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A k-Server Problem with Parallel Requests and Unit Distances

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1 Introduction

A problem in industry, which contains an optimal conversion of machines or moulds (see [3] or [4]), supplied the origin of investigations of the "Stochastic Dynamic Distance Optimal Partitioning (SDDP) problem" (see [6]). Superordinately regarded, SDDP problems are stochastic dynamic programming problems. If we disregard the given probability distributions for SDDP problems k-server problems with parallel requests where several servers can also be located on one point are present. We will distinguish the surplus-situation where the request can be completely fulfilled by means of the k servers and and the scarcity-situation where the request cannot be completely met.

Bartal/Grove showed that the "Harmonic algorithm" is "competitive" for the (usual) k-server problem where at most one server is moved in one step (see [1]). We use the method of the potential function by Bartal/Grove in order to prove that a corresponding Harmonic algorithm is competitive for the more general k-server problem in the case of unit distances. For this we partition the set of points in relation to the online and offline server positions. (The proof in the case of general distances is the aim of further investigations.)

2 The Formulation of the Model

[1] Let $k \underset{(=)}{>} 1$ be an integer, and $M = (M, d)$ be a finite metric space where M is a set of points with $|M| = N$. An algorithm controls k mobiles servers, which are located on points of M. Several servers can be located on one point. The algorithm is presented with a sequence $\sigma = r^1, r^2, \cdots, r^n$ of requests where a request r is defined as an N-ary vector of integers with $r_i \in \{0, 1, \cdots, k\}$. The request means that r_i server are needed on point i $(i = 1, 2, \cdots, N)$. We say a request r is served if $\left\{ \begin{array}{c} \text{at least} \\ \text{at most} \end{array} \right\}$ r_i servers lie on i $(i = 1, 2, \cdots, N)$ in case $\left\{ \begin{array}{c} C[r,k] \\ C[k,r] \end{array} \right\}$. $C[r, k]$ denotes the case

[1]For basic knowledge of (usual) k-server problems see also [2], chapters 10 and 11 for example.

$\sum_{i=1}^{N} r_i \leq k$ (surplus-situation, the request can be completely fulfilled) and $C[k,r]$ denotes the case $\sum_{i=1}^{N} r_i \geq k$ (scarcity-situation, the request cannot be completely met, however they should be met as much as possible). By moving servers, the algorithm must serve the requests $r^1, r^2, \cdots, r^n$ sequentially. For any request sequence σ and any generalized k-server algorithm $ALG_{p(arallel)}$, $ALG_p(\sigma)$ is defined as the total distance (measured by the metric d) moved by the ALG_p's servers in servicing σ.

In this paper we will show that the (generalized) harmonic k-server algorithm attains a competitive ratio of $k(2^{(R(k)-1)}+1)$ (see Theorem 3.1) against an adaptive online adversary in the case of unit distances (for the definitions of competitive ratio and adaptive online adversary see [1] or [2], sections 4.1 and 7.1).

Analogous to [2], p. 152 working with lazy algorithms ALG_p is sufficient. For that reason we define the set of feasible servers positions with respect to s and r in the following way

$$\hat{A}_{N;k}(s,r) = \left\{ s' \in S_{N;k} \;\middle|\; \begin{array}{l} r_i \leq s'_i \leq max\{s_i, r_i\},\ i = 1, \cdots, N,\ \text{in } C[r,k] \\ \\ min\{s_i, r_i\} \leq s'_i \leq r_i,\ i = 1, \cdots, n,\ \text{in } C[k,r] \end{array} \right\} \tag{1}$$

where

$$S_{N;k} := \left\{ s \in \mathbb{Z}_+^n \mid 0 \leq s_i \leq k \ (i = 1, \cdots, n),\ \sum_{i=1}^{N} s_i = k \right\} \tag{2}$$

.

The metric d implies that $S_{N;k} = (S_{N;k}, \hat{d})$ is also a finite metric space where $\hat{d}$ are the optimal values of the classical transportation problems with availabilities s and requirements s' from $S_{N;k}$:

$$\sum_{i=1}^{N} \sum_{j=N}^{N} d(i,j)\, x_{ij} \ \rightarrow \ min$$

subject to

$$\sum_{j=1}^{N} x_{ij} = s_i \ \forall i,\ \ \sum_{i=1}^{N} x_{ij} = s'_j \ \forall j,\ \ x \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$$

(see [5], Lemma 3.6).

The (generalized) $HARMONIC_p$ k-server algorithm operates as follows: Serve a (not completely covered) request r with randomly chosen servers so that for the (new) server positions $s' \in \hat{A}_{N;k}(s,r)$ is valid with respect to the previous server positions s and the request r. More precisely, $HARMONIC_p$ leads to $s' \in \hat{A}_{N;k}(s,r)$ with probability

$$\frac{\frac{1}{\hat{d}(s,s')}}{\sum\limits_{s'':s'' \in \hat{A}_{N;k}(s,r)} \frac{1}{\hat{d}(s,s'')}}. \tag{3}$$

3 The Competitiveness of $HARMONIC_p$ in case of Unit Distances

Theorem 3.1 *.The $HARMONIC_p$ k-server algorithm attains a competitive ratio of $k(2^{(R(k)-1)}+1)$ against an adaptive online adversary in case of unit distances if $\sum\limits_{i \in M} r_i^t \leq R(k)$ $(\forall t)$ for given $R(k) \underset{(=)}{>} k$.* [2]

Proof. We use the method of the potential function (see [1]) in order to prove the statement. In case of unit distances it is sufficient to use the following simple potential function

$$\Phi(s,s') := \hat{f} \sum_{i=1}^{N} \frac{1}{2}|s_i - s_i'| \; (= \hat{f}\; \hat{d}(s,s')), \;\; s,s' \in S_{N;k}. \tag{4}$$

.

At the beginning let $\hat{f} \geq 0$. We will solve for $\hat{f}$ later.

More precisely and analogous to Bartal/Grove, let Φ_t denote the value of Φ at the end of the t th step (corresponding to the t th request r^t in the request sequence) and let $\Phi_t^{\sim}$ denote the value of Φ after the first stage of the t th step (i.e., after the adversary's move and before the algorithm's move).

In cases $C[r,k]$ and $C[k,r]$ we will show the following properties (see [1], pages 4 and 5)

$$\Phi \geq 0. \tag{5}$$

$$\Phi_t^{\sim} - \Phi_{t-1} \leq C(k) D_t, \text{ where} \tag{6}$$

[2]This condition is important for case $C[k,r^t]$. (According to the above model $\sum\limits_{i \in M} r_i^t \leq k$ is true in case $C[r^t,k]$.)

D_t denotes the distance moved by the offline servers (controlled by the adversary) to serve the request in the t th step.

$$E(\Phi_t^{\sim} - \Phi_t) \geq E(Z_t), \text{ where} \tag{7}$$

Z_t represents the cost which incurred by the online algorithm to serve the request in the t th step.

(5) is straightforward if $\hat{f} \geq 0$.

In the following let

$\bar{s}$ ($\in S_{N;k}$) denote the (offline) servers position controlled by the adversary at the end of the t-1 th step (i.e., at the beginning of the t th step)

s ($\in S_{N;k}$) denote the (online) servers position controlled by the algorithm at the beginning of the t th step

s' ($\in \hat{A}_{N;k}(s, r^t)$) denote the (online) servers position at the end of the t th step and

$\bar{s}'$ ($\in S_{N;k}$) denote the (offline) servers position controlled by the adversary after the first stage of the t th step.

Then (6) follows by means of the triangle-equation of the metric $\hat{d}$:

$$\hat{f}\hat{d}(s, \bar{s}') - \hat{f}\hat{d}(s, \bar{s}) \leq \hat{f}\hat{d}(\bar{s}, \bar{s}') = \hat{f}D_t \text{ if } C(k) = \hat{f}.$$

Proof of (7) and determination of $\hat{f}$ in case $C[r^t, k]$:

In this case and for assumed unit distances $d(i,j) = 1, i \neq j$

$$\Phi_t^{\sim}(s, \bar{s}') = \hat{f} \sum_{i:\bar{s}'_i > s_i} (\bar{s}'_i - s_i) = \hat{f} \sum_{i:\bar{s}'_i < s_i} (s_i - \bar{s}'_i) \;\; \forall s' \in \hat{A}_{N;k}(s, r^t) \tag{8}$$

and

$$Z_t(s, s') = \sum_{i:r_i^t > s_i} (r_i^t - s_i) \;\; \forall s' \in \hat{A}_{N;k}(s, r^t) \tag{9}$$

follow and (7) is equivalent to

$$\Phi_t^{\sim} - E(\Phi_t) \geq Z_t. \tag{7a}$$

Now, the set $M = \{i = 1, \cdots, N\}$ of points is partitioned in relation to $s,\ \bar{s}'_i,\ r^t,\ s'_i$ in case $C[r^t, k]$ where $r^t_i \leq s'_i \leq max\{r^t_i, s_i\}$ for $i = 1, \cdots, N$:

$$M_{a1} = \left\{i \in M \mid s_i \leq r^t_i = s'_i \leq \bar{s}'_i\right\} = \left\{i \in M \mid s_i \leq r^t_i\right\},$$

$$M_{a2} = \left\{i \in M \mid r^t_i \leq s'_i < s_i \leq \bar{s}'_i\right\},$$

$$M_{a3} = \left\{i \in M \mid r^t_i < s'_i = s_i \leq \bar{s}'_i\right\},$$

$$M_{b1} = \left\{i \in M \mid s_i > s'_i \geq \bar{s}'_i \geq r^t_i \text{ or } s_i \geq s'_i > \bar{s}'_i \geq r^t_i\right\},$$

$$M_{b2} = \left\{i \in M \mid s_i > \bar{s}'_i > s'_i \geq r^t_i\right\}.$$

$$\begin{aligned} &\Phi^{\sim}_t(s, \bar{s}') - \Phi_t(\bar{s}', s') \\ &\qquad = \hat{f}\left[\sum_{i \in M_{b1} \cup M_{b2}} (s_i - \bar{s}'_i) - \sum_{i \in M_{b1}} (s'_i - \bar{s}'_i)\right] \\ &\qquad = \hat{f}\left[\sum_{i \in M_{b2}} (s_i - \bar{s}'_i) + \sum_{i \in M_{b1}} (s_i - s'_i)\right] \geq 0 \end{aligned} \tag{10}$$

follows from (8), $\Phi_t(\bar{s}', s') = \sum_{i \in M_{b1}} (s'_i - \bar{s}'_i)$ for unit distances and $s_i > \bar{s}'_i$ for $i \in M_{b2}$ and $s_i > s'_i$ for $i \in M_{b1}$.

Furthermore, we show that

$$\exists s' \in \hat{A}_{N;k}(s, r^t) : \ \Phi^{\sim}_t(s, \bar{s}') - \Phi_t(\bar{s}', s') > 0 \text{ if } s \notin \hat{A}_{N;k}(s, r^t). \tag{11}$$

We notice that

$$\begin{aligned} s \notin \hat{A}_{N;k}(s, r^t) \ &\Leftrightarrow \ Z_t(s, s') \neq 0 \ \ \forall s' \in \hat{A}_{N;k}(s, r^t) \\ &\Leftrightarrow \ s \neq s' \ \ \forall s' \in \hat{A}_{N;k}(s, r^t). \end{aligned} \tag{12}$$

If $s \neq s'$ then $\exists\, i_o : \ s_{i_o} > s'_{i_o}$ and hence $i_o \in M_{b2}$ or $i_o \in M_{b1}$ or $i_o \in M_{a2}$.

Furthermore, $\exists\, i_o :\ s_{i_o} \ <\ s'_{i_o}$ and hence $M_{a1} \neq \emptyset$.

Firstly, we show that

$$\exists\, s' \in \hat{A}_{N;k}(s, r^t) :\ M_{a2} = \emptyset \text{ and } M_{b2} = \emptyset. \qquad (13)$$

We set $s'_i = r^t_i\ (\leq \bar{s}'_i)$ if $s_i \leq r^t_i$ according to the conditions from $\hat{A}_{N;k}(s, r^t)$ in case $C[r, k]$ and

$$s'_i \geq\ min\,\{s_i, \bar{s}'_i\}\ \ (\geq r^t_i) \text{ if } s_i > r^t_i \qquad (14)$$

such that $\sum_{i=1}^{N} s'_i = k$. That is possible since $\bar{s}'_i \geq r^t_i\ \forall i$ and $\sum_{i=1}^{N} \bar{s}'_i = k$. (14) implies that $M_{a2} = \emptyset$ and also $M_{b2} = \emptyset$ in relation to $s,\ \bar{s}'_i,\ r^t,\ s'_i$.

If now $s_{i_o} \ >\ s'_{i_o}$ for $i_o \in M_{b1}$ then

$$\Phi^{\sim}_t(s, \bar{s}') - \Phi_t(\bar{s}', s') \geq \hat{f} \cdot 1 \qquad (15)$$

using (10), and (11) is valid since we have above constructed $s' \in \hat{A}_{N;k}(s, r^t)$ with $s_{i_o} \ >\ s'_{i_o}$ for $i_o \in M_{b1}$.

Let

$$\alpha := \left| \left\{ s' \in \hat{A}_{N;k}(s, r^t) \mid (M_{b2} \neq \emptyset)\ \vee\ (\sum_{i \in M_{b1}} (s_i - s'_i) \geq 1) \right\} \right|,$$

$$\beta := \left| \left\{ s' \in \hat{A}_{N;k}(s, r^t) \mid (M_{a2} \neq \emptyset)\ \wedge\ (M_{b2} = \emptyset) \wedge (\sum_{i \in M_{b1}} (s_i - s'_i) = 0) \right\} \right| \text{ and} \qquad (16)$$

$$\gamma := \left| \left\{ s' \in \hat{A}_{N;k}(s, r^t) \mid \Phi^{\sim}_t(s, \bar{s}') - \Phi_t(\bar{s}', s') = 0 \right\} \right|.$$

As we have above shown $\alpha\ \geq\ 1$. (15) and (10) yield that $\gamma \leq \beta$.

Now we want to compute a rough upper bound of β.

According to its definition $M_{a2} \subseteq \{i \in M \mid s_i > s'_i\ (\geq 0)\}$ and

together with $\sum_i s_i = \sum_i s'_i = k$ the relationship

$|M_{a2}| \leq k-1$ follows.

Let $s' \in \hat{A}_{N;k}(s, r^t)$ satisfy the conditions from (16). Then the following relationships are valid for its components s'_i:

$s'_i = r^t_i$ for $i \in M_{a1}$,

$s'_i = s_i$ for $i \in M_{a3}$ or $i \in M_{b1}$,

$0 \leq s'_i < s_i$ for $i \in M_{a2}$.

Hence $0 \leq s'_i \leq s_i$ for $i \in M_{a3}$ or $i \in M_{b1}$ or $i \in M_{a2}$ is true and

$\prod_{i \in \{i \in M \mid s_i > 1\} \setminus M_{a1}} (s_i + 1)$ is a rough upper bound for β.

Since $\sum_{i \in M_{a2}} s_i \leq k-1$ the relationship

$$\beta \leq \prod_{i \in \{i \in M \mid s_i > 1\} \setminus M_{a1}} (s_i + 1)$$

$$\leq \max_{M' \subseteq \{i \in M \mid s_i > 1\} \setminus M_{a1}} \max_{\substack{(\hat{s}_i)_{i \in M'} : \hat{s}_i \in \{0,1,\cdots,k-1\} \\ \sum_{i \in M'} \hat{s}_i = k-1,}} \prod_{i \in M'} (\hat{s}_i + 1)$$

follows and furthermore

$$\beta \leq \max_{M' \subseteq \{i \in M \mid s_i > 1\} \setminus M_{a1}} \left(\frac{k-1}{|M'|} + 1\right)^{|M'|} \leq \left(\frac{k-1}{k-1} + 1\right)^{k-1}$$

since a product is maximal for identical factors subject to the restriction that the sum of the factors is a constant, and $\left(\frac{k-1}{|M'|} + 1\right)^{|M'|}$ is monotone increasing in $|M'|$.

Thus

$$\gamma \leq \beta \leq 2^{k-1}. \tag{17}$$

In case of unit distances the $HARMONIC_p$ k-server algorithm includes that $s' \in \hat{A}_{N;k}(s, r^t)$ are identical distributed.

(3),(17) and (15) in connection with α lead to

$$\underset{s' \in \hat{A}_{N;k}(s,r^t)}{E} \; [\Phi_t^{\sim}(s, \bar{s}') - \Phi_t(\bar{s}', s')] \;\geq\; \hat{f} \cdot 0 \cdot \tfrac{2^{k-1}}{2^{k-1}+\alpha} + \hat{f} \cdot 1 \cdot \tfrac{\alpha}{2^{k-1}+\alpha}$$

$$\underset{s' \in \hat{A}_{N;k}(s,r^t)}{E} \; [\Phi_t^{\sim}(s, \bar{s}') - \Phi_t(\bar{s}', s')] \;\geq \hat{f} \cdot \tfrac{1}{2^{k-1}+1} \quad \text{follows since}$$

$$\tfrac{\alpha}{b+\alpha} \geq \tfrac{1}{b+1} \text{ for } b \geq 0, \alpha \geq 1.$$

Note that $Z_t(s, s') \;\leq\; k$ in case of unit distances.

Then $\hat{f} \;\geq\; k(2^{k-1}+1)$ implies that

$$\underset{s' \in \hat{A}_{N;k}(s,r^t)}{E} \; [\Phi_t^{\sim}(s, \bar{s}') - \Phi_t(\bar{s}', s')] \;\geq k \;\geq\; \underset{s' \in \hat{A}_{N;k}(s,r^t)}{E} \; Z_t(s, s')$$

and the relationship (7) is true for such $\hat{f}$.

Finally, the $HARMONIC_p$ k-server algorithm is $k(2^{k-1}+1)$-competitive in case $C[r^t, k]$ according to [1], Lemma 1.

Proof of (7) and determination of $\hat{f}$ *in case* $C[k, r^t]$*:*

We can use many ideas from case $C[r^t, k]$ in analogous way.

For this we replace

$$\left\{ \begin{matrix} \underset{(-)}{<} \\ \underset{(-)}{>} \end{matrix} \right\} \text{ by } \left\{ \begin{matrix} \underset{(-)}{>} \\ \underset{(-)}{<} \end{matrix} \right\} \text{ in (8), (9) and (14),}$$

$''min''$ by $''max''$ in (14),

and $\bullet - o$ by $o - \bullet$ in (8), (9), (10) and (16)

and in corresponding formulas without numbers.

Thus the considered subsets of M are

$$M_{a1} \;=\; \{i \in M \mid s_i \;\geq\; r_i^t \;=\; s_i' \;\geq\; \bar{s}_i'\} \;=\; \{i \in M \mid s_i \;\geq\; r_i^t\},$$

$$M_{a2} \;=\; \{i \in M \mid r_i^t \;\geq\; s_i' \;>\; s_i \;\geq\; \bar{s}_i'\},$$

$$M_{a3} \;=\; \{i \in M \mid r_i^t \;>\; s_i' \;=\; s_i \;\geq\; \bar{s}_i'\},$$

$$M_{b1} \;=\; \{i \in M \mid s_i \;<\; s_i' \;\leq\; \bar{s}_i' \;\leq\; r_i^t \text{ or } s_i \;\leq\; s_i' \;<\; \bar{s}_i' \;\leq\; r_i^t\},$$

$$M_{b2} \;=\; \{i \in M \mid s_i \;<\; \bar{s}_i' \;<\; s_i' \;\leq\; r_i^t\}$$

in case $C[k, r^t]$.

However the computation of a rough upper bound of β is more different:

If $M_{a2} \neq \emptyset$ then $\bar{s}' \neq s'$ and with $\sum\limits_{i=1}^{N} \bar{s}_i' = \sum\limits_{i=1}^{N} s_i' = k$ the relationship $M_{b1} \neq \emptyset$ follows. Since $r_i^t > 0$ for $i \in M_{b1}$ the inequality

$$\sum_{i \in M_{a2}} r_i^t \;\leq\; R(k) - 1 \tag{18}$$

is valid.

Let $s' \in \hat{A}_{N;k}(s, r^t)$ satisfy the conditions which are analogous to the conditions from (16). Then the following relationships are valid for the components s_i':

$s_i' = r_i^t$ for $i \in M_{a1}$,

$s_i' = s_i$ for $i \in M_{a3}$ or $i \in M_{b1}$,

$r_i^t \geq s_i' > s_i$ for $i \in M_{a2}$.

Hence $r_i^t \geq s_i' \geq s_i$ for $i \in M_{a3}$ or $i \in M_{b1}$ or $i \in M_{a2}$ is true and

since $M_{a2} \subseteq \{i \in M \mid r_i^t > s_i\}$ the numbers

$$\prod_{i \in \{i \in M \mid r_i^t > s_i\} \setminus M_{a1}} (r_i^t - s_i + 1) \;\text{ and also}$$

$$\prod_{i \in \{i \in M \mid r_i^t > 0\} \setminus M_{a1}} (r_i^t + 1)$$

are rough upper bounds for β.

Using (18)

$$\beta \;\le\; \max_{\substack{M' \subseteq \left\{ i \in M \;|\; 0 < r_i^t \right\} \setminus M_{a1} \\ and\; |M'| \le R(k)-1}} \;\; \max_{\substack{(\hat{r}_i^t)_{i \in M'} :\; \hat{r}_i^t \in \{1,2,\cdots,k\} \\ \sum\limits_{i \in M'} \hat{r}_i^t = R(k)-1,}} \;\; \prod_{i \in M'} (\hat{r}_i^t + 1)$$

follows and furthermore analogously to the corresponding inequalities in case $C[r^t, k]$:

$$\beta \;\le\; \max_{\substack{M' \subseteq \left\{ i \in M \;|\; s_i < r_i^t \right\} \setminus M_{a1} \\ and\; |M'| \le R(k)-1}} \left(\frac{R(k)-1}{|M'|} + 1 \right)^{|M'|} \;\le\; \left(\frac{R(k)-1}{R(k)-1} + 1 \right)^{R(k)-1} \;=$$
$2^{R(k)-1}$

Thus

$$\gamma \;\le\; \beta \;\le\; 2^{R(k)-1}. \tag{19}$$

In case of unit distances the $HARMONIC_p$ k-server algorithm includes that $s' \in \hat{A}_{N;k}(s, r^t)$ are identical distributed, and analogously to the case $C[r^t, k]$ follows that

the $HARMONIC_p$ k-server algorithm is $k(2^{R(k)-1} + 1)$-competitive.

■

Conceivable values of $R(k)$ could be $1,1k; ...; 1,3k$ for problems in industry.

Furthermore we give an example that an additional assumption (as $\sum\limits_{i \in M} r_i^t \;\le\; R(k)$ in the above theorem) in case $C[k, r^t]$ is necessary in order to prove the competitiveness.

Let $k = 1$ and $\sum\limits_{i \in M} r_i^t$ not bounded in case $C[k, r^t]$.

The adversary moved his server to another point if and only if the servers of the adversary and of the algorithm are located on the same point.

The adversary produces request sequence $r^t = (1, \cdots, 1, 0, 1, \cdots, 1)$ in steps t where $r_{i_0}^t = 0$ for this point i_0 on which the server of the algorithm is located. Then the cost of the algorithm is equal to 1 in every step.

The $HARMONIC_p$ algorithm moved his server to a point $i \ne i_0$ with the probability $\frac{1}{N-1}$. This also means that then the servers of the adversary and of the algorithm are located on the same point with the probability $\frac{1}{N-1}$. Hence $E[cost(HARMONIC_p\ algorithm)] = (N-1)E[cost(adversary)]$ follows in relation to the expected costs and no $C(k)$ (independent of N) exists such that the $HARMONIC_p$ k-server algorithm is $C(k)$-competitive.

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