$\mathcal{H}_\infty$ Suboptimal Tracking Control for Bilinear Power Converter Systems with Dynamic Feedback - Theory and Experiment

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Zusammenfassung

In der vorliegenden Dissertation werden bilineare Leistungskonvertersysteme untersucht, wie sie für Modellgleichungen mit gemittelten Zuständen im kontinuierlichen Betrieb (engl. „continuous conduction mode“) auftreten. Da eine große Zahl dieser Leistungskonverter nicht eingangs-zustandslinearisierbar hinsichtlich des Regelausgangs und dann oft sogar nicht-minimalphasig sind, zählen sie zur Klasse der schwierig zu regelnden Systeme.

Ein Regelungsziel für die betrachtete Systemklasse ist die Berücksichtigung von Referenztrajektorien für einen Wunschausgang des Systemmodells. Dazu wird ein sogenanntes Fehlersystem eingeführt, das die Differenz zwischen tatsächlichen Größen und Referenzgrößen widerspiegelt. Aufgrund der Bilinearität der ursprünglichen Modellgleichung ist dieses Fehlersystem dann zeitvariant. Ein weiteres Ziel ist das Ausregeln von auftretenden Störungen, Messrauschen, Modellunsicherheiten, usw., was üblicherweise anhand eines Integratoranteils (kurz: I-Anteils) im Regelgesetz berücksichtigt wird. Ein I-Anteil ist eine dynamische Erweiterung der Zustandsgleichungen und führt zu einem zusätzlichen Zustand. Damit die zusätzliche Differentialgleichung nicht entkoppelt vorliegt, muss mit einer geeigneten Eingangstransformation dafür gesorgt werden, dass der Integriererzustand im Regelgesetz vorkommt. Dadurch wird jedoch die ursprüngliche Bilinearität der Gleichungen zerstört, so dass am Ende ein eingangsaffines System vorliegt, das aber natürlich aufgrund der Bilinearität der ursprünglichen Systemgleichungen eine spezifische Struktur aufweist. Eine ähnliche Herangehensweise wie beim I-Anteil ermöglicht die Schätzung und Rückführung der Störung, womit dieselben Regelungsziele verfolgt werden wie bei der Variante mit dem I-Anteil. Hier führt die dynamische Erweiterung mit dem Schätzer im Gegensatz zum I-Anteil allerdings wieder auf eine bilineare Systemgleichung. Allerdings ist dieser Ansatz weniger allgemein und erfordert eine Neuplanung der Referenztrajektorien in Echtzeit, birgt aber mehr Freiheiten in der Wahl der Reglerparameter für den geschlossenen Regelkreis. Als Rückführstrategie wird eine $\mathcal{H}_\infty$-Zustandsregelung gewählt,

um auftretenden Störungen mit möglichst minimalem Stellaufwand auszuregeln. Außerdem soll gleichzeitig der Fehler des Regelausgangs klein gehalten werden. Um schließlich die Stabilität des geschlossenen Regelkreises für nichtverschwindende Störungen untersuchen zu können, wird die sogenannten integral Input-to-State Stability (iISS) verwendet.

Als Ergebnis der Arbeit können Bedingungen formuliert werden, wann eine suboptimale $\mathcal{H}_\infty$-Zustandsregelung gefunden werden kann. Unter Annahme dieser Bedingungen folgt dann sofort die iISS-Eigenschaft des geschlossenen Regelkreises. Die Allgemeinheit des Verfahrens zeigt sich dadurch, dass es sogar möglich ist, den vorgestellten Ansatz auf allgemeine bilineare Systeme mit mehreren Eingängen zu erweitern.

Das experimentelle Beispiel eines Hochsetzstellers in Kombination mit einem Gleichstrommotor wird dann zum Testen des Regelentwurfsverfahrens herangezogen. Dabei ist die Regelungsaufgabe, die Winkelgeschwindigkeit der Motorwelle einer vorgegeben Referenztrajektorie nachfahren zu lassen und auftretende Laststörungen auszuregeln. Dazu wurde die Variante der dynamischen Erweiterung anhand der Rückführung der Störung mit Trajektorienneuplanung verwendet. Mit einer suboptimalen $\mathcal{H}_\infty$-Zustandsregelung wird der Regelkreis geschlossen, so dass iISS gewährleistet werden kann. Für die Echtzeitgenerierung der durch ein Approximationsverfahren ermöglichten Trajektorienneuplanung wird außerdem Beschränktheit gezeigt. Eine Vielzahl von Experimenten dient der genaueren Untersuchung des Verfahrens.

Abstract

In this thesis, bilinear power converters are considered that arise for state-averaged models in continuous conduction mode. Since such power converters are often not feedback linearizable with respect to the output to be controlled, they are an interesting and demanding class of control systems.

One control objective for the considered system class is to include trajectory tracking in the system equations. With a state and input transformation into the so called error system representation, where the error between real variables and reference variables is considered, the error system equations show to be time-varying. Another objective is to cope with disturbances, noise, parameter uncertainties, etc. Therefore, integral feedback is included in the feedback strategy, which leads to input-affine systems with a special structure due to the originally bilinear system equations. A slightly different strategy is a disturbance feedback approach. It addresses the same control objectives, is structurally similar to integral feedback and allows for more freedom in choice of feedback design parameters. However, it is less general and requires online-replanning of the reference trajectory. For state feedback design, we choose $\mathcal{H}_\infty$ control with a quadratic performance functional since we want to have low control effort and want to keep the error of the output to be controlled small in case of appearing disturbances. Finally, so as to address stability properties in the closed-loop, integral Input-to-State Stability (iISS) theory is a good choice to cope with nonzero disturbances.

In order to guarantee stability for the closed-loop system in the presence of disturbances, we link the solution of the nonlinear $\mathcal{H}_\infty$ control problem with iISS. It is possible to derive conditions, when the suboptimal state feedback $\mathcal{H}_\infty$ control problem for the bilinear power converter systems with integral feedback / disturbance feedback and trajectory tracking can be solved. At the same time, it can be shown that the closed-loop systems is iISS. To underline the generality of the approach, the obtained theory for bilinear power converter systems is extended to general bilinear systems and it is even possible to discuss the more demanding multiple-input case.

Equipped with the required theory to solve the posed control problem, we address the experimental setup of a boost converter / DC motor system. Here, the control task is to track the angular velocity of the motor shaft and attenuate appearing load disturbances. Therefore, we implement disturbance feedback and proof boundedness of trajectories for the online-replanning of the approximate trajectory generation method. Various experiments are presented in order to investigate the applicability of the approach.

Chapter 1

Introduction

1.1 Brief Overview of the Topic

It is common in practice that DC-DC converters are used in between a constant voltage source and a consumer load in order to regulate the required power supply of the connected load. For example, DC-DC converters work as drivers for the operation of DC motors so as to adapt the constant input voltage to appearing load changes and guarantee the specified task of the motor. These power converters usually consist of switches (MOSFETs, IGBTs, ...), diodes, storage elements like inductors or capacitors, and dissipative elements such as resistors. Customarily, the switch position of the MOSFET governs the converter currents and the output voltage. Therefore, such converters are often called switching converters, classically controlled via Pulse-Width Modulation (PWM) schemes [1]. In light of the commonly used high switching frequencies of the control switches that are much faster than the system dynamics, so-called averaged models may be employed for the controller design (see e.g. [2]). The duty cycle of the PWM-signal can in this case be viewed as a continuous control variable. For this reason, an actually linear (time-invariant) switched system may become nonlinear (time-invariant) when its average model is considered. More precisely, these systems generally will be bilinear. It is clear that bilinearity of the power converter model is retained in a combination with a linear motor model. An example for such configuration that is also employed in this thesis is a boost converter connected to a DC motor.

What makes the bilinear power converter systems interesting from the control engineering and system theoretic perspective is the fact that most of the standard power converter systems are not feedback linearizable (not differentially flat) with respect to the measurement output and are even nonminimum-phase [2], such that their input-output linearization leads to an unstable internal dynamics. This makes the control of such systems a challenging task and led to immense research activity. Consequently, a large variety of control methods have been tested to address the control problem of power converters. We will discuss only a selection which fits in the context of the proposed approach.

Most strategies cope with time-invariant nonlinear systems and concentrate on set-point control. For time-invariant nonlinear systems, L_2-gain and dissipativity theory [3],[4],[5],[6],[7] ("nonlinear $\mathcal{H}_\infty$ control") is an interesting approach which enables stability in the presence of disturbances and allows one to keep a performance output small in the presence of nonzero disturbances in an (sub-)optimal sense. A prominent approach tackling power converters with nonlinear $\mathcal{H}_\infty$ control can be found in [8],[9],[10]. Therein, a Čuk converter is considered, where an approximate integrator is included (as is common in linear $\mathcal{H}_\infty$ control), which changes the original bilinearity of the power converter system equations. With this integral action, tracking is possible along exogeneous inputs to allow for set-point changes and the integrator copes with noise and parameter variations or disturbances. For the practitioner it is interesting that the control laws are simple and easy to implement, and even coincide with control laws from passivity-based control. However, a general theory is not available, and tracking along exogeneous inputs using integral action is an approximate tracking strategy which often lacks performance and accuracy. One of the authors also presents H_2 control in the book [11], which is solved in general for bilinear time-invariant systems with integral action. The generalization to time-varying systems has not yet been done and is not obvious. Furthermore, this approach leads to complicated control laws which are difficult to implement, especially with the knowledge that power converter systems often are numerically stiff and cause problems in practical realization.

Another prominent approach is passivity-based control which suggests itself to be a good choice since electrical networks are inherently passive due to resistor components. Widely known is the work in [12],[2], where passivity-based control is investigated for different converters, but without integral control or stability considerations with respect to disturbances. The work of [13] uses a specially structured Lyapunov function to add integral action on the voltage output of the boost converter model so as to achieve zero steady-state error for passivity-based control. What makes this approach very interesting is the fact that it is possible to show stability of the closed-loop system for the saturated input, a property that arises naturally in average models: due to the infinite switching frequency, the inputs take values in the interval $[0, 1]$. Again, only time-invariant systems are considered to achieve set-point control, and stability properties with respect to disturbances are not discussed. Furthermore, this approach has been applied merely for a standard circuit and a complete theory is missing. So as to track a smooth set-point transition along a pre-specified output, it is common to introduce a reference system. To obtain the associated reference solution, the system inversion problem has to be solved, wherefrom the exact feedforward control law can be calculated. However, the resulting error dynamics of bilinear systems along such reference solutions is time-varying (see [2, 14, 15, 16]). Thus the controller design and stability analysis becomes much more demanding. For tackling this tracking problem, the authors in [15] use passivity-based control in combination with an algebraic load estimator instead of standard integral control, and an approximate feed-forward control to reject occurring load disturbances in a boost converter / DC motor combination. However, the approximate trajectory tracking strategy and the closed-loop stability properties in the presence of disturbances need further discussion.

Different from the nonlinear strategies, the Jacobi linearization along a reference trajectory leads to a linear time-varying system [16]. This approach therefore allows reference tracking, and using time-varying differential operator methods, it is possible to include integral control and even dynamic output feedback. Since the method is based on pole placement, the closed-loop dynamics is known and performance can be directly addressed in contrast to nonlinear control algorithms. Here the only disadvantages are that first of all, linearization is only locally valid and the region of attraction, especially in the presence of occurring disturbances, is usually not known. Second, deriving the control law including an observer in conjunction with the proof of time-varying controllability and observability along the considered reference trajectory is cumbersome.

Sliding mode control [17], [18], [19], [20], [21], [22], [23] is one of the strategies which match quite well since fast switching can be incorporated in electrical and electromechanical systems much better than in mechanical systems where switching the actuators is more difficult. Here, we do not want to go deeper into detail although sliding mode control is a prominent approach in the control of power converter systems and it is important to mention it. Since the work of the thesis is quite different, we do not feel able to present a competent overview of this research branch.

1.2 Contribution of the Thesis

In this thesis, we address tracking, integral control and stability in the presence of disturbances for the extended class of bilinear power converter systems using nonlinear time-varying state feedback $\mathcal{H}_\infty$ control and iISS theory (integral Input-to-State Stability). We use results from time-varying dissipativity and Lyapunov theory to tackle the posed control problem via nonlinear time-varying $\mathcal{H}_\infty$ control [24, 25]. As a major contribution of this thesis, we are able to provide conditions on solvability of the problem due to the more special class of systems. As shown in [26], it turns out that iISS theory [27] is appropriate for deriving closed-loop stability in the presence of non-zero disturbances for time-invariant systems and has close links with dissipativity theory. This connection is also valid for time-varying systems, which we formulate with the results of [28] for time-varying iISS. In addition, we prove time-varying iISS for the closed-loop system arising from nonlinear time-varying $\mathcal{H}_\infty$ control design in the presence of disturbances. The obtained theoretical results can even be extended to general bilinear systems with multiple inputs. In the end, we show the applicability of the approach with experimental data for tracking the angular velocity of a boost converter / DC motor combination subject to load disturbances. Parts of this thesis, especially the single-input case for bilinear power converter systems, were published in [29], [30].

The thesis is structured as follows: Chapter 2 introduces the considered class of bilinear systems and the closed-loop system structure due to different dynamic feedback strategies. Chapter 3 contains Lyapunov stability for time-invariant and time-varying systems and time-varying ISS (Input-to-State Stability) and iISS theory used in the following text. The $\mathcal{H}_\infty$ control approach and its stability properties are presented in Chapter 4. The main results for the bilinear power converter systems of Chapter 2 are derived in Chapter 5. Experimental results for a boost converter / DC motor combination with load estimation and approximate reference trajectory can be found in Chapter 6. Conclusions are drawn in Chapter 7, while the Appendix A contains numerous figures of experiments which were explained in Chapter 6. Finally, Appendix B discusses system theoretic properties of the boost converter / DC motor equations.

Chapter 2

Motivation

2.1 Power Converters

Power Converters cover a wide field in electrical drive and power engineering and are used for power supplies, rectifiers, etc. with applications in very different scenarios. The important characteristic of such devices is that they can convert power (i.e. current and voltage) almost losslessly and therefore are suited to link between source and load. Further applications are the power management of trams or high performance trains up to hard disk or DVD drives, welding, induction heating, electrical utility applications, etc. [1], [31].

The converters considered here for demonstration experiments are of low power and are well-suited for experimentally validating the proposed control approach. However, it has to be noted that all such power converters exhibit a similar structure of their differential equations, since they all consist of electrical components and obey the Kirchhoff current and voltage laws. We concentrate on converters in continuous conduction mode and use state-averaged models which arise for infinite switching frequency in pulse-width modulation schemes, what will be covered in more detail in the next section.

2.2 Modeling

Power converter systems consist of power supplies (AC or DC), capacitors and inductors as dynamic or storage elements, resistors as passive elements and switches (MOSFETs, IGBTs, etc.). When modeling such systems using the linear Kirchhoff current and voltage laws and only considering linear components, we arrive at switched linear systems. In power converter systems, the switches are often regulated via pulse-width modulation (PWM) [32], [1]. With such PWM schemes, the ideal switch position can always be constrained to have values in a discrete set $\{0, 1\}$, where position 1 is often referred to as the "on" state and postion 0 the "off" state of the switch. When using PWM schemes, sampling is involved with sampled time instants $t_k, k = 0, 1, 2, 3, \dots$ and sampling interval $T > 0$ called duty cycle, such that $t_{k+1} := t_k + T$. In one sampling interval T, the generated PWM signals step down once from 1 to 0 at arbitrary time instants in each interval $[t_k, t_k + T), k = 0, 1, 2, \dots$. The input signal is modeled via the Heaviside function

$$\mu(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

Furthermore, we need the so called duty ratio $0 \leq u \leq 1$, which is fixed for each sampling interval T. In the experimental setup, it is common to use a sawtooth signal[1]

$$\text{tri}(t, T) = \frac{t}{T} - \text{floor}\left(\frac{t}{T}\right)$$

which is compared with the duty ratio u via $\mu(u - \text{tri}(t, T))$, i.e. uT tells how long μ is in the "on"-state in each time interval.

If we sample a nonlinear system $\dot{x}(t) = f(x(t)) + g(x(t))\mu(t)$ with fixed sampling period T and initial state $x(t)$ of the sampling interval at fixed time $t \geq 0$, the solution would be

$$x(t + T) = x(t) + \int_t^{t+T} (f(x(s)) + g(x(s))\mu(s))ds.$$

If we take into account that μ is the Heaviside function, we can write the solution as

$$x(t + T) = x(t) + \int_t^{t+T} f(x(s))ds + \int_t^{t+uT} g(x(s))ds.$$

[1]Another common choice would be a triangular signal. The choice of signal does not affect the final result and is therefore merely an implementation issue.

Rewriting the last equation and dividing by T, we arrive at

$$\frac{1}{T}(x(t+T)-x(t)) = \frac{1}{T}\int_{t}^{t+T} f(x(s))ds + \frac{1}{T}\int_{t}^{t+uT} g(x(s))ds.$$

Taking the limit $T \to 0$ (i.e. assuming an infinite switching frequency) finally leads to

$$\dot{x}(t) = f(x(t)) + g(x(t))u, \quad \text{with fixed } t \geq 0.$$

If we now assume that for each time instant $t \geq 0$ we have an associated duty ratio $0 \leq u(t) \leq 1$ piecewise continuous in t and bounded, the differential equation is given by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t).$$

This approximation is usually met in practice, since already in the design stage the switching frequency is set much higher than the systems dynamics if possible, such that an average model can be used for identification and control design. In addition, this procedure with limit $T \to 0$ can be applied to obtain a closed-loop average model with a state feedback law $u(x)$, too, following the steps of the derivation before[2]. Since the input variable u is a piecewise continuous and bounded function taking values in the interval $[0, 1]$, the resulting system equations are bilinear, see [2],[12].

For the studied bilinear systems, we use the denotation

$$M\dot{x}(t) = Fx(t) + \left(\bar{b} + J_1 x(t)\right) u(t) + \epsilon(t), \quad x(t_0) = x_0, \tag{2.1}$$

with state $x(t) \in \mathbb{R}^n$, piecewise continuous and bounded input $u(t) \in \mathbb{R}$, piecewise continuous and bounded $\epsilon(t) \in \mathbb{R}^n$, vector $\bar{b} \in \mathbb{R}^n$, matrix $F \in \mathbb{R}^{n\times n}$, symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$ and skew-symmetric matrix $J_1 \in \mathbb{R}^{n\times n}$. The state vector $x(t)$ contains the currents and voltages of the circuits, or in the case of attached mechanical parts, for instance a motor, the angular velocity, etc. Matrices F, J_1 and the vector b exhibit the structure of the circuit arising from the Kirchhoff current and voltage laws or from the prinicples of conservation in the mechanical components. Vector $\epsilon(t)$ contains possible voltage or current sources, constant torques, etc.

[2] Modeling of power converter systems can be done via averaging theory [33],[34] for the general case with $T > 0$. A geometric approach for the infinite switching frequency case ($T \to 0$) and the equivalence to sliding mode control is discussed in [32].

For proving stability later, it is advantageous, to use

$$R := -\frac{1}{2}(F + F^{\mathrm{T}}),\; J_0 := \frac{1}{2}(F - F^{\mathrm{T}}),\; J(u(t)) := J_0 + J_1 u(t) \tag{2.2}$$

with matrix $R \in \mathbb{R}^{n\times n}$ symmetric (and positive semi-definite for passive systems) and skew-symmetric matrix $J(u(t))$ to rewrite the system equations as

$$M\dot{x}(t) = (J(u(t)) - R)\, x(t) + \bar{b}\, u(t) + \epsilon(t), \tag{2.3}$$

such that we can distinguish the symmetric and skew-symmetric parts.

Up to now we did not include any disturbances. The possible disturbances $d(t) \in \mathbb{R}$, which are piecewise continuous and bounded functions, naturally keep the affine structure and affect the system via the constant vector $\bar{g} \in \mathbb{R}^n$:[3]

$$M\dot{x}(t) = (J(u(t)) - R)\, x(t) + \bar{b}\, u(t) + \epsilon(t) + \bar{g}d(t). \tag{2.4}$$

Regarding the disturbances in power converter systems which affect the system through a constant vector, we will assume that $\bar{g}$ is a unit vector, i.e. $\bar{g} = 1_j$ with 1_j denoting the j-th n-dimensional unit vector.

Definition 2.1 (Bilinear Power Converter System)
Consider system (2.4), with state $x(t) \in \mathbb{R}^n$, piecewise continuous and bounded input $u(t) \in \mathbb{R}$ and disturbance $d(t) \in \mathbb{R}$, vector $\bar{b} \in \mathbb{R}^n$, piecewise continuous and bounded function $\epsilon(t) \in \mathbb{R}^n$, symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$, symmetric matrix R, skew-symmetric matrix $J(u) := J_0 + J_1 u$, $J(u) \in \mathbb{R}^{n\times n}$ and unit vector $\bar{g} = 1_j, j = 1, \ldots, n$. We will call this system class **B***ilinear* **P***ower* **C***onverter* **S***ystems and use the abbreviation BPCS.*

The bilinear systems arising in power electronics are interesting from a control engineering perspective, since most standard circuits are not feedback linearizable[4] with respect to the measurement outputs and are even nonminimum-phase (cf. Definition B.7), which limits the number of applicable control strategies because of the complexity of the difficult but interesting system

[3] We will loosen this restriction a bit later and allow for the most general class of occurring disturbances $Hx(t)d(t) + \bar{g}d(t)$ in power converter systems and show how those disturbances can be handled in the proposed theory. But in power electronic devices, the described situation is not unrealistic and already covers a large class of disturbances if one looks at the used standard circuits, e.g. input voltage variations or (motor shaft) load variations.

[4] For single-input single-output systems, feedback linearizability is equivalent to differential flatness.

theoretic properties of the converters. These properties were thoroughly investigated in [2]. For the example application in this thesis we also show in Appendix B that the system under investigation is not feedback linearizable, has a high relative degree with respect to the designated control output and is even nonminimum-phase, such that the control problem cannot be solved by standard algorithms.

2.3 Control Strategies

Common tasks for power converter systems are set-point changes of the state variables (voltages, currents), where it is highly desired that peaks are avoided to reduce stress on the circuit components. Furthermore, it is important to maintain the specified set-points in the presence of disturbances, which appear through load changes, voltage source variations or system uncertainties. Avoiding peaks is achieved by tracking fast smooth trajectories between set-points instead of step functions as inputs. To cope with disturbances, noise and parameter uncertainties, integral control[5] should be included, and stability has to be shown for the complete system with nonzero disturbances. Furthermore, it is advantageous, when disturbances do not lead to enormous control input action. Concerning performance, a prespecified performance output should remain small. To achieve the latter two properties, we choose a suboptimal control strategy which contains this information in its optimization functional, which leads to a nonlinear $\mathcal{H}_\infty$ control setting [8],[10]. Let us summarize the control requirements:

- Trajectory tracking of a bounded, but arbitrary reference trajectory that fulfills the system equations.
- A (sub-)optimal state feedback strategy that
  - incorporates integral control in order to cope with disturbances, measurement noise and small parameter deviations.
  - guarantees closed-loop stability in the presence of disturbances.

[5]For linear systems in Laplace domain, we know that integral action (as is the case for PI or PID controllers) can be used to eliminate the effect of constant disturbances [35]. The equivalent formulation in the time domain is to extend the system with an integrator state [36]. Thus, the feedback law incorporates integral as well as proportional feedback of the state and we call this strategy “integral control” or “integral feedback”.

Reference trajectory tracking is considered in what follows introducing a reference system and transforming the system equations into an error system representation. In order to realize integral control, the subsequent sections provide two strategies, first of all via integral feedback (which is more general), second, using a load estimation strategy. Both designs are dynamic feedback strategies. The closed-loop stability in the presence of disturbances will be addressed later in this thesis.

2.3.1 Integral feedback

Adding integral feedback in the design of power converter systems as is done in the work of [8] leads in the case of set-point tracking to time-invariant input-affine error systems. In this section, we extend this approach since we allow for smooth reference trajectory tracking along a reference system which leads to an input-affine time-varying error system.

To achieve trajectory tracking, we first introduce a reference system

$$M\dot{x}^\star(t) \;=\; (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t) \tag{2.5}$$

with the reference solution $x^\star(\cdot, t_0, x_0, u^\star)$. From the practical point of view it directly follows that the solution is supposed to be bounded for bounded reference input $u^\star$.

At this stage, we do not discuss how it is possible to obtain the corresponding reference states and reference input from a specified reference output trajectory, but this problem can be attacked with standard strategies in the literature [37], [38], [39], [40], and, for this reason, will not be thoroughly discussed here. An example is given in the case of the experimental study we discuss later. We want to point out that the occurring disturbance is not included, since it is not known in the reference system in advance. However, we will use this fact later for disturbance feedback.

In the following, we will skip the time argument of state and input for brevity.

With the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ the error system reads

$$M\dot{e}_x = (J_0 - R)\, e_x + \bar{b}\, e_u + J_1\Big(\underbrace{xu - xu^\star}_{=\,x\;e_u} + \underbrace{xu^\star - x^\star u^\star}_{=\,e_x\;u^\star}\Big) + \bar{g}d. \tag{2.6}$$

Finally, since M is invertible, it follows that

$$\dot{e}_x = \underbrace{M^{-1}\left(J(u^\star) - R\right)}_{=:\,A(t)} e_x + \underbrace{M^{-1}\left(\bar{b} + J_1(e_x + x^\star)\right)}_{=:\,\tilde{b}(t,e_x)} e_u + \underbrace{M^{-1}\bar{g}}_{=:\,\tilde{g}}\, d. \tag{2.7}$$

The system output is given by $y = \tilde{c}^{\mathrm{T}} e_x$, with the unit vector $\tilde{c}^{\mathrm{T}}$ singling out the error state to be controlled. With this choice we restrict our considerations to integral control of the error of state variables and do not allow for linear combinations of several error states. This restriction is usually met in power converter systems where it is desired to regulate the error of a system state to zero.

The resulting system

$$\begin{aligned} \dot{e}_x &= A(t)e_x + \tilde{b}(t,e_x)e_u + \tilde{g}d \\ y &= \tilde{c}^{\mathrm{T}} e_x, \end{aligned} \tag{2.8}$$

with $A(t), \tilde{b}(t,e_x)$ as specified in (2.7) is time-varying which shows that we have to study time-varying control strategies which makes the underlying control problem demanding.

Definition 2.2 (Bilinear Power Converter Tracking Error System) *Consider system (2.8) with (2.7), unit vector $\tilde{c} = 1_i, i = 1,\ldots,n$, which is the result of an input transformation $e_u := u - u^\star$ and a state transformation $e_x := x - x^\star$ of BPCS. We will call this system class* **B***ilinear* **P***ower* **C***onverter* **T***racking* **E***rror* **S***ystems and use the abbreviation BPCTES.*

In order to track a given reference trajectory and maintain a zero tracking error, we need integral feedback (as is standard in the linear case). Striving for integral action on the output y of the controlled system (2.8) we need to incorporate dynamic feedback. So we undertake an input transformation with new input $e_{\bar{u}}$

$$e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}} \tag{2.9}$$

and introduce the additional differential equation

$$\dot{z} = -\alpha_4 y - \alpha_1 z \tag{2.10}$$

with integrator state $z(t) \in \mathbb{R}$ and constants $\alpha_i > 0, i = 1\ldots4$. If α_1 would be zero, we would integrate the error of the controlled output $y = \tilde{c}^{\mathrm{T}} e_x$,[6]

[6]Remember again that the output y considered here is not the measurement output, since we only consider full state feedback and we do not want to introduce a variable for the measurement output, since we already have enough variables.

and since $\tilde{c}^{\mathrm{T}}$ is a unit vector, this is equivalent to integrating the error of one of the state variables. The extra term $-\alpha_1 z$ with $\alpha_1 > 0$ leads to a truncated, approximate integrator [8], which is a standard strategy in linear control theory, see [41], [42]. In other words: This kind of integral action is a simple first-order lag $G(s) = Z(s)/\Omega(s) = -\frac{\alpha_4}{s+\alpha_1}$ which converges to a pure integrator via $\lim_{\alpha_1 \to 0} G(s)$. We need $\alpha_1 > 0$, since for a pure integrator, suboptimality and stability cannot be shown under the assumed assumptions, as can be seen from the results in [8] or in the proofs for suboptimality in the chapters below.

The introduced dynamic feedback and input transformation leads to a differential equation driven by the new (transformed) input $e_{\bar{u}}$. Hence, defining the enlarged state $e^{\mathrm{T}} = (e_x^{\mathrm{T}}\; z)^{\mathrm{T}}$, the modified error system has the form

$$\begin{aligned}
\frac{d}{dt}\underbrace{\begin{pmatrix} e_x \\ z \end{pmatrix}}_{=:\, e} &= \underbrace{\begin{pmatrix} A(t)e_x - \alpha_3 \tilde{b}(t,e_x)z \\ -\alpha_4 c^{\mathrm{T}} e - \alpha_1 z \end{pmatrix}}_{=:\, a(t,e)} + \underbrace{\begin{pmatrix} \alpha_2 \tilde{b}(t,e_x) \\ 0 \end{pmatrix}}_{=:\, b(t,e)} e_{\bar{u}} + \underbrace{\begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix}}_{=:\, g} d \\
\dot{e} &= a(t,e) + b(t,e)e_{\bar{u}} + gd \\
y &= c^{\mathrm{T}} e = \left(\tilde{c}^{\mathrm{T}}\; 0\right) e.
\end{aligned} \tag{2.11}$$

Also note that c^{T} is again a unit vector singling out the state to be integrated. Obviously, the modified error system (2.11) is affine-linear in $e_{\bar{u}}$ with a special structure that results from the original bilinear system (2.8) due to the feedback strategy.

Definition 2.3 (Power Converter Integral Feedback Error System)
Consider system (2.11) with elements from BPCTES, integrator state $z(t) \in \mathbb{R}$, new input $e_{\bar{u}}(t) \in \mathbb{R}$ and real constants $\alpha_i > 0, i = 1 \ldots 4$, which results from an input transformation with dynamic feedback from BPCTES. We will call this system class **P***ower* **C***onverter* **I***ntegral* **F***eedback* **E***rror* **S***ystems and use the abbreviation PCIFES.*

2.3.2 Disturbance feedback

Another way to include integral action can be utilized, if we exploit the fact that we know where the disturbance appears in the system equations. Therefore, one inserts a dynamic disturbance feedback with state $\hat{d}(t) \in \mathbb{R}$, with the same structure as the integrator state,

$$\dot{\hat{d}} = -ly - \alpha_5 \hat{d} \tag{2.12}$$

and constants $l, \alpha_5 > 0$. But instead of an input transformation, we include the state $\hat{d}$ as disturbance feedback in our reference system (2.5) taking the exact copy of (2.4)

$$M\dot{x}^\star(t) = (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t) + \bar{g}d^\star(t) \tag{2.13}$$

which requires an online replanning of the reference trajectory, since the solution $x^\star(\cdot, t_0, x_0, u^\star, d^\star)$ is now dependent on the piecewise continuous and bounded "reference disturbance" $d^\star$ which is not known in most cases (except the disturbance can be directly measured, which of course would simplify the posed problem). From the practical point of view it directly follows that the solution is supposed to be bounded for bounded reference input $u^\star$ and bounded reference disturbance $d^\star$. Therefore, the information obtained from the disturbance feedback strategy can be used substituting $d^\star = \hat{d}$.[7] Then the system equations change to

$$M\dot{x}^\star(t) = (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t) + \bar{g}\hat{d}(t). \tag{2.14}$$

In the next step, we again transform to error representation and use the input transformation $e_u = \alpha_2 e_{\bar{u}}, \alpha_2 > 0$ (such that we have an additional tuning parameter) to arrive at

$$\begin{aligned}
\frac{d}{dt}\underbrace{\begin{pmatrix} e_x \\ \hat{d} \end{pmatrix}}_{=:\, \check{e}} &= \underbrace{\begin{pmatrix} A(t)e_x \\ -l\tilde{c}^{\mathrm{T}} e_x - \alpha_5 \hat{d} \end{pmatrix}}_{=:\, \check{A}(t)\check{e}} + \underbrace{\begin{pmatrix} \alpha_2 \tilde{b}(t, e_x) \\ 0 \end{pmatrix}}_{=:\, \check{b}(t,\check{e})} e_{\bar{u}} + \underbrace{\begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix}}_{=:\, \check{g}} \underbrace{(d - \hat{d})}_{=:\, \check{d}}, \\
\dot{\check{e}} &= \check{A}(t)\check{e} + \check{b}(t, \check{e})e_{\bar{u}} + \check{g}\check{d}, \qquad (2.15) \\
\check{y} &= \check{c}^{\mathrm{T}} \check{e} = \begin{pmatrix} \tilde{c}^{\mathrm{T}} & 0 \end{pmatrix} \check{e}, \qquad \check{A}(t) = \begin{pmatrix} A(t) & 0 \\ -l\tilde{c}^{\mathrm{T}} & -\alpha_5 \end{pmatrix}.
\end{aligned}$$

These system equations are again time-varying, but remain bilinear in the sense of (2.8) because of the specific structure in $\check{b}(t, \check{e})$ (see $\tilde{b}(t, e_x)$ in (2.7)).

[7] The online-replanning strategy and occurring obstacles need a closer inspection. At the moment we merely want to know which type of system classes we are facing. Therefore, further discussion on this topic will be devoted to later chapters where we explain a realizable strategy for an example system.

Definition 2.4 (Power Converter Disturbance Feedback System) *Consider system* (2.15) *with estimator state* $\hat{d}(t) \in \mathbb{R}$*, new input* $e_{\bar{u}}(t) \in \mathbb{R}$ *and real constants* $\alpha_2, \alpha_5, l > 0$*. We will call this system class* **P***ower* **C***onverter* **D***isturbance* **F***eedback* **S***ystems and use the abbreviation PCDFS.*

Remark 2.1 *A similar strategy was introduced in [43], but merely for flat systems (i.e. system which are feedback linearizable), not with the approximate integrator as proposed in* (2.12)*, and without stability proof for the trajectory replanning strategy. In contrast, we will prove stability of the replanning strategy for the considered example system in Chapter 6.*

Remark 2.2 *System (2.7) is* C^1 *with respect to* t, e, e_u*. The dynamic feedback is linear, so the overall systems (2.11) and (2.15), respectively, are* C^1 *in all elements. Inputs* $e_{\bar{u}}, d, \check{d}$ *are assumed piecewise continuous, bounded real-valued functions of time. So for given initial condition* $e(t_0) = e_0$*, we have local uniqueness of the solution of (2.11) and (2.15), respectively.*

To sum up, it can be seen that in contrast to linear systems reference tracking leads to a nonlinear time-varying error system due to the nonlinearity of the underlying system equations (here the bilinearity in the obtained differential equations of BPCS during the modeling process). When including disturbance feedback or integral feedback, respectively, the system equations become at least time-varying (PCDFS), or lose their bilinearity and we get a more general class of differential equations (PCICES), which still intrinsically exhibits the structure of the original bilinear error system.

What remains to be shown is how to derive the suboptimal state feedback law that leads to low control effort for occurring disturbances and keeps a prespecified performance output small. Furthermore, we have not yet discussed how to guarantee closed-loop stability in the presence of disturbances. For this reason, we have to introduce the required nonlinear $\mathcal{H}_\infty$ control and stability theory in the following two chapters to be able to tackle the suboptimal state feedback design and its closed-loop stability properties in the presence of disturbances before returning to our more specific problem introduced in this chapter.

Chapter 3

Stability Theory

3.1 Lyapunov Stability

The presented results from Lyapunov Theory are a compilation of what can be found in the texts [44],[45],[46],[47],[3].

3.1.1 Time-invariant systems

Consider the nonlinear system

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}^+ \tag{3.1}$$

with C^1 function $f : D \to \mathbb{R}^n$, where the domain D is an open connected set $D \subset \mathbb{R}^n$ which contains the origin $x = 0$ and $\mathbb{R}^+ := [0, \infty)$. Assume that there exists a unique forward complete solution $x(\cdot, x_0)$ for the initial value problem with initial condition $x(0) = x_0 \in D$, i.e. the solution has domain $[0, \infty)$. The open ball around a given point x_0 with radius $\rho > 0$ is denoted $B_\rho := \{x \in \mathbb{R}^n | \; ||x|| < \rho\}$.

We begin with the definition of Lyapunov stability for time-invariant systems:

Definition 3.1 (Lyapunov Stability) *Let $x = 0$ be an equilibrium point of (3.1), i.e. $f(0) = 0$. The equilibrium point $x = 0$ is called*

- stable, *if* $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x_0 \in B_\delta)(\forall t \in \mathbb{R}^+) \; : \; ||x(t, x_0)|| < \epsilon.$

- asymptotically stable, *if it is stable and* $(\exists \eta > 0)(\forall x_0 \in B_\eta)$: $\lim_{t\to\infty} x(t, x_0) = 0$.
- globally asymptotically stable, *if it is stable and for* $D = \mathbb{R}^n$ *we have* $(\forall \eta > 0)(\forall x_0 \in B_\eta)$: $\lim_{t\to\infty} x(t, x_0) = 0$.
- unstable, *if it is not stable.*

Since the definition of Lyapunov stability requires the knowledge of the solution of the nonlinear differential equation (3.1) which is in general not known, a different strategy is necessary to prove stability. Therefore, we introduce a test function V equipped with appropriate properties, which makes it possible to state a theorem in terms of V and gives links to Lyapunov stability. This theorem proves to be more amenable than the definition of Lyapunov stability itself.

Definition 3.2 *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *is said to be*

- *positive semidefinite, if*

$$V(x) \geq 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V(0) = 0.$$

- *positive definite, if*

$$V(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\} \text{ and } V(0) = 0.$$

- *negative semidefinite, if* $-V(x)$ *is positive semidefinite.*
- *negative definite, if* $-V(x)$ *is positive definite.*
- *radially unbounded, if* $||x|| \to \infty \Rightarrow V(x) \to \infty$.

Then, we can state the following theorem:

Theorem 3.1 *Let* $V : D \to \mathbb{R}$ *be a positive definite* C^1 *function and let* $x = 0$ *be an equilibrium point of* (3.1)*. Then the equilibrium point* $x = 0$ *is*

- *stable, if* $\dot{V}(x)$ *is negative semidefinite for* $x \in D$ *(where* $\dot{V}(x) := V_x(x) f(x)$ *with the row vector* $V_x(x) := \frac{\partial V(x)}{\partial x}$*).*
- *asymptotically stable, if* $\dot{V}(x)$ *is negative definite for* $x \in D$.

- *globally asymptotically stable, if* $\dot{V}(x)$ *is negative definite for* $x \in D = \mathbb{R}^n$ *and* $V(x)$ *is radially unbounded.*

As was done in [48], we call V a *Lyapunov function candidate*, if it is C^1 and positive definite. If it additionally fulfills the conditions for stability, asymptotic stability or global asymptotic stability, we call V a *Lyapunov function.*

Remark 3.1 *Evaluating* V *along the solution* $x(t, x_0)$, *we can use the chain rule and obtain* $\dot{V}(x(t, x_0)) = V_x(x(t, x_0))f(x(t, x_0))$, *which is the special case of the vector-valued definition of* $\dot{V}$ *above.*

Remark 3.2 *We made the assumption of forward completeness on the solutions of the considered systems, which, at first, seems to be a strong assumption. But since we will use Lyapunov functions in order to prove stability, the following argument shows that this assumption is natural: We start with initial condition* $x(0) = x_0 \in D$ *such that* $V(x_0) = c \in \mathbb{R}, c > 0$. *Since* $\dot{V}(x(t, x_0)) \leq 0, \forall t \geq 0$, *we know that* V *is a decreasing function and* $V(x(t, x_0)) \leq V(x_0), \forall t \geq 0$. *But this means, that there exists a compact set* $B := \{x \in D \mid V(x) \leq c\}$ *such that, with* $V(x_0) = c$ *for any* $x_0 \in D$, *the solution remains in* B, *i.e.* $x(t, x_0) \in B, \forall t \geq 0$. *Therefore it is clear that the solution* $x(t, x_0)$ *has domain* $[t_0, \infty)$.

Other interesting properties of solutions involve the notion of positively invariant sets and allow for further investigation of stability of (3.1) via LaSalle's Invariance Principle.

Definition 3.3 *A non-empty subset* $M \subset \mathbb{R}^n$ *is called* positively invariant *for* (3.1), *if for each* $x_0 \in M$ *the solution* $x(t, x_0) \in M$ *for all* $t \geq 0$, *i.e. the solution remains in* M.

Theorem 3.2 (LaSalle's Invariance Principle) *Let* $V : D \to \mathbb{R}$ *be a* C^1 *function and* $\dot{V}(x)$ *be negative semidefinite for all* $x \in D$. *Let* $x(t, x_0), t \geq 0$ *be a solution of* (3.1). *Suppose there exists a compact (closed and bounded) set* B *that is positively invariant with respect to* (3.1). *Then for* $t \to \infty$, *the solution* $x(t, x_0)$ *starting in* B *converges to the largest subset of* $\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\} \cap B$ *that is positively invariant for* (3.1).

With this prerequisite, we can give a corollary that allows us to conclude (global) asymptotic stability although $\dot{V}$ in Theorem 3.1 is merely negative semidefinite.

Corollary 3.1 (Barbashin and Krasovskii) *Let $V : D \to \mathbb{R}$ be a positive definite C^1 function, $\dot{V}(x)$ be negative semidefinite for all $x \in D$ and $x = 0$ be an equilibrium point of (3.1). Suppose that no solution can stay in $\{x \in D \mid \dot{V}(x) = 0\}$ other than the trivial solution $x(t, x_0) = 0, \forall t \geq 0$. Then, the equilibrium point $x = 0$ is asymptotically stable. Furthermore, for $D = \mathbb{R}^n$ and V being radially unbounded, the equilibrium point $x = 0$ is globally asymptotically stable.*

3.1.2 Time-varying systems

We start with the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}^+ \tag{3.2}$$

with C^1 function $f : \mathbb{R}^+ \times D \to \mathbb{R}^n$ (i.e. C^1 in both arguments) with open connected set $D \subset \mathbb{R}^n$ which contains the origin $x = 0$ and $\mathbb{R}^+ := [0, \infty)$. Again, we assume that we have a unique forward complete solution $x(\cdot, t_0, x_0)$ for the initial value problem with initial condition $x(t_0) = x_0 \in D, t_0 \geq 0$, i.e. the solution has domain $[0, \infty)$.

We again start with the definiton of Lyapunov stability, give a theorem in terms of a test function V and discuss some subtleties arising for nonlinear time-varying systems which do not appear in the time-invariant case.

A nonlinear time-varying system has an equilibrium point at $x = 0$ for $t = t_0$ if

$$f(t, 0) = 0, \quad \forall t \geq t_0 \geq 0. \tag{3.3}$$

Definition 3.4 (Lyapunov Stability) *Let $x = 0$ be an equilibrium point of (3.2) for $t = t_0$. The equilibrium point $x = 0$ is called*

- stable, *if* $(\forall \epsilon > 0)(\forall t_0 \in \mathbb{R}^+)(\exists \delta > 0)(\forall x_0 \in B_\delta)(\forall t \geq t_0, t \in \mathbb{R}^+)$: $||x(t, t_0, x_0)|| < \epsilon$.
- uniformly stable, *if* $(\forall \epsilon > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R}^+)(\forall x_0 \in B_\delta)(\forall t \geq t_0, t \in \mathbb{R}^+)$: $||x(t, t_0, x_0)|| < \epsilon$.

- uniformly asymptotically stable, *if it is uniformly stable and* $(\exists\eta > 0)(\forall x_0 \in B_\eta)(\forall t_0 \in \mathbb{R}^+) \; : \; \lim_{t\to\infty} x(t, t_0, x_0) = 0.$

- globally uniformly asymptotically stable, *if it is uniformly stable, and for* $D = \mathbb{R}^n$, $(\forall\eta > 0)(\forall x_0 \in B_\eta)(\forall t_0 \in \mathbb{R}^+) \; : \; \lim_{t\to\infty} x(t, t_0, x_0) = 0.$

- unstable, *if it is not stable.*

Theorem 3.3 *Let* $V : \mathbb{R}^+ \times D \to \mathbb{R}$ *be a* C^1 *function with respect to both arguments and be* $x = 0$ *an equilibrium point of* (3.2) *at* $t = t_0$*. Then the equilibrium point* $x = 0$ *is*

- *uniformly stable, if for all* $t \geq t_0 \in \mathbb{R}^+$ *and for all* $x \in D$

$$\underline{V}(x) \leq V(t, x) \leq \overline{V}(x) \tag{3.4}$$

$$\dot{V}(t, x) \leq 0 \tag{3.5}$$

  and $\dot{V}(t, 0) = 0, \forall t \geq t_0 \in \mathbb{R}^+$ *with continuous positive definite functions* $\underline{V}, \overline{V} : D \to \mathbb{R}$.

- *uniformly asymptotically stable if it is uniformly stable and* $\dot{V}(t, x) \leq -\tilde{V}(x), \forall t \geq t_0, \forall x \in D$ *with continuous positive definite function* $\tilde{V} : D \to \mathbb{R}$.

- *uniformly globally asymptotically stable if it is uniformly asymptotically stable with* $D = \mathbb{R}^n$ *and* $\underline{V}$ *is radially unbounded such that* $||x|| \to \infty \Rightarrow \underline{V}(x) \to \infty$.

Since $\underline{V}$ is radially unbounded and inequality (3.4) holds, the last statement is also valid for both functions $\underline{V}, \overline{V}$ being radially unbounded.

Again, we use the notions from [48]: We call V a *Lyapunov function candidate*, if it is C^1 and $\underline{V}(x) \leq V(t, x) \leq \overline{V}(x), \forall t \geq t_0 \in \mathbb{R}^+, \forall x \in D$ with continuous positive definite functions $\underline{V}, \overline{V} : D \to \mathbb{R}$. If it additionally fulfills the conditions for stability, asymptotic stability or global asymptotic stability, we call V a *Lyapunov function.*

Condition (3.4) is very important in time-varying Lyapunov theory and the weaker condition $V(t, x) > 0, \forall x \in D \setminus \{0\}, V(t, 0) = 0, \forall t \geq 0$ is not sufficient, which will be discussed in the following example from [46]:

Consider the scalar differential equation

$$\dot{x}(t) = x(t), \quad t \in \mathbb{R}^+,$$

$x(t) \in \mathbb{R}, x(0) = x_0$, which is obviously unstable at the origin. Take the function $V(t,x)$ defined by

$$V(t,x) = \begin{cases} x^{e^t} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ V(t,-x) & \text{for } x < 0 \end{cases}.$$

As can be seen, $V(t,x)$ is positive for $x \neq 0$. Moreover, it is differentiable for $x \neq 0$ and the derivative of V for $x > 0$ is given by

$$\dot{V}(t,x) = V_t(t,x) + V_x(t,x)\dot{x} = e^t x^{e^t} \ln x + e^t x^{e^t - 1} x = e^t x^{e^t}(\ln x + 1).$$

The derivative $\dot{V}$ is negative for $x \in (0, \frac{1}{e})$, and hence V is decreasing along nontrivial solutions of the differential equation although we know that the solutions are unstable. This comes from the fact that V does not admit a uniform estimate $\underline{V}$ from below.

In addition, there is an example in [45] on p. 27, where it can be seen that the uniform estimate $\overline{V}$ from above is necessary, too.

3.2 Comparison Functions and Stability

It is possible to rephrase the stability definitions of Section 3.1 in a more elegant way using auxiliary functions, which render the time-varying notions more apparent and are used later to introduce Input-to-State Stability (ISS) and integral Input-to-State Stability (iISS).

Let us first define some basic auxiliary functions introduced by Hahn [49] which prove to be very useful in the context of stability theory. According to the book of Hahn, definitions are given in the standard textbook of Khalil [44] which we use here because of better readability.

Definition 3.5 *[44] A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 3.6 *[44] A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$, is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.*

Consider a time-varying system without inputs, i.e. $\dot{x}(t) = f(t, x(t))$ as was considered in (3.2). Then we can reformulate the stability definitions from Section 3.1 in terms of auxiliary functions within the following lemma:

Lemma 3.1 *The equilibrium point $x = 0$ of (3.2) at $t = t_0$ is*

- *uniformly stable if and only if there exists a class $\mathcal{K}$ function α and $(\exists\gamma > 0)(\forall t_0 \in \mathbb{R}^+)(\forall t \geq t_0, t \in \mathbb{R}^+)(\forall x_0 \in B_\gamma) \ : \ ||x(t, t_0, x_0)|| \leq \alpha(||x_0||)$.*
- *uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function β and $(\exists\gamma > 0)(\forall t_0 \in \mathbb{R}^+)(\forall t \geq t_0, t \in \mathbb{R}^+)(\forall x_0 \in B_\gamma) \ : \ ||x(t, t_0, x_0)|| \leq \beta(||x_0||, t - t_0)$.*
- *globally uniformly asmyptotically stable if and only if for $D = \mathbb{R}^n$, there exists a class $\mathcal{KL}$ function β and $(\forall\gamma > 0)(\forall t_0 \in \mathbb{R}^+)(\forall t \geq t_0, t \in \mathbb{R}^+)(\forall x_0 \in B_\gamma) \ : \ ||x(t, t_0, x_0)|| \leq \beta(||x_0||, t - t_0)$.*

3.3 Input-to-State Stability (ISS)

3.3.1 Conceptual framework

Consider now the time-varying nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \in \mathbb{R}^+ \tag{3.6}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with piecewise continuous, bounded real function $u(\cdot)$ and C^1 function $f(t, x, u)$ (which means C^1 in t, x, u). Assume furthermore that the uniquely defined solution $x(t, t_0, x_0, u(\cdot))$ with $t \geq t_0 \in \mathbb{R}^+, x(t_0) = x_0 \in D$ is forward complete, i.e. has domain $[0, \infty)$. The point $x = 0$ is assumed to be an equilibrium point of (3.6) at t_0, i.e. $f(t, 0, 0) = 0, \forall t \geq t_0 \geq 0$. Again, we try to avoid the time argument in the following where possible to reduce the complexity of notation.

The ISS concept originally was developed by the pionieering work of Eduardo Sontag. The starting point was the paper [50], where the stability notion of Input-to-State Stability was first defined and used, and has led to immense research activities up to now.

A compilation of results can be found in [51], which is short in presentation and focused on summarizing basic concepts and results since it was part of a course at a summer school. The last section on ISS theory in this book contains an overview of research fields where the stability concepts are adopted to e.g. switching systems, nonlinear time-varying systems, etc. For more information we recommend to stick to the original articles, since in the above mentioned book there are often only sketches of the proofs. In addition, some of the mathematical definitions are different from those in the papers they refer to.

The main idea arises from the fact that the standard global asymptotic stability (GAS) property for linear systems does not naturally extend to nonlinear systems when considering inputs. In the linear case we know that an asymptotically stable system will have a bounded state when applying a bounded input. For nonlinear systems, GAS will not result in bounded-input bounded-output (BIBO) or bounded-input bounded-state (BIBS) stability, as easy examples show. Consider the scalar system taken from [51]

$$\dot{x}(t) = -x(t) + (1 + x(t)^2)u(t), \quad x(t_0) = x_0, \quad x(t), u(t) \in \mathbb{R}.$$

For $u \equiv 0$ the system $\dot{x}(t) = -x(t)$ is clearly GAS. But for bounded input $u(t) = (2t+2)^{-\frac{1}{2}}$ and initial condition $x_0 = \sqrt{2}$ the solution is $x(t, x_0) = (2t+2)^{\frac{1}{2}}$ which is unbounded, since $\lim_{t\to\infty} x(t, x_0) = \infty$. In other words, the converging-input converging-state property is not valid in the nonlinear case.

In the sequel, we will present the main idea behind the ISS concept for time-varying systems, which is a straightforward extension of the historically time-invariant formulation. Sketches of the strategy can be found in [27, 51], but a complete argumentation is not available in the literature. For this reason, we attempt a comprehensive presentation for the time-varying case.

The derivation starts with linear time-varying systems and uses nonlinear state and input transformations to render the system nonlinear. Choosing appropriate function spaces for the input functions and assuming asymptotic stability for the linear time-varying system, it is possible to extract appropriate notions for the transformed nonlinear time-varying system. Finally, the same requirements are used for the definition of ISS in the case of the most general class of time-varying systems in order to be an adequate extension of the linear theory.

We start with the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,$$

$A(\cdot), B(\cdot), u(\cdot)$ piecewise continuous and bounded, where the solution $x(t, t_0, x_0, u(\cdot))$ can be written as

$$x(t, t_0, x_0, u(\cdot)) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t - s, t_0)B(s)u(s)ds$$

where $\Phi(t, t_0)$ is the fundamental matrix. Since we assumed that the input functions are piecewise continuous and bounded, we know that the L_∞ norm exists and therefore introduce the notation $||u(\cdot)||_\infty = ||u(\cdot)||_{[t_0,\infty]} = \sup_{s\in[t_0,\infty]} ||u(s)||$ with arbitrary vector norm $||\cdot||$. Furthermore, the homogeneous system is supposed to be uniformly asymptotically stable, such that the norm bound of the input-free solution is finite. Since a linear time-varying system is uniformly asymptotically stable if and only if it is uniformly exponentially stable, the estimate $||\Phi(t, t_0)||_2 \leq ce^{-\lambda(t-t_0)}, \forall t \geq t_0 \geq 0$ with $\lambda, c > 0$ is valid. Herefrom, it follows that there exists a finite constant $\bar{c} > 0$ such that $\int_{t_0}^{t} ||\Phi(t - s, t_0)B(s)||_2 \, ds \leq \bar{c} < \infty, \forall t \geq t_0 \geq 0$, i.e. the integral is finite. Then the norm estimate of the solution

$$||x(t, t_0, x_0, u(\cdot))||_2 \leq ||\Phi(t, t_0)||_2 \, ||x_0||_2 + \int_{t_0}^{t} ||\Phi(t - s, t_0)B(s)||_2 \, ds \, ||u(\cdot)||_\infty$$

is well defined and we finally arrive at

$$||x(t, t_0, x_0, u(\cdot))||_2 \leq ce^{-\lambda(t-t_0)}||x_0||_2 + \bar{c}||u(\cdot)||_\infty, \, \forall t \geq t_0 \geq 0$$

where $\lambda, c, \bar{c} > 0$.

To make the step to nonlinear systems, we define at first the invertible state transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n, \quad \text{with } T, T^{-1} \text{ continuous}, T(0) = 0,$$

and the functions $\underline{\alpha} : \mathbb{R} \to \mathbb{R}, \overline{\alpha} : \mathbb{R} \to \mathbb{R}$ with

$$\underline{\alpha}(r) := \min_{||x||_2 \geq r} ||T(x)||_2 \text{ and } \overline{\alpha}(r) := \max_{||x||_2 \leq r} ||T(x)||_2$$

which are well-defined since T and its inverse are continuous. Both functions are of class $\mathcal{K}_\infty$ and we can write

$$\underline{\alpha}(||x||_2) \leq ||T(x)||_2 \leq \overline{\alpha}(||x||_2), \quad \forall x \in \mathbb{R}^n.$$

For inputs, we introduce additionally a nonlinear invertible input transformation

$$S : \mathbb{R}^m \to \mathbb{R}^m, \qquad \text{with } S, S^{-1} \text{ continuous}, S(0) = 0$$

and functions $\underline{\gamma} : \mathbb{R} \to \mathbb{R}, \overline{\gamma} : \mathbb{R} \to \mathbb{R}$

$$\underline{\gamma}(r) := \min_{||u|| \geq r} ||S(u)|| \text{ and } \overline{\gamma}(r) := \max_{||u|| \leq r} ||S(u)||$$

such that

$$\underline{\gamma}(||u||) \leq ||S(u)|| \leq \overline{\gamma}(||u||), \quad \forall u \in \mathbb{R}^m$$

equipped with an arbitrary vector norm $||\cdot||$ (which does not matter since the different norms are equivalent in finite dimensions).

Now perform the state transformation $x = T(z)$ and the input transformation $u = S(v)$. We identifiy

$$x(t, t_0, x_0, u(\cdot)) = x(t, t_0, T(z_0), S(v(\cdot))) = T(z(t, t_0, z_0, v(\cdot)))$$

such that

$$\underline{\alpha}(||z(t, t_0, z_0, v(\cdot))||_2) \leq ||T(z(t, t_0, z_0, v(\cdot)))||_2 = ||x(t, t_0, x_0, u(\cdot))||_2$$

$$ce^{-\lambda(t-t_0)}||x_0||_2 = c||T(z_0)||_2 e^{-\lambda(t-t_0)} \leq c\overline{\alpha}(||z_0||_2)e^{-\lambda(t-t_0)} = \beta(||z_0||_2, t - t_0)$$

$$\bar{c} \sup_{s \in [t_0, \infty]} ||u(s)|| = \sup_{s \in [t_0, \infty]} \bar{c}||S(v(s))|| \leq \sup_{s \in [t_0, \infty]} \overline{\gamma}(||v(s)||)$$

which results in

$$\underline{\alpha}(||z(t, t_0, z_0, v(\cdot))||_2) \leq \beta(||z_0||_2, t - t_0) + \sup_{s \in [t_0, \infty]} \overline{\gamma}(||v(s)||)$$

$$= \beta(||z_0||_2, t - t_0) + ||\overline{\gamma}(||v(\cdot)||)||_\infty.$$

We want to apply the increasing function $\underline{\alpha}^{-1}$ on both sides of the last equation to have the norm of the solution on the left hand side. Using

$$\underline{\alpha}^{-1}(a+b) \leq \underline{\alpha}^{-1}(2a) + \underline{\alpha}^{-1}(2b)$$

with $a, b \in \mathbb{R}$ and that $||\overline{\gamma}(||u(\cdot)||)||_\infty = \overline{\gamma}(||u(\cdot)||_\infty)$, we get

$$\begin{aligned} ||z(t,t_0,z_0,v(\cdot))||_2 &\leq \underbrace{\underline{\alpha}^{-1}(2\beta(||z_0||_2, t-t_0))}_{=:\, \tilde{\beta}(||z_0||_2, t-t_0)} + \underbrace{\underline{\alpha}^{-1}(2\overline{\gamma}(||v(\cdot)||_\infty))}_{=:\, \tilde{\gamma}(||v(\cdot)||_\infty)} \\ ||z(t,t_0,z_0,v(\cdot))||_2 &\leq \tilde{\beta}(||z_0||_2, t-t_0) + \tilde{\gamma}(||v(\cdot)||_\infty). \end{aligned}$$

To sum up, it can be seen that a nonlinear state and input transformation of a linear time-varying system leads to a state estimate similar to what we know from asymptotically stable linear systems: The first part on the right hand side describes the decay of the solution when starting with a nonzero initial condition, while the second part describes the input-induced system excitation. If this condition holds for the (nonlinearly) transformed solution of the linear time-varying system, the converging-input converging-state property of linear system is preserved for this special class of nonlinear systems.

This calculation provides insight into how a meaningful stability concept for general time-varying system should be posed. It is an elegant combination of input-output stability and Lyapunov theory of nonlinear systems without inputs (which comes into play in the following).

3.3.2 Basic definitons

The result from the above derivation is used to define stability for general time-varying systems with inputs:

Definition 3.7 (ISS) *System (3.6) is Input-to-State Stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that, for each input $u \in L_\infty^m$, each $x_0 \in \mathbb{R}^n$, each $t_0 \geq 0$ and each $t \geq t_0 \geq 0$*

$$||x(t,t_0,x_0,u(\cdot))||_2 \leq \beta(||x_0||_2, t-t_0) + \gamma(||u(\cdot)||_{[t_0,\infty]}). \tag{3.7}$$

Note that ISS systems are always forward complete. In addition, an immediate consequence of ISS is that inputs with $u(t) \to 0$ for $t \to \infty$ lead to solutions $x(t,t_0,x_0,u(\cdot)) \to 0$ for $t \to \infty$.

Remark 3.3 *The original definiton for time-varying systems in [52] for ISS and ISS Lyapunov functions is more involved than we would need to consider in the following. Therefore, we adopt the definitions of [53] since they better fit the conditions derived for the presented control approach.*

Just as the definitions in standard stability theory, this definition is difficult to check, since it requires the knowledge of the solution of a nonlinear time-varying system. Therefore, the definitions are rephrased in terms of test functions, i.e. specific types of Lyapunov-like functions are introduced.

Definition 3.8 (UPPD) *A function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is called uniformly proper and positive definite ($V \in$ UPPD) if $\underline{\alpha}, \overline{\alpha}, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ exist such that*

$$\underline{\alpha}(||x||_2) \leq V(t,x) \leq \overline{\alpha}(||x||_2) \tag{3.8}$$

$$||V_t(t,x)||_2 \leq \gamma_1(||x||_2), ||V_x(t,x)||_2 \leq \gamma_2(||x||_2) \tag{3.9}$$

for each $t \geq 0$ and each $x \in \mathbb{R}^n$.

Definition 3.9 (Strict ISS Lyapunov function) *A function $V \in C^1 \cap$ UPPD is a strict ISS Lyapunov function for (3.6) if there exist $\chi \in \mathcal{K}_\infty$ and $\alpha \in C^1 \cap \mathcal{K}_\infty$ such that for each t, each $x \in \mathbb{R}^n$ and each $u \in \mathbb{R}^m$*

$$||x||_2 \geq \chi(||u||_2) \Longrightarrow V_t(t,x) + V_x(t,x)f(t,x,u) \leq -\alpha(||x||_2). \tag{3.10}$$

Definition 3.10 (Strict DIS Lyapunov function) *A function $V \in C^1 \cap$ UPPD is called strict DIS (strict dissipative) Lyapunov function for (3.6) if there exist $\sigma \in \mathcal{K}_\infty$ and $\alpha \in C^1 \cap \mathcal{K}_\infty$ such that for each $t \geq 0$, each $x \in \mathbb{R}^n$ and each $u \in \mathbb{R}^m$*

$$V_t(t,x) + V_x(t,x)f(t,x,u) \leq -\alpha(||x||_2) + \sigma(||u||_2). \tag{3.11}$$

Theorem 3.4 *The following statements are equivalent:*

1. *System (3.6) admits a strict ISS Lyapunov function.*
2. *System (3.6) admits a strict DIS Lyapunov function.*
3. *System (3.6) is ISS.*

3.4 Integral Input-to-State Stability (iISS)

In practice, it can be demanding to find an $\alpha \in \mathcal{K}_\infty$, while finding an α being merely positive definite is much easier to achieve. Therefore, the following stability concept will be shown to be much more convenient in our context. Assume that the solution $x(t, t_0, x_0, u(\cdot))$ of the linear system $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $A(\cdot), B(\cdot), u(\cdot)$ piecewise continuous and bounded, fulfills

$$||x(t, t_0, x_0, u(\cdot))||_2 \leq ce^{-\lambda(t-t_0)}||x_0||_2 + c\int_{t_0}^{t} ||u(\tau)||_2 d\tau, \quad \forall t \geq t_0 \geq 0$$

where $0 < \lambda, c \in \mathbb{R}$.

Performing the state and input transformation (for which we choose now the 2-norm) from before, we get

$$\underline{\alpha}(||z(t, t_0, z_0, v(\cdot))||_2) \leq ||T(z(t, t_0, z_0, v(\cdot)))||_2 = ||x(t, t_0, x_0, u(\cdot))||_2$$

$$ce^{-\lambda(t-t_0)}||x_0||_2 = c||T(z_0)||_2 e^{-\lambda(t-t_0)} \leq c\overline{\alpha}(||z_0||_2)e^{-\lambda(t-t_0)} = \beta(||z_0||_2, t - t_0)$$

$$c\int_{t_0}^{t} ||u(\tau)||_2 d\tau = \int_{t_0}^{t} c||S(v(\tau))||_2 d\tau \leq \int_{t_0}^{t} \overline{\gamma}(||v(\tau)||_2) d\tau$$

which results in

$$\underline{\alpha}(||z(t, t_0, z_0, v(\cdot))||_2) \leq \beta(||z_0||_2, t - t_0) + \int_{t_0}^{t} \overline{\gamma}(||v(\tau)||_2) d\tau.$$

This leads to the following definition:

Definition 3.11 (iISS) *System (3.6) is called iISS (integral Input-to-State Stable) if there exist $\gamma, \mu \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ for which*

$$\mu(||x(t, t_0, x_0, u(\cdot))||_2) \leq \beta(||x_0||_2, t - t_0) + \int_{t_0}^{t} \gamma(||u(\tau)||_2) d\tau \qquad (3.12)$$

is satisfied for each $t_0 \geq 0$, each $t \geq t_0 \geq 0$, each $x_0 \in \mathbb{R}^n$ and each $u \in \mathbb{R}^m$.

Again, we want to have a Lyapunov-like formulation of the iISS stability definition to be able to derive constructive theorems in order to prove stability. We follow [28].

Definition 3.12 (iISS Lyapunov function) *A function $V \in C^1 \cap UPPD$ with $\underline{\alpha}(||x||_2) \leq V(t,x) \leq \overline{\alpha}(||x||_2), \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ is called an iISS Lyapunov function for (3.6) if there exist $\Delta \in \mathcal{K}_\infty$ and a positive definite function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $t \geq 0$, for each $x \in \mathbb{R}^n$ and each $u \in \mathbb{R}^m$*

$$V_t(t,x) + V_x(t,x)f(t,x,u) \leq -\nu(||x||_2) + \Delta(||u||_2). \tag{3.13}$$

Finally, the link between iISS Lyapunov functions and iISS is given within the following theorem.

Theorem 3.5 *If (3.6) admits an iISS Lyapunov function, then it is iISS.*

In order to highlight the difference between ISS and iISS, we will discuss a simple example that shows that both notions do not coincide in general. As prerequisite, we need an additional theorem which characerizes the stability properties of the class of bilinear systems

$$\dot{x}(t) = \left(A + \sum_{i=1}^{m} A_i u_i(t) \right) x(t) + Bu(t), \quad t \in \mathbb{R}^+ \tag{3.14}$$

with $A, A_i \in \mathbb{R}^{n \times n}, i = 1, \dots m, B \in \mathbb{R}^{n \times m}$ and $u = (u_1, u_2, \dots, u_m)^{\mathrm{T}}$:

Theorem 3.6 *[27] System (3.14) is iISS if and only if the matrix A is Hurwitz.*

Remark 3.4 *The notion "A is Hurwitz" means that all eigenvalues of A have negative real part.*

Consider system [27]

$$\dot{x}(t) = -x(t) + x(t)u(t), \quad t \in \mathbb{R}^+,$$

$x(t) \in \mathbb{R}$, with initial condition $x(0) = x_0$ and bounded, piecewise continuous inputs $u(\cdot)$. This system is clearly iISS, since it is bilinear and the system Matrix $A = -1$ is Hurwitz. But for constant (and therefore bounded) input $u(t) \equiv 2$, the closed-loop system reduces to $\dot{x}(t) = x(t)$ with solution $x(t, x_0) = e^t x_0$, which cannot be bounded as would be required in Definition 3.7 for achieving ISS. From this example it can be seen that iISS allows the right hand side to tend to infinity, which would be not helpful for stability considerations.

To avoid this, the following conclusion similar to the ISS case can be drawn:

For systems which fulfill Definition 3.11 for iISS, input functions with $\int_{t_0}^{\infty} \gamma(||u(\tau)||_2)d\tau < \infty$ induce solutions $x(t, t_0, x_0, u(\cdot)) \to 0$ for $t \to \infty$ (the proof for time-invariant systems is given in [27] and can be extended accordingly).

Chapter 4

$\mathcal{H}_\infty$ Control for Nonlinear Systems

In this chapter we revisit the basic dissipativity and $\mathcal{H}_\infty$ control theory for nonlinear time-invariant and time-varying systems. We will later use this theory to find the suboptimal state feedback control law for the proposed class of power converter systems with integral and disturbance feedback. In addition, we want to discuss stability properties of $\mathcal{H}_\infty$ control systems in the context of what was introduced in the preceding chapter. Finally, we present some additional properties in the time-varying case which we need for stability considerations and control design of the considered power converter systems.

At the beginning, we briefly explain the notion $\mathcal{H}_\infty$ in the context of linear systems so as to be able to discuss the slightly misleading terminology common in the context of nonlinear systems. Then, we introduce dissipativity, L_2-gain and nonlinear $\mathcal{H}_\infty$ control for time-invariant systems and discuss their stability properties which are well-established in the literature [3],[54],[55],[4],[5],[6],[7]. The next part covers the theory for time-varying systems from [24], [25] and links the iISS stability results from the preceding chapter to L_2-gain properties and dissipativity theory.

4.1 Basics of $\mathcal{H}_\infty$ Control for Linear Systems

Before starting with the nonlinear theory itself, we briefly discuss the notion $\mathcal{H}_\infty$ used in the setting of nonlinear systems.

Historically, it was common in linear optimal control to use the Laplace transform and discuss properties of transfer functions. Later on, the equivalent formulation was settled in the time domain [56]. When working with transfer functions, appropriate function spaces for the transfer functions and the arising signals have to be introduced and the induced norms have to be derived. Therefrom, the notion $\mathcal{H}_\infty$, which is a function space for a specific type of transfer functions, gave the developed theory its name. Let us present some basic definitions following [57], to briefly introduce the main idea of the linear theory and pave the way to the nonlinear setting.

Definition 4.1 *A complex valued function $F(s) \in \mathbb{C}$ with complex variable $s \in \mathbb{C}$ is bounded, if there exists a finite real number $b \in \mathbb{R}$ such that*

$$|F(s)| \leq b, \quad \text{for all } s \text{ with } Re(s) > 0.$$

Definition 4.2 (Hardy Space $\mathcal{H}_\infty$) *The Hardy space $\mathcal{H}_\infty$ consists of all complex valued functions $F(s) \in \mathbb{C}$ of a complex variable $s \in \mathbb{C}$ which are analytic and bounded in the open right half-plane, i.e. for $Re(s) > 0$.*

Definition 4.3 ($\mathcal{H}_\infty$-Norm) *The $\mathcal{H}_\infty$-norm of F is defined by*

$$||F||_\infty := \sup\{|F(s)| \ : \ Re(s) > 0\}$$

which is the least upper bound on $|F(s)|$.

Assume now that $F(s)$ is a transfer function with Laplace variable s, which reflects the input-output behavior of the system, since it maps the input signal $U(s)$ to the output signal $Y(s)$ via $Y(s) = F(s)U(s)$, where $U(s)$ is the Laplace transform of the input $u(\cdot)$ and $Y(s)$ the Laplace transform of the output $y(\cdot)$, where $u(\cdot), y(\cdot)$ are supposed to be L_2-functions. Then it can be shown that the induced norm is the $\mathcal{H}_\infty$ norm [41] and the following estimate is valid:

$$||y||_2^2 \leq ||F||_\infty ||u||_2^2. \tag{4.1}$$

For nonlinear systems there are no transfer functions and therefore, instead of Hardy spaces, we need a different function space. Therefore, we look at the input-output behavior in the time-domain, where the appropriate space is the L_2 function space. We get an information on the nonlinear input-output behavior by introducing a constant $\gamma > 0, \gamma \in \mathbb{R}$ reflecting the finite L_2-gain $\leq \gamma$ from input $u(t)$ to output $y(t)$

$$||y||_2^2 \leq \gamma^2 ||u||_2^2. \tag{4.2}$$

If we find the smallest $\gamma^+ > 0, \gamma^+ \in \mathbb{R}$ with $\gamma^+ < \gamma$ such that inequality (4.2) is valid, we finally know the finite L_2-gain of the input-output behavior which is the nonlinear extension of inequality (4.1). Nonetheless, we still use the historically common denotation "nonlinear $\mathcal{H}_\infty$ control", since this term gives the reader the information that the L_2-gain theory is the nonlinear counterpart of linear $\mathcal{H}_\infty$ control.

4.2 Time-Invariant Systems

First of all, we consider dissipativity theory and nonlinear $\mathcal{H}_\infty$ control for time-invariant systems. The content of this section is a summary of what can be found in the standard texts [3],[54],[55] and references therein.

4.2.1 Dissipativity

Consider the general nonlinear system

$$\begin{aligned} \dot{x}(t) =& f(x(t), u(t)), \quad t \in \mathbb{R}^+ \\ y(t) =& h(x(t), u(t)), \end{aligned} \tag{4.3}$$

with C^1 functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, (which means C^1 in x, u), $f(0,0) = 0, h(0,0) = 0$ and piecewise continuous and bounded real function $u(\cdot)$. Furthermore we assume the we have a unique forward complete solution $x(t, x_0, u(\cdot))$ for the initial value problem with initial condition $x(0) = x_0$. As can be seen, the input-free system has an equilibrium point at $x = 0$ because of $f(0,0) = 0$. In the following we try to avoid the time argument where possible to reduce the complexity of the notation.

The function

$$s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}, \quad (u, y) \mapsto s(u, y) \tag{4.4}$$

is called *supply rate* and is assumed to be locally integrable.

Definition 4.4 *System* (4.3) *is said to be* dissipative with respect to the supply rate s *if there exists a positive semidefinite function* $V : \mathbb{R}^n \to \mathbb{R}$*, such that*

$$V(x(T)) \leq V(x_0) + \int_0^T s(u(t), y(t))dt \tag{4.5}$$

for each $x_0 \in \mathbb{R}^n$*, each* $T \geq 0$ *and all real piecewise continuous and bounded input functions* $u(\cdot)$ *with initial condition* $x(0) = x_0$*,* $y(t) = h(x(t, x_0, u(\cdot)), u(t))$ *and where* $x(T) = x(T, x_0, u(\cdot))$ *under abuse of notation. The function* V *is called* storage function *and* (4.5) *is the* dissipation inequality.

Definition 4.5 *The function* $u(\cdot)$ *is an element of* $L_2[0, T]$*, if*

$$\int_0^T ||u(t)||_2^2 \, dt < \infty \tag{4.6}$$

is fulfilled.

Definition 4.6 *Let* $\gamma > 0$*. System* (4.3) *has finite* L_2*-gain* $\leq \gamma$ *from* u *to* y*, if for all* $x_0 \in \mathbb{R}^n$ *there exists a constant* $K(x_0), 0 \leq K(x_0) < \infty$ *with* $K(0) = 0$*, such that*

$$\int_0^T ||y(t)||_2^2 \, dt \leq \gamma^2 \int_0^T ||u(t)||_2^2 \, dt + K(x_0) \tag{4.7}$$

for all $T \geq 0$ *and for all* $u \in L_2[0, T]$ *with* $y(t) = h(x(t, x_0, u(\cdot)), u(t)), x(0) = x_0$*. The finite* L_2*-gain of the system in* (4.3) *is defined as*

$$\gamma^+ := \inf\{\gamma \mid \textit{the system in } (4.3) \textit{ has finite } L_2\textit{-gain} \leq \gamma\}.$$

It is possible to rewrite what is stated in Definition 4.6 in terms of dissipativity:

Lemma 4.1 *System* (4.3) *has finite* L_2*-gain* $\leq \gamma$ *from* u *to* y *if for each* $u \in \mathbb{R}^m$ *and each* $y \in \mathbb{R}^p$ *it is dissipative with respect to the* finite L_2-gain supply rate $s(u, y) = \frac{1}{2}\gamma^2||u||_2^2 - \frac{1}{2}||y||_2^2$.

Proof 4.1 *Let*

$$V(x(T)) - V(x(0)) \leq \frac{1}{2} \int_0^T (\gamma^2||u(t)||_2^2 - ||y(t)||_2^2) \, dt.$$

Since V *is positive semidefinite, we can rephrase the former inequality to*

$$\int_0^T ||y(t)||_2^2 \, dt \leq \gamma^2 \int_0^T ||u(t)||_2^2 \, dt + 2V(x_0)$$

which is the definition of finite L_2*-gain, with* $K(x_0) = 2V(x_0)$ *finite and* $K(0) = 0$ *since* V *is positive semidefinite and therefore* $V(0) = 0$.

Up to now the statements are based on integral inequalities which involve the knowledge of the solution of the nonlinear system (4.3). This would be a big disadvantage of the theory, since the solution is usually not known in the nonlinear case. However, if we require the functions V to be even continuously differentiable (which we assume in the following), it is possible to derive the so called *differential dissipation inequality*:

Lemma 4.2 *Let V be continously differentiable. Then system (4.3) is dissipative with respect to the supply rate s if and only if*

$$\dot{V}(x) \leq s(u, h(x,u)), \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{4.8}$$

Proof 4.2 *One direction of the statement follows directly from differentiation of (4.5). The other direction starts with (4.8) and integration on both sides from 0 to T along a solution $x(t, x_0, u(\cdot))$ of (4.3) with associated input $u(\cdot)$ and resulting output $y(t) = h(x(t, x_0, u(\cdot)), u(t))$ leads to*

$$V(x(T)) - V(x(0)) \leq \int_0^T s(u(t), h(x(t, x_0, u(\cdot)), u(t)))\, dt.$$

However, this is nothing else than the inequality arising in Definition 4.4, and therefore the proof is complete.

The differential dissipation inequality is easier to handle than the dissipation inequality (4.5) and allows the discussion of stability properties in terms of Lyapunov functions, which is investigated in the next section.

4.2.2 Stability

Lemma 4.3 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a positive definite C^1 storage function for (4.3) such that (4.8) is satisfied. Assume that the supply rate s satisfies*

$$s(0, y) \leq 0 \quad \forall y \in \mathbb{R}^p. \tag{4.9}$$

Then $x = 0$ is a stable equilibrium of the unforced system $\dot{x} = f(x, 0)$. Furthermore, suppose that no solution of $\dot{x} = f(x, 0)$ other than $x(t, x_0, 0) \equiv 0$ remains in $\{x \in \mathbb{R}^n \mid s(0, h(x, 0)) = 0\}$ for all $t \geq 0$. Then $x = 0$ is an asymptotically stable equilibrium, which is globally asymptotically stable if V is radially unbounded.

Proof 4.3 *With (4.8) and (4.9) we know that* $V_x(x)f(x,0) \leq s(0,h(x,0)) \leq 0, \forall x \in \mathbb{R}^n$*, i.e. the equilibrium* $x = 0$ *is Lyapunov stable and with the assumption that merely the trivial solution* $x(t,x_0,0) \equiv 0$ *remains in the set* $\{x \in \mathbb{R}^n \mid s(0,h(x,0)) = 0\}$*, we know from Corollary 3.1 that* $x = 0$ *is asymptotically stable, and for* V *radially unbounded globally asymptotically stable.*

4.2.3 State feedback $\mathcal{H}_\infty$ control

Dissipativity theory as introduced above copes with L_2-gain properties of the input-output behavior. But so far we did not discuss how to find a state feedback law that provides the minimal $\gamma^+ > 0, \gamma^+ \in \mathbb{R}$ with $\gamma^+ < \gamma$ which characterizes the L_2-gain. This can be be done via the theory of differential games [7],[58],[59],[60].

Problem 4.1 (Two-Player Zero-Sum Game) *The starting point is the following two-player zero-sum game*

$$\dot{x}(t) = f(x(t), u(t), d(t)), \quad x(0) = x_0, \quad t \in \mathbb{R}^+ \tag{4.10}$$

$$\tilde{V}(x_0, u(\cdot), d(\cdot)) = \int_0^\infty L(x(s), u(s), d(s))ds \tag{4.11}$$

with functional $\tilde{V}$*,* f, L *be* C^1 *in* (x,u,d) *and* $u(\cdot), d(\cdot)$ *piecewise continuous. From a game theoretic perspective, the objective is to minimize the functional with respect to input* u *and maximize it with respect to disturbance* d*. Mathematically speaking, we want to find the value function*

$$V(x_0) = \sup_{d(\cdot)} \inf_{u(\cdot)} \tilde{V}(x_0, u(\cdot), d(\cdot)). \tag{4.12}$$

Of course, at first it is unclear, if such a value function exists, and it is unclear what happens when interchanging sup and inf. Hence, several definitions and statements have to be introduced in order to solve the problem under investigation.

Definition 4.7 *Consider a two-player zero-sum game, with player one playing game* u *in order to minimize* $\tilde{V}$*, and player two playing game* d *in order to* *maximize* $\tilde{V}$. *If*

$$V^+(x_0) = \inf_{u(\cdot)} \sup_{d(\cdot)} \tilde{V}(x_0, u(\cdot), d(\cdot)), \tag{4.13}$$

$$V^-(x_0) = \sup_{d(\cdot)} \inf_{u(\cdot)} \tilde{V}(x_0, u(\cdot), d(\cdot)) \tag{4.14}$$

exist and $V^+(x_0) = V^-(x_0)$, *then we denote this number by* $V(x_0)$ *and say the differential game has value* $V(x_0)$ *and the value of the game exists.*

Definition 4.8 *Suppose the differential game has value* $V(x_0)$. *A pair of strategies* (u^+, d^+) *is said to be a saddle point, if for all* u, d

$$\tilde{V}(x_0, u^+(\cdot), d(\cdot)) \leq \tilde{V}(x_0, u^+(\cdot), d^+(\cdot)) \leq \tilde{V}(x_0, u(\cdot), d^+(\cdot)). \tag{4.15}$$

Then the strategies u^+, d^+ *are called optimal and* $V(x_0) = \tilde{V}(x_0, u^+(\cdot), d^+(\cdot))$.

Theorem 4.1 *Let conditions*

$$f(x, u, d) = f^1(x, u) + f^2(x, d)$$
$$L(x, u, d) = l^1(x, u) + l^2(x, d)$$

hold. Then the differential game of Problem 4.1 has value $V(x_0)$.

Theorem 4.2 *Let conditions*

$$f(x, u, d) = f^0(x) + F^1(x)u + F^2(x)d$$
$$L(x, u, d) = l^0(x) + L^1(x)u + L^2(x)d$$

hold. Then the differential game of Problem 4.1 has a saddle point.

Remark 4.1 *This is an important remark, since it points at a really awkward use of the term saddle point: Saddle point conditions in game theory with respect to the associated Hamiltonian are not the same as they would be in ordinary calculus. The only reference where this is mentioned is the classic book [61].*

Those optimization problems are formulated via the associated Hamilton function and lead to Hamilton-Jacobi-equations, which are nonlinear first order partial differential equations. The solution theory of such partial differential equations would lead to viscosity solutions [62], [63], [64] which are in general not differentiable. Since the main result presented in this thesis is based on an analytically derived classical solution, i.e. a C^1 solution of a Hamilton-Jacobi-equation, we consider this more restrictive case. But we want to point out that non-differentiable solutions could provide better results (but are usually obtained numerically, since it is much more difficult to find them analytically).

In this thesis we do not intend to consider the case of general nonlinear systems. Instead, we focus on nonlinear time-invariant input-affine systems suitable for the considered power converter systems

$$\begin{aligned} \dot{x}(t) &= a(x(t)) + b(x(t))u(t) + g(x(t))d(t), \quad t \in \mathbb{R}^+ \\ w(t) &= \begin{pmatrix} h(x(t)) \\ u(t) \end{pmatrix}, \quad a(0) = h(0) = 0 \end{aligned} \tag{4.16}$$

with functions $a : \mathbb{R}^n \to \mathbb{R}^n, b : \mathbb{R}^n \to \mathbb{R}^{n\times m}, g : \mathbb{R}^n \to \mathbb{R}^{n\times q}, h : \mathbb{R}^n \to \mathbb{R}^p$, $w(t) \in \mathbb{R}^{p+m}$,$u(t) \in \mathbb{R}^m, d(t) \in \mathbb{R}^q$ and $u(\cdot), d(\cdot) \in L_2[0, T]$, while a, b, g, h are C^1 in x.

It is common for nonlinear systems to consider merely the suboptimal state feedback $\mathcal{H}_\infty$ control problem which is easier to solve. Different from the optimal $\mathcal{H}_\infty$ control problem that searches for the smallest $\gamma^+ > 0, \gamma^+ \in \mathbb{R}$ with $\gamma^+ < \gamma$, the suboptimal $\mathcal{H}_\infty$ control problem intends to find a state feedback such that the input-output behavior has a finite L_2-gain $\leq \gamma$ for a prespecified $\gamma > 0$. Note that, with successive iteration, the optimal $\gamma^+ < \gamma$ can be determined to solve the optimal state feedback $\mathcal{H}_\infty$ control problem.

In addition, we have to include the finite L_2-gain property in the two-player zero-sum differential game and choose $L = -s$ with s being the finite L_2-gain supply rate from Lemma 4.1. Then the two-player zero-sum game from Problem 4.1 refers to

$$\begin{aligned} &\dot{x}(t) = a(x(t)) + b(x(t))u(t) + g(x(t))d(t), \quad x(0) = x_0 \quad t \in \mathbb{R}^+ \\ &\tilde{V}(x_0, u(\cdot), d(\cdot)) = \int_0^\infty (||u(x(s))||_2^2 + ||h(x(s))||_2^2 - \gamma^2 ||d(s)||_2^2) ds. \end{aligned} \tag{4.17}$$

Our goal is to find the value function

$$V(x_0) = \sup_{d(\cdot)\in L_2} \inf_{u(\cdot)\in L_2} \tilde{V}(x_0, u(\cdot), d(\cdot)) \tag{4.18}$$

in order to obtain for the worst[1] case disturbance the least possible control effort and small penalty variable w from (4.16). Besides the input effort u, this

[1] "Worst" and "least" in the special case of (4.18) should be understood in the L_2-norm sense.

variable w contains the function $h(x)$, which reflects for example the tracking error between the measurement output and its desired reference value and should be small, too. It immediately follows from Theorem 4.1 and Theorem 4.2, respectively, that this differential game has value $V(t_0, x_0)$ and has a saddle point.

Equivalently, this differential game can be formulated in terms of L_2-gain:

Problem 4.2 (Suboptimal State Feedback $\mathcal{H}_\infty$ control) *In nonlinear $\mathcal{H}_\infty$ suboptimal control we seek for a nonlinear state feedback $u(t) = u^+(x(t))$, $u^+(0) = 0$, such that the closed-loop system of (4.16),*

$$\begin{aligned} \dot{x}(t) &= a(x(t)) + b(x(t))u^+(x(t)) + g(x(t))d(t), \quad t \in \mathbb{R}^+ \\ w^+(x(t)) &= \begin{pmatrix} h(x(t)) \\ u^+(x(t)) \end{pmatrix}, \end{aligned} \tag{4.19}$$

with initial condition $x(0) = x_0$ has L_2-gain $\leq \gamma$ from d to w^+ (cf. Definition 4.6 with $y(t) = w^+(x(t))$ and $u(t) = d(t)$).

To proceed to the solution of this problem, it is common in differential games to introduce the associated Hamilton function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$

$$H(x, u, d) = \frac{1}{2}\left(||h(x)||_2^2 + ||u||_2^2 - \gamma^2 ||d||_2^2\right) + V_x(x)\left(a(x) + b(x)u + g(x)d\right) \tag{4.20}$$

(with $V \in C^1$ and adjoint variable $V_x(x) = \frac{\partial V(x)}{\partial x}$). Necessary conditions for finding the extreme values (in terms of differential games, see Remark 4.1) are $\frac{\partial H}{\partial u} = 0$ and $\frac{\partial H}{\partial d} = 0$, respectively. Solving $\frac{\partial H}{\partial u} = 0$ for u provides the state feedback control law

$$u^+(x) = -b^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x). \tag{4.21}$$

The worst case exogenous input (disturbance) is obtained from solving $\frac{\partial H}{\partial d} = 0$ for d:

$$d^+(x) = \frac{1}{\gamma^2} g^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x). \tag{4.22}$$

Now we have to show that these values are minimal and maximal, respectively. Since the Hamilton function is quadratic in u and d, we can rewrite it using a Taylor series expansion around the expected optimal values u^+, d^+, i.e.

$$H(x, u, d) = H(x, u^+(x), d^+(x)) + \frac{1}{2}||u - u^+(x)||_2^2 - \frac{\gamma^2}{2}||d - d^+(x)||_2^2. \tag{4.23}$$

It can be seen that the Hamilton function shows a saddle point with respect to d and u at the point with optimal values $d^+(x)$, $u^+(x)$

$$H(x, u^+(x), d) \leq H(x, u^+(x), d^+(x)) \leq H(x, u, d^+(x)). \tag{4.24}$$

For brevity, we use $x(\cdot)$ to denote the solution instead of the correct formulation $x(\cdot, x_0, u(\cdot), d(\cdot))$.

Now observe that along any trajectory $x(\cdot)$ of (4.16), we obtain

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w(t)||_2^2 - \frac{\gamma^2}{2}||d(t)||_2^2 = H(x(t), u(t), d(t))$$

with the Hamilton function in (4.20). Assume that $V(x)$ is such that

$$H(x, u^+(x), d^+(x)) \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{4.25}$$

then, with help of (4.23), we get

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w(t)||_2^2 - \frac{\gamma^2}{2}||d(t)||_2^2 \leq \\ -\frac{\gamma^2}{2}||d(t) - d^+(x(t))||_2^2 + \frac{1}{2}||u(t) - u^+(x(t))||^2.$$

Using the last inequality and setting $u(t) = u^+(x(t))$ shows that

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w^+(x(t))||_2^2 - \frac{\gamma^2}{2}||d(t)||^2 \leq 0.$$

In other words, the state feedback law $u^+(x)$ renders the closed-loop of (4.16) dissipative with storage function V and supply rate $s(d, w^+) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+||_2^2$. The feedback law is called optimal, if we find a function V such that

$$H(x, u^+(x), d^+(x)) = 0, \quad \forall x \in \mathbb{R}^n. \tag{4.26}$$

In the case of the inequality (4.25) we call the state feedback law $u^+(x)$ suboptimal and refer to the suboptimal state feedback $\mathcal{H}_\infty$ control problem.

In the next step we want to link the game theoretic results to the dissipativity formulation and discuss its stability properties.

Theorem 4.3 *Let $\gamma > 0$ and V be a positive semidefinite C^1 solution of the Hamilton-Jacobi-Isaacs-inequality (HJIi)*

$$H(x, u^+(x), d^+(x)) \leq 0, \quad \forall x \in \mathbb{R}^n. \tag{4.27}$$

Then the closed-loop of (4.16) with the state feedback control law (4.21)

$$\dot{x} = a(x) - b(x)b^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) + g(x)d \tag{4.28}$$

$$w^+(x) = \begin{pmatrix} h(x) \\ -b^{\mathrm{T}}(x)V_x^{\mathrm{T}}(x) \end{pmatrix} \tag{4.29}$$

has finite L_2-gain $\leq \gamma$ from d to w^+.

Proof 4.4 *Suppose V is a C^1 solution of (4.27). Calculating $\dot{V}$ along the solution of (4.28) with the state feedback control law $u^+(x)$ yields to*

$$\begin{aligned}\dot{V}(x) = V_x(x)\dot{x} &= V_x(x)\big(a(x) + b(x)u^+(x) + g(x)d\big) \\ &= V_x(x)a(x) - ||u^+(x)||_2^2 + V_x(x)g(x)d \\ &\leq \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||u^+(x)||_2^2 - \frac{1}{2}||h(x)||_2^2.\end{aligned} \tag{4.30}$$

For brevity, we use $x(\cdot) = x(\cdot, t_0, x_0, u(\cdot))$ to denote the solution of the considered differential equation.

Setting $s(d, w^+(x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||u^+(x)||_2^2 - \frac{1}{2}||h(x)||_2^2 = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(x)||_2^2$, we recognize the differential dissipation inequality (4.8)

$$\dot{V}(x) \leq s(d, w^+(x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(x)||_2^2. \tag{4.31}$$

Since our system is dissipative with respect to the finite L_2-gain supply rate $s(d, w^+(x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(x)||_2^2$, following Lemma 4.1, we know that the system has a finite L_2-gain $\leq \gamma$ from d to $w^+(x)$. Another way to show this is the following:

Whenever $V(x) \geq 0, \forall x \in \mathbb{R}^n$ then integrating both sides from 0 to T and reordering terms, the result is:

$$\int_0^T \left(||u^+(x(t))||_2^2 + ||h(x(t))||_2^2\right) dt \leq \gamma^2 \int_0^T ||d(t)||_2^2 dt + 2V(x_0). \tag{4.32}$$

Thus,

$$||w^+(x)||_{2,[0,T]} \leq \gamma ||d||_{2,[0,T]} + \sqrt{2V(x_0)} \tag{4.33}$$

with the $L_2[0,T]$-norm $||\cdot||_{2,[0,T]} = \int_0^T |f(t)|^2 dt$, $L_2[0,T]$-function $f(t)$ and using $\sqrt{a^2+b^2} \leq a + b$.

In the unperturbed case $d \equiv 0$ inequality (4.30) reads

$$\dot{V}(x) \leq -\frac{1}{2}||u^+(x)||_2^2 - \frac{1}{2}||h(x)||_2^2 \leq 0 \tag{4.34}$$

which tells us that the closed-loop system for $d \equiv 0$ is at least Lyapunov stable if the function V would be positive definite.

Corollary 4.1 *Choose a positive definite C^1 function V such that the conditions of Theorem 4.3 and Lemma 4.3 are valid. Then system (4.16) has finite L_2-gain $\leq \gamma$ from d to w^+. Furthermore, the closed-loop system for $d \equiv 0$ is asymptotically stable. If V is radially unbounded, the closed-loop system for $d \equiv 0$ is globally asymptotically stable.*

Proof 4.5 *The statement about the finite L_2-gain $\leq \gamma$ from d to w^+ directly follows from Theorem 4.3. Furthermore, since V is positive definite and the conditions of Lemma 4.3 are valid, we know immediately that the closed-loop system for $d \equiv 0$ is asymptotically stable, and globally asymptotically stable for V radially unbounded.*

4.3 Time-Varying Systems

When we consider the time-varying case, dissipativity is the same. However, stability is much more elaborate than in the time-invariant case and requires stronger assumptions, since it is no longer possible to use LaSalle's Invariance Principle (and therefore the Corollary of Barbashin and Krasovskii) in order to prove asymptotic stability.

First let us restate some basic definitions and theorems for the time-varying case.

4.3.1 Dissipativity

Consider the time-varying system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \quad t \in \mathbb{R}^+, \\ y(t) &= h(t, x(t), u(t)). \end{aligned} \tag{4.35}$$

$f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ with piecewise continuous, bounded real functions $u(\cdot)$ and C^1 functions $f(t, x, u), h(t, x, u)$ (which means

C^1 in t, x, u). Remember the assumption that the uniquely defined solution $x(t, t_0, x_0, u(\cdot))$ with $t \geq t_0 \in \mathbb{R}^+, x(t_0) = x_0 \in D$ is forward complete. Again, the point $x = 0$ is supposed to be an equilibrium point of (4.35) at t_0, i.e. $f(t, 0, 0) = 0, \forall t \geq t_0 \geq 0$.

As was done above, we omit the arguments of states and inputs in the following where possible for simplicity and better readability.

Definition 4.9 *System (4.35) is said to be dissipative with respect to the supply rate s if there exists a positive semidefinite function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, such that for each $x_0 \in \mathbb{R}^n$, each $t_0 \geq 0$, each $t_1 \geq t_0 \geq 0$ and each function $u(\cdot)$*

$$V(t_1, x(t_1)) \leq V(t_0, x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt \tag{4.36}$$

with initial condition $x(t_0) = x_0$ and where we use the abbreviation $x(t_1) = x(t_1, t_0, x_0, u(\cdot))$ for convenience.

Definition 4.10 *Let $\gamma > 0$. System (4.35) has finite L_2-gain $\leq \gamma$ from u to y, if for each $x_0 \in \mathbb{R}^n$, each $t_0 \geq 0$ there exists a constant $K(t_0, x_0), 0 \leq K(t_0, x_0) < \infty$, with $K(t_0, 0) = 0$, such that*

$$\int_{t_0}^{t_1} ||y(t)||_2^2 \, dt \leq \gamma^2 \int_{t_0}^{t_1} ||u(t)||_2^2 \, dt + K(t_0, x_0) \tag{4.37}$$

for each $t_1 \geq t_0 \geq 0$ and each $u \in L_2[t_0, t_1]$ with $y(t) = h(t, x(t, t_0, x_0, u(\cdot)), u(t))$, $x(t_0) = x_0$. The finite L_2-gain of the system in (4.35) is defined as

$$\gamma^+ := \inf\{\gamma \mid \textit{the system in (4.35) has finite } L_2\textit{-gain} \leq \gamma\}.$$

Equivalently, Definition 4.10 can be rephrased as is done in the following lemma:

Lemma 4.4 *System (4.35) has finite L_2-gain $\leq \gamma$ from u to y if for each $u \in \mathbb{R}^m$ and each $y \in \mathbb{R}^p$ it is dissipative with respect to the* finite L_2-gain supply rate $s(u, y) = \frac{1}{2}\gamma^2 ||u||_2^2 - \frac{1}{2}||y||_2^2$.

Proof 4.6 *Let*

$$V(t_1, x(t_1)) - V(t_0, x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 ||u(t)||_2^2 - ||y(t)||_2^2)dt.$$

Since V is positive semidefinite, we can rephrase the former inequality to

$$\int_{t_0}^{t_1} ||y(t)||_2^2 dt \leq \gamma^2 \int_{t_0}^{t_1} ||u(t)||_2^2 dt + 2V(t_0, x_0)$$

which is the definition of finite L_2-gain, with $K(t_0, x_0) = 2V(t_0, x_0)$ finite and $K(t_0, 0) = 0$ since $V(t_0, 0) = 0$.

Lemma 4.5 *Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continously differentiable in both arguments. Then system (4.35) is dissipative with respect to the supply rate s if and only if*

$$\dot{V}(t, x) \leq s(u, h(t, x, u)), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m. \tag{4.38}$$

Proof 4.7 *One direction of the statement follows directly from differentiation of (4.36). The other direction starts with (4.38) and integration on both sides from t_0 to t_1 along a solution $x(t, t_0, x_0, u(\cdot))$ of (4.35) with associated input $u(\cdot)$ and resulting output $y(t) = h(t, x(t, t_0, x_0, u(\cdot)), u(t))$ leads to*

$$V(t_1, x(t_1)) - V(t_0, x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), h(t, x(t, t_0, x_0, u(\cdot)), u(t)))\, dt.$$

However, this is nothing else than the inequality arising in Definition 4.9, and therefore the proof is complete.

4.3.2 State feedback $\mathcal{H}_\infty$ control

Again, we pose the control problem in terms of differential games ([7], [58], [59]).

Problem 4.3 (Two-Player Zero-Sum Game) *Starting point is the following two-player zero-sum game*

$$\dot{x}(t) = f(t, x(t), u(t), d(t)), \quad x(t_0) = x_0, \quad t \geq t_0 \in \mathbb{R}^+ \tag{4.39}$$

$$\tilde{V}(t_0, x_0, u(\cdot), d(\cdot)) = \int_{t_0}^{\infty} L(s, x(s), u(s), d(s)) ds \tag{4.40}$$

with functional $\tilde{V}$, f, L be C^1 in (t, x, u, d) and $u(\cdot), d(\cdot)$ piecewise continuous. From a game theoretic perspective, the objective is to minimize the functional *with respect to input u and maximize it with respect to disturbance d. Mathematically speaking, we want to find the value function*

$$V(t_0, x_0) = \sup_{d(\cdot)} \inf_{u(\cdot)} \tilde{V}(t_0, x_0, u(\cdot), d(\cdot)). \tag{4.41}$$

As it was the case for time-invariant systems, we have to provide conditions on the existence of a value function V and some further properties.

Definition 4.11 *Consider a two-player zero-sum game, with player one playing game u in order to minimize $\tilde{V}$, and player two playing game d in order to maximize $\tilde{V}$. If*

$$V^+(t_0, x_0) = \inf_{u(\cdot)} \sup_{d(\cdot)} \tilde{V}(t_0, x_0, u(\cdot), d(\cdot)) \tag{4.42}$$

$$V^-(t_0, x_0) = \sup_{d(\cdot)} \inf_{u(\cdot)} \tilde{V}(t_0, x_0, u(\cdot), d(\cdot)) \tag{4.43}$$

exist and $V^+(t_0, x_0) = V^-(t_0, x_0)$, then we denote this number by $V(t_0, x_0)$ and say the differential game has value $V(t_0, x_0)$ and the value of the game exists.

Definition 4.12 *Suppose the differential game has value $V(t_0, x_0)$. A pair of strategies (u^+, d^+) is said to be a saddle point, if for all u, d*

$$\tilde{V}(t_0, x_0, u^+(\cdot), d(\cdot)) \leq \tilde{V}(t_0, x_0, u^+(\cdot), d^+(\cdot)) \leq \tilde{V}(t_0, x_0, u(\cdot), d^+(\cdot)). \tag{4.44}$$

Then the strategies u^+, d^+ are called optimal and
$V(t_0, x_0) = \tilde{V}(t_0, x_0, u^+(\cdot), d^+(\cdot))$.

Theorem 4.4 *Let conditions*

$$f(t, x, u, d) = f^1(t, x, u) + f^2(t, x, d)$$
$$L(t, x, u, d) = l^1(t, x, u) + l^2(t, x, d)$$

hold. Then the differential game of Problem 4.3 has value $V(t_0, x_0)$.

Theorem 4.5 *Let conditions*

$$f(t, x, u, d) = f^0(t, x) + F^1(t, x)u + F^2(t, x)d$$
$$L(t, x, u, d) = l^0(t, x) + L^1(t, x)u + L^2(t, x)d$$

hold. Then the differential game of Problem 4.3 has a saddle point.

Remark 4.2 *Again, we need to remember that saddle point conditions in game theory are not the same as they would be in ordinary calculus. The only reference where this is mentioned is the classic book [61].*

Similar to the time-invariant case, we stick to the simple case where the value function V is C^1 instead of considering viscosity solutions which arise in the general case.

In this thesis, we do not intend to discuss the case of general nonlinear systems. Instead, we focus on nonlinear time-varying input-affine systems suitable for the considered power converter systems

$$\begin{aligned} \dot{x}(t) =& a(t,x(t)) + b(t,x(t))u(t) + g(t,x(t))d(t), \quad t \in \mathbb{R}^+ \\ w(t) =& \begin{pmatrix} h(t,x(t)) \\ u(t) \end{pmatrix}, \quad a(t,0) = h(t,0) = 0. \end{aligned} \tag{4.45}$$

with functions $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}, g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times q}, h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$ and $w(t) \in \mathbb{R}^{p+m}$, $u(t) \in \mathbb{R}^m, d(t) \in \mathbb{R}^q$ and $u(\cdot), d(\cdot) \in L_2[0,T]$ while a, b, g, h are C^1 in t, x.

Once again, we concentrate on the discussion of the suboptimal state feedback $\mathcal{H}_\infty$ control problem which is easier to solve in the context of nonlinear systems. Different from the optimal $\mathcal{H}_\infty$ control problem that searches for the smallest $\gamma^+ > 0, \gamma^+ \in \mathbb{R}$ with $\gamma^+ < \gamma$, the suboptimal $\mathcal{H}_\infty$ control problem intends to find a state feedback law such that the input-output behavior has a finite L_2-gain $\leq \gamma$ for a prespecified $\gamma > 0$. Therefore, we choose $L = -s$ with s being the finite L_2-gain supply rate from Lemma 4.4 in the two-player zero-sum differential game. Then the two-player zero-sum game from Problem 4.3 refers to

$$\begin{aligned} &\dot{x}(t) = a(t,x(t)) + b(t,x(t))u(t) + g(t,x(t))d(t), \quad x(t_0) = x_0, \quad t \geq t_0 \in \mathbb{R}^+ \\ &\tilde{V}(t_0, x_0, u(\cdot), d(\cdot)) = \int_{t_0}^{\infty} (||u(s,x(s))||_2^2 + ||h(s,x(s))||_2^2 - \gamma^2 ||d(s)||_2^2) ds. \end{aligned} \tag{4.46}$$

Our goal is to find the value function

$$V(t_0, x_0) = \sup_{d(\cdot)\in L_2} \inf_{u(\cdot)\in L_2} \tilde{V}(t_0, x_0, u(\cdot), d(\cdot)) \tag{4.47}$$

in order to obtain for the worst case disturbance the least possible control effort and small penalty variable w from (4.45). This variable w contains besides the input effort u the function $h(t,x)$, which reflects for example the tracking error between the measurement output and its desired reference value and should be kept small. It immediately follows from Theorem 4.4 and Theorem 4.5, respectively, that this differential game has value $V(t_0,x_0)$ and has a saddle point.

Equivalently, this differential game can be formulated in terms of L_2-gain:

Problem 4.4 (Time-Varying Suboptimal State Feedback $\mathcal{H}_\infty$ control) *In nonlinear $\mathcal{H}_\infty$ suboptimal control we seek for a nonlinear time-varying state feedback $u(t) = u^+(t,x(t)), u^+(t,0) = 0, \forall t \geq 0$, such that the closed-loop system of (4.45),*

$$\begin{aligned} \dot{x}(t) &= a(t,x(t)) + b(t,x(t))u^+(t,x(t)) + g(t,x(t))d(t), \quad t \in \mathbb{R}^+ \\ w^+(t,x(t)) &= \begin{pmatrix} h(t,x(t)) \\ u^+(t,x(t)) \end{pmatrix}, \end{aligned} \tag{4.48}$$

with initial condition $x(t_0) = x_0$ has finite L_2-gain $\leq \gamma$ from d to w^+.

To proceed to the solution of this problem, we again stick to the associated Hamilton function so as to solve the problem in terms of differential games. Consider

$$\begin{aligned} H(t,x,u,d) =& \frac{1}{2}\left(||h(t,x)||_2^2 + ||u||_2^2 - \gamma^2||d||_2^2\right) \\ &+ V_t(t,x) + V_x(t,x)\left(a(t,x) + b(t,x)u + g(t,x)d\right) \end{aligned} \tag{4.49}$$

which is minimized with respect to u and maximized with respect to disturbance d. An equivalent argumentation as in the time-invariant case leads to the state feedback control law

$$u^+(t,x) = -b^{\mathrm{T}}(t,x)V_x^{\mathrm{T}}(t,x). \tag{4.50}$$

and the worst case exogenous input (disturbance)

$$d^+(t,x) = \frac{1}{\gamma^2}g^{\mathrm{T}}(t,x)V_x^{\mathrm{T}}(t,x). \tag{4.51}$$

Since the Hamilton function is quadratic in u and d, we can rewrite it using a Taylor series expansion around the expected optimal values $u^+(t,x), d^+(t,x)$, i.e.

$$\begin{aligned} H(t,x,u,d) = H(t,x,u^+(t,x),d^+(t,x)) \\ + \frac{1}{2}||u - u^+(t,x)||_2^2 - \frac{\gamma^2}{2}||d - d^+(t,x)||_2^2. \end{aligned} \tag{4.52}$$

It can be seen that the Hamilton function shows a saddle point with respect to d and u at the point with optimal values $d^+(t,x)$, $u^+(t,x)$

$$H(t,x,u^+(t,x),d) \leq H(t,x,u^+(t,x),d^+(t,x)) \leq H(t,x,u,d^+(t,x)),\quad \forall t \geq t_0 \in \mathbb{R}^+. \tag{4.53}$$

For brevity, we use $x(\cdot) = x(\cdot, t_0, x_0, u(\cdot))$ to denote the solution of the considered differential equation.

Now observe that along any trajectory $x(\cdot)$ of (4.45) and with (4.49),

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w(t)||_2^2 - \frac{\gamma^2}{2}||d(t)||_2^2 = H(x(t),u(t),d(t)).$$

Assume that $V(t,x)$ is such that

$$H(t,x,u^+(t,x),d^+(t,x)) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall t \geq t_0 \in \mathbb{R}^+. \tag{4.54}$$

Then it follows with help of equation (4.52) that

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w(t)||_2^2 - \frac{\gamma^2}{2}||d(t)||_2^2 \leq -\frac{\gamma^2}{2}||d(t) - d^+(t,x(t))||_2^2 + \frac{1}{2}||u(t) - u^+(t,x(t))||^2.$$

Using the last inequality and setting $u(t) = u^+(t,x(t))$ shows that

$$\frac{dV(x(t))}{dt} + \frac{1}{2}||w^+(t,x(t))||_2^2 - \frac{\gamma^2}{2}||d(t)||^2 \leq 0.$$

In other words, the state feedback law $u^+(t,x)$ renders the closed-loop of (4.45) dissipative with storage function V and supply rate $s(d,w^+(x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(x)||_2^2$.

Again, we use the notion suboptimal for the state feedback control law u^+ when we refer to the solution of inequality (4.54).

Let us link this game theoretic result with finite L_2-gain $\leq \gamma$ and discuss stability in the following lines.

Theorem 4.6 *Let $\gamma > 0$ and V be a positive semidefinite C^1 solution of the time-varying Hamilton-Jacobi-Isaacs-inequality (tHJIi)*

$$H(t,x,u^+(t,x),d^+(t,x)) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall t \geq t_0 \in \mathbb{R}^+. \tag{4.55}$$

Then the closed-loop of (4.45) with the state feedback control law (4.50)

$$\dot{x} = a(t,x) - b(t,x)b^{\mathrm{T}}(t,x)V_x^{\mathrm{T}}(t,x) + g(t,x)d \tag{4.56}$$

$$w^+(t,x) = \begin{pmatrix} h(t,x) \\ -b^{\mathrm{T}}(t,x)V_x^{\mathrm{T}}(t,x) \end{pmatrix} \tag{4.57}$$

has finite L_2-gain $\leq \gamma$ from d to $w^+(t,x)$.

Proof 4.8 *Suppose V is a C^1 solution of (4.55). Calculating $\dot{V}$ along the solution of (4.56) with the state feedback control law $u^+(t,x)$ yields to*

$$\begin{aligned}
\dot{V}(t,x) &= V_t(t,x) + V_x(t,x)\dot{x} \\
&= V_t(t,x) + V_x(t,x)\left(a(t,x) + b(t,x)u^+(t,x) + g(t,x)d\right) \\
&= V_t(t,x) + V_x(t,x)a(t,x) - ||u^+(t,x)||_2^2 + V_x(t,x)g(t,x)d \\
&\leq \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||u^+(t,x)||_2^2 - \frac{1}{2}||h(t,x)||_2^2.
\end{aligned} \tag{4.58}$$

For brevity, we use $x(\cdot)$ to denote the solution instead of the correct formulation $x(\cdot, x_0, u(\cdot), d(\cdot))$.

Setting $s(d, w^+(t,x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||u^+(t,x)||_2^2 - \frac{1}{2}||h(x)||_2^2 = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(t,x)||_2^2$, we recognize the time-varying version of the differential dissipation inequality (4.38)

$$\dot{V}(t,x) \leq s(d, w^+(t,x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(t,x)||_2^2. \tag{4.59}$$

Since our system is dissipative with respect to the supply rate $s(d, w^+(t,x)) = \frac{\gamma^2}{2}||d||_2^2 - \frac{1}{2}||w^+(t,x)||_2^2$, following Lemma 4.4, we know that the system has a finite L_2-gain $\leq \gamma$ from d to $w^+(t,x)$.

In the unperturbed case $d \equiv 0$ inequality (4.58) reads

$$\dot{V}(t,x) \leq -\frac{1}{2}||u^+(t,x)||_2^2 - \frac{1}{2}||h(t,x)||_2^2 \leq 0 \tag{4.60}$$

which tells us that the closed-loop system for $d \equiv 0$ is at least uniformly stable if the function V would fulfill $\underline{V}(x) \leq V(t,x) \leq \overline{V}(x), \forall t \geq t_0 \in \mathbb{R}^+, \forall x \in \mathbb{R}^n$ with continuous positive definite functions $\underline{V}, \overline{V} : \mathbb{R}^n \to \mathbb{R}$ (cf. Theorem 3.3).

In contrast to time-invariant dissipativity theory, the difficulty is to guarantee asymptotic stability. The authors of the book [54] mention that there exists a generalization of LaSalle's Invariance Principle and the Barbashin-Krasovskii Corollary presented in [65] which is much more complicated than in the time-invariant case and difficult to handle. Instead of using LaSalle's Invariance Principle, another approach would be to assume the existence of a time-varying Lyapunov function in order to achieve asymptotic stability [24], [25]:

Corollary 4.2 *Choose a C^1 function V which fulfills the conditions of Theorem 4.6. In addition, the function V is supposed to fulfill $\underline{V}(x) \leq V(t,x) \leq \overline{V}(x), \forall t \geq t_0 \in \mathbb{R}^+, \forall x \in \mathbb{R}^n$ with continuous positive definite functions $\underline{V}, \overline{V} : \mathbb{R}^n \to \mathbb{R}$ and $\dot{V}(t,x) \leq -\tilde{V}(x), \forall t \geq t_0 \in \mathbb{R}^+, \forall x \in \mathbb{R}^n$ with continuous positive definite function $\tilde{V} : \mathbb{R}^n \to \mathbb{R}$ for the closed-loop system (4.48) with $d \equiv 0$. Then the closed-loop system has finite L_2-gain $\leq \gamma$ from d to w^+ and is uniformly globally asymptotically stable for $d \equiv 0$.*

Proof 4.9 *Since V fulfills the conditions of Theorem 4.6, we get a suboptimal feedback law $u^+(t,x)$ and the system has a finite L_2-gain $\leq \gamma$ from d to w^+. The further requirements on V are exactly those in Theorem 3.3 for uniform global asymptotic stability, such that the closed-loop system (4.48) is uniformly globally asymptotically stable for $d \equiv 0$.*

As can be seen this corollary requires the knowledge of a solution of the Hamilton-Jacobi-Isaacs inequality, which is at the same time a Lyapunov function for the closed-loop system with zero disturbances. These requirements are not easy to achieve, since already solving the Hamilton-Jacobi-Isaacs inequality, a nonlinear first order partial differential inequality, is a difficult task. The supplementary demand that this V is a time-varying Lyapunov function does not facilitate the solution of this problem.

We can formulate a similar statement in terms of iISS Lyapunov functions:

Corollary 4.3 *Choose a C^1 function V which fulfills the conditions of Theorem 4.6 and is at the same time an iISS Lyapunov function for (4.48). Then (4.48) has a finite L_2-gain $\leq \gamma$ from d to w^+ and is uniformly globally asymptotically stable for $d \equiv 0$. Furthermore, the system is iISS for $d \neq 0$.*

Proof 4.10 *That (4.48) has a finite L_2-gain from d to w^+ is the result of Theorem 4.6. Since V is an iISS Lyapunov function, it follows from Definition 3.12 that the closed-loop system (4.48) is uniformly globally asymptotically stable for $d \equiv 0$. Finally, Theorem 3.5 tells that the closed-loop system (4.48) is iISS.*

Again, this corollary needs a solution V of the Hamilton-Jacobi-Isaacs inequality which is also an iISS Lyapunov function, but we have not merely an input-output property but additionally a link between input and states. In the following chapter, we will address this problem for the specific class of power converter systems with iISS Lyapunov functions which provide deeper insight into the system properties than time-varying Lyapunov functions. In addition, we will even show which function V has to be chosen.

The next proposition links iISS Lyapunov functions with dissipativity theory:

Proposition 4.1 *Assume that there exists an iISS Lyapunov function V for the nonlinear time-varying system (4.35) with $y = x$. Then (4.35) is dissipative with respect to the supply rate $s(u, x) = \Delta(||u||_2) - \nu(||x||_2)$.*

Proof 4.11 *If there exists an iISS Lyapunov function V, we know from Definition 3.12 that*

$$\dot{V}(t, x) \leq -\nu(||x||_2) + \Delta(||u||_2), \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \forall t \geq 0.$$

Integration from t_0 to t_1 on both sides leads to

$$V(t_1, x(t_1)) - V(t_0, x(t_0)) \leq \int\limits_{t_0}^{t_1} (-\nu(||x(t)||_2) + \Delta(||u(t)||_2)\, dt$$

and one can immediately identify $s(u, x) = \Delta(||u||_2) - \nu(||x||_2)$ from Definition 4.9 of dissipativity.

Of course, this proof could be shortened using directly the differential dissipation inequality without using integration.

Corollary 4.4 *Assume that V is an iISS Lyapunov function for (4.35) with $\Delta(||u||_2) = \frac{\gamma^2}{2}||u||_2^2$, $\nu(||x||_2) = \frac{1}{2}||x||_2^2$ and $y = x$. Then (4.35) has L_2-gain $\leq \gamma$ from u to x.*

Proof 4.12 *If there exists an iISS Lyapunov function V, we know from Definition 3.12 that*

$$\dot{V}(t, x) \leq -\nu(||x||_2) + \Delta(||u||_2), \quad \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \forall t \geq 0.$$

Inserting $\Delta(||u||_2) = \frac{\gamma^2}{2}||u||_2^2$ *and* $\nu(||x||_2) = \frac{1}{2}||x||_2^2$ *leads to*

$$\dot{V}(t,x) \leq \frac{\gamma^2}{2}||u||_2^2 - \frac{1}{2}||x||_2^2$$

which shows that (4.35) *is dissipative with respect to the finite* L_2*-gain supply rate* $s(u,x) = \frac{\gamma^2}{2}||u||_2^2 - \frac{1}{2}||x||_2^2$ *from Lemma 4.4 and therefore has finite* L_2*-gain* $\leq \gamma$ *from* u *to* x*.*

Remark 4.3 *First of all, dissipativity reflects a system property in terms of inputs and outputs. Moreover, iISS is an input-to-state property, while* L_2*-gain is an input-output property. Setting* $y = x$*, the last two Corollaries show that iISS is less general than dissipativity, since it represents a subclass of possible supply rates, but more general than the* L_2*-gain concept, since the latter is merely a specific choice of* Δ, ν *in the iISS definition. Furthermore, iISS is at the same time a stability concept, while* L_2*-gain is merely an input-output property and requires additonal statements in order to guarantee uniform asymptotic stability for vanishing inputs* u*. Equivalently, dissipativity requires additional restrictions on the supply rate to guarantee uniform asymptotic stability for zero inputs* u*. In its most general representation, there is no input-output gain information available in case of dissipativity in contrast to iISS and* L_2*-gain.*

Chapter 5

$\mathcal{H}_\infty$ Suboptimal Control for Bilinear Power Converter Systems

In this chapter, we solve the integral and disturbance feedback strategies presented in Chapter 2 and provide the new theoretical results and main contributions of the thesis. The starting point of the time-invariant theory, which has been laid in [8] and [10] for the case of a specific power converter circuit is extended to the time-varying case. The general proofs for the time-invariant theory and set-point tracking which were missing up to now are included as special case in our approach. Some parts of the presented results were published in [29] and [30], but shorter in presentation and therefore less precise due to the page constraints. The results for general bilinear systems and for systems with multiple-inputs later in this chapter have not been discussed before.

5.1 Problem Formulation

In Section 2.3 we derived two control strategies, one including integral feedback, the other involving disturbance feedback for disturbance rejection. We merely discussed the change of structure in the differential equations, but did not discuss feedback design and stability.

For both control strategies, we want to derive a suboptimal state feedback law along the lines of nonlinear $\mathcal{H}_\infty$ control presented in Chapter 4. Furthermore, we want to guarantee stability for the closed-loop system in the presence of disturbances using iISS theory which was presented in Chapter 3, and in connection with nonlinear $\mathcal{H}_\infty$ control at the end of Chapter 4. Of course, the class of differential equations is much more specific so it can be expected that we can give specific conditions for the solution of this problem, which is the main contribution of this chapter.

In summary, we have to find conditions that allow us to fulfill Corollary 4.3, i.e. the following two problems must be solved:

- We have to find a C^1 function $V : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ which solves the time-varying Hamilton-Jacobi-Isaacs-inequality (tHJIi) (4.55), i.e. that Theorem 4.6 is valid. In this setting, we solve the differential game problem addressed with (4.46) and value function (4.47) where we get the least possible state-feedback law u^+ for the worst case disturbance such that function $h(t,x)$ remains small.
- Furthermore, we want to ensure that this function V is at the same time an iISS Lyapunov function for the closed-loop of (4.46), i.e. for (4.48). For stability, we have to fulfill Definition 3.12, i.e. among others find an estimate

$$\dot{V}(t,x) \leq -\nu(||x||_2) + \Delta(||d||_2), \quad \forall t \geq 0, \forall x \in \mathbb{R}^n, \forall d \in \mathbb{R}^q$$

with positive definite function ν and $\Delta \in \mathcal{K}_\infty$.

5.2 Integral feedback

From the modeling process in Chapter 2, we have seen that the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ of BPCS into BPCTES lead to the differential equations (2.8),(2.7)

$$\begin{aligned} \dot{e}_x &= A(t)e_x + \tilde{b}(t,e_x)e_u + \tilde{g}d \\ y &= \tilde{c}^{\mathrm{T}} e_x, \end{aligned}$$

$$A(t) = M^{-1}\left(J_0 - R + J_1 u^\star\right), \tilde{b}(t,e_x) = M^{-1}\left(\bar{b} + J_1(e_x + x^\star)\right), \tilde{g} = M^{-1}\bar{g}$$

with piecewise continuous and bounded functions $u(t), d(t) \in \mathbb{R}$, piecewise continuous and bounded $\epsilon(t) \in \mathbb{R}^n$, vector $\bar{b} \in \mathbb{R}^n$, symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$, symmetric matrix $R \in \mathbb{R}^{n\times n}$, skew-symmetric matrix $J(u(t)) := J_0 + J_1 u(t)$, unit vector $\tilde{c} = 1_i, i = 1, \ldots, n$, listed in Definitions 2.1 and 2.2. The system equations already incorporate the aim that we want to realize trajectory tracking.

Then, in order to allow for integral feedback, the input transformation $e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}}$ with new input $e_{\bar{u}}$ and the additional differential equation $\dot{z} = -\alpha_4 c^{\mathrm{T}} e - \alpha_1 z$ with integrator state z and $\alpha_i > 0, i = 1, \ldots 4$ are introduced, see (2.11) in Definition 2.3 of PCIFES, and we finally obtain

$$\begin{aligned}\dot{e} &= a(t,e) + b(t,e)e_{\bar{u}} + gd\\ y &= c^{\mathrm{T}} e = \begin{pmatrix}\tilde{c}^{\mathrm{T}} & 0\end{pmatrix} e\end{aligned}$$

with

$$e = \begin{pmatrix} e_x \\ z \end{pmatrix}, a(t,e) = \begin{pmatrix} A(t)e_x - \alpha_3 \tilde{b}(t,e_x) z \\ -\alpha_4 c^{\mathrm{T}} e - \alpha_1 z \end{pmatrix}, b(t,e) = \begin{pmatrix} \alpha_2 \tilde{b}(t,e_x) \\ 0 \end{pmatrix}, g = \begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix}.$$

Now, we set $h = y$ with the output y to be controlled which we want to keep small in the presence of disturbances.

Theorem 5.1 *Consider PCIFES with (2.11). Choose*

$$\begin{aligned} V(t,e) \equiv V(e) &= \frac{1}{2} e^{\mathrm{T}} P e, \quad \forall e \in \mathbb{R}^{n+1} \\ P &= \begin{pmatrix} k_1 M & 0 \\ 0 & k_2 \end{pmatrix}, \quad P = P^{\mathrm{T}} > 0, \;\; k_1, k_2 > 0. \end{aligned} \tag{5.1}$$

Assume that matrix R from BPCS in Definition 2.1 is positive definite with minimal eigenvalue $\lambda_{\min}$ and that there exist positive constants $k_1, k_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma$, such that for

1) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x = y^{\mathrm{T}} y$ we have $X_1 > 0, X_2 \geq 0$ with

$$X_1 := k_1 \lambda_{\min} - \frac{1}{2} - \frac{k_1^2}{2\gamma^2} \text{ and } X_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4X_1}\right).$$

2) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x \neq y^{\mathrm{T}} y$ we have $\tilde{X}_1 > 0, \tilde{X}_2 \geq 0, \tilde{X}_3 > 0$ with

$$\tilde{X}_1 := k_1 \lambda_{\min} - \frac{1}{2}, \tilde{X}_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4\tilde{X}_1}\right), \tilde{X}_3 := \left(\lambda_{\min} - \frac{k_1}{2\gamma^2}\right).$$

Then $V(e)$ is a solution of the tHJIi (4.55).

Proof 5.1 *Rewrite the elements of* $a(t,e)$ *as*

$$a(t,e) = \underbrace{\begin{pmatrix} A(t) & 0 \\ 0 & -\alpha_1 \end{pmatrix}}_{\bar{A}_1(t)} e - \frac{\alpha_3}{\alpha_2} b(t,e) z - \underbrace{\begin{pmatrix} 0 \\ \alpha_4 \end{pmatrix}}_{\bar{a}_2} y. \tag{5.2}$$

Insert (5.2) *into the left hand side of* (4.55)

$$V_t + V_e \bar{A}_1(t) e - \frac{1}{2}(e_{\bar{u}}^+)^2 + e_{\bar{u}}^+ \frac{\alpha_3}{\alpha_2} z + \frac{(V_e g)^2}{2\gamma^2} - V_e \bar{a}_2 y + \frac{1}{2} y^2$$

and introduce squares:

$$V_t + V_e \bar{A}_1(t) e - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 + \frac{\alpha_3^2}{2\alpha_2^2} z^2 + \frac{(V_e g)^2}{2\gamma^2} - V_e \bar{a}_2 y + \frac{1}{2} y^2.$$

Use the quadratic function from (5.1) *and simplify to*

$$\begin{aligned} &- k_1 e_x^{\mathrm{T}} R e_x - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 - \alpha_4 k_2 y z \\ &\qquad\qquad + \frac{(e^{\mathrm{T}} P g)^2}{2\gamma^2} - \alpha_1 k_2 z^2 + \frac{\alpha_3^2}{2\alpha_2^2} z^2 + \frac{1}{2} y^2. \end{aligned}$$

The vector g *may be rewritten as* $g = \left((M^{-1} 1_k)^{\mathrm{T}}, 0 \right)^{\mathrm{T}}$ *for some* $k \in \{1, 2, \dots, n\}$ *with* $1_k \in \mathbb{R}^n$ *the* k*-th* n*-dimensional unit vector; the last element of* g *is zero.* Pg *then reduces to* $k_1 \left(1_k^{\mathrm{T}}, 0\right)^{\mathrm{T}}$ *and finally*

$$e^{\mathrm{T}}(Pg) = e_x^{\mathrm{T}} k_1 1_k = k_1 e_{x,k}, \tag{5.3}$$

with $e_{x,k}$ *being the* $k-th$ *component of* e_x*. Since* $\tilde{c}^{\mathrm{T}}$ *is a unit vector, the output* y *to be controlled is the* j*-th component of vector* e_x *which we denote with* $e_{x,j}$ *for some* $j \in \{1, 2, \dots, n\}$*, i.e.* $y = e_{x,j}$ *for some* $j \in \{1, 2, \dots, n\}$*.*

Now let matrix R *from BPCS be positive definite. Since* R *is symmetric, we know that* $\lambda_{\min} e^{\mathrm{T}} e \le e^{\mathrm{T}} R e \le \lambda_{\max} e^{\mathrm{T}} e$*, with* $\lambda_{\min}, \lambda_{\max} > 0$ *the minimal and maximal eigenvalue of* R*, respectively.*

For fixed $j, k \in \{1, 2, \dots, n\}$ *distinguish the following two cases:*

1) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x = y^{\mathrm{T}} y$*: Further introducing squares leads to*

$$\begin{aligned} &-k_1 \sum_{\substack{i=0 \\ i \ne j}}^{n} \lambda_{\min} e_{x,i}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 \\ - &\left(\sqrt{X_1} y + \frac{k_2 \alpha_4}{2\sqrt{X_1}} z \right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4 X_1} \right)}_{=:\, X_2} z^2, \end{aligned}$$

which is clearly negative semidefinite since we assumed that there exist appropriate choices for the unknown constants such that $X_1 > 0$, $X_2 \geq 0$.

2) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x \neq y^{\mathrm{T}} y$:

$$
\begin{aligned}
&- k_1 \sum_{\substack{i=0\\ i\neq j,k}}^{n} \lambda_{\min} e_{x,i}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 - \left(\sqrt{\tilde{X}_1} y + \frac{k_2 \alpha_4}{2\sqrt{\tilde{X}_1}} z \right)^2 \\
&- \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4\tilde{X}_1} \right)}_{=:\, \tilde{X}_2} z^2 - k_1 \underbrace{\left(\lambda_{\min} - \frac{k_1}{2\gamma^2} \right)}_{=:\, \tilde{X}_3} e_{x,k}^2,
\end{aligned}
$$

which is clearly negative semidefinite since we assumed that there exist appropriate choices for the unknown constants such that $\tilde{X}_1, \tilde{X}_3 > 0$, $\tilde{X}_2 \geq 0$.

Remark 5.1 *For bilinear power converter systems, R is usually diagonal. For R diagonal, replace $\lambda_{\min}$ by the appropriate diagonal elements of R as it can be seen later in (6.24). In addition, note that in the case of a pure integrator, i.e. $\alpha_1 = 0$, we would lose negative semidefiniteness. Furthermore, we want to mention that $P = P^{\mathrm{T}} > 0$ immediately follows from $M = M^{\mathrm{T}} > 0$ and $k_1, k_2 > 0$.*

This theorem provides a function V such that we can calculate the suboptimal state feedback control law $e_{\bar{u}}^{+}$. In the next step we want to proof that this V is at the same time an iISS Lyapunov function following Corollary 4.3. Hence, we calculate the time derivative of V subject to the closed-loop of (2.11) (i.e. with the suboptimal state feedback control law $e_{\bar{u}}^{+}$) and introducing squares provides

$$
\begin{aligned}
\dot{V} &= V_t + V_e \left(a + b e_{\bar{u}}^{+} + g d \right) &(5.4)\\
&= V_t + V_e a - (e_{\bar{u}}^{+})^2 - \left(\frac{V_e g}{\sqrt{2}\gamma} - \frac{\gamma d}{\sqrt{2}} \right)^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2} d^2 \\
&\leq V_t + V_e a - (e_{\bar{u}}^{+})^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2} d^2. &(5.5)
\end{aligned}
$$

where we may identify $\Delta(||d||_2) = (\gamma^2/2)d^2$. The problem is to find some match for $\nu(||e||_2)$. Therefore, we rewrite (5.5) with the help of inequality (4.55), which results in

$$
\begin{aligned}
\dot{V} &\leq V_t + V_e a - (e_u^{+})^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2} d^2 \\
&\leq -\frac{1}{2} h^2 - \frac{1}{2} (e_u^{+})^2 + \frac{\gamma^2}{2} d^2. &(5.6)
\end{aligned}
$$

Hence, setting $d \equiv 0$ in (5.6) it is clear that

$$V_t + V_e a - (e_{\bar{u}}^{+})^2 + \frac{(V_e g)^2}{2\gamma^2} \leq 0 \tag{5.7}$$

is at least negative semidefinite. For this reason, we have to show that we can match negative definiteness for the lhs of (5.7), this means that there is a negative definite function $-\nu(||e||_2)$ which bounds (5.7) from above, such that the closed-loop system is iISS.

For the system under investigation this is given in the following theorem:

Theorem 5.2 *Consider the closed-loop of* (2.11) *using the suboptimal control law* (4.50)*. Let* $\Delta(||d||_2) = (\gamma^2/2)d^2$ *and the conditions of Theorem 5.1 be valid. Then, for* $V(e)$ *from (5.1), there exists a positive definite estimate* $\nu(||e||_2)$ *for* (5.7)

$$V_t + V_e a - (e_{\bar{u}}^{+})^2 + \frac{(V_e g)^2}{2\gamma^2} \leq -\nu(||e||_2) < 0, \tag{5.8}$$

such that $V(e)$ *is an iISS Lyapunov function and therefore the closed-loop error system is iISS.*

Proof 5.2 *V is a quadratic form and so it immediately follows that $V \in C^1 \cap UPPD$. Since it is assumed that the conditions of Theorem 5.1 are fulfilled we know that V is a solution of* (4.55) *and therefore $e_{\bar{u}}^{+}$ is the suboptimal state feedback control law. Consider* (2.8),(2.11)*. Use again* (5.2) *and insert it into* (5.7)*. With the suboptimal control law $e_{\bar{u}}^{+}(t,e) = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}}$ from* (4.50) *we get*

$$V_t + V_e\bar{A}_1 e + \frac{\alpha_3}{\alpha_2}\underbrace{(-V_e b)}_{e_{\bar{u}}^{+}} z - V_e\bar{a}_2 y - \underbrace{V_e b b^{\mathrm{T}} V_e^{\mathrm{T}}}_{e_{\bar{u}}^{+2}} + \frac{(V_e g)^2}{2\gamma^2} \leq 0 \tag{5.9}$$

which was already proven to be negative semidefinite. For negative definiteness further introducing squares yields

$$V_t + V_e\bar{A}_1 e - y^2 - \left(\frac{1}{2}V_e\bar{a}_2\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2}z\right)^2 + \left(\frac{\alpha_3}{2\alpha_2}z\right)^2 + \frac{(V_e g)^2}{2\gamma^2} + \left(\frac{V_e\bar{a}_2}{2} - y\right)^2 \leq 0.$$

We may omit the positive quadratic terms without losing the vailidity of the inequality

$$V_t + V_e\bar{A}_1 - y^2 - \left(\frac{1}{2}V_e\bar{a}_2\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2}z\right)^2 \leq 0. \tag{5.10}$$

As can be seen, $V(e)$ from (5.1) is time-invariant such that $V_t \equiv 0$. Inserting into (5.10) and using that M cancels, that J_0, J_1 in $A(t)$ from (2.8) are skew-symmetric and that R is symmetric and positive definite, where the latter was required in Theorem 5.1, we finally arrive at

$$- e^{\mathrm{T}} \begin{pmatrix} k_1 R & 0 \\ 0 & k_2 \alpha_1 \end{pmatrix} e - y^2 - \left(\frac{\bar{a}_2}{2}\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2} z\right)^2 < 0 \quad \text{for } e \neq 0. \qquad (5.11)$$

From this it follows that (5.7) admits an upper bound $-\nu(||e||_2)$ with $\nu(||e||_2)$ positive definite as was claimed. This shows that $V(e)$ is an iISS Lyapunov function and from Theorem 3.5 we can conclude that the considered system is iISS.

Remark 5.2 *Both theorems are based on the assumption that R is positive definite to show suboptimality and iISS, respectively. Since all real-world power converter systems or motors have intrinsically at least some ϵ-losses/friction, this requirement is met in practice. The converse statement, that suboptimality and iISS imply positive definiteness of R need not be true.*

Remark 5.3 *In the proof above, it is also true that $\nu(||e||_2)$ is a $\mathcal{K}_\infty$ function. For quadratic forms (5.1), the boundedness of the gradient, i.e. $||V_t(t,x)||_2 \leq \gamma_1(||x||_2), ||V_x(t,x)||_2 \leq \gamma_2(||x||_2), \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ is automatically fulfilled. So the considered class of systems is ISS following the more restrictive definiton from [53] for ISS Lyapunov functions. For the more general definition of ISS in [52] this would not be the case, since merely ISS implies iISS and the reverse statement is in general not true.*

Up to now, we have made the restriction that $\bar{g}$ is a unit vector, i.e. $\bar{g} = 1_k$ with the k-th n-dimensional unit vector 1_k. If we allow for the more general class of disturbances $Hxd + \bar{g}d$ in (2.4) with unit vector $\bar{g} = 1_k$ and matrix H with one unit row and the rest zero, i.e.

$$H^{\mathrm{T}} = \left(0^{\mathrm{T}}, \; \ldots, \; 0^{\mathrm{T}}, \; 1_i, \; 0^{\mathrm{T}}, \ldots, \; 0^{\mathrm{T}}\right)^{\mathrm{T}}$$

with $h_k = 1_i^{\mathrm{T}}$ denoting the k-th row of H with the i-th n-dimensional unit vector 1_i. The specific choice of H allows the disturbance d to affect the k-th component of the system differential equation via the i-th component of the state vector x. While $\bar{g}d$ comprises disturbances like e.g. input voltage changes or (motor shaft) load variations, H allows disturbances which can occur e.g.

via resistance variations or unknown (resistive) loads. We can use the theory from above, too, but it shows that we need an additional restriction on the state x in order to fulfill the Hamilton-Jacobi-Isaacs inequality (4.55).

The system differential equation changes to

$$M\dot{x}(t) = (J(u(t)) - R)x(t) + \bar{b}u(t) + \epsilon(t) + Hxd(t) + \bar{g}d(t), \tag{5.12}$$

while the reference system remains the same as in (2.5)

$$M\dot{x}^\star(t) \;=\; (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t).$$

With the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ the error system reads

$$M\dot{e}_x = (J_0 - R)\, e_x + \bar{b}\, e_u + J_1\Big(\underbrace{xu - xu^\star}_{= x\ e_u} + \underbrace{xu^\star - x^\star u^\star}_{= e_x\ u^\star}\Big) + Hxd + \bar{g}d.$$

Finally, since M is invertible, it follows that

$$\dot{e}_x = \underbrace{M^{-1}\,(J(u^\star) - R)}_{=:\, A(t)} e_x + \underbrace{M^{-1}\,\big(\bar{b} + J_1(e_x + x^\star)\big)}_{=:\, \tilde{b}(t, e_x)} e_u + \underbrace{M^{-1}\,(H(e_x + x^\star) + \bar{g})}_{=:\, \tilde{g}(t, e_x)} d. \tag{5.13}$$

The system output is given by $y = \tilde{c}^{\mathrm{T}} e_x$, with the unit vector $\tilde{c}^{\mathrm{T}}$ singling out the error state to be controlled. So the resulting system

$$\begin{aligned} \dot{e}_x &= A(t)e_x + \tilde{b}(t, e_x)e_u + \tilde{g}(t, e_x)d \\ y &= \tilde{c}^{\mathrm{T}} e_x, \end{aligned} \tag{5.14}$$

with $A(t), \tilde{b}(t, e_x), g(t, e_x)$ as specified in (5.13) is time-varying.

For integral feedback, use as in (2.9) the input transformation

$$e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}} \tag{5.15}$$

and the additional differential equation

$$\dot{z} = -\alpha_4 y - \alpha_1 z \tag{5.16}$$

with integrator state $z(t) \in \mathbb{R}$ and constants $\alpha_i > 0, i = 1 \ldots 4$. The introduced dynamic feedback and the input transformation lead to a differential equation driven by the new (transformed) input $e_{\bar{u}}$. Hence, defining the enlarged state $e^{\mathrm{T}} = (e_x^{\mathrm{T}}\ z)^{\mathrm{T}}$, the modified error system has the form

$$\underbrace{\frac{d}{dt}\begin{pmatrix} e_x \\ z \end{pmatrix}}_{=:\,e} = \underbrace{\begin{pmatrix} A(t)e_x - \alpha_3\tilde{b}(t,e_x)z \\ -\alpha_4 c^{\mathrm{T}} e - \alpha_1 z \end{pmatrix}}_{=:\,a(t,e)} + \underbrace{\begin{pmatrix} \alpha_2\tilde{b}(t,e_x) \\ 0 \end{pmatrix}}_{=:\,b(t,e)} e_{\bar{u}} + \underbrace{\begin{pmatrix} \tilde{g}(t,e_x) \\ 0 \end{pmatrix}}_{=:\,g(t,e)} d$$

$$\begin{aligned} \dot{e} &= a(t,e) + b(t,e)e_{\bar{u}} + g(t,e)d \\ y &= c^{\mathrm{T}}e = \left(\tilde{c}^{\mathrm{T}}\ 0\right) e \end{aligned} \tag{5.17}$$

Obviously, the modified error system (5.17) is affine-linear in $e_{\bar{u}}$ with a special structure that results from the bilinear original system (5.14) due to the feedback strategy.

Theorem 5.3 *Consider (5.17) with properties as in PCIFES. Choose*

$$\begin{aligned} V(t,e) \equiv V(e) &= \frac{1}{2}e^{\mathrm{T}}Pe, \quad \forall e \in \mathbb{R}^{n+1} \\ P &= \begin{pmatrix} k_1 M & 0 \\ 0 & k_2 \end{pmatrix}, \ P = P^{\mathrm{T}} > 0,\ k_1, k_2 > 0. \end{aligned} \tag{5.18}$$

Assume that matrix R from BPCS is positive definite with minimal eigenvalue $\lambda_{\min}$ and that we have a bounded state x_i for fixed $i \in \{1, 2, \ldots, n\}$ with bound $|x_i| \le x_{\max}, x_{\max} \in \mathbb{R}^+$. Furthermore, assume there exist positive constants $k_1, k_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma, x_{\max}$, such that for

1) $e_x^{\mathrm{T}}(Hx+\bar{g})(Hx+\bar{g})^{\mathrm{T}}e_x = y^{\mathrm{T}}y$ we have $X_1 > 0, X_2 \ge 0$ with

$$X_1 := k_1\lambda_{\min} - \frac{1}{2} - \frac{k_1^2(x_i+1)^2}{2\gamma^2}, X_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2\alpha_4^2}{4X_1}\right).$$

2) $e_x^{\mathrm{T}}(Hx+\bar{g})(Hx+\bar{g})^{\mathrm{T}}e_x \ne y^{\mathrm{T}}y$ we have $\tilde{X}_1 > 0, \tilde{X}_2 \ge 0, \tilde{X}_3 > 0$ with

$$\begin{aligned} \tilde{X}_1 &:= k_1\lambda_{\min} - \frac{1}{2}, \tilde{X}_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2\alpha_4^2}{4\tilde{X}_1}\right), \\ \tilde{X}_3 &:= \left(\lambda_{\min} - \frac{k_1(x_i+1)^2}{2\gamma^2}\right). \end{aligned}$$

Then, $V(e)$ is a solution of the tHJIi (4.55).

Proof 5.3 *Rewrite the elements of $a(t,e)$ as*

$$a(t,e) = \underbrace{\begin{pmatrix} A(t) & 0 \\ 0 & -\alpha_1 \end{pmatrix}}_{\bar{A}_1(t)} e - \frac{\alpha_3}{\alpha_2}b(t,e)z - \underbrace{\begin{pmatrix} 0 \\ \alpha_4 \end{pmatrix}}_{\bar{a}_2} y. \tag{5.19}$$

Insert (5.19) into the left hand side of (4.55) and introduce squares:

$$V_t + V_e\bar{A}_1(t)e - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2}z\right)^2 + \frac{\alpha_3^2}{2\alpha_2^2}z^2 + \frac{(V_e g)^2}{2\gamma^2} - V_e\bar{a}_2 y + \frac{1}{2}y^2.$$

Use the quadratic function from (5.18) and simplify to

$$-k_1 e_x^{\mathrm{T}} R e_x - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2}z\right)^2 - \alpha_4 k_2 yz + \frac{(e^{\mathrm{T}}Pg)^2}{2\gamma^2} - \alpha_1 k_2 z^2 + \frac{\alpha_3^2}{2\alpha_2^2}z^2 + \frac{1}{2}y^2.$$

The vector g may be rewritten as $g = \left((M^{-1}(Hx + 1_k))^{\mathrm{T}}, 0\right)^{\mathrm{T}}$ for some $k \in \{1, 2, \ldots, n\}$ with $1_k \in \mathbb{R}^n$ the k-th n-dimensional unit vector and H the $n \times n$ matrix with the k-th row containing a unit vector 1_i^{T} and the rest zero; the last element of g is zero. Pg then reduces to $k_1\left((Hx + 1_k)^{\mathrm{T}}, 0\right)^{\mathrm{T}}$ and finally

$$e^{\mathrm{T}}(Pg) = e_x^{\mathrm{T}} k_1 (Hx + 1_k) = k_1(e_{x,k}x_i + e_{x,k}) = k_1 e_{x,k}(x_i + 1) \tag{5.20}$$

with $e_{x,k}$ being the k-th component of e_x. Furthermore, we know that $y = e_{x,j}$ for some $j \in \{1, 2, \ldots, n\}$.

Now let R from BPCS be positive definite. Since R is symmetric, we know that $\lambda_{\min} e^{\mathrm{T}}e \leq e^{\mathrm{T}}Re \leq \lambda_{\max} e^{\mathrm{T}}e$, with $\lambda_{\min}, \lambda_{\max} > 0$ the minimal/maximal eigenvalue of R. Furthermore, remember that we have $|x_i| \leq x_{\max}$ for some $i \in \{1, 2, \ldots, n\}$.

For fixed $i, j, k \in \{1, 2, \ldots, n\}$ distinguish the following two cases:

1) $e_x^{\mathrm{T}}(Hx + \bar{g})(Hx + \bar{g})^{\mathrm{T}} e_x = y^{\mathrm{T}} y$: Further introducing squares leads to

$$-k_1 \sum_{\substack{l=0 \\ l \neq j}}^{n} \lambda_{\min} e_{x,l}^2 - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2}z\right)^2 - \left(\sqrt{X_1}y + \frac{k_2\alpha_4}{2\sqrt{X_1}}z\right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2\alpha_4^2}{4X_1}\right)}_{=:\, X_2} z^2,$$

which is clearly negative semidefinite since we assumed that there exist appropriate constants such that $X_1 > 0$, $X_2 \geq 0$.

2) $e_x^{\mathrm{T}}(Hx+\bar{g})(Hx+\bar{g})^{\mathrm{T}}e_x \neq y^{\mathrm{T}}y$*:*

$$-k_1\sum_{\substack{l=0\\ l\neq j,k}}^{n}\lambda_{\min}e_{x,l}^2-\left(\frac{1}{\sqrt{2}}e_{\bar{u}}^{+}-\frac{\alpha_3}{\sqrt{2}\alpha_2}z\right)^2-\left(\sqrt{\tilde{X}_1}y+\frac{k_2\alpha_4}{2\sqrt{\tilde{X}_1}}z\right)^2$$
$$-\underbrace{\left(\alpha_1k_2-\frac{\alpha_3^2}{2\alpha_2^2}-\frac{k_2^2\alpha_4^2}{4\tilde{X}_1}\right)}_{=:\,\tilde{X}_2}z^2-k_1\underbrace{\left(\lambda_{\min}-\frac{k_1(x_i+1)^2}{2\gamma^2}\right)}_{=:\,\tilde{X}_3}e_{x,k}^2,$$

which is clearly negative semidefinite since we assumed that there exist appropriate constants such that $\tilde{X}_1,\tilde{X}_3>0$*,* $\tilde{X}_2\geq 0$*.*

The additional requirement $|x_i|\leq x_{\max}$ restricts the state space such that the state feedback law is only locally valid. If the realistic operating range is known and $x_{\max}$ is set large enough, this side condition is probably fulfilled. But it cannot be guaranteed by the theory itself and therefore this restriction is more difficult to achieve in practice.

Now the next problem is to find some match for $\nu(||e||_2)$, i.e. such that the lhs of (5.7) is negative definite. This is done in the following theorem:

Theorem 5.4 *Consider the closed-loop of* (5.17) *using the suboptimal control law* (4.50)*. Let* $\Delta(||d||_2)=(\gamma^2/2)d^2$ *and the conditions of Theorem 5.3 be valid. Then, for* $V(e)$ *from (5.1), there exists a positive definite estimate* $\nu(||e||_2)$ *for* (5.7)

$$V_t+V_e a-(e_{\bar{u}}^{+})^2+\frac{(V_e g)^2}{2\gamma^2}\leq-\nu(||e||_2)<0, \tag{5.21}$$

such that $V(e)$ *is an iISS Lyapunov function and therefore the closed-loop error system is iISS.*

Proof 5.4 *Follow the lines of Proof 5.2, which does not change here for the more general class of disturbances. The only difference is that the conditions of Theorem 5.3 with state space restriction* $|x_i|\leq x_{\max}$ *are preconditioned instead of (the more special case in) Theorem 5.1.*

To conclude, Theorem 5.3 provides the suboptimal state feedback law in the following way: First of all, extract from BPCS the appropriate variables to build function V. Then decide, which state variable is the output y in the underlying application. With this information, either case 1) or case 2) has to be fulfilled to gurantee suboptimality. The choice of the parameters in 1) or 2) is discussed in Chapter 6 for a specific application example which makes this subject easier to grasp. When all conditions of Theorem 5.3 are fulfilled, we can calculate the state feedback law $e_{\bar{u}}^{+}$ which is hence suboptimal. Finally, we know from Theorem 5.4 that the closed-loop system is iISS and uniformly asymptotically stable for vanishing disturbances.

5.3 Disturbance Feedback

In the case of disturbance feedback, we approach the problem finding the suboptimal $\mathcal{H}_\infty$ state feedback control law according to what was done for integral feedback. Furthermore, we set $h = \check{y}$ with the output $\check{y}$ to be controlled which we want to keep small in the presence of disturbances.

Theorem 5.5 *Consider (2.15) from PCDFS in Definition 2.4. Choose*

$$\begin{aligned} V(t,\check{e}) \equiv V(\check{e}) &= \frac{1}{2}\check{e}^{\mathrm{T}} P \check{e}, \quad \forall \check{e} \in \mathbb{R}^{n+1} \\ P &= \begin{pmatrix} k_1 M & 0 \\ 0 & 1 \end{pmatrix}, \; P = P^{\mathrm{T}} > 0,\; k_1 > 0. \end{aligned} \tag{5.22}$$

Assume that matrix R from BPCS is positive definite with minimal eigenvalue $\lambda_{\min}$ and that there exist positive constants k_1, α_5, l, γ, such that for

1) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x = y^{\mathrm{T}} y$ we have $\check{X} \geq 0$ with

$$\check{X} := \left(k_1 \lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5} - \frac{k_1^2}{2\gamma^2} \right).$$

2) $e_x^{\mathrm{T}} \bar{g} \bar{g}^{\mathrm{T}} e_x \neq y^{\mathrm{T}} y$ we have $\check{X}_1 \geq 0, \check{X}_2 > 0$ with

$$\check{X}_1 := \left(k_1 \lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5} \right), \check{X}_2 := \left(\lambda_{\min} - \frac{k_1}{2\gamma^2} \right).$$

Then $V(\check{e})$ is a solution of tHJIi (4.55).

Proof 5.5 *Rewrite the elements of $\check{a}(t,\check{e})$ as*

$$\check{a}(t,\check{e}) = \begin{pmatrix} A(t) e_x \\ -l c^{\mathrm{T}} e - \alpha_5 \hat{d} \end{pmatrix} = \underbrace{\begin{pmatrix} A(t) & 0 \\ 0 & -\alpha_5 \end{pmatrix}}_{\check{A}_1(t)} \check{e} - \underbrace{\begin{pmatrix} 0 \\ l \end{pmatrix}}_{\check{a}_2} \check{y}. \tag{5.23}$$

Insert (5.23) into the left hand side of (4.55) and introduce squares:

$$V_t + V_{\check{e}}\check{A}_1(t)\check{e} - \frac{1}{2}(e_{\bar{u}}^{+})^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} - V_{\check{e}}\check{a}_2\check{y} + \frac{1}{2}\check{y}^2.$$

Use the quadratic function from (5.22) and simplify to

$$-k_1 e_x^{\mathrm{T}} R e_x - \frac{1}{2}(e_{\bar{u}}^{+})^2 - \left(\sqrt{\alpha_5}\hat{d} + \frac{l\check{y}}{2\sqrt{\alpha_5}}\right)^2 + \frac{l^2}{4\alpha_5}\check{y}^2 + \frac{(\check{e}^{\mathrm{T}} P\check{g})^2}{2\gamma^2} + \frac{1}{2}\check{y}^2. \quad (5.24)$$

The vector $\check{g}$ may be rewritten as $\check{g} = \left((M^{-1}1_k)^{\mathrm{T}}, 0\right)^{\mathrm{T}}$ for some $k \in \{1, 2, \dots, n\}$ with $1_k \in \mathbb{R}^n$ the k-th n-dimensional unit vector; the last element of $\check{g}$ is zero. $P\check{g}$ then reduces to $k_1 1_k$ and finally

$$\check{e}^{\mathrm{T}}(P\check{g}) = e_x^{\mathrm{T}} k_1 1_k = k_1 e_{x,k} \quad (5.25)$$

with $e_{x,k}$ being the $k-$th component of e_x. Furthermore, we know that $y = e_{x,j}$ for some $j \in \{1, 2, \dots, n\}$.

Now let matrix R from BPCS be positive definite. Since R is symmetric, we know that $\lambda_{\min} e^{\mathrm{T}} e \le e^{\mathrm{T}} R\, e \le \lambda_{\max} e^{\mathrm{T}} e$, with $\lambda_{\min}, \lambda_{\max}$ the minimal/maximal eigenvalue of R.

For fixed $j, k \in \{1, 2, \dots, n\}$ distinguish the following two cases:

1) $e_x^{\mathrm{T}}\bar{g}\bar{g}^{\mathrm{T}} e_x = y^{\mathrm{T}} y$: Further introducing squares leads to

$$-k_1 \sum_{\substack{i=0 \\ i\neq j}}^{n} \lambda_{\min} e_{x,i}^2 - \frac{1}{2}(e_{\bar{u}}^{+})^2 - \left(\sqrt{\alpha_5}\hat{d} + \frac{l\check{y}}{2\sqrt{\alpha_5}}\right)^2 - \underbrace{\left(k_1\lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5} - \frac{k_1^2}{2\gamma^2}\right)}_{=: \check{X} \ge 0} \check{y}^2,$$

which is negative semidefinite due to the assumption that there exist appropriate constants such that $\check{X} \ge 0$.

2) $e_x^{\mathrm{T}}\bar{g}\bar{g}^{\mathrm{T}} e_x \neq y^{\mathrm{T}} y$:

$$-k_1 \sum_{\substack{i=0 \\ i\neq j,k}}^{n} \lambda_{\min} e_{x,i}^2 - \frac{1}{2}(e_{\bar{u}}^{+})^2 - \left(\sqrt{\alpha_5}\hat{d} + \frac{l\check{y}}{2\sqrt{\alpha_5}}\right)^2 - \underbrace{\left(k_1\lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5}\right)}_{=: \check{X}_1 \ge 0} \check{y}^2 - k_1 \underbrace{\left(\lambda_{\min} - \frac{k_1}{2\gamma^2}\right)}_{=: \check{X}_2 > 0} e_{x,k}^2,$$

which is negative semidefinite due to the assumption that there exist appropriate constants such that $\check{X}_1 \geq 0, \check{X}_2 > 0$.

Remark 5.4 *As was mentioned in Remark 5.1, we replace* $\lambda_{\min}$ *by the appropriate diagonal element of* R *for matrix* R *diagonal.*

Again, we have to prove in the next step that function V which fulfills Theorem 5.5 is at the same time an iISS Lyapunov function for the closed-loop of PCDFS, i.e. with the suboptimal state feedback control law $e_{\check{u}}^{+}$ inserted into (2.15). We can use the same reasoning as in the case of integral feedback replacing the elements in the derivation with the appropriate notions in terms of PCDFS. This means we get

$$\dot{V} = V_t + V_{\check{e}}\left(\check{A}\check{e} + \check{b}e_{\check{u}}^{+} + \check{g}\check{d}\right) \tag{5.26}$$

$$\leq V_t + V_{\check{e}}\check{A}\check{e} - (e_{\check{u}}^{+})^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} + \frac{\gamma^2}{2}\check{d}^2. \tag{5.27}$$

where we may identify $\Delta(||\check{d}||_2) = (\gamma^2/2)\check{d}^2$. The problem is to find some match for $\nu(||\check{e}||_2)$. Therefore, we rewrite (5.27) with the help of inequality (4.55), which results in

$$\begin{aligned}\dot{V} &\leq V_t + V_{\check{e}}\check{A}\check{e} - (e_{u}^{+})^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} + \frac{\gamma^2}{2}\check{d}^2 \\ &\leq -\frac{1}{2}h^2 - \frac{1}{2}(e_{u}^{+})^2 + \frac{\gamma^2}{2}\check{d}^2.\end{aligned} \tag{5.28}$$

Hence, setting $\check{d} \equiv 0$ in (5.28) it is clear that

$$V_t + V_{\check{e}}\check{A}\check{e} - (e_{u}^{+})^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} \leq 0 \tag{5.29}$$

is at least negative semidefinite. For this reason, we have to show that we can match negative definiteness for the lhs of (5.29), that means that there is a negative definite function $-\nu(||\check{e}||_2)$ which bounds (5.29) from above, such that the closed-loop system is iISS.

Hence, for the system under investigation we can formulate the following theorem:

Theorem 5.6 *Consider the closed-loop of* (2.15) *using the suboptimal control law* (4.50). *Let* $\Delta(||\check{d}||_2) = (\gamma^2/2)\check{d}^2$ *and the conditions of Theorem 5.5 be* *valid. Then, for* $V(\check{e})$ *from (5.22), there exists a positive definite estimate* $\nu(||\check{e}||_2)$ *for (5.7)*

$$V_t + V_{\check{e}}\check{a} - (e_{\bar{u}}^{+})^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} \leq -\nu(||\check{e}||_2) < 0, \tag{5.30}$$

such that $V(\check{e})$ *is an iISS Lyapunov function and therefore the closed-loop error system is iISS.*

Proof 5.6 *Consider (2.8),(2.15).* V *is a quadratic form and so it immediately follows that* $V \in C^1 \cap UPPD$*. Now use again (5.23), insert it into (5.7) and use the suboptimal control law* $e_{\bar{u}}^{+}(t,\check{e}) = -\check{b}^{\mathrm{T}}(t,\check{e})V_{\check{e}}^{\mathrm{T}}$ *from (4.50) to get*

$$V_t + V_{\check{e}}\check{A}_1\check{e} - V_{\check{e}}\check{a}_2\check{y} - \underbrace{V_{\check{e}}\check{b}\check{b}^{\mathrm{T}}V_{\check{e}}^{\mathrm{T}}}_{e_{\bar{u}}^{+2}} + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} \leq 0$$

which was already proven to be negative semidefinite. For negative definiteness further introducing squares yields

$$V_t + V_{\check{e}}\check{A}_1\check{e} - \check{y}^2 - \left(\frac{1}{2}V_{\check{e}}\check{a}_2\right)^2 - \left(e_{\bar{u}}^{+}\right)^2 + \frac{(V_{\check{e}}\check{g})^2}{2\gamma^2} + \left(\frac{V_{\check{e}}\check{a}_2}{2} - \check{y}\right)^2 \leq 0. \tag{5.31}$$

We may omit the positive quadratic terms without losing the validity of the inequality

$$V_t + V_{\check{e}}\check{A}_1\check{e} - \check{y}^2 - \left(\frac{1}{2}V_{\check{e}}\check{a}_2\right)^2 - \left(e_{\bar{u}}^{+}\right)^2 \leq 0. \tag{5.32}$$

As can be seen, $V(\check{e})$ *from (5.22) is time-invariant such that* $V_t \equiv 0$*. Inserting into (5.32) and using that* M *cancels, that* J_0, J_1 *in* $A(t)$ *from (2.8) are skew-symmetric and that* R *is symmetric and positive definite, where the latter arises from the assumptions of Theorem 5.5, we finally arrive at*

$$-\check{e}^{\mathrm{T}}\begin{pmatrix} k_1 R & 0 \\ 0 & \alpha_5 \end{pmatrix}\check{e} - \check{y}^2 - \left(\frac{\check{a}_2}{2}\right)^2 - \left(e_{\bar{u}}^{+}\right)^2 < 0 \text{ for } \check{e} \neq 0. \tag{5.33}$$

From this it follows that (5.7) admits an upper bound $-\nu(||\check{e}||_2)$ *with* $\nu(||\check{e}||_2)$ *positive definite as was claimed. This shows that* $V(\check{e})$ *is an iISS Lyapunov function and from Theorem 3.5 we can conclude that the considered system is iISS.*

Remark 5.5 *The restriction that* R *is positive definite is usually met in practice since all real-world power converter systems have at least some* ϵ*-losses/ friction. The converse statement, that suboptimality and iISS imply positive* *definiteness of R need not be true (see also Remark 5.2). In the proof above, it is also true that $\nu(||\check{e}||_2)$ is a $\mathcal{K}_\infty$-function. So the considered class of systems is (as was pointed out in Remark 5.3) ISS following the more restrictive definition from [53] for ISS Lyapunov functions which was used in Chapter 3.*

In order to calculate the suboptimal state feedback with help of Theorem 5.5, extract from BPCS the appropriate variables to build function V. Then decide which state variable is the output $\check{y}$ in the underlying application. With this information, choose the suitable case 1) or 2) which has to be fulfilled to gurantee suboptimality. The choice of the parameters in 1) or 2) is discussed in Chapter 6 for a specific application example which makes this subject easier to grasp. When all conditions of Theorem 5.5 are fulfilled, we can calculate the state feedback law $e_{\check{u}}^{+}$ which is hence suboptimal. Finally, we know from Theorem 5.6 that the closed-loop system is iISS and uniformly asymptotically stable for vanishing disturbances. What we have not yet discussed is how to replan online the reference trajectory, since for disturbance feedback, the reference system contains $d^\star$ (cf. (2.13)), which is replaced by the additional state $\hat{d}$ (cf. (2.15)) in (2.14). For the specific example of the boost converter / DC motor in Chapter 6, a suitable strategy for replanning is proposed. However, a general solution for the whole system class, which is at the same time practically feasible, cannot be presented in this thesis.

5.4 General Bilinear Systems

Up to now we considered only a special class of bilinear systems because J_1 in (2.1) of BPCS is skew-symmetric. This property arose in the context of power converter systems since it takes account of the Kirchhoff current and voltage laws. We now remove this restriction and consider the general case. We will show that the proposed approach for bilinear power converter systems can be extended accordingly for integral and disturbance feedback, even in the multiple-input case. Since solving the tracking problem later on for the multiple-input case seems to be intractable for disturbance feedback in a real world situation, we concentrate on the integral feedback case and omit the disturbance feedback case.

5.4.1 Single-Input case

Henceforth, we refer to the system differential equation

$$M\dot{x}(t) = Fx(t) + (\bar{b} + Nx(t))u(t) + \epsilon(t), \quad x(t_0) = x_0 \tag{5.34}$$

with all entries as defined in (2.1) for BPCS but with the additional constant matrix $N \in \mathbb{R}^{n\times n}$. Use

$$R_0 := -\frac{1}{2}(F + F^{\mathrm{T}}),\ J_0 := \frac{1}{2}(F - F^{\mathrm{T}}),\ F = J_0 - R_0 \tag{5.35}$$

$$R_1 := -\frac{1}{2}(N + N^{\mathrm{T}}),\ J_1 := \frac{1}{2}(N - N^{\mathrm{T}}),\ N = J_1 - R_1 \tag{5.36}$$

$$J(u(t)) := J_0 + J_1 u(t),\ R(u(t)) := R_0 + R_1 u(t) \tag{5.37}$$

with symmetric matrices $R_0, R_1, R(u(t)) \in \mathbb{R}^{n\times n}$ and skew-symmetric matrices $J_0, J_1, J(u(t)) \in \mathbb{R}^{n\times n}$ to rewrite the system equations as

$$M\dot{x}(t) = (J_0 - R_0)x(t) + \bar{b}u(t) + (J_1 - R_1)x(t)u(t) + \epsilon(t) \tag{5.38}$$

$$= [J(u(t)) - R(u(t))]x(t) + \bar{b}u(t) + \epsilon(t). \tag{5.39}$$

Note that we have not included any disturbances. As was done for the power converter systems, we consider an additive disturbance $d(t) \in \mathbb{R}$, a piecewise continuous and bounded function, which affects the system via $\bar{g} \in \mathbb{R}^n$:

$$M\dot{x}(t) = [J(u(t)) - R(u(t))]x(t) + \bar{b}u(t) + \epsilon(t) + \bar{g}d(t). \tag{5.40}$$

Again, we will assume that $\bar{g}$ is a unit vector, i.e. $\bar{g} = 1_k$ with 1_k denoting the k-th n-dimensional unit vector. To obtain an error dynamics representation, we first introduce a reference system

$$M\dot{x}^\star(t) \ = \ [J(u^\star(t)) - R(u^\star(t))]\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t) \tag{5.41}$$

with the reference solution $x^\star(\cdot, t_0, x_0, u^\star)$. Again, we want to point out that the occurring disturbance is not included, since it is not known in the reference system in advance.

In the following, we will skip the time argument of state and input for brevity.

With the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ the error system reads

$$M\dot{e}_x = (J_0 - R_0)\, e_x + \bar{b}\, e_u + (J_1 - R_1)\Big(\underbrace{xu - xu^\star}_{=\,x\ e_u} + \underbrace{xu^\star - x^\star u^\star}_{=\,e_x\ u^\star}\Big) + \bar{g}d. \tag{5.42}$$

Finally, since M is invertible, it follows that

$$\dot{e}_x = \underbrace{M^{-1}\left(J(u^\star) - R(u^\star)\right)}_{=:\,A(t)} e_x + \underbrace{M^{-1}\left(\bar{b} + (J_1 - R_1)(e_x + x^\star)\right)}_{=:\,\tilde{b}(t,e_x)} e_u + \underbrace{M^{-1}\bar{g}}_{=:\,\tilde{g}}\, d. \tag{5.43}$$

The system output is given by $y = \tilde{c}^{\mathrm{T}} e_x$, with the unit vector $\tilde{c}^{\mathrm{T}}$ singling out the error state to be controlled. So the resulting system

$$\begin{aligned} \dot{e}_x &= A(t)e_x + \tilde{b}(t,e_x)e_u + \tilde{g}d \\ y &= \tilde{c}^{\mathrm{T}} e_x, \end{aligned} \tag{5.44}$$

with $A(t), \tilde{b}(t,e_x)$ as specified in (5.43) is time-varying.

In order to include integral feedback, we make again the same steps as before with the input transformation $e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}}$ and the dynamic feedback strategy. So we arrive at the same structure as in (2.11) with properties as in PCIFES

$$\begin{aligned} \dot{e} &= a(t,e) + b(t,e)e_{\bar{u}} + gd \\ y &= c^{\mathrm{T}} e = \begin{pmatrix} \tilde{c}^{\mathrm{T}} & 0 \end{pmatrix} e \end{aligned} \tag{5.45}$$

with

$$e = \begin{pmatrix} e_x \\ z \end{pmatrix}, a(t,e) = \begin{pmatrix} A(t)e_x - \alpha_3 \tilde{b}(t,e_x) z \\ -\alpha_4 c^{\mathrm{T}} e - \alpha_1 z \end{pmatrix}, b(t,e) = \begin{pmatrix} \alpha_2 \tilde{b}(t,e_x) \\ 0 \end{pmatrix}, g = \begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix}, \tag{5.46}$$

but now with elements from (5.44), (5.43).

What changes in the proofs is the step concerning positive definiteness of matrix R since we now have $R(u^\star)$. Of course, the reference trajectories and the reference input $u^\star$ need to be bounded. Therefore, we assume that we have a bounded $u^\star$ such that $R(u^\star)$ remains positive definite, i.e. there exist real numbers $\bar{c} > \underline{c} > 0$ such that

$$\underline{c}\, e^{\mathrm{T}} e \leq e^{\mathrm{T}} R(u^\star) e \leq \bar{c}\, e^{\mathrm{T}} e.$$

Theorem 5.7 *Consider* (5.45) *with the special structure of* (5.46) *and further properties as in PCIFES. Choose*

$$V(t,e) \equiv V(e) = \frac{1}{2}e^{\mathrm{T}}Pe, \quad \forall e \in \mathbb{R}^{n+1}$$
$$P = \begin{pmatrix} k_1 M & 0 \\ 0 & k_2 \end{pmatrix}, \; P = P^{\mathrm{T}} > 0, \; k_1, k_2 > 0. \tag{5.47}$$

Assume that there exist $\overline{c} > \underline{c} > 0, \underline{c}, \overline{c} \in \mathbb{R}$ *such that*

$$\underline{c}e^{\mathrm{T}}e \leq e^{\mathrm{T}}R(u^{\star}(t))e \leq \overline{c}e^{\mathrm{T}}e, \quad \forall t \geq t_0,$$

and that there exist positive constants $k_1, k_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma$ *such that for*

1) $e_x^{\mathrm{T}}\bar{g}\bar{g}^{\mathrm{T}}e_x = y^{\mathrm{T}}y$ *we have* $X_1 > 0, X_2 \geq 0$ *with*

$$X_1 := k_1\underline{c} - \frac{1}{2} - \frac{k_1^2}{2\gamma^2}, X_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2\alpha_4^2}{4X_1}\right).$$

2) $e_x^{\mathrm{T}}\bar{g}\bar{g}^{\mathrm{T}}e_x \neq y^{\mathrm{T}}y$ *we have* $\tilde{X}_1 > 0, \tilde{X}_2 \geq 0, \tilde{X}_3 > 0$ *with*

$$\tilde{X}_1 := k_1\underline{c} - \frac{1}{2}, \tilde{X}_2 := \left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2\alpha_4^2}{4\tilde{X}_1}\right), \tilde{X}_3 := \left(\underline{c} - \frac{k_1}{2\gamma^2}\right).$$

Then, $V(e)$ *is a solution of the tHJIi* (4.55)*.*

Proof 5.7 *Follow the steps of Proof 5.1.*

Theorem 5.8 *Consider the closed-loop of* (5.46) *using the suboptimal control law* (4.50)*. Let* $\Delta(||d||_2) = (\gamma^2/2)d^2$ *and the conditions of Theorem 5.7 be valid. For* $V(e)$ *from (5.47), there exists a positive definite estimate* $\nu(||e||_2)$ *for* (5.7)

$$V_t + V_e a - (e_{\tilde{u}}^{+})^2 + \frac{(V_e g)^2}{2\gamma^2} \leq -\nu(||e||_2) < 0,$$

such that $V(e)$ *is an iISS Lyapunov function and therefore the closed-loop error system is iISS.*

Proof 5.8 *Follow the steps of Proof 5.2.*

As was discussed for the bilinear power converter systems, the procedure for obtaining the suboptimal state feedback law in the case of general bilinear systems is almost the same. What changes is that we have to check the positive definiteness of $R(u^\star)$. Therefore, we have to calculate first the bounded reference solution and the corresponding bounded reference input $u^\star$ and check explicitly that $R(u^\star(t))$ remains positive definite for each time instant $t \geq t_0$, i.e. calculate the eigenvalues for $R(u^\star)$ for each $t \geq t_0$, investigate the determinant of all principal minors of $R(u^\star)$ for each $t \geq t_0$, etc., which means much more design effort and becomes quickly demanding, especially for increasing dimension.

5.4.2 Multiple-Input case

The step from single-input to multiple-input bilinear systems is not too difficult but increases as usual the notational complexity and readability. Consider

$$M\dot{x}(t) = Fx(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + \bar{B}u(t) + \sum_{j=1}^{q} H_j x(t) d_j(t) + \bar{G}d(t) + \epsilon(t), x(t_0) = x_0, \quad (5.48)$$

with state $x(t) \in \mathbb{R}^n$, piecewise continuous and bounded input $u(t) \in \mathbb{R}^m$ and disturbance $d(t) \in \mathbb{R}^q$, piecewise continuous and bounded function $\epsilon(t) \in \mathbb{R}^n$, matrices $\bar{B} \in \mathbb{R}^{n\times m}, \bar{G} \in \mathbb{R}^{n\times q}$, matrices $F, N_i, H_j \in \mathbb{R}^{n\times n}, i = 1,\dots,m, j = 1,\dots,q$, symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$. Use

$$R_0 := -\frac{1}{2}(F + F^{\mathrm{T}}),\ J_0 := \frac{1}{2}(F - F^{\mathrm{T}}),\ F = J_0 - R_0 \quad (5.49)$$

$$R_{1i} := -\frac{1}{2}(N_i + N_i^{\mathrm{T}}),\ J_{1i} := \frac{1}{2}(N_i - N_i^{\mathrm{T}}),\ N_i = J_{1i} - R_{1i} \quad (5.50)$$

$$R_{2j} := -\frac{1}{2}(H_j + H_j^{\mathrm{T}}),\ J_{2j} := \frac{1}{2}(H_j - H_j^{\mathrm{T}}),\ H_j = J_{2j} - R_{2j} \quad (5.51)$$

$$J(u(t)) := J_0 + \sum_{i=1}^{m} J_{1i} u_i(t),\ R(u(t)) := R_0 + \sum_{i=1}^{m} R_{1i} u_i(t) \quad (5.52)$$

with symmetric matrices $R_0, R_{1i}, R_{2j}, R(u(t)) \in \mathbb{R}^{n\times n}$ and skew-symmetric matrices $J_0, J_{1i}, J_{2j}, J(u(t)) \in \mathbb{R}^{n\times n}$ to rewrite the system equations as

$$M\dot{x}(t) = [J(u(t)) - R(u(t))]x(t) + \bar{B}u(t) + \sum_{j=1}^{q} H_j x(t) d_j(t) + \bar{G}d(t) + \epsilon(t). \quad (5.53)$$

To obtain an error dynamics representation, we first introduce a reference system

$$M\dot{x}^\star(t) \;=\; \left[J(u^\star(t)) - R(u^\star(t))\right] x^\star(t) + \bar{B}u^\star(t) + \epsilon(t) \tag{5.54}$$

with the reference solution $x^\star(\cdot, t_0, x_0, u^\star)$.

In the following, we will skip the time argument of state and input for brevity.

Using the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ and using the fact that M is invertible since it is symmetric and positive definite results in the error system

$$\begin{aligned} \dot{e}_x = & \underbrace{M^{-1}\left(J(u^\star) - R(u^\star)\right)}_{=:\,A(t)} e_x + \sum_{i=1}^{m} \underbrace{M^{-1}\left(\bar{B}_i + (J_{1i} - R_{1i})(e_x + x^\star)\right)}_{=:\,\tilde{B}_i(t,e_x)} e_{u_i} \\ & + \sum_{j=1}^{q} \underbrace{M^{-1}[\bar{G}_j + H_j(e_x + x^\star)]}_{=:\,\tilde{G}_j(t,e_x)} d_j(t) \end{aligned} \tag{5.55}$$

where e_{u_i} denotes the i-th component of the error input vector e_u, d_j the j-th component of the disturbance vector, and $\bar{B} = (\bar{B}_1, \ldots, \bar{B}_m)$,$\bar{G} = (\bar{G}_1, \ldots, \bar{G}_q)$. The outputs to be controlled are given by $y = \tilde{C}e_x, \tilde{C} \in \mathbb{R}^{p\times n}$, where the rows $\tilde{c}_i, i = 1, \ldots, p$ of matrix $\tilde{C}$ are unit vectors and specify the error state to be controlled. So the resulting system

$$\begin{aligned} \dot{e}_x &= A(t)e_x + \tilde{B}(t, e_x)e_u + \tilde{G}(t, e_x)d \\ y &= \tilde{C}e_x, \end{aligned} \tag{5.56}$$

with

$$A(t), \tilde{B}(t, e_x) = \left(\tilde{B}_1(t, e_x), \ldots, \tilde{B}_m(t, e_x)\right), \tilde{G}(t, e_x) = \left(\tilde{G}_1(t, e_x), \ldots, \tilde{G}_q(t, e_x)\right)$$

as specified in (5.55) is time-varying.

As discussed in Section 5.2, it would be possible to use specific H_j to prove suboptimality and stability with appropriate bounds on the system state x. For simplicity, we stick to the simpler case and set $H_j = 0, j = 1 \ldots q$. In addition, $\bar{G}$ is assumed to be a matrix of full rank with unit vectors as columns.

In order to include integral feedback, we use again the same steps as before, but we have to take account of the number p of outputs y to be controlled and the m available control inputs e_u. Let us define the input transformation

$$e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}}, \qquad e_{\bar{u}} \in \mathbb{R}^m, z \in \mathbb{R}^p, \alpha_2 \in \mathbb{R}^{m\times m}, \alpha_3 \in \mathbb{R}^{m\times p}$$

and introduce the integrator state

$$\dot{z} = -\alpha_4 Ce - \alpha_1 z, \qquad \alpha_1 \in \mathbb{R}^{p\times p}, \alpha_4 \in \mathbb{R}^{p\times p}$$

where $y = Ce$ is the new output with $e := \begin{pmatrix} e_x^{\mathrm{T}} & z^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ and $C := \begin{pmatrix} \tilde{C} & 0 \end{pmatrix}$. Then we arrive at the same structure as in (5.44)

$$\begin{aligned} \dot{e} &= a(t,e) + B(t,e)e_{\bar{u}} + Gd \\ y &= Ce = \begin{pmatrix} \tilde{C} & 0 \end{pmatrix} e \end{aligned} \tag{5.57}$$

with

$$e = \begin{pmatrix} e_x \\ z \end{pmatrix}, a(t,e) = \begin{pmatrix} A(t)e_x - \tilde{B}(t,e_x)\alpha_3 z \\ -\alpha_4 Ce - \alpha_1 z \end{pmatrix}, \tag{5.58}$$

$$B(t,e) = \begin{pmatrix} \tilde{B}(t,e_x)\alpha_2 \\ 0 \end{pmatrix}, G = \begin{pmatrix} \tilde{G} \\ 0 \end{pmatrix}, \tag{5.59}$$

but now with elements from (5.56), (5.55) with (5.49)-(5.52).

Then, 3 scenarios can be identified, similar to what can be found in [66],[67] for linear optimal integral control [68]:

1. There are more additional states z than inputs $e_{\bar{u}}$, i.e. $p > m$.
2. The number of additional states z is equivalent to the number of inputs $e_{\bar{u}}$, i.e. $p = m$.
3. There are less additional states z than inputs $e_{\bar{u}}$, i.e. $p < m$.

The first case is practically not very reasonable and makes the problem unsolvable, the third case has too many degrees of freedom, but would be feasible if we add additional states until $p = m$, so we finally restrict ourselves to the case $p = m$.

Theorem 5.9 *Consider* (5.56), (5.55) *with* (5.49)-(5.52) *and* $p = m$. *Choose*

$$\begin{aligned} V(t,e) &\equiv V(e) = \frac{1}{2} e^{\mathrm{T}} P e, \quad \forall e \in \mathbb{R}^{n+p} \\ P &= \begin{pmatrix} k_1 M & 0 \\ 0 & K_2 \end{pmatrix}, \; k_1 \in \mathbb{R}, K_2 \in \mathbb{R}^{p\times p}, K_2 = K_2^{\mathrm{T}}, \end{aligned} \tag{5.60}$$

where $k_1 > 0, K_2 > 0$ *and therefore* $P = P^{\mathrm{T}} > 0$.

Assume that there exist $\overline{c} > \underline{c} > 0, \underline{c}, \overline{c} \in \mathbb{R}$ *such that*

$$\underline{c}e^{\mathrm{T}}e \leq e^{\mathrm{T}}R(u^\star(t))e \leq \overline{c}e^{\mathrm{T}}e, \quad \forall t \geq t_0$$

and that there exist positive constants k_1, γ, *positive definite and diagonal matrix* α_2 *and matrices* $K_2, \alpha_1, \alpha_3, \alpha_4$, *such that for*

1) $e_x^{\mathrm{T}}\bar{G}\bar{G}^{\mathrm{T}}e_x = y^{\mathrm{T}}y$, *we have*

$$X_1 > 0, X_1 := k_1\underline{c} - \frac{1}{2} - \frac{k_1^2}{2\gamma^2}$$
$$\text{and} \quad X_2 \geq 0, X_2 := \left(K_2\alpha_1 - \frac{1}{2}\alpha_3^{\mathrm{T}}\alpha_2^{-\mathrm{T}}\alpha_2^{-1}\alpha_3 - \frac{1}{4X_1}K_2\alpha_4\alpha_4^{\mathrm{T}}K_2\right).$$

2) $e_x^{\mathrm{T}}\bar{G}\bar{G}^{\mathrm{T}}e_x \neq y^{\mathrm{T}}y$, *we have*

$$\tilde{X}_1 > 0, \tilde{X}_3 > 0, \tilde{X}_1 := \frac{k_1}{2}\underline{c} - \frac{1}{2}, \tilde{X}_3 := \left(\frac{\underline{c}}{2} - \frac{k_1}{2\gamma^2}\right),$$
$$\tilde{X}_2 \geq 0, \tilde{X}_2 := \left(K_2\alpha_1 - \frac{1}{2}\alpha_3^{\mathrm{T}}\alpha_2^{-\mathrm{T}}\alpha_2^{-1}\alpha_3 - \frac{1}{4\tilde{X}_1}K_2\alpha_4\alpha_4^{\mathrm{T}}K_2\right).$$

Then, $V(e)$ *is a solution of the tHJIi* (4.55).

Proof 5.9 *Rewrite the elements of* $a(t,e)$ *as*

$$a(t,e) = \underbrace{\begin{pmatrix} A(t) & 0 \\ 0 & -\alpha_1 \end{pmatrix}}_{\bar{A}_1(t)} e - B(t,e)\alpha_2^{-1}\alpha_3 z - \underbrace{\begin{pmatrix} 0 \\ \alpha_4 \end{pmatrix}}_{\bar{a}_2} y. \tag{5.61}$$

which is valid since α_2 *is supposed to be diagonal, positive definite and therefore invertible. Insert* (5.61) *into the left hand side of* (4.55) *and introduce squares:*

$$V_t + V_e\bar{A}_1(t)e - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{1}{\sqrt{2}}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{1}{\sqrt{2}}\alpha_2^{-1}\alpha_3 z\right)$$
$$+ \frac{1}{2}z^{\mathrm{T}}\alpha_3^{\mathrm{T}}\alpha_2^{-\mathrm{T}}\alpha_2^{-1}\alpha_3 z + \frac{V_eGG^{\mathrm{T}}V_e^{\mathrm{T}}}{2\gamma^2} - V_e\bar{a}_2 y + \frac{1}{2}y^{\mathrm{T}}y.$$

Use the quadratic function from (5.60) *and simplify to*

$$-k_1e_x^{\mathrm{T}}Re_x - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{1}{\sqrt{2}}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(\frac{1}{\sqrt{2}}e_{\bar{u}}^+ - \frac{1}{\sqrt{2}}\alpha_2^{-1}\alpha_3 z\right) - z^{\mathrm{T}}K_2\alpha_4 y$$
$$+ \frac{e^{\mathrm{T}}PGG^{\mathrm{T}}Pe}{2\gamma^2} - z^{\mathrm{T}}K_2\alpha_1 z + \frac{1}{2}z^{\mathrm{T}}\alpha_3^{\mathrm{T}}\alpha_2^{-\mathrm{T}}\alpha_2\alpha_3 z + \frac{1}{2}y^{\mathrm{T}}y.$$

The columns of $\bar{G}$ consist of unit vectors and hence $\bar{G}^{\mathrm{T}}$ has the same order of unit vectors in its rows. Then it is clear from the definition of matrix-matrix multiplication that the matrix $\bar{G}\bar{G}^{\mathrm{T}}$ is a diagonal $n \times n$-matrix with q ones and $n-q$ zeros on the main diagonal. Since G (in the considered case here with $H_j = 0, j = 1 \dots q$) equals $G = \left((M^{-1}\bar{G})^{\mathrm{T}},\ 0^{\mathrm{T}}\right)^{\mathrm{T}}$, we know that

$$\begin{aligned} e^{\mathrm{T}} P G G^{\mathrm{T}} P e &= e^{\mathrm{T}} \begin{pmatrix} k_1 M & 0 \\ 0 & K_2 \end{pmatrix} \begin{pmatrix} M^{-1}\bar{G}\bar{G}^{\mathrm{T}} M^{-\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 M & 0 \\ 0 & K_2 \end{pmatrix} e \\ &= e^{\mathrm{T}} \begin{pmatrix} k_1 \bar{G}\bar{G}^{\mathrm{T}} M^{-\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 M & 0 \\ 0 & K_2 \end{pmatrix} e \\ &= e^{\mathrm{T}} \begin{pmatrix} k_1^2 \bar{G}\bar{G}^{\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix} e = k_1^2 e_x^{\mathrm{T}} \bar{G}\bar{G}^{\mathrm{T}} e_x. \end{aligned}$$

Let $\mathcal{N} = \{1, 2, \dots, n\}$. Let us define the set of numbers $\mathcal{G} \subset \mathcal{N}$ which chracterizes the ordering of zeros and ones in $\bar{G}\bar{G}^{\mathrm{T}}$, such that

$$k_1^2 e_x^{\mathrm{T}} \bar{G}\bar{G}^{\mathrm{T}} e_x = k_1^2 \sum_{i \in \mathcal{G}} e_{x,i}^2.$$

Equivalently, use $\mathcal{C} \subset \mathcal{N}$ to denote the set of numbers which characterize the ordering of zeros and ones in $\tilde{C}^{\mathrm{T}}\tilde{C}$, such that

$$y^{\mathrm{T}} y = e_x^{\mathrm{T}} \tilde{C}^{\mathrm{T}} \tilde{C} e_x = \sum_{i \in \mathcal{C}} e_{x,i}^2.$$

Furthermore, we define the sets $\mathcal{R} = \mathcal{N} \setminus \{\mathcal{G} \cup \mathcal{C}\}$ and $\mathcal{S} = \mathcal{G} \cap \mathcal{C}$ and be aware of the fact that in most cases $\mathcal{S} \neq \emptyset$. Finally, let $\mathcal{K} = \{\mathcal{G} \cup \mathcal{C}\} \setminus S$.

Then distinguish the following two cases:

1) $e_x^{\mathrm{T}} \bar{G}\bar{G}^{\mathrm{T}} e_x = y^{\mathrm{T}} y$ ($\mathcal{G} = \mathcal{C}$): Further introducing squares leads to

$$\begin{aligned} &-k_1 \sum_{i \in \mathcal{R}} \underline{c} e_{x,i}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{1}{\sqrt{2}} \alpha_2^{-1} \alpha_3 z\right)^{\mathrm{T}} \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{1}{\sqrt{2}} \alpha_2^{-1} \alpha_3 z\right) \\ &- \left(\sqrt{X_1} y + \frac{1}{2\sqrt{X_1}} \alpha_4^{\mathrm{T}} K_2 z\right)^{\mathrm{T}} \left(\sqrt{X_1} y + \frac{1}{2\sqrt{X_1}} \alpha_4^{\mathrm{T}} K_2 z\right) \\ &- z^{\mathrm{T}} \underbrace{\left(K_2 \alpha_1 - \frac{1}{2} \alpha_3^{\mathrm{T}} \alpha_2^{-\mathrm{T}} \alpha_2^{-1} \alpha_3 - \frac{1}{4X_1} K_2 \alpha_4 \alpha_4^{\mathrm{T}} K_2\right)}_{=: X_2} z, \end{aligned}$$

which is clearly negative semidefinite since we assumed that there exist appropriate choices for the unknown elements such that $X_1 > 0$, $X_2 \geq 0$.

2) $e_x^{\mathrm{T}}\bar{G}\bar{G}^{\mathrm{T}}e_x \neq y^{\mathrm{T}}y$ *(*$\mathcal{G} \neq \mathcal{C}$*):*

$$
\begin{aligned}
&- k_1 \sum_{i\in\mathcal{R}} \underline{c} e_{x,i}^2 - \frac{k_1}{2} \sum_{i\in\mathcal{K}} \underline{c} e_{x,i}^2 - k_1 \underbrace{\left(\frac{\underline{c}}{2} - \frac{k_1}{2\gamma^2}\right)}_{=:\,\tilde{X}_3} \sum_{j\in\mathcal{G}} e_{x,j}^2 \\
&- \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{1}{\sqrt{2}} \alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}} \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{1}{\sqrt{2}} \alpha_2^{-1}\alpha_3 z\right) \\
&- \left(\sqrt{\tilde{X}_1} y + \frac{1}{2\sqrt{\tilde{X}_1}} \alpha_4^{\mathrm{T}} K_2 z\right)^{\mathrm{T}} \left(\sqrt{\tilde{X}_1} y + \frac{1}{2\sqrt{\tilde{X}_1}} \alpha_4^{\mathrm{T}} K_2 z\right) \\
&- z^{\mathrm{T}} \underbrace{\left(K_2\alpha_1 - \frac{1}{2}\alpha_3^{\mathrm{T}}\alpha_2^{-\mathrm{T}}\alpha_2^{-1}\alpha_3 - \frac{1}{4\tilde{X}_1} K_2\alpha_4\alpha_4^{\mathrm{T}} K_2\right)}_{=:\,\tilde{X}_2} z,
\end{aligned}
$$

which is clearly negative semidefinite since we assumed that there exist appropriate choices for the unknown elements such that $\tilde{X}_1, \tilde{X}_3 > 0$*,* $\tilde{X}_2 \geq 0$*.*

Theorem 5.10 *Consider the closed-loop of* (5.56), (5.55) *with* (5.49)-(5.52) *where* $p = m$ *and using the suboptimal control law* (4.50)*. Let* $\Delta(||d||_2) = (\gamma^2/2)d^{\mathrm{T}}d$ *and the conditions of Theorem 5.9 be valid. For* $V(e)$ *from (5.60), there exists a positive definite estimate* $\nu(||e||_2)$ *for*

$$
V_t + V_e a - (e_{\bar{u}}^{+})^{\mathrm{T}} e_{\bar{u}}^{+} + \frac{1}{2\gamma^2} V_e G G^{\mathrm{T}} V_e^{\mathrm{T}} \leq -\nu(||e||_2) < 0 \tag{5.62}
$$

such that $V(e)$ *is an iISS Lyapunov function and therefore the closed-loop error system is iISS.*

Proof 5.10 V *from* (5.60) *is a quadratic form and so it immediately follows that* $V \in C^1 \cap UPPD$*. Since it is assumed that the conditions of Theorem 5.9 are fulfilled we know that* V *is a solution of* (4.55) *and therefore* $e_{\bar{u}}^{+}$ *is the suboptimal state feedback control law. Consider* (5.56), (5.55) *with* (5.49)-(5.52)*. Use again* (5.61) *and insert it into* (5.62)*. With the suboptimal control law* $e_{\bar{u}}^{+}(t,e) = -B^{\mathrm{T}}(t,e)V_e^{\mathrm{T}}$ *from* (4.50) *we get*

$$
V_t + V_e \bar{A}_1 e + \underbrace{(-V_e B)}_{(e_{\bar{u}}^{+})^{\mathrm{T}}} \alpha_2^{-1}\alpha_3 z - V_e \bar{a}_2 y - \underbrace{V_e B B^{\mathrm{T}} V_e^{\mathrm{T}}}_{(e_{\bar{u}}^{+})^{\mathrm{T}} e_{\bar{u}}^{+}} + \frac{1}{2\gamma^2} V_e G G^{\mathrm{T}} V_e^{\mathrm{T}} \leq 0,
$$

which was already proven to be negative semidefinite. For negative definiteness *further introducing squares yields*

$$V_t+V_e\bar{A}_1e-y^{\mathrm{T}}y-\left(\frac{1}{2}V_e\bar{a}_2\right)^{\mathrm{T}}\left(\frac{1}{2}V_e\bar{a}_2\right)-\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)$$
$$+\left(\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)+\frac{1}{2\gamma^2}V_eGG^{\mathrm{T}}V_e^{\mathrm{T}}+\left(\frac{V_e\bar{a}_2}{2}-y\right)^{\mathrm{T}}\left(\frac{V_e\bar{a}_2}{2}-y\right)\leq 0.$$

We may omit the positive quadratic terms without losing the vailidity of the inequality

$$V_t+V_e\bar{A}_1-y^{\mathrm{T}}y-\left(\frac{1}{2}V_e\bar{a}_2\right)^{\mathrm{T}}\left(\frac{1}{2}V_e\bar{a}_2\right)-$$
$$\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)\leq 0. \quad (5.63)$$

As can be seen, $V(e)$ from (5.60) is time-invariant such that $V_t\equiv 0$. Inserting into (5.63) and using that M and the skew-symmetric parts cancel and that $R(u^\star)$ is symmetric and bounded from below and above, where the latter was required in Theorem 5.9, we finally arrive at

$$-e^{\mathrm{T}}\begin{pmatrix}k_1R(u^\star) & 0\\ 0 & K_2\alpha_1\end{pmatrix}e-y^{\mathrm{T}}y-\left(\frac{\bar{a}_2}{2}\right)^{\mathrm{T}}\left(\frac{\bar{a}_2}{2}\right)$$
$$-\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)^{\mathrm{T}}\left(e_{\bar{u}}^{+}-\frac{1}{2}\alpha_2^{-1}\alpha_3 z\right)<0 \quad \textit{for } e\neq 0. \quad (5.64)$$

From this it follows that (5.62) admits an upper bound $-\nu(||e||_2)$ with $\nu(||e||_2)$ positive definite as was claimed. This shows that $V(e)$ is an iISS Lyapunov function and from Theorem 3.5 we can conclude that the considered system is iISS.

5.5 Summary and Discussion

In this section, we summarize the obtained results. For single-input bilinear power converter systems, two theorems were derived for integral feedback and disturbance feedback, respectively, which tell under which conditions it is possible to find the suboptimal $\mathcal{H}_\infty$ state feedback law and guarantee iISS for the closed-loop system in the presence of disturbances. It was crucial for the proof of this statement that the function V is a quadratic form that does not depend on the time t even in the time-varying case. This certain structure of V allows to exploit structural properties of the appearing elements in the differential equations. For example, certain entries of skew-symmetric matrices vanish when pre- and post-multiplied with a vector e^{T} and e, respectively. Then, using the standard method of introducing squares, as it is common in linear and nonlinear $\mathcal{H}_\infty$ control, helps to simplify the equations and show negative definiteness of the tHJIi. In addition, the choice of $\Delta(||d||_2) = \frac{\gamma^2}{2}d^2$ allows us to find an upper bound $-\nu(||e||_2)$ which informs us that the closed-loop system is iISS, using the same arguments as for proving suboptimality. While the calculation of the reference trajectory is not a stumbling block for integral feedback, it is not possible to present a general solution for tracking in the case of disturbance feedback: online-replanning of the reference solution $x^\star(t, x_0, u^\star, \hat{d})$, i.e. calculating a matching $u^\star$, would be necessary since the estimated $\hat{d}$ affects the solution. However, from practical experience, disturbance feedback allows for more freedom in choice of feedback design parameters than integral feedback, while the latter is more general but often comes with less performance for the considered power converter systems.

Furthermore, we have discussed how to solve the control problem in the case of the most general class of disturbances which can occur in bilinear power converter systems. The derivation showed that we need an additional bound on one of the system states, which leads to a local nature of the result in suboptimality and stability, since the possible state space is restricted. In addition, the practicioner has to guarantee that this restriction is met, since it cannot be guaranteed from the derived theory itself.

Moreover, it was possible to extend the proposed theory, even in the case of multiple inputs, to general bilinear systems. The drawback is to check positive definiteness of matrix $R(u^\star)$, since this has to be calculated along the reference input $u^\star$ and gets quickly demanding, even for low dimensions.

Of course, every control design approach has certain disadvantages, and it is important to be aware of the limitations. Although the presented results are mathematically sound and easy to follow, soundness and ease is often not given in an experimental situation, especially for nonlinear systems. Since quadratic Lyapunov functions naturally appear in linear optimal control theory, the use of a quadratic function as the solution of the tHJIi and as the iISS Lyapunov function seems to be a suitable choice and good as a first attempt. But for nonlinear systems, this class of functions is quite restrictive and often comes with performance limitations as will be discussed in Chapter 6.

Furthermore, although we consider the full state-feedback approach, we cannot guarantee that really all states are involved in the feedback because of the specific choice of V. It should be noted that for power converter systems, vector $\tilde{b}$ in BPCS (for simplicity we refer to the single-input case) has several zeros. Consider the calculation of the suboptimal control law $e^{+}_{\tilde{u}}(t,e) = -V_e(t,e)\tilde{b}(t,e) = -e^{\mathrm{T}}P\tilde{b}(t,e)$. Since P is diagonal, it is obvious that with zeros in $\tilde{b}$, several states will be missing in the feedback. A numerical solution of the tHJIi would probably circumvent this leading to a more general class of functions V, but this type of first order nonlinear partial differential equations is known to be difficult to solve, and since we regularly faced numerical problems in solving the often stiff differential equations of power converter systems, this will be really tough for partial differential equations.

Finally we should say that the proposed theory allows for smooth trajectory tracking in the case of integral feedback, while we cannot solve the tracking problem for disturbance feedback in general. With an integral or disturbance feedback strategy, we can attenuate disturbances, noise and cope with small parameter deviations as was demanded in the control requirements. For vanishing disturbances, the closed-loop system shows to be uniformly asymptotically stable, and from iISS we know that there is an upper bound for the error state vector in the case of L_2-disturbances. Finally, the chosen suboptimal $\mathcal{H}_\infty$ state feedback control design minimizes the control effort with respect to disturbances and keeps the output y to be controlled small.

The subsequent chapter addresses an application example, where we apply the results of the obtained theory and present experimental results for further discussion.

Chapter 6

Experiment

6.1 Experimental Setup

In the application we consider now the goal is to track a reference trajectory for the angular shaft velocity of a permanent magnet DC motor attached to a boost converter (see Fig. 6.1). A second motor of the same type is attached to the motor shaft and works as generator. Thus, by closing the switches S_1, S_2 it is possible to apply defined load changes via light bulbs on the generator. The system equations read [69],[15],[16]

$$L\frac{di}{dt} = -v\,u - R_L i + E \tag{6.1}$$

$$C\frac{dv}{dt} = i\,u - G\,v - i_{\mathrm{a}} \tag{6.2}$$

$$L_{\mathrm{m}}\frac{di_{\mathrm{a}}}{dt} = v - R_{\mathrm{m}}\,i_{\mathrm{a}} - K_{\mathrm{e}}\,\omega \tag{6.3}$$

$$J\frac{d\omega}{dt} = -B_{\mathrm{m}}\,\omega + K_{\mathrm{m}}\,i_{\mathrm{a}} - \tau_{\mathrm{l}}. \tag{6.4}$$

The notation is as follows: E is the constant input voltage of the boost converter, C its capacitance, L the coil inductance, R_L the coil resistance, G the resistor conductance, $J = 2J_1 = 2J_2$ the moment of inertia of the motor-generator combination (both motors are of the same type), B_{m} the viscous friction coefficient of the motor shaft in the bearing, τ_{l} the (constant) intrinsic load torque. The parameter L_{m} is the motor inductance, K_{e} coefficient for back emf and K_{m} for mechanical power, respectively. For permanent magnet DC motors, K_{e} and K_{m} have the same value.

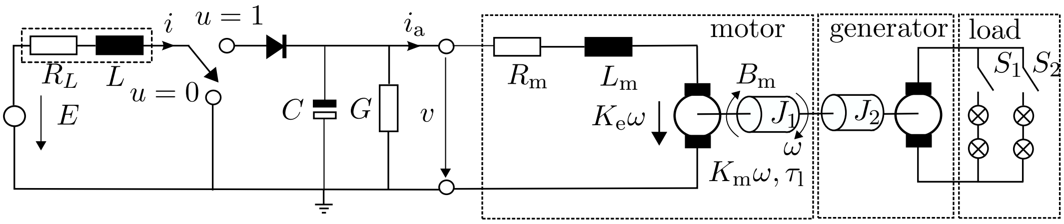

Figure 6.1: Schematic of the boost converter / DC motor combination

Using the system representation BPCS in (2.3), we get

$$M = \mathrm{diag}(L, C, L_{\mathrm{m}}, J),\ R = \mathrm{diag}(R_L, G, R_{\mathrm{m}}, B_{\mathrm{m}}),$$

$$J(u) = \begin{pmatrix} 0 & -u & 0 & 0 \\ u & 0 & -1 & 0 \\ 0 & 1 & 0 & -K_{\mathrm{e}} \\ 0 & 0 & K_{\mathrm{m}} & 0 \end{pmatrix}, \tag{6.5}$$

$$\bar{b}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon(t) = \begin{pmatrix} E & 0 & 0 & -\tau_{\mathrm{l}} \end{pmatrix}^{\mathrm{T}} \equiv \mathrm{const}. \tag{6.6}$$

with state and error state vectors

$$x = \begin{pmatrix} i & v & i_{\mathrm{a}} & \omega \end{pmatrix}^{\mathrm{T}}, \quad e_x = \begin{pmatrix} e_i & e_v & e_{i_{\mathrm{a}}} & e_\omega \end{pmatrix}^{\mathrm{T}}.$$

The possible disturbance is a sudden load change $d = \tau_d$, which means that

$$\bar{g}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.7}$$

in system (2.4).

In addition, we need for the error dynamics formulation (2.8)

$$\tilde{c}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.8}$$

i.e. the output equals the tracking error of the angular velocity, $y = e_\omega$.

6.2 Identification

The starting parameters for the identification were taken from [16], [14] and the "Projektarbeit" of Christoph Schuler with the same type of DC motor and same type of circuit:

$E = 12\,\mathrm{V}$, $C = 470\,\mu\mathrm{F}$, $L = 1.335\,\mathrm{mH}$, $G = 1 \times 10^{-4}\Omega^{-1}$, $J = 15.9 \times 10^{-6}\mathrm{kg\,m}^2$ (two motors), $R_L = 0.3\Omega$, $B_\mathrm{m} = 4.1\,\mu\mathrm{Nm\,sec}$, $\tau_\mathrm{l} = 0.82\,\mathrm{m\,Nm}$, $L_\mathrm{m} = 8.9\,\mathrm{mH}$, $R_\mathrm{m} = 6\,\Omega$, $K_\mathrm{e} = 45.5\,\mathrm{mV\,sec/rad}$ and $K_\mathrm{m} = 45.5\,\mathrm{mV\,rad\,sec}$ [1].

We had some initial guess which parameters in the steady state could be faulty, and after some trial and error we finally found out that the crucial parameters which have to be adapted are $R_L, R_\mathrm{m}, K_\mathrm{e}, \tau_\mathrm{l}$ (and therefore also K_m changes since it has the same value as K_e). Since we have a specific structure in our differential equation which we exploit in the proposed control strategy, we chose the grey-box model approach implemented in the System Identification Toolbox™ of Matlab®.

Note that we did not use the toolbox to adjust the left hand side system parameters L, C, L_m, J, since we think we know these parameters quite well and the experimental data supports our reasoning. Consequently, we concentrate on the discussion of the stationary behavior, i.e. set the left hand side of the system to zero.

Hence, we insert the stationary states of our bilinear system equations and search for better values for the specified parameters. The toolbox algorithm uses a trusted-region reflective newton-method and a nonlinear least-squares solver with trace minimization. If the trace minimization matrix is an identity matrix, we would have the standard sum of least squares optimization criterion $\mathrm{trace}(EE^\mathrm{T})$ with E being the error between measurement and simulation data. Using the weighting matrix W, the criterion changes to a special form of weighted least squares with $\mathrm{trace}(EE^\mathrm{T}W)$. The choice in the identification process was $W = \mathrm{diag}(10,\ 1,\ 100,\ 1)$ in order to weight the coil current in the first component and the armature current in the third component more than the capacitor voltage and the angular shaft velocity, since the currents are the most difficult states to match the model. A practical explanation for this observation is that voltage measuremnts (needed for v, ω) are easier to realize than current measurements which are far more demanding in the required resolution needed here and slight deviations have a large effect on the measurement results.

We finally determined the following parameters:

$E = 12\,\mathrm{V}$, $C = 470\,\mu\mathrm{F}$, $L = 1.335\,\mathrm{mH}$, $G = 1 \times 10^{-4}\Omega^{-1}$, $J = 15.9 \times$

[1] When choosing $[\omega] = \mathrm{rad/sec}$ then K_e and K_m do not have the same units although they have the same value.

10^{-6}kg m^2, $R_L = 77\text{m}\Omega$, $B_\text{m} = 4.1\,\mu\text{Nm sec}$, $\tau_\text{l} = 10.85\,\text{m Nm}$, $L_\text{m} = 8.9\,\text{mH}$, $R_\text{m} = 8.05\,\Omega$, $K_\text{e} = 43.9\,\text{mV sec/rad}$ and $K_\text{m} = 43.9\,\text{mV rad sec}$.

All reference trajectories were calculated with these parameters.

Figure 6.2 shows the used measurement data (solid) and "fit" is the simulation result from the grey-box identification process (dashed). The number in percent in the legend is the result of

$$\text{number}[\%] = 100 \cdot \left(1 - \frac{||x^\star - x||_2}{||(x - \text{mean}(x)||_2}\right),$$

where x are the measured states and $x^\star$ the states simulated by the grey-box model. This number is an indicator for the quality of the identification procedure, i.e. in the case of 100% we have a perfect match of measurement and simulation data. While the states i, v, ω match well following the number in percent, the measurement of the armature current i_a deviates much more in comparison with the other states. A closer look at the y-axis of i_a in Figure 6.2 informs us that the current varies between $0.27 - 0.29$ A, i.e. we would need a measurement resolution of ≈ 5 mA or less to obtain better results. However, since the switching of the MOSFET induces a lot of noise, this indicates that it would be practically really difficult to achieve. When we discuss feedforward and feedback design, we will see that we can cope with this deviation.

6.3 Trajectory Planning for the Combination Boost Converter / DC Motor

We do not only want to track a reference trajectory for the angular shaft velocity of the DC motor, but also want to stabilize the system around the given trajectory when subject to load perturbations. Therefore, we provide a justification why the error system representation and the theory presented before is still valid under the made approximations, meanwhile this is still an open question for the strategies in [14], [15].

For the generation of a nominal feedforward control and the calculation of the respective error signals, a smooth reference trajectory for a set-point to set-point transition with respect to the angular shaft velocity is to be determined (for a transition from non-stationary to stationary set-points cf. [39]). In the case of polynomial reference trajectories, transition polynomials $p(t) \in \mathbb{R}$ are

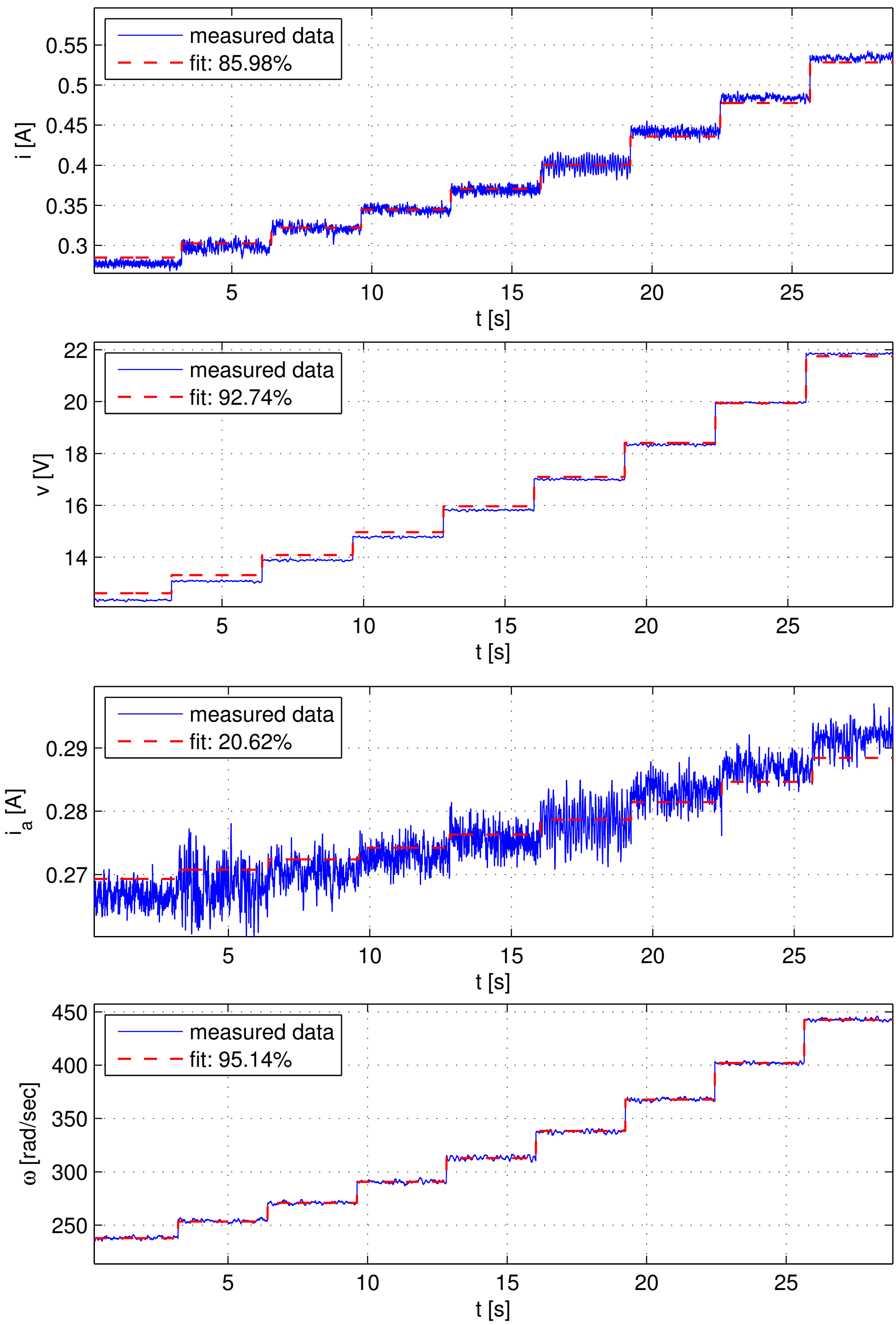

Figure 6.2: Measurement (solid) and reference values (dashed) of the stationary state

introduced. Such polynomials $p(\cdot) \in C^r$ are supposed to fulfill the conditions

$$p(0) = 0,\ p(1) = 1,\ \ p^{(i)}(0) = 0,\ p^{(i)}(1) = 0,\ i = 0, \dots, r, \qquad (6.9)$$

with some finite number r. When we want to track arbitrary set-point transitions, we link those transition polynomials with the tracking output, i.e. in the application considered the angular shaft velocity $\omega^\star$ in the following way:

$$\omega^\star(t) = \begin{cases} \omega_0^\star, & t < t_0, \\ \omega_0^\star + (\omega_\mathrm{f}^\star - \omega_0^\star)\, p\left(\dfrac{t - t_0}{t_\mathrm{f} - t_0}\right), & t_0 \le t \le t_\mathrm{f}, \\ \omega_\mathrm{f}^\star, & t > t_\mathrm{f}, \end{cases} \qquad (6.10)$$

where t_0 denotes the initial time and t_f the final time of the transition, and with initial and final values $\omega^\star(t_0) = \omega_0^\star$ and $\omega^\star(t_\mathrm{f}) = \omega_\mathrm{f}^\star$, respectively. This formulation allows to impose arbitrary initial and final times and arbitrary set-points on the desired tracking output. Furthermore, in case of tracking, we assume that r in (6.9) is at least equivalent to (or greater than) the relative degree of the system with respect to the considered tracking output, see [38]. Then, it follows immediately that also $\omega^\star(\cdot)$ is r-times differentiable and that there are no steps until the r-th derivative of the output signal ($\omega^\star(\cdot) \in C^r$). Since in our application, the "output" is the angular velocity $\omega^\star$ which has relative degree $r = 3$, see Appendix B, we find according to (6.9) that

$$p(t) = 35\, t^4 - 84\, t^5 + 70\, t^6 - 20\, t^7$$

which we finally insert into (6.10).

At first, we write down the equations for trajectory tracking in the case of disturbance feedback, since the equations for tracking in the case of integral feedback are merely the special case $\hat{d} \equiv 0$.

As we have to consider the load disturbance $d(t) = \tau_d(t)$ which enters (2.8) via (6.7), we have to find a strategy to get $u^\star(t)$ in (2.14) which depends upon $\hat{d}(t) = \hat{\tau}_d(t)$:

$$M\dot{x}^\star(t) = (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t) + \bar{g}\hat{d}(t)$$

With the knowledge of $\omega^\star(t)$ and its time derivatives, the system equations (6.3) and (6.4) may be used to obtain $v^\star(t), \dot{v}^\star(t)$ and $i_\mathrm{a}^\star(t)$ in terms of $\omega^\star(t)$. This yields

$$i_\mathrm{a}^\star(t) = \frac{J}{K_\mathrm{m}}\dot{\omega}^\star(t) + \frac{B_\mathrm{m}}{K_\mathrm{m}}\omega^\star(t) + \frac{\hat{d}(t) + \tau_\mathrm{l}}{K_\mathrm{m}} \qquad (6.11)$$

$$v^\star(t) = \left(\frac{J\,L_\mathrm{m}}{K_\mathrm{m}}\right)\ddot{\omega}^\star(t) + \left(\frac{B_\mathrm{m}\,L_\mathrm{m}}{K_\mathrm{m}} + \frac{J\,R_\mathrm{m}}{K_\mathrm{m}}\right)\dot{\omega}^\star(t) \\ + \left(\frac{B_\mathrm{m}\,R_\mathrm{m}}{K_\mathrm{m}} + K_\mathrm{e}\right)\omega^\star(t) + \frac{R_\mathrm{m}}{K_\mathrm{m}}\left(\hat{d}(t) + \tau_\mathrm{l}\right) + \frac{L_\mathrm{m}}{K_\mathrm{m}}\dot{\hat{d}}(t) \tag{6.12}$$

$$\dot{v}^\star(t) = \left(\frac{J\,L_\mathrm{m}}{K_\mathrm{m}}\right)\dddot{\omega}^\star(t) + \left(\frac{B_\mathrm{m}\,L_\mathrm{m}}{K_\mathrm{m}} + \frac{J\,R_\mathrm{m}}{K_\mathrm{m}}\right)\ddot{\omega}^\star(t) \\ + \left(\frac{B_\mathrm{m}\,R_\mathrm{m}}{K_\mathrm{m}} + K_\mathrm{e}\right)\dot{\omega}^\star(t) + \frac{R_\mathrm{m}}{K_\mathrm{m}}\dot{\hat{d}}(t) + \frac{L_\mathrm{m}}{K_\mathrm{m}}\ddot{\hat{d}}(t). \tag{6.13}$$

The inductor current cannot be expressed by the angular velocity $\omega^\star(t)$ and its time derivatives since $\omega^\star(t)$ is not a flat output of (2.14). Hence, the internal dynamics

$$L\frac{di^\star(t)}{dt} = -\frac{v^\star(t)}{i^\star(t)}\left(C\dot{v}^\star(t) + Gv^\star(t) + i_\mathrm{a}^\star(t)\right) - R_L i^\star(t) + E \tag{6.14}$$

has to be solved. The reference input $u^\star(t)$ can be calculated with the solution $i^\star(t)$ of (6.14) via

$$u^\star(t) = \frac{C\dot{v}^\star(t) + Gv^\star(t) + i_\mathrm{a}^\star(t)}{i^\star(t)}. \tag{6.15}$$

What makes the trajectory generation difficult is the fact that an input/output linearization with respect to the angular shaft velocity $\omega^\star$ results in an unstable internal dynamics (6.14), see also the discussion in Appendix B.

For the integral feedback case (i.e. set $\hat{d} \equiv 0$ in (6.11)-(6.13)), we would use the proposed scheme in [40] and references therein, i.e. reformulate the integration of the internal dynamics as a two-point boundary value problem which could be solved offline. For solving the boundary value problem, it is necessary to have an initial guess for the behavior of the solution of the internal dynamics for initialization of the solver. Furthermore, it can happen that numerical problems arise, which appeared from time to time for the power converter systems considered here.

However, in the case of disturbance feedback, striving to accomodate instantaneous load changes it would be necessary to replan the trajectory in an online manner. For the experimental setting, this is practically infeasible because a stiff, parameter sensitive two-point boundary value problem would have to be solved under real-time conditions. Furthermore, the time derivatives $\dot{\hat{d}}(t), \ddot{\hat{d}}(t)$ would be required in (6.12), (6.13) but are unknown.

In the special case of the boost converter / DC motor combination this problem may be circumvented by the following strategy, without loss of stability:

We assume $L\frac{d\bar{\imath}}{dt} \approx 0$, where we use the bar to make clear that we are considering an approximation. Thus, from (6.1) it follows that

$$\bar{u}(t) = \frac{-R_L\bar{\imath}(t) + E}{\bar{v}(t)}.$$

Consequently, we replace $\bar{u}$ in (6.2), assume $C\frac{d\bar{v}}{dt} \approx 0$ and solve for $\bar{\imath}$:

$$\begin{aligned} \bar{\imath}(t) &= \frac{1}{2}\frac{E - \sqrt{\text{val}}}{R_L}, \\ \text{val} &= E^2 - 4R_LC\dot{\bar{v}}(t)\bar{v}(t) - 4R_L\bar{v}(t)^2G - 4R_L\bar{\imath}_\text{a}(t)\bar{v}(t)\,. \end{aligned} \tag{6.16}$$

It is very important to realize that due to these approximations, the desired reference $\omega^\star$ in (6.10) will no longer be exact when using $\bar{u}$ instead of $u^\star$ as input signal. Hence we replace $\omega^\star$ with $\bar{\omega}$ in (6.10) such that the desired reference is now $\bar{\omega}$. Then, (6.11)-(6.13) change to

$$\bar{\imath}_\text{a}(t) = \bar{\imath}_\text{a}(\bar{\omega}(t), \dot{\bar{\omega}}(t), \hat{d}(t)) \tag{6.17}$$

$$\bar{v}(t) = \bar{v}(\bar{\omega}(t), \dot{\bar{\omega}}(t), \ddot{\bar{\omega}}(t)\hat{d}(t), \dot{\hat{d}}(t)) \tag{6.18}$$

$$\dot{\bar{v}}(t) = \dot{\bar{v}}(\dot{\bar{\omega}}(t), \ddot{\bar{\omega}}(t), \dddot{\bar{\omega}}(t), \dot{\hat{d}}(t), \ddot{\hat{d}}(t)), \tag{6.19}$$

which can be inserted into (6.16) in order to obtain $\bar{\imath}(t)$. Finally, the reference input is calculated via

$$\bar{u}(t) = \frac{-R_L\bar{\imath}(t) + E}{\bar{v}(t)}. \tag{6.20}$$

Both assumptions $L\frac{d\bar{\imath}}{dt}, C\frac{d\bar{v}}{dt} \approx 0$ show that this strategy neglects the dynamics of the first two states. The stationary part of the trajectory is exactly matched. In our setup, L, C are very small, and the dynamic behavior of the angular velocity is chiefly based on the motor dynamics. Therefore, we achieve valid values for $\bar{u}$ following this strategy. This strategy might work also for other dynamic systems but it should be checked for each considered system individually.

Concerning the online replanning, we still have to handle the problem of the unknown time derivatives of $\hat{d}(t) = \hat{\tau}_d(t)$ in (6.17)-(6.19). Since we are only interested in the absolute value of the load, not in its dynamic behavior (which would require the knowledge of all necessary time derivatives), we assume that the load is piecewise constant. As a consequence, the time derivatives of the load are zero for each time instant, i.e. $\dot{\tau}_d(t) \equiv \ddot{\tau}_d(t) \equiv 0$. Hence, we use $\bar{\tau}_d(t)$ instead of $\hat{\tau}_d(t)$ to denote the estimate $\hat{d}(t) = \bar{\tau}_d(t)$ with $\dot{\bar{\tau}}(t) \equiv \ddot{\bar{\tau}}(t) \equiv 0$. Again, the stationary part of the trajectory is calculated in an exact manner.

Since we merely allow bounded values d, τ_d, the estimated state $\hat{\tau}_d$ (and therefore $\bar{\tau}_d$) is also bounded. From the stability considerations we then know that the error state e remains bounded in the closed-loop. The specified reference for the angular shaft velocity $\bar{\omega}$ and its time derivatives up to order $r = 3$ are bounded (since the control task is to track a smooth set-point transition). To sum up, it follows that the input reference $\bar{u}$, a function of time generated by the equations above, remains bounded at all times.

Now we need the exact state reference $x^\star$ instead of the approximation $\bar{x}$ in order to be able to use the error dynamics formulation. Therefore, with (2.14), setting $\hat{d}(t) = \bar{\tau}_d(t)$ as discussed above and using the approximate input as reference input $u^\star(t) = \bar{u}(t)$, we have to solve the reference system

$$\begin{aligned} \dot{x}^\star(t) &= \underbrace{M^{-1}(J(\bar{u}(t)) - R)}_{=:\,A(t)} x^\star(t) + \underbrace{M^{-1}[\bar{b}\bar{u}(t) + \epsilon(t) + \bar{g}\bar{\tau}_d(t)]}_{=:\,\beta(t)} \\ &= A(t)x^\star(t) + \beta(t), \quad x^\star(t_0) = x_0^\star. \end{aligned} \tag{6.21}$$

with $\tilde{g} = (0\ 0\ 0\ \ -1/J)^{\mathrm{T}}$. First of all, the linear system is C^1 with respect to $x^\star$. For piecewise continuous bounded $\bar{u}(t), \epsilon(t), \bar{\tau}_d(t)$ we know that $A(t)$ and $\beta(t)$ are piecewise continuous and bounded. So we have a unique solution for given initial conditions. In order to be able to guarantee boundedness of $x^\star$, we derive the following Lemma [30]:

Lemma 6.1 *Let the conditions of Theorem 5.5 and 5.6 be fulfilled. Then the solution of the time-varying (feedforward) system (6.21) is uniformly bounded.*

Proof 6.1 *Theorem 5.5 and 5.6 imply that $\bar{u}, \bar{\tau}_d$ are bounded. As $\epsilon(t)$ is bounded by definition and $\bar{b}, \bar{g}$ are constant vectors we conclude that $\|A(t)\|$ and $\|\beta(t)\|$ are bounded $\forall t \geq t_0$. Furthermore, the origin regarding the homogenous part of (6.21) (that is $\beta(t) \equiv 0$) is uniformly exponentially stable because it admits the quadratic Lyapunov function $V(x^\star) = \frac{1}{2}x^{\star\mathrm{T}}Mx^\star$ with $M > 0$ by definition. This Lyapunov function is associated with a negative definite quadratic derivative $\dot{V}(x^\star) = -x^{\star\mathrm{T}}Rx^\star < 0, \forall x^\star \neq 0, \forall t \geq t_0$ since*

$R > 0$ as is required from Theorem 5.5 and 5.6. Consequently, there are $\mu, \lambda > 0$ such that $\|\Phi(t,t_0)\| \le \mu e^{-\lambda(t-t_0)}, \forall t \ge t_0$, and uniform boundedness of the solution

$$x^\star(t) = \Phi(t,t_0)x_0^\star + \int_{t_0}^{t} \Phi(t-\tau, t_0)\beta(\tau)d\tau$$

is implied by the respective norm bounds for all $t \ge t_0$.

Therefore, we may numerically integrate the differential equation (6.21) with the forward Euler method (which is the standard choice for online evaluation on a real-time platform) for small enough step-time such that the numerical algorithm converges. It has be shown in the experiments that this approximation is applicable and the deviation of $\omega^\star$ from the desired behavior $\bar{\omega}$ remains small in most cases This is investigated in the next section for the experiments conducted.

6.4 Results

In the experimental setup, we used a Laptop with Windows XP® and Matlab® / Simulink® for implementation of the feedforward and feedback controller. With the help of Real-Time Workshop®, this code is loaded to the Dspace® 1103 controller board which is the real-time platform to control the power converter and DC motor. The experimental setup can be seen in Fig. 6.3,6.4.

The sampling time for the Dspace® 1103 controller board microprocessor was set to 50 μsec in order to be able to solve the stiff differential equation for the trajectory replanning. Due to switching, the coil and armature current are very noisy, thus, we utilise first order low pass filters in Simulink® with time constant $\tau_1 = 0.5$ msec for the coil current and $\tau_2 = 3$ msec for the armature current measurement. No filters were used for capacitor voltage and angular velocity measurements. We used a GR $42 \times 25/24$ VDC Dunkermotor with attached tacho generator TG 11 and the same type of motor with tacho generator attached to the shaft in order to enable load changes. Opening and closing the switches S_1, S_2 attaches the 5 W / 12 V light bulbs of type Philips Halotone with type no. 13283.

For the example system, we obtain for (5.1)

$$V(e) = \frac{k_1}{2}\left(L\, e_i^2 + C\, e_v^2 + L_\mathrm{m}\, e_{i_\mathrm{a}}^2 + J\, e_\omega^2\right) + \frac{k_2}{2} z^2 \tag{6.22}$$

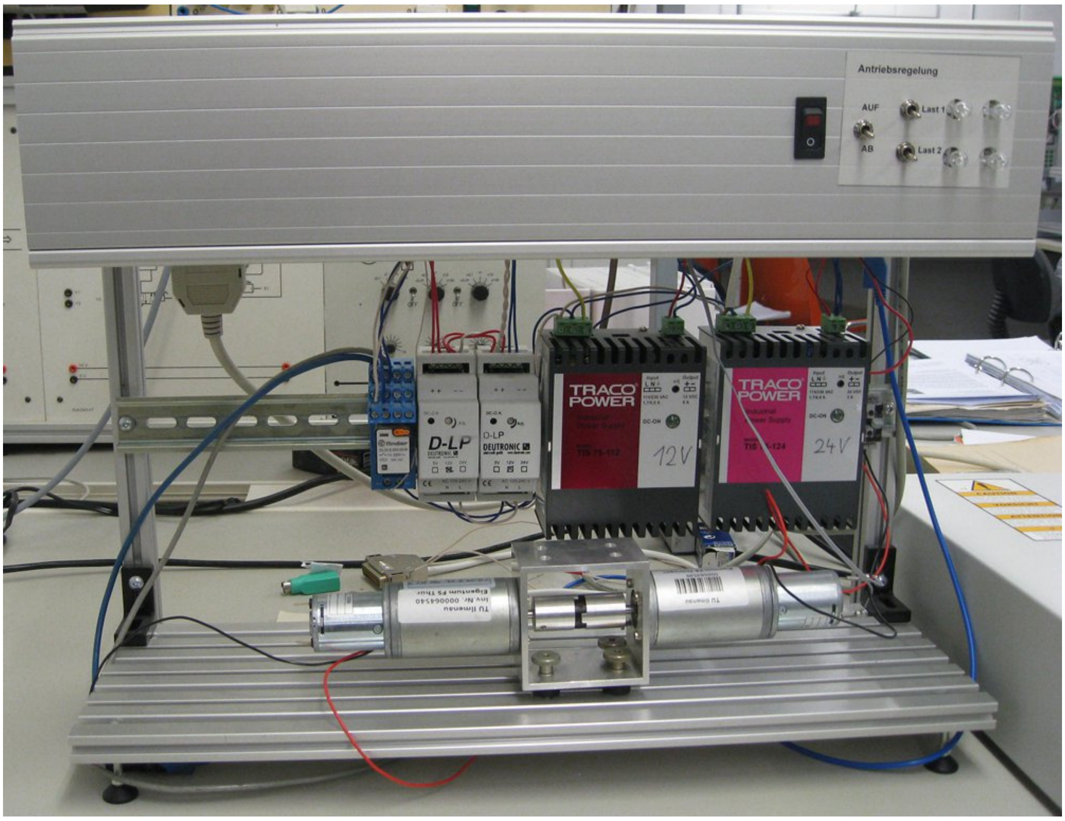

Figure 6.3: Laboratory setup: front view

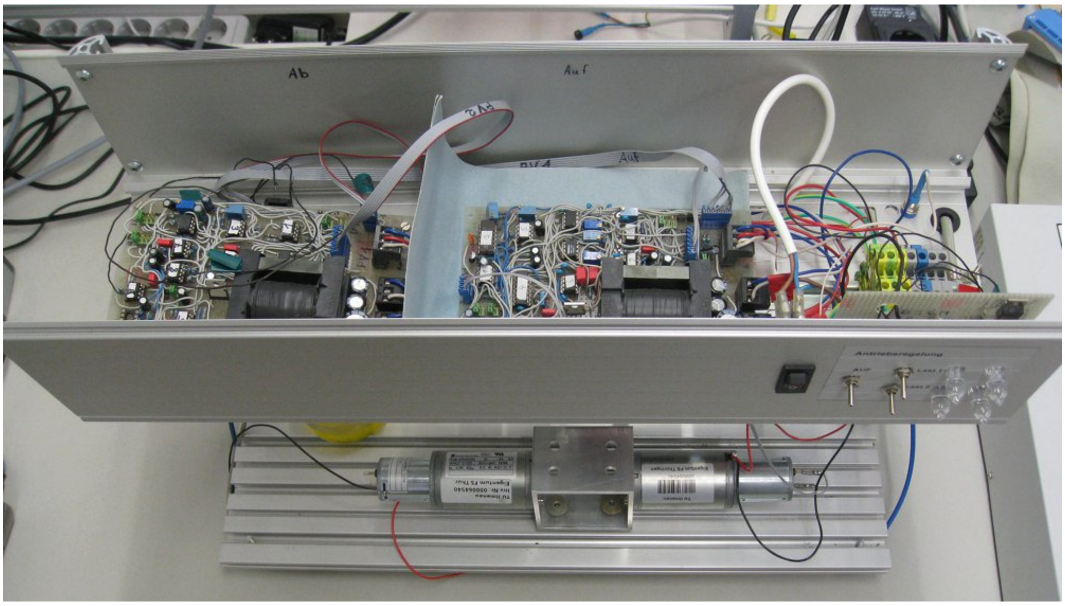

Figure 6.4: Laboratory setup: upper view

with constants $k_1, k_2 > 0$. Since the matrix R is diagonal and of full rank, we know that the time-varying Hamilton-Jacobi-Isaacs inequality and the iISS condition are fulfilled. The suboptimal control law is

$$e_{\bar{u}}^{+} = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}} = -\alpha_2 \tilde{b}(t,e_x)^{\mathrm{T}} k_1 M e_x = k_1 \alpha_2 \left(e_i\, v - e_v\, i\right). \tag{6.23}$$

Since the system output is $y = e_\omega$ and the disturbance enters the e_ω-state, we have case 1) of Proof 5.1 with R diagonal as in Remark 5.1:

$$\begin{aligned} &-k_1 R_L e_i^2 - k_1 G e_v^2 - k_1 R_{\mathrm{m}} e_{i_{\mathrm{a}}}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{\alpha_3}{\sqrt{2}\alpha_2} z\right)^2 \\ &- \left(\sqrt{X_1} y + \frac{k_2 \alpha_4}{2\sqrt{X_1}} z\right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4 X_1}\right)}_{=:\, X_2} z^2 \le 0 \end{aligned} \tag{6.24}$$

with $X_1 := k_1 B_{\mathrm{m}} - \frac{1}{2} - \frac{k_1^2}{2\gamma^2}$ for $X_1 > 0$, $X_2 \ge 0$.

For the example system, we obtain for (5.22)

$$V(e) = \frac{k_1}{2}\left(L\, e_i^2 + C\, e_v^2 + L_{\mathrm{m}}\, e_{i_{\mathrm{a}}}^2 + J\, e_\omega^2\right) + \frac{\hat{\tau}_{\mathrm{d}}^2}{2} \tag{6.25}$$

with constants $k_1, k_2 > 0$ and with new state $\hat{d} = \hat{\tau}_d$ for load disturbance feedback of $\hat{\tau}_d$. Since matrix R is diagonal and of full rank, we know that the time-varying Hamilton-Jacobi-Isaacs inequality and the iISS condition are fulfilled. The suboptimal control law is

$$e_{\bar{u}}^{+} = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}} = -\alpha_2 \tilde{b}(t,e_x)^{\mathrm{T}} k_1 M e_x = k_1 \alpha_2 \left(e_i\, v - e_v\, i\right). \tag{6.26}$$

Since the system output is $y = e_\omega$ and the disturbance enters the e_ω-state, we have case 1) of Proof 5.5 with R diagonal as in Remark 5.1:

$$\begin{aligned} &-k_1 R_L e_i^2 - k_1 G e_v^2 - k_1 R_{\mathrm{m}} e_{i_{\mathrm{a}}}^2 - \frac{1}{2}(e_{\bar{u}}^{+})^2 \\ &- \left(\sqrt{\alpha_5}\hat{\tau}_d + \frac{l\check{y}}{2\sqrt{\alpha_5}}\right)^2 - \underbrace{\left(k_1 B_{\mathrm{m}} - \frac{1}{2} - \frac{l^2}{4\alpha_5} - \frac{k_1^2}{2\gamma^2}\right)}_{=:\, \check{X} \ge 0} \check{y}^2 \le 0. \end{aligned} \tag{6.27}$$

The choice of the values for $k_1, k_2, \gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ depends on X_1, X_2 in (6.24) (or see Proof 5.1). Since α_4 amplifies the noise, the first step is to find a range of appropriate values from simulation data such that the inequalities before are fulfilled. In a second step, α_4 is adjusted in the allowed range via trial and error according to experimental results such that the noise attenuation is satisfactory. Parameter α_1 should be chosen as small as possible so as to keep small a possibly appearing steady state error due to the truncated integrator. The parameters α_2, k_2 are preferably set to small values to maintain X_2 positive [29]. However, α_3 needs to be thoroughly balanced: α_3 should not be too large in order to reject system oscillations, conversely, not too small so as to not reduce the weight of the integral part in the feedback law (6.23), which would substantially limit the closed-loop performance of the integral action.

The choice of the values for k_1, l, γ, α_5 depends on $\check{X}$ in (6.27) (or see Proof 5.5), while α_2 does not depend on any condition, but should not be too small since it would cancel the feedback. The main part is to find an l that does not amplify the noise too much, and an α_5 that keeps a possible estimation error sufficiently small. Parameter k_1 must not be too small as it is part of the control law (6.26), so therefore γ has to be sufficiently large.

The final choices are: $\alpha_1 = 2513$, $\alpha_2 = 0.0001$, $\alpha_3 = 0.0001$, $\alpha_4 = 1$, $k_1 = 125500$, $k_2 = 0.002$, $\gamma = 4.821 \times 10^7$ for integral feedback. As a result of these parameters, it has been shown in simulations and experiments that the closed-loop performance in the presence of disturbances is not satisfactory. This is especially due to α_2 and α_3, which suppress the influence of the feedback. Hence, it was not possible to attenuate load disturbances via integral feedback with acceptable performance for the boost converter / DC motor system.

Therefore, we next considered disturbance feedback and online trajectory replanning in the control algorithm with parameter configuration 1 (PC1) $\alpha_2 = 0.0001$, $k_1 = 125500$, $l = 0.009, \alpha_5 = 0.7$ with $\gamma = 4.821 \times 10^7$ and the less conservative but more sensitive choice (PC2) $\alpha_2 = 0.0001$, $k_1 = 1.3 \times 10^5$, $l = 0.04, \alpha_5 = 0.08$ with $\gamma = 5.5 \times 10^5$.

6.4.1 Set-point transition between stationary states

In the experimental setup, the set-point transition takes place between $t_0 = 4\,\text{sec}$ and $t_\text{f} = 4.5\,\text{sec}$, hence, the transition time is 0.5 sec. The angular shaft velocity changes from $\bar{\omega}(t_0) = 280\,\text{rad/sec}$ to $\bar{\omega}(t_\text{f}) = 380\,\text{rad/sec}$. The reference trajectories and measurements for input and system states in the feedforward control case can be found in Fig. A.1,A.2 with corresponding absolute and relative errors of the measurements in Fig. A.3,A.4, respectively. As can be seen from the absolute errors, the maximal deviations are $\max|i - i^\star| \approx 50\,\text{mA}$, $\max|v - v^\star| \approx 0.7\,\text{V}$, $\max|i_\text{a} - i_\text{a}^\star| \approx 60\,\text{mA}$, $\max|\omega - \omega^\star| \approx 20\,\text{rad/sec}$, while the maximum relative errors are $\max\left|\frac{i-i^\star}{i^\star}\right| \approx 15\%$, $\max\left|\frac{v-v^\star}{v^\star}\right| \approx 4\%$, $\max\left|\frac{i_\text{a}-i_\text{a}^\star}{i_\text{a}^\star}\right| \approx 15\%$, $\max\left|\frac{\omega-\omega^\star}{\omega^\star}\right| \approx 5\%$ (the values from some single peaks were not considered since they come from faulty data acquisition).

For the feedback control law, the parameter configurations (PC1) and (PC2) were implemented. In order to test the performance of the control laws, a severe load step $\tau_d \approx 20 - 30\,\text{m Nm}$ up to 10 times larger than in [16] was imposed on the DC motor during the transition. The results can be found in Fig. A.5,A.6, which depict the system states, the control input and the estimated load for the feedback control case. Two graphs are shown: the reference trajectories with regard to disturbance feedback and online replanning (dashed) and the respective measurements (solid). First of all, the control input u remains within its bounds in both cases, that is $u \in [0, 1]$. Furthermore it can be seen that (PC2) has a better performance than (PC1) but induces (of course) more noise and oscillatory behaviour into the system. In comparison with the results in [16], we have lower performance in the angular velocity ω but less overshoot in the other system states when the disturbance appears. The reason for the better performance of the Jacobi linearization-based design in [16] comes from the fact that this approach allows us to specify the closed-loop behavior close to the reference trajectory directly through pole placement. The nonlinear $\mathcal{H}_\infty$ control merely guarantees dissipativity and finite input-output gain, but addressing directly the performance close to the reference trajectory is not possible. This is a major problem in nonlinear control design different from linear feedback design. However, since [16] is based on linearization along the trajectory, there is no information about the closed-loop stability region and under what circumstances stability can be guaranteed for large load disturbances.

In another experiment, we chose a shorter set-point transition time, i.e. $t_0 = 4\,\text{sec}$ and $t_\text{f} = 4.2\,\text{sec}$, hence $0.2\,\text{sec}$. The angular shaft velocity changes again from $\bar{\omega}(t_0) = 280\,\text{rad/sec}$ to $\bar{\omega}(t_\text{f}) = 380\,\text{rad/sec}$. The reference trajectories and measurements for input and system states in the feedforward control case can be found in Fig. A.9,A.10. As can be seen, a slight overshoot in the angular velocity appears due to the approximate trajectory generation strategy. If one solves the two-point boundary value problem and calculates the exact reference input $u^\star(t)$ for feedforward control, the experimental result changes to what can be found in Fig A.12,A.13. Finally, using (PC2), the overshoot due to the approximate trajectory generation in the angular velocity as was present in Fig. A.10 can be suppressed by disturbance feedback as is shown in Fig. A.11.

6.4.2 Sinusoidal reference trajectory

Another simple bounded reference trajectory is a sinusoid for the angular shaft velocity such that $\omega^\star(t) = \left(50\sin\left(\frac{10}{2\pi}t\right) + 330\right)$ rad/sec. Again, we first consider the feedforward case in Figs. A.14,A.15. Then follow the experiments for the closed-loop control for (PC1) in Fig. A.16 and with appearing load step in Fig. A.17, and for (PC2) in Fig. A.18 and with a load step in Fig. A.19. The disturbance rejection with the parameters (PC1) is slower while for (PC2), the angular velocity shows almost no deviation from its desired behavior.

6.4.3 Relation to other work

We also want to mention that there seems to be a problem with the experimental results in [8], [10], [9], where the Čuk-converter is used as an example application for the presented time-invariant integral feedback and exogeneous tracking approach, are questionable. Let us explain why we believe that the presented results are incorrect. The parameters for stability considerations used in these publications for the Čuk-converter are the following:

1. [8]: $G_L = \frac{1}{33.7}\Omega^{-1}, \alpha_1 = 0.01, k_1\alpha_2^2 = 0.01, \alpha_3 = 1, \alpha_4 = 1.$
2. [10]: $G_L = \frac{1}{22.4}\Omega^{-1}, \alpha_1 = 0.001, k_1\alpha_2^2 = 0.01, \alpha_3 = 8, \alpha_4 = 1.$[2]
3. [9]: $G_L = \frac{1}{22.36}\Omega^{-1}, \alpha_1 = 0.001, k_1\alpha_2^2 = 0.01, \alpha_3 = 1, \alpha_4 = 8.$[3]

Since relevant parameters like $\gamma, k_1, k_2, \alpha_2$ are missing in all three publications, we used the given condition for Lyapunov stability with Lyapunov function

[2] Instead of α, this publication uses δ. Also note that for the $\mathcal{H}_\infty$-condition, a slightly different supply rate was used in the original setting, but this does not change the Lyapunov condition.

[3] Here, the notation in the publication is completely different. It also seems that there has been made a mistake, since $\alpha_3 = 8, \alpha_4 = 1$ in [10] changed later on to $\alpha_3 = 1, \alpha_4 = 8$ in [9], but we did not check this and accepted the values used there.

$P(z)$ to check the validity of the parameters:

$$\dot{P}(z) = \left(\frac{\partial}{\partial z}P(z)\right)(a(z)+b(z)\bar{v}) = \left(\frac{\partial}{\partial z}P(z)\right)a(z) - \underbrace{\left(\left(\frac{\partial}{\partial z}P(z)\right)b(z)\right)^2}_{\bar{v}^2}$$

$$= -\,k_1R_1z_1^2 - \left(\sqrt{k_1G_L}z_4 + \frac{k_2\alpha_4}{2\sqrt{k_1G_L}}z_5\right)^2$$

$$-\,k_1R_2z_2^2 - \frac{1}{2}\bar{v}^2 - \left(\frac{1}{\sqrt{2}}\bar{v} - \frac{\alpha_3}{\sqrt{2}\alpha_2}z_5\right)^2 - \underbrace{\left(k_2\alpha_1 - \frac{k_2^2\alpha_4^2}{4k_1G_L} - \frac{\alpha_3^2}{2\alpha_2^2}\right)}_{=:\,X_3}z_5^2.$$

This shows that the condition on X_3 cannot be fulfilled, since solving for k_2 provides

1. $k_2 = (0.5934718101 \cdot 10^{-3} \pm 2.436127613j)k_1$,
2. $k_2 = (0.8928571429 \cdot 10^{-4} \pm 23.90457219j)k_1$,
3. $k_2 = (0.1397584973 \cdot 10^{-5} \pm 0.3738428778j)k_1$,

which tells us that for real $k_1 > 0$, k_2 will be a complex number and vice versa. Comparing the simulations of [8] with our ones made with the given system equations and parameters, we found out that really these parameters were used although the derived stability conditions are not fulfilled. The reason why we investigated this is, because it was difficult to find good parameters for the boost converter / DC motor which would lead to such excellent performance and we wanted to gain deeper insight from other authors' results. In conclusion, it seems that although the results in [8],[10],[9] allow a good performance for the conducted experiments, they do not fulfill the stability conditions.

6.5 Summary

In this chapter, a boost converter / DC motor combination was considered. The main control task was tracking of the angular shaft velocity of the DC motor in the presence of appearing load disturbances. The model equation shows that the boost converter has bilinear system equations, and since DC motors are linear, the overall system remains bilinear. The first part of this section was devoted to system identification of the boost converter / DC motor which is important, because the proposed control strategy is model-based and we need good system parameters. From the control design it was observed that the performance in the presence of disturbances in the closed-loop system is not sufficient for integral feedback. Due to this reason, we focused on disturbance feedback for our experiments. An online-replanning strategy was proposed and its boundedness was shown to guarantee suboptimality and iISS of the closed-loop system in the presence of disturbances. From the experimental results it can be seen in the case of feedforward control that the system parameters match well and the system identification was successful. When load disturbances occur, the trajectory replanning and disturbance feedback for tracking a smooth set-point transition between stationary states allows the attenuation of the disturbance. For two parameter configurations (PC1) and (PC2), it can be shown that the second one allows better performance but leads to more oscillatory behavior. Finally, it has been shown that the same results can be obtained for a sinusoidal reference trajectory for the angular velocity with the same parameters (PC1) and (PC2), respectively, without change of the feedback design parameters, since the proposed control approach merely requires boundedness of the reference trajectories.

Chapter 7

Conclusion

In this thesis, we considered bilinear power converters as they arise for average models in continuous conduction mode. Since such power converters are often not feedback linearizable with respect to the output to be controlled, they are an interesting and demanding class of control systems.

A major control objective for control of these power converter systems was the inclusion of trajectory tracking, which renders the system differential equations time-varying. Furthermore, in order to cope with disturbances, noise, parameter uncertainties, etc., we used integral feedback, which led to input-affine systems with a special structure due to the originally bilinear system equations. Moreover, we proposed a disturbance feedback design, which is similar to integral feedback but requires online-replanning of the reference trajectory.

At this stage, we did not yet choose the state feedback strategy and studied closed-loop stability properties. Since we wanted to have low control effort and keep a prespecified performance output small in the case of disturbances, we decided to use a $\mathcal{H}_\infty$ control approach.

So as to be able to address stability properties, we recapitulated the basic stability theory for time-invariant and time-varying systems. Since we wanted to discuss stability in the presence of disturbances, we additionally considered Input-to-State Stability (ISS) and integral Input-to-State Stability (iISS).

Then, we discussed standard dissipativiy and $\mathcal{H}_\infty$ control theory, first for time-invariant, later for time-varying systems. Herein, we linked the solution of the nonlinear $\mathcal{H}_\infty$ control problem with iISS in order to guarantee stability for the closed-loop system in the presence of disturbances.

This result was used later on to solve the $\mathcal{H}_\infty$ control problem for the integral and disturbance feedback control with trajectory tracking for the bilinear power converter systems. As the main contribution of this thesis, we were able to present conditions, when the problem is solvable which, at the same time, allows to achieve iISS of the closed-loop system. In addition, it was shown that general bilinear systems can be controlled with the proposed approach, even in the more demanding multi-input case.

Equipped with the required theory to solve the posed control problem, we considered an experimental setup of a boost converter / DC motor system. Here, the control task was to track the angular velocity of the motor shaft. Since the system has relative degree 3 and exhibits an unstable internal dynamics with respect to the angular velocity, standard output feedback approaches are not applicable. At first, we did a thorough system identification since the control strategy used is model-based. The application of our feedback control computation showed that in the case of integral feedback it is difficult to find satisfactory control parameters to achieve a fast disturbance attenuation. Therefore, we decided to use disturbance feedback and proved boundedness of trajectories for the online-replanning of the approximate trajectory generation method which allowed for better results. Various experiments were carried out in order to investigate the applicability of the approach.

The results obtained in this thesis suggest several directions for future research. First, the output feedback problem has to be considered instead of pure state feedback, which would be the next step in the control design. Second, we did not investigate how to handle the input constraints naturally arising due to switching (the input functions are bounded with $u \in [0, 1]$). Third, it remains to find more general functions V which allow for better tuning of the feedback design parameters in the integral feedback case in order to increase performance.

Appendix A

Figures

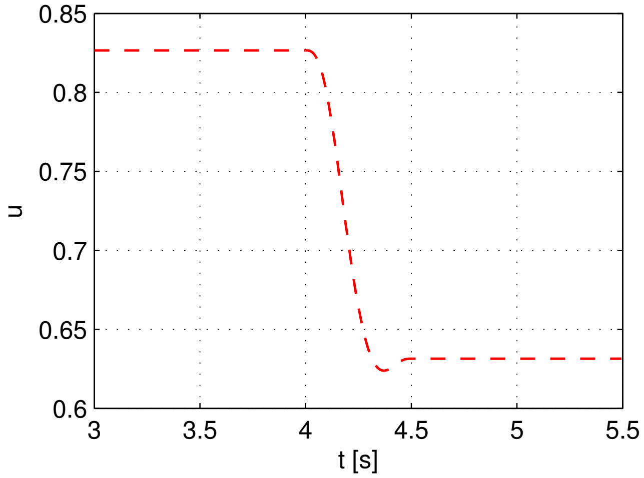

Figure A.1: Reference input $u^{\star}(t)$

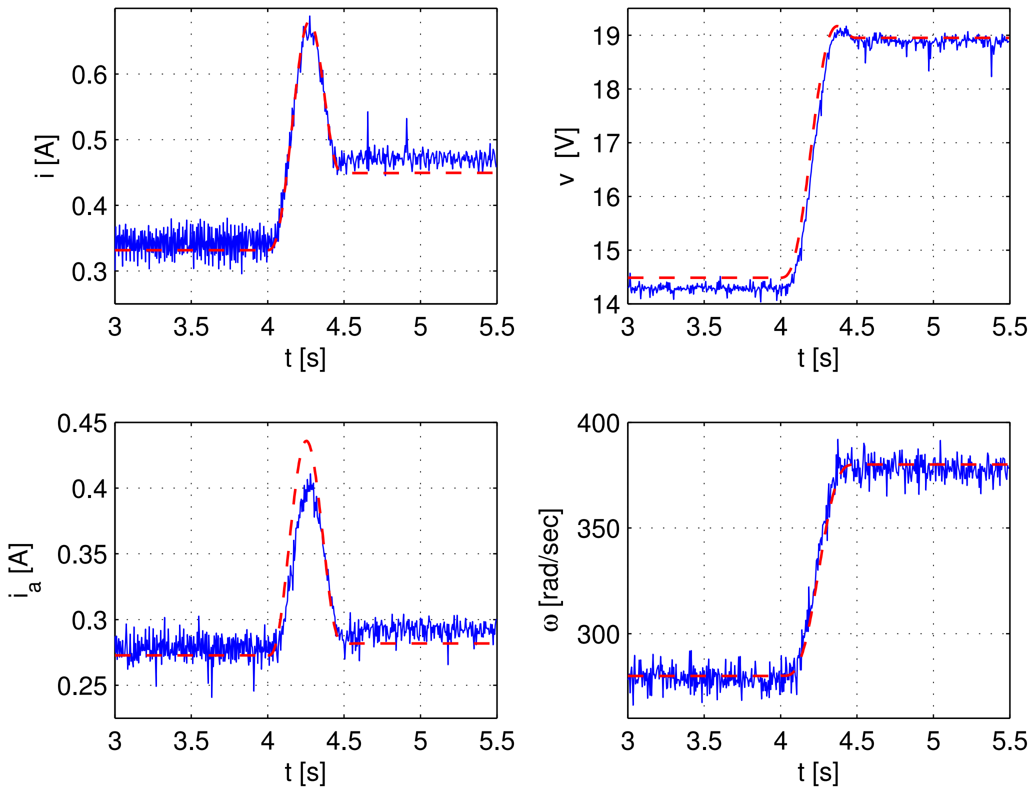

Figure A.2: Feedforward control measurements (solid) and reference trajectories (dashed)

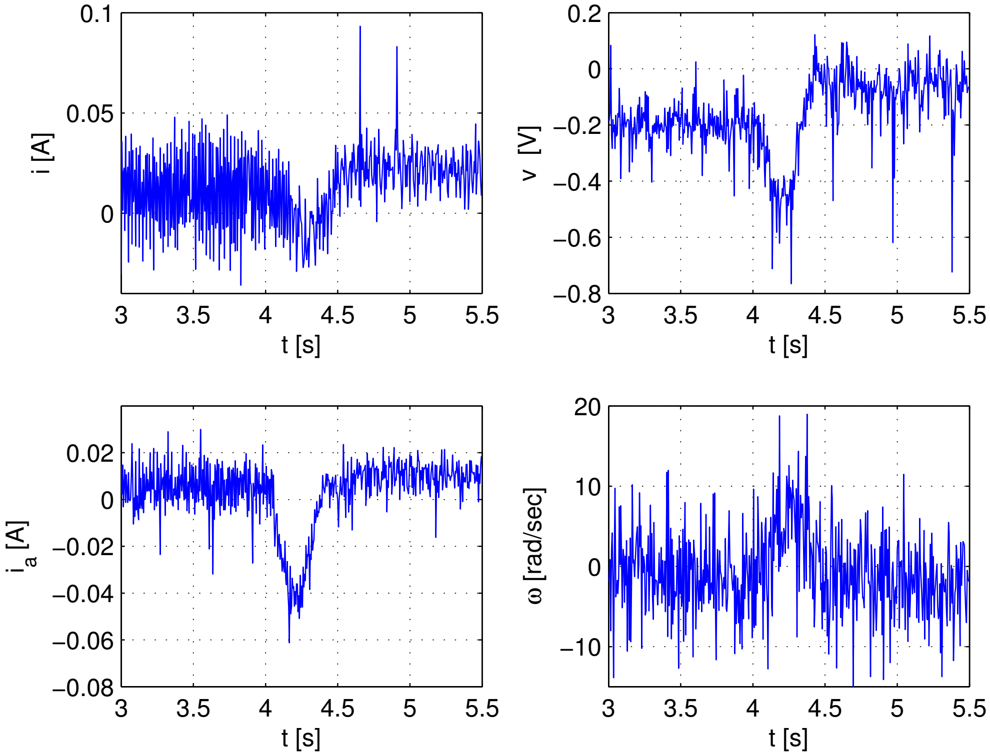

Figure A.3: Absolute error of the state variables.

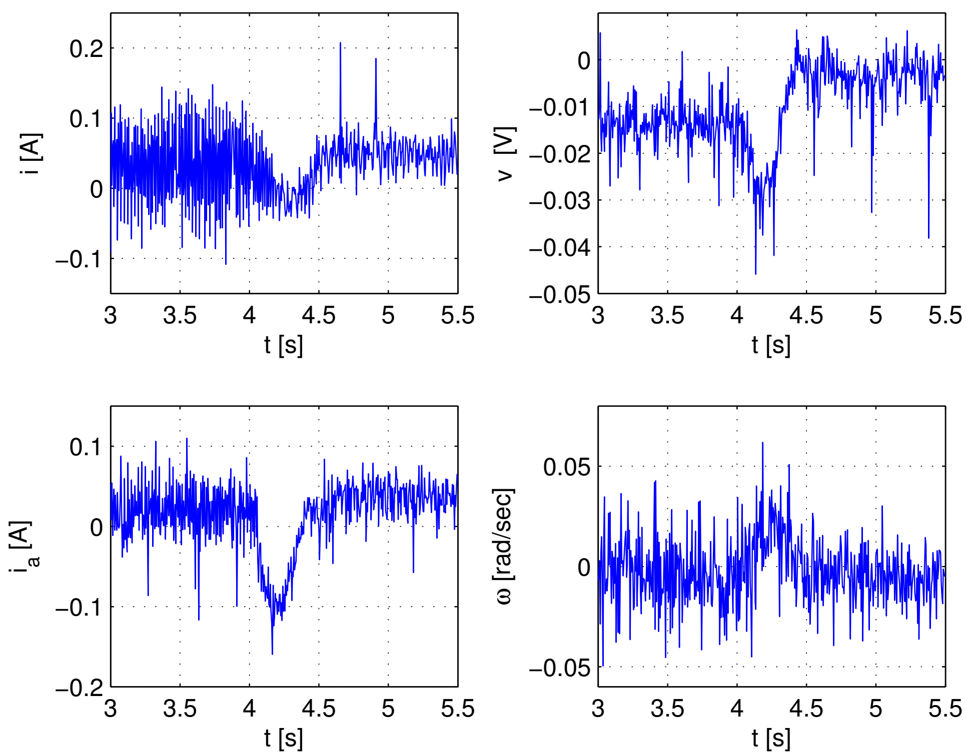

Figure A.4: Relative error of the state variables.

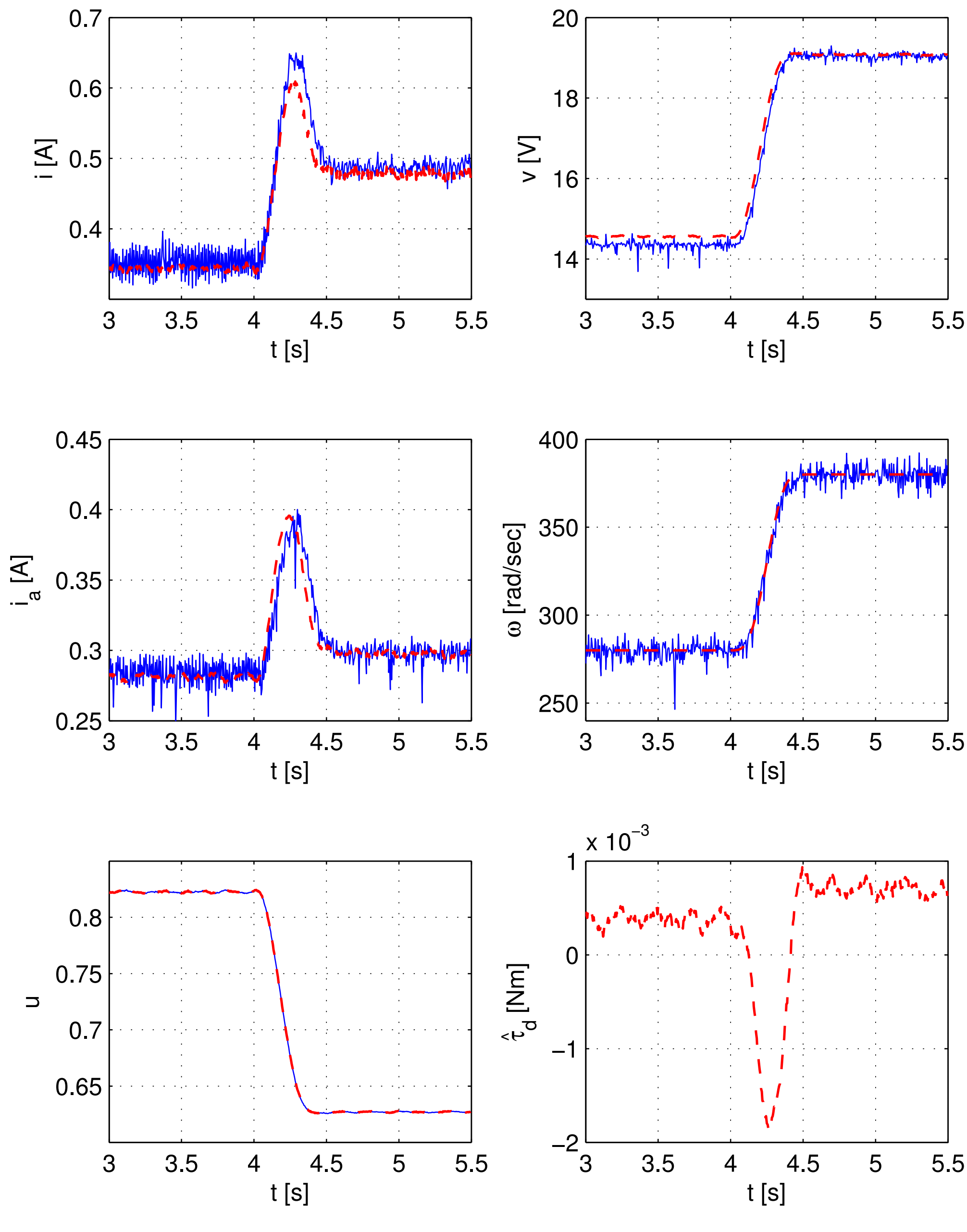

Figure A.5: Laboratory measurements for (PC1): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed).

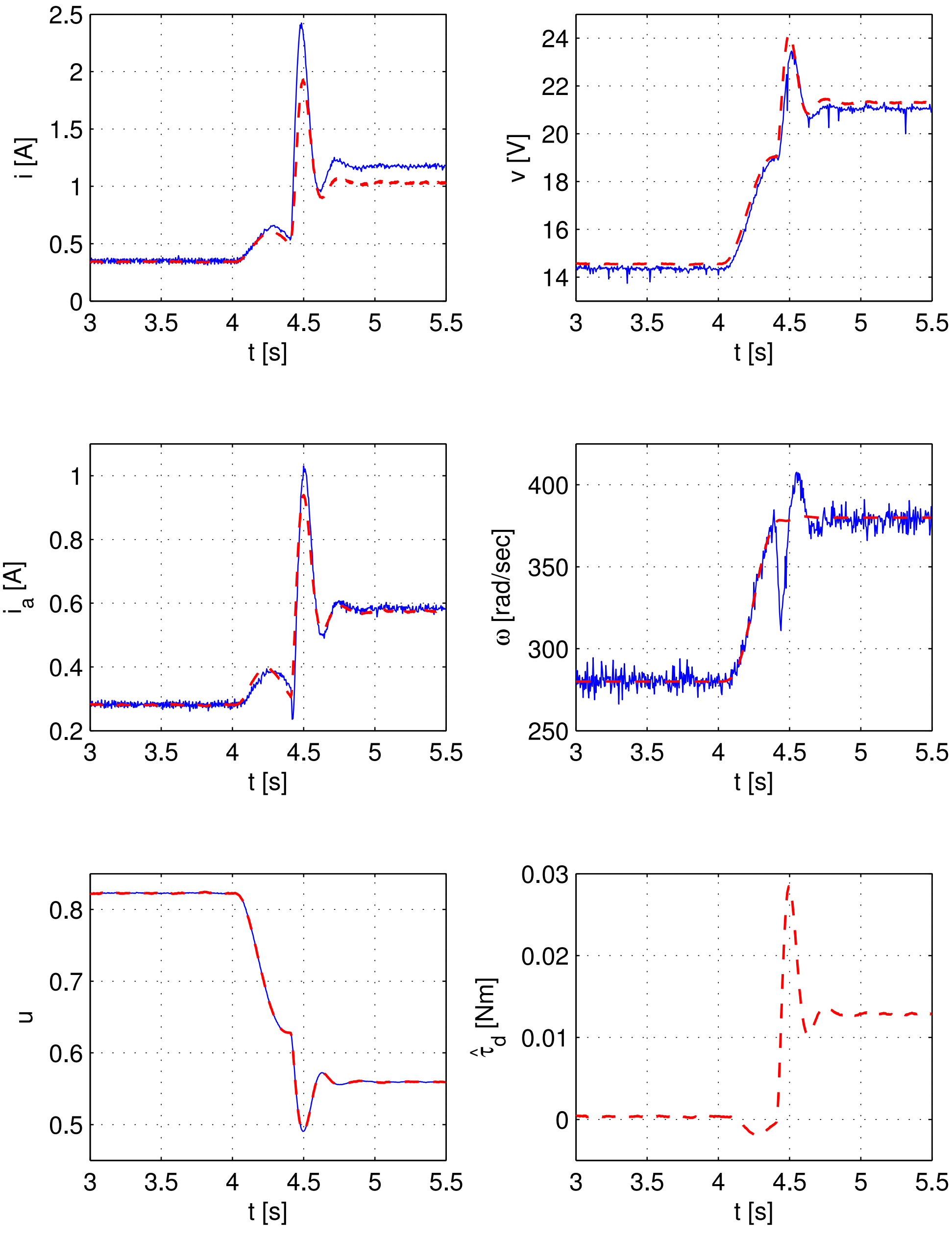

Figure A.6: Laboratory measurements for (PC1): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed), severe load step during transition.

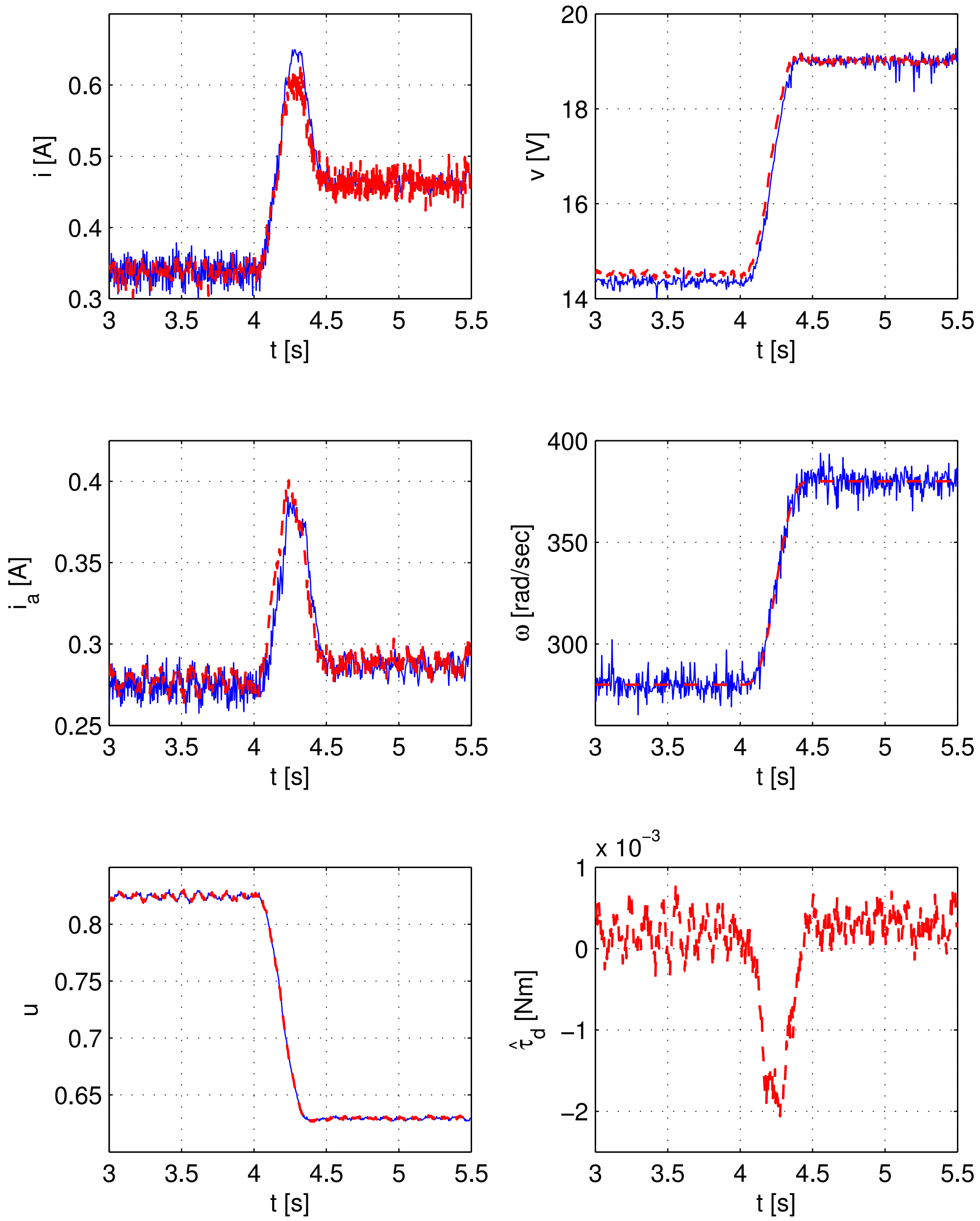

Figure A.7: Laboratory measurements for (PC2): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed).

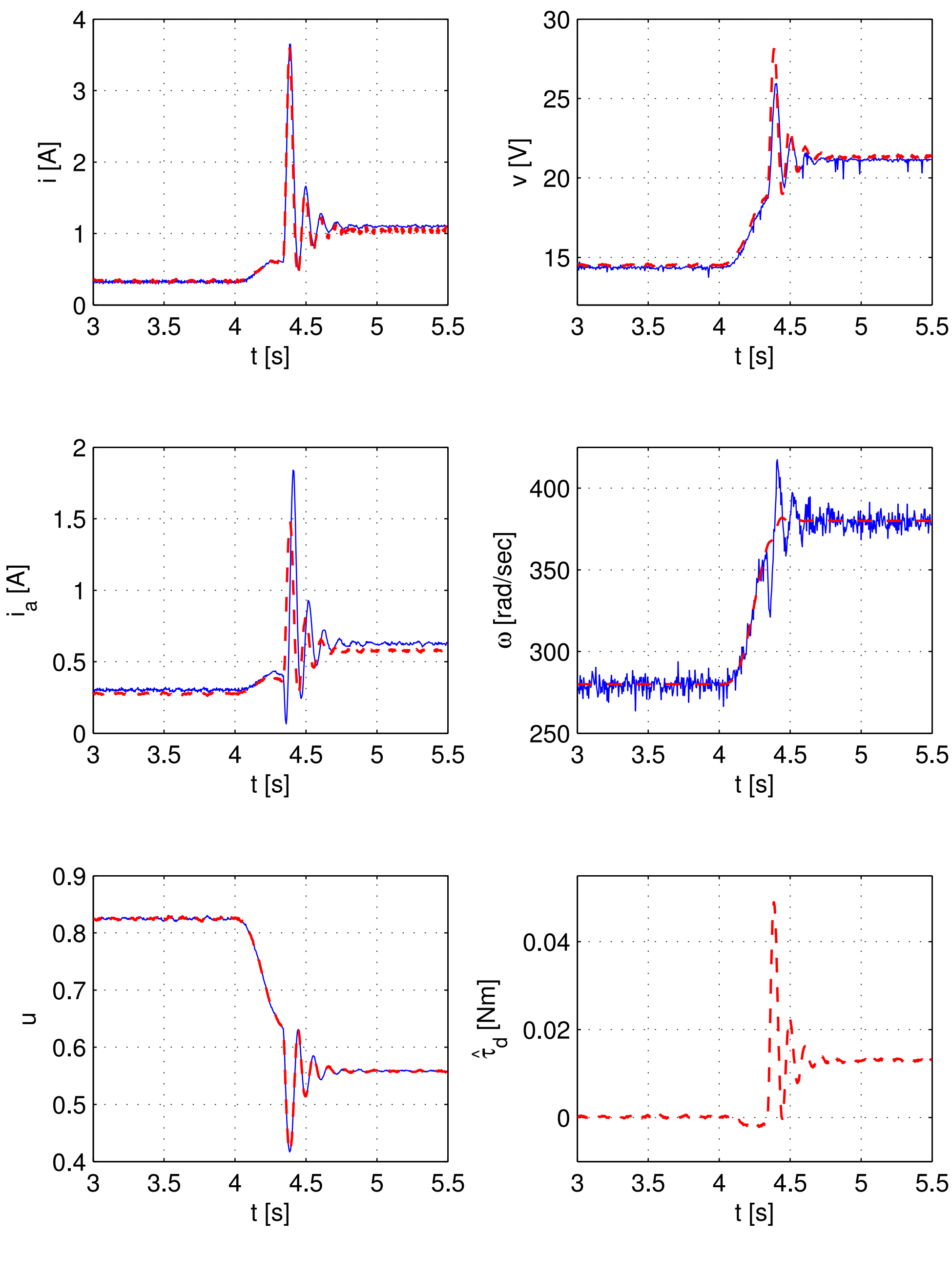

Figure A.8: Laboratory measurements for (PC2): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed), severe load step during transition.

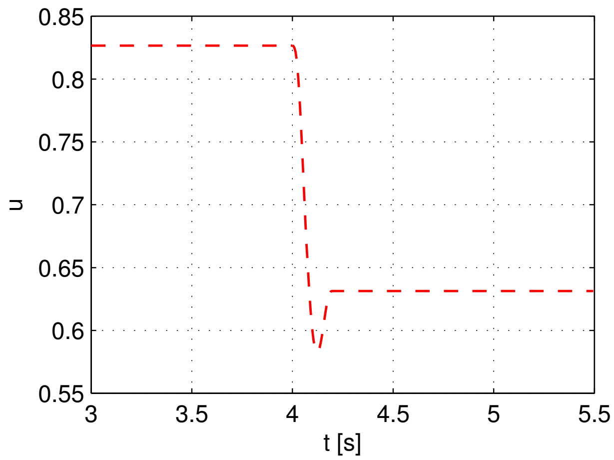

Figure A.9: Reference input $u^{\star}(t)$

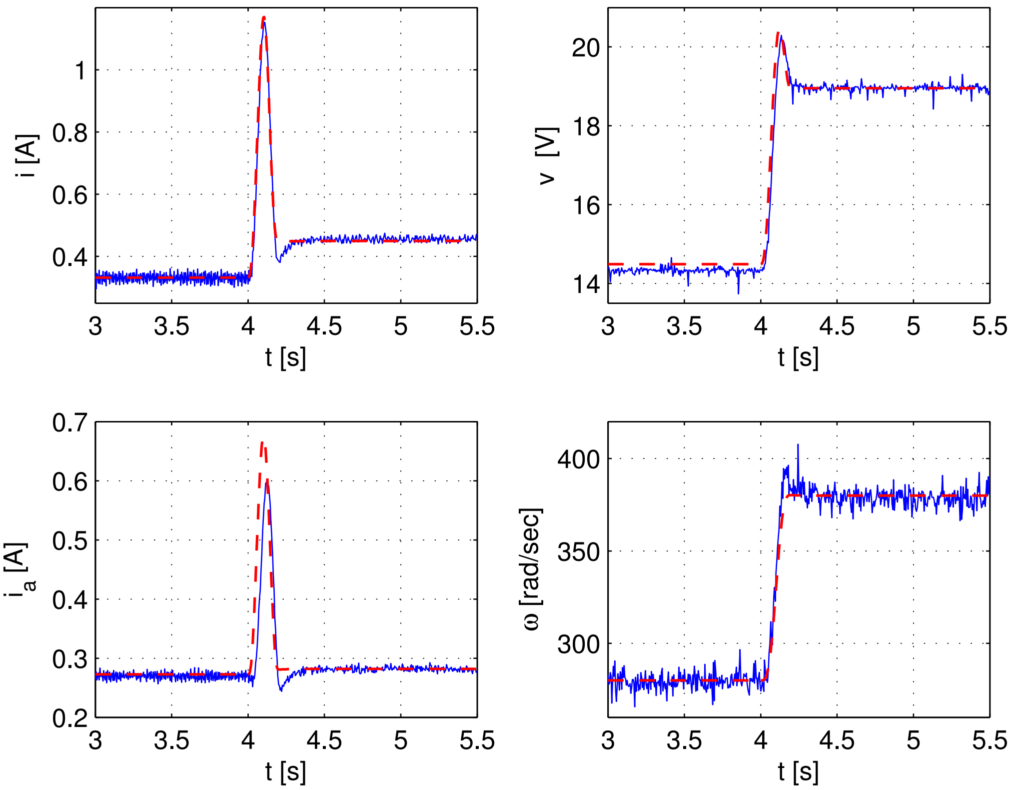

Figure A.10: Feedforward control measurements (solid) and reference trajectories (dashed).

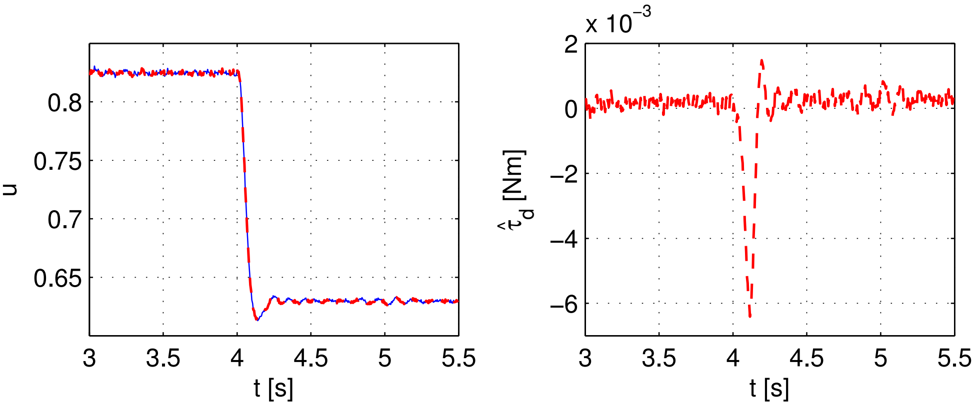

Figure A.11: Laboratory measurements: closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed).

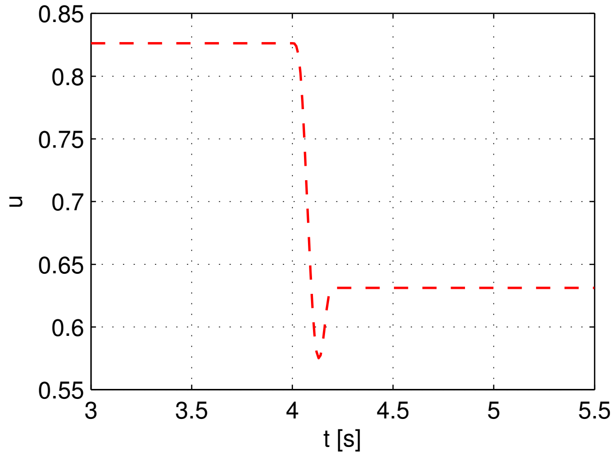

Figure A.12: Reference input $u^{\star}(t)$

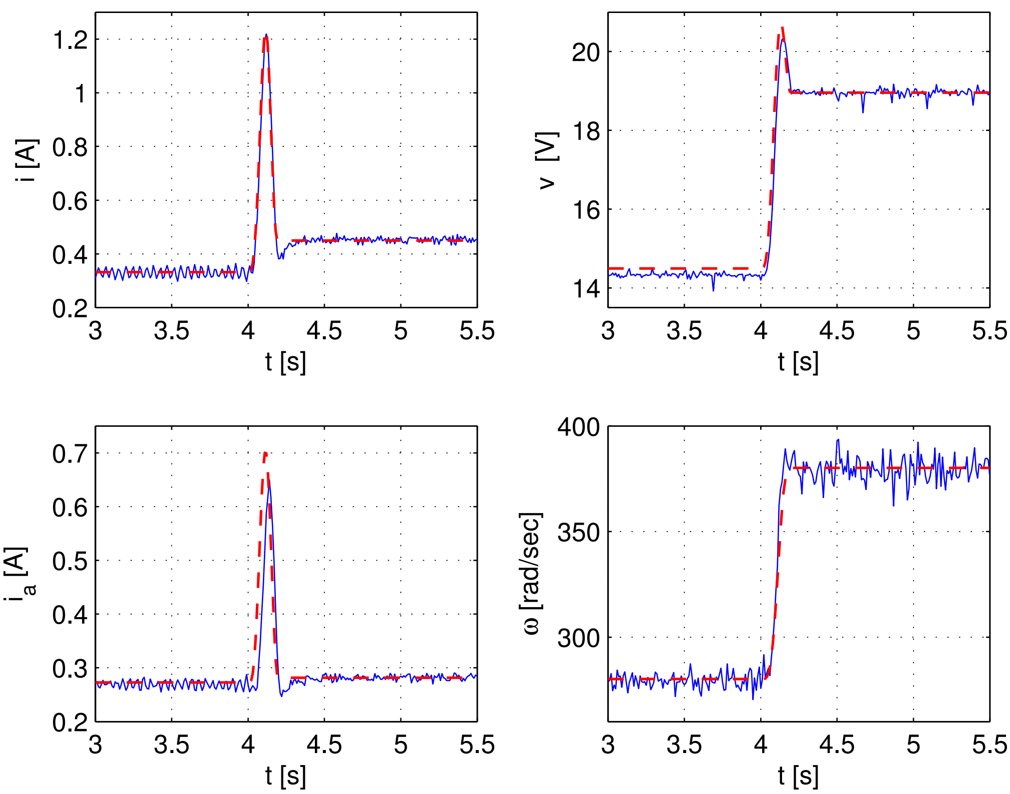

Figure A.13: Feedforward control measurements (solid) and reference trajectories (dashed).

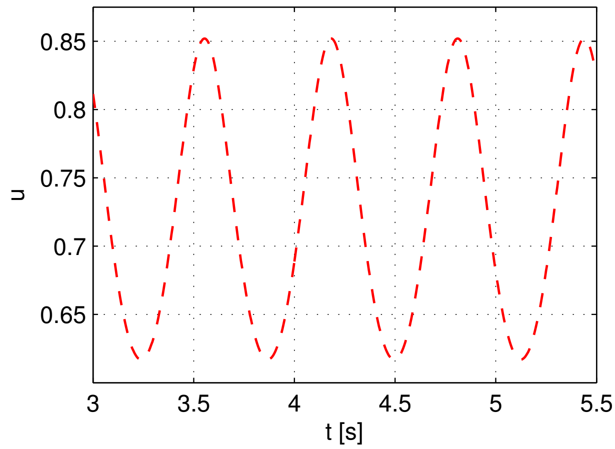

Figure A.14: Reference input $u^{\star}(t)$

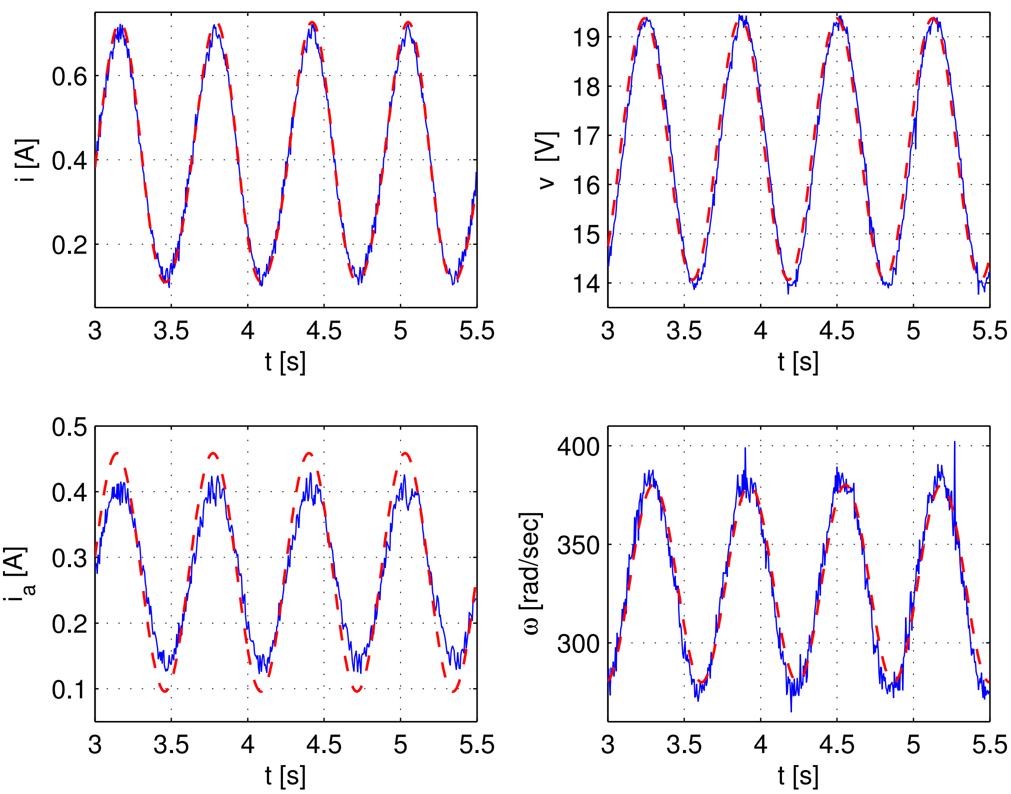

Figure A.15: Feedforward control measurements (solid) and reference trajectories (dashed).

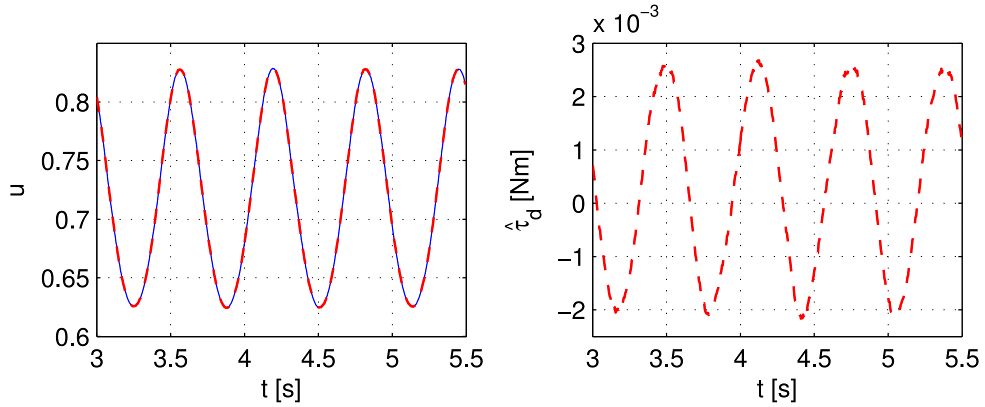

Figure A.16: Laboratory measurements for (PC1): closed-loop with minimum disturbance feedback (solid), reference trajectories in closed-loop case (dashed).

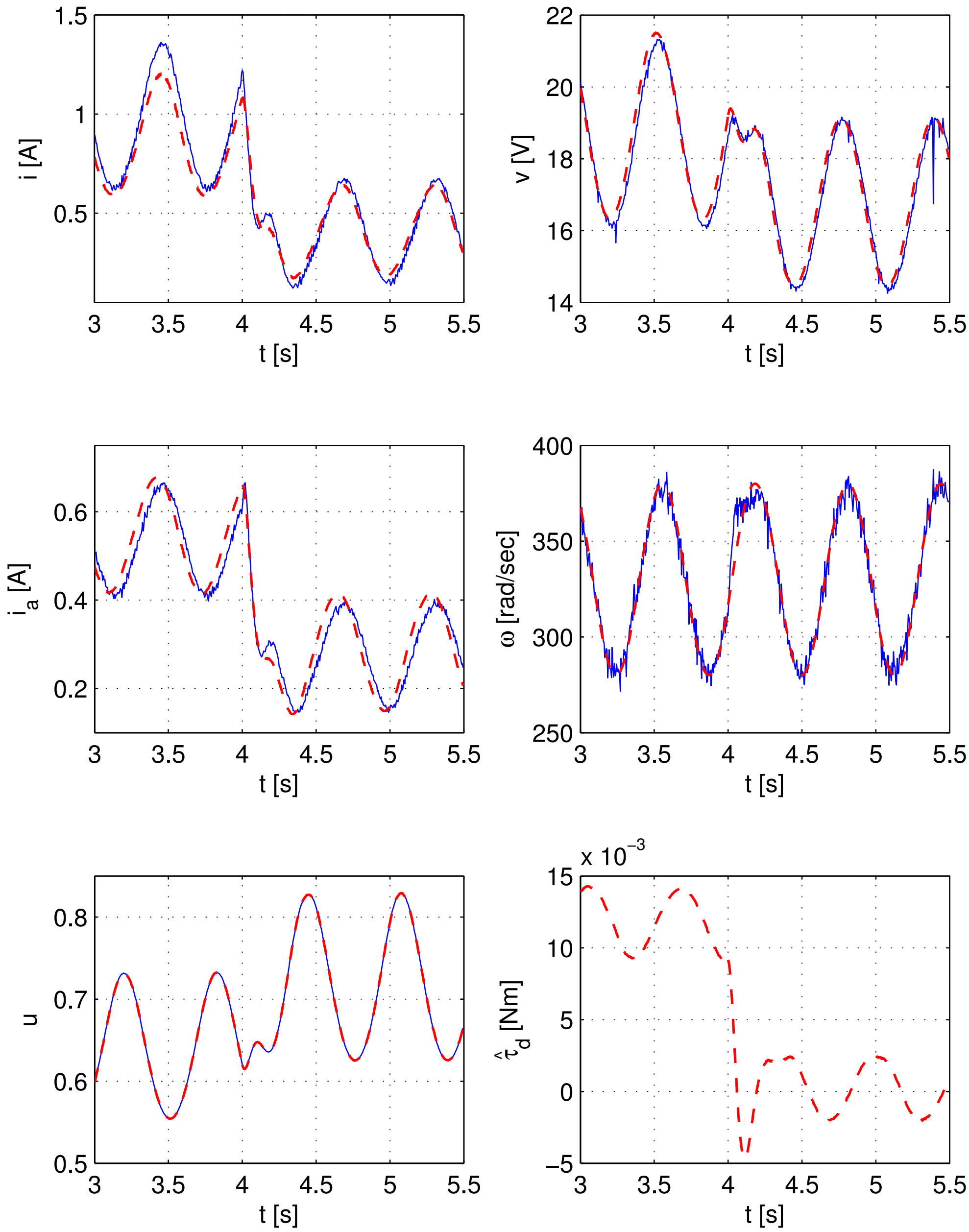

Figure A.17: Laboratory measurements for (PC1): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed), severe load step appearing.

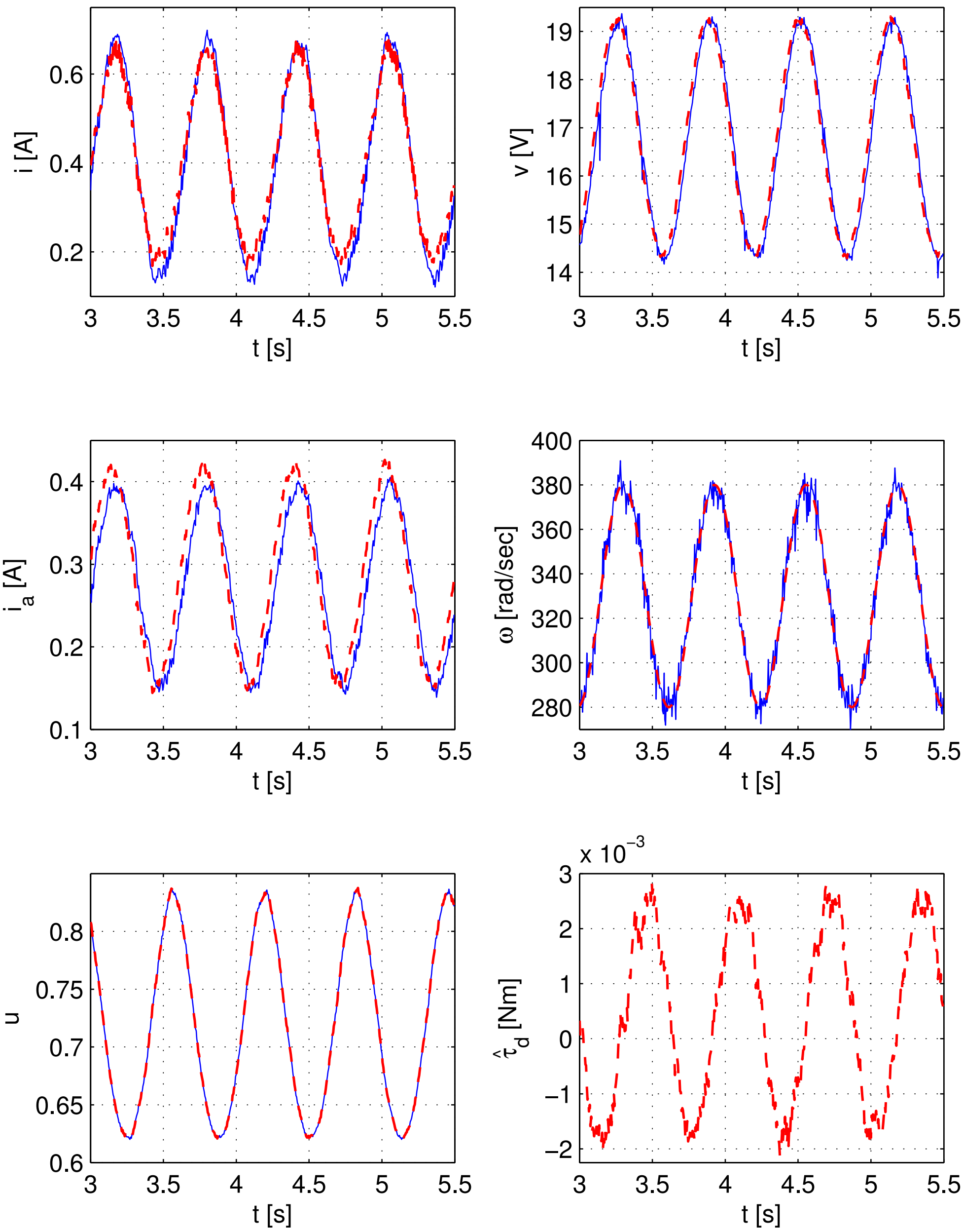

Figure A.18: Laboratory measurements for (PC2): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed).

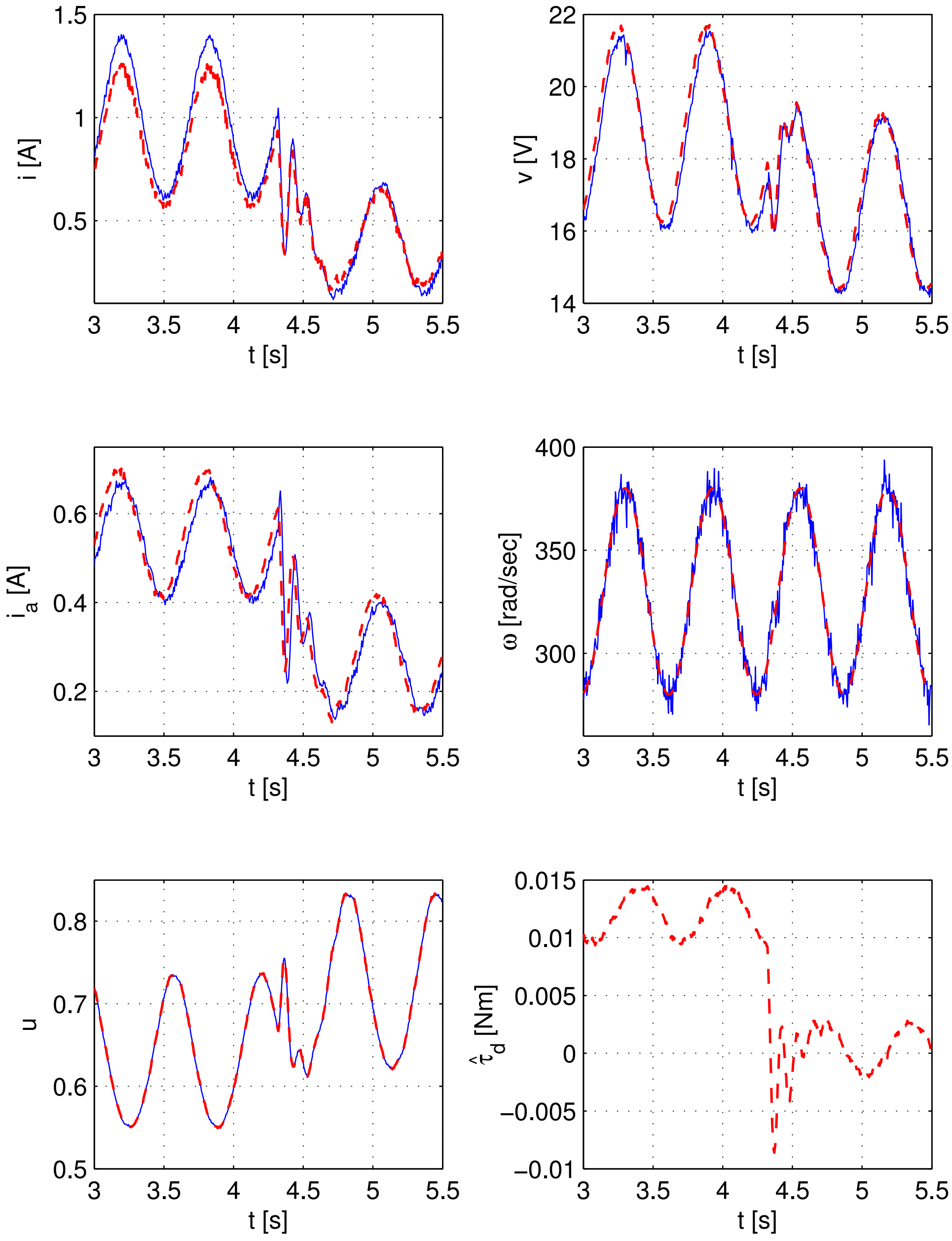

Figure A.19: Laboratory measurements (PC2): closed-loop with disturbance feedback (solid), reference trajectories in closed-loop case (dashed), severe load step appearing.

Appendix B

System Theoretic Properties of the Boost Converter / DC Motor Combination

B.1 Basic Results

The following exposition is taken without proofs from the textbooks [70],[71].

Consider the single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad && x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x), && y \in \mathbb{R} \end{aligned} \tag{B.1}$$

with smooth vector fields f, g, h defined on a domain (open connected set) $D \subset \mathbb{R}^n$ containing the origin $x = 0$.

Definition B.1 (Lie Derivative) *Let f, h be smooth vector fields defined on $D \subset \mathbb{R}^n$ and $x \in D$. The Lie derivative of h in direction of f denoted by $L_f h$ is the mapping $x \mapsto L_f h(x)$ from $\mathbb{R}^n$ to $\mathbb{R}$ where*

$$L_f h(x) := \frac{\partial h(x)}{\partial x} f(x), \quad x \in D. \tag{B.2}$$

Remark B.1 *$\frac{\partial h(x)}{\partial x}$ is a row vector, $\frac{\partial f(x)}{\partial x}$ is the Jacobian of f, and $L_f^0 h(x) := h(x)$,$L_f^k h(x) := \frac{\partial L_f^{k-1} h(x)}{\partial x} f(x), k \geq 1$ is the abbreviation for iterated application of the Lie derivative.*

Definition B.2 (Lie Bracket) *Given two smooth vector fields f, g on $D \in \mathbb{R}^n$, the Lie bracket $[f, g]$ is a vector field defined by*

$$[f, g](x) := \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

Remark B.2 *The following notation is used to simplify repeated bracketing:*

$$\begin{aligned} \mathrm{ad}_f^0 g(x) &:= g(x) \\ \mathrm{ad}_f g(x) &:= [f, g](x) \\ \mathrm{ad}_f^k g(x) &:= [f, \mathrm{ad}_f^{k-1} g](x), \quad k \geq 2. \end{aligned}$$

Definition B.3 (Relative Degree) *The nonlinear system (B.1) is said to have relative degree r at a point $\bar{x} \in \mathbb{R}^n$, if and only if there exists an open neighborhood D of $\bar{x}$, such that*

$$\begin{aligned} L_g L_f^k h(x) &= 0, \quad k = 0, \ldots r-2 \\ L_g L_f^{r-1} h(\bar{x}) &\neq 0 \end{aligned}$$

for all $x \in D$.

Theorem B.1 *Let system (B.1) have relative degree $r = n$. Then there exists a neighborhood D of $\bar{x}$, such that the map*

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = T(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}$$

restricted to D is a diffeomorphism on D and transforms (B.1) to

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= b(z) + a(z)u \end{aligned}$$

with smooth functions a, b defined on $\{z \mid T^{-1}(z) \in D\}$ where $a(z) \neq 0$.

Problem B.1 (State Space Exact Linearization Problem)
System (B.1) is called feedback linearizable (or input-state linearizable), if for

given $\bar{x} \in \mathbb{R}^n$, *there exists a neighborhood* D *of* $\bar{x}$, *a diffeomorphism* $T : D \to \mathbb{R}^n$, *and an input transformation* $u = \alpha(x) + \beta(x)v, x \in D$, *such that* $z = T(x)$ *transforms* (B.1) *into Brunovský normal form*

$$\dot{z} = Az + Bv \tag{B.3}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

An equivalent formulation of the State Space Exact Linearization Problem is the following:

Remark B.3 *System* (B.1) *is called feedback linearizable (or input-state linearizable), if for given* $\bar{x} \in \mathbb{R}^n$, *there exists a neighborhood* D *of* $\bar{x}$, *a diffeomorphism* $T : D \to \mathbb{R}^n$, $x \mapsto z$, *and an input transformation* $u = \alpha(x) + \beta(x)v, x \in D$, *such that*

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v$$

is linear and controllable in z*-coordinates.*

Definition B.4 (Smooth Distribution) *Given a set of smooth vector fields* $f_1(x), \dots, f_m(x)$, *we define the distribution* $\Delta(x)$ *to be*

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \dots, f_m(x)\}, \quad \forall x \in D \subset \mathbb{R}^n.$$

Remark B.4 *In the following, we use*

$$\Delta = \text{span}\{f_1, f_2, \dots, f_m\}$$

to denote the assignment for all $x \in D \subset \mathbb{R}^n$, *while with*

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \dots, f_m(x)\}$$

we mean that the distribution is evaluated at a fixed point $x \in D \subset \mathbb{R}^n$. *In addition,* span *is understood to mean that we consider the ring of smooth functions, i.e. elements of* Δ *at the point* x *are of the form*

$$c_1(x)f_1(x) + c_2(x)f_2(x) + \dots + c_m(x)f_m(x)$$

with smooth functions $c_i(x), i = 1, \dots, m$.

Definition B.5 (Involutive Distribution) *A distribution Δ is called involutive if for any two vector fields $\tau_1, \tau_2 \in \Delta(x)$ their Lie bracket $[\tau_1, \tau_2] \in \Delta(x)$.*

Lemma B.1 *The State Space Exact Linearization Problem is solvable if and only if there exist a neighborhood D of $\bar{x}$ and a real valued function λ defined on D, such that the system*

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= \lambda(x)\end{aligned}$$

has relative degree n at $\bar{x}$.

Theorem B.2 (Feedback Linearization) *System (B.1) is feedback linearizable at a point $\bar{x}$ if and only if*

1. *the matrix $\mathcal{G}(\bar{x}) = \left(g(\bar{x}), ad_f g(\bar{x}), \ldots, ad_f^{n-1} g(\bar{x})\right)$ has rank n;*
2. *the distribution $\mathcal{D} = \operatorname{span}\left\{g, ad_f g, \ldots, ad_f^{n-2} g\right\}$ is involutive near $\bar{x}$.*

Theorem B.3 *Consider (B.1) and assume $r < n$ at $\bar{x} \in \mathbb{R}^n$. Then there exists a diffeomorphism*

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_r \\ \hdashline z_{r+1} \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \xi \\ \hdashline \eta \end{pmatrix} = T(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \hdashline t_{r+1}(x) \\ \vdots \\ t_n(x) \end{pmatrix}$$

defined in a neighborhood D of $\bar{x}$ with

$$L_g t_i(x) = 0 \qquad \text{for all } r+1 \leq i \leq n \text{ and all } x \text{ around } \bar{x} \tag{B.4}$$

which transforms (B.1) to

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= b(\xi, \eta) + a(\xi, \eta)u \\ \dot{\eta} &= q(\xi, \eta).\end{aligned} \tag{B.5}$$

with smooth functions a, b defined on $\{z = (\xi^{\mathrm{T}}, \eta^{\mathrm{T}})^{\mathrm{T}} \mid T^{-1}(z) \in D\}$ where $a(\xi, \eta) \neq 0$.

We will denote the equation $\dot{\eta} = q(\xi, \eta)$ as the *internal dynamics* of system (B.5) since it cannot be directly affected by the input u. Furthermore, the state feedback law $u(\xi, \eta) = \frac{1}{a(\xi,\eta)}(-b(\xi, \eta) + v)$ with new input v leads to $\dot{\xi}_r = y^{(r)} = v$ which is linear. Opposed to the input-state linearization, $y^{(r)} = v$ is merely the *input-output linearization* of system (B.1) with respect to output $y = h(x)$.

Problem B.2 *Find, if possible, an initial state $\bar{x} \in D$ and an input $\bar{u}(\cdot)$ such that the output $y(t) = h(x(t))$ of system*

$$\dot{x} = f(x) + g(x)\bar{u}$$

is identically zero for all $t \geq 0$.

As can be seen from Theorem B.3, we have the output $y = h(x) = z_1$, cf. the definition of the state transformation. If we assume now that the output $y \equiv 0$, we know by differentiation and from the structure of the state transformation that $z_i \equiv 0, i = 1, \ldots, r$, i.e. $\xi \equiv 0$. From (B.5) it follows that

$$\bar{u}(t) = -\frac{b(0, \eta(t))}{a(0, \eta(t))} \tag{B.6}$$

where $\eta(t)$ is any solution of

$$\dot{\eta} = q(0, \eta) \tag{B.7}$$

for arbitrary initial condition $\eta(0)$. The latter equation (B.7) is called *zero dynamics* of the system (B.1). Assume now that $\bar{x} = 0$ is an equilibrium point of the nonlinear system, i.e. $f(0) = 0$ and at the same time $y = h(0) = 0$, and furthermore the transformation $z = T(x)$ preserves the origin, i.e. $T(0) = 0$.

Definition B.6 *The nonlinear system (B.1) is said to be locally asymptotically (exponentially) minimum phase at $\bar{x}$ if the equilibrium point $\eta = 0$ of (B.7) is locally asymptotically (exponentially) stable.*

Definition B.7 *The nonlinear system (B.1) is said to be locally nonminimum-phase at $\bar{x}$ if the equilibrium point $\eta = 0$ of (B.7) is locally unstable.*

B.2 Computations for the Boost Converter / DC Motor

We now check if the given boost converter / DC motor system is feedback linearizable. To this end, we write the state equations in (6.1)-(6.4) in the form of Theorem B.2 and compute the successive Lie brackets:

$$x = \begin{pmatrix} i \\ v \\ i_{\mathrm{a}} \\ \omega \end{pmatrix}, f(x) = \begin{pmatrix} -\frac{R_L}{L}x_1 + \frac{E}{L} \\ -\frac{G}{C}x_2 - \frac{1}{C}x_3 \\ \frac{1}{L_{\mathrm{m}}}x_2 - \frac{R_{\mathrm{m}}}{L_{\mathrm{m}}}x_3 - \frac{K_{\mathrm{e}}}{L_{\mathrm{m}}}x_4 \\ \frac{K_{\mathrm{m}}}{J}x_3 - \frac{B_{\mathrm{m}}}{J}x_4 - \frac{\tau_{\mathrm{l}}}{J} \end{pmatrix}, g(x) = \begin{pmatrix} -\frac{1}{L}x_2 \\ \frac{1}{C}x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$ad_f g(x) = \begin{pmatrix} \frac{x_2 G}{LC} - \frac{x_2 R_L}{L^2} + \frac{x_3}{LC} \\ -\frac{x_1 R_L}{LC} + \frac{x_1 G}{C^2} + \frac{E}{LC} \\ -\frac{x_1}{L_{\mathrm{m}}C} \\ 0 \end{pmatrix}$$

$$ad_f^2 g(x) = \begin{pmatrix} -\frac{x_2 G^2}{C^2 L} + \frac{2x_2 G R_L}{CL^2} + \frac{x_2}{L_{\mathrm{m}}CL} - \frac{x_2 R_L^2}{L^3} - \frac{x_3 G}{C^2 L} + \frac{2x_3 R_L}{L^2 C} - \frac{x_3 R_{\mathrm{m}}}{LCL_{\mathrm{m}}} - \frac{K_{\mathrm{e}} x_4}{LCL_{\mathrm{m}}} \\ \frac{x_1 R_L^2}{L^2 C} - \frac{2x_1 R_L G}{LC^2} + \frac{x_1 G^2}{C^3} - \frac{x_1}{C^2 L_{\mathrm{m}}} - \frac{E R_L}{L^2 C} + \frac{2GE}{C^2 L} \\ \frac{2x_1 R_L}{L_{\mathrm{m}}CL} - \frac{x_1 G}{L_{\mathrm{m}}C^2} - \frac{x_1 R_{\mathrm{m}}}{L_{\mathrm{m}}^2 C} - \frac{2E}{L_{\mathrm{m}}CL} \\ \frac{K_{\mathrm{m}} x_1}{JL_{\mathrm{m}}C} \end{pmatrix}$$

$$ad_f^3 g(x) = \begin{pmatrix} -\frac{2x_2 G}{L_{\mathrm{m}}C^2 L} + \frac{3x_2 R_L}{L_{\mathrm{m}}L^2 C} - \frac{x_2 R_{\mathrm{m}}}{L_{\mathrm{m}}^2 LC} + \frac{x_2 G^3}{C^3 L} - \frac{3x_2 G^2 R_L}{C^2 L^2} + \frac{3x_2 G R_L^2}{CL^3} - \frac{x_2 R_L^3}{L^4} \cdots \\ + \frac{x_3 G^2}{C^3 L} - \frac{3x_3 G R_L}{C^2 L^2} - \frac{x_3}{C^2 L L_{\mathrm{m}}} + \frac{3x_3 R_L^2}{CL^3} - \frac{3x_3 R_L R_{\mathrm{m}}}{L^2 C L_{\mathrm{m}}} + \frac{x_3 R_{\mathrm{m}} G}{L_{\mathrm{m}} C^2 L} + \frac{x_3 R_{\mathrm{m}}^2}{L_{\mathrm{m}}^2 LC} \cdots \\ -\frac{x_3 K_{\mathrm{e}} K_{\mathrm{m}}}{LCL_{\mathrm{m}}J} - \frac{3x_4 R_L K_{\mathrm{e}}}{L^2 C L_{\mathrm{m}}} + \frac{x_4 K_{\mathrm{e}} G}{L_{\mathrm{m}} C^2 L} + \frac{x_4 K_{\mathrm{e}} R_{\mathrm{m}}}{L_{\mathrm{m}}^2 LC} + \frac{x_4 K_{\mathrm{e}} B}{LCL_{\mathrm{m}}J} + \frac{K_{\mathrm{e}} \tau_{\mathrm{l}}}{LCL_{\mathrm{m}}J} \\ -\frac{x_1 R_L^3}{L^3 C} + \frac{3x_1 R_L^2 G}{L^2 C^2} - \frac{3x_1 R_L G^2}{LC^3} + \frac{3x_1 R_L}{LC^2 L_{\mathrm{m}}} + \frac{x_1 G^3}{C^4} - \frac{2x_1 G}{C^3 L_{\mathrm{m}}} - \frac{x_1 R_{\mathrm{m}}}{C^2 L_{\mathrm{m}}^2} \cdots \\ + \frac{E R_L^2}{L^3 C} - \frac{3E R_L G}{L^2 C^2} + \frac{3EG^2}{LC^3} - \frac{3E}{C^2 L_{\mathrm{m}} L} \\ -\frac{3x_1 R_L^2}{L^2 L_{\mathrm{m}} * C} + \frac{3x_1 R_L G}{L L_{\mathrm{m}} C^2} + \frac{3x_1 R_L R_{\mathrm{m}}}{L L_{\mathrm{m}}^2 C} - \frac{x_1 G^2}{L_{\mathrm{m}} C^3} + \frac{x_1}{L_{\mathrm{m}}^2 C^2} - \frac{x_1 R_{\mathrm{m}} G}{L_{\mathrm{m}}^2 C^2} - \frac{x_1 R_{\mathrm{m}}^2}{L_{\mathrm{m}}^3 C} \cdots \\ + \frac{x_1 K_{\mathrm{e}} K_{\mathrm{m}}}{L_{\mathrm{m}}^2 JC} + \frac{3E R_L}{L^2 L_{\mathrm{m}} C} - \frac{3EG}{L L_{\mathrm{m}} C^2} - \frac{3R_{\mathrm{m}} E}{L_{\mathrm{m}}^2 CL} \\ -\frac{3x_1 K_{\mathrm{m}} R_L}{JL_{\mathrm{m}}CL} + \frac{x_1 K_{\mathrm{m}} G}{JL_{\mathrm{m}}C^2} + \frac{x_1 K_{\mathrm{m}} R_{\mathrm{m}}}{JL_{\mathrm{m}}^2 C} + \frac{x_1 B K_{\mathrm{m}}}{J^2 L_{\mathrm{m}} C} + \frac{3K_{\mathrm{m}} E}{JL_{\mathrm{m}}CL} \end{pmatrix}$$

Checking for linear independence of

$$\mathcal{G} = \left(g, ad_f g, ad_f^2 g, ad_f^3 g\right)$$

shows that $\mathrm{rank}(\mathcal{G}) = 4$ for all x, i.e. $\mathcal{G}$ is of full rank.

What concerns the second condition for feedback linearizability, we have to check if the following Lie brackets are elements of $\mathcal{D} = \text{span}\{g, ad_f g, ad_f^2 g\}$, i.e. check involutivity of $\mathcal{D}$

$$[g, \text{ad}_f g], [g, \text{ad}_f^2 g], [\text{ad}_f g, \text{ad}_f^2 g] \in \mathcal{D}.$$

First, for completeness, we state the results for the Lie brackets:

$$[g, \text{ad}_f g](x) = \begin{pmatrix} -\frac{2x_1 R_L}{L^2 C} + \frac{2x_1 G}{LC^2} + \frac{E}{L^2 C} \\ \frac{2x_2 R_L}{L^2 C} - \frac{2x_2 G}{LC^2} - \frac{x_3}{C^2 L} \\ \frac{x_2}{L_\text{m} C L} \\ 0 \end{pmatrix}$$

$$[g, \text{ad}_f^2 g](x) = \begin{pmatrix} -\frac{E R_L}{L^3 C} + \frac{2GE}{L^2 C^2} \\ -\frac{2x_3 R_L}{C^2 L^2} + \frac{x_3 G}{C^3 L} + \frac{x_3 R_\text{m}}{C^2 L L_\text{m}} + \frac{K_\text{e} x_4}{C^2 L L_\text{m}} \\ -\frac{2x_2 R_L}{L_\text{m} L^2 C} + \frac{x_2 G}{L_\text{m} C^2 L} + \frac{x_2 R_\text{m}}{L_\text{m}^2 L C} \\ -\frac{K_\text{m} x_2}{J L_\text{m} C L} \end{pmatrix}$$

$$[\text{ad}_f g, \text{ad}_f^2 g](x) = \begin{pmatrix} -\frac{6x_1 R_L}{L^2 C^2 L_\text{m}} + \frac{4x_1 G}{LC^3 L_\text{m}} + \frac{2x_1 R_\text{m}}{LC^2 L_\text{m}^2} - \frac{6x_1 R_L^2 G}{L^3 C^2} + \frac{2x_1 R_L^3}{L^4 C} + \frac{6x_1 R_L G^2}{L^2 C^3} \cdots \\ -\frac{2x_1 G^3}{C^4 L} - \frac{3EG^2}{C^3 L^2} + \frac{5EGR_L}{C^2 L^3} + \frac{3E}{L^2 C^2 L_\text{m}} - \frac{2ER_L^2}{CL^4} \\ \frac{6x_2 R_L^2 G}{L^3 C^2} - \frac{2x_2 R_L^3}{L^4 C} - \frac{6x_2 R_L G^2}{L^2 C^3} + \frac{2x_2 G^3}{C^4 L} - \frac{2x_2 G}{LC^3 L_\text{m}} + \frac{2x_2 R_L}{L^2 C^2 L_\text{m}} \cdots \\ + \frac{3x_3 R_L^2}{L^3 C^2} - \frac{5x_3 R_L G}{L^2 C^3} + \frac{2x_3 G^2}{LC^4} - \frac{x_3}{LC^3 L_\text{m}} - \frac{x_3 R_L R_\text{m}}{C^2 L^2 L_\text{m}} + \frac{x_3 G R_\text{m}}{C^3 L L_\text{m}} \cdots \\ -\frac{K_\text{e} x_4 R_L}{L^2 C^2 L_\text{m}} + \frac{K_\text{e} x_4 G}{LC^3 L_\text{m}} \\ \frac{5x_2 R_L G}{L_\text{m} C^2 L^2} - \frac{3x_2 R_L^2}{L_\text{m} C L^3} - \frac{2x_2 G^2}{L_\text{m} C^3 L} - \frac{x_2 R_\text{m} G}{L_\text{m}^2 C^2 L} + \frac{x_2 R_\text{m} R_L}{L_\text{m}^2 C L^2} + \frac{x_2}{L_\text{m}^2 C^2 L} \cdots \\ + \frac{4x_3 R_L}{L^2 C^2 L_\text{m}} - \frac{2x_3 G}{LC^3 L_\text{m}} - \frac{2x_3 R_\text{m}}{LC^2 L_\text{m}^2} - \frac{K_\text{e} x_4}{L_\text{m}^2 C^2 L} \\ \frac{K_\text{m} x_2 G}{J L_\text{m} C^2 L} - \frac{K_\text{m} x_2 R_L}{J L_\text{m} C L^2} + \frac{K_\text{m} x_3}{J L_\text{m} C^2 L} \end{pmatrix}$$

This shows that $\text{rank}(\mathcal{D}, [g, \text{ad}_f g]) = \text{rank}(\mathcal{D}, [g, \text{ad}_f^2 g]) = \text{rank}(\mathcal{D}, [\text{ad}_f g, \text{ad}_f^2 g]) = 4$ but $\text{rank}(\mathcal{D}) = 3$. Therefore it follows that $\mathcal{D}$ is not involutive and the system (6.1)-(6.4) is not feedback linearizable.

For further discussion, note that the boost converter / DC motor system (6.1)-(6.4) has no stationary state at the origin and in the derivations below we do not transform the system equations. That means, that we consider equilibrium points $f(\bar{x}) = 0, \bar{x} \neq 0$ with $y = h(\bar{x}) = 0$ and $T(\bar{x}) = \bar{z}$.

Since the boost converter / DC motor system is not feedback linearizable, we now investigate the zero dynamics of the system with respect to the angular velocity $x_4 = \omega$ which is the output y. Calculating the derivatives of $y = x_4 = z_1$, the input u appears at the third derivative, such that system (6.1)-(6.4) has relative degree $r = 3 < n = 4$ for output $y = x_4$:

$$\dot{x}_4 = \frac{K_\mathrm{m} x_3}{J} - \frac{B x_4}{J} - \frac{\tau_\mathrm{l}}{J} = L_f x_4 \tag{B.8}$$

$$\ddot{x}_4 = \frac{K_\mathrm{m} x_2 - K_\mathrm{m} R_\mathrm{m} x_3 - K_\mathrm{m} K_\mathrm{e} x_4}{J L_\mathrm{m}} - \frac{B K_\mathrm{m} x_3}{J^2} + \frac{B^2 x_4}{J^2} + \frac{B \tau_\mathrm{l}}{J^2} = L_f^2 x_4 \tag{B.9}$$

$$\begin{aligned} x_4^{(3)} = & -\frac{K_\mathrm{m} G x_2}{L_\mathrm{m} J C} - \frac{K_\mathrm{m} x_3}{L_\mathrm{m} J C} - \frac{x_2 R_\mathrm{m} K_\mathrm{m}}{L_\mathrm{m}^2 J} - \frac{x_2 K_\mathrm{m} B}{L_\mathrm{m} J^2} + \frac{R_\mathrm{m}^2 x_3 K_\mathrm{m}}{L_\mathrm{m}^2 J} + \frac{R_\mathrm{m} x_3 K_\mathrm{m} B}{L_\mathrm{m} J^2} \\ & + \frac{K_\mathrm{e} x_4 R_\mathrm{m} K_\mathrm{m}}{L_\mathrm{m}^2 J} + \frac{2 K_\mathrm{e} x_4 K_\mathrm{m} B}{L_\mathrm{m} J^2} - \frac{K_\mathrm{m}^2 x_3 K_\mathrm{e}}{J^2 L_\mathrm{m}} + \frac{K_\mathrm{m} x_3 B^2}{J^3} - \frac{B^3 x_4}{J^3} \\ & + \frac{\tau_\mathrm{l} K_\mathrm{e} K_\mathrm{m}}{J^2 L_\mathrm{m}} - \frac{\tau_\mathrm{l} B^2}{J^3} + \frac{x_1 K_\mathrm{m}}{C L_\mathrm{m} J} u = L_f^3 x_4 + L_g L_f^2 x_4 u \end{aligned} \tag{B.10}$$

Because of relative degree $r = 3$, the internal dynamics is of first order. In order to calculate the internal dynamics, we have to find a solution for

$$L_g t_4(x) = \frac{\partial t_4(x)}{\partial x} \begin{pmatrix} -\frac{1}{L} x_2 \\ \frac{1}{C} x_1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{L} \frac{\partial t_4(x)}{\partial x_1} x_2 + \frac{1}{C} \frac{\partial t_4(x)}{\partial x_2} x_1 = 0$$

from (B.4), such that the transformation $z = T(x)$ in Theorem B.3 is (at least) a local diffeomorphism and the differential equation in z-coordinates looks like (B.5). From inspection or a short calculation using separation of variables for $t_4(x)$, it shows that one possible solution of this partial differential equation is $t_4(x) = \frac{1}{2}(L x_1^2 + C x_2^2)$. The transformation matrix

$$z = T(x) = \begin{pmatrix} x_4 \\ L_f x_4 \\ L_f^2 x_4 \\ \frac{1}{2}(L x_1^2 + C x_2^2) \end{pmatrix}.$$

is a local diffeomorphism because

$$\operatorname{rank}\left(\frac{\partial T(x)}{\partial x}\right) = \operatorname{rank} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_\mathrm{m}}{J} & \frac{B}{J} \\ 0 & \frac{K_\mathrm{m}}{L_\mathrm{m} J} & -\frac{R_\mathrm{m} K_\mathrm{m}}{L_\mathrm{m} J} - \frac{K_\mathrm{m} B}{J^2} & -\frac{K_\mathrm{e} K_\mathrm{m}}{L_\mathrm{m} J} + \frac{B^2}{J^2} \\ L x_1 & C x_2 & 0 & 0 \end{pmatrix} = 4$$

for $x_1 \neq 0$. This condition is always met since the states of the boost converter / DC motor are required to be positive for all time. Moreover, there exists no stationary state with $x_1 = i = 0$ because of the constant input voltage E. Then, the internal dynamics reads as

$$\dot{\eta} = \left(-R_L x_1^2 - G x_2^2 + x_1 E - x_2 x_3\right)_{x = T^{-1} z}. \tag{B.11}$$

In order to get the zero dynamics, remember that the output $y = x_4$. Therefore, set $x_4 \equiv 0$, and from relative degree $r = 3$ follows that $\dot{x}_4 \equiv \ddot{x}_4 \equiv x_4^{(3)} \equiv 0$. Then, from (B.8)-(B.10) with the control law $\bar{u}$ for the zero dynamics as in (B.6), we get

$$\begin{aligned} x_3 &= \frac{\tau_{\mathrm{l}}}{K_{\mathrm{m}}}, \\ x_2 &= \frac{R_{\mathrm{m}}\tau_{\mathrm{l}}}{K_{\mathrm{m}}}, \\ \bar{u} &= \frac{\tau_{\mathrm{l}}(GR_{\mathrm{m}}+1)}{x_1 K_{\mathrm{m}}}. \end{aligned}$$

If we substitute this into the equation of the internal dynamics (B.11) and solve the left hand side $\dot{\eta} = Lx_1\dot{x}_1 + Cx_2\dot{x}_2$ for $\dot{x}_1$ and replace $\dot{x}_2$ with its right hand side (6.2) to finally obtain

$$\dot{x}_1 = -\frac{R_L}{L}x_1 + \frac{E}{L} - \frac{GR_{\mathrm{m}}^2\tau_{\mathrm{l}}^2 - R_{\mathrm{m}}\tau_{\mathrm{l}}^2}{LK_{\mathrm{m}}^2}\frac{1}{x_1}.$$

The linearization about a stationary point $\bar{x}_1$ with $\Delta x_1 = x_1 - \bar{x}_1$ is

$$\Delta\dot{x}_1 = \frac{-R_L\bar{x}_1^2K_{\mathrm{m}}^2 + GR_{\mathrm{m}}^2\tau_{\mathrm{l}}^2 + R_{\mathrm{m}}\tau_{\mathrm{l}}^2}{K_{\mathrm{m}}^2L\bar{x}_1^2}\Delta x_1$$

which will be exponentially stable if $-R_L\bar{x}_1^2K_{\mathrm{m}}^2 + GR_{\mathrm{m}}^2\tau_{\mathrm{l}}^2 + R_{\mathrm{m}}\tau_{\mathrm{l}}^2 < 0$. Considering the identified parameters in Section 6.2, it shows that for $\bar{u} = 1$ the current would be around $\bar{x}_1 \approx 0.269\,\mathrm{A}$ and would lead to an unstable zero dynamics. The zero dynamics gets exponentially stable for $\bar{u} \approx 0.16$ or $\bar{x}_1 \approx 2.528\,\mathrm{A}$, so that the zero dynamics is unstable almost all over the whole range of $\bar{u} \in [0, 1]$. Finally, we can conclude that the nonlinear boost converter / DC motor system is not minimum-phase but *nonminimum-phase* such that the standard control approaches using input-output linearization for trajectory tracking control fail.

Bibliography

[1] R. W. Erickson and D. Maksimović, *Fundamentals of Power Electronics*, 2nd ed. London: Springer Science + Business Media, Inc., 2001.

[2] H. Sira Ramírez and R. Silva Ortigoza, *Control Design Techniques in Power Electronics Devices.* Berlin: Springer, 2006.

[3] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control.* London: Springer, 2000.

[4] A. Isidori, "Dissipation inequalities in nonlinear H_∞-control," in *Proc. of the IEEE Conference on Decision and Control (CDC)*, 1992, pp. 3265–3270.

[5] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear control systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1283–1293, 1992.

[6] A. Isidori, *New Results on Nonlinear H_∞ Control via Measurement Feedback*, ser. Annals of the International Society of Dynamic Games: Advances in Dynamic Games and Applications. Boston: Birkhäuser, 1994, vol. 1.

[7] T. Başar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Boston: Birkhäuser, 1995.

[8] A. Kugi and K. Schlacher, "Nonlinear H_∞ controller design for a dc-to-dc power converter," *IEEE Transactions on Control Systems Technology*, vol. 7, no. 2, pp. 230–237, 1999.

[9] K. Schlacher and A. Kugi, "Automatic control of mechatronic systems," *International Journal of Applied Mathematics and Computer Science*, vol. 11, no. 1, pp. 131–164, 2001.

[10] A. Kugi, "Regelung eines Čuk-Konverters," *at-Automatisierungstechnik*, vol. 48, no. 3, pp. 116–123, 2000.

[11] A. Kugi, *Non-linear control based on physical models*, ser. Lecture Notes in Control and Information Science, M. Thoma and M. Morari, Eds. London Limited: Springer-Verlag, 2001, vol. 260.

[12] R. Ortega, A. Loría, P. J. Nicklasson, and H. Sira Ramírez, *Passivity-Based Control of Euler-Lagrange-Systems.* New York: Springer, 1998.

[13] R. Leyva, A. C. Pastor, C. Alonso, I. Queinnec, S. Tarbouriech, and L. M. Salamero, "Passivity-based integral control of a boost converter for large-signal stability," *IEE Proceedings - Control Theory and Applications*, vol. 153, no. 2, pp. 139–146, 2006.

[14] J. Linares Flores, J. Reger, and H. Sira Ramírez, "A time-varying linear state feedback tracking controller for a boost-converter driven dc motor," in *Proc. of the 4th IFAC-Symposium on Mechatronic Systems*, Heidelberg, Germany, 2006.

[15] J. Linares Flores, J. Reger, and H. Sira Ramírez, "Load torque estimation and passivity-based control of a boost-converter/dc-motor combination," *IEEE Transaction on Control Systems Technology*, vol. 18, no. 6, pp. 1398–1405, 2010.

[16] F. Antritter and J. Tautz, "Tracking control of the angular velocity of a dc-motor via a boost-converter," in *Proc. NOLCOS*, Pretoria, South Africa, 2007.

[17] V. I. Utkin, *Sliding modes in control and optimization.* Berlin, New York: Springer-Verlag, 1992.

[18] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and Applications.* London, Bristol: Taylor & Francis Ltd, 1998.

[19] A. S. I. Zinober, Ed., *Deterministic Control of Uncertain Systems.* London: Peter Peregrinus Press, 1990.

[20] A. S. I. Zinober, Ed., *Variable Structure and Lyapunov Control.* London: Springer-Verlag, 1994.

[21] C. M. Dorling and A. S. I. Zinober, "Two approaches to hyperplane design in multivariable variable structure control systems," *International Journal of Control*, vol. 44, no. 1, pp. 65–82, 1986.

[22] C. M. Dorling and A. S. I. Zinober, "Robust hyperplane design in multivariable variable structure control systems," *International Journal of Control*, vol. 48, pp. 2043–2054, 1988.

[23] C. Edwards, E. Fossas Colet, and L. Fridman, Eds., *Advances in Variable Structure and Sliding Mode Control*, ser. Lecture Notes in Control and Information Science. Berlin: Springer-Verlag, 2006, vol. 334.

[24] Y. Orlov, L. Acho, and V. Solis, "Nonlinear H_∞-control of time-varying systems," in *Proc. of the IEEE Conference on Decision and Control (CDC)*, Phoenix, Arizona, 1999, pp. 3764–3769.

[25] L. Acho, Y. Orlov, and L. Aguilar, "Global H_∞ control design for tracking control of robot manipulators," in *Proc. of the American Control Conference (ACC)*, Anchorage, AK, 2002, pp. 3986–3990.

[26] D. Angeli, E. D. Sontag, and Y. Wang, "A characterization of integral input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1082–1097, 2000.

[27] E. D. Sontag, "Comments on integral variants of ISS," *Systems & Control Letters*, vol. 34, no. 1-2, pp. 93–100, 1998.

[28] F. Mazenc, M. Malisoff, and M. S. de Queiroz, "Further results on strict Lyapunov functions for rapidly time-varying nonlinear systems," *Automatica*, vol. 42, pp. 1663–1671, 2006.

[29] B. Schmidt and J. Reger, "H_∞ suboptimal tracking control with integral action and load estimation applied on a boost-converter/dc-motor combination," in *Proc. of the IEEE International Conference on Control Applications*, St. Petersburg, Russia, 2009, pp. 147–152.

[30] B. Schmidt and J. Reger, "H_∞-suboptimal tracking control for bilinear power systems with dynamic feedback - theory and experiment," in *Proc.*

of the IEEE International Conference on Control Applications, Yokohama, Japan, 2010, pp. 578–584.

[31] N. Mohan, T. M. Undeland, and W. P. Robbins, *Power Electronics: Converters, Applications, and Design*, 3rd ed. John Wiley & Sons, Inc., 2003.

[32] H. Sira Ramírez, "A geometric approach to pulse-width modulated control in nonlinear dynamical systems," *IEEE Transactions on Automatic Control*, vol. 34, no. 2, pp. 184–187, 1989.

[33] P. T. Krein, J. Bentsman, R. M. Bass, and B. L. Lesieutre, "On the use of averaging for the analysis of power electronic systems," *IEEE Transactions on Power Electronics*, vol. 5, no. 2, pp. 182–190, 1990.

[34] B. Lehman and R. M. Bass, "Extensions of averaging theory for power electronic systems," *IEEE Transactions on Power Electronics*, vol. 11, no. 4, pp. 542–553, 1996.

[35] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. New Jersey: Prentice-Hall, Inc., 2001.

[36] H. M. Power and B. Porter, "Necessary and sufficient conditions for controllability of multivariable systems incorporating integral feedback," *Electronic Letters*, vol. 6, no. 25, pp. 816–817, 1970.

[37] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Transactions on Automatic Control*, vol. 41, no. 7, pp. 930–942, 1996.

[38] A. Piazzi and A. Visioli, "Optimal noncausal set-point regulation of scalar systems," *Automatica*, vol. 37, no. 1, pp. 121–127, 2001.

[39] J. Reger, "Planning a finite time transition from a non-stationary to a stationary point without overshoot," in *Proc. of the American Control Conference (ACC)*, Seattle, Washington, USA, 2008, pp. 5127–5129.

[40] K. Graichen and M. Zeitz, "Feedforward control design for finite-time transition problems of nonlinear systems with input and output constraints," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1273–1278, 2008.

[41] K. Zhou, K. Glover, and J. Doyle, *Robust and Optimal Control.* Upper Saddle River, N.J.: Prentice Hall, 1995.

[42] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control - Analysis and Design*, 2nd ed. Chichester West Sussex, England: John Wiley & Sons, Ltd, 2005.

[43] V. Hagenmeyer and E. Delaleau, "Continuous-time non-linear flatness-based predictive control: an exact feedforward linearisation setting with an induction drive example," *International Journal of Control*, vol. 81, pp. 1645–1663, 2008.

[44] H. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey: Prentice Hall, 2002.

[45] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method.* New York Berlin Heidelberg: Springer-Verlag, 1977.

[46] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory.* Berlin Heidelberg: Springer-Verlag, 2005.

[47] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory I.* Berlin Heidelberg: Springer-Verlag, 2005.

[48] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Philadelphia: Society for Industrial and Applied Mathematics, 2002.

[49] W. Hahn, *Stability of Motion.* Berlin Heidelberg: Springer-Verlag, 1967.

[50] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.

[51] E. D. Sontag, *Nonlinear and Optimal Control Theory.* Berlin: Springer-Verlag, 2007, ch. Input to state stability: Basic concepts and results, pp. 163–220.

[52] H. A. Edwards, Y. Lin, and Y. Wang, "On input-to-state stability for time varying nonlinear systems," in *Proc. of the IEEE Conference on Decision and Control (CDC)*, Sydney, Australia, 2000, pp. 3501–3506.

[53] M. Malisoff and F. Mazenc, "Further remarks on strict input-to-state stable Lyapunov functions for time-varying systems," *Automatica*, vol. 41, pp. 1973–1978, 2005.

[54] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*, 2nd ed. London: Springer, 2007.

[55] A. van der Schaft, *Nonlinear State Space H_∞ Control Theory*, ser. Progress in Systems and Control Theory, Essays on Control: Perspectives in the Theory and its Applications. Boston: Birkhäuser, 1993, vol. 14.

[56] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, pp. 831–847, 1989.

[57] B. A. Francis, *A course in H_∞ Control Theory*, ser. Lecture Notes in Control and Information Science. Berlin, Heidelberg: Springer-Verlag, 1987, vol. 88.

[58] A. Friedman, *Differential Games.* Mineola, New York: Dover Publications, 2006.

[59] J. Lewin, *Differential Games: Theory and Methods for Solving Game Problems with Singular Surfaces.* London Limited: Springer-Verlag, 1994.

[60] J. A. Ball and J. W. Helton, "Viscosity solutions of Hamilton-Jacobi equations arising in nonlinear H_∞-control," *Journal of Mathematical Systems, Estimation, and Control*, vol. 6, no. 1, pp. 1–22, 1996.

[61] J. Arthur E. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation, and Control.* Washington, DC: Hemisphere Publishing Corporation, 1975.

[62] M. G. Crandall and P. L. Lions, "Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordre," *Comptes Rendus de l'Académie des Sciences, Série I, Mathématique*, vol. 292, pp. 183–186, 1981.

[63] P. L. Lions, *Generalized solutions of Hamilton-Jacobi equations.* Boston: Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program, 1982.

[64] L. C. Evans and P. E. Souganidis, "Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations," *Indiana University Mathematics Journal*, vol. 33, pp. 773–797, 1984.

[65] T. Lee and Z. Jiang, "A generalization of Krasovskii-LaSalle theorem for nonlinear time-varying systems: Converse results and applications," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1147–1163, 2005.

[66] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: John Wiley & Sons, Inc., 1972.

[67] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Englewood Cliffs, New Jersey: Prentice-Hall International, Inc., 1989.

[68] B. Porter, "Optimal control of multivariable linear systems incorporating integral feedback," *Electronic Letters*, vol. 7, no. 8, pp. 170–172, 1971.

[69] J. Chiasson, *Modeling and High-Performance Control of Electric Machines*. Hoboken, New Jersey: John Wiley & Sons, 2005.

[70] A. Isidori, *Nonlinear Control Systems: An Introduction*, 3rd ed. London: Springer, 1995.

[71] S. Sastry, *Nonlinear Systems: Analysis, Stability and Control*. New York, Inc.: Springer-Verlag, 1999.

Theses

1. State-averaged models of power converters in continuous conduction mode result in bilinear system equations.
2. Frequently, power converter systems are not feedback linearizable with respect to the desired control output and therefore are an interesting application for various control strategies.
3. In order to allow for smooth reference trajectory tracking for a specified output, a reference system is introduced.
4. The error dynamics for reference tracking of bilinear power converter systems is time-varying.
5. In order to cope with disturbances, noise and parameter uncertainty, integral feedback and disturbance feedback strategies are considered.
6. The control strategies lead to nonlinear input-affine systems with a specific structure resulting from the original bilinear system equations.
7. In order to keep the control effort and the output to be controlled small with respect to occurring disturbances, a nonlinear $\mathcal{H}_\infty$ approach is chosen to find a state feedback law.
8. Integral Input-to-State Stability (iISS) shows to be a good choice to guarantee closed-loop stablity in the presence of nonzero disturbances for the considered systems.
9. For bilinear power converter systems with trajectory tracking, integral feedback / disturbance feedback, conditions for the solvability of the suboptimal nonlinear state feedback $\mathcal{H}_\infty$ control problem are derived.
10. With these conditions, iISS directly follows for the closed-loop system.
11. The obtained results can be generalized to the most general class of bilinear systems and also extended to the multi-input case.
12. For integral feedback, the reference tracking problem can be addressed via the solution of a two-point boundary value problem.
13. In the case of disturbance feedback, the reference trajectory has to be replanned online when a disturbance occurs.
14. For the boost converter / DC motor system, boundedness of a specific replanning strategy is proven.
15. Experimental results support the applicability of the proposed control design method.