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Facial non-repetitive vertex colouring of some families of 2-connected plane graphs

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Abstract. A sequence $a_1a_2 \dots a_{2n}$ such that $a_i = a_{n+i}$ for all $1 \leq i \leq n$ is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition. Let G be a graph whose vertices are coloured. A path in G is a called *non-repetitive* if the sequence of colours of its vertices is non-repetitive. If G is a plane graph, a *facial non-repetitive vertex colouring* of G is a vertex colouring such that any facial path (i.e. a path of consecutive vertices on the boundary walk of a face) is non-repetitive. We denote $\pi_f(G)$ the minimum number of colours of a facial non-repetitive vertex colouring of G. In this article, we show that $\pi_f(G) \leq 16$ for any plane hamiltonian graph G and $\pi_f(G) \leq 112$ for any 2-connected cubic plane graph G. These bounds are improved for some subfamilies of 2-connected plane graphs. All proofs are constructive.

1 Introduction

Let $\mathbb{A}$ be a set of symbols and let $S = a_1a_2 \dots a_{2n}$ be a sequence with $a_i \in \mathbb{A}$, $n \geq 1$. S is called a *repetition* if $a_i = a_{i+n}$ for all $i = 1, \dots, n$ and *non-repetitive* if no block of S (i.e. subsequence of consecutive terms of S) is a repetition. On the alphabet $\{0, 1\}$ the longest sequences not containing repetitions are 010 and 101. On the other hand, more than hundred years ago, Thue [21] proved that over $\{0, 1, 2\}$ there is an infinite sequence not containing any repetition.

The first problem involving graphs and non-repetitive sequences was studied by Currie in 1987, see [7]. The idea to study non-repetitive colourings of finite graphs comes from the paper of Alon, Grytczuk, Haluszczak and Riordan [1].

Let G be a simple graph and let f be a colouring of the vertices of G by symbols (colours) of $\mathbb{A}$. We say that f is *non-repetitive* if for any simple path $v_1v_2\dots v_{2n}$ in G the associated sequence of colours $f(v_1)f(v_2)\dots f(v_{2n})$ is not a repetition. The minimum number of colours in a non-repetitive colouring of G is the *Thue chromatic number* and it will be denoted by $\pi(G)$. Let P_n and C_n be a path and a cycle, respectively, on n vertices. Then Thue's result can be reformulated as follows.

Theorem 1.1 (Thue [21]). *$\pi(P_1) = 1$, $\pi(P_2) = \pi(P_3) = 2$, and $\pi(P_n) = 3$ for every $n \geq 4$.* □

This implies that $\pi(C_n) \leq 4$. The precise value of the Thue chromatic number for cycles has been determined recently.

Theorem 1.2 (Currie [8]). *$\pi(C_n) = 4$ for $n \in \{5, 7, 9, 10, 14, 17\}$. Otherwise $\pi(C_n) = 3$.*

Let $\pi(\Delta)$ denote the supremum of $\pi(G)$ where G ranges over graphs of maximum degree at most Δ. Thus, $\pi(2) = 4$.

The proofs of the following two theorems relay on probabilistic method and at moment no constructive argument is known for any $\Delta \geq 3$, see [1] and [12].

Theorem 1.3 (Alon et al. [1]). *There are absolute positive constants c_1 and c_2 such that*

$$c_1 \frac{\Delta^2}{\log \Delta} \leq \pi(\Delta) \leq c_2 \Delta^2 .$$

Theorem 1.4 (Grytczuk [12]). *If G is a graph of maximum degree at most Δ, then $\pi(G) \leq 16\Delta^2$.*

Alon et al. in [1] posed the following

Problem 1.5 (Alon et al. [1]). *Determine $\pi(3)$.*

A major problem in this area concerning planar graphs is due to Grytczuk.

Conjecture 1.6 (Grytczuk [12]). *There exists an integer K such that $\pi(G) \leq K$ for every planar graph G.*

Motivated by Conjecture 1.6, Brešar, Grytczuk, Klavžar, Niwczyk and Peterin proved

Theorem 1.7 (Brešar et al. [5]). *Let T be a tree. Then $\pi(T) \leq 4$. Moreover, the bound 4 is tight.*

The following results have been proved independently by Kündgen and Pelsmajer [15] and Barát and Varjú [2].

Theorem 1.8 ([2], [15]). *Let G be an outerplanar graph. Then*

$$\pi(G) \leq 12\,.$$

The *tree-width* of a graph G is the minimal k such that G can be embedded into a k-tree. See the book of Bondy and Murty [4] for a definition of the tree-width.

Theorem 1.9 (Kündgen and Pelsmajer [15]). *Let G be a graph of tree-width $t \geq 0$. Then $\pi(G) \leq 4^t$.*

For some other results concerning non-repetitive vertex colourings of graphs see [1]–[3], [5], [7], [10]–[13], [16], [17]. There is a rich literature concerning non-repetitive edge colourings, see e.g. [1], [10] or [14] where the reader can find surveys of the results on the topic.

2 Facial non-repetitive vertex-colouring of plane graphs

In this article, we study a variant of non-repetitive vertex colourings for plane graphs which is less constrained. We use standard terminology according to Bondy and Murty [4] except for few notations defined throughout this article. However, we recall some frequently used terms.

Analogously as in Havet et al. [14], we introduce in this paper a facial non-repetitive colouring. If G is a connected plane graph, a *facial non-repetitive vertex colouring* of G is a vertex colouring such that any facial path (i.e. a path of consecutive vertices on the boundary walk of a face) is non-repetitive. The *facial Thue chromatic number* of G, denoted by $\pi_f(G)$, is the minimum number of colours of a facial non-repetitive vertex colouring of G. Note that the facial Thue chromatic number depends on the embedding of the graph.

Observe that a facial non-repetitive vertex colouring is less constrained than a (general) non-repetitive vertex colouring. Since a facial non-repetitive vertex colouring is also proper, we immediately have

Proposition 2.1. *Let G be a 2-connected plane graph and let $\chi_0(G)$ be its (usual) chromatic number. Then*

$$\chi_0(G) \leq \pi_f(G) \leq \pi(G)\,.$$

□

On the other hand, any usual vertex colouring of a plane triangulation uses on each triangular face three different colours. It follows

Proposition 2.2. *Let G be a plane triangulation. Then*

$$3 \leq \pi_f(G) = \chi_0(G) \leq 4\,.$$

□

Motivated by the Problems 1.5 and Conjecture 1.6 we consider relaxations of them. More precisely, we restrict ourselves to the following two problems.

Problem 2.3. *Let $\pi_f(\Delta)$ denotes the supremum of $\pi_f(G)$ where G ranges over plane graphs of maximum degree at most Δ. Determine $\pi_f(3)$.*

Conjecture 2.4. *There is an integer K such that $\pi_f(G) \leq K$ for every 2-connected plane graph.*

We give an upper bound on $\pi_f(3)$ in the case of 2-connected cubic plane graphs. We show that a constant K exists for the family of Halin graphs and the family of hamiltonian plane graphs. All our proofs are constructive.

3 Lemmas on Thue sequences

To be able to prove the theorems below we need some preliminary lemmas.

For a sequence of symbols $S = a_1a_2 \dots a_n$ with $a_i \in \mathbb{A}$ and for all $1 \leq k \leq l \leq n$ let us denote by $S_{k,l}$ the block $a_ka_{k+1} \dots a_l$

Lemma 3.1 ([14]). *Let $A = a_1a_2 \dots a_m$ be a non-repetitive sequence with $a_i \in \mathbb{A}$ for all $i = 1, 2, \dots, m$. Let $B^i = b^i_1 b^i_2 \dots b^i_{m_i}$, $0 \leq i \leq r+1$, be a non-repetitive sequences with $b^i_j \in \mathbb{B}$ for all $i = 0, 1, \dots, r+1$, and $j = 1, 2, \dots, m_i$. If $\mathbb{A} \cap \mathbb{B} = \emptyset$ then $S = B^0A_{1,n_1}B^1A_{n_1+1,n_2} \dots B^rA_{n_r+1,m}, B^{r+1}$ with $1 < n_1 < n_2 < \dots < n_r < m$ is a non-repetitive sequence.* □

The second lemma is an immediate implication of Theorem 1.1.

Lemma 3.2. *Every cycle admits a non-repetitive vertex-colouring in $\{1, 2, 3, 4\}$ such that exactly one vertex obtains colour 4.* □

4 Halin graphs

A *Halin graph* is a plane graph which is a union of a tree T of order $p \geq 3$ having no vertex of degree 2 and a cycle C connecting the leaves of T in a cyclic order determined by a plane embedding of T.

Theorem 4.1. *Let G be a Halin graph. Then*

$$\pi_f(G) \leq 7\,.$$

Proof. Let G be a union of a tree T and a cycle C. Let T_1 be a subtree of T obtained by deleting all leaves up to one, say v, from T. By Theorem 1.7 there is a non-repetitive 4-colouring φ_1 of T_1 with colours in $\{1, 2, 3, 4\}$. By Lemma 3.2 there is a non-repetitive 4-coloring of vertices of C with the colour $\varphi_1(v)$ of v and other colours in $\{5, 6, 7\}$. If we use these colourings also for G then, by Lemma 3.1, we obtain a required facial non-repetitive vertex 7-colouring. □

5 Hamiltonian plane graphs

Lemma 5.1. *Let G be a 2-connected outerplanar graph. Then the vertices of G can be coloured with 4 colours in such a way that no interior face contains a repetition.*

Proof. The proof is by induction on the number m of interior faces. If $m = 1$ then the statement follows from the Currie's Theorem 1.2. Let G have m faces $\alpha_1, \alpha_2, \ldots, \alpha_m$ and assume that the face α_m is a face sharing exactly one edge with exactly one of the remaining faces. Let $e = uv$ be this edge. Let $G^{'}$ be the subgraph of G consisting of faces $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$. By induction hytothesis, the graph $G^{'}$ has the desired coloring $\varphi^{'}$ with at most four colours from $\{1, 2, 3, 4\}$ and let $\varphi'(u) = a$ and $\varphi'(v) = b$. By Theorem 1.2, the vertices of α_m can be coloured with four colours from the same set $\{1, 2, 3, 4\}$ such that the vertex u gets colour a and the vertex v receives colour b. The union of these two colourings gives a suitable vertex 4-colouring of G. □

Theorem 5.2. *Let G be a plane hamiltonian graph G then*

$$\pi_f(G) \leq 16\,.$$

Proof. Let H be a Hamilton cycle in G and G_1 (and G_2) be the outerplanar graph induced by the edges of H and the edges in the interior (in the exterior) of H in G. By Lemma 5.1, there is a vertex-colouring φ_1 of G_1 and a vertex colouring φ_2 of G_2 that is facially non-repetitive on the interior faces of G_1 and G_2, respectively. If we associate to each vertex v_i of G the ordered pair $(\varphi_1(v_i), \varphi_2(v_i))$ we obtain a suitable facial non-repetitive vertex 16-colouring of G. □

Using Theorem 1.2 and the ideas of the proof of the previous theorem, one can prove the following

Theorem 5.3. *If a hamiltonian plane graph G does not contain any r-gonal face for $r \in \{5, 7, 9, 10, 14, 17\}$ then*

$$\pi_f(G) \le 9\,.$$

□

Because every 4-connected planar graph is hamiltonian, see [22] , we immediately have

Corollary 5.4. *If G is a 4-connected plane graph then*

$$\pi_f(G) \le 16\,.$$

□

6 Cubic plane graphs – basic results

In this Section we deal with 2-connected cubic (i.e. 3-regular) plane graphs. From Theorem 1.4 it follows for every 2-connected cubic plane graph G that $\pi_f(G) \le 144$. In this and the next sections the better bounds for different families of 2-connected cubic graphs will be given. To present the main idea of our proofs below we first prove the following proposition.

Proposition 6.1. *Let G be a 2-connected cubic plane graph. Then*

$$\pi_f(G) \le 256\,.$$

Proof. We have to show that there is a vertex colouring of G with 256 colours in which any facial path is non-repetitive. Let α be a face of G. By Theorem 1.2, there is a vertex 4-colouring φ_α of the vertices of the facial cycle of α which is non-repetitive.

By the Four Colour Theorem [19], [23] there exists a proper face 4-colouring ξ of the faces of G with colour from the set $\{1, 2, 3, 4\}$. To obtain a required vertex colouring of G which is facial non-repetitive we associate each vertex v with colour - the ordered quadruple - (x_1, x_2, x_3, x_4) defined in the following way. Let a vertex v be incident with the face α. If $\xi(\alpha) = i$ then we put $x_i = \varphi_\alpha(v)$. If for some i no face incident with v is coloured with colour i we put $x_i = 0$. (Note that such i exists for every vertex because the graph G is cubic.) It is easy to see that there are $4 \times 4^3 = 4^4$ ordered quadruples of the required type and that any facial path in G is non-repetitive. □

If a 2-connected cubic plane graph G does not contain any k-gonal face for $k \in \{5, 7, 9, 10, 14, 17\}$ then for any face α there is, by the Currie result, a vertex 3-colouring φ_α that is non-repetitive on the facial cycle of α. Hence, by a slight modification of the proof of the previous theorem, we immediately have

Proposition 6.2. *Let G be a 2-connected cubic plane graph that does not contain any k-gonal face for $k \in \{5, 7, 8, 10, 14, 17\}$. Then*

$$\pi_f(G) \leq 108\,.$$

□

It is well known that any cubic plane graph all faces of which are even-gonal has a proper face 3-colouring ξ, (see [15], [20]). Using this fact for ξ in the proof of Proposition 6.1 we are able to prove

Proposition 6.3. *Let G be a 2-connected bipartite cubic plane graph. Then*

$$\pi_f(G) \leq 64\,.$$

□

Proof. We follow the idea of the proof of Theorem 6.1. By Theorem 1.2 there is a vertex 4-colouring φ_α of the vertices of the facial cycle of α which is non-repetitive.

By the Heawood theorem [15], (see also [20]) there exists a proper face 3-colouring ξ of the faces of G with colour from the set $\{1, 2, 3\}$. To obtain a required vertex 64-colouring of G which is facial non-repetitive we associate each vertex v with the ordered triple - (x_1, x_2, x_3) defined in the following way. Let a vertex v be incident with the face α. If $\xi(\alpha) = i$ then we put $x_i = \varphi_\alpha(v)$. It is easy to see that there is at most 4^3 ordered triples of the required type and that any facial path in G is non-repetitive. □

The proof of the next theorem is left to the reader

Theorem 6.4. *Let G be a 2-connected cubic plane graph all faces of which are multi-4-gonal. Then*

$$\pi_f(G) \leq 27\,.$$

7 Cubic plane graphs – main result

We are going to improve Proposition 6.1. To do this we need two more notions. A set of vertices (edges) of a plane graph G is *face independent* if no two vertices (edges) of this set are incident with the same face. Our improvement is the following theorem.

Theorem 7.1. *Let G be a 2-connected cubic plane graph. Then*

$$\pi_f(G) \leq 112\,.$$

Proof. The proof consists of two parts. First we choose some vertices that receive extra colours. Then we show that the remaining vertices in any face can be coloured with three colours such that the resulting vertex colouring of vertices on facial cycles will be non-repetitive. At the beginning, let us call all faces of G *white faces* and all the vertices *white* ones.

First we find a maximum set A of face independent vertices. Let us call the vertices of the set A *green vertices*. Each face which is incident with a green vertex is called a *green* face. Observe that now every white vertex is incident with at least one green face. Now an edge is called *white* if it is incident with two white faces. Let B be a maximum set of face independent white edges. Let us call the edges from B *blue edges*. Next let us choose exactly one end vertex from every blue edge and call it a *blue vertex*. Rename each white face that is incident with a blue vertex to be a *blue* face. From the remaining white faces we select those which have sizes from the set $\{5, 7, 9, 10, 14, 17\}$ and call them *pink faces*. Observe that the vertices of any pink face are white. Let F be the subgraph of G induced by the vertices at pink faces. The maximum degree of F is at most 3 therefore there is a (usual) proper 3-colouring of the vertices of F with a colour from the set { pink, red, black }. If possible then on each pink face choose a vertex coloured with pink colour and call this vertex *pink*. If such a vertex on a pink face α does not exist then choose a vertex of α coloured red and call this vertex *red*.

Hence, after the above described procedure we have green, blue, pink and white faces. Similarly we have green, blue, pink, red and white vertices. All vertices on white faces are white, exactly one vertex on any pink face is either pink or red and all other are white. Each blue face contains exactly one blue vertex, some white vertices and no or some pink and/or red vertices, but these are at mutual distance at least two except possibly when a red vertex and a pink vertex are adjacent. On the boundary of green faces there can be a more complicated situation. It contains exactly one green vertex, no two consecutive vertices (except of white) are of the same colour, no green vertex is adjacent with any blue vertex, and among any four consecutive vertices at least one is white. Vertices, that are not white will be refereed as precoloured.

Now we are going to show that for any face α there exists a colouring of white vertices on the boundary of α by three colours from the set $\{1, 2, 3\}$ which together with precolored vertices on α gives a non-repetitive colouring φ_α.

If α is a white face then, due the Currie theorem, there exists a non-repetitive vertex 3-colouring φ_α of vertices of the facial cycle of α with colours from $\{1, 2, 3\}$. Next let α be not a white face and u be the green vertex if α is green, the blue vertex if α is blue, and the pink or red vertex if α is pink. Choose an orientation of α. Let $v_1, v_2, \ldots, v_k$ be the sequence of all white vertices of α in order around α chosen in such a way that the vertex u is on the segment of the facial cycle between v_k and v_1 containing no other white vertex.

By Thue Theorem 1.1, there is a non-repetitive 3-colouring with colours $1, 2$, and 3 of the vertices of the sequence of white vertices of α. By Lemma 3.1, this colouring together with colourings of precolored vertices incident with α provides a non-repetitive vertex colouring φ_α of the facial cycle of α.

The remaining part of the procedure of the colouring is analogous to those in the proof of Proposition 6.1. By the Four Colour Theorem, there is a proper face 4-colouring ξ of the faces of G with colours from the set $\{1, 2, 3, 4\}$. To obtain a required vertex colouring of G which is facially non-repetitive we associate to each vertex v a colour – the ordered quadruple – $\{x_1, x_2, x_3, x_4\}$ defined in the following way: If v is a precoloured vertex with a colour a, $a \in \{green, blue, pink, red\}$, we put $x_1 = x_2 = x_3 = x_4 = a$. Let v be a white vertex incident with the face α. If $\xi(\alpha) = i$ then we put $x_i = \varphi_\alpha(v)$. If for some i no face α incident with v is coloured with colour i we put $x_i = 0$.

It is an easy calculation to get that the number of suitable quadruples is bounded by $4 \times 3^3 + 4 = 112$. This finishes the proof. □

Combining the methods of the proofs of Theorem 7.1 and Proposition 6.3 one can prove the following strengthening of the latter one. We omit the proof.

Theorem 7.2. *Let G be a 2-connected bipartite cubic plane graph. Then*

$$\pi_f(G) \leq 31\,.$$

8 Concluding remark

One can ask how good are the bounds in our theorems. In [2] Barát and Varjú provide examples of outer planar graphs G with $\pi(G) \geq 7$ and examples of plane graphs H with $\pi(H) \geq 10$. Unfortunately we do not know any plane graph G with $\pi_f(G) \geq 5$. We strongly believe that a constant K in our Conjecture 2.4 is smaller than 10.

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References

[1] N. Alon, J. Grytczuk, M. Haluszczak, O. Riordan, *Non-repetitive colourings of graphs*, Random Structures Algorithms 21 (2002), 336–346.

[2] J. Barát, P. P. Varjú, *On square-free vertex colorings of graphs*, Studia Sci. Math. Hungar. 44 (2007), 411–422.

[3] J. Barát, D. Wood, *Notes on square-free colourings of graphs*, Electron. J. Combin. 15 (2008), Research Paper 99, 13 pp.

[4] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, 2008.

[5] B. Brešar, J. Grytczuk, S. Klavžar, S. Niwczyk, I. Peterin, *Non-repetitive colourings of trees*, Discrete Math. 307 (2007), 163–172.

[6] B. Brešar, S. Klavžar, *Square-free colorings of graphs*, Ars Combin. 70 (2004), 3–13.

[7] J. D. Currie, *Open problems in pattern avoidance*, The Amer. Math. Monthly 100 (1993), 790–793.

[8] J. D. Currie, *There are ternary circular square-free words of length n for $n \geq 18$*, Electron. J. Combin. 9 (2002), #N10.

[9] J. D. Currie, *Pattern avoidance: themes and variations*, Theoret. Comput. Sci. 339 (2005), 7–18.

[10] J. Grytczuk, *Thue type problems of graphs, points and numbers*, Discrete Math. 308 (2008), 4419–4429.

[11] J. Grytczuk, *Pattern avoidance on graphs*, Discrete Math. 307 (2007), 1341–1346.

[12] J. Grytczuk, *Nonrepetitive colorings of graphs – A survey*, Int. J. Math. Math. Sci. (2007), Article ID 74639.

[13] J. Grytczuk, W. Śliwa, *Non-repetitive colorings of infinite sets*, Discrete Math. 265 (2003), 365–373.

[14] F. Havet, S. Jendrol', R. Soták, E. Škrabuľáková, *Facial non-repetitive edge-colouring of plane graphs*, J. Graph Theory 66 (2011), 38–48.

[15] P. J. Heawood, *On the four-color map theorem*, Quart. J. Pure Math. 29 (1898), 270–285.

[16] A. Kündgen, M. J. Pelsmajer, *Non-repetitive colourings of graphs of bounded tree-width*, Discrete Math. 308 (2008), 4473–4478.

[17] A. Pezarski, M. Zmarz, *Non-repetitive 3-coloring of subdivided graphs*, Electron. J. Combin. 16 (2009), #N15.

[18] N. Rampersad, *A note on non-repetitive colourings of planar graphs*, arXiv:math/0307365 v 1, (2003), 1–3.

[19] N. Robertson, D. P. Sanders, P. Seymour, R. Thomas, *The four-color theorem*, J. Combin. Theory Ser. B 70 (1997), 2–44.

[20] R. Steinberg, *The state of the three color problem*, Annals of Discrete Mathematics 55 (1993), 211–248.

[21] A. Thue, *Über unendliche Zeichenreichen*, Norske Vid. Selsk. Skr., I Mat. Nat. Kl., Christiana 7 (1906), 1–22 (In German).

[22] W. T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. 82 (1956), 99–116.

[23] R. A. Wilson, *Graphs, colourings and the four-colour theorem*, Oxford University Press, New York, 2002.