Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 10/03

Selfadjoint operators in S-spaces

Philipp, Friedrich; Szafraniec, Franciszek Hugon; Trunk, Carsten

December 10, 2009

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Selfadjoint Operators in S-Spaces

Friedrich Philipp

Technische Universität Ilmenau, Institut für Mathematik, Postfach 100565, D-98684 Ilmenau, Germany

Franciszek Hugon Szafraniec

Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków, Poland

Carsten Trunk

Technische Universität Ilmenau, Institut für Mathematik, Postfach 100565, D-98684 Ilmenau, Germany

Abstract

We study S-spaces and operators therein. An S-space is a Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$ with an additional inner product given by $[\,\cdot\,, -] := (U\,\cdot\,, -)$, where U is a unitary operator in $(\mathfrak{S}, (\,\cdot\,, -))$. We investigate spectral properties of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. As a main result we prove that for each selfadjoint operator A in an S-space with $\rho(A) \neq \emptyset$ we find an inner product which turns $\mathfrak{S}$ into a Krein space and A into a selfadjoint operator therein. In addition, we give a simple condition for the existence of invariant subspaces.

Keywords: S-space, Krein space, indefinite inner products, selfadjoint operators, invariant subspaces
2000 MSC: 47B50, 46C50, 46C20

Email addresses: `friedrich.philipp@tu-ilmenau.de` (Friedrich Philipp), `umszafra@cyf-kr.edu.pl` (Franciszek Hugon Szafraniec), `carsten.trunk@tu-ilmenau.de` (Carsten Trunk)

1. Introduction

A complex linear space $\mathcal{H}$ with a hermitian sesquilinear form $[\,\cdot\,,-]$ is called a *Krein space* if there exists a fundamental decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\,\cdot\,,-]$ such that $(\mathcal{H}_\pm, \pm[\,\cdot\,,-])$ are Hilbert spaces. If $\mathcal{H}_-$ or $\mathcal{H}_+$ is finite dimensional, then $(\mathcal{H}, [\,\cdot\,,-])$ is called a *Pontryagin space*. To each decomposition (1) there corresponds a Hilbert space inner product $(\,\cdot\,,-)$ and a selfadjoint operator J (the fundamental symmetry) with $JJ^* = I$, $J = J^*$ such that

$$[x, y] = (Jx, y) \quad \text{for } x, y \in \mathcal{H}, \tag{2}$$

see, e.g. [2, 5, 13].

Conversely, every bounded and boundedly invertible selfadjoint operator G in a Hilbert space $(\mathcal{H}, (\,\cdot\,,-))$ defines an inner product via

$$[\,\cdot\,,-] := (G\,\cdot\,,-) \tag{3}$$

and $(\mathcal{H}, [\,\cdot\,,-])$ becomes a Krein space. In particular, if the spectrum of G consists on the positive (or negative) semiaxis only of finitely many isolated eigenvalues of finite multiplicity, then $(\mathcal{H}, [\,\cdot\,,-])$ is a Pontryagin space.

Equation (3) is the starting point for various generalizations. E.g., if G is a bounded selfadjoint operator (but no more boundedly invertible) in $\mathcal{H}$ such that $\sigma(G) \cap (-\infty, \varepsilon)$ consists of finitely many eigenvalues of G with finite multiplicities for some $\varepsilon > 0$, then $(\mathcal{H}, [\,\cdot\,,-])$, where $[\,\cdot\,,-]$ is defined by (3), is called an Almost Pontryagin space, see [9]. Observe that in this case zero is allowed to be an eigenvalue of G with finite multiplicity. Almost Pontryagin spaces and operators therein were considered in various situations, we mention only [1, 9, 10, 11, 12, 16, 20, 26, 27]. The more general case that G is a bounded selfadjoint operator in $\mathcal{H}$ such that zero is an isolated eigenvalue of G with finite multiplicity gives rise to Almost Krein spaces, see [3]. Spaces with an inner product given by an arbitrary bounded selfadjoint operator were studied, e.g., in [15, 21, 22]. For applications we refer to [4, 6, 8, 9, 10, 11, 12, 14, 16, 17, 18, 19, 26, 27].

In all the above-mentioned generalizations of (3) the selfadjointness of the operator G in $\mathcal{H}$ is maintained and the bounded invertibility is dropped. Obviously, this is the same as generalizing (2) by dropping $JJ^* = I$ and preserving $J = J^*$. From this point of view, it seems natural to generalize (2) the other way: dropping selfadjointness and preserving unitarity of J. The inner product space $(\mathcal{H}, [\,\cdot\,,-])$, where $[\,\cdot\,,-]$ is defined by (2) with a unitary operator J is called an *S-space*, cf. [23] and also Definition 2.1 below. Moreover, the pair $((\,\cdot\,,-), J)$ is called a *Hilbert space realization* of the S-space $(\mathcal{H}, [\,\cdot\,,-])$. Evidently, by definition every Krein space is a special case of an S-space.

In this paper we continue the study of S-spaces and operators therein started in [23, 24]. It is known from [24] that the inner products of two Hilbert space realizations $((\,\cdot\,,-)_1, U_1)$ and $((\,\cdot\,,-)_2, U_2)$ define the same topology. Here, we show in particular that U_1 and U_2 are similar operators with respect to this topology, cf. Proposition 2.4. In Section 3 we introduce the notion of selfadjoint operators in S-spaces. We show that their spectrum is symmetric with respect to the real axis. As a main result we prove that to each selfadjoint operator A in an S-space $(\mathfrak{S}, [\,\cdot\,,-])$ with $\rho(A) \neq \emptyset$ we find an inner product $\langle\,\cdot\,,-\rangle$ on $\mathfrak{S}$ such that $(\mathfrak{S}, \langle\,\cdot\,,-\rangle)$ is a Krein space with the same topology as $(\mathfrak{S}, [\,\cdot\,,-])$ and A is a selfadjoint operator in the Krein space $(\mathfrak{S}, \langle\,\cdot\,,-\rangle)$, cf. Theorem 3.13.

Moreover, if $((\,\cdot\,,-), U)$ is a Hilbert space realization, we show in Theorem 3.13 below that each spectral subspace of U related to a Borel subset Δ of the unit circle which is symmetric with respect to the origin (i.e. $x \in \Delta$ implies $-x \in \Delta$) is invariant under A. Hence, in this paper we obtain the rather unexpected result: Each selfadjoint operator in an S-space with $\rho(A) \neq \emptyset$ is a selfadjoint operator in a Krein space with many invariant subspaces, provided the spectrum of the operator U from some Hilbert space realization $((\,\cdot\,,-), U)$ of $(\mathfrak{S}, [\,\cdot\,,-])$ is sufficiently rich, i.e., if it consists of more than two points.

2. Definition and Basic Properties

The following definition is taken from [23].

Definition 2.1. *A complex linear space $\mathfrak{S}$ with an inner product $[\,\cdot\,,-]$, that is a mapping from $\mathfrak{S} \times \mathfrak{S}$ into $\mathbb{C}$ which is linear in the first variable and conjugate linear in the other, is said to be an* S-space *if there is a Hilbert space structure in $\mathfrak{S}$ given by a positive definite inner product $(\,\cdot\,,-)$ and if there is a unitary operator U in the Hilbert space $(\mathfrak{S}, (\,\cdot\,,-))$ such that*

$$[f, g] = (Uf, g) \quad \textit{for all } f, g \in \mathfrak{S}.$$

We refer to $[\cdot,-]$ *as the inner product of* $\mathfrak{S}$. *The pair* $((\cdot,-),U)$ *is called a Hilbert space realization of* $(\mathfrak{S},[\cdot,-])$.

Note, that the inner product $[\cdot,-]$ is not Hermitian, in general. An S-space is a Krein space if and only if the operator U in Definition 2.1 is in addition selfadjoint in the Hilbert space $(\mathfrak{S},(\cdot,-))$. For the theory of operators in Krein spaces we refer to [2, 5].

Proposition 2.2. *Let* $\mathfrak{S}$ *be a complex linear space with an inner product* $[\cdot,-]$. *Then the pair* $(\mathfrak{S},[\cdot,-])$ *is an S-space if and only if there exists a Hilbert space inner product* $(\cdot,-)$ *on* $\mathfrak{S}$ *and a bounded and boundedly invertible normal operator* T *in* $(\mathfrak{S},(\cdot,-))$ *such that*

$$[f,g]=(Tf,g)\quad \textit{for all } f,g\in\mathfrak{S}.$$

Proof. We define the operator $U:=T(T^*T)^{-1/2}$ and the inner product

$$\langle x,y\rangle := ((T^*T)^{1/2}x,y),\quad x,y\in\mathfrak{S}.$$

Since T is bijective, this is a Hilbert space inner product on $\mathfrak{S}$. From the relation $(T(T^*T)^{-1/2}T^*)^2=TT^*$ it follows that

$$(TT^*)^{1/2}=(T^*T)^{1/2}=T(T^*T)^{-1/2}T^*. \tag{4}$$

Hence, for $x,y\in\mathfrak{S}$ we obtain

$$\langle Ux,y\rangle = ((T^*T)^{1/2}T(T^*T)^{-1/2}x,y) = (T(T^*T)^{-1/2}T^*T(T^*T)^{-1/2}x,y)=[x,y]$$

and

$$\begin{aligned}\langle Ux,Uy\rangle &= ((T^*T)^{1/2}T(T^*T)^{-1/2}x,T(T^*T)^{-1/2}y)\\ &= (T(T^*T)^{-1/2}T^*T(T^*T)^{-1/2}x,T(T^*T)^{-1/2}y)\\ &= (Tx,T(T^*T)^{-1/2}y)=((T^*T)^{-1/2}T^*Tx,y)\\ &= ((T^*T)^{1/2}x,y)=\langle x,y\rangle,\end{aligned}$$

which shows that U is unitary in $(\mathfrak{S},\langle\cdot,\cdot\rangle)$ and $(\mathfrak{S},[\cdot,-])$ is an S-space. □

Lemma 2.3. *Let* $(\mathfrak{S},[\cdot,-])$ *be an S-space. Then there exists a uniquely defined linear operator* $D:\mathfrak{S}\to\mathfrak{S}$ *such that*

$$[x,y]=\overline{[y,Dx]}\qquad \textit{for all }\ x,y\in\mathfrak{S}. \tag{5}$$

If $((\cdot,-),U)$ *is a Hilbert space realization of* $(\mathfrak{S},[\cdot,-])$, *then* $D=U^2$.

Proof. Let $((\cdot, -), U)$ be a Hilbert space realization of $(\mathfrak{S}, [\cdot, -])$. Then it is easily seen that U^2 satisfies the relation (5) (with D replaced by U^2). Let $D : \mathfrak{S} \to \mathfrak{S}$ be a linear operator satisfying (5). Then from $[y, Dx] = [y, U^2 x]$ for all $x, y \in \mathfrak{S}$ we conclude $(Uy, Dx - U^2 x) = 0$ for all $x, y \in \mathfrak{S}$. And since U is bijective, it follows that $D = U^2$. $\square$

The topology of an S-space $(\mathfrak{S}, [\cdot, -])$ is given by the topology induced by the Hilbert space inner product $(\cdot, -)$ of some Hilbert space realization of $(\mathfrak{S}, [\cdot, -])$. The following proposition states in particular that it does not depend on the choice of the Hilbert space realization, see also [24].

Proposition 2.4. *Let $(\mathfrak{S}, [\cdot, -])$ be an S-space and assume that there are two Hilbert space realizations $((\cdot, -)_1, U_1)$ and $((\cdot, -)_2, U_2)$ with*

$$[f, g] = (U_1 f, g)_1 = (U_2 f, g)_2 \quad \textit{for all } f, g \in \mathfrak{S}.$$

Then $(\cdot, -)_1$ and $(\cdot, -)_2$ are equivalent and the Gram operator S, defined by

$$(f, g)_2 = (Sf, g)_1 \quad \textit{for } f, g \in \mathfrak{S},$$

is bounded, boundedly invertible and selfadjoint with respect to $(\cdot, -)_1$ and with respect to $(\cdot, -)_2$. Moreover, the following statements holds.

(i) $U_1^2 = U_2^2$.

(ii) *The spectral measures of S in $(\mathfrak{S}, (\cdot, -)_1)$ and $(\mathfrak{S}, (\cdot, -)_2)$ coincide and we have*

$$S = U_1 U_2^{-1} = U_1^{-1} U_2, \quad \textit{and} \quad U_1^{-1} S U_1 = S^{-1} = U_2^{-1} S U_2. \tag{6}$$

Hence, the operator S is unitarily equivalent to its inverse.

(iii) *The operators U_1 and U_2 are similar. We have*

$$U_1 = S^{1/2}\, U_2\, S^{-1/2}.$$

Hence

$$\sigma(U_1) = \sigma(U_2).$$

Proof. Denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the norms induced by $(\cdot\,,-)_1$ and $(\cdot\,,-)_2$, respectively, and set $B_1 := \{y \in \mathfrak{S} : \|y\|_1 = 1\}$. Then, for $y \in B_1$ the linear functional

$$F_y := [\cdot, y] = (U_1 \cdot, y)_1 = (U_2 \cdot, y)_2$$

is continuous on both $(\mathfrak{S}, (\cdot\,,-)_1)$ and $(\mathfrak{S}, (\cdot\,,-)_2)$. For its corresponding operator norms $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_1),\mathbb{C})}$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})}$, respectively, we obtain $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_1),\mathbb{C})} = 1$ and $\|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})} = \|y\|_2$. For all $x \in \mathfrak{S}$ we have $\sup_{y \in B_1} |F_y(x)| \leq \|x\|_1 < \infty$. Due to the principle of uniform boundedness there exists some $c \in (0, \infty)$ with

$$\sup_{y \in B_1} \|F_y\|_{\mathcal{L}((\mathfrak{S},(\cdot\,,-)_2),\mathbb{C})} \leq c.$$

This yields $\|y\|_2 \leq c\|y\|_1$ for all $y \in \mathfrak{S}$. By interchanging the roles of $\|\cdot\|_1$ and $\|\cdot\|_2$ we obtain that these two norms are equivalent. Hence, by the well-known Lax-Milgram Theorem there exists a unique bounded linear operator S, selfadjoint in $(\mathfrak{S}, (\cdot\,,-)_1)$, such that

$$(f, g)_2 = (Sf, g)_1 \quad \text{for} \quad f, g \in \mathfrak{S}.$$

It is boundedly invertible since $\|Sf_n\|_1 \to 0$ and $\|f_n\|_1 = 1$ would imply $\|f_n\|_2^2 = (Sf_n, f_n)_1 \to 0$ which contradicts the above proven fact that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. For $f, g \in \mathfrak{S}$ we have

$$(Sf, g)_2 = (S^2 f, g)_1 = (Sf, Sg)_1 = (f, Sg)_2.$$

Thus, S is also selfadjoint with respect to $(\cdot\,,-)_2$. Moreover, as $(\cdot\,,-)_1$ and $(\cdot\,,-)_2$ are positive definite, the operator S is uniformly positive.

Now we will show (i)-(iii). Statement (i) follows directly from Lemma 2.3. The equality of the spectral measures E_1 and E_2 of S in $(\mathfrak{S}, (\cdot\,,-)_1)$ and $(\mathfrak{S}, (\cdot\,,-)_2)$ follows from the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and Stone's formula (see, e.g., [7, XII.2]),

$$\begin{aligned} E_1((a,b)) &= \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \frac{1}{2\pi \mathrm{i}} \int_{a+\delta}^{b-\delta} \left((S - (\lambda + \epsilon \mathrm{i}))^{-1} - (S - (\lambda - \epsilon \mathrm{i}))^{-1} \right) d\lambda \\ &= E_2((a,b)). \end{aligned}$$

As

$$(Sf, g)_1 = (f, g)_2 = (U_2 U_2^{-1} f, g)_2 = [U_2^{-1} f, g] = (U_1 U_2^{-1} f, g)_1,$$

where the limit is taken in the strong operator topology. we have $S = U_1 U_2^{-1}$ and, with (i), we conclude $S = U_1^{-1} U_1^2 U_2^{-1} = U_1^{-1} U_2$. We will denote the adjoint with respect to $(\,\cdot\,, -)_1$ by the symbol $*_1$ and the adjoint with respect to $(\,\cdot\,, -)_2$ by $*_2$. For $f, g \in \mathfrak{S}$ we have

$$\begin{aligned}(U_2 f, g)_2 &= (SU_2 f, g)_1 = (U_2 f, Sg)_1 = (f, U_2^{*_1} Sg)_1 \\ &= (S^{-1} f, U_2^{*_1} Sg)_2 = (f, S^{-1} U_2^{*_1} Sg)_2,\end{aligned}$$

thus

$$U_2^{*_2} = S^{-1} U_2^{*_1} S. \tag{7}$$

This implies

$$S = S^{*_1} = (U_1 U_2^{-1})^{*_1} = (U_2^{-1})^{*_1} U_1^{-1} = (S U_2^{*_2} S^{-1})^{-1} U_1^{-1} = S U_2 S^{-1} U_1^{-1},$$

hence, with $S = U_1 U_2^{-1}$ we get $S^{-1} = U_1^{-1} S U_1$. Replacing U_1 by U_2 and U_2 by U_1 also $S^{-1} = U_2^{-1} S U_2$ holds and formula (6) and (ii) are proved.

By (ii) the square roots of S in $(\mathfrak{S}, (\,\cdot\,, -)_1)$ and $(\mathfrak{S}, (\,\cdot\,, -)_2)$ coincide. We denote this operator by $S^{1/2}$. Since, by (6), $(U_1^{-1}\, S^{-1/2}\, U_1)^2 = U_1^{-1}\, S^{-1}\, U_1 = S$, we have the relation

$$S^{1/2} = U_1^{-1}\, S^{-1/2}\, U_1,$$

which yields

$$S^{1/2}\, U_2\, S^{-1/2} = S^{1/2}\, S^{-1} U_1\, S^{-1/2} = S^{-1/2}\, U_1\, S^{-1/2} = U_1$$

and (iii) is proved. □

3. Linear Operators in S-spaces

For the rest of this paper let $(\mathfrak{S}, [\,\cdot\,, -])$ be an S-space and let $((\,\cdot\,, -), U)$ be a fixed Hilbert space realization of $(\mathfrak{S}, [\,\cdot\,, -])$. In the following all topological notions are related to the Hilbert space topology given by $(\,\cdot\,, -)$. Its topology is independent of the particular choice of a Hilbert space realization (see Proposition 2.4).

The adjoint with respect to the Hilbert space inner product $(\,\cdot\,, -)$ will be denoted by *.

Definition 3.1. *Let A be a closed, densely defined operator in an S-space. An adjoint $A^\natural$ with respect to $[\,\cdot\,,-]$ is defined via the following relations:*

$$\operatorname{dom} A^\natural := \{g \in \mathfrak{S} : \exists h \in \mathfrak{S} \text{ with } [Af, g] = [f, h] \text{ for all } f \in \operatorname{dom} A\},$$

$$[Af, g] = [f, A^\natural g] \quad \text{for all } f \in \operatorname{dom} A \quad \text{and} \quad g \in \operatorname{dom} A^\natural.$$

Analogously, we define ${}^\natural A$ via

$$\operatorname{dom} {}^\natural A := \{f \in \mathfrak{S} : \exists h \in \mathfrak{S} \text{ with } [f, Ag] = [h, g] \text{ for all } g \in \operatorname{dom} A\},$$

$$[f, Ag] = [{}^\natural A f, g] \quad \text{for all } g \in \operatorname{dom} A \quad \text{and} \quad f \in \operatorname{dom} {}^\natural A.$$

In the following lemma we collect some of the properties of $A^\natural$ and ${}^\natural A$. For a proof we refer to [24].

Proposition 3.2. *The operators $A^\natural$ and ${}^\natural A$ are closed, densely defined and satisfy*

$$\operatorname{dom} A^\natural = U \operatorname{dom} A^* = \operatorname{dom}(A^* U^*) \quad \text{and} \quad A^\natural = U A^* U^* \tag{8}$$

and

$$\operatorname{dom} {}^\natural A = U^* \operatorname{dom} A^* = \operatorname{dom}(A^* U) \quad \text{and} \quad {}^\natural A = U^* A^* U. \tag{9}$$

Recall that for for a densely defined operator T and a bounded operator X in a Hilbert space we have (see [25, Section 4.4])

$$(XT)^* = T^* X^* \text{ and, if } X \text{ is boundedly invertible, } (TX)^* = X^* T^*. \tag{10}$$

Proposition 3.3. *If ${}^\natural A = A^\natural$ then $AD = DA$ where $D = U^2$.*

Proof. If ${}^\natural A = A^\natural$, then from Proposition 3.2 and (10) we conclude

$${}^\natural\big(A^\natural\big) = {}^\natural\big(U A^* U^*\big) = U^*(U A^* U^*)^* U = A,$$

and hence, with ${}^\natural A = A^\natural$,

$$A = {}^\natural{}^\natural A = U^*({}^\natural A)^* U = U^*(U^* A^* U)^* U = (U^*)^2 A U^2 = D^* A D.$$

And since D is unitary, the assertion follows. □

Corollary 3.4. *If ${}^\natural A = A^\natural$ and U has no eigenvalues, then A does not have eigenvalues with finite geometric multiplicity.*

Proof. By Proposition 3.3 we have $AD = DA$. Assume that λ is an eigenvalue of A with finite geometric multiplicity. From $AD = DA$ it follows that $\ker(A - \lambda)$ is invariant under D. Therefore, D (and hence U) has eigenvalues. □

Definition 3.5. *A densely defined operator A in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$ is called* selfadjoint *if*

$$A = A^\natural.$$

We have the following characterization for selfadjointness of operators in S-spaces.

Proposition 3.6. *For a densely defined operator A in $\mathfrak{S}$ the following assertions are equivalent.*

(i) $A = A^\natural$, *i.e., A is selfadjoint in $(\mathfrak{S}, [\,\cdot\,, -])$.*

(ii) $U^*A = A^*U^*$.

(iii) $UA = A^*U$.

(iv) $A = {}^\natural\! A$.

If one of these equivalent statements holds true we have

$$f \in \operatorname{dom} A \Leftrightarrow U^*f \in \operatorname{dom} A^* \Leftrightarrow Uf \in \operatorname{dom} A^*. \tag{11}$$

Proof. The equivalence of (i) and (ii) follows from (8), the equivalence of (iii) and (iv) follows from (9).

Assume that (ii) holds. For $f \in \operatorname{dom} A$ we conclude $U^*f \in \operatorname{dom} A^*$. This implies for $f, g \in \operatorname{dom} A$:

$$(f, UAg) = (A^*U^*f, g) = (U^*Af, g) = (Af, Ug)$$

and we have $Ug \in \operatorname{dom} A^*$, hence $UA \subset A^*U$. For the other inclusion, we observe by (ii) that $\operatorname{dom} A^* = U^*\operatorname{dom} A$. For $Ug \in \operatorname{dom} A^*$ and $f \in \operatorname{dom} A$ we have $U^*f \in \operatorname{dom} A^*$ and

$$(U^*f, U^*A^*Ug) = (f, A^*Ug) = (Af, Ug) = (U^*Af, g) = (A^*U^*f, g),$$

thus $g \in \operatorname{dom}(A^*)^* = \operatorname{dom} A$. This gives $U^*A^*Ug = Ag$ and $A^*U \subset UA$. This proves (iii).

Assume that (iii) holds. For $f \in \operatorname{dom} A$ we conclude $Uf \in \operatorname{dom} A^*$. This gives for $f, g \in \operatorname{dom} A$

$$(U^*Ag, f) = (Ag, Uf) = (g, A^*Uf) = (g, UAf) = (U^*g, Af)$$

and we have $U^*g \in \operatorname{dom} A^*$, hence $U^*A \subset A^*U^*$. For the other inclusion, we observe by (iii) that $\operatorname{dom} A^* = U\operatorname{dom} A$. For $U^*g \in \operatorname{dom} A^*$ and $f \in \operatorname{dom} A$ we have $Uf \in \operatorname{dom} A^*$ and

$$(Uf, UA^*U^*g) = (f, A^*U^*g) = (Af, U^*g) = (UAf, g) = (A^*Uf, g),$$

thus $g \in \operatorname{dom}(A^*)^* = \operatorname{dom} A$. This gives $A^*U^*g = U^*Ag$ and $A^*U^* \subset U^*A$. This proves (ii). Moreover, we have shown that (11) holds. $\square$

Proposition 3.7. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. Then the spectrum of A is symmetric with respect to the real axis.*

Proof. Since $A = A^\natural = UA^*U^*$, cf. Proposition 3.2, the operator A is unitarily equivalent to its adjoint. Hence, $\sigma(A) = \sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$. $\square$

Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. If $(\mathfrak{S}, [\,\cdot\,, -])$ is a Krein space, then U is selfadjoint and thus $\sigma(U) = \sigma_p(U) \subset \{-1, 1\}$. It is well-known that the spectrum of A may be rather arbitrary. For example, it can happen that $\sigma(A) = \mathbb{C}$.

Example 3.8. Assume that – in contrast to the Krein space case – $\sigma(U)$ consists of two eigenvalues λ_1, λ_2 with $\lambda_1 \neq -\lambda_2$, e.g. $\sigma(U) = \{1, \mathrm{i}\}$. Then $\sigma(U^2) = \{1, -1\}$, and since A commutes with $D = U^2$ by Proposition 3.3 the spectral subspaces of D are A-invariant. Since these coincide with the eigenspaces of U corresponding to 1 and i, respectively, we have $A = A_1 \oplus A_{\mathrm{i}}$ and $U = I \oplus \mathrm{i}I$ with respect to the decomposition $\mathfrak{S} = \ker(U-1) \oplus \ker(U-\mathrm{i})$. From the selfadjointness of A in $(\mathfrak{S}, [\,\cdot\,, -])$ we conclude that both A_1 and A_{i} are selfadjoint with respect to the Hilbert space scalar product $(\,\cdot\,, -)$ in $\ker(U-1)$ and $\ker(U-\mathrm{i})$, respectively. Hence, A is selfadjoint in $(\mathfrak{S}, (\,\cdot\,, -))$. In particular its spectrum is real.

This simple example shows that it is not necessarily "better" to know that an operator is selfadjoint in a Krein space than in an S-space. In fact, we will show in the following that every selfadjoint operator in an S-space is also selfadjoint in some Krein space (if only the resolvent set of the operator is nonempty). However, in general (if $\sigma(U) \neq \{\mathrm{e}^{\mathrm{i}t}, -\mathrm{e}^{\mathrm{i}t}\}$ for some $t \in [0, \pi)$) the selfadjointness in the S-space gives us more information about the operator. E.g. we automatically know a whole bunch of invariant subspaces of the operator – namely the spectral subspaces of D.

Definition 3.9. *Let G be a bounded selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$. A closed and densely defined linear operator T in $\mathfrak{S}$ will be called G-symmetric if $GT \subset (GT)^*$. The operator T is called G-selfadjoint if $GT = (GT)^*$.*

In the following we will deal with the operators

$$G(t) := \frac{1}{2i}\left(\mathrm{e}^{\mathrm{i}t} U - \mathrm{e}^{-\mathrm{i}t} U^*\right), \quad t \in [0, \pi).$$

It is easily seen that all these operators are bounded selfadjoint operators in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$. We have $G(0) = \operatorname{Im} U$ and $G(\pi/2) = \operatorname{Re} U$. Moreover, the operator $G(t)$ can be factorized in the following way

$$G(t) = \frac{\mathrm{e}^{\mathrm{i}t}}{2i}\, U^* \left(U^2 - \mathrm{e}^{-2\mathrm{i}t}\right) = \frac{\mathrm{e}^{\mathrm{i}t}}{2i}\, U^* \, (U - \mathrm{e}^{-\mathrm{i}t})(U + \mathrm{e}^{-\mathrm{i}t}).$$

Therefore, $G(t)$ is boundedly invertible if and only if $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$. In this case $(\mathfrak{S}, (G(t)\,\cdot\,, -))$ is a Krein space.

Proposition 3.10. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. Then A is $G(t)$-symmetric for all $t \in [0, \pi)$. If for some $t \in [0, \pi)$ we have $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$ and $\rho(A) \neq \emptyset$, then the operator A is $G(t)$-selfadjoint.*

Proof. Let $t \in [0, \pi)$. Then by Proposition 3.6 we have

$$\begin{aligned} G(t)A &= \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t} U - \mathrm{e}^{-\mathrm{i}t} U^*\big)A = \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t} UA - \mathrm{e}^{-\mathrm{i}t} U^* A\big) \\ &= \frac{1}{2i}\big(\mathrm{e}^{\mathrm{i}t} A^* U - \mathrm{e}^{-\mathrm{i}t} A^* U^*\big) \\ &\subset A^* G(t) = (G(t)A)^*. \end{aligned}$$

This shows that A is $G(t)$-symmetric.

Assume $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$ and $\rho(A) \neq \emptyset$. Then by Proposition 3.7 there exist $\lambda, \overline{\lambda} \in \rho(A)$. In order to see that A is $G(t)$-selfadjoint we have to show that $\operatorname{dom}(A^* G(t)) \subset \operatorname{dom}(G(t)A)$. To this end let $g \in \operatorname{dom}(A^* G(t))$ and set $f := (A-\lambda)^{-1}(G(t)^{-1}A^*G(t)-\lambda)g$. As A is $G(t)$-symmetric we have $A \subset G(t)^{-1}A^*G(t)$ and thus

$$(G(t)^{-1}A^*G(t)-\lambda)f = (A-\lambda)f = (G(t)^{-1}A^*G(t)-\lambda)g.$$

Therefore

$$0 = (G(t)^{-1}A^*G(t)-\lambda)(f-g) = G(t)^{-1}(A^*-\lambda)G(t)(f-g),$$

and thus $G(t)(g-f)$ is in the kernel of $A^*-\lambda$. By $\lambda, \overline{\lambda} \in \rho(A)$, we conclude $f-g=0$ and $g=f\in \mathrm{dom} A$. □

Note that in general the operator A in Proposition 3.11 is not $G(t)$-selfadjoint. For example let $U := iI$ and suppose that A is unbounded. Then $G(\pi/2) = 0$ and $G(\pi/2)A$ is the restriction of the zero operator to $\mathrm{dom}\, A$, whereas $(G(\pi/2)A)^*$ equals the zero operator on $\mathfrak{S}$. Hence, in this case, A is not $G(\pi/2)$-selfadjoint.

If $G(t)$ is boundedly invertible, then the space $\mathfrak{S}$ equipped with the inner product $(G(t)\,\cdot\,,-)$ is a Krein space. The following theorem follows immediately from Proposition 3.10.

Theorem 3.11. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,,-])$. If for some $t \in [0,\pi)$ we have $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$ and $\rho(A) \neq \emptyset$, then the operator A is selfadjoint in the Krein space $(\mathfrak{S}, (G(t)\,\cdot\,,-))$.*

If in the situation of Theorem 3.11 the operator U satisfies some additional assumptions, more can be said about the spectrum of A.

Theorem 3.12. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,,-])$ with $\rho(A) \neq \emptyset$ and assume that there is some $t \in [0,\pi)$ such that $\mathrm{e}^{-\mathrm{i}t}, -\mathrm{e}^{-\mathrm{i}t} \in \rho(U)$. Let $\mathbb{T} = \mathbb{T}_1 \dot{\cup} \mathbb{T}_2$ be a decomposition of the unit circle, where*

$$\mathbb{T}_1 := \{\mathrm{e}^{\mathrm{i}s} : -t \le s < -t+\pi\} \quad \text{and} \quad \mathbb{T}_2 := \{\mathrm{e}^{\mathrm{i}s} : -t+\pi \le s < -t+2\pi\}.$$

If $\mathbb{T}_1 \cap \sigma(U) = \emptyset$ or $\mathbb{T}_2 \cap \sigma(U) = \emptyset$ then A is selfadjoint in the Hilbert space $(\mathfrak{S}, (G(t)\,\cdot\,,-))$. In particular,

$$\sigma(A) \subset \mathbb{R}.$$

If $\mathbb{T}_1 \cap \sigma(U)$ or $\mathbb{T}_2 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U with finite multiplicities then the non-real spectrum of A consists of finitely many isolated eigenvalues with finite algebraic multiplicities,

$$\sigma(A) \setminus \mathbb{R} = \{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, \dots \lambda_n, \overline{\lambda_n}\} \subset \sigma_p(A).$$

Proof. We define

$$\widetilde{U} := \mathrm{e}^{\mathrm{i}t} U.$$

Then $\pm 1 \in \rho(\widetilde{U})$. The operator A is selfadjoint in the S-space $(\mathfrak{S}, [\,\cdot\,, -]_\sim)$, where $[\,\cdot\,, -]_\sim)$ is given by

$$[f, g]_\sim := (\widetilde{U} f, g) \quad \text{for all } f, g \in \mathfrak{S}.$$

By Theorem 3.11, A is selfadjoint in the Krein space $(\mathfrak{S}, (\operatorname{Im} \widetilde{U}\,\cdot\,, \,\cdot\,))$. If $\mathbb{T}_1 \cap \sigma(U) = \emptyset$ then $\operatorname{Im} \widetilde{U}$ is a uniformly negative operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$, and hence A is a selfadjoint operator in the Hilbert space $(\mathfrak{S}, -(\operatorname{Im} \widetilde{U}\,\cdot\,, \,\cdot))$. A similar argument holds for the case $\mathbb{T}_2 \cap \sigma(U) = \emptyset$ and the first assertion of the theorem is proved.

If $\mathbb{T}_1 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U with finite multiplicity then $\operatorname{Im} \widetilde{U}$ is a bounded and boundedly invertible selfadjoint operator in the Hilbert space $(\mathfrak{S}, (\,\cdot\,, -))$. Moreover, the spectral subspace of $\operatorname{Im} \widetilde{U}$ corresponding to the positive real numbers is finite dimensional. Therefore A is a selfadjoint operator in the Pontryagin space $(\mathfrak{S}, (\operatorname{Im} \widetilde{U}\,\cdot\,, \,\cdot\,))$ and the second assertion of the theorem follows from well-known properties of selfadjoint operators in Pontryagin spaces, see, e.g., [5]. Similar arguments apply if $\mathbb{T}_2 \cap \sigma(U)$ consists of finitely many isolated eigenvalues of U. $\square$

The following theorem is the main result of this paper. It shows that for operators with nonempty resolvent set the notions of S-space selfadjointness and Krein space selfadjointness coincide.

Theorem 3.13. *Let A be a selfadjoint operator in the S-space $(\mathfrak{S}, [\,\cdot\,, -])$. If $\rho(A) \neq \emptyset$ then there exists a Krein space inner product $\langle\,\cdot\,, -\rangle$ such that A is selfadjoint in the Krein space $(\mathfrak{S}, \langle\,\cdot\,, -\rangle)$. Moreover, if E_U denotes the spectral measure of U and if Δ is a Borel subset of the unit circle $\mathbb{T}$ with the property that $x \in \Delta$ implies $-x \in \Delta$, then the spectral subspace $E_U(\Delta)\mathfrak{S}$ is an invariant subspace for $(A - \lambda)^{-1}$, $\lambda \in \rho(A)$.*

Proof. We choose some $\varepsilon \in (0, \pi/2)$ and define

$$\Delta_1 := \{\mathrm{e}^{\mathrm{i}t} : t \in (-\varepsilon, \varepsilon)\} \cup \{-\mathrm{e}^{\mathrm{i}t} : t \in (-\varepsilon, \varepsilon)\}, \quad \Delta_2 := \mathbb{T} \setminus \Delta_1.$$

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be the spectral subspaces of U corresponding to Δ_1 and Δ_2, respectively, i.e.

$$\mathfrak{S}_1 = E_U(\Delta_1)\mathfrak{S} \quad \text{and} \quad \mathfrak{S}_2 = E_U(\Delta_2)\mathfrak{S}.$$

Then we have

$$\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2.$$

We define the sets

$$\Delta_1^2 := \{\mathrm{e}^{\mathrm{i}t} : t \in (-2\varepsilon, 2\varepsilon)\} \quad \text{and} \quad \Delta_2^2 := \mathbb{T} \setminus \Delta_1^2 = \{z^2 : z \in \Delta_2\}.$$

If E_{U^2} denotes the spectral measure of U^2 and $h : \mathbb{C} \to \mathbb{C}$ denotes the function given by $h(z) = z^2$, then we deduce from the properties of the functional calculus for unitary operators for $j = 1, 2$

$$E_{U^2}(\Delta_j^2) = \mathbf{1}_{\Delta_j^2}(U^2) = (\mathbf{1}_{\Delta_j^2} \circ h)(U) = \mathbf{1}_{h^{-1}(\Delta_j^2)}(U) = E_U(\Delta_j),$$

where $\mathbf{1}_\Delta$ is the indicator function corresponding to a Borel set Δ and $h^{-1}(\Delta_j^2)$ denotes the pre-image of Δ_j^2 under h. Therefore, the spectral subspace of $D = U^2$ corresponding to Δ_j^2 coincides with $\mathfrak{S}_j$, $j = 1, 2$. By Proposition 3.3 we have $AD = DA$. Hence, for $\lambda \in \rho(A)$ the operator $(A - \lambda)^{-1}$ commutes with D which implies that the spectral subspaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ of D are invariant under $(A - \lambda)^{-1}$. In particular, we have

$$\operatorname{dom} A = (\operatorname{dom} A \cap \mathfrak{S}_1) \oplus (\operatorname{dom} A \cap \mathfrak{S}_2),$$

and $\mathfrak{S}_1$ and $\mathfrak{S}_2$ both are invariant under A. Thus, with respect to the decomposition $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$ the operators A and U decompose as $A = A_1 \oplus A_2$ and $U = U_1 \oplus U_2$, where $A_j = A|\mathfrak{S}_j$ and $U_j = U|\mathfrak{S}_j$, $j = 1, 2$. It is easy to see that A_1 is selfadjoint in the S-space $(\mathfrak{S}_1, (U_1 \cdot, -))$ and that A_2 is selfadjoint in the S-space $(\mathfrak{S}_2, (U_2 \cdot, -))$. And, as $\rho(A) \neq \emptyset$, it follows that also the resolvent sets of A_1 and A_2 are nonempty. Since $i, -i \in \rho(U_1)$ and $1, -1 \in \rho(U_2)$, it follows from Theorem 3.11 that there are Krein space inner products $\langle \cdot, - \rangle_1$ and $\langle \cdot, - \rangle_2$ in $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively, such that A_j is selfadjoint in the Krein space $(\mathfrak{S}_j, \langle \cdot, - \rangle_j)$, $j = 1, 2$. Hence, A is obviously selfadjoint in the Krein space $(\mathfrak{S}, \langle \cdot, - \rangle)$, where $\langle \cdot, - \rangle$ is given by

$$\langle x, v \rangle := \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2,$$

$x = x_1 + x_2$, $y = y_1 + y_2$, $x_1, x_2 \in \mathfrak{S}_1$, $x_2, y_2 \in \mathfrak{S}_2$. $\square$

Remark 3.14. *Each Krein space is also an S-space, hence, obviously, every selfadjoint operator in a Krein space is simultaneously selfadjoint in an S-space. Theorem 3.13 shows that the contrary is true, as long as the resolvent* *set is nonempty. Hence, for each selfadjoint operator A in an S-space $\mathfrak{S}$ with $\rho(A) \neq \emptyset$ we find an inner product which turns $\mathfrak{S}$ into a Krein space and A into a selfadjoint operator with respect to this inner product. In addition, as revealed in the proof of Theorem 3.13, all spectral subspaces of U which correspond to Borel sets symmetric with respect to $z \mapsto -z$ are invariant subspaces of A.*

Example 3.15. As an illustration of Theorem 3.13 we consider a simple example with 2×2 matrices. Let U be unitary in $\mathbb{C}^2$ and choose an orthonormal basis of $\mathbb{C}^2$ such that the corresponding matrix is diagonal with entries $z_1, z_2 \in \mathbb{T}$. A matrix with entries $a, b, c, d \in \mathbb{C}$ which is selfadjoint in the S-space given by U has to satisfy

$$\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix},$$

cf. Proposition 3.6, Part (iii). We assume $cb \neq 0$. From this we see that a and d are real, $z_1 = \pm z_2$ and $b = \pm \overline{c}$. Hence, either the matrix is selfadjoint (in the case $z_1 = z_2$) or, if $z_1 = -z_2$, we have $b = -\overline{c}$ and the matrix is selfadjoint in the (finite dimensional) Krein space with fundamenal symmetry

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Concluding Remarks

S-spaces are Hilbert spaces with an additional inner product given by an unitary Gramian U. Krein spaces are special cases of S-spaces as their Gramian can be choosen to be selfadjoint and simultaneously unitary.

From this point of view, the class of S-spaces is larger then the class of Krein spaces. It is the main result of this paper that the class of selfadjoint operators in S-spaces having nonempty resolvent set is not larger than the correponding class in Krein spaces. The nonemptiness of the resolvent set is crucial for the proof presented here, the corresponding question for selfadjoint operators with empty resolvent set is not considered.

Moreover, Theorem 3.13 reveals an interesting fact: A selfadjoint operator in an S-space having a Gramian U with spectrum larger than the set $\{-1, 1\}$ has invariant subspaces - a fact which is not known a priori for selfadjoint operators in Krein spaces. An interesting, and so far unanswered, question is: Which class of selfadjoint operators in Krein spaces are selfadjoint in an S-space with a Gramian U which has a spectrum larger than $\{-1, 1\}$ (and, hence, gives rise to many invariant subspaces)?

Acknowledgements

The first and the second author would like to acknowledge an assistance of the EU Sixth Framework Programme for the Transfer of Knowledge "Operator theory methods for differential equations" (TODEQ) # MTKDCT-2005-030042. The first author is grateful to the Deutsche Forschungsgesellschaft (DFG) for their support under grant BE 3765/5-1 TR 904/4-1. Moreover, the work of the second author was partially supported by the MNiSzW grant N201 026 32/1350.

[1] T.Ya. Azizov, "About compact operators which are selfadjoint with respect to a degenerated indefinite inner product", Matematicheskie Issledovaniya, Kishinyov, UII:4 Vol. 26, pg. 237-240, 1972 (Russian).

[2] T.Ya. Azizov and I.S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley & Sons, Ltd., Chichester, 1989.

[3] T.Ya. Azizov and L.I. Soukhotcheva, "Linear operators in almost Krein spaces", Oper. Theory Adv. Appl., Vol. 175, pg. 1-11, 2007.

[4] P. Binding and R. Hryniv, "Full and partial range completeness", Oper. Theory Adv. Appl., Vol. 130, pg. 121-133, 2002.

[5] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag, New York-Heidelberg, 1974.

[6] V. Bolotnikov, C.-K. Li, P. Meade, C. Mehl and L. Rodman, "Shells of matrices in indefinite inner product spaces", Electron. J. Linear Algebra, Vol. 9, pg. 67-92, 2002.

[7] Dunford N. and Schwartz J. T., *Linear Operators. Part II: Spectral Theory*, John Wiley & Sons, 1963.

[8] M.-A. Henn, C. Mehl and C. Trunk, "Hyponormal and strongly hyponormal matrices in inner product spaces", submitted.

[9] M. Kaltenbäck, H. Winkler and H. Woracek, "Almost Pontryagin spaces", Oper. Theory Adv. Appl., Vol. 160, pg. 253-371, 2005.

[10] M. Kaltenbäck and H. Woracek, "Selfadjoint extensions of symmetric operators in degenerated inner product spaces", Integral Equations Operator Theory, Vol. 28, pg. 289-320, 1997.

[11] M. Kaltenbäck and H. Woracek, "On extensions of Hermitian functions with a finite number of negative squares", J. Operator Theory, Vol. 40, pg. 147-183, 1998.

[12] M. Kaltenbäck and H. Woracek, "The Krein formula for generalized resolvents in degenerated inner product spaces", Monatsh. Math., Vol. 127, pg. 119-140, 1999.

[13] M.G. Krein, "Introduction to the theory of indefinite J-spaces and to the theory of operators in those spaces", Amer. Math. Soc. Transl., Vol. 93, pg. 103-176, 1970.

[14] P. Lancaster, A.S. Markus and P. Zizler, "The order of neutrality for linear operators on inner product spaces", Linear Algebra Appl., Vol. 259, pg. 25-29, 1997.

[15] H. Langer, A.S. Markus and V.I. Matsaev, "Locally definite operators in indefinite inner product spaces", Math. Ann., Vol. 308, pg. 405-424, 1997.

[16] H. Langer, R. Mennicken and C. Tretter, "A self-adjoint linear pencil $Q - \lambda P$ of ordinary differential operators", Methods Funct. Anal. Topology, Vol. 2, pg. 38-54, 1996.

[17] C. Mehl, A.C.M. Ran and L. Rodman, "Semidefinite invariant subspaces: degenerate inner products", Oper. Theory Adv. Appl., Vol. 149, pg. 467-486, 2004.

[18] C. Mehl and L. Rodman, "Symmetric matrices with respect to sesquilinear forms", Linear Algebra Appl., Vol. 349, pg. 55-75, 2002.

[19] C. Mehl and C. Trunk, "Normal matrices in inner product spaces", Oper. Theory Adv. Appl., Vol. 175, pg. 193-209, 2007.

[20] F. Philipp and C. Trunk, "G-selfadjoint operators in Almost Pontryagin spaces", Oper. Theory Adv. Appl., Vol. 188, pg. 207-235, 2008.

[21] F. Philipp and C. Trunk, "Spectral points of type π_+ and type π_- of closed operators in indefinite inner product spaces", manuscript.

[22] B. C. Ritsner, *The Theory of Linear Relations* (Russian), Voronezh, Dep. VINITI, No. 846-82, 1982.

[23] F.H. Szafraniec, "Two-sided weighted shifts are almost Krein normal", Oper. Theory Adv. Appl., Vol. 188, pg. 245-250, 2008.

[24] F.H. Szafraniec, "Dissymmetrising inner product spaces", manuscript, in preparation.

[25] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer-Verlag, New York-Heidelberg-Berlin, 1980.

[26] H. Woracek, "An operator theoretic approach to degenerated Nevanlinna-Pick interpolation", Math. Nachr., Vol. 176, 335-350, 1995.

[27] H. Woracek, "Resolvent matrices in degenerate inner product spaces", Math. Nachr., Vol. 213, pg. 155-175, 2000.