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Harant, Jochen; Rautenbach, Dieter

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Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
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Independence in Connected Graphs

Jochen Harant and Dieter Rautenbach

Institut für Mathematik
TU Ilmenau, Postfach 100565
D-98684 Ilmenau, Germany
emails: {jochen.harant,dieter.rautenbach}@tu-ilmenau.de

Abstract

We prove that if $G = (V_G, E_G)$ is a finite, simple, and undirected graph with κ components and independence number $\alpha(G)$, then there exist a positive integer $k \in \mathbb{N}$ and a function $f : V_G \to \mathbb{N}_0$ with non-negative integer values such that $f(u) \leq d_G(u)$ for $u \in V_G$, $\alpha(G) \geq k \geq \sum\limits_{u \in V_G} \frac{1}{d_G(u)+1-f(u)}$, and $\sum\limits_{u \in V_G} f(u) \geq 2(k-\kappa)$. This result is a best-possible improvement of a result due to Harant and Schiermeyer (On the independence number of a graph in terms of order and size, *Discrete Math.* **232** (2001), 131-138) and implies that $\frac{\alpha(G)}{n(G)} \geq \frac{2}{\left(d(G)+1+\frac{2}{n(G)}\right)+\sqrt{\left(d(G)+1+\frac{2}{n(G)}\right)^2-8}}$ for connected graphs G of order $n(G)$, average degree $d(G)$, and independence number $\alpha(G)$.

Keywords: Independence; stability; connected graph
AMS subject classification: 05C69

1 Introduction

We consider *finite*, *simple*, and *undirected graphs* G with *vertex set* V_G and *edge set* E_G. For a graph G, we denote its *order* by $n(G)$ and its *size* by $m(G)$, respectively. The *neighbourhood* of a vertex $u \in V_G$ in a graph G is denoted by $N_G(u)$. The *degree* of u in G is $d_G(u) = |N_G(u)|$ and the *closed neighbourhood* of u in G is $N_G[u] = \{u\} \cup N_G(u)$. The *minimum degree*, *average degree*, and *maximum degree* of G are denoted by $\delta(G)$, $d(G)$, and $\Delta(G)$, respectively. For a set $U \subseteq V_G$, the subgraph of G induced by $V_G \setminus U$ is denoted by $G - U$. A set of vertices $I \subseteq V_G$ in a graph G is *independent*, if no two vertices in I are adjacent. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set of G.

The independence number is one of the most fundamental and well-studied graph parameters [8]. In view of its computational hardness [7] various bounds on the independence number have been proposed. The following classical bound holds for every graph G and is due to Caro and Wei [4, 13]

$$\alpha(G) \geq \sum_{u \in V_G} \frac{1}{d_G(u)+1}. \tag{1}$$

Since the only graphs for which (1) is best-possible are the disjoint unions of cliques, additional structural assumptions excluding these graphs allow improvements of (1). Natural candidates for such assumptions are triangle-freeness or — more generally — K_r-freeness as well as connectivity.

For triangle-free graphs, Shearer [10, 11] proved

$$\alpha(G) \geq \sum_{u \in V_G} f(d_G(u))$$

where $f(d) = \Omega\left(\frac{\log(d)}{d}\right)$ has the best-possible order of magnitude (cf. also [2, 3, 12] and for similar results concerning K_r-free graphs [1, 9]).

For connected graphs, Harant and Schiermeyer proved [5] (cf. also [6])

$$\frac{\alpha(G)}{n(G)} \geq \frac{2}{\left(d(G)+1+\frac{1}{n(G)}\right)+\sqrt{\left(d(G)+1+\frac{1}{n(G)}\right)^2-4}}. \tag{2}$$

Considering

$$\alpha_{\mathcal{P}}(d) = \lim_{n\to\infty} \inf \left\{ \frac{\alpha(G)}{n(G)} \mid G \in \mathcal{P},\ d(G) \leq d,\ n(G) \geq n \right\}$$

where $d \in \mathbb{R}_{\geq 0}$ and $\mathcal{P}$ denotes an infinite class of graphs allows a simpler comparison of (1) and (2). If $\mathcal{G}$ denotes the class of all graphs, then (1) implies $\alpha_{\mathcal{G}}(d) \geq \frac{1}{d+1}$. Similarly, if $\mathcal{G}_{\text{conn}}$ denotes the class of all connected graphs, then (2) implies

$$\alpha_{\mathcal{G}_{\text{conn}}}(d) \geq \left(\frac{2}{1+\sqrt{1-\frac{4}{(d+1)^2}}}\right)\frac{1}{d+1}. \tag{3}$$

The goal of the present paper are best-possible improvements of (2) and (3).

2 Results

In [5] Harant and Schiermeyer analyse the performance of a simple greedy algorithm — similar to Algorithm 1 below — for the construction of an independent set in a given graph. They show that applied to a connected graph G, the algorithm produces an independent set I of G with

$$|I| \geq k \geq \sum_{u \in V_G} \frac{1}{d_G(u)+1-g(u)} \tag{4}$$

where k is some positive integer and $g : V_G \to \mathbb{N}_0$ is a function with $g(u) \leq d_G(u)$ for $u \in V_G$ — which we will abbreviate as "$g \leq d_G$" in the following — and

$$\sum_{u \in V_G} g(u) \geq k-1. \tag{5}$$

Applying Jensen's inequality to (4) and (5) easily yields (2) (cf. the proof of Corollary 3 below).

We achieve our best-possible improvements of (2) and (3) by preprocessing the input graph for the greedy algorithm, restricting the behaviour of the algorithm, and refining its analysis. Altogether, this allows to improve (5) by a factor of 2. Our main result is the following.

Theorem 1 *If G is a graph with κ components, then there exist a positive integer $k \in \mathbb{N}$ and a function $f : V_G \to \mathbb{N}_0$ with non-negative integer values such that $f \leq d_G$,*

$$\alpha(G) \geq k \geq \sum_{u \in V_G} \frac{1}{d_G(u)+1-f(u)},$$

and

$$\sum_{u \in V_G} f(u) \geq 2(k-\kappa).$$

Note that Theorem 1 is best-possible for the connected graphs which arise by adding bridges to disjoint unions of cliques, i.e. it is best-possible for the intuitively most natural candidate of a connected graph with small independence number.

Before we proceed to the proof of Theorem 1, we show the desired conclusion under stronger assumptions. (These assumptions correspond to preprocessing the input graph accordingly as will become clear in the proof of Theorem 1 below.)

Lemma 2 *If G is a connected graph such that*

($\mathcal{G}_1$) $\delta(G) \geq 3$,

($\mathcal{G}_2$) *there is no vertex whose neighbourhood induces a complete subgraph,*

($\mathcal{G}_3$) *there are no $2\delta(G)$ distinct vertices $u_1, u_2, \ldots, u_{\delta(G)}$ and $v_1, v_2, \ldots, v_{\delta(G)}$ such that*

$d_G(u_i) = \delta(G)$ *for* $1 \leq i \leq \delta(G)$,

$d_G(v_i) = \delta(G) + 1$ *for* $1 \leq i \leq \delta(G)$,

$\{u_i \mid 1 \leq i \leq \delta(G)\}$ *induces a complete subgraph,*

$\{v_i \mid 1 \leq i \leq \delta(G)\}$ *is independent, and*

$u_i v_i \in E_G$ *for* $1 \leq i \leq \delta(G)$,

then there exist $k \in \mathbb{N}$ and $f : V_G \to \mathbb{N}_0$ such that $f \leq d_G$,

$$\alpha(G) \geq k \geq \sum_{u \in V_G} \frac{1}{d_G(u) + 1 - f(u)}$$

and

$$\sum_{u \in V_G} f(u) \geq 2(k-1).$$

Proof: The proof relies on the analysis of the greedy Algorithm 1 below. In order to complete the description of Algorithm 1, we need to specify the set $\mathcal{S}(G_i)$: For a subgraph H of G let $\mathcal{S}(H)$ denote the set of vertices u of H such that

($\mathcal{S}_1$) $d_H(u) = \delta(H)$,

($\mathcal{S}_2$) subject to condition ($\mathcal{S}_1$),

$$\sum_{v \in N_H[u]} (d_G(v) - d_H(v))$$

is maximum,

($\mathcal{S}_3$) subject to conditions ($\mathcal{S}_1$) and ($\mathcal{S}_2$),

$$\sum_{v \in N_H[u]} (d_H(v) - \delta(H))$$

is maximum,

($\mathcal{S}_4$) subject to conditions ($\mathcal{S}_1$), ($\mathcal{S}_2$), and ($\mathcal{S}_3$),

$$\delta(H - N_H[u])$$

is minimum.

Input: A graph G satisfying $(\mathcal{G}_1)$, $(\mathcal{G}_2)$, and $(\mathcal{G}_3)$.
Output: An independent set $\{u_1, u_2, \ldots, u_k\}$.
$i := 1$;
$G_i := G$;
while $V_{G_i} \neq \emptyset$ **do**
  Select $u_i \in \mathcal{S}(G_i)$;
  Set

$$\begin{aligned}
\delta_i &:= \delta(G_i);\\
V_i &:= N_{G_i}[u_i];\\
\gamma(u) &:= d_G(u) - d_{G_i}(u) \forall u \in V_i;\\
\Gamma_i &:= \sum_{u \in V_i} \gamma(u);\\
\beta(u) &:= d_{G_i}(u) - \delta_i \forall u \in V_i;\\
B_i &:= \sum_{u \in V_i} \beta(u);\\
G_{i+1} &:= G_i - V_i;\\
i &:= i + 1;
\end{aligned}$$

end
$k := i - 1$;

Algorithm 1

In view of Algorithm 1, we obtain

$$\begin{aligned}
\alpha(G) &\geq k\\
&= \sum_{i=1}^{k} \sum_{u \in V_i} \frac{1}{\delta_i + 1}\\
&= \sum_{i=1}^{k} \sum_{u \in V_i} \frac{1}{d_G(u) + 1 - (d_G(u) - d_{G_i}(u)) - (d_{G_i}(u) - \delta_i)}\\
&= \sum_{i=1}^{k} \sum_{u \in V_i} \frac{1}{d_G(u) + 1 - (\gamma(u) + \beta(u))}\\
&= \sum_{u \in V_G} \frac{1}{d_G(u) + 1 - (\gamma(u) + \beta(u))}
\end{aligned}$$

and

$$\gamma(u) + \beta(u) = d_G(u) - \delta_i \leq d_G(u)$$

for $1 \leq i \leq k$ and $u \in V_i$.

Therefore, in order to complete the proof it suffices to show that

$$\sum_{u \in V_G} (\gamma(u) + \beta(u)) = \sum_{i=1}^{k} (\Gamma_i + B_i) \geq 2(k-1).$$

Claim 1 *If* $(\Gamma_i, B_i) = (0, 0)$ *for some* $1 \leq i \leq k$*, then* $i = 1$.

Proof of Claim 1: By the definition of $\mathcal{S}(G_i)$, we obtain that for every vertex u in G_i which is of minimum degree δ_i all vertices v in the closed neighbourhood $N_{G_i}[u]$ of u in G_i satisfy $d_G(v) = d_{G_i}(v) = \delta_i$. Since G is connected, this implies $d_G(v) = d_{G_i}(v)$ for all vertices of G which implies $G = G_i$, i.e. $i = 1$. $\Box$

Claim 2 *If* $(\Gamma_i, B_i) = (0, 1)$ *for some* $1 \leq i \leq k$*, then* $i < k$ *and* $\Gamma_{i+1} + B_{i+1} \geq 3$.

Proof of Claim 2: By the definition of Γ_i, we obtain that all vertices v in V_i satisfy $d_G(v) = d_{G_i}(v)$ which implies $\delta_i = d_G(u_i) \geq \delta(G) \geq 3$. Furthermore, by the definition of B_i, there is exactly one vertex, say v_i, in V_i which is of degree $\delta_i + 1$ and all other vertices in V_i are of degree δ_i. Since v_i has a neighbour which is not contained in V_i, we obtain $V_{G_{i+1}} \neq \emptyset$, i.e. $i < k$.

If $\delta_{i+1} < \delta_i$, then

$$\begin{aligned}
\Gamma_{i+1} + B_{i+1} &= \sum_{u \in V_{i+1}} (d_G(u) - \delta_{i+1}) \\
&= (d_G(u_{i+1}) - \delta_{i+1}) + \sum_{u \in V_{i+1} \setminus \{u_{i+1}\}} (d_G(u) - \delta_{i+1}) \\
&\geq (\delta_i - \delta_{i+1}) + \sum_{u \in V_{i+1} \setminus \{u_{i+1}\}} (\delta_i - \delta_{i+1}) \\
&\geq (\delta_i - \delta_{i+1}) + |V_{i+1} \setminus \{u_{i+1}\}| \\
&= (\delta_i - \delta_{i+1}) + \delta_{i+1} \\
&= \delta_i \\
&\geq 3.
\end{aligned}$$

Hence, we may assume that $\delta_{i+1} \geq \delta_i$.

By $(\mathcal{G}_2)$, some vertex u' in $V_i \setminus \{u_i, v_i\}$ has a neighbour v' which is not contained in V_i. By $(\mathcal{S}_2)$, $d_G(v') = d_{G_i}(v')$. Since $\delta_{i+1} \geq \delta_i$, $d_G(v') \geq \delta_i + 1$. By $(\mathcal{S}_3)$, $d_G(v') = d_{G_i}(v') = \delta_i + 1$ and all neighbours of u' different from v' are of degree δ_i in G as well as G_i. This implies that u' is non-adjacent to v_i and that v' is the unique neighbour of u' which is not contained in V_i, i.e. $N_{G_i}[u'] = (V_i \setminus \{v_i\}) \cup \{v'\}$.

If some vertex u'' in $V_i \setminus \{u_i, v_i\}$ is adjacent to v_i, then $(\mathcal{S}_2)$ and $(\mathcal{S}_3)$ together with $\delta_{i+1} \geq \delta_i$ imply that u'' has no neighbour which is not contained in V_i and hence $N_{G_i}[u''] = V_i$. Now,

$$\delta(G_i - N_{G_i}[u']) \leq d_{G_i}(v_i) - 2 < \delta_i$$

which, by $(\mathcal{S}_4)$, implies the contradiction $u_i \notin \mathcal{S}(G_i)$, i.e. Algorithm 1 would have selected u' rather than u_i. Therefore, no vertex in $V_i \setminus \{u_i, v_i\}$ is adjacent to v_i which implies that they all have neighbours which are not contained in V_i. Arguing as for u' above, we obtain that every vertex in $V_i \setminus \{u_i, v_i\}$ is adjacent to all vertices of V_i except for v_i and itself and has a unique neighbour which is not contained in V_i. Furthermore, this unique neighbour not contained in V_i is of degree $\delta_i + 1$ in G as well as G_i.

Let x and y be two distinct vertices in $V_i \setminus \{u_i, v_i\}$ and let x' and y' denote their unique neighbours which are not contained in V_i, respectively. If $x' = y'$, then

$$\delta_{i+1} \leq d_{G_{i+1}}(x') \leq d_{G_i}(x') - 2 = \delta_i + 1 - 2 < \delta_i$$

which is a contradiction. Hence $x' \neq y'$. If x' and y' are adjacent, then

$$\delta(G_i - N_{G_i}[x]) \leq d_{G_i}(y') - 2 = \delta_i + 1 - 2 < \delta_i$$

which, by $(\mathcal{S}_4)$, implies the contradiction $u_i \notin \mathcal{S}(G_i)$, i.e. Algorithm 1 would have selected x rather than u_i. By symmetry, this implies that G does not satisfy $(\mathcal{G}_3)$ which is a contradiction and completes the proof of the claim. $\Box$

Claim 3 *If* $(\Gamma_i, B_i) = (1, 0)$ *for some* $1 \leq i \leq k$*, then* $i < k$ *and* $\Gamma_{i+1} + B_{i+1} \geq 3$*.*

Proof of Claim 3: By the definition of Γ_i, we obtain that there is a unique vertex u' in V_i such that $d_G(u') = d_{G_i}(u') + 1$ and $d_G(v) = d_{G_i}(v)$ for $v \in V_i \setminus \{u'\}$. By the definition of B_i, $d_{G_i}(v) = \delta_i$ for $v \in V_i$. This implies that

$$\delta_i = \max\{d_{G_i}(v) \mid v \in V_i\} \geq \max\{d_{G_i}(v) \mid v \in V_i \setminus \{u'\}\} = \max\{d_G(v) \mid v \in V_i \setminus \{u'\}\} \geq \delta(G) \geq 3.$$

By $(\mathcal{G}_2)$, V_i does not induce a complete graph. This implies that some vertex u'' in $V_i \setminus \{u'\}$ has a neighbour v'' which is not contained in V_i and hence $V_{G_{i+1}} \neq \emptyset$, i.e. $i < k$.

If $\delta_{i+1} < \delta_i$, then exactly the same calculation as in the proof of Claim 2 yields $\Gamma_{i+1}+B_{i+1} \geq \delta_i \geq 3$. Hence, we may assume that $\delta_{i+1} \geq \delta_i$.

If u' and u'' are adjacent, then $(\mathcal{S}_2)$ and $(\mathcal{S}_3)$ imply $d_{G_i}(v'') = \delta_i$ which yields the contradiction

$$\delta_{i+1} \leq d_{G_{i+1}}(v'') \leq d_{G_i}(v'') - 1 = \delta_i - 1.$$

This implies that u' and u'' are non-adjacent and hence $u' \neq u_i$. Since $d_{G_i}(u') = \delta_i$, u' has a neighbour v' which is not contained in V_i. Now $(\mathcal{S}_2)$ and $(\mathcal{S}_3)$ imply that $d_G(v') = d_{G_i}(v') = \delta_i$ which yields the contradiction

$$\delta_{i+1} \leq d_{G_{i+1}}(v') \leq d_{G_i}(v') - 1 = \delta_i - 1.$$

This completes the proof of Claim 3. $\square$

Since Claims 1, 2, and 3 immediately imply $\sum\limits_{u \in V_G} (\gamma(u) + \beta(u)) \geq 2(k-1)$, the proof is complete. $\square$

With Lemma 2 at hand, we can now proceed to the

Proof of Theorem 1: For contradiction, we assume that G is a counterexample of minimum order. Clearly, G is connected and not complete. By Lemma 2, G does not satisfy either $(\mathcal{G}_1)$, or $(\mathcal{G}_2)$, or $(\mathcal{G}_3)$. Accordingly, we will consider three cases.

Case 1 G *does not satisfy* $(\mathcal{G}_2)$*.*

Let u be a vertex of G whose neighbourhood induces a complete subgraph. The number m' of edges of G between $N_G(u)$ and $V_G \setminus N_G[u]$ is exactly $\sum\limits_{v \in N_G(u)} (d_G(v) - d_G(u))$ and the number κ' of components of $G' = G - N_G[u]$ satisfies $\kappa' \leq m'$.

By the choice of G, there exist $k' \in \mathbb{N}$ and $f' : V_{G'} \to \mathbb{N}_0$ with $f' \leq d_{G'}$ such that

$$\alpha(G') \geq k' \geq \sum_{v \in V_{G'}} \frac{1}{d_{G'}(v) + 1 - f'(v)}$$

and

$$\sum_{v \in V_{G'}} f'(v) \geq 2(k' - \kappa').$$

Clearly, $\alpha(G) \geq \alpha(G') + 1 \geq k' + 1$. If $k = k' + 1$ and $f : V_G \to \mathbb{N}_0$ is such that

$$f(v) = \begin{cases} 0 & , \text{if } v = u, \\ d_G(v) - d_G(u) & , \text{if } v \in N_G(u), \\ f'(v) + (d_G(v) - d_{G'}(v)) & , \text{if } v \in V_{G'} = V_G \setminus N_G[u], \end{cases}$$

then $f \leq d_G$,

$$\begin{aligned}
\alpha(G) &\geq k \\
&= 1+k' \\
&\geq \sum_{v\in N_G[u]} \frac{1}{d_G(u)+1} + \sum_{v\in V_{G'}} \frac{1}{d_{G'}(v)+1-f'(v)} \\
&= \sum_{v\in N_G[u]} \frac{1}{d_G(v)+1-f(v)} + \sum_{v\in V_{G'}} \frac{1}{d_G(v)+1-f(v)} \\
&= \sum_{v\in V_G} \frac{1}{d_G(v)+1-f(v)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{v\in V_G} f(v) &= \sum_{v\in N_G(u)} (d_G(v)-d_G(u)) + \sum_{v\in V_{G'}} (f'(v)+(d_G(v)-d_{G'}(v))) \\
&= m' + \sum_{v\in V_{G'}} (d_G(v)-d_{G'}(v)) + \sum_{v\in V_{G'}} f'(v) \\
&= 2m' + \sum_{v\in V_{G'}} f'(v) \\
&\geq 2m' + 2(k'-\kappa) \\
&\geq 2k' \\
&= 2(k-1).
\end{aligned}$$

This contradiction completes the proof for Case 1.

Case 2 G *does not satisfy* $(\mathcal{G}_1)$, *i.e.* $\delta(G) \leq 2$.

By Case 1, we may assume that $\delta(G) = 2$ and that u is a vertex of degree 2 in G with the two non-adjacent neighbours v and w.

Let G' arise from $G - \{u, w\}$ by adding new edges between v and all vertices in $N_G(w) \setminus N_G(v)$. Clearly, G' is connected. Let I' be a maximum independent set of G'. If I' contains v, then let $I = I' \cup \{w\}$, otherwise, let $I = I' \cup \{u\}$. Clearly, I is an independent set of G which implies $\alpha(G) \geq \alpha(G') + 1$.

By the choice of G, there exist $k' \in \mathbb{N}$ and $f' : V_{G'} \to \mathbb{N}_0$ with $f' \leq d_{G'}$ such that

$$\alpha(G') \geq k' \geq \sum_{v\in V_{G'}} \frac{1}{d_{G'}(v)+1-f'(v)}$$

and

$$\sum_{v\in V_{G'}} f'(v) \geq 2(k'-1).$$

If $k = k'+1$ and $f : V_G \to \mathbb{N}_0$ is such that

$$f(x) = \begin{cases} 1 & \text{, if } x = u, \\ d_G(w)-1 & \text{, if } x = w, \\ f'(v) - (|N_G(w) \setminus N_G(v)| - 1) & \text{, if } x = v, \\ f'(x)+1 & \text{, if } x \in (N_G(w) \cap N_G(v)) \setminus \{u\}, \\ f'(v) & \text{, if } x \in V_G \setminus (\{v, w\} \cup (N_G(w) \cap N_G(v))). \end{cases}$$

then $f \leq d_G$,

$$\begin{aligned}
\alpha(G) &\geq k \\
&= 1 + k' \\
&\geq \frac{1}{2} + \frac{1}{2} + \sum_{x \in V_{G'}} \frac{1}{d_{G'}(x) + 1 - f'(x)} \\
&= \frac{1}{d_G(u) + 1 - 1} + \frac{1}{d_G(w) + 1 - (d_G(w) - 1)} + \sum_{x \in V_{G'}} \frac{1}{d_G(x) + 1 - f(x)} \\
&= \sum_{x \in V_G} \frac{1}{d_G(x) + 1 - f(x)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{v \in V_G} f(v) &= 1 + (d_G(w) - 1) - (|N_G(w) \setminus N_G(v)| - 1) + |(N_G(w) \cap N_G(v)) \setminus \{u\}| + \sum_{x \in V_{G'}} f'(x) \\
&\geq 1 + (d_G(w) - 1) - (d_G(w) - 2) + \sum_{x \in V_{G'}} f'(x) \\
&= 2 + \sum_{x \in V_{G'}} f'(x) \\
&\geq 2 + 2(k' - 1) \\
&= 2(k - 1).
\end{aligned}$$

This contradiction completes the proof for Case 2.

Case 3 *G does not satisfy $(\mathcal{G}_3)$.*

For contradiction, we assume that the vertices $u_1, u_2, \ldots, u_{\delta(G)}$ and $v_1, v_2, \ldots, v_{\delta(G)}$ are as specified in $(\mathcal{G}_3)$. By Case 2, $\delta(G) \geq 3$. Let G' arise from $G - \{u_1, u_2, \ldots, u_{\delta(G)}\}$ by adding $\delta(G) - 1$ new edges between v_1 and the vertices in $\{v_2, v_3, \ldots, v_{\delta(G)}\}$. Clearly, G' is connected.

Let I' be a maximum independent set of G'. If I' contains v_1, then let $I = I' \cup \{u_2\}$, otherwise, let $I = I' \cup \{u_1\}$. Clearly, I is an independent set of G which implies $\alpha(G) \geq \alpha(G') + 1$.

By the choice of G, there exist $k' \in \mathbb{N}$ and $f' : V_{G'} \to \mathbb{N}_0$ with $f' \leq d_{G'}$ such that

$$\alpha(G') \geq k' \geq \sum_{v \in V_{G'}} \frac{1}{d_{G'}(v) + 1 - f'(v)}$$

and

$$\sum_{v \in V_{G'}} f'(v) \geq 2(k' - 1).$$

If $k = k' + 1$ and $f : V_G \to \mathbb{N}_0$ is such that

$$f(x) = \begin{cases} 1 & \text{, if } x \in \{u_1, u_2, \ldots, u_{\delta(G)}\}, \\ f'(v) - (\delta(G) - 2) & \text{, if } x = v_1, \\ f'(v) & \text{, if } x \in V_{G'} \setminus \{v_1\}. \end{cases}$$

then $f \leq d_G$,

$$\begin{aligned}
\alpha(G) &\geq k \\
&= 1 + k'
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\delta(G)}{\delta(G)+1-1} + \sum_{x\in V_{G'}} \frac{1}{d_{G'}(x)+1-f'(x)}\\
&= \sum_{v\in V_G} \frac{1}{d_G(v)+1-f(v)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{v\in V_G} f(v) &= \delta(G) - (\delta(G)-2) + \sum_{x\in V_{G'}} f'(x)\\
&= 2 + \sum_{x\in V_{G'}} f'(x)\\
&\geq 2 + 2(k'-1)\\
&= 2(k-1).
\end{aligned}
$$

This contradiction completes the proof. □

Corollary 3 *(i) If G is a connected graph, then*

$$
\frac{\alpha(G)}{n(G)} \geq \frac{2}{\left(d(G)+1+\frac{2}{n(G)}\right) + \sqrt{\left(d(G)+1+\frac{2}{n(G)}\right)^2 - 8}}.
$$

(ii) If $d \in \mathbb{R}_{\geq 0}$, then

$$
\alpha_{\mathcal{G}_{\rm conn}}(d) \geq \left(\frac{2}{1+\sqrt{1-\frac{8}{(d+1)^2}}}\right)\frac{1}{d+1}.
$$

Proof: (i) By the convexity of the function $x \mapsto \frac{1}{x}$ and Jensen's inequality (J), we obtain from Theorem 1

$$
\begin{aligned}
k &\geq \sum_{x\in V_G} \frac{1}{d_G(x)+1-f(x)}\\
&\stackrel{(J)}{\geq} \frac{n(G)}{\frac{1}{n(G)}\sum\limits_{x\in V_G}(d_G(x)+1-f(x))}\\
&\geq \frac{n(G)}{d(G)+1-\frac{2(k-1)}{n(G)}}
\end{aligned}
$$

which is equivalent to $\left(\frac{k}{n(G)}\right)^2 + \frac{1}{2}\left(d(G)+1+\frac{2}{n(G)}\right)\frac{k}{n(G)} - 1 \geq 0$. Since $\frac{\alpha(G)}{n(G)} \geq \frac{k}{n(G)}$, this easily implies (i).

Since (ii) follows immediately from (i), the proof is complete. □

For integer values of $d \geq 0$, (1) together with the consideration of disjoint unions of complete graphs of order $d+1$ actually yields $\alpha_{\mathcal{G}}(d) = \frac{1}{d+1}$. Similarly, if $d \in \mathbb{R}_{\geq 0}$ is such that $d = \frac{2\left(\binom{r}{2}+1\right)}{r} = r-1+\frac{2}{r}$ for some integer $r \geq 2$, then the connected graphs $G_{r,s}$ which arise by adding s new edges to the disjoint union of s complete graphs of order r satisfy $d(G_{r,s}) = d = r-1+\frac{2}{r}$ and $\frac{\alpha(G_{r,s})}{n(G_{r,s})} = \frac{s}{rs} = \frac{1}{r}$. Since $\left(\frac{2}{1+\sqrt{1-\frac{8}{(d(G_{r,s})+1)^2}}}\right)\frac{1}{d(G_{r,s})+1} = \frac{1}{r}$ we obtain $\alpha_{\mathcal{G}_{\rm conn}}(d) = \left(\frac{2}{1+\sqrt{1-\frac{8}{(d+1)^2}}}\right)\frac{1}{d+1}$ for these values of d.

For $d \in \mathbb{R}_{\geq 0} \setminus \mathbb{N}_0$, the convexity of $x \mapsto \frac{1}{x+1}$ implies that the right hand side of (1) is smallest possible for a graph G, if all vertices of G have degree either $\lfloor d(G) \rfloor$ or $\lceil d(G) \rceil$. Since the disjoint union of cliques of orders $\lfloor d(G) \rfloor + 1$ and $\lceil d(G) \rceil + 1$ has this property and gives equality in (1), it follows easily that $\alpha_{\mathcal{G}}(d) = \frac{\lceil d(G) \rceil - d(G)}{\lfloor d(G) \rfloor + 1} + \frac{d(G) - \lfloor d(G) \rfloor}{\lceil d(G) \rceil + 1}$, i.e. $\alpha_{\mathcal{G}}(d)$ is the linear interpolation of the values $\frac{1}{d+1}$ assumed for integer values of d. Using similar arguments, it is straightforward to show that the exact value of $\alpha_{\mathcal{G}_{\rm conn}}(d)$ also is the linear interpolation of the values $\frac{1}{r}$ assumed for values of $d = r - 1 + \frac{2}{r}$ for integer $r \geq 2$.

References

[1] M. Ajtai, P. Erdős, J. Komlós, and E. Szemerédi, On Turán's theorem for sparse graphs, *Combinatorica* **1** (1981), 313-317.

[2] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, *J. Comb. Theory, Ser. A* **29** (1980), 354-360.

[3] M. Ajtai, J. Komlós, and E. Szemerédi, A dense infinite Sidon sequence, *Eur. J. Comb.* **2** (1981), 1-11.

[4] Y. Caro, New Results on the Independence Number, Technical Report, Tel-Aviv University, 1979.

[5] J. Harant and I. Schiermeyer, On the independence number of a graph in terms of order and size, *Discrete Math.* **232** (2001), 131-138.

[6] J. Harant and I. Schiermeyer, A lower bound on the independence number of a graph in terms of degrees, *Discuss. Math., Graph Theory* **26** (2006), 431-437.

[7] J. Håstad, Clique is hard to approximate within $n^{1-\epsilon}$, in: Proceedings of 37th Annual Symposium on Foundations of Computer Science (FOCS), 1996, 627-636.

[8] V. Lozin and D. de Werra, Foreword: Special issue on stability in graphs and related topics, *Discrete Appl. Math.* **132** (2003), 1-2.

[9] I. Schiermeyer, Approximating Maximum Independent Set in k-Clique-Free Graphs, in: Proceedings of Approximation Algorithms for Combinatorial Optimization (APPROX98), *Lecture Notes in Computer Science* **1444** (1998) , 159-168.

[10] J.B. Shearer, A note on the independence number of triangle-free graphs, *Discrete Math.* **46** (1983), 83-87.

[11] J.B. Shearer, A note on the independence number of triangle-free graphs. II, *J. Comb. Theory, Ser. B* **53** (1991), 300-307.

[12] W. Staton, Some Ramsey-type numbers and the independence ratio, *Trans. Amer. Math. Soc.* **256** (1979), 353-370.

[13] V.K. Wei, A Lower Bound on the Stability Number of a Simple Graph, Technical memorandum, TM 81 - 11217 - 9, Bell laboratories, 1981.