - \delta| \qquad [7.9]$$

After multiplication and neglecting small 2nd order variables within the product (**Δ*R*.Δ*G***) we then get

$$S(\omega)\Delta R(\omega) + R(\omega)\Delta S(\omega) = 0$$

That is $\dfrac{\Delta R(\omega)}{R(\omega)} = -\dfrac{\Delta S(\omega)}{S(\omega)}$

A change in phase in the relation will results in:

$$\varphi(\omega) + \Delta\varphi(\omega) + \psi(\omega) + \Delta\psi(\omega) = \varphi(\omega) + \psi(\omega) + \pi$$

where:

$$\Delta\psi(\omega) = \pi - \Delta\varphi(\omega)$$

***From the aforementioned, it follows that the stability reserve is retained if the relative change in the module of the system's frequency transfer will meet an equally sized (but oppositely oriented) relative change in the control module and the phase change of the controller's transfer will be in reverse to the change in phase of the system's transfer***.

It is obvious (without detailed analysis) that the general fulfillment of the mentioned conditions is practically impossible through a simple control circuit. However it is achievable by using an adaptive system (at least for the controller), which corrects its parameters in accordance to the formulated conditions depending on the change in system parameters (or transfer). In this case it is necessary to add more members to the control circuit which would determine generated changes in parameters of the system and adapt the parameters of the controller depending on the change in parameters of the controlled system (then we are talking about an adaptive system). ***Without the use of adaptation it is possible to obtain, to an extent, the robustness of the control circuit, mainly in the appropriate choice of control parameters. Solving this problem, that is, to design such a control structure which for any given control system, ensures the robust control (for example: robust stability or quality of the process) belongs to the problem of*** **robust control (1, 4, 6, 9, 19).** Systems which are resistant to changes in parameters (caused either by the change in surrounding effects, aging and degradation, as well as other undesirable effects) must be designed especially where these changes can result in extensive material losses or life endangering (nuclear plants, chemical complexes, navigation and control systems in aviation and transportation, space programs, etc…).

When designing robust control systems, with the assumption that the robustness of the controlled object cannot be increased, as a result of technical reasons, then the solution of the problem depends only on the ***robustness of the control system***.

## 7.4 Perturbation of Linear Dynamic Systems

A process operating within the control circuit with permissible changes in action and fault variables as well as its input depends on the dynamic properties of the control system and properties of the controller. If the properties of the system change then the change in waveform of the control process and control deviations are inevitable. On the change caused by the ***perturbation*** (small change or deviation) value of ***parameters*** (the characteristic of the controlled system within certain limits) the controller should react in such a way, that ***we limit the undesirable effect of these changes on the required waveform***. Then we can consider a controller to be robust ***when its sets (family) of possible controllers is capable of best satisfying the desired task within the defined (considered) range of the control systems operational conditions***. As a criterion of the control process, we frequently consider the stability and desired quality of the control process. For the ***family of perturbation systems*** resulting from the ***original system*** (for given constant, original parameter values) we consider all systems belonging to the bounded region of the expected change in parameters (characteristic of the original system). It is clear that the control of the perturbation system can be secured only if it is possible ***to create a controller, that is to say a control system, which is capabl***e, for given or expected region of change in the properties of the system, to ***implement a process which ensures that the conditions are met*** (*without adaptation of parameters!*).

Although the ***problem of robust*** control has recently been widely elaborated, in terms of prior considerations focusing on the area of sensitivity analysis of the system, we introduce only some remarks, procedures and solutions where information from previous sections can be used.

Change (perturbation) in properties of the control system can be understood as the uncertainty of immediate parameter values of its structure, or the uncertainty of the response (behavior, characteristic). In this respect, we are then referring to the ***structured and unstructured uncertainty***.

**Interval of uncertainty**

In the analysis of structured uncertainty, we usually base off of the system's description in the form of transfer functions, whose coefficients can change continuously within definite interval values (we assume upper and lower limits) – ***interval of uncertainty***. Such a perturbation system is then described by an infinite number of transfer functions within the finite region. To verify the stability of such a system in respect to classical criteria, we must perform an analysis of an infinite number of transfer functions, which is of course, impossible. Solving problems of stability for a finite number of operations can be performed using ***Kharitonov's theorem*** **(4)**, which states that: for the assessment of stability of perturbed system, it is enough to have ***four Kharitonov polynoms (complex)*** prepared from coefficients of the polynomial in the denominator of the perturbed system which ***satisfies the algebraic conditions for stabili***ty.

Let us assume the ***characteristic transfer polynomial*** of the perturbed system in the form

$$N(p) = a_0 + pa_1 + p^2 a_2 + p^3 a_3 + \ldots + p^i a_i \ldots = \sum_{i=0}^{n} p^i a_i \qquad [7.10]$$

where $a_i = \in \langle a_i^-, a_i^+ \rangle$ $\quad a_i^-$ is the min of $a_i$; $\quad a_i^+$ is the max of $a_i$

***Kharitonov's polynomial*** for the 6th order system will have the form

$$\begin{aligned}
N^{++}(p) &= a_0^+ + a_1^+ p + a_2^- p^2 + a_3^- p^3 + a_4^+ p^4 + a_5^+ p^5 + a_6^- p^6 + \ldots \\
N^{--}(p) &= a_0^- + a_1^- p + a_2^+ p^2 + a_3^+ p^3 + a_4^- p^4 + a_5^- p^5 + a_6^+ p^6 + \ldots \\
N^{+-}(p) &= a_0^+ + a_1^- p + a_2^- p^2 + a_3^+ p^3 + a_4^+ p^4 + a_5^- p^5 + a_6^- p^6 + \ldots \\
N^{-+}(p) &= a_0^- + a_1^+ p + a_2^+ p^2 + a_3^- p^3 + a_4^- p^4 + a_5^+ p^5 + a_6^+ p^6 + \ldots
\end{aligned} \qquad [7.11]$$

If we denote

$$\begin{aligned}
K_1(j\omega) &= \{N^{++}(p)\} \mapsto p = j\omega \\
K_2(j\omega) &= \{N^{--}(p)\} \mapsto p = j\omega \\
K_3(j\omega) &= \{N^{+-}(p)\} \mapsto p = j\omega \\
K_4(j\omega) &= \{N^{-+}(p)\} \mapsto p = j\omega
\end{aligned}$$

then the coordinates of points $K_1(j\omega_i),..,K_4(j\omega_i)$ for frequencies $\boldsymbol{\omega_i}$ create rectangles in the gauss complex plane (**GCP**) and for the changing frequency of $\boldsymbol{\omega}$, sets of rectangles (area) are created, shown in **fig 7.1**

**[www.polyx.com/_robust]**, where all possible values (end points) of the polynomials in the interval of uncertainty of coefficient $a_i$ lay.

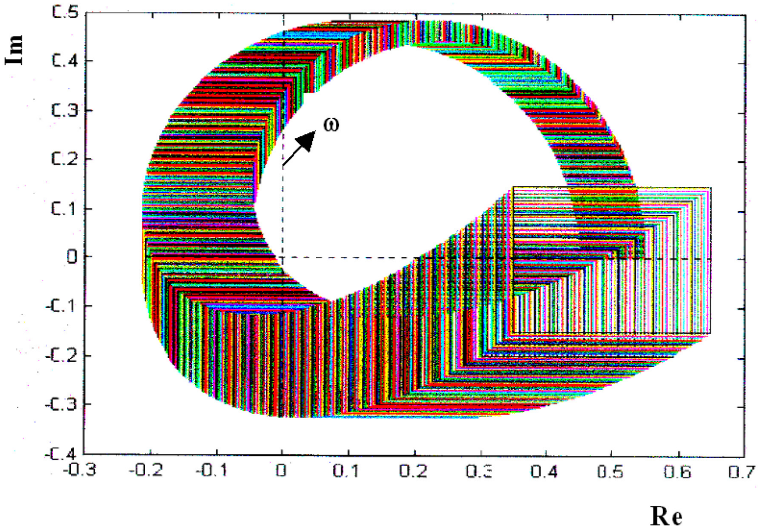



*__Fig. 7.1__ Region of values of polynomials with an interval of uncertainty within the GCP*

***Polynomials of type (10) are stabile if their roots lay in the left quadrant of the gauss complex plane. In this case all polynomial coefficients $a_i$ are non zero and have the same sign. The phase of the polynomials is continuous and the frequency ω is a monotonously growing function.***

***If all four Kharitonov polynomials are stable within a specific range of the systems function, then the system is*** **robust and stable** ***within this range*** (for example: within the frequency transfer band).

The problem of designing a controller which accommodates a robust control process, within a desired band of its dynamics, requires the design of such a controller (if feasible) which will, for given (desired) perturbation coefficients of the system's transfer, ensures a stable and quality course of the control process. In solving such problems we use knowledge from the areas of stability, quality and optimal control, as well as the properties of the algebraic polynomials. A more detailed solution of these problems are presented at, for example: www.polyx.com/robust.

**Multiplicative uncertainty**

Intervals of uncertainty, displayed in the intervals of changing coefficients of the transfer, is not possible to describe if we do not have a model of the system's behavior, i.e. in the form of the frequency characteristic. It is then advantageous to display the perturbation of the characteristic i.e. relative change in amplitude of the frequency characteristic within the frequency band of the system's operation.

We describe the frequency transfer of the perturbed system S(jω) depending on the frequency ω

$$S(j\omega) = S_0(j\omega)\left[1 + \delta(j\omega)\right] \qquad [7.12]$$

where $G_0(j\omega)$ is the frequency transfer of the original system and δ(jω) is its relative change

$$\delta(j\omega) = \frac{S(j\omega) - S_0(j\omega)}{S_0(j\omega)} \qquad [7.13]$$

If, for example, we assume the relative change in amplitude $\delta(j\omega)=0.5$ for $G_0(j\omega)=\dfrac{2.5}{(j\omega+1)^3}$ then the area bound by circles for individual values of frequency **ω** displayed within the **GCP** creates the pattern in **fig. 7.2 [Polyx]** in which lay the frequency characteristics of the perturbed system.

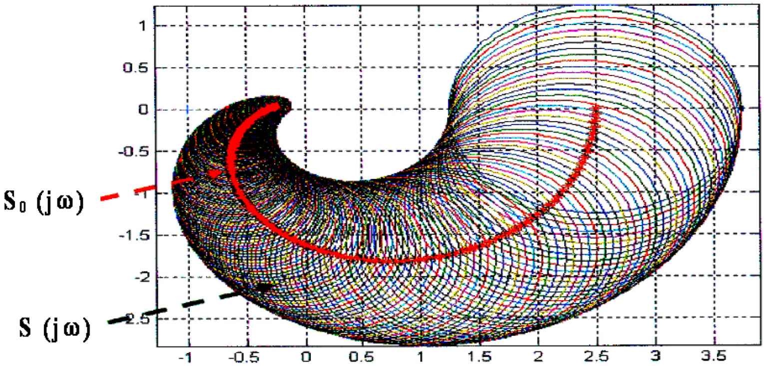



*Fig. 7.2 Region of the frequency characteristic in the GCP*

In the analysis and synthesis of robust control, different types of descriptions for the unstructured uncertainty are used. Yet they always point to the solution of ***designing control circuits such, that for the widest band of parameter perturbation, to ensure the course of the control process within allowable tolerances***. The design of such control requires knowledge from control theory, mainly in the field of stability, quality, optimization, filtering, algorithms, sensitivity and robustness of continuous and discrete control systems. The design of a robust control cannot be achieved without the intensive use of computers and appropriate software such as **MATLAB, Mathcad SIMULINK** etc…

## 7.5 Robust controller with a simple system model

Based on the assumption that if, for the given control circuit, a change in parameters of the system occurs then its dynamic characteristics also change and this results in a change in its behavior (feedback).

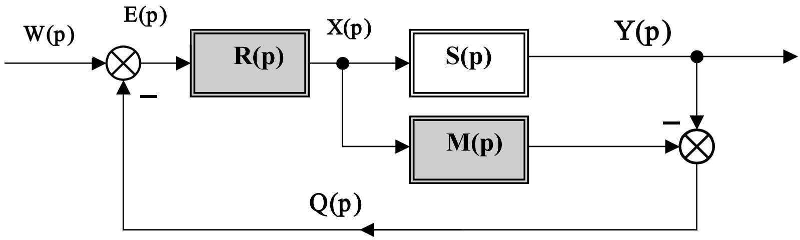



***Fig. 7.3*** *Control circuit with the model of the system*

To improve the control we offer the solution where the controller has information about the change in transfer of the system against the nominal state. This idea brings about the proposal of a ***controller with a nominal model of the system***.

The structure of the control circuit will be composed from the control system $\boldsymbol{S(p)}$, the nominal model $\boldsymbol{M(p)=S_0(p)}$ and controller $\boldsymbol{R(p)}$ (**fig. 7.3**)

The structure of the control circuit is expressed by the model $\boldsymbol{M(p)=S_0(p)}$ of the nominal transfer of the controlled system (part of the control circuit) at the original values of its parameters.

With changes in parameters of the system the difference between the input of the perturbation system $\boldsymbol{S(p)}$ and model $\boldsymbol{M(p)}$ is subtracted from the control signal $\boldsymbol{W}$. the feedback signal **Q** for the control transfer will be zero if $\boldsymbol{S(p)=S_0(p)}$**.** The controller then reacts to differences between the output of the perturbed system and its model. The following relations then apply:

$$Y(p) = S(p)X(p)$$
$$Q(p) = [S(p) - M(p)]X(p$$
$$Y(p) = \frac{S(p)R(p)}{1 + R(p)[S(p) - M(p)]}W(p)$$

The control transfer of the control circuit for the perturbed system $S(p)$ will be

$$G^M(p) = \frac{Y(p)}{W(p)} = \frac{S(p)R(p)}{1 + R(p)[S(p) - M(p)]} \qquad [7.14]$$

Transfer of the perturbed system with simple controller (without model) is

$$G(p) = \frac{S(p)R(p)}{1 + S(p)R(p)} \qquad [7.15]$$

At ***nominal transfer*** of the system $\boldsymbol{S_0(p) = M(p)}$ the ***control transfer of the control circuit with a model*** will be

$$G_0^M(p) = \frac{Y(p)}{W(p)} = S_0(p)R(p)$$

which represents the ***frequency transfer*** within the frequency domain

$$G_0^M(j\omega) = \{G_0^M(p)\} \mapsto p = j\omega = S_0(j\omega)R(j\omega) \qquad [7.16]$$

We will now consider the uncertainty of the perturbed ***system with a multiplicative model*** according to [7.12] and in the form

$$S(j\omega) = \{S_0(p)[1 + \delta(p)]\}, p \mapsto j\omega$$

where $\boldsymbol{S_0}$**(p)** is the transfer of the nominal system and **δ(p)** is the relative change in transfer of the perturbed system against the transfer of the nominal system. The task of the controller using this model is ensure, with sufficient quality, ***meeting the requirements for control, that is Y(jω)→W(jω),*** within the required band of desired function.

Frequency ***transfer of the controller*** within the control circuit for a system with multiplicative uncertainty, subbing [**7.12**] into [**7.14**] obtains

$$G_\delta^M(j\omega) = \frac{Y(j\omega)}{W(j\omega)} = \frac{S_0(j\omega)R(j\omega)[1 + \delta(j\omega)]}{1 + R(j\omega)S_0(j\omega)\delta(j\omega)} \qquad [7.17]$$

Let us analyze some important properties of the control circuit with a model.

**<u>Sensitivity of transfer</u>**

The absolute transfer sensitivity of the control circuit with a model on the change in transfer ***S*(jω)** of the perturbed system will be

$$C_S^M(j\omega) = \frac{\partial G^M(j\omega)}{\partial S(j\omega)} = \frac{R(j\omega)[1 - R(j\omega)M(j\omega)]}{\{1 + R(j\omega)[S(j\omega) - M(j\omega)]\}^2} \qquad [7.18]$$

In terms of expression [**7.18**] it turns out that when the condition

$$R(j\omega)M(j\omega) = R(j\omega)S_0(j\omega) = 1 \qquad [7.18*]$$

is met then the ***control sensitivity*** on the change in transfer of the system will be ***zero***.

***However, this condition is difficult to meet because, for the feasible system, often times the controller is unfeasible*** (and vice versa). Although it is possible to request its ***compliance to a steady state***, which leads to the requirement, that the sum of the amplified controller and model must equal **1**. ***From this we can determine the required gain of the controller***.

For the ***perturbed system S(p)*** we can express the ***transfer sensitivity of the control circuit on the transfer of the controller*** by relation

$$C_\delta^M(j\omega) = \frac{S(j\omega)}{\{1 + R(j\omega)[S(j\omega - M(j\omega)]\}^2} \qquad [7.19]$$

For the ***multiplicative perturbed model*** [7.12] the **control sensitivity** will be

$$C_\delta^M(j\omega) = \frac{S_0(j\omega)[1 + \delta(j\omega]}{[1 + R(j\omega)S_0(j\omega)\delta(j\omega)]^2}$$

***From this relation, and for known situations (the original system, perturbation) we can assess the choice of the controller when respecting the conditions for stability.***

**<u>Control error</u>**

The image of the ***real error*** **e(t)** (difference between the real and desired value of the control circuit with system model output) will be

$$E(p) = Y(p) - W(p) = \frac{R(p)M(p) - 1}{1 + R(p)[S(p) - M(p)]} W(p)$$

Or $$E(j\omega) = Y(j\omega) - W(j\omega) = \frac{R(j\omega)M(j\omega) - 1}{1 + R(j\omega)\left[S(j\omega) - M(j\omega)\right]} W(j\omega) \qquad [7.20]$$

for the system with multiplicative model, the control error will be

$$E_\delta(p) = -\frac{R(p)M(p) - 1}{1 + R(j\omega)M(j\omega)\delta(j\omega)} W(p) \qquad [7.21]$$

and the steady value of the control error

$$E_u = \lim E_\delta(p)_{p=0} = \lim_{p=0} \left\{ \frac{R(p)M(p) - 1}{1 + M(p)R(p)\delta(p)} W(p) \right\} \qquad [7.22]$$

From relation [**7.22**] it follows that choosing such a controller where **R(p).M(p) = 1**, the value of the steady control error will then be zero.

**Stability**

The ***control stability*** is the most important property of the control process and a necessary condition for its implementation. Because system's with parametric uncertainty the dynamic properties of the system in a certain interval change, it is necessary to study the conditions for control stability in the region of these changes. ***The general criterion for stability, whether it be algebraic of frequency (Ljapunov criteria) for control process are true for the instantaneous transfer of the perturbed system***. But it is necessary to study the conditions to fulfill these criteria for the considered range of perturbed parameters, or in other words; characteristic of the system for a concrete control process.

For known values in the interval of change of the coefficients transfer function (that is, the differential equation) of the perturbed system, the analysis of the algebraic criteria was derived by **Kharitonov** and is known as the ***algebraic condition for stability*** **(4)**. From this we can verify the stability of the system as well as the control process by the stability of the four ***Kharitonov polynomials*** **[7.11].**

The analysis of the stability for a system with multiplicative uncertainty according to [**7.12**] will be described in more detail

From relation [**7.17**] and according to the **Nyquist** frequency criteria, for the transition of the frequency characteristic of an open control circuit with perturbed system with negative real axis in the **GCP**, the following condition must be met

$$\left|R(j\omega)S_0(j\omega).\left[1+\delta(j\omega)\right]\right| < 1 \qquad [7.23]$$

If we indicate $\boldsymbol{S_0(j\omega).R(j\omega)=F_0(j\omega)}$ which is the transfer of the open control loop for the original system, then the control process will be stable with the transition of the frequency characteristic $\boldsymbol{F_0(j\omega)}$ in the axis of the GCP (Gauss Complex Plane) then (**fig. 7.4)** will apply

$$\mathrm{Re}\left\{F_0(j\omega\left[1+\delta(j\omega\right]\right\} > -1 \qquad [7.24]$$

for all ω from the transmission band of the control system (see **fig. 7.4**)

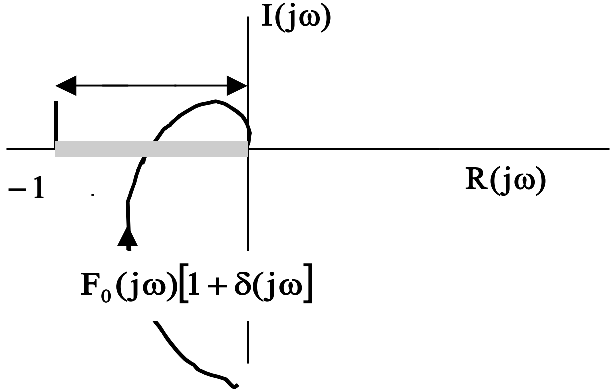



*Fig. 7.4 Control stability for the perturbed system*

**Illustration:**

For the stable system of the third order with transfer function

$$S(p) = \frac{2p+6}{p^3+6p^2+11p+6} \qquad [7.24\text{-P}]$$

and multiplicative uncertainty according to **[7.12]** with a value of **|δ(*j*ω)|=0.5** are shown in **fig. 7.5** illustrating its frequency characteristic while **fig. 7.6** illustrates the waveform of its bode characteristic. **Fig. 7.7** illustrated the waveform of the controller on the required value ***W*=1** of the same perturbed system using a classic statically proportional controller with gain ***K*=1** and **fig. 7.8** shows the waveform of the controller with an internal model ***M***.

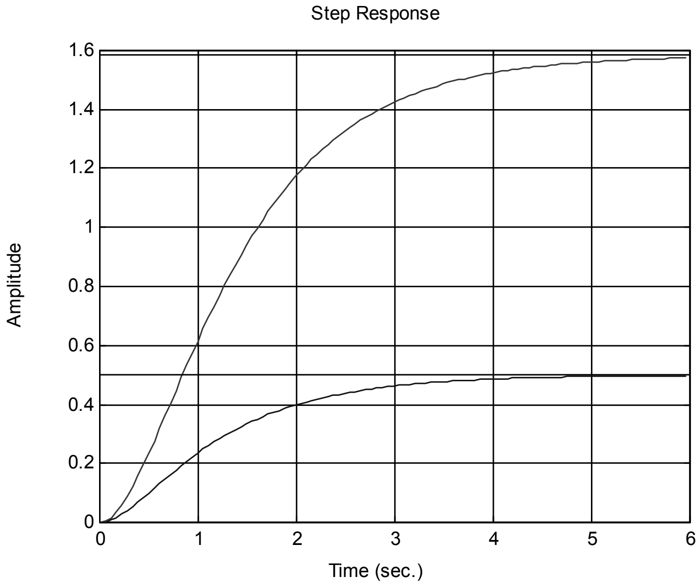



***Fig. 7.5*** *Transfer characteristic of the perturbed system*

From **fig. 7.5** is can be seen that the perturbation of the system manifests itself at **60%** of the steady value of its transfer characteristic. For the control of a classical controller, the relative change of the control error's steady value against the nominal shows at about **20%**.

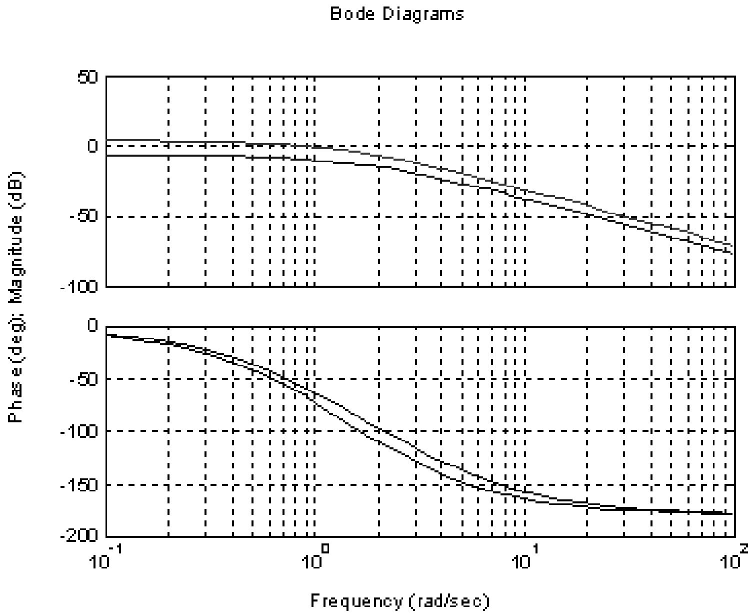



***Fig. 7.6*** *Bode characteristic of the perturbed system*

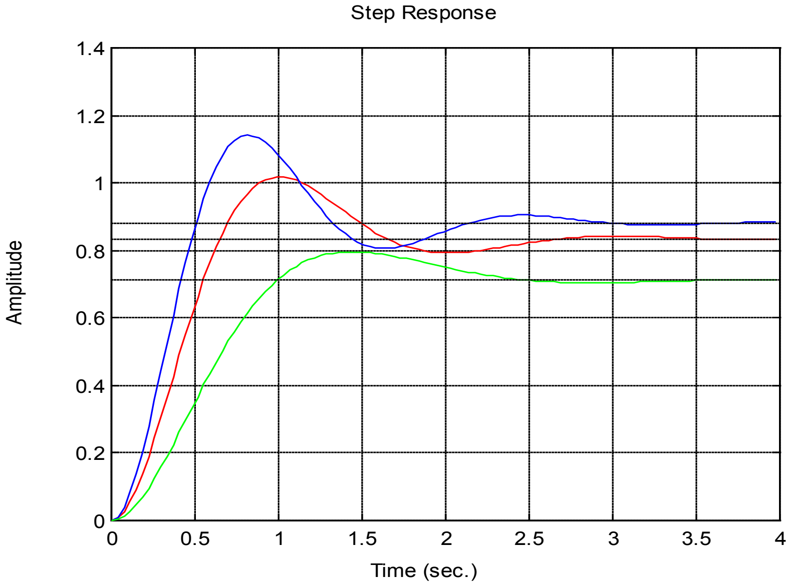



***Fig. 7.7a*** *Control characteristics for classical control*

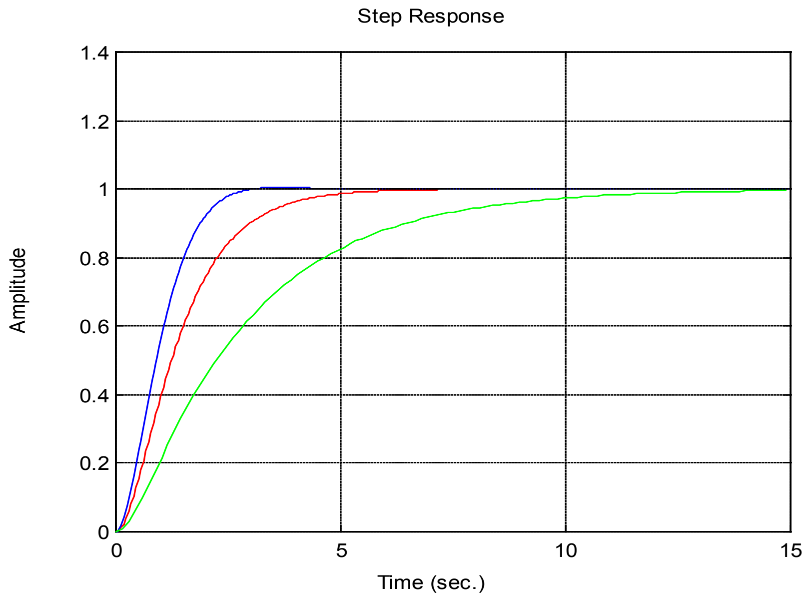



***Fig. 7.7b*** *Control characteristic for control with a model*

In terms of the waveform of the control characteristic shown on fig. **7.7a** (classic controller) and on fig. **7.7b** (controller with internal model), the ***significant change in control quality can be seen, mainly in the decrease in relative change of the continuous control error***.

### 7.6 Control circuit with model and additional controller

Another type of control circuit which decreases the sensitivity of changes in the perturbed system against the classical controller is the control circuit with a model and additional controller, whose structure can be seen in **fig.**

**7.8**. The additional controller **H** can change the properties of the control system **S** that even with the perturbation of the system , the transfer of the bounded region (dashed line) of the control circuit follows as closely as possible to the desired original transfer of the system $\mathbf{S_0}$.

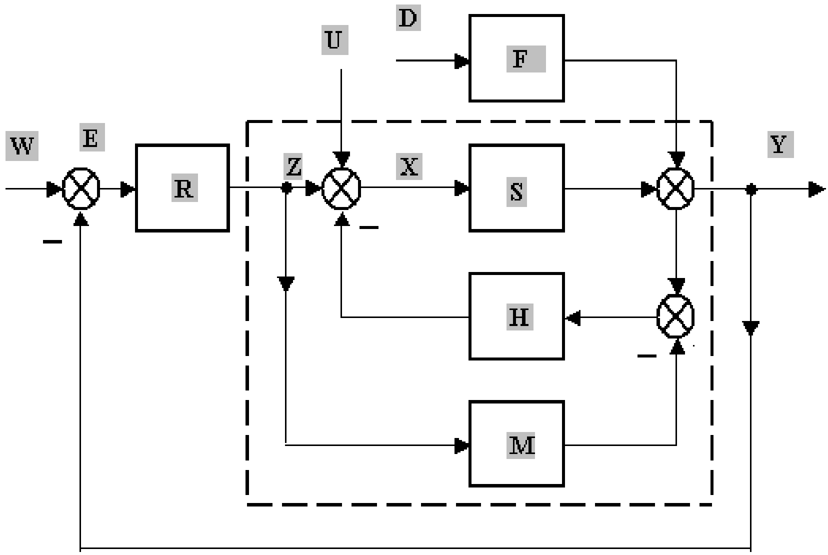



***Fig. 7.8*** *Control circuit with an additional controller*

For the control transfer we can derive a relation from the analysis of the circuit structure

$$G_w(p) = \frac{Y(p)}{W(p)} = \frac{S(p)R(p)[1 + H(p)M(p)]}{1 + S(p)[R(p) + H(p) + M(p)R(p)H(p)]} \quad [7.25]$$

In terms of the structure of the transfer function we can assess and solve a range of problems with the selection of individual circuit elements such, that for the desired waveform of the perturbation parameters (characteristic) system reaches the best possible waveform of the control process.

**Sensitivity of the circuit:**

In terms of our reason for monitoring the properties of the circuit it is important to determine (derive relations) for the absolute ***sensitivity of the circuit on the change in transfer of the control system S*(p)**

$$C_s(p) = \frac{R(p)[1 + M(p)H(p)]}{\{1 + S(p)[H(p) + R(p) + M(p)H(p)R(p)]\}^2} \qquad [7.26]$$

The relation can be used in the design of the control circuit and in choosing a controller **R** and **H** in the expectation to achieve the smallest amount of sensitivity (greatest robustness) for the allowable change in transfer of the system and maintaining the quality control process.

**Requirements for controller H:**

If the control circuit (**fig. 7.8**) is to insure robust control, then it should also partially compensate for change in the transfer of the system ***S***, which brings us to the requirement that the transfer ***Z*(*p*)** on ***Y*(*p*)** be as close as possible to the model of the original system ***M*(*p*)=*S*$_0$(*p*)**, so

$$G_{y,z}(p) = \frac{S(p)[1 + H(p)M(p)]}{1 + S(p)H(p)} S_0 \rightarrow S_0(p) \qquad [7.27]$$

After substituting for ***S*(*p*)** from **[7.12]** for the selection of the controller's frequency transfer ***H*(*j*ω)** results in the requirement

$$H(j\omega) \to \frac{\delta(j\omega)}{\left[1+\delta(j\omega)\right]\cdot\left[S_0(j\omega)-M(j\omega)\right]} \qquad [7.28]$$

Equation [**7.28**] results in a paradoxical (but real) requirement, to ensure condition [**7.27**], small differences between the nominal transfer of the system ***S*₀(*j*ω)** and the model ***M*(*j*ω)** requires a larger gain from the control ***H*(*j*ω)** such that the condition for process stability is conserved. Larger gain values of the controller ***H*** (permissible in terms of stability) also causes a decrease in steady value of the control error.

**<u>Process Stability:</u>**

From the transfer of the control circuit [7.25] and considering the Nyquist criteria for stability results in the condition that

$$Re\{S(j\omega)[R(j\omega)+H(j\omega)+M(j\omega)R(j\omega)H(j\omega)]\}\rangle -1 \qquad [7.29]$$

throughout the complete activity within the frequency band ω. We can then use relation [7.29] as a criterium for stability for the control process with a robust controller illustrated in the schematic (**fig. 7.8).**

***<u>Illustration:</u>***

The waveform of the control characteristic of a perturbed system with transfer according to [**7.24-P**], the model ***M*=*S*₀(*p*)**, controller gain

***R*(*p*)=*K*=3**, ***H*=5** and perturbation **δ(*p*)=0.4**, ***for a controller with structure according to*** **fig. 7.8** is illustrated on fig 7.9 with the use of **MATLAB**.

From the waveform of the control characteristic, it is ***possible to interpret*** that the relative change of controlled variables for the significant relative change (perturbation) of the amplified system **(80%)**, is only **10%**.

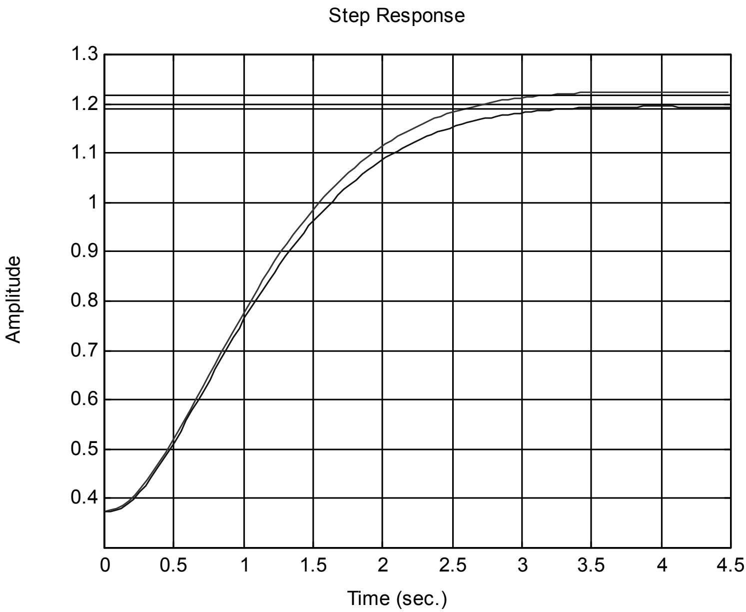



***Fig. 7.9*** *Control characteristic of the control circuit in fig. 7.8*

## 7.7 Modification of the internal model of the control system

The purpose of the controller with internal model is to monitor the change in transfer of the controlled system against the nominal model, and from its instantaneous values, create an additional control signal. We will present the controller with internal model in fig. 7.10.

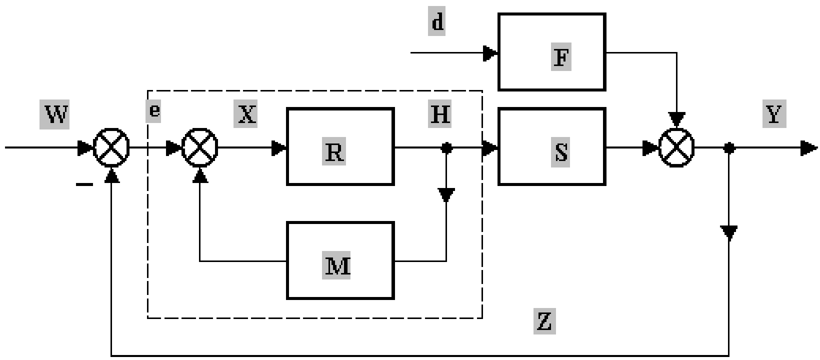



*Fig. 7.10 Controller with internal model*

**Meaning of the symbols in the scheme above:**

**R – controller**
**S – controlled object (system)**
**M – model of the nominal system**
**F – fault transfer**
**Z - feedback**

**W – requested output value**
**e – control error**
**X – control signal**
**d – fault signal**
**Y – output variable**

A robust controller is created by a controller R, bridged by the positive feedback by means of the nominal system model **M.** ***Input of the controller is created by the difference between the input of the system's nominal model and the difference between the desired and actual value of the control system's output.***

For the input of the image *Y*(*p*) depending on the requested value *W*(*p*) and error variable *D*(*p*) we can derive the relation

$$Y(p)=\frac{S(p)R(p)}{1+R(p)[S(p)-M(p)]}.W(p)\frac{1-M(p)R(p)}{1+R(p)[S(p)-M(p)]}.D(p) \quad [7.30]$$

If we neglect the effects of the error variable (input noise), then for the determination of important properties of the controller for the perturbed system, it is enough to analyze just the ***transfer* G(p) *of the desired value* W(p) *of input (controlled) variables* Y(p)**

$$G(p)=\frac{S(p)R(p)}{1+R(p)[S(p)-M(p)]} \quad [7.31]$$

Comparing the transfers of **7.31** and **7.14** it can be seen that the controller on **fig. 7.3** and **7.10** are, in terms of control, ***identical*** even though their schematics represent different structures. Both schematics ***represent a robust internal model control***.

If we consider the model of the perturbed system in the form of [**7.12**], and the transfer of this model to be equal to the nominal transfer of the system (${\boldsymbol{S_0}}$=***M***) then the frequency transfer of the internal model control circuit ***M*** will be

$$G(j\omega) = \frac{S_0(j\omega)R(j\omega)[1+\partial(j\omega)]}{1+R(j\omega)S_0(j\omega)\partial(j\omega)} \qquad [7.32]$$

**Process stability**

For the ***stable course*** of the control process with the transfer function (7.31) the following conditions ***must*** be met

$$1 + R(j\omega)M(j\omega)\delta(j\omega) > 0 \qquad [7.33a]$$

That is $Re[R(j\omega)S_0(j\omega)\delta(j\omega)]\rangle -1$ [7.33b]

***Considering the*** **Nyquist criteria for stability**, ***with increasing ω in the transmission of the frequency characteristics*** $\boldsymbol{N(j\omega)=R(j\omega)S_0(j\omega)\delta(j\omega)}$ ***in the negative real axis of the gauss plane, then the point*** **(-1+*j*0)** ***must lay on the left side of*** **fig. [7.11].**

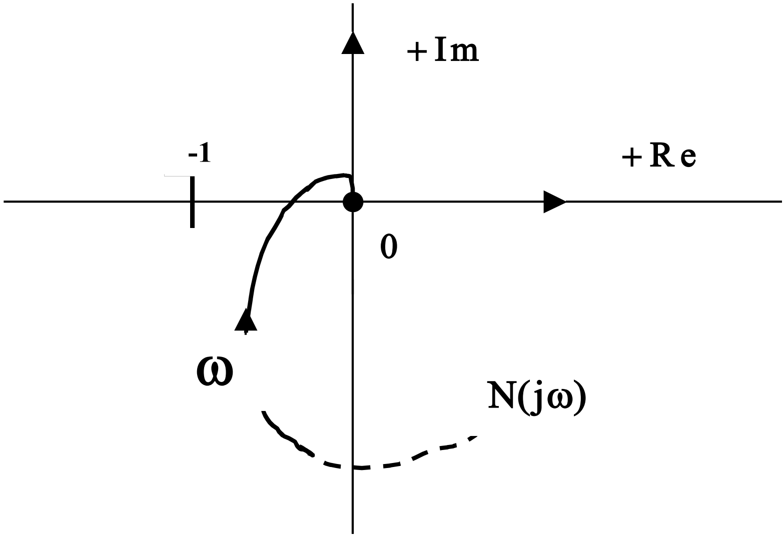



***Fig. 7.11*** *Nyquist characteristic of the stable control circuit*

According to relation [**7.33**], for known transfer of the original system **$S_0$(jω)** and model of perturbation **δ(*j*ω)** can control the transfer of the controller **R(jω)**.

**<u>Control error</u>**

The image of the actual values of the control error in the control of perturbed systems, controllers given by **fig. 7.10** are described by the difference between the output image and desired values of the control variables, ***E*(*p*) = *Z*(*p*) - *W*(*p*)**. After substitution into [**7.29**] and neglecting external noise, we assuming the perturbation with respect to (7.12] and selecting the model $M(p)=S_0(p)$ we get

$$E(p)=\frac{S_0(p)R(p)-1}{1+S_0(p)R(p)\delta(p)}W(p) \qquad [7.34]$$

***The steady value of the control error*** with respect to [**7.34**] will be

$$\lim_{t\to\infty}\varepsilon(t)=\lim_{t\to\infty}\frac{S_0(p)R(p)-1}{1+S_0(p)R(p)\delta(p)}W(p)=\varepsilon_u \qquad [7.35]$$

From the conditions for ***zero control error*** and from the transfer [**7.34**]

$$R(j\omega)S_0(j\omega)\to 1 \qquad [7.36]$$

Relation [**7.36**] can be ***useful for choosing the controller*** which must ***control the stable*** process with respect to [**7.33**].

***We will illustrate the function of this modification to the structure of the internal model controller on a specific case of the control process.***

**Illustration of the control process:**

The waveform of the classical control circuit response (control characteristic) for the system with original transfer

$$S_0(p)=\frac{p+6}{p^3+6p^2+11p+6}$$

amplified by controller R=1, and gain of the system's perturbation $\delta=\pm 0.4$, is illustrated in **fig. 7.1**

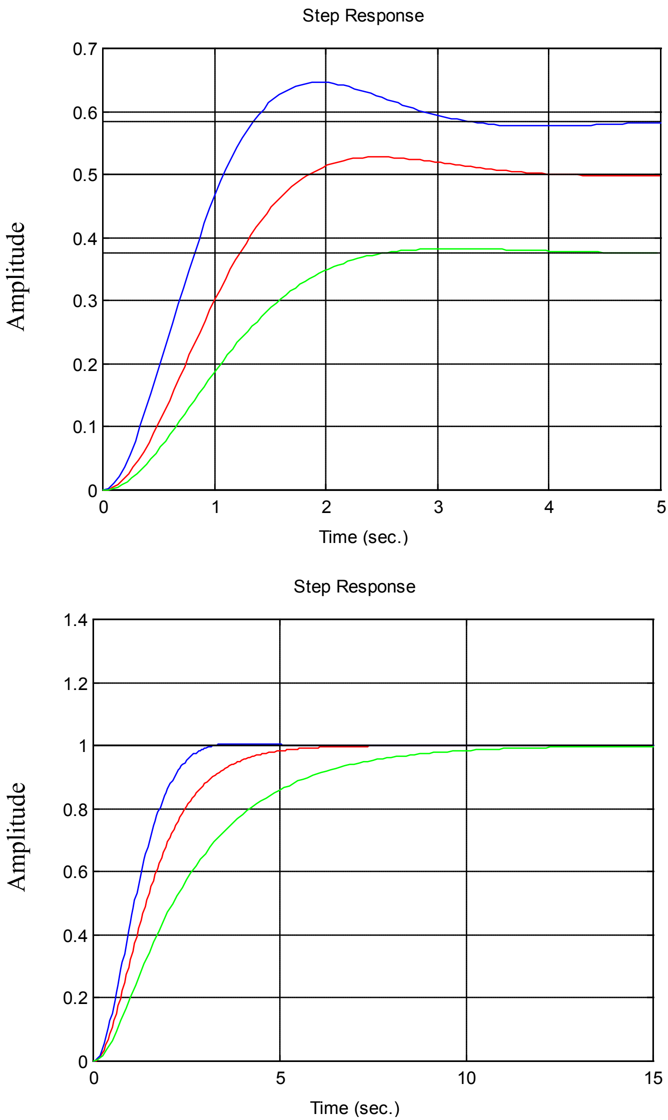



***Fig. 7.12*** *Control process in the limits of perturbation for the classical controller*

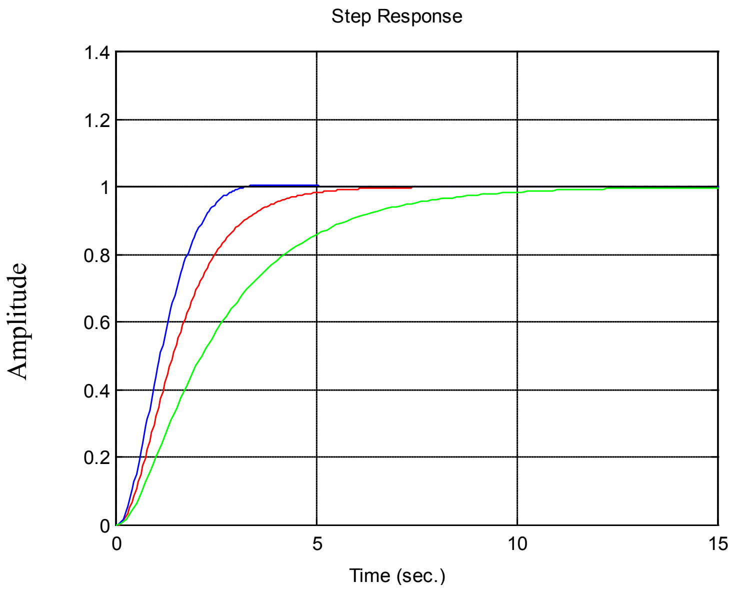



***Fig. 7.13** Control characteristics of the IMC controller*

In **fig. 7.13** is the waveform of the control of the same **IMC *controller*** for optimally selected controller parameters and model ***R*(*p*) = 1, *M*(*p*) = 1.** ***(steady state control error is zero!)***

From the analysis performed in **section 7.5 and 7.6**, as well as the comparison of schematics on **fig. 7.3 and 7.10** shows that the structure of the control circuit in terms of its function (transfer properties) are equivalent.

## 7.8 Problems of Robust control

The time course of physical or technological processes within real dynamic systems and environments is influenced by a range of legitimate and random effects occurring within the elements themselves. That is to say, within the mutual connections between system elements as well as effects in the close and distant environment, as well as the legitimate change in element properties as a result of aging, wear, overloading etc… In doing so we usually require that the course of events and processes when these effects are in action remain within the requested limits (the product of the technological process shows properties within the required tolerance parameters. The desired course of the process is ensured mainly in automatic control in simpler cases and automatic control of feedback control systems (controllers). If we succeed in directing the course of the control system process, even with small (not very large) changes in the system structure's parameters or environment, we are referring to the ***robust control of the system parameters or process***. It is evident that achieving robust control depends primarily on the robustness of the control system itself as well as the control process. Only after this does it depend on the ability of the robust controller (control system). It is also evident that the concept, or ***definition of robustness*** is determined by the behavioral properties of the control system, characterizing its functionality, process quality or usefulness. In the control of the dynamic system's process, we typically try to achieve, or maintain, the required (allowable) course of the dynamic characteristics, which show the "quality" of the controlled process. ***Robustness of the control systems behavior*** is one of the basic requirements for the design of systems with automatic control. The ***design of an automatically controlled system*** is based upon knowledge ***(analysis) of the structure***, static and dynamic properties (characteristics and their tolerance) of the controlled system, continues with the analysis of the internal and external ***environment*** in terms of their effects on the change in parameters of the system, studies the ***tolerance region of the control process of the control characteristic***, for which the process meets the specified requirements on its course. The next phase in the design is to propose a suitable ***structure for the controller and it parameters*** such, that it is capable of ensuring the course of the process even with perturbation of the parameters of the system, as well as perturbation of the controller's parameters themselves. The final phase of the design should be a reliable ***simulation of the process*** in the proposed system (***correction*** of the structure and parameters of the controller),

Even in the aforementioned procedures we can criticize a range of problems in the design of robust control. Situations may arise that for a given system it is not possible, with known structure of the controller, to ***achieve the requirements*** for the course of the dynamic process. In this case it is necessary to carry out either a revision of the given requirements, or (more frequently) proceed with the reconstruction of the controlled system.

Within the design itself of robust control there are many influences whose effects are not describable by definite values, in other words the value of the process variables can change within certain (allowable) limits. In the control of such processes it is preferable ***to use fuzzy controllers***, which are oftentimes much easier to implement as opposed to a controller with concrete characteristics.

The development of theory and application in the areas of robust control represents a new philosophy for control, which is an inevitable continuation of previous approaches. We can remember situations where, for the implementation of proper machine operation, waveform of the process and even the correct computer operation (relay and electronic tube structure) use a strict set of "***operational conditions***" which must be met. Robustness of the system within a limited area of parameter changes of the internal and external environment must ***guarantee relatively independent behavior of the system from these changes.***

## 8. CONCLUSIONS

The analysis of a dynamic system's properties, in its own definitions of "***sensitivity***", "***tolerance***" and "***robustness***" evokes the comparison with identical concepts used to describe the properties of organisms (generally), as well as the properties which characterize the behavior of a human being. It is interesting to follow these interdependencies of properties defined for mechanisms and organisms alike (understood in cybernetics). If we abstract from specifications that include these concepts in organisms and humans (i.e. change in structure and properties of organisms over their life, fertility, metabolism, homeostasis, natural intelligence and a range of properties effecting a living organism, and are dependent on it), then we can find common general rules applicable to mechanisms and organisms which must be known, and with expected (desired) behavior (its guidance and control) to respect and benefit from.

Amongst these properties we can include, i.e. the ***principle of equilibrium state,*** which the mechanism or organism is trying to achieve for the given environment and conditions for its activity. In this, the laws of equilibrium and conservation are achieved (by the simple laws of mechanics, conservation laws throughout regions in the matter of the elementary structure, all the way to the conservation laws in the universe).Disturbing the equilibrium state, whether statically but mainly dynamically, leads to the sudden change in behavior of the system, in nonlinear systems to the unclear change in state, bifurcation (***without process control***) all the way to the chaotic state. Awareness of these facts in terms of the properties and possible behavior of living or social systems and complexes, considering the negative effects of the internal and external structure (environmental and existence), the environment requires ***not only to accept these laws but also to respect and use them for the control of social systems.***

***In this respect, the study of mutual dependencies of mechanical and organic properties characterized by concepts of sensitivity, tolerance and robustness can be of benefit in the areas of controlling living and social systems.***

***Sensitivity and tolerance***, in terms of behavior of living systems, is given by its genetic structure and development within the existing environment and can be affected by the healthy development of individuals and creation of a suitable existential condition. The area of ***control has generally valid principles, especially in social systems***. However, in many cases they are not respected or even deliberately ignored (i.e. environmental disasters, inhumane living conditions, etc…). In these cases, the behavior of the control structure can be compared to the ***behavior of an insensitive and intolerant robot and while acting "robustly" it is not within the interests of preservation*** (increase in quality of existing living conditions for the society as a whole).

One of the basic control principles in the area of control processes within systems and complexes, is to consider ***principle of necessary variety*** (Ashby W. R.) which says: to cope with the control of the system (limiting its variety) the available variety of control actions must be equal or greater than the variety of possible undesirable states of the controlled system. Ignoring this generally applicable principle, either in the ***creation of the***

***control structure or in its operation***, will always lead to faults within the control system. In order for the controlled system to function or “viable” even for the time bound imperfect control, it must also hold an ***acceptable dose of tolerance and robustness***.

**Sensitivity analysis**, which should result in the determination of the most sensitive system elements, allows for the more advanced knowledge on the behavior when exposed to the external, but mainly internal, environment. When designing the system, such an analysis will uncover the weak and critical areas whose negative effects on the behavior of the system is possible to reconstruct (to weaken system reorganization or eliminate altogether, this way we can improve its robustness). Such an analysis should be applied more intensively also in the control process of organisms (area of medicine) and also in the control of social systems.

**Tolerance** of the non-living dynamic system depends on the ability of its elements (material and surroundings) to retain their parameters within a definite range of change of the external and internal environmental properties. As far as we consider the effects of the property changes in the external environment, these changes reflect on the change in behavior of the system (response). If this change in behavior, for non-negligible and bounded fluctuation of the external environments properties, is relatively small, then we are referring to the ***tolerance of the system with respect to its surroundings***. If the change in the systems behavior for significant changes in parameter values of the system’s elements and constraints results in only a slight change in behavior, we are referring to a ***robust system***. It is evident that these characteristics for non-living systems actually convene with similar characteristics of the organism and also of the social system. Even if in living systems we use not only the physical meaning of these concepts and relations, but we must also consider the effects of the emotional, spiritual, intellectual and control properties and abilities (homeostasis, adaptation). Significant findings which say, that ***every system can have tolerant behavior within a bound range of negative effects, can also have general validity.***

In the study of general dynamic systems, it is necessary to examine the individual properties and their mutual dependencies, in respect to their compliance with the requirements for their determination and behavior within the internal environment and existential surroundings. It is therefore necessary to apply a systematic approach in a narrower or wider scale. While in a closed physical (non-living) system, the law of ***entropy growth*** applies and the system is directed to the "*numb*" equilibrium state, organisms also within social systems as a result of energetic and informational interactions with the surrounding environment, results in the existence of a definite degree of awareness and the resulting guidance (control) of its own behavior, the effect of this law are applied significantly less. In living and social systems, in the perspective of different goals for the control of one's own life, result in the concepts of sensitivity, tolerance and robustness as well as other strange dimensions (i.e. individualism, desire for power etc…) for which the laws of statistical dynamics, as well as the laws of preservation and other laws, disagree and do not apply. ***In any case it is necessary to consider and respect the synergistic effect and the consequences of violating the laws of nature, not only in mechanisms and***

***organisms, but also in dynamic development (within the very existence of human society).***

When considering relations between the properties of mechanisms, problems amongst organisms and people also emerge, such as formation, realization, the use and "***cooperation of robots***" as "***helpers***" to substitute human actions where the properties of the dynamic systems, which this publication deals with, results in particular importance. The concepts of ***sensitivity, robustness and tolerance*** transfer to the area of neural networks in solving problems of artificial intelligence. ***Therefore, our contribution to processing these problems can be considered useful and even inspirational for a wide variety of readers and students.***

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