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# Constructing a Dominating Set for bipartite graphs in several Rounds

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## Abstract

Using the probabilistic method, new upper bounds on the domination number of a bipartite graph in terms of the cardinalities and the minimum degrees of the two colour classes are established.

**Keywords.** Domination, bipartite graph, probabilistic method, multilinear function  
**2010 MSC:** 05C69

We consider finite, undirected and simple graphs without isolated vertices. The domination number  $\gamma = \gamma(G)$  of a bipartite graph  $G = (A, B, E)$  is the minimum cardinality of a set  $D \subseteq V$  of vertices such that every vertex in  $V \setminus D$  has a neighbour in  $D$ . This parameter is one of the most well-studied in graph theory, and the two volume monograph [5, 6] provides an impressive account of the research related to this concept.

Based on the results of [3] we adopt the approach to create a dominating set for an arbitrary graph in several rounds to bipartite graphs. The idea is to choose vertices for a dominating set  $D$  at random. In every round we only want to choose those which are not dominated by the previous ones. Extensive calculations cause a modified idea.

We choose  $k$  sets  $X_1, \dots, X_k$  independently at random and add from every set  $X_i$  only those vertices to  $D$  that are not dominated by the first  $(i - 1)$  sets. Contrary to the results in [3] for arbitrary graphs, we use different probabilities  $p_i$  and  $q_i$  in  $A$  and  $B$  for the vertices to belong to a set  $X_i$ . With this we are able to prove the following theorem.

**Theorem 1** *Let  $G = (A, B, E)$  be a bipartite graph of maximum degrees  $\Delta_1$  and  $\Delta_2$  of the two colour classes  $A$  and  $B$  and girth at least six.*

*For some  $k \in \mathbb{N}$  let  $p_1, \dots, p_k$  and  $q_1, \dots, q_k \in [0, 1]$ . If  $p_{<1} = q_{<1} = 0$ ,  $p_{<i} = 1 - \prod_{j=1}^{i-1} (1 - p_j)$*

*and  $q_{<i} = 1 - \prod_{j=1}^{i-1} (1 - q_j)$  for  $2 \leq i \leq k$ , then*

$$\gamma(G) \leq \sum_{v \in A} \left( \prod_{i=1}^k (1 - p_i)(1 - q_i)^{d_G(v)} + \sum_{i=1}^k \left[ p_i \cdot (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(v)} + \right. \right. \\ \left. \left. (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(v)} \cdot (1 - p_i) \cdot \left( \left( 1 - q_i(1 - p_{<i})^{(\Delta_B - 1)} \right)^{d_G(v)} - (1 - q_i)^{d_G(v)} \right) \right] \right)$$

$$+ \sum_{v \in B} \left( \prod_{i=1}^k (1 - q_i)(1 - p_i)^{d_G(v)} + \sum_{i=1}^k \left[ q_i \cdot (1 - q_{<i}) \cdot (1 - p_{<i})^{d_G(v)} + (1 - q_{<i}) \cdot (1 - p_{<i})^{d_G(v)} \cdot (1 - q_i) \cdot \left( (1 - p_i(1 - q_{<i})^{(\Delta_A - 1)})^{d_G(v)} - (1 - p_i)^{d_G(v)} \right) \right] \right)$$

*Proof:* For  $1 \leq i \leq k$  let  $X_{A_i}$  ( $X_{B_i}$ ) be a subset of  $A$  ( $B$ ) which arises by choosing every vertex of  $A$  ( $B$ ) independently at random with probability  $p_i$  ( $q_i$ ). Let  $Y_{A_1} = X_{A_1}$ ,  $Y_{B_1} = X_{B_1}$  and  $Z_{A_1} = Z_{B_1} = \emptyset$ . For  $2 \leq i \leq k$  let

$$X_{A_{<i}} = \bigcup_{j=1}^{i-1} X_{A_j}, \quad X_{B_{<i}} = \bigcup_{j=1}^{i-1} X_{B_j},$$

$$Y_{A_i} = X_{A_i} \setminus (X_{A_{<i}} \cup N(X_{B_{<i}})), \quad Y_{B_i} = X_{B_i} \setminus (X_{B_{<i}} \cup N(X_{A_{<i}}))$$

and

$$Z_{A_i} = N(X_{B_i}) \setminus (X_{A_{<i}} \cup Y_{A_i} \cup N(X_{B_{<i}} \cup Y_{B_i}))$$

$$Z_{B_i} = N(X_{A_i}) \setminus (X_{B_{<i}} \cup Y_{B_i} \cup N(X_{A_{<i}} \cup Y_{A_i})).$$

Let

$$R_A = A \setminus \left( \bigcup_{j=1}^k X_{A_j} \cup N \left( \bigcup_{j=1}^k X_{B_j} \right) \right), \quad R_B = B \setminus \left( \bigcup_{j=1}^k X_{B_j} \cup N \left( \bigcup_{j=1}^k X_{A_j} \right) \right).$$

**Claim 1** For  $1 \leq i \leq k$  is

$$N_G(X_{A_1} \cup \dots \cup X_{A_i}) \cup X_{B_1} \cup \dots \cup X_{B_i}$$

$$\subseteq (Y_{B_1} \cup Z_{B_1}) \cup \dots \cup (Y_{B_i} \cup Z_{B_i}) \cup N_G((Y_{A_1} \cup Z_{A_1}) \cup \dots \cup (Y_{A_i} \cup Z_{A_i}))$$

and

$$N_G(X_{B_1} \cup \dots \cup X_{B_i}) \cup X_{A_1} \cup \dots \cup X_{A_i}$$

$$\subseteq (Y_{A_1} \cup Z_{A_1}) \cup \dots \cup (Y_{A_i} \cup Z_{A_i}) \cup N_G((Y_{B_1} \cup Z_{B_1}) \cup \dots \cup (Y_{B_i} \cup Z_{B_i})).$$

*Proof of Claim 1:* We only prove the first equation of Claim 1 by induction. The second part follows analogously. For  $i = 1$  we get

$$N[X_{A_1} \cup X_{B_1}] \cap A \subseteq (Y_{A_1} \cup Z_{A_1}) \cup N(Y_{B_1} \cup Z_{B_1})$$

$$\Leftrightarrow X_{A_1} \cup N(X_{B_1}) \subseteq (Y_{A_1} \cup Z_{A_1}) \cup N(Y_{B_1} \cup Z_{B_1})$$

and this is easy to see, because  $X_{A_1} = Y_{A_1} \cup Z_{A_1}$  and  $X_{B_1} = Y_{B_1} \cup Z_{B_1}$ . For  $i \geq 2$ , by induction,

$$N(X_{A_1} \cup \dots \cup X_{A_{i-1}}) \cup X_{B_1} \cup \dots \cup X_{B_{i-1}}$$

$$\subseteq (Y_{B_1} \cup Z_{B_1}) \cup \dots \cup (Y_{B_i} \cup Z_{B_i}) \cup N((Y_{A_1} \cup Z_{A_1}) \cup \dots \cup (Y_{A_i} \cup Z_{A_i}))$$

and it suffices to show

$$X_{B_i} \cup N(X_{A_i}) \subseteq (Y_{B_1} \cup Z_{B_1}) \cup \dots \cup (Y_{B_i} \cup Z_{B_i}) \cup N((Y_{A_1} \cup Z_{A_1}) \cup \dots \cup (Y_{A_i} \cup Z_{A_i})).$$

Case 1) If  $x \in X_{Ai}$ , then either  $x \in Y_{Bi}$  or  $x \in N(X_{A<i}) \cup X_{B<i}$ . In both cases we are done.

Case 2) If  $x \in N(X_{Ai})$ , then either  $x \in N(X_{A<i}) \cup X_{B<i}$  or  $x \in Y_{Bi} \cup N[Y_{Ai}]$  or, by definition,  $x \in Z_{Bi}$ . Again in all cases we are done and the proof of the claim is complete.  $\square$

Note that, by the claim and the definition of  $R_A$  and  $R_B$ , the set

$$D = R_A \cup R_B \cup \left( \bigcup_{i=1}^k (Y_{Ai} \cup Z_{Ai}) \right) \cup \left( \bigcup_{i=1}^k (Y_{Bi} \cup Z_{Bi}) \right)$$

is a dominating set of  $G$ .

The expected cardinality of  $Y_{A1}$  is  $p_1|A| = p_1a$ . Now let  $2 \leq i \leq k$ . Since the sets  $X_1, \dots, X_{i-1}$  are chosen independently, the set  $X_{<i}$  arises by choosing every vertex of  $G$  independently at random with probability

$$p_{<i} = 1 - \prod_{j=1}^{i-1} (1 - p_j).$$

Hence

$$\mathbb{P}[x \in Y_{Ai}] = p_i \cdot (1 - p_{<i}) \cdot (1 - q_{<i})^{d_A(x)}$$

for every vertex  $x \in A$ .

Analogous we get  $E(|Y_{B1}|) = q_1|B| = q_1b$ ,  $q_{<i} = 1 - \prod_{j=1}^{i-1} (1 - q_j)$  and

$$\mathbb{P}[x \in Y_{Bi}] = q_i \cdot (1 - q_{<i}) \cdot (1 - p_{<i})^{d_B(x)}$$

for every vertex  $x \in B$ .

Furthermore, a vertex  $x \in A$  is in  $Z_{Ai}$  if and only if  $x \notin X_{A<i} \vee x \in N_G[X_{B<i}]$  and  $x \notin X_{Ai}$  and there is some non-empty set  $U \subseteq N_G(x)$  with  $N_G(x) \cap (N_G(X_{A<i}) \cap X_{Bi}) = U$  and  $N_G(x) \cap (B \setminus X_{Bi}) = N_G(x) \setminus U$ .

For some specific set  $U$  let

$$N_G(x) \setminus U = \{x_1, x_2, \dots, x_{d_G(x)-l}\}$$

and

$$U = \{x_{d_G(x)-l+1}, x_{d_G(x)-l+2}, \dots, x_{d_G(x)}\}.$$

By the independence of the choice of the elements of the sets  $X_j$  and by the girth condition, we obtain - in what follows we indicate the use of the independence by “(i)” and the use of

the girth condition by “(g)”

$$\begin{aligned}
& \mathbb{P}[v \in Z_{A_i} \cap (N(v) \cap N(X_{A<i}) \cap X_{B_i} = U) \wedge (N(v) \cap (B \setminus X_{B_i}) = (N(v) \setminus U))] \\
&= \mathbb{P} \left[ (v \notin X_{A<i}) \wedge (v \notin N(X_{B<i})) \wedge (v \notin X_{A_i}) \wedge \left( \bigwedge_{j=1}^{d(v)-l} (v_j \notin X_{B_i}) \right) \right. \\
&\quad \left. \wedge \left( \bigwedge_{j=d^A(v)-l+1}^{d^A(v)} (v_j \in N(X_{A<i}) \cap X_{B_i}) \right) \right], \\
&\stackrel{(i)}{=} (1 - p_{<i}) \cdot (1 - q_{<i})^{d(v)} \cdot (1 - p_i) \cdot (1 - q_i)^{(d(v)-l)} \\
&\quad \cdot \mathbb{P} \left[ \left( \bigwedge_{j=d(v)-l+1}^{d(v)} (v_j \in N(X_{A<i}) \cap X_{B_i}) \right) \mid (v \notin X_{A<i} \wedge v \notin N(X_{B<i})) \right], \\
&\stackrel{(i)}{=} (1 - p_{<i}) \cdot (1 - q_{<i})^{d(v)} \cdot (1 - p_i) \cdot (1 - q_i)^{(d(v)-l)} \\
&\quad \cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P} \left[ (v_j \in N(X_{A<i}) \cap X_{B_i}) \mid \left( \bigwedge_{r=d(v)-l+1}^{j-1} (v_r \in N(X_{A<i}) \cap X_{B_i}) \right) \right. \\
&\quad \left. \wedge (v \notin X_{A<i} \wedge v \notin N(X_{B<i})) \right], \\
&\stackrel{(i)}{=} (1 - p_{<i}) \cdot (1 - q_{<i})^{d(v)} \cdot (1 - p_i) \cdot (1 - q_i)^{(d(v)-l)} \cdot q_i^l \\
&\quad \cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P} \left[ (v_j \in N(X_{A<i})) \mid \left( \bigwedge_{r=d(v)-l+1}^{j-1} v_r \in N(X_{A<i}) \right) \wedge (v \notin X_{A<i} \wedge v \notin N(X_{B<i})) \right], \\
&\stackrel{(g)}{=} (1 - p_{<i}) \cdot (1 - q_{<i})^{d(v)} \cdot (1 - p_i) \cdot (1 - q_i)^{(d(v)-l)} \cdot q_i^l \\
&\quad \cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P} [(v_j \in N(X_{A<i})) \mid (v \notin X_{A<i} \wedge v \notin N(X_{B<i}))], \\
&\stackrel{(g)}{=} (1 - p_{<i})(1 - q_{<i})^{d(v)}(1 - p_i)(1 - q_i)^{(d(v)-l)} q_i^l \prod_{j=d(v)-l+1}^{d(v)} \left( 1 - (1 - p_{<i})^{(d(v_j)-1)} \right) \\
&\leq (1 - p_{<i}) \cdot (1 - q_{<i})^{d(v)} \cdot (1 - p_i) \cdot (1 - q_i)^{(d(v)-l)} \cdot q_i^l \cdot \left( 1 - (1 - p_{<i})^{(\Delta_B - 1)} \right)^l.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \mathbb{P}[v \in Z^{A_i}] \\
&\leq (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(x)} \cdot (1 - p_i) \cdot \sum_{l=1}^{d_G(x)} \binom{d_G(x)}{l} \cdot (1 - q_i)^{(d_G(x)-l)} \cdot q_i^l \cdot \left( 1 - (1 - p_{<i})^{(\Delta_{V_2} - 1)} \right)^l \\
&= (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(x)} \cdot (1 - p_i) \cdot \left( \left( (1 - q_i) + q_i \left( 1 - (1 - p_{<i})^{(\Delta_B - 1)} \right) \right)^{d_G(x)} - (1 - q_i)^{d_G(x)} \right) \\
&= (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(x)} \cdot (1 - p_i) \cdot \left( \left( 1 - q_i (1 - p_{<i})^{(\Delta_{V_2} - 1)} \right)^{d_G(x)} - (1 - q_i)^{d_G(x)} \right)
\end{aligned}$$

for every vertex  $x \in A$ .

Finally,

$$\mathbb{P}[x \in R_A] = \prod_{i=1}^k (1 - p_i)(1 - q_i)^{d_G(x)}$$

for every vertex  $x \in A$  and

$$\mathbb{P}[x \in R_B] = \prod_{i=1}^k (1 - q_i)(1 - p_i)^{d_G(x)}$$

for every vertex  $x \in B$ .

By linearity of expectation, we obtain

$$\begin{aligned} \gamma(G) &\leq \mathbb{E}[|D|] \\ &= \mathbb{E}[|R^{V_1}|] + \mathbb{E}[|R^{V_2}|] + \sum_{i=1}^k \left( \mathbb{E}[|Y_i^{V_1}|] + \mathbb{E}[|Y_i^{V_2}|] \right) + \sum_{i=1}^k \left( \mathbb{E}[|Z_i^{V_1}|] + \mathbb{E}[|Z_i^{V_2}|] \right) \\ &\leq \sum_{x \in A} \left( \prod_{i=1}^k (1 - p_i)(1 - q_i)^{d_G(x)} + \sum_{i=1}^k \left[ p_i \cdot (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(x)} + \right. \right. \\ &\quad \left. \left. (1 - p_{<i}) \cdot (1 - q_{<i})^{d_G(x)} \cdot (1 - p_i) \cdot \left( \left( 1 - q_i(1 - p_{<i})^{(\Delta_B - 1)} \right)^{d_G(x)} - (1 - q_i)^{d_G(x)} \right) \right] \right) \\ &\quad + \sum_{x \in B} \left( \prod_{i=1}^k (1 - q_i)(1 - p_i)^{d_G(x)} + \sum_{i=1}^k \left[ q_i \cdot (1 - q_{<i}) \cdot (1 - p_{<i})^{d_G(x)} + \right. \right. \\ &\quad \left. \left. (1 - q_{<i}) \cdot (1 - p_{<i})^{d_G(x)} \cdot (1 - q_i) \cdot \left( \left( 1 - p_i(1 - q_{<i})^{(\Delta_A - 1)} \right)^{d_G(x)} - (1 - p_i)^{d_G(x)} \right) \right] \right) \end{aligned}$$

and the proof is complete.  $\square$

Theorem 1 still leaves the task to find good values for the probabilities  $p_1, \dots, p_k$  and  $q_1, \dots, q_k$ . In order to compare it for instance to the bound of Alon and Spencer ( $\frac{\gamma(G)}{n} \leq \frac{\ln(\delta+1)+1}{\delta+1}$  short AS, vgl. [1]), we present some numerical results for  $r$ - $s$ -regular graphs in two rounds.

Table 1 gives the numerically optimal value for the bound on  $\frac{\gamma(G)}{|V|}$  in Theorem 1. For comparison we also list the result by Harant and Pruchnewski in [4] for bipartite graphs (short HP). To show the improvement compared with the result in [3] for regular graphs, that bound is also shown.

r	s	<i>AS</i>	<i>HP</i>	<i>k</i> rounds (regular)	<i>k</i> rounds (bipartite)
1	2	0,84657359	0,333333	0,650898	0,333333
1	3		0,25	0,573343	0,25
1	5		0,166666	0,464088	0,166666
1	10		0,090909	0,323649	0,090909
2	2	0,69953743	0,5	0,596325	0,426062
2	3		0,4	0,541691	0,37984
2	5		0,285714	0,457228	0,285714
2	10		0,166666	0,337451	0,166666
3	3	0,59657359	0,5	0,499870	0,367340
3	5		0,375	0,432477	0,320023
3	10		0,230769	0,333735	0,230769
5	5	0,465293245	0,417649	0,385762	0,292534
5	10		0,319350	0,311052	0,244798
10	10	0,30889957	0,285899	0,256895	0,203927

**Table 1** Numerical results for Theorem 1

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