Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **11/03**

Closed linear relations and their regular points

Jean-Philippe Labrousse, Adrian Sandovici,
Henk de Snoo, Henrik Winkler

2011

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

CLOSED LINEAR RELATIONS AND THEIR REGULAR POINTS

J.-PH. LABROUSSE, A. SANDOVICI, H.S.V. DE SNOO, AND H. WINKLER

Abstract. For a closed linear relation A in a Hilbert space $\mathfrak{H}$ the notions of resolvent set and set of points of regular type are extended to the set of regular points. Such points are defined in terms of quasi-Fredholm relations of degree 0. The set of regular points is open and for $\lambda \in \mathbb{C}$ in this set the spaces $\ker(A-\lambda)$ and $\operatorname{ran}(A-\lambda)$ are continuous in the gap metric.

1. Introduction

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. A point $\lambda \in \mathbb{C}$ is said to belong to the *resolvent set* $\rho(A)$ of A if

(R1) $\operatorname{ran}(A-\lambda) = \mathfrak{H}$;
(R2) $\ker(A-\lambda) = \{0\}$.

The set $\rho(A)$ is open and $(A-\lambda)^{-1}$, $\lambda \in \rho(A)$, is a holomorphic family of bounded everywhere defined linear operators on $\mathfrak{H}$. Furthermore, $\lambda \in \mathbb{C}$ is said to belong to the set of *points of regular type* $\gamma(A)$ of A if

(T1) $\operatorname{ran}(A-\lambda)$ is closed in $\mathfrak{H}$;
(T2) $\ker(A-\lambda) = \{0\}$.

The set $\gamma(A)$ is open and $(A-\lambda)^{-1}$, $\lambda \in \gamma(A)$, is a family of bounded linear operators on $\operatorname{ran}(A-\lambda)$; see for instance [9].

The purpose of the present paper is to extend the notion of points of regular type. A point $\lambda \in \mathbb{C}$ is said to belong to the set $\operatorname{reg}(A)$ of *regular points* of A if

(F1) $\operatorname{ran}(A-\lambda)$ is closed in $\mathfrak{H}$;
(F2) $\ker(A-\lambda)^n \subset \operatorname{ran}(A-\lambda)$, $n \in \mathbb{N}$,

or, equivalently, if

(F3) $\operatorname{ran}(A-\lambda)$ is closed in $\mathfrak{H}$;
(F4) $\ker(A-\lambda) \subset \operatorname{ran}(A-\lambda)^n$, $n \in \mathbb{N}$.

It wil be shown that the set $\operatorname{reg}(A)$ is open and that for $\lambda \in \operatorname{reg}(A)$ the mapping $\lambda \to \ker(A-\lambda)$ is continous in the gap-metric (for closed linear subspaces of $\mathfrak{H}$). Moreover, it will be shown that $\lambda \in \operatorname{reg}(A)$ if and only if $\operatorname{ran}(A-\lambda)$ is closed and there exists a neighborhood $\mathcal{U}$ of λ such that $\ker(A-\zeta)$ is close to $\ker(A-\lambda)$ in the gap-metric for all $\zeta \in \mathcal{U}$. Finally, a characterization of $\operatorname{reg}(A)$ is given in terms of generalized resolvents of A. For the case where A is an operator, these results can

2000 *Mathematics Subject Classification.* Primary 47A57, 47B25; Secondary 47A55, 47B65.

Key words and phrases. Quasi-Fredholm relation, ascent, descent, generalized resolvent, gap metric, range space.

The research of the first two authors was supported by NWO. The work of the second author was supported by AM-POSDRU, project number: POSDRU/89/1.5/S/49944. The third author thanks the Deutsche Forschungsgemeinschaft (DFG) for the Mercator visiting professorship at the Technische Universität Berlin.

be found in Labrousse's paper [12], and it turns out that the results in [12] remain valid in the context of relations. However, all the previous arguments require an interpretation and an adaptation to make them work for relations. The present paper can be seen as a natural continuation of [13] and the notations introduced in [13] will be used here as well.

As to the extension of previous results from the case of operators to the case of relations, recall the following. For a closed relation A the adjoint A^* is automatically a closed linear relation and $\operatorname{ran}(A-\lambda)$ is closed if and only if $\operatorname{ran}(A^*-\bar\lambda)$ is closed. More generally, λ is a regular point of A if and only if $\bar\lambda$ is a regular point of A^*. In other words, there is a complete symmetry in the results for a closed linear relation A and its adjoint A^*.

The paper is organized as follows. Section 2 contains a short introduction to relations in Hilbert spaces. In particular, the notions of operator part, minimum modulus, and generalized resolvent are introduced. Furthermore, there is a brief review of the opening and gap between closed linear subspaces of a Hilbert space, which play a fundamental role in the later arguments. Finally, a useful estimate for the gap between eigenspaces of A in terms of the minimum modulus is given. Section 3 contains the characterization of points in $\operatorname{reg}(A)$ in terms of a gap estimate. In Section 4 it is shown that the set $\operatorname{reg}(A)$ is open and that various spaces are continuous on $\operatorname{reg}(A)$ in terms of the gap metric. In Sections 5 there is a characterization of $\operatorname{reg}(A)$ in terms of *generalized resolvents* of A. For the convenience of the reader Section 6 returns to the notions of the opening and gap between closed linear subspaces of a Hilbert space. The various connections for gaps are illustrated.

2. Preliminaries

In this section some basic material is presented concerning linear relations in Hilbert spaces, operator parts of linear relations, and of the minimum modulus of relations. Estimates for the distance between eigenspaces of a linear relation will be presented in terms of the gap or opening between such subspaces.

2.1. **Relations, minimum moduli, and operator parts.** Let A be a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$; i.e., A is a closed linear subspace of the product space $\mathfrak{H}\times\mathfrak{K}$. Then A is the graph of a linear operator if and only if $\operatorname{mul} A=\{0\}$. Here $\operatorname{mul} A$ stands for the multivalued part of A; since A is closed, it is automatically closed. The *orthogonal operator part* $A_{\rm s}$ of A is defined by

$$A_{\rm s}=\{\,\{f,g\}:\,\{f,g\}\in A,\ (I-Q)g=0\,\}=A\cap(\mathfrak{H}\oplus(\operatorname{mul} A)^{\perp}),$$

where Q be the orthogonal projection from $\mathfrak{K}$ onto $(\operatorname{mul} A)^{\perp}$. In the sense of relations one then has $A_{\rm s}=QA$. Clearly A_s is a closed operator contained in A. Note that

$$\operatorname{dom} A \quad \text{closed} \quad \Leftrightarrow \quad A_{\rm s} \quad \text{bounded}.$$

The adjoint A^* of A is a closed linear relation from $\mathfrak{K}$ to $\mathfrak{H}$, defined by

$$A^*=\{\,\{f,f'\}\in\mathfrak{K}\times\mathfrak{H}:\,(f',h)=(f,h'),\ \{h,h'\}\in A\,\}.$$

The orthogonal operator part $(A^*)_{\rm s}$ of A^* is defined as above. Then $A_{\rm s}$ is a densely defined operator from the Hilbert space $\overline{\operatorname{dom}}\,A$ to the Hilbert space $\overline{\operatorname{dom}}\,A^*$. Likewise $(A^*)_{\rm s}$ is a densely defined operator from $\overline{\operatorname{dom}}\,A^*$ to $\overline{\operatorname{dom}}\,A$. It is clear that

$$(A_{\rm s})^{\times}=(A^*)_{\rm s},$$

where $A^\times$ denotes the adjoint of the densely defined operator $A_{\rm s}$ (as defined between $\overline{\rm dom}\, A$ and $\overline{\rm dom}\, A^*$). It is obvious that $A_{\rm s}$ is bounded if and only if $(A^*)_{\rm s}$ is bounded, and in this case

$$\|A_{\rm s}\| = \|(A^*)_{\rm s}\|, \tag{2.1}$$

which follows from the usual identity $\|A_{\rm s}\| = \|(A_{\rm s})^\times\|$. Equivalently one has

$$\operatorname{dom} A \quad \text{closed} \quad \Leftrightarrow \quad \operatorname{dom} A^* \quad \text{closed}. \tag{2.2}$$

For different proofs of this equivalence, see [9].

Let A be a closed relation from $\mathfrak{H}$ to $\mathfrak{K}$. Then the *minimum modulus* of A is defined by

$$r(A) = \inf \left\{ \frac{\|h'\|}{\|h\|} : \{h, h'\} \in A,\ h \perp \ker\, A \right\}.$$

This number belongs to $[0, \infty]$. Note that $r(A) > 0$ if and only if $(A^{-1})_{\rm s}$ is bounded, in which case

$$r(A) = \frac{1}{\|(A^{-1})_{\rm s}\|},$$

cf. [6]. Moreover, it is clear from (2.1) that $r(A) = r(A^*)$, and that

$$\operatorname{ran} A \quad \text{closed} \quad \Leftrightarrow \quad \operatorname{ran} A^* \quad \text{closed},$$

which of course is also clear from (2.2) by going over to inverses.

The multivalued part $\operatorname{mul} A$ is a closed linear subspace of $\mathfrak{H}$ which induces the following closed restriction of A:

$$A_{\rm mul} = \{0\} \times \operatorname{mul} A.$$

An *operator part* B of A is an operator from $\mathfrak{H}$ to $\mathfrak{K}$ which satisfies

$$A = B \mathbin{\widehat{+}} A_\infty, \quad \text{direct sum},$$

where $\widehat{+}$ stands for a componentwise sum. The orthogonal operator part $A_{\rm s}$ of A is an example of an operator part. Note that $A_{\rm s}$ and A are related by $A_{\rm s} = (I - Q)A$, where the product is in the sense of relations. The orthogonal operator part is based on the orthogonal decomposition $\mathfrak{K} = (\operatorname{mul} A)^\perp \oplus \operatorname{mul} A$. For a different approach to operator parts, see [9]. Now consider a closed linear subspace $\mathfrak{X}$ of $\mathfrak{K}$, such that

$$\mathfrak{K} = \mathfrak{X} + \operatorname{mul} A, \quad \text{direct sum}, \tag{2.3}$$

and let $Q_{\mathfrak{X}}$ be the projection onto $\mathfrak{X}$ parallel to $\operatorname{mul} A$.

Lemma 2.1. *The relation $A_{\mathfrak{X}}$ defined by*

$$A_{\mathfrak{X}} = \{\, \{f, g\} : \{f, g\} \in A,\ g \in \mathfrak{X} \,\} = A \cap (\mathfrak{H} \oplus \mathfrak{X}) \tag{2.4}$$

is a closed operator part of A and $A_{\mathfrak{X}} = Q_{\mathfrak{X}} A$, so that

$$A = A_{\mathfrak{X}} \mathbin{\widehat{+}} A_{\rm mul}, \quad \textit{direct sum}. \tag{2.5}$$

Moreover, $A_{\mathfrak{X}}$ is bounded if and only if $\operatorname{dom} A$ *is closed.*

Proof. The identity (2.4) shows that $A_{\mathfrak{X}}$ is closed. Furthermore $A_{\mathfrak{X}} \subset A$ and $A_{\rm mul} \subset A$ show that $A_{\mathfrak{X}} \mathbin{\widehat{+}} A_{\rm mul} \subset A$. For the converse inclusion $\{h, h'\} \in A$. Then according to (2.3) $h' = k + \varphi$ with $k \in \mathfrak{X}$ and $\varphi \in \operatorname{mul} A$, so that

$$\{h, h'\} = \{h, k\} + \{0, \varphi\}.$$

This shows that $\{h,k\} \in A$, since $\{0,\varphi\} \in A_{\rm mul} \subset A$. Hence $\{h,k\} \in A_{\mathfrak{X}}$ and thus $A \subset A_{\mathfrak{X}} \widehat{+} A_{\rm mul}$. To see that $A_{\mathfrak{X}}$ is an operator, let $\{0,k\} \in A_{\mathfrak{X}}$, so that $k \in \mathfrak{X} \cap {\rm mul}\, A$ and $k=0$; cf. (2.3). Finally $A_{\mathfrak{X}} = Q_{\mathfrak{X}} A$ is straightforward. □

2.2. **Generalized resolvents.** Let A be a closed relation in a Hilbert space $\mathfrak{H}$ and let $\lambda \in \mathbb{C}$. Then the formal inverse $(A-\lambda)^{-1}$ is a closed relation in $\mathfrak{H}$ defined by

$$(A-\lambda)^{-1} = \{\, \{h'-\lambda h, h\} : \{h,h'\} \in A \,\}.$$

Clearly ${\rm mul}\,(A-\lambda)^{-1} = \ker\,(A-\lambda)$ and the orthogonal operator part $((A-\lambda)^{-1})_s$ of $(A-\lambda)^{-1}$ is given by

$$((A-\lambda)^{-1})_{\rm s} = \{\, \{h'-\lambda h, h\} : \{h,h'\} \in A, h \perp \ker\,(A-\lambda) \,\}.$$

The *minimum modulus* of $A-\lambda$ is given by

$$r(A-\lambda) = \inf \left\{ \frac{\|h'-\lambda h\|}{\|h\|} : \{h,h'\} \in A,\ h \perp \ker\,(A-\lambda),\ h \neq 0 \right\}. \tag{2.6}$$

Hence, ${\rm ran}\,(A-\lambda)$ is closed if and only if $r(A-\lambda) > 0$, and in this case

$$r(A-\lambda) = \frac{1}{\|((A-\lambda)^{-1})_{\rm s}\|}.$$

In order to associate an everywhere defined closed operator with $(A-\lambda)^{-1}$ some direct sum decompositions of the Hilbert space $\mathfrak{H}$ will be introduced.

Let $\mathfrak{X}(\lambda)$ be a closed linear subspace of $\mathfrak{H}$ such that

$$\mathfrak{H} = \mathfrak{X}(\lambda) + \ker\,(A-\lambda), \quad \text{direct sum.} \tag{2.7}$$

Note that the special choice $\mathfrak{X}(\lambda) = \overline{\rm ran}\,(A^*-\bar\lambda)$ corresponds to an orthogonal decomposition. Let Q_λ be the projection onto $\mathfrak{X}(\lambda)$ parallel to $\ker\,(A-\lambda)$. Clearly, $\ker\, Q_\lambda = \ker\,(A-\lambda)$ and Q_λ maps ${\rm dom}\, A$ into itself. The relation $Q_\lambda(A-\lambda)^{-1}$ corresponding to the decomposition (2.7) is a closed operator and it satisfies

$$Q_\lambda (A-\lambda)^{-1}(k-\lambda h) = Q_\lambda h, \quad \{h,k\} \in A. \tag{2.8}$$

Moreover, parallel to (2.5) one has the direct sum decomposition

$$(A-\lambda)^{-1} = Q_\lambda(A-\lambda)^{-1} \,\widehat{+}\, (\{0\} \times \ker\,(A-\lambda)), \quad \text{direct sum.} \tag{2.9}$$

Hence if $r(A-\lambda) > 0$ or, equivalently, if ${\rm ran}\,(A-\lambda)$ is closed, then $Q_\lambda(A-\lambda)^{-1}$ is a bounded operator; cf. Lemma 2.1.

Now assume that $r(A-\lambda) > 0$ or, equivalently, that ${\rm ran}\,(A-\lambda)$ is closed. Let $\mathfrak{Y}(\lambda)$ be a closed linear subspace of $\mathfrak{H}$ for which

$$\mathfrak{H} = \mathfrak{Y}(\lambda) + {\rm ran}\,(A-\lambda), \quad \text{direct sum.} \tag{2.10}$$

Note that the special choice $\mathfrak{Y}(\lambda) = \ker\,(A^*-\bar\lambda)$ corresponds to an orthogonal decomposition. Let P_λ be the projection onto ${\rm ran}\,(A-\lambda)$ parallel to $\mathfrak{Y}(\lambda)$. Clearly $\ker\, P_\lambda = \mathfrak{Y}(\lambda)$.

Corresponding to the direct sum decompositions (2.7) and (2.10) the operator $\mathcal{R}(\lambda)$ is defined by

$$\mathcal{R}(\lambda) = Q_\lambda(A-\lambda)^{-1}P_\lambda.$$

Clearly, it belongs to $\mathbf{B}(\mathfrak{H})$, the Hilbert space of all bounded linear operators defined on all of $\mathfrak{H}$. Note that if $\lambda \in \rho(A)$, then ${\rm ran}\,(A-\lambda) = \mathfrak{H}$ and $\ker\,(A-\lambda) = \{0\}$, and $\mathcal{R}(\lambda)$ coincides with the usual resolvent. For $\lambda \in \mathbb{C}$ the following notation is useful:

$$\mathfrak{N}_\lambda(A) = \ker\,(A-\lambda), \quad \widehat{\mathfrak{N}}_\lambda(A) = \{\, \{h,\lambda h\} : h \in \mathfrak{N}_\lambda(A) \,\}.$$

Lemma 2.2. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$, let $\lambda \in \mathbb{C}$, and assume that* $\operatorname{ran}(A-\lambda)$ *is closed. Then*

$$A = \{\, \{\mathcal{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathcal{R}(\lambda)\varphi\} : \varphi \in \mathfrak{H} \,\} \mathbin{\widehat{+}} \widehat{\mathfrak{N}}_\lambda(A), \quad \textit{direct sum.} \tag{2.11}$$

Moreover,

$$\operatorname{mul} A = \{\, P_\lambda\varphi : \mathcal{R}(\lambda)\varphi \in \ker\,(A-\lambda) \,\}. \tag{2.12}$$

Proof. The proof will be given in several steps.

Step 1. First it will be shown that the righthand side of (2.11) belongs to A or, equivalently, it will be shown that

$$\{\mathcal{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathcal{R}(\lambda)\varphi\} \in A. \tag{2.13}$$

Clearly, $P_\lambda\varphi = k - \lambda h$ for some $\{h,k\} \in A$. With the projection P_λ (2.8) reads as

$$Q_\lambda(A-\lambda)^{-1}P_\lambda(k-\lambda h) = Q_\lambda h, \quad \{h,k\} \in A.$$

Next observe that $\ker Q_\lambda = \ker\,(A-\lambda)$ implies

$$\{0, (I-Q_\lambda)h\} \in (A-\lambda)^{-1}.$$

Together with $\{k-\lambda h, h\} \in (A-\lambda)^{-1}$, this gives

$$\{k-\lambda h, Q_\lambda h\} \in (A-\lambda)^{-1}$$

But with (2.8) this shows

$$\{k-\lambda h, Q_\lambda(A-\lambda)^{-1}P_\lambda(k-\lambda h\} \in (A-\lambda)^{-1}$$

or, equivalently,

$$\{P_\lambda\varphi, \mathcal{R}(\lambda)\varphi\} \in (A-\lambda)^{-1}, \quad \varphi \in \mathfrak{H},$$

which is equivalent to (2.13).

Step 2. Next it will be shown that

$$A \subset \{\, \{\mathcal{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathcal{R}(\lambda)\varphi\} : \varphi \in \mathfrak{H} \,\} \mathbin{\widehat{+}} \mathfrak{N}_\lambda(A). \tag{2.14}$$

Let $\{h,k\} \in A$. Then $\ker Q_\lambda = \ker\,(A-\lambda)$ implies that

$$\{h,k\} - \{(I-Q_\lambda)h, \lambda(I-Q_\lambda)h\} = \{Q_\lambda h, k-\lambda h + \lambda Q_\lambda h\} \in A,$$

and observe that with $\varphi = k - \lambda h$

$$\{Q_\lambda h, k-\lambda h+\lambda Q_\lambda h\} = \{\mathcal{R}(\lambda)\varphi, P_\lambda\varphi + \lambda\mathcal{R}(\lambda)\varphi\}.$$

Step 3. Write (2.11) as

$$A = \{\, \{\mathcal{R}(\lambda)\varphi + h, \lambda(\mathcal{R}(\lambda)\varphi + h) + P_\lambda\varphi\} : \varphi \in \mathfrak{H},\, h \in \ker\,(A-\lambda) \,\}.$$

Then it is clear that $P_\lambda\varphi \in \operatorname{mul} A$ if and only $\mathcal{R}(\lambda)\varphi + h = 0$. This completes the proof of (2.12). □

Corollary 2.3. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$, let $\lambda \in \mathbb{C}$, and assume that* $\operatorname{ran}(A-\lambda)$ *is closed. Let P be the orthogonal projection onto* $\operatorname{mul} A$. *Then the orthogonal operator part $A_{\rm s}$ acts as follows:*

$$(A_{\rm s}-\lambda)\mathcal{R}(\lambda) = (I-P)P_\lambda - \lambda P\mathcal{R}(\lambda). \tag{2.15}$$

and

$$\mathcal{R}(\lambda)(A_{\rm s}-\lambda)h = Q_\lambda h, \quad h \in \operatorname{dom} A. \tag{2.16}$$

2.3. **Special properties of generalized resolvents.** Assume that there is an open set $\mathcal{U} \subset \mathbb{C}$, such that for all $\lambda \in \mathcal{U}$ the subspace $\operatorname{ran}(A-\lambda)$ is closed. Furthermore, assume that for all $\lambda \in \mathcal{U}$ there exist closed linear subspaces $\mathfrak{X}$ and $\mathfrak{Y}$ of $\mathfrak{H}$, such that the following decompositions hold:

$$\mathfrak{H} = \ker\,(A-\lambda) + \mathfrak{X}, \quad \text{direct sum}, \tag{2.17}$$

and

$$\mathfrak{H} = \operatorname{ran}(A-\lambda) + \mathfrak{Y}, \quad \text{direct sum}. \tag{2.18}$$

In other words, it is assumed that the closed linear subspaces $\mathfrak{X}(\lambda)$ and $\mathfrak{Y}(\lambda)$ in (2.7) and (2.10) are independent of $\lambda \in \mathcal{U}$. The projection Q_λ onto $\mathfrak{X}$ parallel to $\ker\,(A-\lambda)$ and the projection P_λ onto $\operatorname{ran}(A-\lambda)$ parallel to $\mathfrak{Y}$ then satisfy some special properties.

Lemma 2.4. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Assume that for all λ in an open set $\mathcal{U}$ $\operatorname{ran}(A-\lambda)$ is closed and that the direct sum decompositions* (2.17) *and* (2.18) *hold. Then for all $\lambda, \mu \in \mathcal{U}$ one has*

$$Q_\lambda Q_\mu = Q_\mu, \quad P_\mu P_\lambda = P_\mu.$$

Proof. Let $h \in \mathfrak{H}$, then

$$h = (I-Q_\lambda)h + Q_\lambda h = (I-Q_\mu)h + Q_\mu h,$$

so that

$$(I-Q_\mu)h = (I-Q_\lambda)h + [Q_\lambda h - Q_\mu h] \in \ker\,(A-\lambda) + \mathfrak{X}.$$

Hence

$$(I-Q_\lambda)(I-Q_\mu) = I - Q_\lambda,$$

which leads to $Q_\lambda Q_\mu = Q_\mu$. Similarly, for $h \in \mathfrak{H}$,

$$h = P_\lambda h + (I-P_\lambda)h = P_\mu h + (I-P_\mu)h,$$

so that

$$P_\lambda h = P_\mu h + [(I-P_\mu)h - (I-P_\lambda)h] \in \operatorname{ran}(A-\mu) + \mathfrak{Y},$$

which leads to $P_\mu P_\lambda = P_\mu$. □

Lemma 2.5. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Assume that for all λ in an open set $\mathcal{U}$ $\operatorname{ran}(A-\lambda)$ is closed and that the direct sum decompositions* (2.17) *and* (2.18) *hold. Then for all $\lambda, \mu \in \mathcal{U}$, $\lambda \neq \mu$, one has*

$$\mathcal{R}(\lambda) - \mathcal{R}(\mu) = (\lambda-\mu)\mathcal{R}(\lambda)\mathcal{R}(\mu). \tag{2.19}$$

Proof. Recall that $\{\mathcal{R}(\mu)u, P_\mu u + \mu\mathcal{R}(\mu)u\} \in A$. Furthermore it follows from (2.8) that

$$\mathcal{R}(\lambda)(k - \lambda h) = Q_\lambda h, \quad \{h,k\} \in A.$$

A combination leads to

$$\mathcal{R}(\lambda)(P_\mu u + \mu\mathcal{R}(\mu)u - \lambda\mathcal{R}(\mu)u) = Q_\lambda \mathcal{R}(\mu)u,$$

or, equivalently,

$$(\lambda-\mu)\mathcal{R}(\lambda)\mathcal{R}(\mu) = \mathcal{R}(\lambda)P_\mu - Q_\lambda\mathcal{R}(\mu) = \mathcal{R}(\lambda) - \mathcal{R}(\mu),$$

which follows from Lemma 2.4. □

2.4. **The opening between subspaces.** This subsection contains a collection of results concerning the various openings between closed linear subspaces of a Hilbert space. A further discussion can be found in Section 6.

Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of $\mathfrak{H}$. Denote the corresponding orthogonal projections by $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$. Define the *opening* $\delta(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ by

$$\delta(\mathfrak{M},\mathfrak{N}) = \|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}\|, \tag{2.20}$$

so that clearly $0 \le \delta(\mathfrak{M},\mathfrak{N}) \le 1$ and also

$$\delta(\mathfrak{N}^\perp,\mathfrak{M}^\perp) = \delta(\mathfrak{M},\mathfrak{N}). \tag{2.21}$$

Moreover, observe that

$$\delta(\mathfrak{M},\mathfrak{N}) < 1 \quad \Leftrightarrow \quad \mathfrak{M}+\mathfrak{N}^\perp \text{ closed}, \quad \mathfrak{M}\cap\mathfrak{N}^\perp = \{0\}. \tag{2.22}$$

The *opening* $\varepsilon(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined by

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \|(I-P_{\mathfrak{N}})P_{\mathfrak{M}\ominus(\mathfrak{M}\cap\mathfrak{N}^\perp)}\|. \tag{2.23}$$

This leads to $\varepsilon(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{M}\ominus(\mathfrak{M}\cap\mathfrak{N}^\perp),\mathfrak{N})$ and also to

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \varepsilon(\mathfrak{N},\mathfrak{M}), \quad \varepsilon(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = \varepsilon(\mathfrak{M},\mathfrak{N}). \tag{2.24}$$

Due to the symmetry in (2.24) it follows that

$$\varepsilon(\mathfrak{M},\mathfrak{N}) < 1 \quad \Leftrightarrow \quad \mathfrak{M}+\mathfrak{N}^\perp \text{ closed} \quad \Leftrightarrow \quad \mathfrak{M}^\perp+\mathfrak{N} \text{ closed}. \tag{2.25}$$

The *gap* $g(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined by

$$g(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}} - P_{\mathfrak{N}}\|, \tag{2.26}$$

so that $g(\mathfrak{M},\mathfrak{N}) \le 1$. Moreover, it is clear that

$$g(\mathfrak{M},\mathfrak{N}) = g(\mathfrak{N},\mathfrak{M}), \quad g(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = g(\mathfrak{M},\mathfrak{N}). \tag{2.27}$$

The gap in (2.26) provides a metric on the space $\mathbf{S}(\mathfrak{H})$ of all closed linear subspaces of $\mathfrak{H}$.

Recall that for any pair of not necessarily orthogonal projections $Q_{\mathfrak{M}}$ and $Q_{\mathfrak{N}}$ in $\mathbf{B}(\mathfrak{H})$ such that $\operatorname{ran} Q_{\mathfrak{M}} = \mathfrak{M}$ and $\operatorname{ran} Q_{\mathfrak{N}} = \mathfrak{N}$ one has

$$g(\mathfrak{M},\mathfrak{N}) \le \|Q_{\mathfrak{M}} - Q_{\mathfrak{N}}\|. \tag{2.28}$$

There is a chain of (in)equalities satisfied by the gap and the openings between the subspaces $\mathfrak{M}$ and $\mathfrak{N}$:

$$\varepsilon(\mathfrak{M},\mathfrak{N}) \le \min(\delta(\mathfrak{M},\mathfrak{N}),\delta(\mathfrak{N},\mathfrak{M})) \le \max(\delta(\mathfrak{M},\mathfrak{N}),\delta(\mathfrak{N},\mathfrak{M})) = g(\mathfrak{M},\mathfrak{N}). \tag{2.29}$$

Observe that

$$g(\mathfrak{M},\mathfrak{N}) < 1 \;\Leftrightarrow\; \mathfrak{H} = \mathfrak{M}+\mathfrak{N}^\perp \text{ direct sum} \;\Leftrightarrow\; \mathfrak{H} = \mathfrak{M}^\perp+\mathfrak{N}, \text{ direct sum}. \tag{2.30}$$

If $\mathfrak{M}\cap\mathfrak{N}^\perp = \{0\}$ and $\mathfrak{M}^\perp\cap\mathfrak{N} = \{0\}$, then

$$\varepsilon(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{M},\mathfrak{N}) = \delta(\mathfrak{N},\mathfrak{M}) = g(\mathfrak{M},\mathfrak{N}). \tag{2.31}$$

Hence, if $\delta(\mathfrak{M},\mathfrak{N}) < 1$ and $\delta(\mathfrak{M}^\perp,\mathfrak{N}^\perp) < 1$, and, in particular, if $g(\mathfrak{M},\mathfrak{N}) < 1$, then the identities in(2.31) are satisfied.

Finally, note that if $\mathfrak{H} = \mathfrak{M}+\mathfrak{N}^\perp$, $\mathfrak{M}\cap\mathfrak{N}^\perp = \{0\}$, and P is the projection onto $\mathfrak{M}$ parallel to $\mathfrak{N}^\perp$, then

$$\|P\| = \frac{1}{\sqrt{1-g(\mathfrak{M},\mathfrak{N})^2}}, \tag{2.32}$$

cf. Corollary 6.11. Furthermore, if $g(\mathfrak{M},\mathfrak{N})<1$, then

$$\dim \mathfrak{M} = \dim \mathfrak{N},$$

see [8] or [10].

2.5. **Minimum modulus, opening, and gap.** Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then $\operatorname{ran}(A-\lambda_0)$ is closed if and only if $r(A-\lambda_0)>0$. In this case the disk $|\lambda-\lambda_0|<r(A-\lambda_0)$ will play a role in estimating openings and gaps of closed linear subspaces associated with A.

Lemma 2.6. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Assume that $r(A-\lambda_0)>0$ for some $\lambda_0\in\mathbb{C}$. Then for all $\lambda\in\mathbb{C}$:*

$$\delta(\ker\,(A-\lambda),\ker\,(A-\lambda_0)) \le \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}. \tag{2.33}$$

Proof. The proof will be given in two steps.

Step 1. It will be shown that for all $\lambda\in\mathbb{C}$:

$$\|(I-P_{\ker\,(A-\lambda_0)})h\| \le \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}\,\|h\|, \quad h\in\ker\,(A-\lambda). \tag{2.34}$$

To see this, let $h\in\ker\,(A-\lambda)$, so that $\{h,\lambda h\}\in A$. Decompose the element h as follows:

$$h=h_0+h_1, \quad h_0\in\ker\,(A-\lambda_0), \quad h_1\in\ker\,(A-\lambda_0)^{\perp}.$$

Then $\{h_0,\lambda h_0\}\in A$ and clearly

$$\{h_1,\lambda h-\lambda_0 h_0\}=\{h,\lambda h\}-\{h_0,\lambda_0 h_0\}\in A, \quad h_1\perp\ker\,(A-\lambda_0).$$

Since $\lambda h-\lambda_0 h_0-\lambda_0 h_1=(\lambda-\lambda_0)h$, it follows from (2.6) (with λ_0 instead of λ) that

$$r(A-\lambda_0)\le \frac{|\lambda-\lambda_0|\,\|h\|}{\|h_1\|}.$$

Observe that $h_1=(I-P_{\ker\,(A-\lambda_0)})h$, where $P_{\ker\,(A-\lambda_0)}$ is the orthogonal projection onto $\ker\,(A-\lambda_0)$. Hence (2.34) follows.

Step 2. Apply (2.34) with $h=P_{\ker\,(A-\lambda)}\varphi$, $\varphi\in\mathfrak{H}$, where $P_{\ker\,(A-\lambda)}$ stands for the orthogonal projection onto $\ker\,(A-\lambda)$. This gives

$$\begin{aligned}\|(I-P_{\ker\,(A-\lambda_0)})P_{\ker\,(A-\lambda)}\varphi\| &\le \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}\,\|P_{\ker\,(A-\lambda)}\varphi\|\\ &\le \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}\,\|\varphi\|, \quad \varphi\in\mathfrak{H}.\end{aligned}$$

Now apply the definition in (2.20) to obtain (2.33). □

Let A be a closed linear relation and assume that $r(A-\lambda_0)>0$ for some $\lambda_0\in\mathbb{C}$. Then it follows from Lemma 2.6 and (2.22) that there is a direct sum decomposition:

$$\mathfrak{H}=\ker\,(A-\lambda)+(\ker\,(A-\lambda_0))^{\perp}, \quad \text{direct sum}, \tag{2.35}$$

valid for each $\lambda\in\mathbb{C}$ with $|\lambda-\lambda_0|<r(A-\lambda_0)$ This observation will be used to prove the following lemma.

Lemma 2.7. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and assume that $r(A-\lambda_0)>0$. Then for all $\lambda \in \mathbb{C}$ which satisfy $|\lambda-\lambda_0|<r(A-\lambda_0)$:*

$$r(A-\lambda) \geq r(A-\lambda_0) - |\lambda-\lambda_0| > 0, \tag{2.36}$$

so that, in particular, $\operatorname{ran}(A-\lambda)$ *is closed.*

Proof. It follows from (2.36) that $\operatorname{ran}(A-\lambda)$ is closed for $|\lambda-\lambda_0|<r(A-\lambda_0)$. Hence, it suffices to show (2.36).

To show (2.36), let $\{h,k\} \in A-\lambda$ and assume without loss of generality that $h \in \ker\,(A-\lambda)^\perp$. Then, according to the direct sum decomposition (2.35) one has

$$h = h_1 + h_2, \quad h_1 \in \ker\,(A-\lambda), \quad h_2 \in \ker\,(A-\lambda_0)^\perp,$$

and since $h \in \ker\,(A-\lambda)^\perp$ it follows for $h_2 = h - h_1$ that

$$\|h_2\|^2 = \|h\|^2 + \|h_1\|^2 \geq \|h\|^2. \tag{2.37}$$

Due to $\{h,k\} \in A-\lambda$ and $\{h_1,0\} \in A-\lambda$ it follows that

$$\{h_2,k\} \in A-\lambda \quad \text{or} \quad \{h_2, k+\lambda h_2\} \in A.$$

Hence, one sees that

$$\{h_2, k+(\lambda-\lambda_0)h_2\} \in A-\lambda_0, \quad h_2 \in (\ker\,(A-\lambda_0))^\perp,$$

so that from (2.6) (with λ_0 instead of λ) it follows that

$$\|k+(\lambda-\lambda_0)h_2\| \geq r(A-\lambda_0)\|h_2\|. \tag{2.38}$$

Therefore, via the triangle inequality, (2.37), and (2.38) one obtains

$$\begin{aligned} \|k\| &\geq \big|\|k+(\lambda-\lambda_0)h_2\| - |\lambda-\lambda_0|\,\|h_2\|\big| \\ &\geq (r(A-\lambda_0) - |\lambda-\lambda_0|)\,\|h_2\| \\ &\geq (r(A-\lambda_0) - |\lambda-\lambda_0|)\,\|h\|, \end{aligned} \tag{2.39}$$

where use has been made of $|\lambda-\lambda_0|<r(A-\lambda_0)$. Since the inequality (2.39) holds for all $\{h,k\} \in A-\lambda$ with $h \perp \ker\,(A-\lambda)$, it follows that (2.36) holds. □

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. It follows from Lemma 2.7 that the set

$$\operatorname{reg}_0(A) = \{\, \lambda \in \mathbb{C} : \operatorname{ran}(A-\lambda) \text{ is closed} \,\}$$

is open. Note that $\rho(A) \subset \gamma(A) \subset \operatorname{reg}_0(A)$.

Lemma 2.8. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and assume that $r(A-\lambda_0)>0$. Then for all $\lambda \in \mathbb{C}$ which satisfy $|\lambda-\lambda_0|<r(A-\lambda_0)$:*

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) \leq \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)} \ (<1). \tag{2.40}$$

Proof. Assume that $\lambda \in \mathbb{C}$ satisfies $|\lambda-\lambda_0|<r(A-\lambda_0)$. The proof will be given in three steps.

Step 1. It will be shown that

$$\operatorname{ran}(A-\lambda) \cap \operatorname{ran}(A-\lambda_0)^\perp = \{0\}. \tag{2.41}$$

To see this, let $k \in \operatorname{ran}(A-\lambda) \cap \operatorname{ran}(A-\lambda_0)^\perp$. Since $k \in \operatorname{ran}(A-\lambda)$ there exists some $h \in \mathfrak{H}$ with $\{h,k\} \in A-\lambda$. Decompose h according to the direct sum decomposition (2.35) as

$$h = h_1 + h_2, \quad h_1 \in \ker\,(A-\lambda), \quad h_2 \in \ker\,(A-\lambda_0)^\perp.$$

Then $\{h_2, k\} = \{h, k\} - \{h_1, 0\} \in A - \lambda$, so that

$$\{h_2, k + (\lambda - \lambda_0)h_2\} \in A - \lambda_0, \quad h_2 \in (\ker\, (A - \lambda_0))^\perp. \tag{2.42}$$

Hence, if $h_2 \neq 0$ it follows from (2.42) and (2.6) (with λ_0 instead of λ) that

$$r(A - \lambda_0) \leq \frac{\|k + (\lambda - \lambda_0)h_2\|}{\|h_2\|}. \tag{2.43}$$

Recall that also $k \in (\mathrm{ran}\,(A - \lambda_0))^\perp$, so that

$$(k + (\lambda - \lambda_0)h_2, k) = 0 \quad \text{or} \quad (\lambda - \lambda_0)(h_2, k) = -\|k\|^2. \tag{2.44}$$

This implies that

$$(k + (\lambda - \lambda_0)h_2, k + (\lambda - \lambda_0)h_2) = |\lambda - \lambda_0|^2\|h_2\|^2 - \|k\|^2 \leq |\lambda - \lambda_0|^2\|h_2|^2. \tag{2.45}$$

If $h_2 \neq 0$, then (2.43) and (2.45) lead to

$$r(A - \lambda_0) \leq \frac{\|k + (\lambda - \lambda_0)h_2\|}{\|h_2\|} \leq |\lambda - \lambda_0|.$$

Since $|\lambda - \lambda_0| < r(A - \lambda_0)$, it follows that $h_2 = 0$, and thus $k = 0$ by (2.44). Hence (2.41) is valid.

Step 2. It is shown that

$$\delta(\mathrm{ran}\,(A - \lambda_0), \mathrm{ran}\,(A - \lambda)) \leq \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)} < 1. \tag{2.46}$$

To see this, let $k \in \mathrm{ran}\,(A - \lambda_0)$. Then $\{h, k\} \in A - \lambda_0$ for some $h \in \mathfrak{H}$ and one may choose $h \in (\ker\, (A - \lambda_0))^\perp$. Hence

$$\{h, k\} \in A - \lambda_0, \quad h \in (\ker\, (A - \lambda_0))^\perp,$$

so that, by (2.6) (with λ_0 instead of λ)

$$r(A - \lambda_0) \leq \frac{\|k\|}{\|h\|}. \tag{2.47}$$

From $k \in \mathrm{ran}\,(A - \lambda_0)$ and the inclusion $\{h, k + (\lambda_0 - \lambda)h\} \in A - \lambda$ it follows that

$$\begin{aligned}(I - P_{\mathrm{ran}\,(A-\lambda)})&P_{\mathrm{ran}\,(A-\lambda_0)}k \\ &= (I - P_{\mathrm{ran}\,(A-\lambda)})k \\ &= (\lambda - \lambda_0)(I - P_{\mathrm{ran}\,(A-\lambda)})h,\end{aligned}$$

where $P_{\mathrm{ran}\,(A-\lambda)}$ and $P_{\mathrm{ran}\,(A-\lambda_0)}$ are the orthogonal projections onto $\mathrm{ran}\,(A - \lambda)$ and $\mathrm{ran}\,(A - \lambda_0)$, respectively. Therefore, with (2.47), one obtains

$$\|(I - P_{\mathrm{ran}\,(A-\lambda)})P_{\mathrm{ran}\,(A-\lambda_0)}k\| \leq |\lambda - \lambda_0|\,\|h\| \leq \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)}\,\|k\|,$$

for all $k \in \mathrm{ran}\,(A - \lambda_0)$ and hence for all $k \in \mathfrak{H}$. This shows that (2.46) holds.

Step 3. Recall that Lemma 2.7 shows that $\mathrm{ran}\,(A - \lambda)$ is closed. It follows from (2.46) in Step 2 and (2.22) that

$$\mathrm{ran}\,(A - \lambda_0) \cap (\mathrm{ran}\,(A - \lambda))^\perp = \{0\}. \tag{2.48}$$

Then (2.48) and (2.41) in Step 1 together with (2.31) and (2.46) lead to (2.40). $\square$

3. Regular points and quasi-Fredholm relations of degree 0

A closed linear relation A with $\operatorname{ran} A$ closed and $\ker A^n \subset \operatorname{ran} A$, $n \in \mathbb{N}$, or, equivalently, with $\operatorname{ran} A$ closed and $\ker A \subset \operatorname{ran} A^n$, $n \in \mathbb{N}$, is said to be a *quasi-Fredholm relation of degree* 0; cf. (F1), (F2), and (F3), (F4). Such relations and, more generally, quasi-Fredholm relations of order d, $d \in \mathbb{N}$, have been studied in [13], building on the work presented for closed linear operators in [12].

Definition 3.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. A point $\lambda_0 \in \mathbb{C}$ is said to be a regular point of A if $A - \lambda_0$ is a quasi-Fredholm relation of degree* 0*. The set of regular points of A is denoted by* $\operatorname{reg}(A)$.

The following theorem is the basic result of this paper: it characterizes regular points of a closed linear relation.

Theorem 3.2. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$.*

(i) *Assume that $\lambda_0 \in \operatorname{reg}(A)$. Then $r(A-\lambda_0) > 0$ and for all $\lambda \in \mathbb{C}$ for which $|\lambda - \lambda_0| < r(A - \lambda_0)$:*

$$g(\ker (A - \lambda), \ker (A - \lambda_0)) < 1. \tag{3.1}$$

(ii) *Assume that $r(A-\lambda_0) > 0$ and that* (3.1) *holds for all λ in a neighborhood of λ_0. Then $\lambda_0 \in \operatorname{reg}(A)$.*

Proof. (i) Assume that $\lambda_0 \in \operatorname{reg}(A)$. Then $A-\lambda_0$ is quasi-Fredholm of degree 0; in other words, $\operatorname{ran}(A-\lambda_0)$ is closed and $\ker (A-\lambda_0)^n \subset \operatorname{ran}(A-\lambda_0)$, $n \in \mathbb{N}$; cf. (F1) and (F2). The condition that $\operatorname{ran}(A-\lambda_0)$ is closed is equivalent to $r(A-\lambda_0) > 0$. The rest of the proof will be given in two steps.

Step 1. It will be shown that for all $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < r(A - \lambda_0)$:

$$\mathfrak{H} = \ker (A - \lambda) + (\ker (A - \lambda_0))^{\perp}, \tag{3.2}$$

or, equivalently, that for all $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < r(A - \lambda_0)$:

$$\ker (A - \lambda_0) \subset \ker (A - \lambda) + \ker (A - \lambda_0)^{\perp}, \tag{3.3}$$

In order to show (3.3), let $h_0 \in \ker (A - \lambda_0)$ and assume $|\lambda - \lambda_0| < r(A - \lambda_0)$. Since $\ker (A - \lambda_0) \subset \operatorname{ran}(A - \lambda_0)$, there exists an element $h_1 \in \mathfrak{H}$ such that $\{h_1, u_0\} \in A - \lambda_0$. In fact, one may choose h_1 such that $h_1 \perp \ker (A - \lambda_0)$. Note that $\{h_1, 0\} \in (A - \lambda_0)^2$ and $h_1 \in \ker (A - \lambda_0)^2$. Thus

$$\{h_0, h_1\} \in A - \lambda_0, \quad h_1 \in \ker (A - \lambda_0)^2 \cap \ker (A - \lambda_0)^{\perp}.$$

Continuing by induction it is shown that there is a sequence $(h_j)_{j=0}^{\infty}$ such that for all $j \in \mathbb{N} \cup \{0\}$:

$$\{h_{j+1}, h_j\} \in A - \lambda_0, \quad h_j \in \ker (A - \lambda_0)^{j+1} \cap \ker (A - \lambda_0)^{\perp}. \tag{3.4}$$

It follows from (3.4) and (2.6) that $r(A - \lambda_0)\|h_{j+1}\| \leq \|h_j\|$, and therefore

$$\|h_j\| \leq \frac{\|u_0\|}{(r(A - \lambda_0))^j}, \quad j \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

Hence it follows from (3.4) and (3.5) that $|\lambda - \lambda_0| < r(A - \lambda_0)$ the sequence in

$$\left\{ \sum_{j=1}^{n} (\lambda - \lambda_0)^j h_j, -\sum_{j=1}^{n} (\lambda - \lambda_0)^j h_{j-1} \right\} \in A - \lambda_0, \tag{3.6}$$

converges for $n \to \infty$ to some element

$$\{\varphi, (\lambda - \lambda_0)(\varphi + h_0)\} \in \mathfrak{H} \times \mathfrak{H},$$

where the element φ is defined by the convergent series

$$\varphi = \sum_{j=1}^{\infty} (\lambda - \lambda_0)^j h_j \in \ker A - \lambda_0)^{\perp}, \tag{3.7}$$

cf. (3.4). Since A is closed, it follows that $\{\varphi, (\lambda - \lambda_0)(\varphi + h_0)\} \in A - \lambda_0$. Due to $\{h_0, 0\} \in A - \lambda_0$ it follows that

$$\{\varphi + h_0, (\lambda - \lambda_0)(\varphi + h_0)\} \in A - \lambda_0,$$

or, in other words,

$$\{\varphi + h_0, 0\} \in A - \lambda \quad \text{or} \quad \varphi + h_0 \in \ker (A - \lambda).$$

Recall that $\varphi \in \ker (A - \lambda_0)^{\perp}$, so that

$$h_0 \in \ker (A - \lambda) + \ker (A - \lambda_0)^{\perp}.$$

This proves (3.3).

Step 2. It will be shown that (3.1) holds. Since $r(A - \lambda_0) > 0$, it follows from Lemma 2.6 that with $|\lambda - \lambda_0| < r(A - \lambda_0)$:

$$\delta(\ker (A - \lambda), \ker (A - \lambda_0)) < 1. \tag{3.8}$$

Hence (3.8) together with (2.22) lead to

$$\ker (A - \lambda) \cap \ker (A - \lambda_0)^{\perp} = \{0\}. \tag{3.9}$$

Furthermore, it follows from (3.2) in Step 1 that

$$(\ker (A - \lambda))^{\perp} \cap \ker (A - \lambda_0) = \{0\}. \tag{3.10}$$

Hence (3.9) and (3.10) together with (2.30) and (2.31) show that (3.1) holds.

(ii) Assume that $r(A - \lambda_0) > 0$ and that (3.1) holds for all λ in a neighborhood $\mathcal{U}(\lambda_0)$ of λ_0. Then $\operatorname{ran}(A - \lambda_0)$ is closed and by (F3) and (F4) it suffices to show that

$$\ker (A - \lambda_0) \subset \operatorname{ran}(A - \lambda_0)^n, \quad n \in \mathbb{N}.$$

In fact, the following statement will be proved by induction:

$$\ker (A - \lambda_0) \subset \operatorname{ran}(A - \lambda_0)^n, \quad \operatorname{ran}(A - \lambda_0)^n \text{ is closed }, \quad n \in \mathbb{N}. \tag{3.11}$$

For $n = 0$ this is clearly satisfied. Assume that (3.11) is valid for some $n \in \mathbb{N}$. The argument will be given in two steps.

Step 1. It will be shown that $\operatorname{ran}(A - \lambda_0)^{n+1}$ is closed. Let $k \in \overline{\operatorname{ran}}\,(A - \lambda_0)^{n+1}$. Then there exist elements $k_j \in \operatorname{ran}(A - \lambda_0)^{n+1}$ such that $k_j \to k$ in $\mathfrak{H}$, and there are elements $h_j \in \mathfrak{H}$ such that

$$\{h_j, k_j\} \in (A - \lambda_0)^{n+1}.$$

Since $(A - \lambda_0)^{n+1} = (A - \lambda_0)(A - \lambda_0)^n$, there are elements $\chi_j \in \mathfrak{K}$ such that

$$\{h_j, \chi_j\} \in (A - \lambda_0)^n, \quad \{\chi_j, k_j\} \in A - \lambda_0. \tag{3.12}$$

Decompose these elements $\chi_j j$ by

$$\chi_j = \varphi_j + \psi_j, \quad \varphi_j \in \ker (A - \lambda_0), \quad \psi_j \in \ker (A - \lambda_0)^{\perp}. \tag{3.13}$$

Note that $\{\varphi_j, 0\} \in A - \lambda_0$, and it follows from (3.12) and (3.13) that

$$\{\psi_j, k_j\} = \{\chi_j, k_j\} - \{\varphi_j, 0\} \in A - \lambda_0, \quad \psi_j \perp \ker (A - \lambda_0). \tag{3.14}$$

Therefore, by (2.6), it follows from (3.14) that

$$\|\psi_j - \psi_l\| \le r(A-\lambda_0)\|k_j - k_l\|.$$

Hence, (ψ_j) is a Cauchy sequence, and thus $\psi_j \to \psi$ for some $\psi \in \mathfrak{H}$. Therefore $\{\psi_j, k_j\}$ is a sequence in $A-\lambda_0$ with the property

$$\{\psi_j, k_j\} \to \{\psi, k\} \in A - \lambda_0,$$

since A is closed.

Now recall that $\psi_j = \chi_j - \varphi_j$. Here by (3.12) one has $\chi_j \in \operatorname{ran}(A-\lambda_0)^n$ and by (3.13) and the induction hypothesis one has

$$\varphi_j \in \ker\,(A-\lambda_0) \quad \text{and} \quad \varphi_j \in \operatorname{ran}(A-\lambda_0)^n.$$

Therefore $\psi_j \in \operatorname{ran}(A-\lambda_0)^n$ and by the induction hypothesis that $\operatorname{ran}(A-\lambda_0)^n$ is closed, it follows that $\psi \in \operatorname{ran}(A-\lambda_0)^n$. Together with $\{\psi, k\} \in A-\lambda_0$ this shows that $k \in \operatorname{ran}(A-\lambda_0)^{n+1}$. Hence $\overline{\operatorname{ran}}\,(A-\lambda_0)^{n+1} \subset \operatorname{ran}(A-\lambda_0)^{n+1}$ and $\operatorname{ran}(A-\lambda_0)^{n+1}$ is closed.

Step 2. It will be shown that

$$\ker\,(A-\lambda_0) \subset \operatorname{ran}(A-\lambda_0)^{n+1}. \tag{3.15}$$

But first observe that

$$\ker\,(A-\lambda) \subset \operatorname{ran}(A-\lambda_0)^n \text{ for } \lambda \neq \lambda_0,\ n \in \mathbb{N}, \tag{3.16}$$

since $h \in \ker\,(A-\lambda)$ implies that $\{h, (\lambda-\lambda_0)h\} \in A-\lambda_0$.

Now let $h \in \ker\,(A-\lambda_0)$, so that $h = P_{\ker\,(A-\lambda_0)}h$, where $P_{\ker\,(A-\lambda_0)}$ stands for the orthogonal projection onto $\ker\,(A-\lambda_0)$. The induction hypothesis (3.11) gives

$$P_{\ker\,(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0)^{n+1},$$

and it follows that

$$h - (P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h = P_{\ker\,(A-\lambda)}h \in \operatorname{ran}(A-\lambda_0)^{n+1}. \tag{3.17}$$

By Step 1 the space $\operatorname{ran}(A-\lambda_0)^{n+1}$ is closed and let $P_{\operatorname{ran}(A-\lambda_0)^{n+1}}$ be the corresponding orthogonal projection. From (3.17) it follows that

$$\begin{aligned} &h - (P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h \\ &\qquad = P_{\operatorname{ran}(A-\lambda_0)^{n+1}}(h - (P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h), \end{aligned}$$

or, equivalently,

$$(I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})h = (I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})(P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h. \tag{3.18}$$

Hence, (3.18) and (2.26), give

$$\begin{aligned} \|(I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})h\| &\le \|(P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)})h\| \\ &\le \|P_{\ker\,(A-\lambda_0)} - P_{\ker\,(A-\lambda)}\|\,\|h\| \\ &= g(\ker\,(A-\lambda), \ker\,(A-\lambda_0))\,\|h\|. \end{aligned} \tag{3.19}$$

Due to (2.30) and (2.31) it follows that

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) = \delta(\ker\,(A-\lambda), \ker\,(A-\lambda_0)). \tag{3.20}$$

Therefore (3.19) and Lemma 2.6 give

$$\|(I - P_{\operatorname{ran}(A-\lambda_0)^{n+1}})h\| \le \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}\,\|h\| \tag{3.21}$$

for all $\lambda \in \mathcal{U}(\lambda_0)$ with $|\lambda - \lambda_0| < r(A - \lambda_0)$. The inequality (3.21) implies

$$h = P_{\mathrm{ran}\,(A-\lambda_0)^{n+1}} h \in \mathrm{ran}\,(A - \lambda_0)^{n+1}.$$

Hence (3.15) has been shown. □

The following corollary is a useful restatement of Theorem 3.2, which parallels Corollary 2.8.

Corollary 3.3. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then $\lambda_0 \in \mathrm{reg}\,(A)$ if and only if $r(A-\lambda_0) > 0$ and there exists a neighborhood $\mathcal{U}(\lambda_0)$ of λ_0 such that*

$$g(\ker\,(A - \lambda), \ker\,(A - \lambda_0)) < 1, \quad \lambda \in \mathcal{U}(\lambda_0). \tag{3.22}$$

The neighborhood $\mathcal{U}(\lambda_0)$ contains the disk

$$\{\, \lambda \in \mathbb{C} : |\lambda - \lambda_0| < r(A - \lambda_0) \,\}, \tag{3.23}$$

and on that disk

$$g(\ker\,(A - \lambda), \ker\,(A - \lambda_0)) \leq \frac{|\lambda - \lambda_0|}{r(A - \lambda_0)}. \tag{3.24}$$

The following result is a direct consequence of Theorem 3.2; see [13]. Due to the formal level of relations there is no need anymore to require A to be a densely defined operator, cf. [12, Corollaire 4.12].

Corollary 3.4. *Let A be a closed relation in a Hilbert space. Then*

$$\lambda \in \mathrm{reg}\,(A) \quad \Leftrightarrow \quad \bar{\lambda} \in \mathrm{reg}\,(A^*).$$

Note also that $\lambda \in \rho(A)$ if and only if $\bar{\lambda} \in \rho(A^*)$. However, when $\lambda \in \gamma(A)$, then, in general, $\bar{\lambda} \in \mathrm{reg}\,(A^*) \setminus \gamma(A^*)$.

Another direct consequence of Theorem 3.2 is the following result, cf. [12, Corollaire 4.11].

Corollary 3.5. *Let A be a closed relation in a Hilbert space for which*

$$\ker\,(A^* - \bar{\lambda}) = \ker\,(A - \lambda), \quad \lambda \in \mathbb{C}. \tag{3.25}$$

Then $\mathrm{reg}\,(A) = \rho(A)$.

Proof. Since $\rho(A) \subset \mathrm{reg}\,(A)$, it suffices to show $\mathrm{reg}\,(A) \subset \rho(A)$. Let $\lambda \in \mathrm{reg}\,(A)$, then (3.25) implies that

$$\ker\,(A^* - \bar{\lambda}) = \ker\,(A - \lambda) \subset \mathrm{ran}\,(A - \lambda) = \ker\,(A^* - \bar{\lambda})^{\perp},$$

so that $\ker\,(A^* - \bar{\lambda}) = \{0\}$. Hence $\ker\,(A - \lambda) = \{0\}$ and $\mathrm{ran}\,(A - \lambda) = \mathfrak{H}$, which shows that $\lambda \in \rho(A)$. □

4. Regular points and continuity

Let A be a closed relation for which $r(A - \lambda_0) > 0$. Then for all $\lambda \in \mathbb{C}$ which satisfy $|\lambda - \lambda_0| < r(A - \lambda_0)$ one has the direct sum decomposition

$$\mathfrak{H} = \mathrm{ran}\,(A - \lambda) + (\mathrm{ran}\,(A - \lambda_0))^{\perp}, \quad \text{direct sum}, \tag{4.1}$$

and, likewise,

$$\mathfrak{H} = \mathrm{ran}\,(A - \lambda)^{\perp} + \mathrm{ran}\,(A - \lambda_0), \quad \text{direct sum}, \tag{4.2}$$

as follows from Lemma 2.8. Now assume that $\lambda_0 \in \mathrm{reg}\,(A)$. Then for all λ which satisfy $|\lambda - \lambda_0| < r(A - \lambda_0)$ one has the direct sum decompositions

$$\mathfrak{H} = \ker\,(A-\lambda) + (\ker\,(A-\lambda_0))^{\perp}, \quad \text{direct sum.} \tag{4.3}$$

and, likewise

$$\mathfrak{H} = \ker\,(A-\lambda)^{\perp} + \ker\,(A-\lambda_0), \quad \text{direct sum,} \tag{4.4}$$

as follows from Corollary 3.3. The dependence on λ of the first summands in the direct sum decompositions (4.1), (4.2), (4.3), and (4.4) is studied in the following proposition.

Proposition 4.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$. Then the set* $\mathrm{reg}\,(A)$ *is open, and the mappings*

(i) $\lambda \in \mathrm{reg}\,(A) \to \ker\,(A-\lambda)$*;*
(ii) $\lambda \in \mathrm{reg}\,(A) \to (\ker\,(A-\lambda))^{\perp}$*;*
(iii) $\lambda \in \mathrm{reg}\,(A) \to \mathrm{ran}\,(A-\lambda)$*;*
(iv) $\lambda \in \mathrm{reg}\,(A) \to \mathrm{ran}\,((A-\lambda))^{\perp}$*,*

from $\mathrm{reg}\,(A)$ *into the space* $\mathbf{S}(\mathfrak{H})$ *of closed linear subspaces of* $\mathfrak{H}$ *provided with the gap-metric, are continuous.*

Proof. It suffices to show that the set $\mathrm{reg}\,(A)$ is open. It then follows from Lemma 2.8, Corollary 3.3, and (2.27) that the indicated mappings are continuous from $\mathrm{reg}\,(A)$ into the space $\mathbf{S}(\mathfrak{H})$, provided with the gap-metric.

In order to show that the set $\mathrm{reg}\,(A)$ is open, let $\lambda_0 \in \mathrm{reg}\,(A)$. Then let the point $\lambda \in \mathbb{C}$ satisfy

$$|\lambda - \lambda_0| < r(A-\lambda_0). \tag{4.5}$$

It has been shown in Lemma 2.7 that $\mathrm{ran}\,(A-\lambda)$ is closed. Now it will be shown that there exists a neighborhood $\mathcal{V}$ of λ so that for all μ in that neighborhood one has $g(\ker\,(A-\mu), \ker\,(A-\lambda)) < 1$; in other words that λ is also a regular point of the relation A.

Let $\lambda \in \mathbb{C}$ satisfy (4.5). The neighborhood $\mathcal{V}$ of λ is defined by

$$\mathcal{V} = \{\, \mu \in \mathbb{C} :\, 2|\mu-\lambda| < r(A-\lambda_0) - |\lambda-\lambda_0| \,\}. \tag{4.6}$$

For any $\mu \in \mathcal{V}$ it follows from the definition in (4.6) and the assumption (4.5) that

$$\begin{aligned} |\mu-\lambda_0| &\leq |\mu-\lambda| + |\lambda-\lambda_0| \\ &< (r(A-\lambda_0) - |\lambda-\lambda_0|)/2 + |\lambda-\lambda_0| \\ &= (r(A-\lambda_0) + |\lambda-\lambda_0|)/2. \end{aligned} \tag{4.7}$$

Due to (4.5), the inequality (4.7) shows that any $\mu \in \mathcal{V}$ also satisfies:

$$|\mu-\lambda_0| < r(A-\lambda_0), \tag{4.8}$$

Hence $\mathcal{V}$ is contained in the disk in (4.5). In particular, it follows from Lemma 2.7 that $\mathrm{ran}\,(A-\mu)$ is closed for all $\mu \in \mathcal{V}$.

For $\mu \in \mathcal{V}$ the definition in (4.6) and the inequality in (2.36) imply that

$$|\mu-\lambda| < 2|\mu-\lambda| \leq r(A-\lambda_0) - |\lambda-\lambda_0| \leq r(A-\lambda). \tag{4.9}$$

Since $\mathrm{ran}\,(A-\lambda)$ is closed, Lemma 2.6 may be applied, which gives with (4.9)

$$\delta(\ker\,(A-\mu), \ker\,(A-\lambda)) \leq \frac{|\mu-\lambda|}{r(A-\lambda)} < 1, \quad \mu \in \mathcal{V}. \tag{4.10}$$

Furthermore, (4.8) shows that (2.36) holds with λ replaced by μ:

$$r(A-\mu) \geq r(A-\lambda_0) - |\mu - \lambda_0| > 0, \tag{4.11}$$

Hence, in (4.11) an application of (4.7) and the definition of $\mathcal{V}$ lead to

$$\begin{aligned} r(A-\mu) &\geq r(A-\lambda_0) - |\mu-\lambda_0| \\ &> r(A-\lambda_0) - |\lambda-\lambda_0|)/2 \\ &> |\mu-\lambda|. \end{aligned} \tag{4.12}$$

Since $\operatorname{ran}(A-\mu)$ is closed, Lemma 2.6 may be applied, which gives with (4.12)

$$\delta(\ker\,(A-\lambda), \ker\,(A-\mu)) \leq \frac{|\lambda-\mu|}{r(A-\mu)} < 1, \quad \mu \in \mathcal{V}. \tag{4.13}$$

It follows from (4.10), (4.13), and (2.31) that $g(\ker\,(A-\lambda), \ker\,(A-\mu)) < 1$ for $\mu \in \mathcal{V}$. In particular, this leads to $\lambda \in \operatorname{reg}(A)$. Therefore it has been shown that $\operatorname{reg}(A)$ is open in $\mathbb{C}$. □

5. Regular points and generalized resolvents

Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ for which $\lambda_0 \in \operatorname{reg}(A)$. Then by Lemma 2.7, Lemma 2.8, and Corollary 3.3 there is a neighborhood

$$\mathcal{U} = \{\, \lambda \in \mathbb{C} : |\lambda-\lambda_0| < r(A-\lambda_0) \,\} \tag{5.1}$$

of λ_0, such that

$$r(A-\lambda) \geq r(A-\lambda_0) - |\lambda-\lambda_0| > 0, \quad \lambda \in \mathcal{U}, \tag{5.2}$$

so that $\operatorname{ran}(A-\lambda)$, $\lambda \in \mathcal{U}$, is closed,

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) \leq \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}, \quad \lambda \in \mathcal{U}, \tag{5.3}$$

and

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) \leq \frac{|\lambda-\lambda_0|}{r(A-\lambda_0)}, \quad \lambda \in \mathcal{U}. \tag{5.4}$$

Hence, by (5.3) and (5.4), one has for all $\lambda \in \mathcal{U}$ the direct sum decompositions

$$\mathfrak{H} = \ker\,(A-\lambda) + \ker\,(A-\lambda_0)^{\perp}, \quad \text{direct sum}, \tag{5.5}$$

and

$$\mathfrak{H} = \operatorname{ran}(A-\lambda) + \operatorname{ran}(A-\lambda_0)^{\perp}, \quad \text{direct sum}. \tag{5.6}$$

Let Q_λ be the projection onto $(\ker\,(A-\lambda_0))^{\perp}$ parallel to $\ker\,(A-\lambda)$ defined by (5.5), so that

$$\ker Q_\lambda = \ker\,(A-\lambda).$$

Likewise, let P_λ be the projection onto $\operatorname{ran}(A-\lambda)$ parallel to $\operatorname{ran}(A-\lambda_0)^{\perp}$ defined by (5.6), so that

$$\operatorname{ran} P_\lambda = \operatorname{ran}(A-\lambda).$$

With these projections define the generalized resolvent $\mathcal{R}(\lambda)$, $\lambda \in \mathcal{U}$:

$$\mathcal{R}(\lambda) = Q_\lambda (A-\lambda)^{-1} P_\lambda, \quad \lambda \in \mathcal{U}. \tag{5.7}$$

and recall that $\mathcal{R}(\lambda) \in \mathbf{B}(\mathfrak{H})$. Due to (5.5) and (5.6) the generalized resolvent $\mathcal{R}(\lambda)$ satisfies the resolvent identity (2.19).

In the following theorem the notion of graph norm will be used. Recall that a closed linear relation A in a Hilbert space $\mathfrak{H}$ induces a "graph norm" on $\operatorname{dom} A$:

$$\|h\|_{\mathfrak{D}}^2 = \|h\|^2 + \|A_{\rm s}h\|^2, \quad h \in \operatorname{dom} A,$$

so that the pair $(\operatorname{dom} A, \|\cdot\|_{\mathfrak{D}})$ is a Hilbert space. With these simple preparations the following theorem may be stated.

Theorem 5.1. *Let A be a closed linear relation in a Hilbert space $\mathfrak{H}$ and let $\lambda_0 \in \mathbb{C}$. Then the following statements are equivalent:*

(i) $\lambda_0 \in \operatorname{reg}(A)$*;*
(ii) *there is a generalized resolvent of A, holomorphic in a neighborhood of λ_0, in the sense of the graph norm.*

Proof. (i) ⇒ (ii) Let $\lambda_0 \in \operatorname{reg}(A)$ and let the neighborhood $\mathcal{U}$ of λ_0 be defined in (5.1). Define $\mathcal{R}(\lambda)$ as in (5.7), then by (2.15)

$$A_{\rm s}\mathcal{R}(\lambda) = (I-P)P_\lambda + \lambda(I-P)\mathcal{R}(\lambda).$$

Hence, for all $h \in \mathfrak{H}$ this leads to

$$\begin{aligned}\|\mathcal{R}(\lambda)h\|_{\mathfrak{D}}^2 &= \|A_s\mathcal{R}(\lambda)h\|^2 + \|\mathcal{R}(\lambda)h\|^2 \\ &= \|(I-P)P_\lambda h + \lambda(I-P)\mathcal{R}(\lambda)h\|^2 + \|\mathcal{R}(\lambda)h\|^2 \\ &\le 2\|P_\lambda h\|^2 + (2|\lambda|^2+1)\|\mathcal{R}(\lambda)h\|^2.\end{aligned} \tag{5.8}$$

Each of these terms will be estimated. First observe that

$$\{P_{\ker(A-\lambda)}\mathcal{R}(\lambda)h, 0\} \in A-\lambda,$$

and thus (2.11) leads to

$$\begin{aligned}&\{(I-P_{\ker(A-\lambda)})\mathcal{R}(\lambda)h, P_\lambda h\} \in A-\lambda, \\ &\qquad (I-P_{\ker(A-\lambda)})\mathcal{R}(\lambda)h \perp \ker(A-\lambda).\end{aligned} \tag{5.9}$$

It follows from (5.9) and (2.6) that

$$r(A-\lambda)\|(I-P_{\ker(A-\lambda)})\mathcal{R}(\lambda)h\| \le \|P_\lambda h\|, \quad u \in \mathfrak{H}. \tag{5.10}$$

Furthermore, it follows from the definition of Q_λ that for all $h \in \mathfrak{H}$:

$$\begin{aligned}\|P_{\ker(A-\lambda)}\mathcal{R}(\lambda)h\| &= \|P_{\ker(A-\lambda)}(I-P_{\ker(A-\lambda_0)})\mathcal{R}(\lambda)h\| \\ &\le \|P_{\ker(A-\lambda)}(I-P_{\ker(A-\lambda_0)})\|\,\|\mathcal{R}(\lambda)h\| \\ &\le g(\ker(A-\lambda), \ker(A-\lambda_0))\,\|\mathcal{R}(\lambda)h\|\end{aligned} \tag{5.11}$$

Note that (5.11) shows

$$\begin{aligned}&(1-g(\ker(A-\lambda), \ker(A-\lambda_0))^2)\|\mathcal{R}(\lambda)h\|^2 \\ &\qquad \le \|\mathcal{R}(\lambda)u\|^2 - \|P_{\ker(A-\lambda)}\mathcal{R}(\lambda)h\|^2 \\ &\qquad = \|(I-P_{\ker(A-\lambda)})\mathcal{R}(\lambda)h\|^2.\end{aligned} \tag{5.12}$$

Combine (5.10) and (5.12) to obtain

$$\|\mathcal{R}(\lambda)h\|^2 \le \frac{1}{(1-g(\ker(A-\lambda), \ker(A-\lambda_0))^2)(r(A-\lambda)^2)}\|P_\lambda h\|^2. \tag{5.13}$$

Recall that

$$\|P_\lambda\|^2 = \frac{1}{1-g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0))^2}, \tag{5.14}$$

as follows from (2.32).

Now choose $0 < c < r(A-\lambda_0)$ and consider a compact disk $\mathcal{U}_c$ of the form

$$\mathcal{U}_c = \{\, \lambda \in \mathcal{U} : |\lambda - \lambda_0| \leq c \,\}$$

inside $\mathcal{U}$. Then one obtains from (5.2), (5.3), and (5.4) the uniform bounds

$$r(A-\lambda) \geq r(A-\lambda_0) - c > 0, \quad \lambda \in \mathcal{U}_c, \tag{5.15}$$

$$g(\operatorname{ran}(A-\lambda), \operatorname{ran}(A-\lambda_0)) \leq \frac{c}{r(A-\lambda_0)} < 1, \quad \lambda \in \mathcal{U}_c, \tag{5.16}$$

and

$$g(\ker\,(A-\lambda), \ker\,(A-\lambda_0)) \leq \frac{c}{r(A-\lambda_0)} < 1, \quad \lambda \in \mathcal{U}_c. \tag{5.17}$$

Hence (5.8), (5.13), and (5.14) together with (5.15), (5.16), and (5.17) lead to the existence of K_c for which

$$\|\mathcal{R}(\lambda)h\|_{\mathfrak{D}} \leq K_c \|h\|, \quad h \in \mathfrak{H},$$

for all $\lambda \in \mathcal{U}_c$. Since the family $\mathcal{R}(\lambda)$, $\lambda \in \mathcal{U}$, forms a pseudo-resolvent, this implies analyticity on $\mathcal{U}$.

(ii) $\Rightarrow$ (i) Assume that there exists a generalized resolvent $\mathcal{R}(\lambda)$ of A which is holomorphic in a neighborhood of λ_0. In order to show that $\lambda_0 \in \operatorname{reg}(A)$ it suffices to show that $A-\lambda_0$ is quasi-Fredholm of degree 0. By assumption, $\operatorname{ran}(A-\lambda_0)$ is closed. Hence, it remains to show that for all $n \in \mathbb{N}$

$$\ker\,(A-\lambda_0)^n \subset \operatorname{ran}(A-\lambda_0), \tag{5.18}$$

recall the equivalence between (F1), (F2) and (F3), (F4).

Step 1. It will be shown by induction that for $\lambda \neq \lambda_0$ and all $n \in \mathbb{N}$

$$\ker\,(A-\lambda_0)^n \subset \operatorname{ran}(A-\lambda).$$

If $u \in \ker\,(A-\lambda_0)$ then $\{u, (\lambda_0-\lambda)u\} \in A-\lambda$, and it follows that $u \in \operatorname{ran}(A-\lambda)$. If $u \in \ker\,(A-\lambda_0)^n$ and $\ker\,(A-\lambda_0)^{n-1} \subset \operatorname{ran}(A-\lambda)$ then $\{u,v\} \in A-\lambda_0$ for some $v \in \ker\,(A-\lambda_0)^{n-1}$, implying that $v + (\lambda_0-\lambda)u \in \operatorname{ran}(A-\lambda)$ and hence $u \in \operatorname{ran}(A-\lambda)$.

Step 2. Observe that $\operatorname{ran}(A-\lambda)$ is closed for λ in a neighborhood of λ_0. Let P denote the orthogonal projection onto $\operatorname{mul} A$. As $\operatorname{mul} A \subset \operatorname{ran}(A-\lambda)$ for all $\lambda \in \mathbb{C}$, it follows that both P and $I-P$ commute with $P_{\operatorname{ran}(A-\lambda)}$ and also with P_λ for $\lambda \in \operatorname{reg}(A)$. Note that $(I-P)P_{\operatorname{ran}(A-\lambda)}$ is an orthogonal projection.

Let $h \in \ker\,(A-\lambda_0)^n$, then clearly $P_{\operatorname{ran}(A-\lambda)}h = h$ and $Ph = PP_{\operatorname{ran}(A-\lambda_0)}h$, and it follows that

$$\begin{aligned}
\|h - P_{\operatorname{ran}(A-\lambda_0)}h\| &= \|(I-P)(P_{\operatorname{ran}(A-\lambda)} - P_{\operatorname{ran}(A-\lambda_0)})h\| \\
&\leq \|(I-P)P_{\operatorname{ran}(A-\lambda)} - P_{\operatorname{ran}(A-\lambda_0)}\|\,\|h\| \\
&\leq g((I-P)\operatorname{ran}(A-\lambda), (I-P)\operatorname{ran}(A-\lambda_0))\,\|h\| \\
&\leq \|(I-P)(P_\lambda - P_{\lambda_0})\|\|h\|,
\end{aligned}$$

where (2.26) and (2.28) have been used. Now observe that

$$(I-P)P_\lambda = A_{\mathrm{s}}\mathcal{R}(\lambda) - \lambda\mathcal{R}(\lambda) + \lambda P\mathcal{R}(\lambda).$$

Since $\mathcal{R}(\lambda)$ is holomorphic in the graph norm, it follows that $\|(I-P)(P_\lambda - P_{\lambda_0})\|$ tends to 0 for $\lambda \to \lambda_0$. Hence one concludes that

$$h = P_{\operatorname{ran}(A-\lambda_0)}h \in \operatorname{ran}(A-\lambda_0),$$

which shows (5.18). □

6. On the opening between subspaces

Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of $\mathfrak{H}$. In general, the sum $\mathfrak{M}+\mathfrak{N}$ need not be closed (see [16] for an interesting example). This section presents of review of necessary and sufficient conditions under which $\mathfrak{M}+\mathfrak{N}$ is closed.

The intersection $\mathfrak{M}\cap\mathfrak{N}$, the *overlapping* of $\mathfrak{M}$ and $\mathfrak{N}$, is a closed linear subspace. Hence the Hilbert space $\mathfrak{H}$ has the following orthogonal decomposition

$$\mathfrak{H} = (\mathfrak{M}\cap\mathfrak{N})^{\perp} \oplus (\mathfrak{M}\cap\mathfrak{N}). \tag{6.1}$$

Introduce the 'reduced' subspaces $\mathfrak{M}_0$ and $\mathfrak{N}_0$ by

$$\mathfrak{M}_0 = \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}, \quad \mathfrak{N}_0 = \mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}. \tag{6.2}$$

Then $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are closed linear subspaces of $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ and

$$\mathfrak{M}_0\cap\mathfrak{N}_0 = \{0\}. \tag{6.3}$$

Denote the orthogonal complements of $\mathfrak{M}_0$ and $\mathfrak{N}_0$ in $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ by $\mathfrak{M}_0^{\perp}$ and $\mathfrak{N}_0^{\perp}$, respectively.

Lemma 6.1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be defined by* (2.20)*. Then, corresponding to* (6.1)*, $\mathfrak{M}$ and $\mathfrak{N}$ have the orthogonal decompositions*

$$\mathfrak{M} = \mathfrak{M}_0 \oplus (\mathfrak{M}\cap\mathfrak{N}), \quad \mathfrak{N} = \mathfrak{N}_0 \oplus (\mathfrak{M}\cap\mathfrak{N}). \tag{6.4}$$

Moreover, the space $(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ has the following decompositions

$$(\mathfrak{M}\cap\mathfrak{N})^{\perp} = \mathfrak{M}_0 \oplus \mathfrak{M}^{\perp}, \quad (\mathfrak{M}\cap\mathfrak{N})^{\perp} = \mathfrak{N}_0 \oplus \mathfrak{N}^{\perp}, \tag{6.5}$$

in other words $\mathfrak{M}^{\perp} = \mathfrak{M}_0^{\perp}$ and $\mathfrak{N}^{\perp} = \mathfrak{N}_0^{\perp}$.

Corollary 6.2. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be defined by* (2.20)*. Then the following statements are equivalent:*

(i) *$\mathfrak{M}+\mathfrak{N}$ is closed;*
(ii) *$\mathfrak{M}_0+\mathfrak{N}_0$ is closed.*

Moreover, the orthogonal complements satisfy

$$\mathfrak{M}^{\perp}+\mathfrak{N}^{\perp} = \mathfrak{M}_0^{\perp}+\mathfrak{N}_0^{\perp},$$

so that both sums are closed simultaneously.

To see whether the linear subspace $\mathfrak{M}+\mathfrak{N}$ is closed or not, it suffices, according to Lemma 6.1, to assume that $\mathfrak{M}\cap\mathfrak{N}=\{0\}$. If the subspace $\mathfrak{M}+\mathfrak{N}$ is closed and $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, then $\mathfrak{M}+\mathfrak{N}$ may be considered as a Hilbert space in its own right with corresponding projections from $\mathfrak{M}+\mathfrak{N}$ onto $\mathfrak{M}$ or $\mathfrak{N}$. This leads to the following simple characterization, based on parallel projections and the closed graph theorem.

Lemma 6.3. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *$\mathfrak{M}+\mathfrak{N}$ is closed and $\mathfrak{M}\cap\mathfrak{N}=\{0\}$;*
(ii) *there exists $\rho>0$ such that*

$$\rho\sqrt{\|f\|^2+\|g\|^2} \le \|f+g\|, \quad f\in\mathfrak{M},\ g\in\mathfrak{N}. \tag{6.6}$$

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$ and let $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ denote the corresponding orthogonal projections. The *opening* $c_0(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as

$$c_0(\mathfrak{M},\mathfrak{N}) = \sup\{\, |(f,g)| : \, f \in \mathfrak{M}, \ \|f\| \leq 1, \ g \in \mathfrak{N}, \ \|g\| \leq 1 \,\}. \tag{6.7}$$

It is clear from this definition that $c_0(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{N},\mathfrak{M})$. Moreover, since

$$c_0(\mathfrak{M},\mathfrak{N}) = \sup\{\, |(P_{\mathfrak{M}}f, P_{\mathfrak{N}}g)| : \, \|f\| \leq 1, \ \|g\| \leq 1 \,\},$$

it follows that

$$c_0(\mathfrak{M},\mathfrak{N}) = \|P_{\mathfrak{M}}P_{\mathfrak{N}}\|,$$

which characterizes $c_0(\mathfrak{M},\mathfrak{N})$ in terms of the orthogonal projections $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$.

Proposition 6.4. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $c_0(\mathfrak{M},\mathfrak{N}) < 1$;
(ii) $\mathfrak{M}+\mathfrak{N}$ *is closed and* $\mathfrak{M}\cap\mathfrak{N} = \{0\}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. The *opening* $c(\mathfrak{M},\mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M}_0,\mathfrak{N}_0), \tag{6.8}$$

where $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are defined as in (2.20). It is clear from this definition that $c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{N},\mathfrak{M})$. Moreover, it follows that

$$\begin{aligned} c(\mathfrak{M},\mathfrak{N}) &= \sup\{\, |(P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}f, P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}g)| : \, \|f\| \leq 1, \ \|g\| \leq 1 \,\} \\ &= \sup\{\, |(P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{M}}f, P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}g)| : \, \|f\| \leq 1, \ \|g\| \leq 1 \,\} \\ &= \sup\{\, |P_{\mathfrak{M}}f, P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}g)| : \, \|f\| \leq 1, \ \|g\| \leq 1 \,\} \\ &= \sup\{\, |(P_{\mathfrak{M}}f, P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}g)| : \, \|f\| \leq 1, \ \|g\| \leq 1 \,\}, \end{aligned}$$

which leads to

$$c(\mathfrak{M},\mathfrak{N}) = c_0(\mathfrak{M},\mathfrak{N}_0) = c_0(\mathfrak{M}_0,\mathfrak{N}), \tag{6.9}$$

where the last equality follows by symmetry. In terms of orthogonal projections one has

$$\begin{aligned} c(\mathfrak{M},\mathfrak{N}) &= \|P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}\| = \|P_{\mathfrak{M}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\| \\ &= \|P_{\mathfrak{M}}P_{\mathfrak{N}}P_{(\mathfrak{M}\cap\mathfrak{N})^\perp}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}}(I-P_{\mathfrak{M}\cap\mathfrak{N}})\| \\ &= \|P_{\mathfrak{M}}P_{\mathfrak{N}} - P_{\mathfrak{M}}P_{\mathfrak{N}}P_{\mathfrak{M}\cap\mathfrak{N}}\| = \|P_{\mathfrak{M}}P_{\mathfrak{N}} - P_{\mathfrak{M}\cap\mathfrak{N}}\|, \end{aligned}$$

which characterizes $c(\mathfrak{M},\mathfrak{N})$ in terms of the orthogonal projections $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$.

Proposition 6.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $c(\mathfrak{M},\mathfrak{N}) < 1$;
(ii) $\mathfrak{M}+\mathfrak{N}$ *is closed.*

Let $\mathfrak{H}$ be a Hilbert space and let $A \in \mathbf{B}(\mathfrak{H})$ (the bounded linear operators, defined on all of $\mathfrak{H}$). The *minimum modulus* $r(A)$ of A is now

$$r(A) = \inf\left\{ \frac{\|Ah\|}{\|h\|} : \, h \in \mathfrak{H} \ominus \ker\, A \right\}. \tag{6.10}$$

Then $\operatorname{ran} A$ is closed if and only if $r(A) > 0$ and, furthermore, $r(A^*) = r(A)$.

Theorem 6.6. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then*

$$c(\mathfrak{M},\mathfrak{N})^2 + r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1. \tag{6.11}$$

In particular,

$$c(\mathfrak{M}^\perp,\mathfrak{N}^\perp) = c(\mathfrak{M},\mathfrak{N}). \tag{6.12}$$

Proof. First observe that the following identity holds:

$$\ker\,((I-P_{\mathfrak{N}})P_{\mathfrak{M}}) = (\mathfrak{M}\cap\mathfrak{N}) \oplus \mathfrak{M}^\perp. \tag{6.13}$$

To see this, note that the righthand side is contained in the lefthand side. For the reverse inclusion, assume that $(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h = 0$ and write $h = f + g$ with $f \in \mathfrak{M}$ and $g \in \mathfrak{M}^\perp$. Then $f = P_{\mathfrak{N}}f$, so that $f \in \mathfrak{M}\cap\mathfrak{N}$. Hence, $h \in (\mathfrak{M}\cap\mathfrak{N})\oplus\mathfrak{M}^\perp$. This completes the proof of the reverse inclusion. It follows from (6.13) and (6.5) that

$$(\ker\,((I-P_{\mathfrak{N}})P_{\mathfrak{M}}))^\perp = \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp. \tag{6.14}$$

Hence, by means of (6.10) and (6.14), it can be seen that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}}) = \inf\left\{ \frac{\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h\|}{\|h\|} \,:\, h \in \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp \right\}. \tag{6.15}$$

The following straightforward identity

$$\frac{\|(I-P_{\mathfrak{N}})P_{\mathfrak{M}}h\|^2}{\|h\|^2} = \frac{\|P_{\mathfrak{M}}h\|^2}{\|h\|^2} - \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2}, \quad h \in \mathfrak{H}\setminus\{0\},$$

and (6.15) lead to

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1 - \sup\left\{ \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2} \,:\, h \in \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp \right\}. \tag{6.16}$$

It follows from $\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp \subset \mathfrak{M}$ and the identity (6.5) that

$$\begin{aligned}
&\sup\left\{ \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}}h\|^2}{\|h\|^2} \,:\, h \in \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp \right\}\\
&\quad = \sup\left\{ \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}h\|^2}{\|h\|^2} \,:\, h \in \mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp \right\}\\
&\quad = \sup\left\{ \frac{\|P_{\mathfrak{N}}P_{\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp}h\|^2}{\|h\|^2} \,:\, h \in \mathfrak{H} \right\}.
\end{aligned} \tag{6.17}$$

Hence, (6.16) and (6.17) show that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})^2 = 1 - c_0(\mathfrak{N},\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^\perp)^2. \tag{6.18}$$

This identity (6.18) together with (6.9) and the symmetry of c_0 lead to (6.11).

Since the minimum modulus is invariant under taking adjoints, it follows that

$$r((I-P_{\mathfrak{N}})P_{\mathfrak{M}})) = r(P_{\mathfrak{M}}(I-P_{\mathfrak{N}})). \tag{6.19}$$

The identity (6.19), Theorem 6.6, and the symmetry property of $c(\mathfrak{M},\mathfrak{N})$ lead to

$$c(\mathfrak{M},\mathfrak{N}) = c(\mathfrak{N}^\perp,\mathfrak{M}^\perp) = c(\mathfrak{M}^\perp,\mathfrak{N}^\perp),$$

in other words (6.12) has been shown. □

The next result is a direct consequence of Theorem 6.6, when it is combined with the characterization in Proposition 6.5.

Theorem 6.7. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\mathfrak{M} + \mathfrak{N}$ *is closed;*
(ii) $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ *is closed.*

Morover, the following statements are equivalent:

(iii) $\mathfrak{M} + \mathfrak{N}$ *is closed and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$*;*
(iv) $\mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{H}$.

In particular, the following statements are equivalent:

(v) $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$ *and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$*;*
(vi) $\mathfrak{M}^\perp + \mathfrak{N}^\perp = \mathfrak{H}$ *and* $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be closed linear subspaces of a Hilbert space $\mathfrak{H}$. The *gap* $g(\mathfrak{M}, \mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ is defined as (2.26), where $P_\mathfrak{M}$ and $P_\mathfrak{N}$ are the orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}$, respectively. The identity

$$P_\mathfrak{M} - P_\mathfrak{N} = P_\mathfrak{M}(I - P_\mathfrak{N}) - (I - P_\mathfrak{M})P_\mathfrak{N}$$

shows that $g(\mathfrak{M}, \mathfrak{N}) \leq 1$.

Proposition 6.8. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be closed linear subspaces in* $\mathfrak{H}$. *Then*

$$\max(c_0(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)) = g(\mathfrak{M}, \mathfrak{N}^\perp). \tag{6.20}$$

In particular, if $c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)$*, then* $c_0(\mathfrak{M}, \mathfrak{N}) = g(\mathfrak{M}, \mathfrak{N}^\perp)$.

Corollary 6.9. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be closed linear subspaces in* $\mathfrak{H}$. *Then*

$$\begin{aligned} c(\mathfrak{M}, \mathfrak{N}) &\leq \min(c_0(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) \\ &\leq \max(c_0(\mathfrak{M}, \mathfrak{N}), c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)) = g(\mathfrak{M}, \mathfrak{N}^\perp). \end{aligned} \tag{6.21}$$

Moreover, if $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ *and* $\mathfrak{M}^\perp \cap \mathfrak{N}^\perp = \{0\}$*, then*

$$c(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp) = g(\mathfrak{M}, \mathfrak{N}^\perp). \tag{6.22}$$

Theorem 6.10. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be closed linear subspaces of a Hilbert space* $\mathfrak{H}$. *Then the following statements are equivalent:*

(i) $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$*;*
(ii) $\mathfrak{M} + \mathfrak{N} = \mathfrak{H}$ *and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$.

If either of these equivalent conditions holds, then the chain of equalities in (6.22) *is satisfied.*

Corollary 6.11. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be closed linear subspaces of a Hilbert space* $\mathfrak{H}$ *such that* $g(\mathfrak{M}, \mathfrak{N}^\perp) < 1$*, or equivalently,* $\mathfrak{H} = \mathfrak{M} + \mathfrak{N}$ *and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$. *Then*

$$g(\mathfrak{M}, \mathfrak{N}^\perp) = \sqrt{1 - \frac{1}{\|P\|^2}} \quad (= c(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)), \tag{6.23}$$

where P *is the projection onto* $\mathfrak{M}$*, parallel to* $\mathfrak{N}$.

Proof. Observe that the condition $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ implies that

$$\begin{aligned} r((I - P_\mathfrak{N})P_\mathfrak{M}) &= \inf \left\{ \frac{\|(I - P_\mathfrak{N})f\|}{\|f\|} : f \in \mathfrak{M} \right\} \\ &= \left(\sup \left\{ \frac{\|h\|}{\|(I - P_\mathfrak{N})f\|} : f \in \mathfrak{M} \right\} \right)^{-1}. \end{aligned} \tag{6.24}$$

Hence (6.23) follows from Theorem 6.6, Theorem 6.10, and (6.24) once the following identity has been established:

$$\|P\| = \sup \left\{ \frac{\|f\|}{\|(I - P_{\mathfrak{N}})f\|} \,:\, f \in \mathfrak{M} \right\}. \tag{6.25}$$

In order to show (6.25), note that

$$\|P\| = \sup \left\{ \frac{\|f\|}{\|f + g\|} \,:\, f \in \mathfrak{M},\, g \in \mathfrak{N} \right\}.$$

The decomposition $f + g = (I - P_{\mathfrak{N}})f + h$ with $h = P_{\mathfrak{N}}f + g$ belonging to $\mathfrak{N}$ gives

$$\|f + g\|^2 = \|(I - P_{\mathfrak{N}})f\|^2 + \|P_{\mathfrak{N}}f + g\|^2, \quad f \in \mathfrak{M}, \quad g \in \mathfrak{N},$$

and it follows that

$$\|P\| = \sup \left\{ \frac{\|f\|}{\sqrt{\|(I - P_{\mathfrak{N}})f\|^2 + \|h\|^2}} \,:\, f \in \mathfrak{M},\, h \in \mathfrak{N} \right\}.$$

This representation clearly implies (6.25). □

The opening $c_0(\mathfrak{M}, \mathfrak{N})$ and the opening $c(\mathfrak{M}, \mathfrak{N})$ have been introduced by J. Dixmier [3] and by K. Friedrichs [5], respectively. For related treatments, see [2] and [12]; note that in [12] the notations

$$\varepsilon(\mathfrak{M}, \mathfrak{N}) = c(\mathfrak{M}, \mathfrak{N}^\perp) \quad \text{and} \quad \delta(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}, \mathfrak{N}^\perp)$$

have been used. The results in Propositions 6.4 and 6.5 go back to J.-Ph. Labrousse [12] and to F. Deutsch [2, Theorem 12]. Theorem 6.6 goes back to Labrousse [12]. According to [2] the identity (6.12) was originally found by D.C. Salmon [15]; a different proof of it was provided in [2]. Note that a similar result does not hold for the opening $c_0(\mathfrak{M}, \mathfrak{N})$. Theorem 6.7 can be found, for instance, in [10]. Proposition 6.8 has a long history; see [1] and [12]. The result in Corollary 6.11 goes back to V.E. Lyantse [14]. In this particular case the identity $c_0(\mathfrak{M}, \mathfrak{N}) = c_0(\mathfrak{M}^\perp, \mathfrak{N}^\perp)$ (< 1) goes back to M.G. Kreĭn, M.A. Krasnoselskiĭ, and D.P. Milman [11]; for a different proof see [2], [8].

References

[1] N.I. Achieser and I.M. Glasmann, *Theorie der linearen Operatoren im Hilbert-Raum*, 8th edition, Akademie-Verlag, Berlin, 1981.

[2] F. Deutsch, "The angle between subspaces of a Hilbert space", in S. P. Singh (editor) *Approximation Theory, Wavelets and Applications*, Kluwer Academic Publishers, 1995, 107–130.

[3] J. Dixmier, "Études sur les varietés et les opérateurs de Julia avec quelques applications", Bull. Soc. Math. France, 77 (1949), 11–101.

[4] P.A. Fillmore and J.P. Williams, "On operator ranges", Adv. Math., 7 (1971), 254–281.

[5] K. Friedrichs, "On certain inequalities and characteristic value problems for analytic functions and for functions of two variables", Trans. Amer. Math. Soc., 41 (1937), 321–364.

[6] S. Goldberg, *Unbounded linear operators*, Mc Graw Hill, New York, 1966.

[7] I.C. Gohberg and M.G. Kreĭn, "The basic propositions on defect numbers, root numbers and indices of linear operators", Uspekhi Mat. Nauk., 12 (1957), 43-118 (Russian) [English translation: Transl. Amer. Math. Soc. (2), 13 (1960), 185-264].

[8] I.C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators*, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., 1969.

[9] S. Hassi, H.S.V. de Snoo, and F.H. Szafraniec, "Componentwise and canonical decompositions of linear relations", Dissertationes Mathematicae, 465, 2009 (59 pages).

[10] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1980.

[11] M. G. Kreĭn, M. A. Krasnoselskiĭ, and D.P. Milman, "On the defect numbers of linear operators in a Banach space and on some geometric questions", Akad. Nauk Ukrain RSR. Zbirnik Prac Inst. Mat., 11 (1948), 97–112 (Ukrainian).

[12] J.-Ph. Labrousse, "Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm", Rend. Circ. Mat. Palermo, 29 (1980), 161-258.

[13] J.-Ph. Labrousse, A. Sandovici, H.S.V. de Snoo, and H. Winkler, "The Kato decomposition of quasi-Fredholm relations", Operators and Matrices, 4 (2010), 1–51.

[14] V.E. Lyantse, "Some properties of idempotent operators", Teor. i Prikl. Mat. L'vov, 1 (1959), 16–22 (Russian).

[15] D.C. Salmon, *Mathematical aspects of computed tomography*, Handwritten lecture notes, 1976–1977.

[16] M.H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, American Mathematical Society Colloquium Publications, 15, Providence, RI, 1932.

Département de Mathematiques, Université de Nice, Laboratoire J. A. Dieudonné, UMR du CNRS 6621 Parc Valrose, 06108 Nice Cedex 02, France

E-mail address: labro@math.unice.fr

Department of Sciences, University "Al. I. Cuza", Lascăr Catargi 54, 700107, Iaşi, Romania

E-mail address: adrian.sandovici@uaic.ro

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, Nederland

E-mail address: desnoo@math.rug.nl

Institut für Mathematik, Technische Universität Ilmenau, Curiebau, Weimarer Str. 25, 98693 Ilmenau, Germany

E-mail address: henrik.winkler@tu-ilmenau.de