Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 09/32

Independence, odd girth, and average degree

Löwenstein, Christian; Pedersen, Anders Sune; Rautenbach, Dieter; Regen, Friedrich

November 2009

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Independence, Odd Girth, and Average Degree

Christian Löwenstein[1],
Anders Sune Pedersen[2],
Dieter Rautenbach[1],
and
Friedrich Regen[1]

[1] Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany, emails: {christian.loewenstein, dieter.rautenbach, friedrich.regen}@tu-ilmenau.de

[2] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark, email: asp@imada.sdu.dk

Abstract

We prove several best-possible lower bounds in terms of the order and the average degree for the independence number of graphs which are connected and/or satisfy some odd girth condition. Our main result is the extension of a lower bound for the independence number of triangle-free graphs of maximum degree at most 3 due to Heckman and Thomas [A New Proof of the Independence Ratio of Triangle-Free Cubic Graphs, *Discrete Math.* **233** (2001), 233-237] to arbitrary triangle-free graphs. For connected triangle-free graphs of order n and size m, our result implies the existence of an independent set of order at least $(4n-m-1)/7$.

Keywords: Independence; stability; triangle-free graph; odd girth
AMS subject classification: 05C69

1 Introduction

We consider *finite*, *simple*, and *undirected graphs* G with *vertex set* $V(G)$ and *edge set* $E(G)$. For a graph G, we denote its *order* by $n(G)$ and its *size* by $m(G)$, respectively. The *open neighbourhood* of a vertex $u \in V(G)$ in a graph G is denoted by $N_G(u)$. The *degree* of u in G is $d_G(u) = |N_G(u)|$ and the *closed neighbourhood* of u in G is $N_G[u] = \{u\} \cup N_G(u)$. The *minimum degree*, *average degree*, and *maximum degree* of G are denoted by $\delta(G)$, $d(G)$, and $\Delta(G)$, respectively. The *odd girth* $g_{\rm odd}(G)$ of a graph G is the minimum length of a cycle of odd length in G. For a set $U \subseteq V(G)$, the subgraph of G *induced* by U is denoted by $G[U]$. A *cutvertex* of a connected graph G is a vertex whose removal disconnects G. A *block* of a graph is a maximal induced subgraph without a cutvertex. An *endblock* of a connected graph G is a block which contains at most one cutvertex of G. A set of vertices $I \subseteq V(G)$ in a graph G is *independent*, if no two vertices in I are adjacent. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set of G. An independent set of G of cardinality $\alpha(G)$ is called *maximum*. For undefined notation and terminology please refer to [3].

The independence number is one of the most fundamental and well-studied graph parameters [13]. In view of its computational hardness, various bounds on the independence number have been proposed. Caro [4] and Wei [18] proved

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{1}{d_G(u)+1} \tag{1}$$

for every graph G. Since the only graphs for which this is best-possible are the disjoint unions of cliques, additional structural assumptions excluding these graphs allow improvements. Natural candidates for such assumptions are triangle-freeness or — more generally — odd girth conditions as well as connectivity.

For triangle-free graphs G, Shearer [15] proved

$$\alpha(G) \geq \sum_{u \in V(G)} f_{\rm Sh}(d_G(u)) \tag{2}$$

where $f_{\rm Sh}(0) = 1$ and $f_{\rm Sh}(d) = \frac{1+(d^2-d)f_{\rm Sh}(d-1)}{d^2+1}$ for $d \in \mathbb{N}$. The function $f_{\rm Sh}$ has the best-possible order of magnitude $f_{\rm Sh}(d) = \Omega\left(\frac{\log d}{d}\right)$. For graphs with a specified odd girth, Denley [5] and Shearer [16] gave best-possible bounds in terms of the vertex degrees.

For triangle-free graphs G with maximum degree at most 3, Heckman and Thomas [11] gave an elegant proof for the best-possible inequality

$$\alpha(G) \geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right)$$

where $\lambda(G)$ counts the number of so-called *difficult components* of G which will be defined later. Their result implies

$$\alpha(G) \geq \frac{5}{14}n(G)$$

for triangle-free graphs G of maximum degree at most 3 which was originally conjectured by Albertson, Bollobás, and Tucker [1] and first proved by Staton [17] (cf. also [2, 6, 7, 10, 12]).

For connected graphs G, Harant and Rautenbach [8] proved the existence of a positive integer $k \in \mathbb{N}$ and a function $f : V(G) \to \mathbb{N}_0$ with non-negative integer values such that $f(u) \leq d_G(u)$ for $u \in V(G)$,

$$\alpha(G) \geq k \geq \sum_{u \in V(G)} \frac{1}{d_G(u)+1-f(u)}, \text{ and } \sum_{u \in V(G)} f(u) \geq 2(k-1). \tag{3}$$

Their result is a best-possible improvement of an earlier result due to Harant and Schiermeyer [9].

The purpose of the present paper is to study independence in graphs with additional structural properties. We want to prove bounds on the independence number of connected graphs subject to odd girth conditions and are mainly interested in bounds depending on the order and the size of the graph.

For a comparison of the results, it is convenient to introduce the following notion. For a class $\mathcal{P}$ of graphs and a $d \in \mathbb{R}_{\geq 0}$, let

$$\alpha(\mathcal{P}, d) = \lim_{n \to \infty} \inf \left\{ \frac{\alpha(G)}{n(G)} \,\middle|\, G \in \mathcal{P}, d(G) \leq d, \text{ and } n(G) \geq n \right\}.$$

Let $\mathcal{G}$, $\mathcal{G}_{\Delta-\rm free}$, and $\mathcal{G}_{\rm conn}$ denote the class of all graphs, all triangle-free graphs, and all connected graphs, respectively.

Using the convexity of the function $x \mapsto \frac{1}{x+1}$, the bound (1) implies that $\alpha(\mathcal{G}, d)$ is at least the linear interpolation of the values $\alpha(\mathcal{G}, d) = \frac{1}{d+1}$ for integral d. For $d \in \mathbb{R}_{\geq 0}$, suitable unions of cliques of orders $\lfloor d+1 \rfloor$ and $\lceil d+1 \rceil$ imply that $\alpha(\mathcal{G}, d)$ is exactly this lower bound. Similarly, a convexity property of f_{Sh} (cf. Lemma 1 in [15]) implies that $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ is at least the linear interpolation of the values $\alpha(\mathcal{G}_{\Delta-\text{free}}, d) = f_{\text{Sh}}(d)$ for integral d. Here suitable unions of complete graphs of orders 1 and 2 and cycles of length 5 imply that $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ is exactly this lower bound for $d \leq 2$. For $d > 2$, the exact value of $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ is unknown. Finally, (3) implies that $\alpha(\mathcal{G}_{\text{conn}}, d)$ is at least the linear interpolation of the values $\alpha(\mathcal{G}_{\text{conn}}, d) = \frac{1}{r}$ for d of the form $d = r - 1 + \frac{2}{r}$ for integral $r \geq 2$. In this case, suitable connected graphs for which the removal of all bridges results in a union of cliques imply that $\alpha(\mathcal{G}_{\text{conn}}, d)$ is exactly this lower bound.

Figure 1 illustrates these bounds.

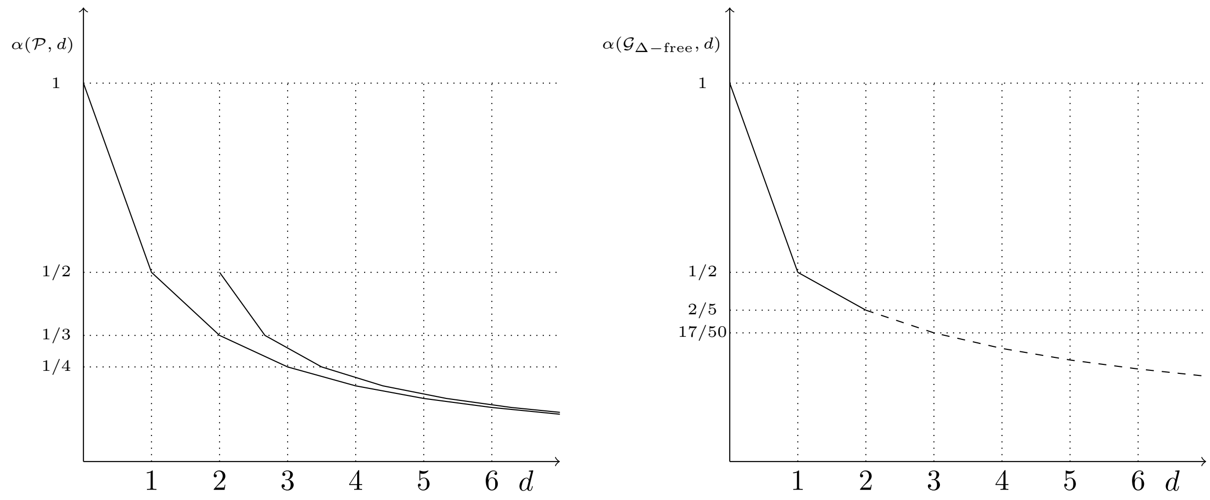

Figure 1 The left graph shows the exact values of $\alpha(\mathcal{P}, d)$ for $\mathcal{P} \in \{\mathcal{G}, \mathcal{G}_{\text{conn}}\}$. The right graph shows the exact value of $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ for $d \leq 2$ and the lower bound based on (2).

Our results are as follows. In Section 2, we prove a lower bound on the independence number of connected graphs of specified odd girth. This result relies on a very simple argument but is best-possible for small average degrees. Still in Section 2, we give a first improvement for arbitrary odd girth and larger average degrees. In Section 3, we prove as our main result that — after a suitable modification — the above-mentioned bound due to Heckman and Thomas [11] still holds even if we drop the maximum degree condition. As a consequence we determine the exact value of $\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d)$ for $d \leq \frac{10}{3}$ and improve the estimate of $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ for $d \in \left]2, \frac{107}{30}\right[$.

2 Connected Graphs with Specified Odd Girth

We need the following two notions related to the independence number. If G is a graph and H is a spanning bipartite subgraph of G with a fixed bipartition $V(G) = A \cup B$, then let

$$\begin{aligned} \alpha\alpha(G) &= \max\{|I_1| + |I_2| \mid I_1 \text{ and } I_2 \text{ are disjoint independent sets in } G\}, \text{ and} \\ \alpha\alpha(G, H) &= \max\{|I_1| + |I_2| \mid (I_1 \subseteq A) \wedge (I_2 \subseteq B) \wedge (I_1 \text{ and } I_2 \text{ are independent sets in } G)\}. \end{aligned}$$

Clearly,

$$2\alpha(G) \geq \alpha\alpha(G) \geq \alpha\alpha(G, H).$$

The basic idea of our approach in this section is captured by the following very simple lemma.

Lemma 1 *If G is a graph and H is a spanning bipartite subgraph of G, then*

$$\alpha\alpha(G,H) \geq n(G) - |E(G) \setminus E(H)|.$$

Proof: Starting with $(I_1, I_2) = (A, B)$ where $V(G) = A \cup B$ is the fixed bipartition of H and adding the edges of $E(G) \setminus E(H)$ one by one to H, we have to remove at most one vertex from either I_1 or I_2 for every added edge. Therefore, after adding all edges from $E(G) \setminus E(H)$ into H, we obtain two disjoint independent sets of G respecting the bipartition of H which are of total cardinality at least $n(G) - |E(G) \setminus E(H)|$. $\Box$

The next result is a first application of this idea.

Proposition 2 *If G is a connected graph and T is a spanning tree of G, then the following statements hold.*

(i)

$$\alpha\alpha(G,T) \geq 2n(G) - m(G) - 1 \tag{4}$$

with equality if and only if $E(G) \setminus E(T)$ is a matching and $T + e = (V(G), E(T) \cup \{e\})$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$.

(ii)

$$\alpha\alpha(G) \geq 2n(G) - m(G) - 1 \tag{5}$$

with equality if and only if all cycles of G are odd and vertex disjoint.

Proof: The lower bounds in (i) and (ii) follow immediately from Lemma 1. It remains to characterize the extremal graphs for (4) and (5).

(i) Let $V(G) = A \cup B$ denote the bipartition of T. If $E(G) \setminus E(T)$ is a matching and $T + e$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$, then $G' = (V(G), E(G) \setminus E(T))$ is the union of complete graphs of orders 1 and 2. Since $\alpha\alpha(G,T) = \alpha(G')$, this easily implies equality in (4).

Conversely, we assume that equality holds in (4). If $T + e$ has no odd cycle for some edge $e \in E(G) \setminus E(T)$, then

$$\alpha\alpha(G,T) = \alpha\alpha(G - e, T) \geq 2n(G) - (m(G) - 1) - 1 = 2n(G) - m(G)$$

which is a contradiction. Hence $T + e$ has an odd cycle for every edge $e \in E(G) \setminus E(T)$.

If $E(G) \setminus E(T)$ contains two distinct edges e and f which are both incident with a common vertex u, then T is a spanning tree of $G' = G - \{e, f\} = (V(G), E(G) \setminus \{e, f\})$. For every pair (I_1', I_2') of disjoint independent sets of G' with $I_1' \subseteq A$ and $I_2' \subseteq B$, $(I_1' \setminus \{u\}, I_2' \setminus \{u\})$ is a pair of disjoint independent sets of G with $I_1 \subseteq A$ and $I_2 \subseteq B$ which implies the contradiction

$$\alpha\alpha(G,T) \geq \alpha\alpha(G',T) - 1 \geq 2n(G') - m(G') - 1 - 1 = 2n(G) - m(G). \tag{6}$$

This completes the proof of (i).

(ii) Let G be a connected graph such that all cycles of G are odd and vertex disjoint. If G contains a vertex u of degree 1, then, by an inductive argument,

$$\begin{aligned}\alpha\alpha(G) &= \alpha\alpha(G[V(G) \setminus \{u\}]) + 1 = 2n(G[V(G) \setminus \{u\}]) - m(G[V(G) \setminus \{u\}]) - 1 + 1 \\ &= 2(n(G) - 1) - (m(G) - 1) - 1 + 1 = 2n(G) - m(G) - 1.\end{aligned}$$

Hence, we may assume that G has an endblock which is an odd cycle C. Clearly, for every pair (I_1, I_2) of disjoint independent sets of G, the set $I_1 \cup I_2$ contains at most $n(C) - 1$ many vertices of C. Let $G' = G[V(G) \setminus V(C)]$. If G' is empty, then G is an odd cycle and equality in (5) is trivial. Otherwise, by an inductive argument,

$$\begin{aligned}\alpha\alpha(G) &\leq \alpha\alpha(G') + n(C) - 1 = 2n(G') - m(G') - 1 + n(C) - 1\\ &= 2(n(G) - n(C)) - (m(G) - (n(C) + 1)) - 1 + n(C) - 1 = 2n(G) - m(G) - 1,\end{aligned}$$

i.e. equality in (5) holds.

Conversely, let G be a connected graph with equality in (5). If G contains two incident edges whose removal does not disconnect the graph, then we obtain a similar contradiction as in (6). Therefore, removing any pair of incident edges disconnects G which immedately implies that all cycles of G are vertex disjoint. In view of this restricted structure of G, the assumption of the existence of an even cycle easily leads to the contradiction $\alpha\alpha(G) \geq 2n(G) - m(G)$ which completes the proof. □

Proposition 2 immediately implies the following.

Corollary 3 *If G is a connected graph, then*

$$\alpha(G) \geq n(G) - \frac{m(G)}{2} - \frac{1}{2} \tag{7}$$

with equality only if all cycles of G are odd and vertex disjoint.

In view of the extremal graphs, the estimates (5) and (7) are best-possible for graphs G of odd girth $g \in 2\mathbb{N} + 1$ if and only if their size is at most $\frac{(g+1)n(G)}{g} - 1$. Intuitively speaking, up to this maximum possible size, the "*price*" of an additional edge is 1 for $\alpha\alpha(G)$ and $1/2$ for $\alpha(G)$. Our next result shows that beyond this maximum possible size, additional edges are at least "50% *off*".

If T is a tree and e is such that $T + e = (V(T), E(T) \cup \{e\})$ is not bipartite, then e is called *T-unfaithful.*

Theorem 4 *Let G be a connected graph. If $m(G) \geq \left\lfloor \frac{(g_{\text{odd}}(G)+1)n(G)}{g_{\text{odd}}(G)} \right\rfloor - 1$, then*

$$2\alpha(G) \geq \alpha\alpha(G) \geq \left\lceil \frac{(g_{\text{odd}}(G) - 1)n(G)}{g_{\text{odd}}(G)} \right\rceil - \frac{1}{2}\left(m(G) - \left(\left\lfloor \frac{(g_{\text{odd}}(G) + 1)n(G)}{g_{\text{odd}}(G)} \right\rfloor - 1\right)\right). \tag{8}$$

Proof: We consider a finite sequence

$$G = G_0, G_1, \ldots, G_k$$

of connected graphs defined as follows. Let $G_0 = G$. If for some $i \in \mathbb{N}_0$, the graph G_i is defined, then let T_i be a spanning tree of G_i. Let m_i denote the number of T_i-unfaithful edges of G_i. Note that all cycles created in T_i by adding a T_i-unfaithful edge of G_i have length at least $g_{\text{odd}}(G)$. If $m_i \leq \left\lfloor \frac{n(G)}{g_{\text{odd}}(G)} \right\rfloor$, then set $k = i$ and terminate the sequence. If $m_i > \left\lfloor \frac{n(G)}{g_{\text{odd}}(G)} \right\rfloor$, then there are two T_i-unfaithful edges of G_i such that the two cycles created in T_i by adding these edges intersect. Clearly, this implies the existence of two incident edges e_i and f_i of G_i such that $G_i - \{e_i, f_i\}$ is connected. Let $G_{i+1} = G_i - \{e_i, f_i\}$. Since, for $i \geq 0$, the graph G_{i+1} arises from G_i by deleting two edges, this process necessarily terminates. By the choice of k, we have $m_{k-1} \geq \left\lfloor \frac{n(G)}{g_{\text{odd}}(G)} \right\rfloor + 1$. Furthermore, since

G_{k-1} has exactly $m(G)-2(k-1)$ edges, we have $m_{k-1} \leq m(G)-(n(G)-1)-2(k-1)$. Combining these two estimates yields

$$k \leq \frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)+\frac{1}{2}$$

with equality if and only if

$$\left\lfloor\frac{n(G)}{g_{\rm odd}(G)}\right\rfloor+1=m_{k-1}=|E(G_{k-1})\setminus E(T_{k-1})|=m(G)-(n(G)-1)-2(k-1), \tag{9}$$

which implies that all edges in $E(G_{k-1})\setminus E(T_{k-1})$ are T_{k-1}-unfaithful.

Let G'_k arise from G_k by deleting all non-T_k-unfaithful edges of G_k which do not belong to T_k. By definition,

$$\alpha\alpha(G_k,T_k)=\alpha\alpha(G'_k,T_k).$$

By Lemma 1,

$$\begin{aligned}\alpha\alpha(G'_k,T_k) &\geq n(G)-|E(G'_k)\setminus E(T_k)|=n(G)-m_k\geq n(G)-\left\lfloor\frac{n(G)}{g_{\rm odd}(G)}\right\rfloor\\ &= \left\lceil\frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)}\right\rceil. \end{aligned}\tag{10}$$

Since, for $0\leq i\leq k-1$, the graph G_{i+1} arises from G_i by deleting two incident edges, we have $\alpha\alpha(G_i)\geq\alpha\alpha(G_{i+1})-1$ which implies

$$\alpha\alpha(G) = \alpha\alpha(G_0)\geq\alpha\alpha(G_k)-k\geq\alpha\alpha(G_k,T_k)-k=\alpha\alpha(G'_k,T_k)-k. \tag{11}$$

If $k<\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)+\frac{1}{2}$, then, by (10) and (11),

$$\begin{aligned}\alpha\alpha(G) &\geq \alpha\alpha(G'_k,T_k)-k\\ &\geq \left\lceil\frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)}\right\rceil-\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right).\end{aligned}$$

If $k=\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)+\frac{1}{2}$, then

$$\begin{aligned}\alpha\alpha(G) &\stackrel{(11)}{\geq} \alpha\alpha(G'_k,T_k)-k\\ &\stackrel{(10)}{\geq} n(G)-m_k-k\\ &= n(G)-(m_{k-1}-2)-k\\ &\stackrel{(9)}{=} n(G)-\left(\left\lfloor\frac{n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)-\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)-\frac{1}{2}\\ &> \left\lceil\frac{(g_{\rm odd}(G)-1)n(G)}{g_{\rm odd}(G)}\right\rceil-\frac{1}{2}\left(m(G)-\left(\left\lfloor\frac{(g_{\rm odd}(G)+1)n(G)}{g_{\rm odd}(G)}\right\rfloor-1\right)\right)\end{aligned}$$

which completes the proof. □

3 Triangle-free Graphs

In order to state the result of Heckman and Thomas [11], we need to define $\lambda(G)$ for triangle-free graphs G of maximum degree at most 3.

Heckman and Thomas call a graph a *difficult block* if it is one of the two graphs G_2 and G_3 in Figure 2. Furthermore, they call a graph G *difficult* if every block of G is either difficult or is an edge between two difficult blocks. For a graph G, $\lambda(G)$ counts the number of components of G which are difficult.

Theorem 5 (Heckman and Thomas [11]) *If G is a triangle-free graph of maximum degree at most 3, then*

$$\alpha(G) \geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right).$$

We will show that, in a suitably modified form, Theorem 5 remains true without the bound on the maximum degree. Our approach will closely follow the method from [11]. A main ingredient of our proof are further difficult blocks. Before we proceed to the formulation of our result, we define a sequence $G_2, G_3, G_4, \ldots$ of graphs. The first six members of this sequence are shown in Figure 2.

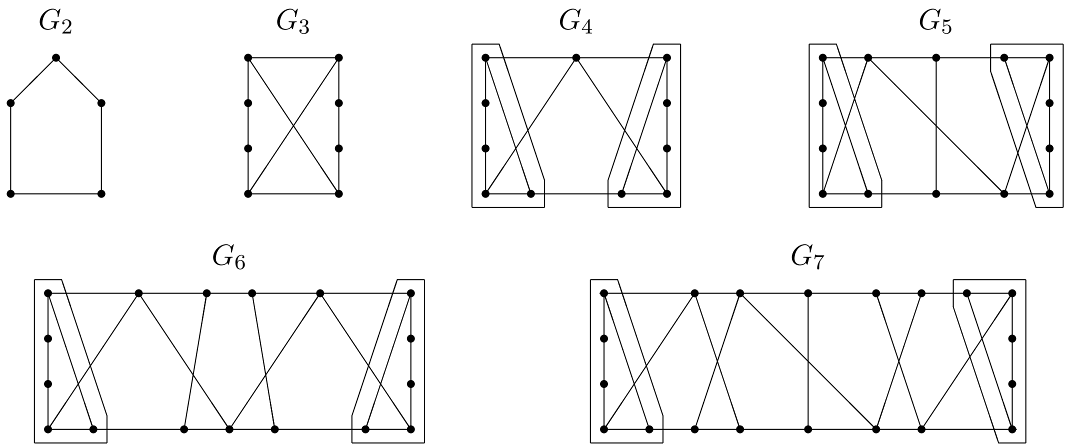

Figure 2 Some difficult blocks.

For $k \geq 4$, each graph G_k in this sequence arises from the graph G_{k-1} by applying the *extension operation* illustrated in Figure 3.

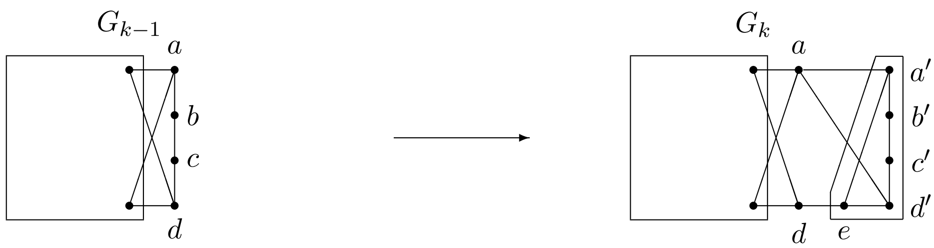

Figure 3 The extension operation.

More formally, assuming inductively that G_{k-1} contains an induced path $abcd$ such that the vertices a and d are of degree 3 and the vertices b and c are of degree 2, then G_k arises from G_{k-1} by deleting the two vertices b and c, adding the new vertices a', b', c', d', and e and new edges as in Figure 3.

It follows inductively that, for $k \geq 3$, the graph G_k contains exactly two vertex disjoint paths $abcd$ such that the vertices a and d are of degree 3 and the vertices b and c are of degree 2. Obviously, applying the extension operation to the reverse path $dcba$ yields an isomorphic graph. Furthermore, we will show in Lemma 7 (i) that it does not matter to which one of the two possible paths we apply the extension operation, i.e. the graphs in the sequence are well-defined.

Furthermore, it follows inductively that, for $k \geq 4$, the graph G_k contains two unique vertex disjoint cycles of length five which contain all vertices of degree 2 but no vertex of degree 4. We call these two cycles the two *ends* of G_k. In Figure 2, the ends of G_4, G_5, G_6, and G_7 are indicated.

The graphs in $\{G_k \mid k \geq 2\}$ are called *difficult blocks*. A graph is called *difficult*, if the removal of all its bridges results in a graph whose components are all difficult blocks. For a graph G, let $\lambda(G)$ denote the number of difficult components of G. Our main result in this section is the following.

Theorem 6 *If G is a triangle-free graph, then*

$$\alpha(G) \geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right).$$

The bound in Theorem 6 is best-possible for all difficult graphs (cf. Claim 1 below). Furthermore, it is clearly also best-possible for all graphs for which the bound in Theorem 5 is best-possible. These graphs have been characterized by Heckman [10].

Before we prove Theorem 6 we establish some useful properties of the difficult blocks.

Lemma 7 *Let $k \geq 2$.*

(i) For $k \geq 4$, G_k has an automorphism which exchanges the ends.

(ii) G_k has order $3k-1$, size $5k-5$, and independence number k.

(iii) For every two vertices u and v of G_k, the graph G_k has a maximum independent set containing neither u nor v.

(iv) If $abcd$ is an induced path of G_k such that the vertices b and c have degree 2 and $u \notin \{a, d\}$ is a vertex of G_k, then the graph G_k has a maximum independent set containing a and d but not u.

Proof: (i) We prove the result by induction on k. For $k \leq 7$, the statement is obvious from Figure 2. Therefore, let $k \geq 8$. By induction, G_k arises by applying the extension operation to both ends of G_{k-2} (cf. Figure 4).

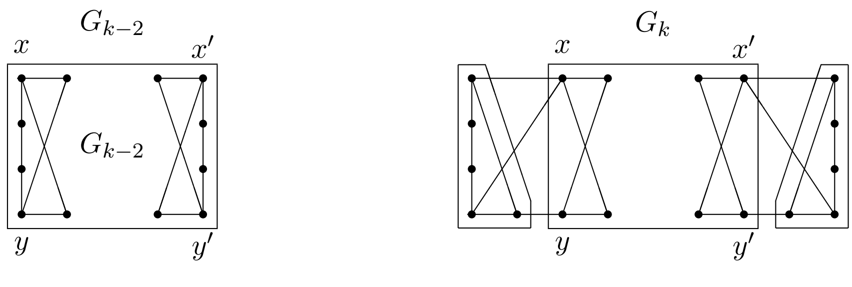

Figure 4 G_{k-2} extended twice.

By induction, G_{k-2} has an automorphism π which exchanges the ends. Clearly, $\{\pi(x), \pi(y)\} = \{x', y'\}$ and also $\{\pi(x'), \pi(y')\} = \{x, y\}$. Since $x \leftrightarrow y$ and $x' \leftrightarrow y'$ are automorphisms of G_{k-2}, we may assume $\pi(x) = x'$, $\pi(x') = x$, $\pi(y) = y'$, and $\pi(y') = y$ which easily implies the existence of the desired automorphism for G_k.

(ii) The order and size of G_k are obvious in view of the definition of the extension operation. We prove that G_k has independence number k by induction on k. For $k \leq 4$, this is obvious. Therefore, let $k \geq 5$. By (i), G_{k-1} arises by applying the extension operation to one end of G_{k-2} and G_k arises by applying the extension operation to the end of G_{k-1} which is disjoint from the vertex set of G_{k-2} (cf. Figure 5).

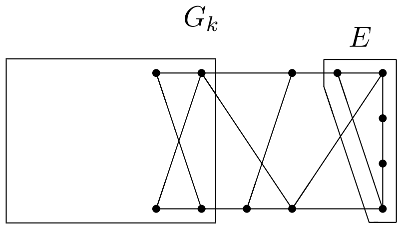

Figure 5 G_{k-2} extended twice together with an end E.

Let I be a maximum independent set of G_k. Clearly, I intersects the set E (cf. Figure 5) in at least 1 and at most 2 vertices. If $|I \cap E| = 1$, then we may assume that I contains a vertex u of degree 2 from E and deleting $N_{G_k}[u]$ from G_k results in G_{k-1}. Hence, by induction, $\alpha(G_k) = |I| = 1 + \alpha(G_{k-1}) = k$. If $|I \cap E| = 2$, then deleting $N_{G_k}[I \cap E]$ from G_k results in G_{k-2}. Hence, by induction, $\alpha(G_k) = |I| = 2 + \alpha(G_{k-2}) = k$, by induction. This completes the proof of (ii).

(iii) We prove this by induction on k. For $k \leq 4$, the statement is easily verified. Therefore, let $k \geq 5$. Let E be an end of G_k which contains as few elements from $\{u, v\}$ as possible. Since the two ends of G_k are vertex disjoint, E contains at most one of the two vertices u and v. By (i), G_k arises by applying the extension operation to an end of G_{k-1} such that E is created as a new end, i.e. using the notation in Figure 3 we have $E = \{a', b', c', d', e\}$. Since $k \geq 5$, the choice of E implies that if $e \in \{u, v\}$, then $d \notin \{u, v\}$.

By induction, G_{k-1} has a maximum independent set I containing neither u nor v. Let $I' = I \backslash \{b, c\}$. If $a, d \in I$, then one of the two sets $I' \cup \{b'\}$ and $I' \cup \{c'\}$ yields the desired independent set. If $b, d \in I$, then one of the two sets $I' \cup \{a', c'\}$ and $I' \cup \{b', d'\}$ yields the desired independent set. If $a, c \in I$, then one of the four sets $I' \cup \{b', e\}$, $I' \cup \{c', e\}$, $I' \cup \{b', d\}$, and $I' \cup \{c', d\}$ yields the desired independent set. If $I \cap \{a, b, c, d\} \subseteq \{b, c\}$, then one of the two sets $I' \cup \{a', c'\}$ and $I' \cup \{b', d'\}$ yields the desired independent set. This completes the proof of (iii).

(iv) Since the result is easy to check for $k \leq 4$, we assume that $k \geq 5$. By (i), we may assume that G_k is as in Figure 5 such that a and d belong to E. Since deleting $N_{G_k}[a] \cup N_{G_k}[d]$ from G_k results in G_{k-2}, the desired statement follows from (iii). This completes the proof. $\Box$

Proof of Theorem 6: In order to obtain a contradiction, we assume that G is a counterexample of mininum order. Clearly, G is connected. Analogously to Heckman and Thomas [11], we prove a series of claims.

Claim 1 *G is not difficult, i.e.* $\lambda(G) = 0$.

Proof of Claim 1: In order to obtain a contradiction, we assume that G is difficult. Let G have b_k blocks isomorphic to G_k for $k \geq 2$. Note that G has $\sum\limits_{k \geq 2} b_k - 1$ bridges. By Lemma 7 (ii) and (iii), we obtain

$$\alpha(G) = \sum_{k \geq 2} k b_k$$

$$= \frac{1}{7}\left(4\sum_{k\geq 2}(3k-1)b_k - \left(\sum_{k\geq 2}(5k-5)b_k + \sum_{k\geq 2}b_k - 1\right) - 1\right)$$
$$= \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right)$$

which is a contradiction. □

Claim 2 *The graph G has no vertex of degree 1.*

Proof of Claim 2: In order to obtain a contradiction, we assume that u is a vertex of degree 1. Let v denote the neighbour of u. If $G' = G[V(G) \setminus \{u, v\}]$, then $\alpha(G) \geq \alpha(G') + 1$ and $\lambda(G') \leq d_G(v) - 1$. We obtain

$$\begin{aligned} & (\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \\ \geq\ & 1 + \frac{1}{7}\big(- 8 + d_G(v) - (d_G(u) - 1)\big) \geq 0. \end{aligned} \tag{12}$$

Now, by the choice of G,

$$\begin{aligned} \alpha(G) &= (\alpha(G) - \alpha(G')) + \alpha(G') \\ &\geq (\alpha(G) - \alpha(G')) + \frac{1}{7}\left(4n(G') - m(G') - \lambda(G')\right) \\ &= \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right) \\ &\quad + (\alpha(G) - \alpha(G')) + \frac{1}{7}\left(-4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\right) \\ &\geq \frac{1}{7}\left(4n(G) - m(G) - \lambda(G)\right) \end{aligned} \tag{13}$$

which is a contradiction. □

For a subgraph H of G, let $\phi(H)$ denote the number of edges of G with exactly one end in $V(H)$.

Claim 3 *If H is a difficult induced subgraph of G, then $\phi(H) \geq 3$. Furthermore, if $\phi(H) = 3$, then H is a difficult block.*

Proof of Claim 3: In order to obtain a contradiction, we assume that H is a difficult induced subgraph of G with $\phi(H) \leq 2$. Clearly, $\phi(B) \leq 2$ for some endblock B of H. Let $k \geq 2$ be such that B is isomorphic to G_k. Let $G' = G[V(G) \setminus V(B)]$. By Lemma 7 (ii) and (iii), $\alpha(G) \geq \alpha(G') + k$. By the definition of a difficult graph, $\lambda(G') \leq \phi(B) - 1$. By Lemma 7 (ii),

$$\begin{aligned} & (\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \\ \geq\ & k + \frac{1}{7}\big(- 4(3k - 1) + (5k - 5 + \phi(B)) - (\phi(B) - 1)\big) = 0. \end{aligned}$$

Arguing as with (12) and (13), we obtain a contradiction. This proves the first part of the desired statement.

If H is not a difficult block, then H has two endblocks B_1 and B_2, and so $\phi(H) \geq (\phi(B_1) - 1) + (\phi(B_2) - 1) \geq 4$. This completes the proof of Claim 3. □

Claim 4 *If v is a vertex of degree 2 and u and w are the neighbours of v, then $d_G(u) + d_G(w) \leq 5$. Furthermore, if $d_G(u) + d_G(w) = 5$, then $G' = G[V(G) \setminus \{u, v, w\}]$ is a difficult block and $\alpha(G) = \alpha(G') + 1$.*

Proof of Claim 4: In order to obtain a contradiction, we assume that $d_G(u) + d_G(w) \geq 6$. Clearly, $\alpha(G) \geq \alpha(G') + 1$. By Claim 3, $\lambda(G') \leq \left\lfloor \frac{d_G(u)+d_G(w)-2}{3} \right\rfloor$. Now

$$\begin{aligned}
& (\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \\
\geq \; & 1 + \frac{1}{7}\left(-12 + (d_G(u) + d_G(w)) - \left\lfloor \frac{d_G(v) + d_G(w) - 2}{3} \right\rfloor\right) \\
= \; & 1 + \frac{1}{7}\left(-10 + \left\lceil \frac{2(d_G(v) + d_G(w) - 2)}{3} \right\rceil\right) \\
\geq \; & 0.
\end{aligned}$$

Arguing as with (12) and (13), we obtain a contradiction.

If $d_G(u) + d_G(w) = 5$ and either $\alpha(G) \geq \alpha(G') + 2$ or $\lambda(G') = 0$, we obtain a similar contradiction as above. Hence $\alpha(G) = \alpha(G') + 1$ and $\lambda(G') = 1$. This, together with Claim 3, implies that G' is a difficult block. □

Claim 5 *If abcd is an induced path such that $d_G(b) = d_G(c) = 2$, then a and d have a common neighbour.*

Proof of Claim 5: In order to obtain a contradiction, we assume that a and d have no common neighbour. This implies that

$$G' = (V(G) \setminus \{b, c\}, (E(G) \setminus \{ab, bc, cd\}) \cup \{ad\}).$$

is triangle-free. Clearly, $\alpha(G) \geq \alpha(G') + 1$. Since G' is connected, $\lambda(G') \leq 1$. Now

$$(\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \geq 1 + \frac{1}{7}(-8 + 2 - 1) = 0.$$

Arguing as with (12) and (13), we obtain a contradiction. □

Claim 6 *If v is a vertex of degree 2 and u and w are the neighbours of v, then $d_G(u) + d_G(w) = 5$.*

Proof of Claim 6: In view of Claim 2 and Claim 4, we assume, for contradiction, that $d_G(u)+d_G(w) = 4$. Let u' be the neighbour of u different from v and let w' be the neighbour of w different from v. By Claim 5, u' and w' are adjacent. Let $G' = G[V(G) \setminus \{u, v, w, u', w'\}]$. Clearly, $\alpha(G) \geq \alpha(G') + 2$. By the definition of a difficult graph, $\lambda(G') \leq (d_G(u') - 2) + (d_G(w') - 2) - 1$. Now

$$\begin{aligned}
& (\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \\
\geq \; & 2 + \frac{1}{7}\big(- 20 + (d_G(u') + d_G(w') + 1) - (d_G(u') + d_G(w') - 5)\big) \\
= \; & 0.
\end{aligned}$$

Arguing as with (12) and (13), we obtain a contradiction. □

Claim 7 *The graph G has no vertex of degree 2.*

Proof of Claim 7: In order to obtain a contradiction, we assume that b is a vertex of degree 2 and that a and c are the neighbours of b. By Claim 6, we may assume that a has degree 3 and c has degree 2. By Claim 4, $G' = G[V(G) \setminus \{a, b, c\}]$ is a difficult block and satisfies $\alpha(G) = \alpha(G') + 1$. If $G' = G_2$, then it follows easily that $G = G_3$ which is a contradiction. Hence, we may assume that $G' = G_k$ for some $k \geq 3$.

By Claim 6, the neighbour d of c different from b has degree 2 in G'. Since G' is a difficult block, d has a neighbour e of degree 2 in G'. Let f denote the neighbour of d in G' different from e and let g denote the neighbour of e in G' different from f. Since g is of degree at least 3, Claim 6 implies that a is adjacent to e in G. Since G is triangle-free, a is not adjacent to g in G. If a is not adjacent to f in G, then Lemma 7 (iv) applied to G', the induced path $gedf$ of G', and the unique vertex u in $N_G(a) \setminus \{b, e\}$ implies the existence of a maximum independent set I' of G' containing g and f but not u. Now $I' \cup \{a, c\}$ is an independent set of G which implies the contradiction $\alpha(G) \geq 2 + \alpha(G')$. Hence a is adjacent to f in G. Now G arises by applying the extension operation to the difficult block G' (cf. Figure 6), i.e., by definition, G is a difficult block which is a contradiction. □

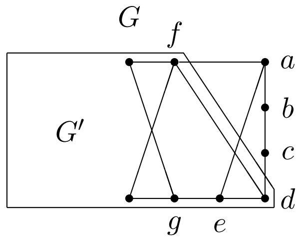

Figure 6 G arises by applying the extension operation to G'.

Claim 8 *If H is a difficult induced subgraph of G, then $\phi(H) \geq 4$.*

Proof of Claim 8: This follows immediately from the fact that every difficult graph has at least four vertices of degree 2 and G has no vertex of degree 2. □

Claim 9 *The minimum degree of G is* 3.

Proof of Claim 9: In order to obtain a contradiction, we assume that the minimum degree of G is $d \geq 4$. Let u be a vertex of minimum degree. Let D denote the degree sum of the neighbours of u. Clearly, $D \geq d^2$. Let $G' = G[V(G) \setminus N_G[u]]$. Clearly, $\alpha(G) \geq \alpha(G') + 1$. By Claim 8, $\lambda(G') \leq \lfloor \frac{D-d}{4} \rfloor$. Now

$$\begin{aligned}
& (\alpha(G) - \alpha(G')) + \frac{1}{7}\big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\big) \\
\geq\ & 1 + \frac{1}{7}\left(-4(d+1) + D - \left\lfloor \frac{D-d}{4} \right\rfloor\right) \\
=\ & 1 + \frac{1}{7}\left(-4(d+1) + d + \left\lceil \frac{3(D-d)}{4} \right\rceil\right) \\
\geq\ & 1 + \frac{1}{7}\left(-4(d+1) + d + \left\lceil \frac{3(d^2-d)}{4} \right\rceil\right) \\
\geq\ & 0.
\end{aligned}$$

Arguing as with (12) and (13), we obtain a contradiction. □

Claim 10 *The graph G is cubic.*

Proof of Claim 10: In order to obtain a contradiction, we assume that G is not cubic. By Claim 9, this implies the existence of a vertex u of degree 3 such that the degree sum D of the neighbours of u

satisfies $D \geq 10$. Let $G' = G[V(G) \setminus N_G[u]]$. Clearly, $\alpha(G) \geq \alpha(G') + 1$. By Claim 8, $\lambda(G') \leq \left\lfloor \frac{D-3}{4} \right\rfloor$. Now

$$\begin{aligned}
& (\alpha(G) - \alpha(G')) + \frac{1}{7}\Big(- 4(n(G) - n(G')) + (m(G) - m(G')) - \lambda(G')\Big) \\
\geq \;& 1 + \frac{1}{7}\left(-16 + D - \left\lfloor \frac{D-3}{4} \right\rfloor\right) \\
= \;& 1 + \frac{1}{7}\left(-13 + \left\lceil \frac{3(D-3)}{4} \right\rceil\right) \\
\geq \;& 1 + \frac{1}{7}\left(-13 + \left\lceil \frac{3(10-3)}{4} \right\rceil\right) \\
= \;& 0.
\end{aligned}$$

Arguing as with (12) and (13), we obtain a contradiction. □

Since, by Theorem 5, the desired result holds for cubic graphs, the proof is complete. (The only argument missing for an independent proof not relying on Theorem 5 corresponds exactly to Claim 6 in [11]. Since G is cubic, the very same proof as in [11] works.) □

Heckman and Thomas [11] described a linear time algorithm which determines an independent set of an order as guaranteed by Theorem 5 in a given triangle-free graph of maximum degree at most 3. The proof of Theorem 6 easily yields a polynomial time algorithm which determines an independent set of an order as guaranteed by Theorem 6 in a given triangle-free graph. In fact, Claims 1 through 10 correspond to reduction steps in an obvious recursive procedure. Since one can check in polynomial time whether a given graph is difficult or a difficult block, the reduction steps corresponding to Claims 1 and 3 can be implemented in polynomial time. Since all other claims correspond to purely local operations and the problem can be solved in linear time for cubic graphs, the polynomial running time of the overall procedure follows.

We close with some consequences of our results. We are able to determine the exact value of $\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d)$ for $d \leq \frac{10}{3}$ and to improve the estimate of $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ based on (2) (cf. Figure 1) for $d \in \left]2, \frac{107}{30}\right[$.

Corollary 8 *(i)*

$$\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d) \geq \begin{cases} \frac{4-d}{4} & , \text{ if } 2 \leq d \leq \frac{12}{5}, \\ \frac{8-d}{14} & , \text{ if } \frac{12}{5} < d \end{cases}$$

with equality for $2 \leq d \leq \frac{10}{3}$.

(ii) If G is a triangle-free connected graph, then

$$\frac{\alpha(G)}{n(G)} \geq \begin{cases} \frac{10-d(G)}{20} & , \text{ if } 2 \leq d(G) < \frac{10}{3}, \\ \frac{8-d(G)}{14} & , \text{ if } \frac{10}{3} \leq d(G). \end{cases}$$

(iii) If G is a triangle-free connected graph, then

$$\frac{\alpha(G)}{n(G)} \geq \frac{119}{235} - \frac{5}{94} d(G).$$

Furthermore, $\alpha(\mathcal{G}_{\Delta-\text{free}}, d) \geq \frac{119}{235} - \frac{5}{94} d(G)$.

Proof: (i) Corollary 3 and Theorem 6 imply that $\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d)$ has at least the given values. That $\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d)$ is not larger follows by considering connected graphs whose cycles are all of length 5 and vertex disjoint for $2 \le d \le \frac{12}{5}$ and connected difficult graphs for $\frac{12}{5} \le d \le \frac{10}{3}$.

(ii) If $d(G) \ge \frac{10}{3}$, then $\lambda(G) = 0$ and Theorem 6 implies the desired result. Hence, we assume that $2 \le d(G) < \frac{10}{3}$. If G is not difficult, then Theorem 6 implies the desired result, because $\frac{4n(G)-m(G)}{7} > \frac{5n(G)-m(G)}{10}$. If G is a difficult block, then Lemma 7 (ii) implies $\alpha(G) = \frac{5n(G)-m(G)}{10}$. Finally, if G is a difficult graph but not a block, then Lemma 7 (ii) and (iii) imply $\alpha(G) > \frac{5n(G)-m(G)}{10}$.

(iii) Let G be a triangle-free connected graph. Let $g(d) = \frac{119}{235} - \frac{5}{94}d$. Since, for $d = 2$ and $d = \frac{107}{30}$, the value of $g(d)$ coincides with the convex linear interpolation of the values $f_{\text{Sh}}(x)$ for integral x, Shearer's bound (2) implies $\frac{\alpha(G)}{n(G)} \ge \frac{119}{235} - \frac{5}{94}d(G)$ for $d(G) \le 2$ or $d(G) \ge \frac{107}{30}$. For $2 < d \le \frac{10}{3}$, we have $g(d) \le \frac{10-d}{20}$. Finally, for $\frac{10}{3} < d < \frac{107}{30}$, we have $g(d) \le \frac{8-d}{14}$. Therefore, (ii) implies that $\frac{\alpha(G)}{n(G)} \ge \frac{119}{235} - \frac{5}{94}d(G)$ for $2 < d(G) < \frac{107}{30}$.

Since $\frac{\alpha(G)}{n(G)} \ge g(d(G))$ holds for all values of $d(G)$, the lower bound on $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ follows. $\square$

Figure 7 summarizes the results from Corollary 8.

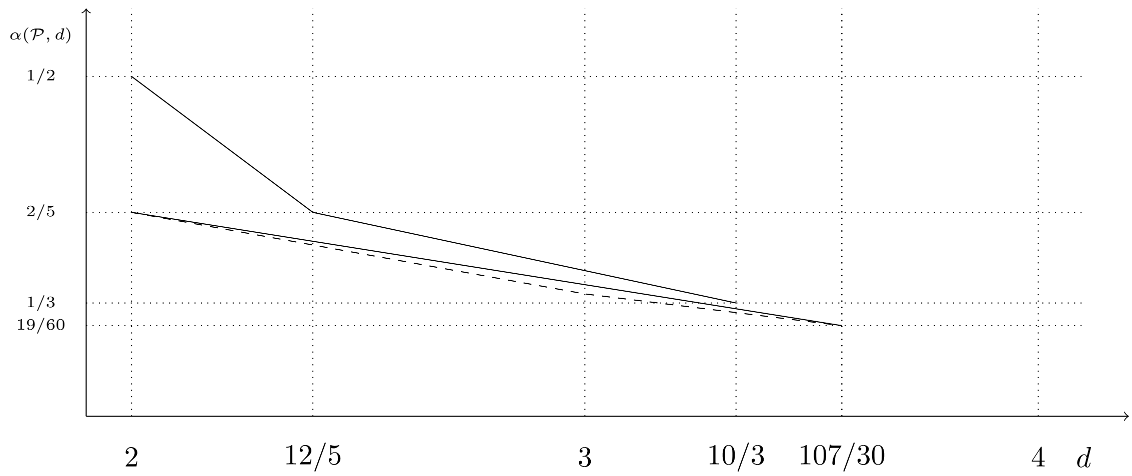

Figure 7 The upper line shows the exact value of $\alpha(\mathcal{G}_{\Delta-\text{free}} \cap \mathcal{G}_{\text{conn}}, d)$ from Corollary 8 (i). The middle line shows the lower bound on $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ from Corollary 8 (iii). The lower dashed line is the lower bound on $\alpha(\mathcal{G}_{\Delta-\text{free}}, d)$ based on (2) (cf. the right graph in Figure 1).

References

[1] M. Albertson, B. Bollobás, and S. Tucker, The independence ratio and the maximum degree of a graph, *Congressus Numerantium* **17** (1976),43-50.

[2] B. Bajnok and G. Brinkmann, On the Independence Number of Triangle-Free Graphs with Maximum Degree Three, *J. Combin. Math. Combin. Comput.* **26** (1998), 237-254.

[3] J.A. Bondy and U.S.R Murty, Graph theory. Graduate Texts in Mathematics 244. Berlin: Springer (2008). xii, 651 p.

[4] Y. Caro, New Results on the Independence Number, Technical Report, Tel-Aviv University, 1979.

[5] T. Denley, The Independence Number of Graphs with Large Odd Girth, *Electron. J. Comb.* **1** (1994), R9.

[6] S. Fajtlowicz, On the size of independent sets in graphs, *Congressus Numerantium* **21** (1978), 269-274.

[7] J. Harant, M.A. Henning, D. Rautenbach, and I. Schiermeyer, Independence Number in Graphs of Maximum Degree Three, *Discrete Math.* **308** (2008), 5829-5833.

[8] J. Harant and D. Rautenbach, Independence in Connected Graphs, manuscript 2009.

[9] J. Harant and I. Schiermeyer, On the independence number of a graph in terms of order and size, *Discrete Math.* **232** (2001), 131-138.

[10] C.C. Heckman, On the Tightness of the 5/14 Independence Ratio, *Discrete Math.* **308** (2008), 3169-3179.

[11] C.C. Heckman and R. Thomas, A New Proof of the Independence Ratio of Triangle-Free Cubic Graphs, *Discrete Math.* **233** (2001), 233-237.

[12] K.F. Jones, Size and independendence in triangle-free graphs with maximum degree three, *J. Graph Theory* **14** (1990), 525-535.

[13] V. Lozin and D. de Werra, Foreword: Special issue on stability in graphs and related topics, *Discrete Appl. Math.* **132** (2003), 1-2.

[14] S.C. Locke, Bipartite Density and the Independence Ratio, *J. Graph Theory* **10** (1986), 47-53.

[15] J.B. Shearer, A note on the independence number of triangle-free graphs. II, *J. Comb. Theory, Ser. B* **53** (1991), 300-307.

[16] J.B. Shearer, The independence number of dense graphs with large odd girth, *Electron. J. Comb.* **2** (1995), N2.

[17] W. Staton, Some Ramsey-type numbers and the independence ratio, *Trans. Amer. Math. Soc.* **256** (1979), 353-370.

[18] V.K. Wei, A Lower Bound on the Stability Number of a Simple Graph, Technical memorandum, TM 81 - 11217 - 9, Bell laboratories, 1981.