Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **11/08**

A perturbation approach to differential operators with indefinite weights

Jussi Behrndt, Friedrich Philipp, Carsten Trunk

2011

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

A PERTURBATION APPROACH TO DIFFERENTIAL OPERATORS WITH INDEFINITE WEIGHTS

JUSSI BEHRNDT, FRIEDRICH PHILIPP, AND CARSTEN TRUNK

Abstract. Ordinary and partial differential operators with indefinite weight functions can be regarded as perturbations of non-negative operators in Krein spaces. Under the additional assumption that the points 0 and ∞ are regular critical points of the unperturbed operator it is shown that a bounded additive perturbation leads to an operator which admits a decomposition into a direct sum of a bounded operator and an unbounded selfadjoint operator in a Hilbert space. In particular, this leads to estimates for the non-real spectrum in terms of the operator norm of the perturbation term and the resolvent of the unperturbed non-negative operator. The general results are illustrated for Sturm-Liouville operators and second order elliptic partial differential operators on unbounded domains.

1. Introduction

We consider ordinary and partial differential operators associated with

$$\mathcal{L} = \frac{1}{r}\,\ell, \tag{1.1}$$

where $r \neq 0$ is a real-valued, locally integrable weight function which changes its sign and ℓ is a second order differential expression of the form

$$\ell = -\frac{d}{dx}p\frac{d}{dx} + q \qquad \text{or} \qquad \ell = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\, a_{jk}\, \frac{\partial}{\partial x_k} + a \tag{1.2}$$

acting on an unbounded real interval or domain $\Omega \subset \mathbb{R}^n$, respectively. In the first case p^{-1} and q are assumed to be real-valued and (locally) integrable over the interval Ω. In the second case $r, r^{-1} \in L^\infty(\Omega)$ and ℓ is assumed to be formally symmetric and uniformly elliptic on $\Omega \subset \mathbb{R}^n$ with C^∞-coefficients a_{jk} and $a \in L^\infty(\Omega)$ is real-valued. Together with appropriate boundary conditions (if necessary) the differential expression ℓ gives rise to a selfadjoint operator T in a weighted L^2-space. Multiplication with $1/r$ lead to the corresponding indefinite differential operator A associated with $\mathcal{L}$ in (1.1).

Most of the existing literature for differential operators with indefinite weights focuses on regular or left-definite problems. The spectral properties of the operators associated to $\mathcal{L}$ in the case of a regular Sturm-Liouville expression ℓ are investigated in a great extend. We refer only to [17], the monograph [40] and the detailed references therein. The spectral properties of indefinite elliptic partial differential operators in the case of a bounded domain have been investigated in, e.g., [20, 21, 22, 23, 35, 36, 37]. We mention that in the case of a bounded interval or domain Ω the spectrum of the indefinite differential operator A associated with $\mathcal{L}$ in (1.1) consists only of eigenvalues with at most finitely many non-real eigenvalues.

2000 *Mathematics Subject Classification.* Primary 47B50, 47A55; Secondary 47A10, 34B24.

Key words and phrases. Krein space, definitizable operator, critical point.

The authors gratefully acknowledge the support of the Deutsche Forschungsgemeinschaft (DFG) under the grants BE 3765/5-1 and TR 903/4-1.

Also the case of singular left-definite Sturm-Liouville problems is well studied. Here the selfadjoint operator T associated with ℓ is uniformly positive and, hence, the corresponding indefinite differential operator A associated with $\mathcal{L}$ in (1.1) has real spectrum with a gap around zero, cf. e.g., [10, 11, 12, 33, 32] and the monograph [40] for further references. In the case $T \geq 0$ it is of particular interest whether the operator A is similar to a selfadjoint operator. A set of necessary and sufficient similarity criteria can be found in [27, 28, 29].

The slightly more general situation where the indefinite (Sturm-Liouville or elliptic) differential operator A has either a finite number of negative squares or is quasi-uniformly positive is discussed in, e.g., [7, 9, 15, 16]. The general non-left-definite situation is much more difficult to treat, especially the situation where the essential spectrum of the selfadjoint operator T associated with ℓ is no longer contained in $\mathbb{R}^+$. In this case subtle problems appear, as, e.g., accumulation of non-real eigenvalues to the real axis, see [4, 8, 6, 30].

In the present paper a new general perturbation approach is provided which is applicable for singular non-left-definite ordinary and partial differential operators, and which moreover leads to quantitative estimates for the non-real spectrum. Although the analysis carried out relies on sophisticated Krein space methods for so-called locally definitizable operators, the basic idea is simple: If the selfadjoint operator T associated with ℓ is not uniformly positive but semibounded from below, then the operator $T + \gamma$ becomes non-negative or uniformly positive for a suitably large $\gamma > 0$. Hence the spectrum of $A_0 := A + \gamma/r = 1/r(T + \gamma)$ is real. In general, a bounded perturbation of A_0 may lead to unbounded non-real spectrum, but under the additional assumption that 0 and ∞ are no so-called singular critical points the influence of the perturbation on the spectrum can be controlled. In fact, our main result Theorem 3.1 provides a bound for the non-real spectrum of A in terms of the resolvent of A_0 and the norm of the bounded perturbation.

Theorem 3.1 is applied to ordinary and partial differential operators with indefinite weights in Section 4. We first investigate a singular indefinite Sturm-Liouville operator with an essentially bounded, real-valued potential and the particularly simple weight function $r(x) = \operatorname{sgn}(x)$. Our second example is a second order uniformly elliptic operator defined on an unbounded domain $\Omega \subset \mathbb{R}^n$ with bounded coefficients and an essentially bounded weight function r having an essentially bounded inverse. To the best of our knowledge the estimates obtained here for the non-real spectrum of singular indefinite differential operators are the first ones in the mathematical literature.

2. Locally definitizable operators

Throughout this section let $(\mathcal{H}, [\cdot\,,\cdot])$ be a Krein space. For a detailed treatment of Krein spaces and operators therein we refer to the monographs [1] and [13]. In the following all topological notions are understood with respect to some fixed Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot\,,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent, see e.g. [1, Ch 1, Theorem 7.19]. The Hilbert space scalar product induced by $\|\cdot\|$ will be denoted by $(\cdot\,,\cdot)$. By the Lax-Milgram theorem there exists a bounded linear operator G in $\mathcal{H}$ such that

$$[f, g] = (Gf, g) \quad \text{for all } f, g \in \mathcal{H}.$$

This operator is boundedly invertible and selfadjoint in $(\mathcal{H}, (\cdot\,,\cdot))$.

Let A be a densely defined linear operator in $\mathcal{H}$. The adjoint of A in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ is defined by

$$A^+ := G^{-1}A^*G,$$

where A^* denotes the adjoint of A in the Hilbert space $(\mathcal{H},(\cdot\,,\cdot))$. We have

$$[Af,g]=[f,A^+g]\quad \text{for all } f\in\operatorname{dom}A,\ g\in\operatorname{dom}A^+.$$

The operator A is called *selfadjoint* (in the Krein space $(\mathcal{H},[\cdot\,,\cdot])$) if $A=A^+$. A selfadjoint operator A in $(\mathcal{H},[\cdot\,,\cdot])$ is called *non-negative* if $\rho(A)\neq\varnothing$ and $[Af,f]\geq 0$ for all $f\in\operatorname{dom}A$. In the sequel, if not otherwise stated, properties like "selfadjoint" or "non-negative" of an operator always refer to the corresponding property of the operator with respect to the Krein space inner product $[\cdot\,,\cdot]$.

Let T be a closed and densely defined linear operator in $\mathcal{H}$. Recall that the *approximate point spectrum* of T consists of those $\lambda\in\mathbb{C}$ for which there exists a sequence $(f_n)\subset\operatorname{dom}T$ with $\|f_n\|=1$ for $n\in\mathbb{N}$ and $(T-\lambda)f_n\to 0$ as $n\to\infty$. Evidently, $\sigma_{ap}(T)$ is a subset of $\sigma(T)$. If T is selfadjoint, it is not difficult to see that $\sigma(T)\cap\mathbb{R}\subset\sigma_{ap}(T)$. The *extended spectrum* $\widetilde{\sigma}(T)$ of T is defined by $\widetilde{\sigma}(T):=\sigma(T)$ if T is bounded and $\widetilde{\sigma}(T):=\sigma(T)\cup\{\infty\}$ if T is unbounded.

Let us recall the notions of spectral points of positive and negative type of a selfadjoint operator. The following definition was given in [34] for bounded selfadjoint operators.

Definition 2.1. Let A be a selfadjoint operator in the Krein space $(\mathcal{H},[\cdot\,,\cdot])$. A point $\lambda\in\sigma_{ap}(A)$ is called a *spectral point of positive* (*negative*) *type* of A if for every sequence $(f_n)\subset\operatorname{dom}A$ with $\|f_n\|=1$ and $(A-\lambda)f_n\to 0$ as $n\to\infty$ we have

$$\liminf_{n\to\infty}\,[f_n,f_n]>0\quad\Big(\limsup_{n\to\infty}\,[f_n,f_n]<0,\ \text{respectively}\Big).$$

The point ∞ is called a spectral point of positive (negative) type if A is unbounded and if for every sequence $(f_n)\subset\operatorname{dom}A$ with $\|Af_n\|=1$ and $f_n\to 0$ as $n\to\infty$ we have

$$\liminf_{n\to\infty}\,[Af_n,Af_n]>0\quad\Big(\limsup_{n\to\infty}\,[Af_n,Af_n]<0,\ \text{respectively}\Big).$$

The set of all spectral points of positive (negative) type of A will be denoted by $\sigma_+(A)$ ($\sigma_-(A)$, respectively). A set $\Delta\subset\overline{\mathbb{C}}$ is said to be of positive (negative) type with respect to A if

$$\Delta\cap\widetilde{\sigma}(A)\subset\sigma_+(A)\quad\Big(\Delta\cap\widetilde{\sigma}(A)\subset\sigma_-(A),\ \text{respectively}\Big).$$

The set Δ is said to be of *definite type* with respect to A if it is either of positive or of negative type with respect to A.

It is easily seen that the sets $\sigma_+(A)$ and $\sigma_-(A)$ are contained in $\mathbb{R}$. Moreover, they are open in $\sigma_{ap}(A)$, see [2]. This implies in particular that the non-real spectrum of A cannot accumulate to $\sigma_+(A)\cup\sigma_-(A)$. In [34] it was proved that a bounded selfadjoint operator A possesses a local spectral function E on intervals which are of positive type with respect to A. The spectral subspaces defined by E are then Hilbert spaces with respect to the inner product $[\cdot\,,\cdot]$. A similar statement holds for intervals which are of negative type. An extension of these statements to unbounded operators can be found in in [26]. Due to these properties, the spectrum of positive and negative type is of particular interest in the analysis of selfadjoint operators in Krein spaces.

In the next definition we recall the notion of locally definitizable operators, see e.g. [25, Definition 2.3], see also [26]. As usual we denote the open half planes by $\mathbb{C}^\pm:=\{z\in\mathbb{C}:\pm\operatorname{Im}z>0\}$ and the one-point compactifications of $\mathbb{C}$ and $\mathbb{R}$ by $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$, respectively. Moreover, for a set $\Delta\subset\overline{\mathbb{C}}$ we define $\Delta^*:=\{\overline{z}:z\in\Delta\}$, where $\overline{\infty}:=\infty$.

Definition 2.2. Let $\Omega = \Omega^*$ be a domain in $\overline{\mathbb{C}}$ with $\Omega \cap \overline{\mathbb{R}} \neq \varnothing$, such that $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ are simply connected. A selfadjoint operator A in $\mathcal{K}$ is called *definitizable over* Ω, if the following conditions are satisfied.

(i) The set $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ does not have any accumulation point in Ω and consists of poles of the resolvent of A.

(ii) For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighborhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$, such that

$$\|(A-\lambda)^{-1}\| \leq M\,\frac{(1+|\lambda|)^{2m-2}}{|\mathrm{Im}\,\lambda|^m} \tag{2.1}$$

holds for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

(iii) For every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ there exist an open neighborhood I_λ in $\overline{\mathbb{R}}$ of λ, such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type with respect to A.

The points in $\widetilde{\sigma}(A) \cap \Omega \cap \overline{\mathbb{R}}$ which are not spectral points of definite type of A are called the *critical points* of A (in Ω).

We emphasize that a selfadjoint operator A is definitizable over $\overline{\mathbb{C}}$ if and only if it is definitizable, cf. [26], i.e. $\rho(A) \neq \varnothing$ and there exists a polynomial $p \neq 0$ with real coefficients such that $p(A)$ is non-negative. Moreover, a selfadjoint operator A which is definitizable over a domain Ω (as in Definition 2.2) has a local spectral function E on $\Omega \cap \overline{\mathbb{R}}$, see [26]. The projection $E(\Delta)$ is defined for all Borel subsets Δ of $\Omega \cap \overline{\mathbb{R}}$ with $\overline{\Delta} \subset \Omega$ the boundary points of which are not critical points of A. We denote this system of sets by $\mathfrak{B}(A;\Omega)$. A critical point λ of A in $\Omega \cap \overline{\mathbb{R}}$ is called *regular* if there exists $\delta > 0$ such that

$$\sup\{\|E(\Delta)\| : \Delta \in \mathfrak{B}(A;\Omega),\ \Delta \subset (\lambda-\delta, \lambda+\delta)\} \;<\; \infty.$$

If the critical point λ of A in $\Omega \cap \overline{\mathbb{R}}$ is not regular, it is called *singular*. The following proposition is a generalization of a result of Ćurgus. For $r > 0$ and $\lambda \in \mathbb{C}$ we set $B_r(\lambda) := \{z \in \mathbb{C} : |z-\lambda| < r\}$.

Proposition 2.3. *Let the selfadjoint operator A in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ be definitizable over a domain $\Omega \subset \overline{\mathbb{C}}$ with $\infty \in \Omega$. Then ∞ is not a singular critical point of A if and only if there exists a bounded and boundedly invertible non-negative operator W in $\mathcal{H}$ such that $W\,\mathrm{dom}\,A \subset \mathrm{dom}\,A$.*

Proof. It follows directly from Definition 2.2 that there exists $r > 0$ such that the following statements hold:

(a) A is definitizable over $\overline{\mathbb{C}} \setminus \overline{B_r(0)}$.

(b) $\sigma(A) \setminus \mathbb{R} \,\subset\, \overline{B_r(0)}$.

(c) (r, ∞) is of definite type with respect to A.

(d) $(-\infty, -r)$ is of definite type with respect to A.

Choose some $r_1 > r$, set $\Delta := \overline{\mathbb{R}} \setminus (-r_1, r_1)$ and denote by E the local spectral function of A on $\overline{\mathbb{R}} \setminus [-r, r]$. Define the operators

$$B := A{\upharpoonright}E(\Delta)\mathcal{H} \quad \text{and} \quad C := A{\upharpoonright}(I - E(\Delta))\mathcal{H}.$$

Then the space $\mathcal{H}$ admits the decomposition $\mathcal{H} = E(\Delta)\mathcal{H}\,[\dotplus]\,(I - E(\Delta))\mathcal{H}$ and with respect to this decomposition the operator A is decomposed as $A = B\,[\dotplus]\,C$. Here, $[\dotplus]$ denotes the direct $[\cdot\,,\cdot]$-orthogonal sum. As the operator B is definitizable over $\overline{\mathbb{C}}$, it is definitizable. Evidently, the point ∞ is a critical point of A if and only if it is a critical point of B. Moreover, it is easily seen that in this case ∞ is a regular critical point of A if and only if it is a regular critical point of B.

Assume that ∞ is not a singular critical point of A. Then the same is true for the definitizable operator B. Hence, by [14, Theorem 3.2] there exists a bounded and boundedly invertible non-negative operator V in $E(\Delta)\mathcal{H}$ such that $V \operatorname{dom} B \subset \operatorname{dom} B$. With an arbitrary fundamental symmetry J_b in $(I - E(\Delta))\mathcal{H}$ the operator $W := V\,[\dotplus]\,J_b$ is a bounded and boundedly invertible non-negative operator in $\mathcal{H}$. Moreover, since $\operatorname{dom} A = \operatorname{dom} B\,[\dotplus]\,(I - E(\Delta))\mathcal{H}$, we have $W \operatorname{dom} A \subset \operatorname{dom} A$. Conversely, assume that such an operator W exists. Then there exists some $\delta > 0$ such that $[Wf, f] \geq \delta\|f\|^2$ holds for all $f \in \mathcal{H}$. Define the bounded operator V in $E(\Delta)\mathcal{H}$ by

$$V := E(\Delta)(W \restriction E(\Delta)\mathcal{H}).$$

Then for $f \in E(\Delta)\mathcal{H}$ we have

$$[Vf, f] = [E(\Delta)Wf, f] = [Wf, E(\Delta)f] = [Wf, f] \geq \delta\|f\|^2,$$

which shows that V is non-negative and boundedly invertible. If $f \in \operatorname{dom} B$, then also $f \in \operatorname{dom} A$ and thus $Wf \in \operatorname{dom} A$ which implies

$$Vf = E(\Delta)Wf \in E(\Delta) \operatorname{dom} A = \operatorname{dom} B.$$

This and [14, Theorem 3.2] imply that ∞ is not a singular critical point of B. $\square$

3. The main result

Let $(\mathcal{H}, [\cdot\,,\cdot])$ be a Krein space. Recall that for any Hilbert space scalar product $(\cdot\,,\cdot)$ on $\mathcal{H}$ such that $[\cdot\,,\cdot]$ is continuous with respect to $\|\cdot\| := (\cdot\,,\cdot)^{1/2}$ there exists a bounded and selfadjoint operator G in $(\mathcal{H}, (\cdot\,,\cdot))$ such that

$$[f, g] = (Gf, g) \quad \text{for all } f, g \in \mathcal{H}.$$

The operator G is called the *Gram operator* of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$. The following theorem is the main result of this paper.

Theorem 3.1. *Let A_0 be a non-negative selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ and let $(\cdot\,,\cdot)$, $\|\cdot\|$ and G be as above. Furthermore, assume that 0 and ∞ are not singular critical points of A_0 and $0 \notin \sigma_p(A_0)$. Set*

$$\tau_0 := \frac{1}{\pi} \limsup_{N\to\infty} \left\| \int_{\frac{1}{N}}^{N} \left((A_0 + it)^{-1} + (A_0 - it)^{-1} \right) dt \right\| < \infty\,.$$

Then for every bounded selfadjoint operator V in $(\mathcal{H}, [\cdot\,,\cdot])$ and

$$\delta := \frac{3}{2}\,(1 + \tau_0) \max\{1, \|V\|\}$$

the following statements hold:

(i) *The operator $A_0 + V$ is definitizable over $\overline{\mathbb{C}} \setminus \overline{B_\delta(0)}$.*
(ii) *∞ is not a singular critical point of $A_0 + V$.*
(iii) *(δ, ∞) is of positive type with respect to $A_0 + V$.*
(iv) *$(-\infty, -\delta)$ is of negative type with respect to $A_0 + V$.*
(v) *$\sigma(A_0 + V) \setminus \mathbb{R} \subset \overline{B_\delta(0)}$.*

If $GA_0 \subset A_0G$ or $A_0G \subset GA_0$, then $\tau_0 = 1$ and hence $\delta = 3\max\{1, \|V\|\}$.

Proof. The proof is divided into three steps. In the first step we prove that indeed $\tau_0 < \infty$. In the second step it is shown that (i)–(v) hold with some $r > 0$ instead of δ. The proof is completed by showing $\delta \geq r$ in step 3.

1. By E denote the spectral function of the non-negative operator A_0. By assumption, 0 and ∞ are not singular critical points of A_0. Therefore, the spectral projections $E(\mathbb{R}^+)$ and $E(\mathbb{R}^-)$ exist, where $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^- := (-\infty, 0)$. Let us define the operator

$$\widetilde{J} := E(\mathbb{R}^+) - E(\mathbb{R}^-).$$

Since zero is not an eigenvalue of A_0, the space $\mathcal{H}$ decomposes as

$$\mathcal{H} = E(\mathbb{R}^+)\mathcal{H}\,[\dot{+}]\,E(\mathbb{R}^-)\mathcal{H}, \tag{3.1}$$

and with respect to this decomposition of $\mathcal{H}$ the operators A_0 and $\widetilde{J}$ admit the following matrix representations:

$$A_0 = \begin{pmatrix} A_0^+ & 0 \\ 0 & A_0^- \end{pmatrix} \quad \text{and} \quad \widetilde{J} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where $A_0^\pm := A_0 \upharpoonright E(\mathbb{R}^\pm)\mathcal{H}$. Note that the operator $A_0^\pm$ is selfadjoint in the Hilbert space $(E(\mathbb{R}^\pm)\mathcal{H}, \pm[\cdot\,,\cdot])$ and that $\sigma(A_0^\pm) \subset \mathbb{R}_0^\pm := \mathbb{R}^\pm \cup \{0\}$.

In the following we shall prove that

$$\frac{1}{\pi}\, s\!-\!\lim_{N\to\infty} \int_{\frac{1}{N}}^{N} \big((A_0^+ + it)^{-1} + (A_0^+ - it)^{-1}\big)\, dt = I. \tag{3.2}$$

To this end we first of all observe that

$$(A_0^+ + it)^{-1} + (A_0^+ - it)^{-1} = 2A_0^+((A_0^+)^2 + t^2)^{-1}.$$

Clearly, the spectral measure of A_0^+ is given by $E_+(\Delta)f = E(\Delta \cap \mathbb{R}^+)f$, $f \in E(\mathbb{R}^+)\mathcal{H}$ and $\Delta \subset \mathbb{R}$ Borel-measurable. Therefore,

$$\begin{aligned} 2\int_{\frac{1}{N}}^{N} A_0^+((A_0^+)^2 + t^2)^{-1}\, dt &= 2\int_{\frac{1}{N}}^{N} \int_{\mathbb{R}^+} \frac{s}{s^2+t^2}\, dE(s)\, dt \\ &= 2\int_{\mathbb{R}^+} \int_{\frac{1}{N}}^{N} \frac{s}{s^2+t^2}\, dt\, dE(s) \\ &= 2\int_{\mathbb{R}^+} \big(\arctan(N/s) - \arctan(1/Ns)\big)\, dE(s) \\ &= 2\arctan(N(A_0^+)^{-1}) - 2\arctan(N^{-1}(A_0^+)^{-1}). \end{aligned}$$

Fubini's theorem can be applied here since the integrand is bounded on $\mathbb{R}^+ \times [1/N, N]$. On $\mathbb{R}^+$, the functions

$$x \mapsto 2\arctan(Nx) \quad \text{and} \quad x \mapsto 2\arctan(N^{-1}x)$$

tend pointwise to π and 0 as $N \to \infty$, respectively. Therefore, the operator sequence $2\arctan(N(A_0^+)^{-1})$ tends strongly to πI and $2\arctan(N^{-1}(A_0^+)^{-1})$ tends strongly to the zero operator as $N \to \infty$, cf. [38, Theorem VIII.5]. This proves (3.2). Similarly, one shows that

$$\frac{1}{\pi}\, s\!-\!\lim_{N\to\infty} \int_{\frac{1}{N}}^{N} \big((A_0^- + it)^{-1} + (A_0^- - it)^{-1}\big)\, dt = -I_{E(\mathbb{R}^-)\mathcal{H}}.$$

Hence, we have

$$\frac{1}{\pi}\, s\!-\!\lim_{N\to\infty} \int_{\frac{1}{N}}^{N} \big((A_0 + it)^{-1} + (A_0 - it)^{-1}\big)\, dt = \widetilde{J}.$$

The Banach-Steinhaus theorem now yields $\tau_0 < \infty$ and

$$\|\widetilde{J}\| \le \tau_0. \tag{3.3}$$

2. Let $A := A_0 + V$. With respect to the decomposition (3.1) of $\mathcal{H}$ we have

$$\operatorname{dom} A = \operatorname{dom} A_0 = \big(\operatorname{dom} A_0 \cap E(\mathbb{R}^+)\mathcal{K}\big)\,[\dot{+}]\,\big(\operatorname{dom} A_0 \cap E(\mathbb{R}^-)\mathcal{K}\big).$$

As $(E(\mathbb{R}^\pm)\mathcal{H}, \pm[\cdot\,,\cdot])$ are Hilbert spaces, the operator $\widetilde{J}$ is a fundamental symmetry of the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$, so that the norm $\|\cdot\|_\sim$, induced by the scalar product

$$(f,g)_\sim := [\widetilde{J}f, g] = (G\widetilde{J}f, g), \quad f, g \in \mathcal{H},$$

is equivalent to the fixed norm $\|\cdot\|$, cf. [1]. Therefore, the operator V is also bounded in the Hilbert space $(\mathcal{H},(\cdot\,,\cdot)_\sim)$ and thus admits a representation

$$V=\begin{pmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}$$

with bounded entries V_{ij}, $i,j=1,2$.

Let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geq 0$ and

$$|\lambda|\;>\;\tau_+:=\|V_{22}\|_\sim+2\max\{1,\|V_{21}\|_\sim\}.$$

From $\operatorname{dist}(\lambda,\sigma(A_0^-))\;>\;\tau_+$ and the selfadjointness of A_0^- in the Hilbert space $(E(\mathbb{R}^-)\mathcal{H},(\cdot\,,\cdot)_\sim)$ we conclude

$$\|(A_0^--\lambda)^{-1}\|_\sim=\frac{1}{\operatorname{dist}(\lambda,\sigma(A_0^-))}<\frac{1}{\tau_+}.$$

This, in particular, implies

$$\|V_{22}(A_0^--\lambda)^{-1}\|_\sim<\frac{\|V_{22}\|_\sim}{\tau_+}<1.$$

Therefore the operator $I+V_{22}(A_0^--\lambda)^{-1}$ is boundedly invertible in $E(\mathbb{R}^-)\mathcal{H}$, and it follows that

$$\left\|\big(I+V_{22}(A_0^--\lambda)^{-1}\big)^{-1}\right\|_\sim\;\leq\;\frac{1}{1-\|V_{22}(A_0^--\lambda)^{-1}\|_\sim}\;\leq\;\frac{1}{1-\frac{\|V_{22}\|_\sim}{\tau_+}}\,.$$

Hence, also the operator

$$A_0^-+V_{22}-\lambda=\big(I+V_{22}(A_0^--\lambda)^{-1}\big)\,(A_0^--\lambda)$$

is boundedly invertible in $E(\mathbb{R}^-)\mathcal{H}$. The definition of τ_+ yields

$$\big\|(A_0^-+V_{22}-\lambda)^{-1}\big\|_\sim\;\leq\;\frac{1}{\tau_+}\cdot\frac{1}{1-\frac{\|V_{22}\|_\sim}{\tau_+}}=\frac{1}{\tau_+-\|V_{22}\|_\sim}\;\leq\;\frac{1}{2}\tag{3.4}$$

and

$$\big\|(A_0^-+V_{22}-\lambda)^{-1}V_{21}\big\|_\sim\;\leq\;\frac{\|V_{21}\|_\sim}{\tau_+-\|V_{22}\|_\sim}\;\leq\;\frac{1}{2}\,.\tag{3.5}$$

Let $f\in\operatorname{dom}A$ be arbitrary. We set $g:=(A-\lambda)f=(A_0+V-\lambda)f$ as well as $f_\pm:=E(\mathbb{R}^\pm)f\in\operatorname{dom}A$, $g_\pm:=E(\mathbb{R}^\pm)g$. Then $g_-=V_{21}f_++(A_0^-+V_{22}-\lambda)f_-$, or, equivalently,

$$f_-=(A_0^-+V_{22}-\lambda)^{-1}(g_--V_{21}f_+).$$

By (3.4) and (3.5) and due to $\|v\|_\sim^2=\|E(\mathbb{R}^+)v\|_\sim^2+\|E(\mathbb{R}^-)v\|_\sim^2$ for $v\in\mathcal{H}$ this implies

$$\|f_-\|_\sim\;\leq\;\frac12\|g_-\|_\sim+\frac12\|f_+\|_\sim\;\leq\;\frac12\|(A-\lambda)f\|_\sim+\frac12\|f\|_\sim\,.$$

Squaring this gives

$$\|f_-\|_\sim^2\;\leq\;\frac14\|(A-\lambda)f\|_\sim^2+\frac12\|(A-\lambda)f\|_\sim\|f\|_\sim+\frac14\|f\|_\sim^2\,,$$

and thus

$$\begin{aligned}[f,f]&=\|f_+\|_\sim^2-\|f_-\|_\sim^2=\|f\|_\sim^2-2\|f_-\|_\sim^2\\ &\geq\frac12\|f\|_\sim^2-\frac12\|(A-\lambda)f\|_\sim^2-\|(A-\lambda)f\|_\sim\|f\|_\sim\,.\end{aligned}\tag{3.6}$$

Now, we set $\varepsilon:=\sqrt5-2$. It is then easily seen with the help of the above inequality that for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\geq0$ and $|\lambda|>\tau_+$ the following implication holds:

$$f\in\operatorname{dom}A,\;\|(A-\lambda)f\|_\sim\leq\varepsilon\|f\|_\sim\quad\Longrightarrow\quad[f,f]\geq\varepsilon\|f\|_\sim^2.$$

By definition of $\sigma_+(A)$ we obtain

$$\{\lambda\in\mathbb{C} : |\lambda| > \tau_+,\ \operatorname{Re}\lambda \geq 0\}\cap\sigma_{ap}(A)\ \subset\ \sigma_+(A).$$

Analogously, one shows that for $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\leq 0$,

$$|\lambda|\ >\ \tau_- := \|V_{11}\|_\sim + 2\max\{1, \|V_{12}\|_\sim\}$$

and $f\in\operatorname{dom}A$ we have

$$[f,f]\leq -\frac{1}{2}\|f\|_\sim^2 + \frac{1}{2}\|(A-\lambda)f\|_\sim^2 + \|(A-\lambda)f\|_\sim\|f\|_\sim\,, \tag{3.7}$$

and thus

$$\|(A-\lambda)f\|_\sim\leq\varepsilon\|f\|_\sim \quad\Longrightarrow\quad [f,f]\leq -\varepsilon\|f\|_\sim^2.$$

Consequently,

$$\{\lambda\in\mathbb{C} : |\lambda| > \tau_-,\ \operatorname{Re}\lambda \leq 0\}\cap\sigma_{ap}(A)\ \subset\ \sigma_-(A).$$

We set

$$r := \max\{\tau_+,\tau_-\}$$

and conclude from $\sigma_+(A_0)\cup\sigma_-(A_0)\ \subset\ \overline{\mathbb{R}}$ that the operator A has the following spectral properties

(iii') (r,∞) is of positive type with respect to A.
(iv') $(-\infty,-r)$ is of negative type with respect to A.
(v') $\sigma(A)\setminus\mathbb{R}\subset\overline{B_r(0)}$.

We will now show that A is definitizable over $\overline{\mathbb{C}}\setminus\overline{B_r(0)}$. To this end it remains to prove that the resolvent of A has finite order growth near $\overline{\mathbb{R}}\setminus[-r,r]$. Let $\lambda\in(\mathbb{C}\setminus\mathbb{R})\setminus\overline{B_r(0)}$ and $f\in\operatorname{dom}A$. Then (see (3.6) and (3.7))

$$\big|[f,f]\big|\ \geq\ \frac{1}{2}\|f\|_\sim^2 - \frac{1}{2}\|(A-\lambda)f\|_\sim^2 - \|(A-\lambda)f\|_\sim\|f\|_\sim\,.$$

On the other hand, we also have

$$\big|(\operatorname{Im}\lambda)[f,f]\big| = \big|\operatorname{Im}[\lambda f,f]\big| = \big|\operatorname{Im}[(\lambda-A)f,f]\big|\ \leq\ \|(A-\lambda)f\|_\sim\|f\|_\sim\,,$$

and it follows that

$$\|(A-\lambda)f\|_\sim^2 + 2\left(1+\frac{1}{|\operatorname{Im}\lambda|}\right)\|(A-\lambda)f\|_\sim\|f\|_\sim - \|f\|_\sim^2\ \geq\ 0.$$

We set $t := 1+\frac{1}{|\operatorname{Im}\lambda|}$ and observe that the quadratic equation

$$x^2 + 2t\|f\|_\sim x - \|f\|_\sim^2 = 0$$

has a negative root and the positive root $x_+ = c(\lambda)\|f\|_\sim$, where $c(\lambda) := \sqrt{1+t^2}-t$. Hence, $\|(A-\lambda)f\|_\sim\geq c(\lambda)\|f\|_\sim$ holds for all $f\in\operatorname{dom}A$. Thus,

$$\|(A-\lambda)^{-1}\|_\sim\ \leq\ \frac{1}{c(\lambda)}.$$

And since

$$\begin{aligned}\frac{1}{c(\lambda)} &= \frac{1}{\sqrt{1+t^2}-t} = \frac{|\operatorname{Im}\lambda|(\sqrt{1+t^2}+t)}{|\operatorname{Im}\lambda|}\\ &= \frac{\sqrt{|\operatorname{Im}\lambda|^2+(1+|\operatorname{Im}\lambda|)^2}+|\operatorname{Im}\lambda|+1}{|\operatorname{Im}\lambda|}\\ &\leq \frac{|\operatorname{Im}\lambda|+(1+|\operatorname{Im}\lambda|)+|\operatorname{Im}\lambda|+1}{|\operatorname{Im}\lambda|}\\ &\leq 3\,\frac{1+|\lambda|}{|\operatorname{Im}\lambda|}\ \leq\ 3\,\frac{(1+|\lambda|)^2}{|\operatorname{Im}\lambda|^2},\end{aligned}$$

we have proved that (2.1) is satisfied with $M = 3$ and $m = 2$. Since $\operatorname{dom} A = \operatorname{dom} A_0$, it follows from Proposition 2.3 that ∞ is not a singular critical point of A.

3. In this step we show $\delta \geq r$. Set $\mathcal{H}_1 := E(\mathbb{R}^+)\mathcal{H}$ and $\mathcal{H}_2 := E(\mathbb{R}^-)\mathcal{H}$. By $T^{\tilde{*}}$ we denote the adjoint of $T \in L(\mathcal{H}_i, \mathcal{H}_j)$, $i, j \in \{1, 2\}$, with respect to the scalar product $(\cdot\,,\cdot)_\sim$. We have

$$V_{11}^{\tilde{*}} = V_{11}, \quad V_{22}^{\tilde{*}} = V_{22} \quad \text{and} \quad V_{12}^{\tilde{*}} = -V_{21}.$$

This is a direct consequence of the selfadjointness of V in $(\mathcal{H}, [\cdot\,,\cdot])$. By $W_{(\cdot\,,\cdot)}(T)$ denote the numerical range of $T \in L(\mathcal{H}_j)$, $j \in \{1, 2\}$, with respect to the scalar product $(\cdot\,,\cdot)$, i.e.

$$W_{(\cdot\,,\cdot)}(T) = \overline{\{(Tf, f) : f \in \mathcal{H}_j,\ \|f\| = 1\}}.$$

Recall that $\sigma(T)$ is always a subset of $W_{(\cdot\,,\cdot)}(T)$. Therefore,

$$\begin{aligned} \|V_{11}\|_\sim &= \sup\{|\lambda| : \lambda \in \sigma(V_{11})\} \\ &\leq \sup\{|\lambda| : \lambda \in W_{(\cdot\,,\cdot)}(V_{11})\} \;\leq\; \|V_{11}\|. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \|V_{12}\|_\sim^2 &= \|V_{12}^{\tilde{*}} V_{12}\|_\sim = \sup\{|\lambda| : \lambda \in \sigma(V_{12}^{\tilde{*}} V_{12})\} \\ &\leq \sup\{|\lambda| : \lambda \in W_{(\cdot\,,\cdot)}(V_{12}^{\tilde{*}} V_{12})\}. \end{aligned}$$

Hence, from

$$|(V_{12}^{\tilde{*}} V_{12} f_2, f_2)| = |(V_{21} V_{12} f_2, f_2)| \;\leq\; \|V_{12}\|\,\|V_{21}\|\,\|f_2\|^2$$

for $f_2 \in \mathcal{H}_2$ we conclude $\|V_{12}\|_\sim \;\leq\; \|V_{12}\|^{1/2}\|V_{21}\|^{1/2}$. This gives

$$\begin{aligned} \tau_- &= \|V_{11}\|_\sim + 2\max\{1, \|V_{12}\|_\sim\} \\ &\leq \|V_{11}\| + 2\max\{1, \|V_{12}\|^{1/2}\|V_{21}\|^{1/2}\}. \end{aligned}$$

From the identities $V_{11} = E(\mathbb{R}^+)(V \restriction \mathcal{H}_1)$, $V_{12} = E(\mathbb{R}^+)(V \restriction \mathcal{H}_2)$ and $V_{21} = E(\mathbb{R}^-)(V \restriction \mathcal{H}_1)$ we obtain

$$\tau_- \;\leq\; \|E(\mathbb{R}^+)\|\|V\| + 2\max\{1, \|E(\mathbb{R}^+)\|^{1/2}\|E(\mathbb{R}^-)\|^{1/2}\|V\|\}.$$

Hence, with $c := \max\{\|E(\mathbb{R}^+)\|, \|E(\mathbb{R}^-)\|\} \geq 1$ it holds

$$\begin{aligned} \tau_- &\leq c\|V\| + 2\max\{1, c\|V\|\} \;\leq\; c\big(\|V\| + 2\max\{1, \|V\|\}\big) \\ &= 3c\max\{1, \|V\|\}. \end{aligned}$$

A similar reasoning shows that $\tau_+ \;\leq\; 3c\max\{1, \|V\|\}$ and hence

$$r \;\leq\; 3c\max\{1, \|V\|\}.$$

Note now that

$$2E(\mathbb{R}^+) = I + \widetilde{J} \quad \text{and} \quad 2E(\mathbb{R}^-) = I - \widetilde{J}.$$

This yields

$$r \;\leq\; \frac{3}{2}\,(1 + \|\widetilde{J}\|)\,\max\{1, \|V\|\},$$

and hence $r \leq \delta$, see (3.3).

Assume $GA_0 \subset A_0 G$. Then $A_0^* = G(A_0)^+ G^{-1} = G A_0 G^{-1} \subset A_0$. Thus, A_0^* is symmetric. As $\pm i \in \rho(A_0)$, we have $\mp i \in \rho(A_0^*)$ and thus $A_0 = A_0^*$. Hence, A_0 is selfadjoint in the Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$. Therefore, the same holds for the operator $\widetilde{J} = E(\mathbb{R}^+) - E(\mathbb{R}^-)$. And as $\widetilde{J}^2 = I$, it follows that $\widetilde{J}$ is also unitary in $(\mathcal{H}, (\cdot\,,\cdot))$. Thus, $\tau_0 = \|\widetilde{J}\| = 1$. A similar reasoning applies to the case $A_0 G \subset G A_0$. The theorem is proved. □

Remark 3.2. In step 1 of the proof of Theorem 3.1 it is proved that $\tau_0 < \infty$ if both 0 and ∞ are not singular critical points of A_0. For the case that A_0 is boundedly invertible this was shown in [24, Lemma 1] (see also [39]). There, also the converse of this statement was proved.

Remark 3.3. In the proof of Theorem 3.1 it turns out that under the assumptions of Theorem 3.1 the limit

$$\widetilde{J} = \frac{1}{\pi}\, \text{s-}\lim_{N\to\infty} \int_{1/N}^{N} \left((A_0 + it)^{-1} + (A_0 - it)^{-1}\right)\, dt$$

exists and that $\tau_0 = \|\widetilde{J}\|$ can be chosen in the formulation of the theorem.

Remark 3.4. Clearly, the number δ in Theorem 3.1 is not optimal for (i)–(v) to hold. For example, $\delta = 0$ can be chosen if V is non-negative in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$.

Let T be a selfadjoint operator in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ and let B be a bounded, but maybe non-selfadjoint operator in $\mathcal{K}$. Then it is easy to see that the spectrum of $T + B$ is contained in the strip $\{x + iy : x \in \mathbb{R}, |y| \le \|B\|\}$. But the non-real spectrum of $T + B$ might be unbounded, e.g. if $B = i$. The following corollary of Theorem 3.8 shows that the non-real spectrum of $T + B$ is bounded if B is connected to T in a certain way. By $B^{\langle * \rangle}$ we denote the adjoint of B with respect to the scalar product $\langle\cdot,\cdot\rangle$.

Corollary 3.5. *Let T be a selfadjoint operator in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ and let $\varphi \in L^\infty(\mathbb{R})$ be real-valued. Assume that there exist $r, \varepsilon > 0$ such that*

$$|\varphi(x)| \ge \varepsilon \quad \text{for a.e. } x \in \mathbb{R} \text{ and}$$
$$\pm\varphi(x) > 0 \quad \text{for a.e. } x \in \mathbb{R}^\pm \setminus (-r, r).$$

Then for every $B \in L(\mathcal{K})$ satisfying $B^{\langle * \rangle}\varphi(T) = \varphi(T)B$ the non-real spectrum of $T + B$ is bounded. More precisely, we have $\sigma(T + B) \setminus \mathbb{R} \subset \overline{B_\delta(0)}$, where

$$\delta := 3(1 + r + \|B\|).$$

Proof. Set $G := \varphi(T)$. From the properties of φ it follows that G is selfadjoint in the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$, bounded and boundedly invertible. Hence, the inner product space $(\mathcal{K}, [\cdot\,,\cdot])$, where

$$[f, g] := \langle Gf, g\rangle, \quad f, g \in \mathcal{K},$$

is a Krein space. For $f, g \in \operatorname{dom} T$ we have

$$[Tf, g] = \langle\varphi(T)Tf, g\rangle = \langle T\varphi(T)f, g\rangle = \langle\varphi(T)f, Tg\rangle = [f, Tg].$$

Together with $\pm i \in \rho(T)$ this shows that T is selfadjoint in the Krein space $(\mathcal{K}, [\cdot\,,\cdot])$. The same holds for the operator B:

$$B^+ = G^{-1}B^{\langle * \rangle}G = B.$$

Now, define the function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) := \begin{cases} \operatorname{sgn}(\varphi(x)) & \text{for } |x| \le r \\ x & \text{for } |x| > r. \end{cases}$$

From

$$(\varphi\psi)(x) = \begin{cases} |\varphi(x)| & \text{for } |x| \le r \\ x\varphi(x) & \text{for } |x| > r \end{cases} \ge \begin{cases} \varepsilon & \text{for } |x| \le r \\ r\varepsilon & \text{for } |x| > r \end{cases} \ge \varepsilon \min\{1, r\}$$

we obtain for $f \in \operatorname{dom}\psi(T) = \operatorname{dom} T$:

$$[\psi(T)f, f] = ((\varphi\psi)(T)f, f) \ \ge \varepsilon \min\{1, r\}\|f\|^2.$$

Hence, $\psi(T)$ is boundedly invertible and non-negative in the Krein space $(\mathcal{K},[\cdot\,,\cdot])$. With the function $\phi(x) := x - \psi(x)$ we have $T = \psi(T) + \phi(T)$. And as $\psi(T)G = G\psi(T)$, by Theorem 3.1 the non-real spectrum of $T + B = \psi(T) + (\phi(T) + B)$ is contained in $B_{\delta_1}(0)$, where $\delta_1 = 3\max\{1, \|\phi(T) + B\|\}$. The assertion now follows from $\|\phi(T)\| = \|\phi\|_\infty \le 1 + r$. □

Corollary 3.6. *Let $(\mathcal{H},[\cdot\,,\cdot])$ be a Krein space and let $(\cdot\,,\cdot)$, $\|\cdot\|$ and G be fixed as in Theorem* 3.1. *Let A be a selfadjoint operator in $(\mathcal{H},[\cdot\,,\cdot])$ and assume that there exists some $\gamma > 0$ such that*

$$[Af, f] \ge -\gamma\|f\|^2 \quad \textit{for all } f \in \operatorname{dom} A.$$

Assume furthermore that for some (and hence for all) $\nu > \gamma$

$$\tau_\nu := \frac{1}{\pi}\limsup_{\eta\uparrow\infty}\left\|\int_{-\eta}^{\eta}(A + \nu G^{-1} - it)^{-1}\,dt\right\| < \infty$$

and set

$$\delta := \frac{3}{2}\,(1 + \tau_\nu)\max\{1, \nu\|G^{-1}\|\}.$$

Then the following holds:

(i) *The operator A is definitizable over $\overline{\mathbb{C}} \setminus \overline{B_\delta(0)}$.*
(ii) *∞ is not a singular critical point of A.*
(iii) *(δ, ∞) is of positive type with respect to A.*
(iv) *$(-\infty, -\delta)$ is of negative type with respect to A.*
(v) *$\sigma(A) \setminus \mathbb{R} \subset \overline{B_\delta(0)}$.*

If $GA \subset AG$ or $AG \subset GA$, then $\tau_\nu = 1$ for each $\nu > \gamma$ and hence (i)–(v) *hold with $\delta = 3\max\{1, \nu\|G^{-1}\|\}$.*

Proof. The operator $A_\nu := A + \nu G^{-1}$ is boundedly invertible and non-negative:

$$[A_\nu f, f] = [Af, f] + \nu[G^{-1}f, f] \ge -\gamma\|f\|^2 + \nu\|f\|^2 = (\nu - \gamma)\|f\|^2.$$

Moreover, $\tau_\nu < \infty$ implies that ∞ is not a singular critical point of A_ν, see Remark 3.2. With $V := -\nu G^{-1}$ we have $A = A_\nu + V$, and all statements follow from Theorem 3.1. □

Remark 3.7. If, in addition to the assumptions in Corollary 3.6, zero is neither an eigenvalue nor a singular critical point of the non-negative operator $A_\gamma := A + \gamma G^{-1}$, the assertions of Corollary 3.6 hold with

$$\delta := \frac{3}{2}\,(1 + \tau_\gamma)\max\{1, \gamma\|G^{-1}\|\},$$

where

$$\tau_\gamma := \frac{1}{\pi}\limsup_{N\to\infty}\left\|\int_{\frac{1}{N}}^{N}\big((A_\gamma + it)^{-1} + (A_\gamma - it)^{-1}\big)\,dt\right\|.$$

Indeed, the assumptions of Corollary 3.6 imply that ∞ is not a singular critical point of A_γ, see also Proposition 2.3. Therefore, the operator A_γ satisfies the conditions in Theorem 3.1.

A selfadjoint operator A in the Krein space $(\mathcal{H},[\cdot\,,\cdot])$ is called *non-negative in a neighborhood of* ∞ if it is definitizable over a domain Ω as in Definition 2.2, $\infty \in \Omega$, such that $\Omega \cap \mathbb{R}^+$ is of positive type and $\Omega \cap \mathbb{R}^-$ is of negative type with respect to A. The following theorem shows that a subclass of these operators is stable under bounded perturbations.

Theorem 3.8. *Let the selfadjoint operator A in the Krein space $\mathcal{H}$ be non-negative in a neighborhood of ∞ such that ∞ is not a singular critical point of A. Then for every bounded selfadjoint operator B in $\mathcal{H}$ the operator $A + B$ is non-negative in a neighborhood of ∞, and ∞ is not a singular critical point of $A + B$.*

Proof. Let $r > 0$ such that $[r, \infty)$ is of positive type and $(-\infty, -r]$ is of negative type with respect to A. By E denote the local spectral function of A on $\overline{\mathbb{R}} \setminus (-r, r)$. Set $\Delta := \overline{\mathbb{R}} \setminus [-r, r]$. Since ∞ is not a singular critical point of A, the projections $E((-\infty, -r))$ and $E((r, \infty))$ are defined, and with

$$\mathcal{K}_- := E((-\infty, -r))\mathcal{H}, \quad \mathcal{K}_b := (I - E(\Delta))\mathcal{H} \quad \text{and } \mathcal{K}_+ := E((r, \infty))\mathcal{H}$$

we have

$$\mathcal{H} = \mathcal{K}_- \,[\dot{+}]\, \mathcal{K}_b \,[\dot{+}]\, \mathcal{K}_+.$$

Corresponding to this decomposition of $\mathcal{H}$ the operator A decomposes as

$$A = A_- \,[\dot{+}]\, A_b \,[\dot{+}]\, A_+,$$

where the operators $\pm A_\pm$ are uniformly positive selfadjoint operators in the Hilbert spaces $(\mathcal{K}_\pm, \pm[\cdot\,,\cdot])$ and A_b is a bounded operator in $\mathcal{K}_b$. With some fundamental symmetry J_b in the Krein space $(\mathcal{K}_b, [\cdot\,,\cdot])$ define the operator

$$\widetilde{A} := A_- \,[\dot{+}]\, J_b \,[\dot{+}]\, A_+.$$

Then $\widetilde{A}$ is easily seen to be boundedly invertible and non-negative. From $\operatorname{dom} \widetilde{A} = \operatorname{dom} A$ and Proposition 2.3 we conclude that ∞ is not a singular critical point of $\widetilde{A}$. As every bounded perturbation of A is at the same time a bounded perturbation of $\widetilde{A}$, the assertion follows from Theorem 3.1. □

4. Applications to differential operators with indefinite weights

In this section we apply our main theorem from the previous section to ordinary and partial differential operators.

4.1. Indefinite Sturm-Liouville operators

In this subsection we consider Sturm-Liouville differential expressions of the form

$$\mathcal{L}(f)(x) = \operatorname{sgn}(x)\big(- f''(x) + q(x)f(x)\big), \quad x \in \mathbb{R},$$

with an essentially bounded and real-valued potential $q \in L^\infty(\mathbb{R})$. The corresponding differential operator in $L^2(\mathbb{R})$ is defined by

$$Af := \ell(f), \quad f \in \operatorname{dom} A := H^2(\mathbb{R}).$$

By $(\cdot\,,\cdot)$ and $\|\cdot\|$ we denote the usual scalar product and norm in $L^2(\mathbb{R})$. Let R be the operator of multiplication with the function $\operatorname{sgn}(x)$. This operator is obviously selfadjoint and unitary in $L^2(\mathbb{R})$. Since the operator T, defined by

$$Tf := RAf = -f'' + qf, \quad f \in \operatorname{dom} T := H^2(\mathbb{R}),$$

is selfadjoint in $L^2(\mathbb{R})$, it immediately follows that the operator A is selfadjoint in the Krein space $(L^2(\mathbb{R}), [\cdot\,,\cdot])$, where

$$[f, g] := (Rf, g) = \int_{\mathbb{R}} f(x)g(x)\operatorname{sgn}(x)\,dx, \quad f, g \in L^2(\mathbb{R}).$$

In the following for real numbers a and b we set

$$[a, b] := \{x \in \mathbb{R} : a \le x \le b\},$$

which might be a singleton or the empty set. The following proposition is in essence well-known, see [8, Theorem 4.2]. For the convenience of the reader we give a short proof here.

Proposition 4.1. *The operator A is definitizable over $\overline{\mathbb{C}} \setminus [m_+, m_-]$, where*

$$m_+ := \inf_{x \in \mathbb{R}^+} q(x) \quad \text{and} \quad m_- := -\inf_{x \in \mathbb{R}^-} q(x).$$

Moreover, A is non-negative in a neighborhood of ∞.

Proof. Define the selfadjoint operators $T_\pm$ in $L^2(\mathbb{R}^\pm)$ by

$$T_\pm f_\pm := -f_\pm'' + q f_\pm, \quad f \in \operatorname{dom} T_\pm := \{f_\pm \in H^2(\mathbb{R}^\pm) : f_\pm(0) = 0\}$$

and the operator $A_p := T_+ \oplus (-T_-)$, where the orthogonal sum is to be seen with respect to the decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^-)$ of $L^2(\mathbb{R})$. We have $RA_p = T_+ \oplus T_-$. Hence, the operator A_p is selfadjoint in the Krein space $(L^2(\mathbb{R}), [\cdot\,,\cdot])$. The operator T_+ is bounded from below by m_+ and $-T_-$ is bounded from above by m_-. From this it is easily seen that (m_-, ∞) is of positive type and $(-\infty, m_+)$ is of negative type with respect to A_p. And since A_p is also selfadjoint in the Hilbert space $(L^2(\mathbb{R}), (\cdot\,,\cdot))$, it follows that A_p is definitizable over $\overline{\mathbb{C}} \setminus [m_+, m_-]$. Note that the operators A and A_p coincide on $\{f \in H^2(\mathbb{R}) : f(0) = 0\}$. Therefore, the difference of their resolvents is one-dimensional. And since the reolvent set of the operator A is non-empty (see, e.g., [30, Proposition 2.4]) all assertions follow from [3, Theorem 2.2]. □

Proposition 4.1 in particular implies that the non-real spectrum of the operator A is discrete and can only accumulate to $[m_+, m_-]$. Moreover, it is bounded by Definition 2.2(i). But a bound for the non-real spectrum has not been given, yet. By means of numerical examples (see, e.g., [6]) one might conjecture that there is a close relationship between the maximal magnitude of the non-real eigenvalues of A and the lower bound of the selfadjoint operator $T = RA$ in $L^2(\mathbb{R})$ (see also [40, Remark and Example 11.4.1]). As $-\|q\|_\infty$ is the smallest possible value of $\inf \sigma(T)$, the following theorem confirms this conjecture to some extent.

Theorem 4.2. *The non-real spectrum of A is contained in $B_\delta(0)$, where*

$$\delta := \frac{3}{2}\,(1 + \tau_0) \max\{1, \|q\|_\infty\},$$

$$\tau_0 := \frac{1}{\pi} \limsup_{N \to \infty} \left\| \int_{\frac{1}{N}}^{N} \left((A_0 + it)^{-1} + (A_0 - it)^{-1} \right) dt \right\|,$$

and A_0 is the differential operator defined by

$$(A_0 f)(x) := -\operatorname{sgn}(x) f''(x), \quad f \in \operatorname{dom} A_0 := H^2(\mathbb{R}),\ x \in \mathbb{R}.$$

Moreover, (δ, ∞) is of positive type and $(-\infty, -\delta)$ is of negative type with respect to A.

Proof. Define the operator V by

$$(Vf)(x) := \operatorname{sgn}(x) q(x) f(x), \quad f \in L^2(\mathbb{R}),\ x \in \mathbb{R}.$$

Then we have $A = A_0 + V$. The operator V is obviously bounded in $L^2(\mathbb{R})$ with $\|V\| = \|q\|_\infty$, and A_0 is non-negative in the Krein space $(L^2(\mathbb{R}), [\cdot\,,\cdot])$. Its spectrum covers the entire real axis, i.e. $\sigma(A_0) = \mathbb{R}$. Moreover, $0 \notin \sigma_p(A_0)$. Owing to [18] the operator A_0 is similar to a selfadjoint operator in a Hilbert space. Equivalently, the points 0 and ∞ are regular critical points of A_0. Hence, all assertions follow from Theorem 3.1. □

4.2. Second order elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be a domain and let ℓ be the "formally symmetric" uniformly elliptic second order differential expression

$$(\ell f)(x) := -\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k} \right)(x) + a(x) f(x), \quad x \in \Omega, \tag{4.1}$$

with bounded coefficients $a_{jk} \in C^\infty(\Omega)$ satisfying $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \Omega$ and $j, k = 1, \dots, n$, the function $a \in L^\infty(\Omega)$ is real valued and

$$\sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k \geq C \sum_{k=1}^{n} \xi_k^2$$

holds for some $C > 0$, all $\xi = (\xi_1, \dots, \xi_n)^\top \in \mathbb{R}^n$ and $x \in \Omega$. To the differential expression ℓ we associate the elliptic differential operator

$$Tf := \ell(f), \qquad \operatorname{dom} T = \bigl\{ f \in H_0^1(\Omega) : \ell(f) \in L^2(\Omega) \bigr\}, \tag{4.2}$$

where $H_0^1(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$. It is well known that T is an unbounded selfadjoint operator in the Hilbert space $(L^2(\Omega), (\cdot,\cdot))$ with spectrum semibounded from below by $\operatorname{essinf} a$; cf. [19].

Let r be a real valued function such that $r, r^{-1} \in L^\infty(\Omega)$ and each of the sets

$$\Omega_+ := \bigl\{ x \in \Omega : r(x) > 0 \bigr\} \quad \text{and} \quad \Omega_- := \bigl\{ x \in \Omega : r(x) < 0 \bigr\} \tag{4.3}$$

has positive Lebesgue measure. We define a second order elliptic differential expression $\mathcal{L}$ with the indefinite weight r by

$$(\mathcal{L} f)(x) := \frac{1}{r(x)} (\ell f)(x), \qquad x \in \Omega.$$

The multiplication operator $Rf = rf$, $f \in L^2(\Omega)$, is an isomorphism in $L^2(\Omega)$ with inverse $R^{-1} f = r^{-1} f$, $f \in L^2(\Omega)$, and gives rise to the Krein space inner product

$$[f, g] := (Rf, g) = \int_\Omega f(x) \overline{g(x)}\, r(x)\, dx, \qquad f, g \in L^2(\Omega).$$

As in the previous subsection the Gram operator is denoted by R instead of G. The differential operator associated with $\mathcal{L}$ is defined as

$$Af = \mathcal{L}(f), \qquad \operatorname{dom} A = \bigl\{ f \in H_0^1(\Omega) : \mathcal{L}(f) \in L^2(\Omega) \bigr\}. \tag{4.4}$$

Since for $f \in H_0^1(\Omega)$ we have $\ell(f) \in L^2(\Omega)$ if and only if $\mathcal{L}(f) \in L^2(\Omega)$ it follows that $\operatorname{dom} A = \operatorname{dom} T$ and $A = R^{-1} T$ hold. Hence A is a selfadjoint operator in the Krein space $(L^2(\Omega), [\cdot,\cdot])$.

In order to illustrate Theorem 3.1 for the indefinite elliptic operator A we assume from now on that

$$\min \sigma_{\text{ess}}(T) \leq 0$$

holds. This also implies that the domain Ω is necessarily unbounded as otherwise $\sigma_{\text{ess}}(T) = \varnothing$. A discussion of the cases $\sigma_{\text{ess}}(T) = \varnothing$ and $\min \sigma_{\text{ess}}(T) > 0$ is contained in [5], see also [21, 22, 35]. Fix some $\gamma > 0$ such that $-\gamma < \min \sigma(T)$ and define the spaces $\mathcal{H}_s$, $s \in [0, 2]$, as the domains of the $\frac{s}{2}$-th powers of the uniformly positive operator $T + \gamma$ in $L^2(\Omega)$,

$$\mathcal{H}_s := \operatorname{dom}\bigl((T + \gamma)^{\frac{s}{2}} \bigr), \qquad s \in [0, 2].$$

Note that $\mathcal{H} = \mathcal{H}_0$, $\operatorname{dom} T = \mathcal{H}_2$ and the form domain of T is $\mathcal{H}_1$. The spaces $\mathcal{H}_s$ become Hilbert spaces when they are equipped with the usual inner products, the induced topologies do not depend on the particular choice of γ; cf. [31].

The following theorem is a consequence of Theorem 3.1 and the considerations in [15], see also [5].

Theorem 4.3. *Let A be the indefinite elliptic operator in* (4.4) *and assume that there exists a bounded uniformly positive operator W in $(L^2(\Omega), [\cdot,\cdot])$ such that $W\mathcal{H}_s \subset \mathcal{H}_s$ holds for some $s \in (0,2]$. Then $A_0 = A + \gamma R^{-1}$ is uniformly positive in $(L^2(\Omega), [\cdot,\cdot])$,*

$$\tau_0 := \frac{1}{\pi} \limsup_{N\to\infty} \left\| \int_{\frac{1}{N}}^{N} \left((A_0 + it)^{-1} + (A_0 - it)^{-1} \right) dt \right\| < \infty \,,$$

and with $\delta := \frac{3}{2}(1+\tau_0)\max\{1, \gamma\|r^{-1}\|_\infty\}$ the following statements hold:

(i) *The operator A is definitizable over $\overline{\mathbb{C}} \setminus \overline{B_\delta(0)}$.*
(ii) *(δ, ∞) is of positive type with respect to A.*
(iii) *$(-\infty, -\delta)$ is of negative type with respect to A.*
(iv) *$\sigma(A) \setminus \mathbb{R} \subset \overline{B_\delta(0)}$.*

Moreover, ∞ is not a singular critical point of A and A_0.

Proof. Observe that first that $\gamma < \min \sigma(T)$ implies that the operator

$$A_0 = A + \gamma R^{-1} = R^{-1}T + \gamma R^{-1} = R^{-1}(T + \gamma)$$

is uniformly positive in $(L^2(\Omega), [\cdot,\cdot])$, and, in particular, 0 is not a singular critical point of A_0. The assumption $W\mathcal{H}_s \subset \mathcal{H}_s$ for some $s \in (0,2]$ together with [15, Theorem 2.1 (iii)] (see also [14]) implies that also ∞ is not a singular critical point of A_0. Hence Theorem 3.1 can be applied to the operator A_0 and $V = -\gamma R$, that is,

$$A = A_0 + V.$$

As $\|V\| = \gamma\|r^{-1}\|_\infty$ the assertions in Theorem 4.3 follow. □

As in [5] we consider the special case $\Omega = \mathbb{R}^n$, where $\Omega_\pm = \{x \in \mathbb{R}^n : \pm r(x) > 0\}$ consist of finitely many connected components with compact smooth boundaries. Hence one of the sets $\Omega_\pm$ is bounded and one is unbounded. Since the weight function satisfies $r, r^{-1} \in L^\infty(\mathbb{R}^n)$ the restrictions $r_\pm, r_\pm^{-1}$ belong to $L^\infty(\Omega_\pm)$ and hence the multiplication operators $R_\pm f_\pm = r_\pm f_\pm$ are isomorphisms in $L^2(\Omega_\pm)$ with inverses $R_\pm^{-1} f_\pm = r_\pm^{-1} f_\pm$, $f_\pm \in L^2(\Omega_\pm)$.

Let us now assume that the coefficients $a_{jk} \in C^\infty(\mathbb{R}^n)$ in (4.1) and their derivatives are uniformly continuous and bounded, and that (as before) $a \in L^\infty(\mathbb{R}^n)$ is real valued. Then by elliptic regularity and interpolation

$$\operatorname{dom} A = \operatorname{dom} T = H^2(\mathbb{R}^n) \qquad \text{and} \qquad \mathcal{H}_s = H^s(\mathbb{R}^n), \qquad s \in [0,2],$$

holds; cf. [5] for more details. Here $H^s(\mathbb{R}^n)$ is the Sobolev space or order s. The spaces consisting of restrictions of functions from $H^s(\mathbb{R}^n)$ onto $\Omega_\pm$ are denoted by $H^s(\Omega_\pm)$. The following corollary is a consequence of [5, Lemma 5.1] and Theorem 4.3; cf. [5, Theorem 5.4].

Corollary 4.4. *Assume that for some $s \in (0, \frac{1}{2})$ the spaces $H^s(\Omega_+)$ and $H^s(\Omega_-)$ are invariant subspaces of the multiplication operators R_+ and R_-, respectively. Then the assertions in Theorem 4.3 are true.*

References

[1] T.Ya. Azizov and I.S. Iokhvidov, Linear operators in spaces with an indefinite metric, John Wiley Sons, Ltd., Chichester, 1989.

[2] T.Ya. Azizov, P. Jonas, and C. Trunk, Spectral points of type π_+ and π_- of selfadjoint operators in Krein spaces, J. Funct. Anal. **226** (2005), 115–137.

[3] J. Behrndt, Finite rank perturbations of locally definitizable self-adjoint operators in Krein spaces, J. Operator Theory **58** (2), 101–126, 2007.

[4] J. Behrndt, On the spectral theory of singular indefinite Sturm-Liouville operators, J. Math. Anal. Appl. **334** (2007), 1439–1449.

[5] J. Behrndt, Spectral theory of elliptic differential operators with indefinite weights, submitted.

[6] J. Behrndt, Q. Katatbeh, and C. Trunk, Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators, J. Phys. A: Math. Theor. **41** (2008), 244003.
[7] J. Behrndt, Q. Katatbeh, and C. Trunk, Non-real eigenvalues of singular indefinite Sturm-Liouville operators, Proc. Amer. Math. Soc. 137 (2009), 3797–3806.
[8] J. Behrndt and F. Philipp, Spectral analysis of singular ordinary differential operators with indefinite weights, J. Differential Equations **248** (8), 2015–2037, 2010.
[9] J. Behrndt and C. Trunk, On the negative squares of indefinite Sturm-Liouville operators, J. Differential Equations **238** (2007), 491–519.
[10] C. Bennewitz and W.N. Everitt, On second-order left-definite boundary value problems, Ordinary differential equations and operators (Dundee, 1982), 31–67, Lecture Notes in Math. **1032**, Springer, 1983.
[11] P. Binding and P. Browne, Left definite Sturm-Liouville problems with eigenparameter dependent boundary conditions, Differential Integral Equations **12** (1999), 167–182.
[12] P. Binding, P. Browne, and B. Watson, Inverse spectral problems for left-definite Sturm-Liouville equations with indefinite weight, J. Math. Anal. Appl. **271** (2002), 383–408.
[13] J. Bognar, Indefinite inner product spaces, Springer, 1974.
[14] B. Ćurgus, On the regularity of the critical point infinity of definitizable operators, Integral Equations Operator Theory **8** (1985), 462–488.
[15] B. Ćurgus and B. Najman, A Krein space approach to elliptic eigenvalue problems with indefinite weights, Differential Integral Equations **7** (1994), 1241–1252.
[16] B. Ćurgus and B. Najman, Quasi-uniformly positive operators in Krein spaces, Oper. Theory Adv. Appl. **80** (1995), 90–99.
[17] B. Ćurgus and H. Langer, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, J. Differential Equations **79** (1989), 31–61.
[18] B. Ćurgus and B. Najman, The operator $\operatorname{sgn}(x)\frac{d^2}{dx^2}$ is similar to a selfadjoint operator in $L^2(\mathbb{R})$, Proc. Amer. Math. Soc. **123** (1995), 1125–1128.
[19] D.E. Edmunds and W.D. Evans, Spectral theory and differential operators, Oxford Mathematical Monographs, Oxford University Press, New York, 1987.
[20] M. Faierman, Elliptic problems involving an indefinite weight, Trans. Amer. Math. Soc. **320** (1990), 253–279.
[21] M. Faierman, Nonselfadjoint elliptic problems involving an indefinite weight, Comm. Partial Differential Equations **15** (1990), 939–982.
[22] M. Faierman, On the spectral theory of an elliptic boundary value problem involving an indefinite weight, Oper. Theory Adv. Appl. **80**, Birkhäuser, Basel (1995), 137–154.
[23] M. Faierman and H. Langer, Elliptic problems involving an indefinite weight function, Oper. Theory Adv. Appl. **87**, Birkhäuser, Basel (1996), 105–124,
[24] P. Jonas, Compact perturbations of definitizable operators II, J. Oper. Theory **8** (1982), 3–18.
[25] P. Jonas, On a class of selfadjoint operators in Krein space and their compact perturbations, Integral Equations Operator Theory **11** (1988), 351–384.
[26] P. Jonas, On locally definite operators in Krein spaces, in: Spectral Theory and its Applications, Ion Colojoară Anniversary Volume, Theta (2003), 95–127.
[27] I.M. Karabash and A.S. Kostenko, Indefinite Sturm-Liouville operators with the singular critical point zero, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 801–820.
[28] I.M. Karabash, A.S. Kostenko and M.M. Malamud, The similarity problem for J-nonnegative Sturm-Liouville operators, J. Differential Equations **246** (2009), 964–997.
[29] I.M. Karabash and M.M. Malamud, Indefinite Sturm-Liouville operators $(\operatorname{sgn} x)(-\frac{d^2}{dx^2}+q(x))$ with finite-zone potentials, Oper. Matrices **1** (2007), 301–368.
[30] I.M. Karabash and C. Trunk, Spectral properties of singular Sturm-Liouville operators, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), 483–503.
[31] T. Kato, Perturbation Theory for Linear Operators, Second Edition, Springer, 1976.
[32] Q. Kong, H. Wu and A. Zettl, Left-definite Sturm-Liouville problems, J. Differential Equations **177** (2001), 1–26.
[33] Q. Kong, H. Wu and A. Zettl, Singular left-definite Sturm-Liouville problems, J. Differential Equations **206** (2004), 1–29.
[34] H. Langer, A. Markus, and V. Matsaev, Locally definite operators in indefinite inner product spaces, Math. Ann. **308** (1997), 405–424.
[35] S.G. Pyatkov, Some properties of eigenfunctions of linear pencils, Sibirian Math. Journal **30** (1989), 587–597.
[36] S.G. Pyatkov, Elliptic eigenvalue problems involving an indefinite weight function, Siberian Adv. Math. **10** (2000), 134–150.
[37] S.G. Pyatkov, Interpolation of Sobolev spaces and indefinite elliptic problems, Oper. Theory Adv. Appl. **198**, Birkhäuser, Basel (2009), 265–290.

[38] M. Reed and B. Simon, Methods of modern mathematical physics I. Functional Analysis, Academic Press, 1980.

[39] K. Veselić, On spectral properties of a class of J-selfadjoint operators I, Glas. Math., III. Ser. **7** (1972), 229–248.

[40] A. Zettl, Sturm-Liouville theory, AMS, Providence, RI, 2005.

Jussi Behrndt, Institut für numerische Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
E-mail address: behrndt@tugraz.at

Friedrich Philipp, Institut für Mathematik, MA 6-4, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany
E-mail address: philipp@math.tu-berlin.de

Carsten Trunk, Institut für Mathematik, Technische Universität Ilmenau, Postfach 10 05 65, 98684 Ilmenau, Germany
E-mail address: carsten.trunk@tu-ilmenau.de