Edge Colourings of Multigraphs

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1 Introduction

The Edge Colouring Problem (ECP) is to find the chromatic index of a given graph G, that is, the minimum number of colours needed to colour the edges of G such that no two adjacent edges receive the same colour. Edge colouring problems occur in various scheduling applications, typically in conjunction with task processing or network communication, minimizing the number of time-slots needed for completing a given set of tasks or data transfers. Good scheduling algorithms become more and more important, e.g., due to the increased use of multiprocessor environments or the increasing complexity in the wide field of logistics. Most papers on edge colouring deal mainly with edge colourings of simple graphs. In many scheduling problems, however, (multi)graphs occur in a natural way, and (multi)graph edge colouring is a topic where further work should be done. One of the major unsolved problems in this area is Goldberg's conjecture (see. Section 1.4).

Goldberg suggested an upper bound for the chromatic index χ' in terms of the maximum degree Δ and the density $\mathcal{W}$, namely $\chi' \leq \max\{\Delta + 1, \mathcal{W}\}$. Since Δ is a lower bound for the chromatic index, Goldberg's conjecture implies that χ' is equal to Δ, or to $\Delta + 1$, or to $\mathcal{W}$.

Upper bounds for the chromatic index of a graph come often from algorithms that produce edge colourings. As discussed in Chapter 2, a typical algorithm colours the edges of an input graph sequentially, that is, the edges are coloured one at a time with respect to a given edge order. The core of such an algorithm is a subroutine that extends a given partial colouring of the input graph to the next uncoloured edge. In each call of the subroutine, we have to decide whether we will use a fresh colour or not. To make this decision, we shall use so-called test objects. A classical kind of test objects, called Vizing fans, was introduced by Vizing [39] in 1964. Vizing used these test objects to establish an upper bound for the chromatic index χ' in terms of the maximum degree Δ and the maximum multiplicity μ, namely $\chi' \leq \Delta + \mu$. Vizing's bound is rather generous. In most graphs there will be scope for an improvement of Vizing's bound by choosing a suitable edge order to start with. This leads to a new graph parameter, the fan number. The fan number, introduced in Section 2.4, resembles the colouring number, and seems to be the best upper bound for the chromatic index that can be obtained by the fan argument. Another type of test objects, called Kierstead paths, was introduced by Kierstead [19] in 1984. Recently, Tashkinov [38] obtained a common generalization, Tashkinov trees, of the Vizing fans and the Kierstead paths, see Section 2.6. As we shall see in Section 2.7, some new upper bounds for the chromatic index in terms of the maximum degree Δ and the density $\mathcal{W}$ can be obtained from Tashkinov methods. These results generalize an earlier result of Kahn [18] proved by probabilistic methods, and improve earlier bounds for χ' by Sanders and Steurer [25].

In Chapter 3 we extend an earlier result of Favrholdt, Stiebitz and Toft [8], and show that $\chi' \leq \max\{\frac{15}{14}\Delta + \frac{12}{14}, \mathcal{W}\}$. The proof of this result is based on an extension of Tashkinov's methods. The results of Chapter 3 imply that Goldberg's conjecture holds for all graphs with at most 15 vertices as well as for all graphs with maximum degree at most 15. If Goldberg's conjecture is true, we can also give a complete answer to the following natural question. For which value of Δ and μ does there exist a graph G satisfying $\Delta(G) = \Delta$, $\mu(G) = \mu$, and $\chi'(G) = \Delta + \mu$?

All colouring algorithms presented in the first three chapters have execution time polynomial in $|E|$ and $|V|$, but are only pseudo-polynomial in the number of bits needed to describe the graph $G = (V, E)$, since the size of the input graph G may be of order $\log |E|$. In Chapter 4 we use ideas from Sanders and Steurer [25] to develop a scheme for polynomial-time edge colouring algorithms that can realize several upper bounds of the chromatic index.

1.1 Graphs

By a **graph** we mean a finite undirected graph without loops, but possibly with multiple edges. The **vertex set** and the **edge set** of a graph G are denoted by $\boldsymbol{V(G)}$ and $\boldsymbol{E(G)}$, respectively. For a vertex $x \in V(G)$, let $\boldsymbol{E_G(x)}$ denote the set of all edges of G that are incident with x. Two distinct edges of G incident to the same vertex will be called **adjacent edges**. Furthermore, for $X, Y \subseteq V(G)$, let $\boldsymbol{E_G(X, Y)}$ denote the set of all edges of G joining a vertex of X with a vertex of Y. We write $E_G(x, y)$ instead of $E_G(\{x\}, \{y\})$. Two distinct vertices $x, y \in V(G)$ with $E_G(x, y) \neq \emptyset$ will be called **adjacent vertices** or **neighbours**. Let $\boldsymbol{N_G(x)}$ denote the set of all neighbours of x in G, that is, $N_G(x) = \{y \in V(G) \mid E_G(x, y) \neq \emptyset\}$.

The **degree** of a vertex $x \in V(G)$ is $\boldsymbol{d_G(x)} = |E_G(x)|$, and the **multiplicity** of two distinct vertices $x, y \in V(G)$ is $\boldsymbol{\mu_G(x, y)} = |E_G(x, y)|$. Let $\boldsymbol{\delta(G)}$, $\boldsymbol{\Delta(G)}$ and $\boldsymbol{\mu(G)}$ denote the **minimum degree**, **maximum degree** and the **maximum multiplicity** of G, respectively. A graph G is called **simple** if $\mu(G) \leq 1$. A graph G is called **regular** or ***r*-regular** if $d_G(x) = r$ for every vertex x of G, where $r \geq 0$ is an integer.

For H is a **subgraph** of G, we write briefly $H \subseteq G$. For a graph G and a set $X \subseteq V(G)$, let $\boldsymbol{G[X]}$ denote the subgraph of G **induced** by X, that is, $V(G[X]) = X$ and $E(G[X]) = E_G(X, X)$. Further, let $\boldsymbol{G - X} = G[V(G) \setminus X]$. We also write $G - x$ instead of $G - \{x\}$. For $F \subseteq E(G)$, let $\boldsymbol{G - F}$ denote the subgraph H of G satisfying $V(H) = V(G)$ and $E(H) = E(G) \setminus F$. If $F = \{e\}$ is a singleton, we write $G - e$ rather than $G - \{e\}$. For the graph $H = G - e$, we further define $H + e$ to be the graph G.

The **line graph** of G, denoted by $\mathbf{L(G)}$, is the simple graph defined as follows. The vertex set $V(L(G))$ equals the edge set of G, and between two vertices $e, f \in V(L(G))$ there is an edge iff e and f are adjacent edges in G.

If S is a sequence consisting of edges and vertices of a given graph G, then we denote by $\boldsymbol{V(S)}$, respectively $\boldsymbol{E(S)}$, the set of all elements of $V(G)$, respectively $E(G)$, that belong to the sequence S. Let G be a graph, and let $S = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ be a sequence such that $v_0, \ldots, v_p$ are distinct vertices of G, and $e_1, \ldots, e_p$ are edges of G. For a vertex $v_i \in V(S)$, we define $\boldsymbol{Sv_i} = (v_0, e_1, \ldots, e_i, v_i)$ and $\boldsymbol{v_i S} = (v_i, e_{i+1}, \ldots, v_p)$.

By a path, a cycle, or a tree we usually mean a graph or subgraph rather than a sequence consisting of edges and vertices. The only exceptions will be the Kierstead path and the Tashkinov tree, we consider both as sequences. If P is a path of length $p \geq 0$ with $V(P) = \{v_0, \ldots, v_p\}$ and $E(P) = \{e_1, \ldots, e_p\}$ such that $e_i \in E_P(v_{i-1}, v_i)$ for $i = 1, \ldots, p$, then we also write $P = \boldsymbol{P(v_0, e_1, v_1, \ldots, e_p, v_p)}$. Clearly, the vertices $v_0, \dots, v_p$ are distinct, and we say that v_0 and v_p are the **endvertices** of P or that P is a path **joining** v_0 and v_p. For $x, y \in V(P)$, the subpath of P joining x and y is denoted by $\boldsymbol{xPy}$ or $\boldsymbol{yPx}$. If u is an endvertex of P, then we obtain a linear order $\boldsymbol{\preceq_{(u,P)}}$ of the vertex set of P in a natural way, where $x \preceq_{(u,P)} y$ if the vertex x belongs to the subpath uPy.

1.2 Edge Colourings

By a $\boldsymbol{k}$**-edge colouring** of a graph G we mean a map $\varphi : E(G) \to \{1, \dots, k\}$ that assigns to every edge e of G a colour $\varphi(e) \in \{1, \dots, k\}$ such that no two adjacent edges receive the same colour. The set of all k-edge colourings of G is denoted by $\boldsymbol{\mathcal{C}_k(G)}$. The **chromatic index** or **edge chromatic number** $\boldsymbol{\chi'(G)}$ is the smallest integer $k \geq 0$ such that $\mathcal{C}_k(G) \neq \emptyset$.

In the classic papers by Shannon [36] and Vizing [39, 40] a simple, but very useful recolouring technique was developed, dealing with edge colouring problems in graphs. Suppose that G is a graph, and φ is a k-edge colouring of G. To obtain a new colouring, choose two distinct colours α, β, and consider the subgraph H with $V(H) = V(G)$ and $E(H) = \{e \in E(G) \mid \varphi(e) \in \{\alpha, \beta\}\}$. Then every component of H is either a path or an even cycle, and we refer to such a component as an $\boldsymbol{(\alpha,\beta)}$**-chain** of G with respect to φ. For every vertex $v \in V(G)$, we denote by $\boldsymbol{P_v(\alpha, \beta, \varphi)}$ the unique (α, β)-chain with respect to φ that contains the vertex v. For two vertices $v, w \in V(G)$, the chains $P_v(\alpha, \beta, \varphi)$ and $P_w(\alpha, \beta, \varphi)$ are either equal or vertex disjoint. Now choose an arbitrary (α, β)-chain C of G with respect to φ. If we interchange the colours α and β on C, then we obtain a k-edge colouring φ' of G satisfying

$$\varphi'(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G) \setminus E(C), \\ \beta & \text{if } e \in E(C) \text{ and } \varphi(e) = \alpha, \\ \alpha & \text{if } e \in E(C) \text{ and } \varphi(e) = \beta. \end{cases}$$

In what follows, we briefly say that the colouring φ' is obtained from φ by **recolouring** C, and we write $\boldsymbol{\varphi' = \varphi/C}$. This recolouring operation is also called a **Kempe change**. Kempe changes have been introduced by Kempe in his false proof of the four colour theorem. They have proved to be an utmost useful tool in graph colouring theory; it remains one of the basic and most powerful tools.

Consider a graph G and a colouring $\varphi \in \mathcal{C}_k(G)$. For a vertex $v \in V(G)$, we define the two colour sets

$$\varphi(v) = \{\varphi(e) \mid e \in E_G(v)\}$$

and

$$\bar{\varphi}(v) = \{1, \dots, k\} \setminus \varphi(v).$$

We call $\varphi(v)$ the set of colours **present** at v, and $\bar{\varphi}(v)$ the set of colours **missing** at v with respect to φ. For a vertex set $X \subseteq V(G)$, we define $\bar{\varphi}(X) = \bigcup_{x \in X} \bar{\varphi}(x)$. If $\alpha, \beta \in \{1, \dots, k\}$ are two distinct colours and u, v are two distinct vertices of G satisfying $\alpha \in \bar{\varphi}(u)$ and $\beta \in \bar{\varphi}(v)$, then (u, v) is called an $\boldsymbol{(\alpha,\beta)}$**-pair** with respect to φ.

Let $\alpha, \beta \in \{1, \dots, k\}$ be two distinct colours. Moreover, let $v \in V(G)$, and let $P = P_v(\alpha, \beta, \varphi)$. If exactly one of the two colours α or β is missing at v with respect to φ, then P is a path where one endvertex is v, and the other endvertex is some vertex $u \neq v$ such that either α or β is missing at u. For the colouring $\varphi' = \varphi/P$, we have $\varphi' \in \mathcal{C}_k(G)$. Moreover, if w is an endvertex of P then we have

$$\bar{\varphi}'(w) = \begin{cases} (\bar{\varphi}(w)\setminus\{\beta\}) \cup \{\alpha\} & \text{if } \bar{\varphi}(w) \cap \{\alpha,\beta\} = \{\beta\}, \\ (\bar{\varphi}(w)\setminus\{\alpha\}) \cup \{\beta\} & \text{if } \bar{\varphi}(w) \cap \{\alpha,\beta\} = \{\alpha\}, \\ \bar{\varphi}(w) & \text{if } \bar{\varphi}(w) \cap \{\alpha,\beta\} = \{\alpha,\beta\}. \end{cases}$$

For all other vertices w beside the endvertices of P, we have $\bar{\varphi}'(w) = \bar{\varphi}(w)$. These facts shall be used quite often without mentioning it explicitly.

Consider a graph G, an edge $e \in E(G)$, a colouring $\varphi \in \mathcal{C}_k(G-e)$, and a vertex set $X \subseteq V(G)$. Then X is called **elementary** with respect to φ if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. The set X is called **closed** with respect to φ if, for every edge $f \in E_G(X, V(G)\setminus X)$, the colour $\varphi(f)$ is present at every vertex of X, that is, $\varphi(f) \in \varphi(v)$ for every $v \in X$. Finally, the set X is called **strongly closed** with respect to φ if X is closed with respect to φ, and $\varphi(f) \neq \varphi(f')$ for every two distinct edges $f, f' \in E_G(X, V(G)\setminus X)$.

Figure 1: A graph G with an chain $P_x(\alpha,\beta,\varphi)$ (bold edges)

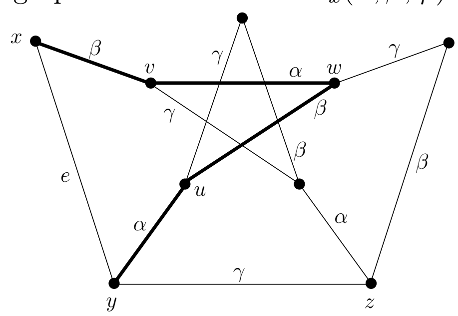

Figure 1 shows the graph G obtained from the Petersen graph by deleting one vertex as well as a 3-edge-colouring φ of $G-e$ where the three colours are α, β, γ. The graph G itself has chromatic index 4. Furthermore, $\bar{\varphi}(x) = \{\alpha, \gamma\}$, $\bar{\varphi}(y) = \{\beta\}$, $\bar{\varphi}(u) = \bar{\varphi}(v) = \bar{\varphi}(w) = \bar{\varphi}(z) = \emptyset$, and $P_x(\alpha,\beta,\varphi)$ is a path of length 4 with vertex set $X = \{x, v, w, u, y\}$. The set X is elementary with respect to φ, but not closed.

1.3 Critical Graphs

By a **graph parameter** we mean a function ρ that assigns to each graph G a real number $\rho(G)$ such that $\rho(G) = \rho(H)$ whenever G and H are isomorphic graphs. A graph parameter ρ is called **monotone** if $\rho(H) \leq \rho(G)$ whenever H is a subgraph of G. Clearly, the set of all graph parameters form a real vector space with respect to the addition of functions and the multiplication of a function by a real number. Let ρ and ρ' be two graph parameters. If $\rho'(G) = c$ for every graph G, then instead of $\rho + \rho'$ we also write $\rho + c$. If $\rho(G) \leq \rho'(G)$ holds for every graph G, then we say that ρ' is an **upper bound** for ρ and ρ is a **lower bound** for ρ'.

Criticality is a general concept in graph theory and can be defined with respect to various graph parameters. The importance of the notion of criticality is that problems for graphs in general may often be reduced to problems for critical graphs whose structure is more restricted. Critical graphs (with respect to the chromatic number) were first defined and used by Dirac [5] in 1951.

Let ρ be a monotone graph parameter. A graph G is called **ρ-critical**, if $\rho(H) < \rho(G)$ for every proper subgraph H of G. We say that $e \in E(G)$ is a **ρ-critical edge** if $\rho(G - e) < \rho(G)$. Evidently, in a ρ-critical graph every edge is ρ-critical.

Proposition 1.1 *Let ρ and ρ' be two monotone graph parameters. Then the following statements hold:*

(a) *Every graph G contains a ρ-critical subgraph H with $\rho(H) = \rho(G)$.*

(b) *If $\rho(H) \leq \rho'(H)$ for all ρ-critical graphs H, then ρ' is an upper bound of ρ.*

Proof: Since ρ is monotone, every graph G contains a minimal subgraph H with $\rho(H) = \rho(G)$. Obviously, the graph H is ρ-critical which proves (a).

Now let G be an arbitrary graph. By (a), G contains a ρ-critical subgraph H with $\rho(H) = \rho(G)$. Then $\rho(H) \leq \rho'(H)$ and, since ρ' is monotone, we have $\rho(G) = \rho(H) \leq \rho'(H) \leq \rho'(G)$. This proves (b). ■

For convenience, we allow a graph G to be **empty**, that is, $V(G) = E(G) = \emptyset$. In this case we also write $G = \emptyset$. For the empty graph G, define $\chi'(G) = \Delta(G) = \delta(G) = \mu(G) = 0$. If ρ is a monotone graph parameter, then the empty graph is ρ-critical; it is the only ρ-critical graph H with $\rho(H) = \rho(\emptyset)$.

By a **critical graph** we always mean a χ'-critical graph, and by a **critical edge** we always mean a χ'-critical edge. Clearly, a graph G is critical if and only if G is connected and every edge of G is critical. Moreover, $\chi'(G) \leq k$ if and only if G does not contain a $(k+1)$-critical subgraph.

1.4 Elementary Graphs

Consider a graph G and a colouring $\varphi \in \mathcal{C}_k(G)$. Clearly, for every colour $\gamma \in \{1, \ldots, k\}$ and every subgraph H of G with $|V(H)| \geq 2$, the edge set $E_\gamma(H) = \{e \in E(H) \mid \varphi(e) = \gamma\}$ is a matching of H. Consequently, we have $|E_\gamma(H)| \leq \frac{1}{2}|V(H)|$ for every colour γ and, therefore, $|E(H)| \leq k \left\lfloor \frac{1}{2}|V(H)| \right\rfloor$. This observation leads to the following parameter for a graph G with $|V(G)| \geq 2$, namely the **density**

$$\boldsymbol{\mathcal{W}(G)} = \max_{\substack{H \subseteq G \\ |V(H)| \geq 2}} \left\lceil \frac{|E(H)|}{\left\lfloor \frac{1}{2}|V(H)| \right\rfloor} \right\rceil.$$

For a graph G with $|V(G)| \leq 1$, define $\mathcal{W}(G) = 0$. Then, clearly, we have $\chi'(G) \geq \mathcal{W}(G)$ for every graph G. A graph G satisfying $\chi'(G) = \mathcal{W}(G)$ is called an **elementary graph**. The following conjecture seems to have been thought of first by Goldberg [9] around 1970 and, independently, by Seymour [35] in 1977.

Conjecture 1.2 (Goldberg [9] 1973 and Seymour [35] 1979) *Every graph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

The density $\mathcal{W}$ is related to the so-called fractional chromatic index. A **fractional edge colouring** of a graph G is an assignment of a non-negative weight w_M to each matching M of G such that, for every edge $e \in E(G)$, we have

$$\sum_{M: e \in M} w_M \geq 1.$$

Then the **fractional chromatic index** $\chi'_f(G)$ is the minimum value of $\sum_M w_M$, where the sum is over all matchings M of G, and the minimum is over all fractional edge colourings w of G. In case of $|E(G)| = 0$ we have $\chi'_f(G) = 0$. From the definition it follows that $\chi'_f(G) \leq \chi'(G)$ for every graph G. The computation of the chromatic index is NP-hard, but with matching techniques one can compute the fractional chromatic index in polynomial time, see [33, 34] for a proof.

From Edmonds' matching polytope theorem the following characterization of the fractional chromatic index of an arbitrary graph G can be obtained (see [33, 34] for details):

$$\chi'_f(G) = \max\{\Delta(G), \max_{\substack{H \subseteq G \\ |V(H)| \geq 2}} \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor}\}. \tag{1.1}$$

As an immediate consequence of this characterization we obtain $\mathcal{W}(G) \leq \Delta(G)$ if $\chi'_f(G) = \Delta(G)$, and $\mathcal{W}(G) = \lceil \chi'_f(G) \rceil$ if $\chi'_f(G) > \Delta(G)$. This implies, that Goldberg's conjecture is equivalent to the claim that $\chi'(G) = \lceil \chi'_f(G) \rceil$ for every graph G with $\chi'(G) \geq \Delta(G) + 2$.

The following result due to Kahn [18] shows that the fractional chromatic index asymptotically approximates the chromatic index.

Theorem 1.3 (Kahn [18] 1996) *For every $\epsilon \geq 0$, there is a Δ_ϵ such that any graph G with $\chi'_f(G) > \Delta_\epsilon$ satisfies $\chi'(G) < (1 + \epsilon)\chi'_f(G)$.*

This result was proved by probabilistic methods. We will extend this result and show, using constructive colouring arguments, that, for every $\epsilon > 0$, every graph G with $\Delta(G) \geq \frac{1}{2\epsilon^2}$ satisfies $\chi'(G) \leq \max\{(1 + \epsilon)\Delta(G), \mathcal{W}(G)\}$, see Section 2.7.

The concept of elementary graphs is closely related to the concept of elementary sets. The relations between elementary graphs and sets being elementary and strongly closed are shown by the following result, which is implicitly contained in the papers by Andersen [1] and Goldberg [11]. A proof of this theorem can be found in [8].

Theorem 1.4 (Favrholdt, Stiebitz and Toft [8] 2006) *Let G be a graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)$. If G is critical, then the following conditions are equivalent:*

(a) *G is elementary.*

(b) *For every edge $e \in E(G)$ and every colouring $\varphi \in \mathcal{C}_k(G - e)$, the set $V(G)$ is elementary with respect to φ.*

(c) *There is an edge* $e \in E(G)$ *and a colouring* $\varphi \in \mathcal{C}_k(G-e)$ *such that* $V(G)$ *is elementary with respect to* φ.

(d) *There is an edge* $e \in E(G)$, *a colouring* $\varphi \in \mathcal{C}_k(G-e)$, *and a set* $X \subseteq V(G)$ *such that* X *contains the two endvertices of* e, *and* X *is elementary as well as strongly closed with respect to* φ.

The following result provides some basic facts about elementary sets which will be useful for our further investigations.

Proposition 1.5 (Favrholdt, Stiebitz and Toft [8] 2006) *Let* G *be a graph with* $\chi'(G) = k+1$ *for an integer* $k \geq \Delta = \Delta(G)$, *and let* $e \in E(G)$ *be a critical edge of* G. *If* $X \subseteq V(G)$ *is an elementary set with respect to a colouring* $\varphi \in \mathcal{C}_k(G-e)$ *such that both endvertices of* e *are contained in* X, *then the following statements hold:*

(a) $|X| \leq \frac{|\bar{\varphi}(X)|-2}{k-\Delta} \leq \frac{k-2}{k-\Delta}$ *provided that* $k \geq \Delta + 1$

(b) $\sum_{v \in X} d_G(v) \geq k(|X|-1)+2$

(c) *Suppose that*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

for an integer $m \geq 3$. *Then* $|X| \leq m-1$ *and, moreover,* $|\bar{\varphi}(X)| \geq \Delta + 1$ *provided that* $|X| = m-1$.

2 Edge Colouring Algorithms

2.1 How to Colour a Graph?

The **Edge Colouring Problem** (**ECP**) asks for an optimal edge colouring of a graph G, that is, an edge colouring with $\chi'(G)$ colours. Holyer [15] proved that the determination of the chromatic index is NP-hard even for simple 3-regular graphs, where the chromatic index is either 3 or 4. Hence it is reasonable to search for upper bounds of the chromatic index, in particular for those bounds that can be efficiently realized by a colouring algorithm. A graph parameter ρ is said to be an **efficiently realizable** upper bound of χ' if there exists an algorithm that computes, for every graph $G = (V, E)$, an edge colouring of G using at most $\rho(G)$ colours, where the algorithm has time complexity bounded from above by a polynomial in $|V|$ and $|E|$.

Edge colouring algorithms often have an execution time polynomial in $|E|$, but are only pseudo-polynomial in the number of bits needed to describe the graph, since edge multiplicities may be encoded as binary numbers, and the size of the input graph therefore may be just of order $\log |E|$. In scheduling applications usually it is necessary to really assign the colours to the edges instead of just computing a description of a colouring. Hence the number of edges is the relevant input parameter which justifies the definition of efficiently realizability.

One obvious way to find an edge colouring of an arbitrary graph G with at least one edge is the following **greedy algorithm**. The algorithm starts from an arbitrary fixed edge order $e_1, \ldots, e_m$ of G. Then it subsequently colours each edge

e_i with the smallest positive integer not already used to colour any adjacent edge e_j with $j < i$. If $\Delta(G) \le 2$, then G contains only cycles, paths, and isolated vertices and, therefore, the greedy algorithm computes an optimal colouring with either $\Delta(G)$ or $\Delta(G) + 1$ colours. If $\Delta(G) \ge 3$ then the greedy algorithm still works within a constant relative error. Since every edge is adjacent to at most $2(\Delta(G) - 1)$ other edges, this algorithm colours G using at most $2\Delta(G) - 1$ colours. Consequently, 2Δ is an efficiently realizable upper bound of χ' (including the case $E(G) = \emptyset$).

This upper bound is rather generous, and in most graphs there will be scope for an improvement of this bound by choosing a particularly suitable edge order to start with. Let us say that an **edge order** of a graph G is of **depth** p if each edge in this order is preceded by fewer than p of its adjacent edges. Clearly, if we start the greedy algorithm with an edge order of depth p, then the algorithm terminates with a p-edge colouring. The least number $p \ge 1$ such that G has an edge order of depth p is called the **colouring index** $\mathbf{col'(G)}$ of G. Obviously, every graph G with at least one edge satisfies $\mathrm{col}'(G) \le 2\Delta(G) - 1$. For an edgeless graph G, we have $\mathrm{col}'(G) = 1$. It is also known, see for instance [17], that an edge order $e_1, \dots, e_m$ of depth $\mathrm{col}'(G)$ can be obtained by letting e_i be an edge having a minimum number of adjacent edges in the subgraph $G_i = G - \{e_{i+1}, \dots, e_m\}$ for $i = m, m-1, \dots, 1$, where $G_m = G$. Hence, col' is an efficiently realizable upper bound of χ', obviously the best upper bound that can be realized by the greedy strategy.

Observe that the colouring index of a graph is nothing else than the colouring number of its line graph. The **colouring number** of a simple graph H is defined by the smallest integer $d \ge 1$ such that in some linear ordering of $V(H)$ every vertex of H is preceded by at most $d-1$ of its neighbours. This parameter was introduced and studied by Erdős and Hajnal [6] in 1966. The mentioned greedy algorithm using the edge order of depth col' corresponds to a similar vertex colouring algorithm on the line graph using a similar vertex order. In general, every k-edge colouring of a graph G is equivalent to a k-vertex colouring of the line graph $L(G)$. Hence, in order to find an edge colouring of a graph G, one can simply take a vertex colouring algorithm and run it on the line graph of G. Although the mentioned greedy strategy on the special edge ordering is such an example, we will not concentrate on this general approach.

Of course, other bounds better than 2Δ can be attained. Some classical efficiently realizable upper bounds of χ' are $\lfloor \frac{3}{2}\Delta \rfloor$ (Shannon [36] 1949), $\Delta+\mu$ (Vizing [39] 1964), and Δ_μ (Ore [23] 1967), where

$$\boldsymbol{\Delta_\mu(G)} = \max\{d_G(x) + \mu_G(x,y) \mid E_G(x,y) \neq \emptyset\}$$

if $E(G) \neq \emptyset$, and $\Delta_\mu(G) = 0$ if G is edgeless. An even better bound, the fan number, is presented in Section 2.4.

Like the greedy algorithm, a typical approximation algorithm for ECP colours the edges of the input graph sequentially. Hence such an algorithm first fixes an edge order of the input graph, either an arbitrarily order or one that satisfies a certain property. The core of the algorithm is a subroutine **Ext** that extends a given partial colouring of the input graph. The input of Ext is a tuple (G, e, x, y, k, φ), where G is the graph consisting of all edges that are already coloured as well as the next uncoloured edge $e \in E_G(x,y)$ with respect to the given edge order and a colouring

$\varphi \in \mathcal{C}_k(G-e)$. The output of Ext is a tuple (k', φ') where $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}_{k'}(G)$.

Now, to explain how Ext works, a well defined set $\boldsymbol{\mathcal{O}(G, e, \varphi)}$ of so-called **test objects** will be introduced. A test object $T \in \mathcal{O}(G, e, \varphi)$ is usually a labeled subgraph of G that fulfills a certain property with respect to the uncoloured edge e and the colouring $\varphi \in \mathcal{C}_k(G-e)$. In most cases, we start with the test object that only consists of the uncoloured edge e. The investigation of a test object $T \in \mathcal{O}(G, e, \varphi)$ may have three possible outcomes. First, a colouring $\varphi' \in \mathcal{C}_k(G)$ can be obtained from φ by a sufficiently small number of recolouring steps, for example Kempe changes. Then Ext returns (k, φ'). Second, the test object T can be "improved", possibly with respect to a new colouring $\varphi' \in \mathcal{C}_k(G-e')$ obtained from φ. Then the new test object $T' \in \mathcal{O}(G, e', \varphi')$ has to be investigated. Third, the edge e is coloured with a new colour, resulting in a colouring $\varphi' \in \mathcal{C}_{k+1}(G)$. Then Ext returns $(k+1, \varphi')$. To ensure that the subroutine Ext works correctly, we need statements about the test objects of the following type.

(T) *Let G be a graph, let $e \in E_G(x, y)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for some integer $k \geq \Delta(G)$. If the vertex set of a test object $T \in \mathcal{O}(G, e, \varphi)$ does not have the property $\mathcal{P}$ with respect to φ, then there is a colouring $\varphi' \in \mathcal{C}_k(G)$.*

Constructive proofs of such statements can be directly transformed into an algorithm Ext. Additionally, they provide some information about the time complexity of the algorithm. For complex test objects it may be useful to consider a somewhat weaker version of (T).

(T′) *Let G be a graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)$, let $e \in E_G(x, y)$ be a critical edge of G, and let $\varphi \in \mathcal{C}_k(G-e)$. Then each test object $T \in \mathcal{O}(G, e, \varphi)$ has the property $\mathcal{P}$ with respect to φ.*

Clearly, (T) implies (T′) but not the other way round. However, often the proof of (T′) is constructive in the way that it derives a contradiction to $\chi'(G) = k+1$ by constructing a colouring $\varphi' \in \mathcal{C}_k(G)$. In this case the proof can usually be translated into an equivalent proof of (T) or an algorithm Ext.

The advantage of the second version are shorter proofs which are easier to read and easier to understand, especially if the proof requires many steps changing the test object and/or the partial colouring. For example, suppose that the proof uses similar statements for some properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$. Then, every time the test object or the partial colouring changes, the proof of version (T) has to mention the cases where some of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are not fulfilled anymore, and the desired colouring φ' can be obtained in one of the 'old' ways. In version (T′) these properties are automatically fulfilled in the new situation, and the proof can concentrate on the significant new arguments.

Of course, when it comes to construct the algorithm Ext from the proof of (T′), then we have to translate it to version (T), at least notionally. We also have to be more careful when estimating the time complexity based on a proof of (T′).

For the basic test objects we will start with the first version, because the statement is stronger, and easier to adapt algorithmically. However, as the complexity of the test objects and recolouring strategies increases, we will change to the second version at some point.

A typical test in many colouring algorithms is whether the vertex set of a test object $T \in \mathcal{O}(G, e, \varphi)$ is elementary with respect to φ. Most of the classical kinds of test objects fulfil the following condition.

(1) *Let G be a graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)$, let $e \in E_G(x, y)$ be a critical edge of G, and let $\varphi \in \mathcal{C}_k(G - e)$ be a colouring. Then the vertex set of each test object $T \in \mathcal{O}(G, e, \varphi)$ is elementary with respect to φ.*

To control the number of colours used by a colouring algorithm of the above type, we need some further information about **maximal test objects**, that means test objects $T \in \mathcal{O}(G, e, \varphi)$ which cannot be "improved" by certain techniques. For the proof of Goldberg's conjecture, a statement of the following type would be sufficient.

(2) *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$, let $e \in E_G(x, y)$ be a critical edge of G, and let $\varphi \in \mathcal{C}_k(G - e)$ be a colouring. Then the vertex set of each maximal test object $T \in \mathcal{O}(G, e, \varphi)$ is elementary and strongly closed both with respect to φ.*

Suppose our test objects satisfy (1) and (2), and we start our colouring algorithm with $k = \Delta(G) + 1$ colours. If the algorithm never uses a new colour, then $\chi'(G) \leq \Delta(G)+1$. Otherwise, let us consider the last call of Ext where we use a new colour. The input is a tuple $(G', e, x, y, k, \varphi)$ where G' is a subgraph of G, $e \in E_{G'}(x, y)$, and $\varphi \in \mathcal{C}_k(G' - e)$. Since Ext returns $(k + 1, \varphi')$ with $\varphi' \in \mathcal{C}_{k+1}(G')$, there exist a maximal test object $T \in \mathcal{O}(G', e, \varphi)$ such that $X = V(T)$ is elementary and strongly closed both with respect to φ. Clearly, the colouring algorithm terminates with a $(k + 1)$-edge colouring of G implying $\chi'(G) \leq k + 1$. Now, let H be the subgraph of G with $V(H) = X$ and $E(H) = E(G[X]) \cap E(G')$. Then $E(H)$ consists of the uncoloured edge e and all edges of G that are already coloured and have both endvertices in X. Since $\bar{\varphi}(X) \supset \bar{\varphi}(x) \neq \emptyset$ and X is elementary as well as closed with respect to $\varphi \in \mathcal{C}_k(G' - e)$, it follows that $|X|$ is odd and, for every colour $\gamma \in \bar{\varphi}(X)$, there are $\lfloor \frac{1}{2}|X| \rfloor$ edges in $E_G(X, X)$ coloured with γ. Moreover, since $|X|$ is odd and X is strongly closed both with respect to φ, it follows that, for every colour $\alpha \notin \bar{\varphi}(X)$, there is exactly one edge in $E_G(X, V(G) \backslash X)$ coloured with α. Since X is elementary with respect to φ, this implies that there are $\lfloor \frac{1}{2}|X| \rfloor$ edges in $E_G(X, X)$ coloured with α. Consequently, $|V(H)|$ is odd, and H contains $\lfloor \frac{1}{2}|V(H)| \rfloor$ edges of each colour plus the uncoloured edge e, implying that $|E(H)| = 1 + k \lfloor \frac{1}{2}|V(H)| \rfloor$. Hence we have

$$\mathcal{W}(G) \geq \mathcal{W}(H) \geq \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil \geq k + 1 \geq \chi'(G) \geq \mathcal{W}(G)$$

and, therefore, $\chi'(G) = \mathcal{W}(G)$. As a consequence, our algorithm colours the edges of G with at most $\max\{\Delta(G)+1, \mathcal{W}(G)\}$ colours. Since both Δ and $\mathcal{W}$ are lower bounds of χ', this would imply that this algorithm works within an absolute approximation error of 1. For a polynomial-time algorithm, this would be best possible unless P equals NP.

Classical kinds of test objects are the fans first used by Shannon [36] and by Vizing [39], the critical chains introduced, independently, by Andersen [1] and by Goldberg [10, 11], and the Kierstead paths introduced by Kierstead [19]. A more recent kind of test objects, namely Tashkinov trees, were invented by Tashkinov [38]. All these kinds of test objects satisfy (1), but up to now test objects that fulfil both conditions (1) and (2) are not known. A possible way out of this disaster is to modify the subroutine Ext and/or to add further heuristics before using a fresh colour. If the vertex set X of a maximal test object $T \in \mathcal{O}(G, e, \varphi)$ is both elementary and strongly closed with respect to φ, then we just colour e with a new colour. However, if X is elementary, but not strongly closed with respect to φ, it might be reasonable to use again a small number of Kempe changes to obtain a better test object $T' \in \mathcal{O}(G, e', \varphi')$, and to continue with T' instead of T. We shall use this approach to get some partial results related to Goldberg's conjecture.

2.2 Implementation Details

Before constructing algorithms we will say something about time and space complexities. As usual in many algorithmic publications, our basic computer model matches a real computer rather than a Turing machine, but with the additionally assumption that memory and bandwidth is always big enough to handle the actual problem properly. Due to this model, we will generally assume that all basic operations with basic data types and memory access operations, especially pointer operations, have constant time costs O(1). In practice this is usually true up to a certain problem size which depends on the machine. For practical reasons, often only such problem sizes are relevant, this also supports our assumption. Note that this is a difference to the more theoretical definition using the Turing machine model, where time costs always rely on the length of the data. Since time complexity of an implementation on a Turing machine differs only by a polynomial factor, any polynomial-time algorithm in our model still is in P.

The time complexity T of our colouring algorithms has the form $T = T_1 + |E|T_2$, where T_1 is the time complexity for computing the required edge order of the input graph $G = (V, E)$, and T_2 is the (worst case) time complexity for one call of the subroutine Ext.

The running time T_2 depends much on the manner in which the partial colouring is stored. As long as we are satisfied with an overall running time T that is polynomial in $|E|$ and $|V|$, we can use the following ideas from [14]. In addition to the standard adjacency list representing the graph G, a **same-colour list** is stored for each colour. Each same-colour list contains all of the edges assigned a particular colour. In addition, there is a list of all uncoloured edges. An edge $e \in E_G(x, y)$ appears in the two adjacency lists for x and y as well as in a third list, which is either the same-colour list for the colour of e, or the list of uncoloured edges. These elements of the three lists are linked to each other by pointers so that each can be directly accessed from another. Each element of the same-colour list has also a pointer to the beginning of the list. If these lists are doubly-linked, then the basic operations, like inserting or deleting elements, have constant time costs. Since every vertex and every edge of G is stored constant times, we need $O(|V(G)| + |E(G)|)$ space for this representation.

Even the simple greedy algorithm uses at most $2\Delta(G)$ colours. Hence, for the number of colours k, we can assume that $k \in O(\Delta)$, where $\Delta = \Delta(G)$. Then, as explained in [14], the set $\bar{\varphi}(x)$ can be found in time $O(\Delta)$ by simply scanning the adjacency list of x and, therefore, one can decide in time $O(\Delta)$ whether two vertices have a common missing colour. Furthermore, it takes time $O(|V|)$ to find an (α, β)-chain $P = P_x(\alpha, \beta, \varphi)$. This can be done in two steps. First, by scanning the two same-colour lists, one can construct the subgraph H of G consisting of all edges coloured α or β in time $O(|V|)$. Second, one can just start in x and follow the chain in H. Since the length of P is at most $|V|$, this gives the mentioned time complexity. Performing a Kempe change on the chain P also costs time $O(|V|)$. This task can be easily accomplished. During the second step of finding P, one may simply recolour the edges of P. This gives the mentioned time complexity, because the length of P is at most $|V|$, and any given edge can be recoloured in constant time by deleting it from one and inserting it to the other same-colour list.

In most of our recolouring routines, Kempe changes are applied after a constant number of operations of lower cost. Consequently, in this case, the costs for the Kempe changes will dominate the other costs, and we will only count the number of Kempe changes to derive the time costs for this routine.

2.3 The Vizing Fan

One non-trivial test object for the edge colouring problem is the Vizing fan. The fan argument was introduced by Vizing [39] in order to prove that $\Delta + \mu$ is an upper bound for the chromatic index. However, the appropriate conclusion of the fan argument is not just Vizing's bound, but the so-called fan equation from which all the classical bounds can easily be derived.

Definition 2.1 *Let G be a graph, and let $\varphi \in \mathcal{C}_k(G - e)$ for an edge $e \in E(G)$ and an integer k. Further, let x be an endvertex of e. A* **multi-fan** *at x with respect to e and φ is a sequence $(e_1, y_1, \dots, e_p, y_p)$ consisting of edges $e_1, \dots, e_p$ and vertices $y_0, \dots, y_p$ satisfying the following two conditions:*

(F1) *The edges $e_1, \dots, e_p$ are distinct, $e_1 = e$, and $e_i \in E_G(x, y_i)$ for $i = 1, \dots, p$.*

(F2) *For every edge e_i with $2 \le i \le p$, there is a vertex y_j with $1 \le j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.*

Our definition of a fan differs slightly from the classical definition going back to Vizing [39, 40]. We allow multiple edges and only require the colour of an edge of the fan to be missing at some previous vertex of the fan (instead of missing exactly at the previous vertex). This change makes proofs easier and is essential for obtaining the fan equation in Theorem 2.2(d).

Let G be a graph, and let $F = (e_1, y_1, \dots, e_p, y_p)$ be a multi-fan at x with respect to an edge $e \in E(G)$ and a colouring $\varphi \in \mathcal{C}_k(G - e)$. Since the vertices of F need not be distinct, the set $V(F) = \{y_1, \dots, y_p\}$ may have a cardinality smaller than p. For $z \in V(F)$, let $\boldsymbol{\mu_F(x,z)} = |E_G(x,z) \cap \{e_1, \dots, e_p\}|$. Further, we call F a **maximal multi-fan** at x with respect to e and φ if there is no edge-vertex pair (f, v) such that (F, f, v) is a multi-fan at x with respect to e and φ.

Figure 2: A multi-fan $F = (e, y, e_2, z, e_3, v, e_4, z)$

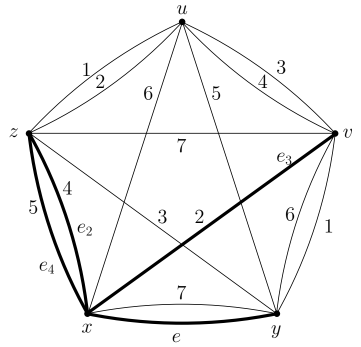

Figure 2 shows a graph G together with a colouring $\varphi \in \mathcal{C}_7(G - e)$. For the sets of missing colours, we obtain $\bar{\varphi}(x) = \{1,3\}$, $\bar{\varphi}(y) = \{2,4\}$, $\bar{\varphi}(z) = \{6\}$, $\bar{\varphi}(u) = \{7\}$, and $\bar{\varphi}(v) = \{5\}$. Hence $V(G) = \{x, y, z, u, v\}$ is elementary and strongly closed with respect to φ. Furthermore, $F = (e, y, e_2, z, e_3, v, e_4, z)$ is a multi-fan at x with respect to e and φ. Clearly, $\chi'(G) \leq 8$ and it is easy to show that equality holds. This follows from the simple fact that $\chi'(G) \geq \mathcal{W}(G) \geq \lceil \frac{|E(G)|}{2} \rceil = 8$. Hence, G is elementary. It is also easy to check that G is critical.

Theorem 2.2 *Let G be a graph, let $e \in E_G(x,y)$, and let $\varphi \in \mathcal{C}_k(G - e)$ for an integer $k \geq \Delta(G)$. Furthermore, let $F = (e_1, y_1, \dots, e_p, y_p)$ be a multi-fan at x with respect to e and φ. Then the following statements hold:*

(a) *If $\bar{\varphi}(x) \cap \bar{\varphi}(y_i) \neq \emptyset$ for an $i \in \{1, \dots, p\}$, then a colouring $\varphi^* \in \mathcal{C}_k(G)$ can be derived from φ by recolouring at most $i - 1$ edges and colouring the edge e.*

(b) *If $\alpha \in \bar{\varphi}(x)$, $\beta \in \bar{\varphi}(y_i)$ for an $i \in \{1, \dots, p\}$, and x is not an endvertex of the (α, β)-chain $P_{y_i}(\alpha, \beta, \varphi)$, then a colouring $\varphi^* \in \mathcal{C}_k(G)$ can be derived from φ by at most one Kempe change, recolouring at most $i - 1$ edges and colouring the edge e.*

(c) *If $y_i \neq y_j$ and $\bar{\varphi}(y_i) \cap \bar{\varphi}(y_j) \neq \emptyset$ for $1 \leq i < j \leq p$, then a colouring $\varphi^* \in \mathcal{C}_k(G)$ can be derived from φ by at most one Kempe change, recolouring at most $j - 1$ edges and colouring the edge e.*

(d) *If F is a maximal multi-fan at x with respect to e and φ such that $V(F) \cup \{x\}$ is elementary with respect to φ, then $|V(F)| \geq 2$ and*

$$\sum_{z \in V(F)} (d_G(z) + \mu_F(x,z) - k) = 2. \tag{2.1}$$

Proof: To prove (a), we assume that there is a colour $\alpha \in \bar{\varphi}(x) \cap \bar{\varphi}(y_i)$. Then, by definition of a multi-fan, there are indices $i_1 < \ldots < i_\ell$ such that $i_1 = 1$, $i_\ell = i$ and, for $j = 1, \ldots, \ell - 1$ the colour $\varphi(e_{i_{j+1}})$ is missing at y_{i_j} with respect to φ. Now we define

$$\varphi'(f) = \begin{cases} \alpha & \text{if } f = e_\ell\,, \\ \varphi(e_{i_{j+1}}) & \text{if } f = e_j,\ j \in \{1, \ldots, \ell - 1\}\,, \\ \varphi(f) & \text{otherwise.} \end{cases}$$

Evidently, we have $\varphi' \in \mathcal{C}_k(G)$. Only the edges $e_{i_2}, \ldots, e_{i_\ell}$ had to be recoloured to colour $e_{i_1} = e$. Since we have $\ell \le i$, this proves (a).

To prove (b), we assume that there are two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y_i)$ such that x is not an endvertex of $P = P_{y_i}(\alpha, \beta, \varphi)$. Since $\alpha \in \bar{\varphi}(x)$, this implies that x does not belong to $V(P)$ at all. Now let j be the smallest integer such that y_j is an endvertex of P. Since y_i is an endvertex of P, we have $j \le i$ and $P = P_{y_j}(\alpha, \beta, \varphi)$. Note that even in the case $j < i$ the vertices y_j and y_i may be the same. If $\alpha \in \bar{\varphi}(y_j)$ then we are done, using (a) and $j \le i$. If $\alpha \notin \bar{\varphi}(y_j)$ then $\beta \in \bar{\varphi}(y_j)$. Let $\varphi' = \varphi/P$ be the colouring obtained from φ by recolouring P. Since $\{x, y_1, \ldots, y_{j-1}\}$ does not contain an endvertex of P, we conclude that $\bar{\varphi}'(z) = \bar{\varphi}(z)$ for all vertices $z \in \{x, y_1, \ldots, y_{j-1}\}$. Moreover, since $x \notin V(P)$, and since all edges of F are incident with x, it follows that $E(P) \cap E(F) = \emptyset$. Hence, we have $\varphi'(f) = \varphi(f)$ for all edges $f \in E(F) \setminus \{e\}$. Consequently, $F' = (e_1, y_1, \ldots, e_j, y_j)$ is a multi-fan at x with respect to e and φ' where $\alpha \in \bar{\varphi}'(x) \cap \bar{\varphi}'(y_j)$. Then, by (a), a colouring $\varphi^* \in \mathcal{C}_k(G)$ can be derived form $\varphi' = \varphi/P$ by recolouring at most $j - 1$ edges and colouring the edge e. Since $j \le i$, this proves (b).

To prove (c), we assume that there are two indices $i < j$ a colour β such that $y_i \ne y_j$ and $\beta \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_j)$. From $k \ge \Delta(G)$ and $e \in E_G(x)$ we infer that there is a colour $\alpha \in \bar{\varphi}(x)$. If $\alpha = \beta$ then we are done, using (a). If $\alpha \ne \beta$ then let $P_1 = P_{y_i}(\alpha, \beta, \varphi)$. If x is not an endvertex of P_1 then we are done, using (b). If x is an endvertex of P_1 then let $P_2 = P_{y_j}(\alpha, \beta, \varphi)$. Since $y_j \notin \{x, y_i\}$ is an endvertex of P_2, we conclude that $P_2 \ne P_1$ and, therefore, x is not an endvertex of P_2. Hence, we are done, using (b) again. This proves (c).

For the proof of (d), we assume that F is maximal with respect to e and φ. By definition, we have $y \in V(F)$. From $k \ge \Delta(G)$ we infer that there is a colour $\beta \in \bar{\varphi}(y)$. Since $V(F) \cup \{x\}$ is elementary with respect to φ, this implies that there is an edge $e' \in E_G(x, y')$ with $\varphi(e') = \beta$ where $y' \ne y$. Since F is maximal with respect to e and φ, we have $y' \in V(F)$ and, therefore, $|V(F)| \ge 2$.

Next, we claim that the colour sets $\Gamma = \{\varphi(e_2), \ldots, \varphi(e_p)\}$ and $\Gamma' = \bigcup_{z \in V(F)} \bar{\varphi}(z)$ are equal. By definition, we have $\Gamma \subseteq \Gamma'$. Conversely, if $\beta \in \Gamma'$ then, since $V(F) \cup \{x\}$ is elementary with respect to φ, we have $\beta \in \varphi(x)$. Since we allow a multi-fan to have multiple edges, the maximality of F implies that $\beta \in \Gamma$. This proves the claim that $\Gamma = \Gamma'$. Since $y \in V(F)$, and $V(F) \cup \{x\}$ is elementary with respect to φ, we infer that

$$p - 1 = |\Gamma| = |\Gamma'| = \sum_{z \in V(F)} |\bar{\varphi}(z)| = 1 + \sum_{z \in V(F)} (k - d_G(z)).$$

Since $p = \sum_{z \in V(F)} \mu_F(x,z)$, this implies

$$\sum_{z \in V(F)} (d_G(z) + \mu_F(x,z) - k) = 2.$$

This completes the proof of Theorem 2.2. ■

Given a graph G, an edge $e \in E_G(x,y)$, and a colouring $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)$, Theorem 2.2 shows a way to colour the uncoloured edge e using multi-fans as test objects. This leads to an algorithm **VizExt** that runs on the input (G,e,x,y,k,φ) and returns an edge colouring for G. The algorithm VizExt is a first non-trivial version of Ext and works as follows. It successively builds a multi-fan F at x with respect to e and φ until either a k-edge colouring of G can be derived by means of Theorem 2.2, or F is maximal and the set $V(F) \cup \{x\}$ is elementary with respect to φ. In the latter case a new colour is used for e, and the so-called **fan equation** (2.1) holds.

VizExt(G,e,x,y,k,φ):

1) $p \leftarrow 1$, $e_p \leftarrow e$, $y_p \leftarrow y$, $F \leftarrow (e_p, y_p)$.

2) `If` $\bar{\varphi}(x) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  2a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.2(a).`

  2b) `Return` (k, φ').

3) `If` $\exists i \in \{1, \ldots, p-1\} : \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  3a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.2(c).`

  3b) `Return` (k, φ').

4) `If` $\exists e_{p+1} \in E_G(x) \backslash E(F) : \varphi(e_{p+1}) \in \bar{\varphi}(V(F))$ `then`

  4a) `Let` y_{p+1} `be the endvertex of` e_{p+1} `that is not` x.

  4b) $F \leftarrow (F, e_{p+1}, y_{p+1})$, $p \leftarrow p+1$.

  4c) `Goto 2.`

5) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k+1$.

6) `Return` $(k+1, \varphi')$.

Theorem 2.3 *Let G be a graph, let $e \in E_G(x,y)$ be an edge, and let $\varphi \in \mathcal{C}_k(G)$ be a colouring for an integer $k \geq \Delta(G)$. Then, on the input (G,e,x,y,k,φ), the algorithm* VizExt *returns a tuple (k', φ') such that $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}_{k'}(G)$. Moreover, if $k' = k+1$ then there is a maximal multi-fan F at x with respect to e and φ such that $V(F) \cup \{x\}$ is elementary with respect to φ.*

Proof: The algorithm VizExt contains one goto-statement, so it may enter step 2 many times. We claim that when VizExt enters step 2 for the j-th time then we have $p = j$, $F = (e_1, y_1, \ldots, e_p, y_p)$ is a multi-fan at x with respect to e and φ, and $(V(F) \cup \{x\}) \backslash \{y_p\}$ is elementary with respect to φ. We prove this by induction.

For $j = 1$, the algorithm comes from step 1, and we have $p = 1$ and $F = (e_1, y_1) = (e, y)$. Clearly, F is a multi-fan at x with respect to e and φ and, trivially,

$(V(F) \cup \{x\}) \setminus \{y_p\} = \{x\}$ is elementary with respect to φ. This settles the basic case.

For $j > 1$, the algorithm comes from step 4c and must have past steps 2-4 before. When VizExt enters step 2 the $(j-1)$th time then, by induction, we have $p = j - 1$, $F = (e_1, y_1, \ldots, e_p, y_p)$ is a multi-fan at x with respect to e and φ, and $(V(F) \cup \{x\}) \setminus \{y_p\}$ is elementary with respect to φ. To reach step 4c the conditions in 2 and 3 must fail. This implies that $V(F) \cup \{x\}$ is elementary with respect to φ. Further, the condition of 4 must be true. Hence, after step 4b we have $p = j$, $F = (e_1, y_1, \ldots, e_p, y_p)$ is a multi-fan at x with respect to e and φ, and $(V(F) \cup \{x\}) \setminus \{y_p\}$ is elementary with respect to φ. Then the algorithm enters step 2 the jth time. This completes the induction and proves the claim.

If, at some point, the conditions of statement 2 or 3 are fulfilled, it then follows that the corresponding conditions of Theorem 2.2 are fulfilled. Hence, in both cases, VizExt works correctly, computes a colouring $\varphi' \in \mathcal{C}_k(G)$ and returns (k, φ'). Now consider the case that neither of these two conditions is fulfilled. Since in step 4b the size of F increases but is bounded by $d_G(x) \leq \Delta(G)$, the algorithm cannot stay in this loop forever. Hence, at one point, the condition of statement 4 has to fail. In this case VizExt extends φ to φ' by using an additional colour for the edge e and returns $(k+1, \varphi')$. This proves the first part of the theorem.

If VizExt returns $(k+1, \varphi')$ then, at some point, the condition of statement 4 fails. At this point, F is a maximal multi-fan at x with respect to e and φ, and $V(F) \cup \{x\}$ is elementary with respect to φ. This completes the proof. ■

Note that, for any input of VizExt, the number of colours k must be at least the maximum degree of the input graph. Since Δ is a lower bound for χ', this is not a real limitation. An edge colouring algorithm that uses VizExt as kernel may simply start with $k = \Delta(G)$, then VizExt always gets a valid input.

Theorem 2.3 shows that VizExt works as intended, so let us now take a look at the running time of VizExt on a valid input (G, e, x, y, k, φ), where $G = (V, E)$ and $\Delta(G) = \Delta$. Further, we can assume that $k \in \mathrm{O}(\Delta)$, see Section 2.2. The first initializing step clearly needs only constant time. Due to the goto-statement in step 4c we have a loop. In every sweep through the loop the multi-fan F is extended by an edge-vertex pair. Since the edges of F are pairwise distinct and incident to the vertex x, the loop is repeated at most $d_G(x) \leq \Delta$ times.

Let us analyse the time costs inside the loop. The steps 2a and 3a result in a new colouring and ending the algorithm, so they are executed at most once and do not count to the loop. The check of step 2 can be done in time $\mathrm{O}(\Delta)$, see Section 2.2. The check of step 3 can also be done in time $\mathrm{O}(\Delta)$ by keeping some additional entries, one for every colour. For every colour α, we keep IND$[\alpha]$ the smallest index i such that this colour is missing at $y_i \in V(F)$. Then, in step 3, we can at the same time maintain this mapping and make the necessary check by simply scanning the colours of $\bar{\varphi}(y_p)$. For any colour with no entry IND so far, we simply set it to p. This way we get the time $\mathrm{O}(\Delta)$ for this step. Using this mapping and additionally marking the edges already used in F, we can also do the check of step 4 in time $\mathrm{O}(\Delta)$ by scanning the edges in the adjacency list of x and looking, for every non-marked edge, whether the colour of the edge has an IND-entry or not. If we found an edge with an entry then simply stop scanning and mark the new edge when adding to F. Getting y_{p+1} and extending F only needs constant time. Consequently, since the loop repeats at most $O(\Delta)$ times, we need time $O(\Delta^2)$ in total for the loop.

From Theorem 2.2(a) it follows that in step 2a at most $p-1 \le \Delta$ edges have to be recoloured. From the check in step 2 there is already a colour $\alpha \in \bar{\varphi}(x) \cap \bar{\varphi}(y_p)$ at hand. Then, using the IND-entries, all necessary edges and colours can be found in time $O(p)$, see proof of Theorem 2.2(a). The recolouring itself only needs time $O(1)$ for every edge, so we have a total time of $O(\Delta)$ for this step.

From Theorem 2.2(c) it follows that in step 3a at most one Kempe change is performed, and at most $p-1$ edges are recoloured. To do this, see the proof of Theorem 2.2(c), the algorithm has to find a colour $\alpha \in \bar{\varphi}(x)$ and a colour $\beta \in \varphi(y_i) \cap \bar{\varphi}(y_p)$ (which is already given by the check in step 3), find at most two (α, β)-chains, recolour one of it, and then recolour at most $p-1$ edges similar to step 2a. This can be done in time $O(|V| + \Delta)$, see Section 2.2.

In step 5 a new colour is assigned to the edge e. To do this, a new same-colour list with just one element has to be generated; this only requires constant time. Consequently, the whole algorithm VizExt has a time complexity of $O(|V| + \Delta^2)$. Further, note that VizExt can work on the data structures from the input and extend them. The notation φ' in the algorithm is just for convenience, it is contained in the same structures as φ.

The following edge colouring algorithm **Vizing** chooses an arbitrary edge ordering and then successively colours the edges of the input graph G, using VizExt.

Vizing(G):

1) Let G' be the edgeless graph with $V(G') = V(G)$,
let φ be the empty colouring of G', and let $k = \Delta(G)$.

2) For every edge $e \in E(G)$ do

- **2a)** Let x, y be the two endvertices of e.
- **2b)** $E(G') \leftarrow E(G') \cup \{e\}$
- **2c)** $(k, \varphi) \leftarrow$ VizExt$(G', e, x, y, k, \varphi)$

3) Return φ.

On an input $G = (V, E)$ with $\Delta(G) = \Delta$, the algorithm starts with $k = \Delta$ and a proper k-edge colouring of an edgeless graph. Since the routine VizExt never decreases the value of k, it always gets a valid input. Consequently, the algorithm Vizing terminates and computes a proper edge colouring of G. Further, the subroutine VizExt can work on the given data structures and simply extend them, and Vizing starts VizExt exactly $|E|$ times. Hence, the algorithm Vizing has a running time in $O\big(|E|(|V| + \Delta^2)\big)$.

In principle, Vizing is Vizing's classical colouring algorithm. He used it to prove the classical bounds $\frac{3}{2}\Delta$ and $\Delta+\mu$. These can easily be derived from Theorem 2.2(d), the fan equation. If G is a graph and $k \ge \Delta(G) + \mu(G)$ then the fan equation (2.1) fails for G as well as for any subgraph of G. Since Vizing starts with $k = \Delta(G)$, it follows that the value of k is never increased above $\Delta(G) + \mu(G)$ by the subroutine

VizExt. Consequently, Vizing uses not more than $\Delta(G) + \mu(G)$ colours, which proves Vizing's bound. Shannon's bound can be proved in the same way, because the fan equation also fails for every $k > \frac{3}{2}\Delta(G) - 1$. This can be seen as follows. Using $\sum_{z \in V(F)} \mu_F(x, z) \leq d_G(x)$, the fan equation implies $(|V(F)|+1)\Delta(G) - k|V(F)| \geq 2$, which fails for all $k > \Delta(G) + \frac{\Delta(G)-2}{|V(F)|}$. Since $\Delta(G) \geq d_G(x) \geq |V(F)| \geq 2$, the fan equation also fails for every $k > \frac{3}{2}\Delta(G) - 1$. Since Vizing starts with $k = \Delta(G)$, it follows that the value of k is never increased above $\frac{3}{2}\Delta(G)$ by the subroutine VizExt, which proves Shannon's bound.

The fan equation is a unifying result from which all the classical results on edge colourings seem easily derivable. The fan equation appears in several earlier papers as part of proofs, rather than as a separate result of interest in its own right; versions of it can be found in the papers by Andersen [1], Goldberg [11], Hilton and Jackson [13], and Choudum and Kayathri [4]. There have been three independent papers that have explicitly mentioned the fan equation as an important result and tool, namely, in chronological order, the M.Sc. thesis by Favrholdt [7] (see also the paper of Favrholdt, Stiebitz and Toft [8]), the paper by Reed and Seymour [24], and the Ph.D. thesis by Cariolaro [2]. The next section shows how the fan equation can be used to improve the colouring algorithm Vizing.

2.4 The Fan Number

One way to improve Vizing's colouring algorithm is to leave the kernel VizExt intact, but to choose an appropriate edge ordering. In [32] this was done, resulting in a new graph parameter, the fan number. The fan number is based on an ordering of vertex pairs that takes the fan equation into account.

Let G be a graph, and let $k \geq 0$ be an integer. By $\boldsymbol{\mathcal{F}_k(G)}$ we denote the set of all triples (x, y, Z) such that $x, y \in V(G)$, $y \in Z \subseteq N_G(x)$, $|Z| \geq 2$,

$$d_G(x) + d_G(y) - \mu_G(x, y) \geq k + 1,$$

and

$$\sum_{z \in Z} (d_G(z) + \mu_G(x, z) - k) \geq 2.$$

Obviously, for $k \geq \ell \geq 0$, we have

$$\mathcal{F}_k(G) \subseteq \mathcal{F}_\ell(G) \tag{2.2}$$

and, moreover, we have

$$H \subseteq G \Rightarrow \mathcal{F}_k(H) \subseteq \mathcal{F}_k(G) \tag{2.3}$$

For a pair (x, y) of distinct vertices of G, let $\mathbf{deg}_{\boldsymbol{G}}\boldsymbol{(x, y)}$ be the smallest integer $k \geq 0$ such that there is no vertex set Z with $(x, y, Z) \in \mathcal{F}_k(G)$. We call $\deg_G(x, y)$ the **fan-degree** of the vertex pair (x, y).

Note that if $E_G(x, y) = \emptyset$ or if x has at most one neighbour in G, then the fan-degree satisfies $\deg_G(x, y) = 0$.

Now, suppose that $E_G(x, y) \neq \emptyset$ and $|N_G(x)| \geq 2$. For a vertex $z \in N_G(x)$, let $d(z) = d_G(z) + \mu_G(x, z)$. Then $d(z) \geq 2$ for every neighbour z of x, and $N_G(x) \setminus \{y\}$

consists of $p \geq 1$ vertices $z_1, \ldots, z_p$, where the order is chosen such that $d(z_1) \geq d(z_2) \geq \ldots \geq d(z_p)$. For integers k, ℓ with $1 \leq \ell \leq p$, let

$$m_{k,\ell} = (d(y) - k) + \sum_{i=1}^{\ell} (d(z_i) - k).$$

Now, let Q denote the set of all integers k such that $0 \leq k \leq d_G(x) + d_G(y) - \mu_G(x, y) - 1$ and $m_{k,\ell} \geq 2$ for some integer $\ell \in \{1, \ldots, p\}$. Obviously, there is a set Z such that $(x, y, Z) \in \mathcal{F}_k(G)$ if and only if $k \in Q$. If $q = \max Q$ then $Q = \{0, 1, \ldots, q\}$ and $\deg_G(x, y) = q + 1 \geq 2$. Since we have $m_{k,\ell} = d(y) + \left(\sum_{i=1}^{\ell} d(z_i)\right) - (\ell + 1)k$, we can explicitly compute q by

$$q = \min \left\{ d_G(x) + d_G(y) - \mu_G(x, y) - 1, \ \max_{1 \leq \ell \leq p} \left\lfloor \frac{d(y) - 2 + \sum_{i=1}^{\ell} d(z_i)}{\ell + 1} \right\rfloor \right\}.$$

Note that, while increasing ℓ, the value of $\sum_{i=1}^{\ell} d(z_i)$ can always be calculated by just one simple addition to the former value. Hence, finding the necessary maximum term can be done in $\mathrm{O}(p)$ steps. Consequently, given all neighbours z of x, already sorted by $d(z)$, and given the values $d_G(z)$ and $\mu_G(x, z)$, we can compute $\deg_G(x, y)$ in $\mathrm{O}(|N_G(x)|)$ steps. Observe that $\deg_G(x, y)$ and $\deg_G(y, x)$ might be different.

From (2.2) and (2.3) it follows that if $H \subseteq G$ then every pair $x, y \in V(H)$ of distinct vertices satisfies

$$\deg_H(x, y) \leq \deg_G(x, y).$$

For a graph G with at least one edge, let

$$\boldsymbol{\delta^f(G)} = \min\{\deg_G(x, y) \mid x, y \in V(G),\ E_G(x, y) \neq \emptyset\}$$

be the **minimum fan-degree** of G. If G is an edgeless graph, we define $\delta^f(G) = 0$. Note that the parameter δ^f is not monotone. However, the **fan number** defined by

$$\mathbf{fan}\boldsymbol{(G)} = \max_{H \subseteq G} \delta^f(H)$$

is a monotone graph parameter.

By the above remark, it follows that there is a polynomial-time algorithm to compute the minimum fan-degree $\delta^f(G)$. In order to show that this is also true for the fan number, we describe an alternative way for computing the graph parameter $\mathrm{fan}(G)$.

Suppose that the graph G has at least one edge. Let $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ be a sequence of pairs consisting of distinct vertices of G. For such a sequence S, we define a sequence $(G_i \mid i = 1, \ldots, \ell + 1)$ of subgraphs of G by letting $G_1 = G$ and $G_{i+1} = G_i - E_{G_i}(x_i, y_i)$ for $i = 1, \ldots, \ell$. We call S a **feasible sequence** for G if $G_{\ell+1}$ is an edgeless graph and $E_{G_i}(x_i, y_i) \neq \emptyset$ for $i = 1, \ldots, \ell$. Now, for a graph G with at least one edge, we define $\boldsymbol{\sigma(G)}$ to be the smallest integer p such that there exists a feasible sequence $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ for G satisfying

$$\deg_{G_i}(x_i, y_i) \leq p \tag{2.4}$$

for $i = 1, \ldots, \ell$. We say that $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ is an **optimal sequence** for G if S is a feasible sequence for G and

$$\deg_{G_i}(x_i, y_i) = \delta^f(G_i) \tag{2.5}$$

for $i = 1, \ldots, \ell$, where $(G_i \mid i = 1, \ldots, \ell + 1)$ is the corresponding sequence of subgraphs. For an edgeless graph G, we define $\sigma(G) = 0$.

Theorem 2.4 (Scheide and Stiebitz [32]) *Every graph G satisfies*

$$\mathrm{fan}(G) = \sigma(G).$$

Furthermore, if G has at least one edge, and if $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ is an optimal sequence for G, then

$$\mathrm{fan}(G) = \max_{1 \le i \le \ell} \deg_{G_i}(x_i, y_i),$$

where $(G_i \mid i = 1, \ldots, \ell + 1)$ is the corresponding sequence of subgraphs.

Proof: First, consider the case that $\mathrm{fan}(G) = 0$. Then $\delta^f(H) = 0$ for every subgraph H of G. If G is edgeless, then $\mathrm{fan}(G) = \sigma(G) = 0$, and we are done. Otherwise, every subgraph H of G with at least one edge contains two adjacent vertices x, y such that $\deg_H(x, y) = 0$. This implies that there is a feasible sequence $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ for G such that $\deg_{G_i}(x_i, y_i) = 0$ for $i = 1, \ldots, \ell$, where $(G_i \mid i = 1, \ldots, \ell+1)$ is the corresponding sequence of subgraphs. Hence we are done, too.

Now, consider the case that $\mathrm{fan}(G) \ge 1$. Then G has at least one edge, and there is a feasible sequence $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ for G such that $\deg_{G_i}(x_i, y_i) \le \sigma(G)$ holds for $i = 1, \ldots, \ell$, where $(G_i \mid i = 1, \ldots, \ell + 1)$ is the corresponding sequence of subgraphs of G, that is, $G_1 = G$ and $G_{i+1} = G_i - E_{G_i}(x_i, y_i)$ for $i = 1, \ldots, \ell$. By definition, there is a subgraph H of G such that $\mathrm{fan}(G) = \delta^f(H)$. Since $\mathrm{fan}(G) \ge 1$, this implies that H has at least one edge. Consequently, since S is feasible, there is a smallest index i such that $E_H(x_i, y_i) \ne \emptyset$. Then H is a subgraph of G_i, and we conclude that

$$\mathrm{fan}(G) = \delta^f(H) \le \deg_H(x_i, y_i) \le \deg_{G_i}(x_i, y_i) \le \sigma(G).$$

Now let $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ be an optimal sequence for G, and let $(G_i \mid i = 1, \ldots, \ell + 1)$ be the corresponding sequence of subgraphs of G. Then, since S is feasible, we obtain

$$\sigma(G) \le \max_{1 \le i \le \ell} \deg_{G_i}(x_i, y_i) = \max_{1 \le i \le \ell} \delta^f(G_i) \le \max_{H \subseteq G} \delta^f(H) = \mathrm{fan}(G).$$

This completes the proof of the theorem. ∎

In particular, Theorem 2.4 implies, that there is a polynomial-time algorithm that computes, for a given graph G, the fan number of G as well as an optimal sequence S for G. This algorithm starts with $i = 1$ and $G_1 = G$ and, as long as G_i contains an edge, it chooses a vertex pair (x_i, y_i) such that $\deg_{G_i}(x_i, y_i)$ is minimal, computes $G_{i+1} = G_i - E_{G_i}(x_i, y_i)$ and increases i. Assuming that G_i is represented by an adjacency list containing edge multiplicities, this can be done as follows. First the vertex degrees $d_{G_i}(x)$ of all vertices $x \in V(G_i)$ are computed. This can be done in time $\mathrm{O}(|V(G_i)|^2)$, by simply adding multiplicities. Then for every vertex $x \in V(G_i)$ the neighbours z in the adjacency list of x are sorted by $d_{G_i}(z) + \mu_{G_i}(x,z)$, this needs time $\mathrm{O}(|N_{G_i}(x)| \log |N_{G_i}(x)|)$ for every x. Now, by the above remark, for every neighbour y of x the value of $\deg_{G_i}(x,y)$ can be computed in time $\mathrm{O}(|N_{G_i}(x)|)$. Thus, we need time $\mathrm{O}(|N_{G_i}(x)|^2)$, sorting inclusive, for each vertex x and, therefore, time $\mathrm{O}(|V(G_i)|^3)$ for the whole graph G_i. Eventually, the computation of the graph G_{i+1} only needs updating the adjacency list for x_i and y_i. The time cost for this depends a bit on the list representation, but definitely does not exceed the time $\mathrm{O}(|V(G_i)|^3)$ already needed. Since the length of the computed sequence S is bounded by $|V(G)|^2$, the algorithm has a running time $\mathrm{O}(|V(G)|^5)$.

For a graph G, let

$$\textbf{Fan}(\boldsymbol{G}) = \max\{\Delta(G), \mathrm{fan}(G)\}.$$

Clearly, this graph parameter is monotone and can be computed in $\mathrm{O}(|V(G)|^5)$ steps, too. We are now ready to design an efficient edge colouring algorithm VIZING2 that colours any graph G with at most $\mathrm{Fan}(G)$ colours.

VIZING2(G):

1) `If` G `is edgeless, then return the empty colouring` φ.

2) `Compute Fan`(G) `and an optimal sequence` $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$ `for` G.

3) `Let` G' `be the edgeless graph with` $V(G') = V(G)$, `and let` φ `be the empty colouring of` G'.

4) $k \leftarrow \Delta(G)$, $i \leftarrow \ell$.

5) `While` $i > 0$ `do:`

  5a) `For every edge` $e \in E_G(x_i, y_i)$ `do`

    5a-1) $E(G') \leftarrow E(G') \cup \{e\}$

    5a-2) $(k, \varphi) \leftarrow$ `VIZEXT`$(G', e, x_i, y_i, k, \varphi)$

  5b) $i \leftarrow i - 1$

6) `Return` φ.

The fan number in some way resembles the colouring number. While the definition of the colouring number relies on the degree of the vertices, the definition of the fan number relies on the fan-degree of the edges, or vertex pairs joined by at least one edge. Then the colouring algorithm VIZING2 picks edges of large fan-degree early and edges of small fan-degree last. Now we have to show that, for any graph G, the algorithm VIZING2 computes an edge colouring of G using at most $\mathrm{Fan}(G)$ colours.

Lemma 2.5 *Let G be a graph, let $e \in E_G(x,y)$, and let $\varphi \in \mathcal{C}_k(G)$ for an integer $k \geq \Delta(G)$. If $\deg_G(x,y) \leq k$ then, on the input (G, e, x, y, k, φ), the algorithm* VIZEXT *returns a tuple (k', φ') where $\varphi' \in \mathcal{C}_{k'}(G)$ and $k' = k$.*

Proof: From Theorem 2.3 we infer that $\varphi' \in \mathcal{C}_{k'}(G)$ and $k' \in \{k, k+1\}$. So we only have to show that $k' = k$. Assume, on the contrary, that $k' = k+1$. Then Theorem 2.3 implies that there is a maximal multi-fan F at x with respect to e and φ such that $V(F) \cup \{x\}$ is elementary with respect to φ. From Theorem 2.2(d) it then follows that the vertex set $Z = V(F)$ satisfies $|Z| \geq 2$, $y \in Z$, and

$$\sum_{z \in Z}(d_G(z) + \mu_G(x,z) - k) \geq \sum_{z \in Z}(d_G(z) + \mu_F(x,z) - k) = 2.$$

Since $Z \cup \{x\}$ is elementary with respect to φ, and since $y \in Z$, we obtain $\bar{\varphi}(x) \cap \bar{\varphi}(y) = \emptyset$. Hence all k colours are present at x or y with respect to φ and, therefore, we have $k = |\varphi(x) \cup \varphi(y)| = |\varphi(x)| + |\varphi(y)| - |\varphi(x) \cap \varphi(y)| \leq (d_G(x) - 1) + (d_G(y) - 1) - (\mu_G(x,y) - 1)$. Hence, we have $d_G(x) + d_G(y) - \mu_G(x,y) \geq k+1$. Consequently, we obtain $(x, y, Z) \in \mathcal{F}_k(G)$ and, therefore, $\deg_G(x,y) \geq k+1$, a contradiction. This completes the proof. ■

Theorem 2.6 *Let G be an arbitrary graph. Then, on the input G, the algorithm* VIZING2 *returns an edge colouring of G that uses at most* $\mathrm{Fan}(G)$ *colours.*

Proof: If G is edgeless then $\mathrm{Fan}(G) = 0$, and VIZING2 instantly returns the empty colouring. Hence, we are done. If G contains at least one edge then VIZING2 first computes $\mathrm{Fan}(G)$ and an optimal sequence $S = ((x_i, y_i) \mid i = 1, \ldots, \ell)$. Let $(G_i \mid i = 1, \ldots, \ell+1)$ be the corresponding sequence of subgraphs of G. Then the algorithm initializes G', φ, k, and i. Now VIZING2 enters step 5 and, since $i = \ell > 0$, enters the loop 5a-5b. We claim that after $j \geq 0$ times running this loop, we have $i = \ell - j$, $G' = G_{i+1}$, $\Delta(G) \leq k \leq \mathrm{Fan}(G)$, and $\varphi \in \mathcal{C}_k(G')$. We prove this by induction.

For the basic case $j = 0$, we have the initial state $i = \ell$, $G' = G_{\ell+1}$ the edgeless graph, $k = \Delta(G) \leq \mathrm{Fan}(G)$, and $\varphi \in \mathcal{C}_k(G')$ the empty colouring. This settles the case.

Now let $j > 0$. By induction, when VIZING2 enters the loop for the jth time, we have $i = \ell - j + 1$, $G' = G_{i+1}$, $\Delta(G) \leq k \leq \mathrm{Fan}(G)$, and $\varphi \in \mathcal{C}_k(G')$. Then VIZING2 executes step 5a. In this step, the algorithm subsequently extends G' by the edges in $E_G(x_i, y_i)$, and uses VIZEXT to extend the colouring φ. Consequently, while executing this loop, at any time we have $E(G') \subseteq E(G_{i+1}) \cup E_G(x_i, y_i)$ and, therefore, $G' \subseteq G_i$. Moreover, at the end we have $G' = G_i$. To analyse the development of the value of k during the loop, we consider two cases. In the case that $k \leq \mathrm{Fan}(G) - 1$ before step 5a-2 in this loop, we conclude from Theorem 2.3 that $k \leq \mathrm{Fan}(G)$ after step 5a-2. Now consider the case that $k = \mathrm{Fan}(G)$ before step 5a-2. From Theorem 2.4 it then follows that $\deg_{G'}(x,y) \leq \deg_{G_i}(x,y) \leq \mathrm{fan}(G) \leq \mathrm{Fan}(G) = k$. Then, by Lemma 2.5, the value of k remains the same during execution of step 5a-2. Since we had $k \leq \mathrm{Fan}(G)$ before step 5a, the value of k never exceeds $\mathrm{Fan}(G)$. Hence, at the beginning of step 5b, we have $G' = G_i$, $\Delta(G) \leq k \leq \mathrm{Fan}(G)$, and $\varphi \in \mathcal{C}_k(G')$. Then the value of i is decreased by 1, resulting in $i = \ell - j$ and $G' = G_{i+1}$. This completes the induction step.

After running the loop ℓ times we have $i = 0$, and the loop ends. Then we have $G' = G$, and VIZING2 returns the colouring $\varphi \in \mathcal{C}_k(G)$ where $k \leq \mathrm{Fan}(G)$. This completes the proof. ■

Now we will look at the time complexity of VIZING2. Let G be a graph, and let $n = |V(G)|$, $m = |E(G)|$, and $\Delta = \Delta(G)$. Then VIZING2 needs the following time to compute a colouring for G. If not already given, the data structures used to compute S (see above), can be constructed in time $\mathrm{O}(n + m)$, and then S can be computed in time $\mathrm{O}(n^5)$. Note that the algorithm does not have to compute and store the whole sequence $(G_i \mid i = 1, \ldots, \ell + 1)$ of graphs; it was only used for notation in the proof. Clearly, after computing the optimal sequence S, the rest of the algorithm VIZING2 runs in time $\mathrm{O}\big(m(n + \Delta^2)\big)$, just like VIZING. This gives a total running time of $\mathrm{O}\big(n^5 + m(n + \Delta^2)\big)$ for the algorithm VIZING2 for the input G. A simple consequence of this is the following result.

Theorem 2.7 (Scheide and Stiebitz [32]) *The parameter* Fan *is an efficiently realizable upper bound for the chromatic index* χ'*.*

Theorem 2.7 implies, in particular, that $\chi'(G) \leq \mathrm{fan}(G)$ whenever G is a graph with $\mathrm{fan}(G) \geq \Delta(G)$. The next result provides some information about the structure of fan-critical graphs. This result enables us to show that various known upper bounds for the chromatic index are also upper bounds for the fan number. The advantage of using the concept of criticality is that the existence of bounds usually can be shown much easier this way than by constructing algorithms. Since we already have the algorithm VIZING2 which efficiently realizes the parameter Fan, this algorithm will also efficiently realize all upper bounds of Fan.

Theorem 2.8 (Scheide and Stiebitz [32]) *Let G be a* fan*-critical graph such that* $\mathrm{fan}(G) = k + 1$ *for an integer $k \geq 0$, and let $x, y \in V(G)$ with $E_G(x, y) \neq \emptyset$. Then there is a vertex set Z such that*

(a) $(x, y, Z) \in \mathcal{F}_k(G)$.

Furthermore, there are two vertices $z_1, z_2 \in Z$ such that

(b) $d_G(z_1) + \mu_G(x, z_1) \geq \mathrm{fan}(G)$,

(c) $d_G(z_1) + \mu_G(x, z_1) + d_G(z_2) + \mu_G(x, z_2) \geq 2\mathrm{fan}(G)$, *and*

(d) $d_G(x) + d_G(z_1) + d_G(z_2) \geq 2\mathrm{fan}(G)$.

Proof: Since fan as well as Fan are monotone graph parameters and G is critical with respect to the fan number, we infer that $\mathrm{fan}(G) = \delta^f(G)$. Therefore, we have $\deg_G(x, y) \geq \delta^f(G) = k + 1 \geq 1$. This implies that there is a vertex set Z such that $(x, y, Z) \in \mathcal{F}_k(G)$, this shows (a). Consequently, we have $|Z| \geq 2$, $y \in Z \subseteq N_G(x)$,

$$d_G(x) + d_G(y) - \mu_G(x, y) \geq k + 1,$$

and

$$\sum_{z \in Z} (d_G(z) + \mu_G(x, z) - k) \geq 2.$$

If $a_1, \ldots, a_\ell$ is a non-increasing sequence of $\ell \geq 2$ integers with $\sum_{i=1}^{\ell} a_i \geq 2$ then, clearly, $a_1 \geq 1$ and $a_1 + a_2 \geq 2$. This proves (b) as well as (c). Since $d_G(x) \geq \mu_G(x, z_1) + \mu_G(x, z_2)$, (c) implies (d). ■

Theorem 2.9 *The colouring index* col′ *is an upper bound for the fan number and hence an efficiently realizable upper bound for* χ'.

Proof: The aim is to show that every graph G satisfies $\text{fan}(G) \leq \text{col}'(G)$. Since both parameters are monotone, it follows from Proposition 1.1(b) that it is sufficient to prove this inequality for all fan-critical graphs G. If $\text{fan}(G) = 0$, this is evident. Otherwise, G is a fan-critical graph with $\text{fan}(G) \geq 1$ and, therefore, $E(G) \neq \emptyset$. Let $e \in E_G(x, y)$ be an arbitrary edge. Clearly, exactly $d_G(x) + d_G(y) - \mu_G(x, y) - 1$ edges of G are adjacent to e. From Theorem 2.8 we infer that there is a set Z such that $(x, y, Z) \in \mathcal{F}_k(G)$ where $k = \text{fan}(G) - 1$. In particular, this implies that $d_G(x) + d_G(y) - \mu_G(x, y) \geq k + 1$. Consequently, at least k edges of G are adjacent to e. Since this holds for every edge $e \in E(G)$, this implies that $\text{col}'(G)$ is greater than k and, therefore, $\text{col}'(G) \geq k + 1 = \text{fan}(G)$. ■

Theorem 2.10 (Scheide and Stiebitz [32]) *Every parameter* $\rho \in \{\Delta+\mu, \frac{3}{2}\Delta, \Delta_\mu\}$ *is an upper bound for the fan number and hence an efficiently realizable upper bound for* χ'.

Proof: Let $\rho \in \{\Delta + \mu, \frac{3}{2}\Delta, \Delta_\mu\}$. The aim is to show that every graph G satisfies $\text{fan}(G) \leq \rho(G)$. Since both parameters are monotone, it follows from Proposition 1.1(b) that it is sufficient to prove this inequality for all fan-critical graphs G. If $\text{fan}(G) = 0$, this is evident. Otherwise, G is a fan-critical graph with $\text{fan}(G) \geq 1$. Then G has an edge, and we easily infer from Theorem 2.8 that there are three vertices $x, z_1, z_2 \in V(G)$ satisfying $\text{fan}(G) \leq d_G(z_1)+\mu_G(x, z_1) \leq \Delta_\mu(G) \leq \Delta(G)+\mu(G)$ and $2\text{fan}(G) \leq d_G(x) + d_G(z_1) + d_G(z_2) \leq 3\Delta(G)$. Consequently, we have $\text{fan}(G) \leq \rho(G)$. Since $\rho(G) \geq \Delta(G)$ for every graph G, it then follows from Theorem 2.7 that ρ is an efficiently realizable upper bound for the chromatic index χ'. ■

For an arbitrary graph G, we define

$$\boldsymbol{\mu^-(G)} = \min_{v \in V(G)} \mu(G - v)$$

if G has at least one vertex, and $\mu^-(G) = 0$ otherwise. Favrholdt, Stiebitz and Toft [8] proved that $\Delta + \mu^-$ is an upper bound for χ', thus generalizing an earlier result of Chetwynd and Hilton [3] about **almost simple graphs**, that are graphs G for which there exists a vertex v such that the graph $G - v$ is simple.

Theorem 2.11 (Scheide and Stiebitz [32]) $\Delta + \mu^-$ *is an upper bound for the fan number and hence an efficiently realizable upper bound for* χ'.

Proof: Clearly, $\Delta + \mu^-$ is a monotone graph parameter and, by Proposition 1.1(b), it is sufficient to show that $\text{fan}(G) \leq \Delta(G) + \mu^-(G)$ for every fan-critical graph G. If $\text{fan}(G) = 0$, this is evident.

Now assume that $\text{fan}(G) = k + 1$ with $k \geq 0$. Then $|V(G)| \geq 3$, and there is a vertex $v \in V(G)$ such that $\mu^-(G) = \mu(G - v)$. We have to show that $\text{fan}(G) \leq \Delta(G) + \mu^-(G) = \Delta(G) + \mu(G - v)$. Suppose on the contrary that $\text{fan}(G) \geq \Delta(G) + \mu(G - v) + 1$. Let $X = V(G) \setminus \{v\}$, and let $s = |X|$. Evidently, we have $s \geq 2$. By

Theorem 2.8(c), every vertex $x \in X$ has two distinct neighbours in G, denoted by $z_1 = z_1(x)$ and $z_2 = z_2(x)$, such that

$$d_G(z_1) + \mu_G(x, z_1) + d_G(z_2) + \mu_G(x, z_2) \geq 2\text{fan}(G) \geq 2\Delta(G) + 2\mu(G - v) + 2.$$

We may assume that $z_2 \neq v$, hence we have $\mu_G(x, z_1) \geq \mu(G - v) + 2$, implying that $z_1 = v$ and $\mu_G(x, v) \geq \mu(G - v) + 2$. From the above inequality it then follows that

$$\mu_G(x, v) + d_G(z_2) \geq \Delta(G) + \mu(G - v) + 2 \geq d_G(v) + \mu(G - v) + 2.$$

Since $s = |X| = |V(G)| - 1$, this implies

$$\begin{aligned}\mu_G(x, v) + \mu_G(z_2, v) + (s - 1)\mu(G - v) &\geq \mu_G(x, v) + d_G(z_2) \\ &\geq d_G(v) + \mu(G - v) + 2 \\ &= \left(\sum_{y \in X} \mu_G(y, v)\right) + \mu(G - v) + 2.\end{aligned}$$

Since $s \geq 2$ and $\mu_G(y, v) \geq \mu(G - v) + 2$ for all $y \in X$, we then obtain

$$\begin{aligned}(s - 1)\mu(G - v) &\geq \left(\sum_{y \in X - \{x, z_2\}} \mu_G(y, v)\right) + \mu(G - v) + 2 \\ &\geq (s - 2)(\mu(G - v) + 2) + \mu(G - v) + 2 \\ &\geq (s - 1)\mu(G - v) + 2.\end{aligned}$$

This, however, gives the desired contradiction. ■

Eventually, let us mention two other upper bounds for the chromatic index known in the literature. The first parameter was independently introduced by Andersen [1] and Goldberg [10], namely

$$\boldsymbol{\mathrm{ag}(G)} = \max\{\Delta(G), \max_{\mathcal{P}} \lfloor \tfrac{1}{2}(d_G(y) + \mu_G(x, y) + d_G(z) + \mu_G(x, z)) \rfloor\}$$

where $\mathcal{P} = \{(x, y, z) \mid x, y, z \in V(G),\ z \neq y,\ E_G(x, y) \neq \emptyset,\ E_G(x, z) \neq \emptyset\}$.

Another monotone parameter, the so-called supermultiplicity, was introduced by Kochol, Krivoňáková and Smejová [20]. For a graph G and two distinct vertices $x, y \in V(G)$, let

$$\boldsymbol{\mathrm{sm}_G(x, y)} = \min\{d_G(y) + \mu_G(x, y), d_G(x) + d_G(y) - \mu_G(x, y)\}.$$

Let $k \geq \Delta(G)$ be an integer. We call x a ***k*-reducible vertex** of G if every neighbour y of x satisfies $\mathrm{sm}_G(x, y) \leq k$. Then the **supermultiplicity sm(G)** is the smallest integer $k \geq \Delta(G)$ for which there exists a labeling $x_1, \ldots, x_n$ of the vertices of G such each x_i is a k-reducible vertex of the graph $G - \{x_1, \ldots, x_{i-1}\}$. If G is edgeless, then $\mathrm{sm}(G) = 0$. It is not difficult to show that sm is a monotone graph parameter.

Theorem 2.12 (Scheide and Stiebitz [32]) *Both graph parameters* ag *and* sm *are upper bounds for the fan number and hence efficiently realizable upper bounds for* χ'.

Proof: From Theorem 2.8(b) we conclude that every fan-critical graph G satisfies $\text{fan}(G) \le \text{ag}(G)$. Since both parameters are monotone, Proposition 1.1(b) then implies that ag is an upper bound for the fan number and hence an efficiently realizable upper bounds for χ'.

Now assume that G is a fan-critical graph. If $\text{fan}(G) = 0$ then $\text{fan}(G) \le \text{sm}(G)$. Otherwise, $\text{fan}(G) \ge 1$ and we conclude from Theorem 2.8(a),(b) that each vertex x has a neighbour z such that $d_G(z) + \mu_G(x, z) \ge \text{fan}(G)$ and $d_G(x) + d_G(z) - \mu_G(x, z) \ge \text{fan}(G)$, implying that $\text{sm}_G(x, z) \ge \text{fan}(G)$. Consequently, we obtain $\text{fan}(G) \le \text{sm}(G)$. By Proposition 1.1(b) this implies that sm is an upper bound for the fan number and hence an efficiently realizable upper bounds for χ'. ■

The fan number plays a similar role for the chromatic index as the colouring number does for the chromatic number. Several results concerning the chromatic number were originally proved using the colouring number. The most famous example is Heawood's bound for the chromatic number of graphs embedded on a surface. As long as a result about the chromatic index can be proved using just Vizing's fan argument, and hence the fan equation, the result has a natural counterpart for the fan number.

For simple graphs, Vizing's bound implies that there are only two types of graphs, **class one graphs** where χ' equals Δ, and **class two graphs** where χ' equals $\Delta + 1$. Clearly, since Vizing's bound is also an upper bound of the fan number, we also have only two types of simple graphs with respect to the parameter Fan. In [40] Vizing used the fan argument to establish conditions for a simple graph G to be class one. These results can be generalized for the fan number. To do this we first prove the following adjacency lemma, which is a counterpart to Vizing's famous adjacency lemma for critical, simple graphs, see [40].

Lemma 2.13 *Let G be a* fan*-critical, simple graph with* $\text{fan}(G) = \Delta(G) + 1$*, and let $e \in E_G(x, y)$ be an edge of G. Then x is adjacent to at least $\Delta(G) - d_G(y) + 1$ vertices $z \neq y$ such that $d_G(z) = \Delta(G)$.*

Proof: Since G is fan-critical and $\text{fan}(G) = \Delta(G)+1$, it follows from Theorem 2.8(a) that there is a set $Z \subseteq N_G(x)$ such that $(x, y, Z) \in \mathcal{F}_{\Delta(G)}(G)$, that is, $|Z| \ge 2$, $y \in Z$, and $\sum_{z \in Z}(d_G(z) + 1 - \Delta(G)) \ge 2$. Consequently, we have

$$\sum_{z \in Z \setminus \{y\}} (d_G(z) + 1 - \Delta(G)) \ge \Delta(G) - d_G(y) + 1.$$

Let $Z' = \{z \in Z \setminus \{y\} \mid d_G(z) = \Delta(G)\}$. Then, evidently, we have

$$\sum_{z \in Z \setminus \{y\}} (d_G(z) + 1 - \Delta(G)) \le \sum_{z \in Z'} (d_G(z) + 1 - \Delta(G)) = |Z'|.$$

Hence $|Z'| \ge \Delta(G) - d_G(y) + 1$, which completes the proof. ■

Using this lemma, we can prove two results for the fan number, similar to results for the chromatic index in [40]. The first result is a relation between the fan number and the colouring number col. To prove this, we will use the fact that the colouring number equals the **Szekeres-Wilf number**, introduced by Szekeres and Wilf [37] in 1968. This means that every simple graph G satisfies

$$\mathrm{col}(G) = 1 + \max_{H \subseteq G} \delta(H).$$

The second result is about planar, simple graphs with maximum degree at least 8. The proofs of both results are nearly the same as of the corresponding results for the chromatic index in [40].

Theorem 2.14 *Let G be a simple graph. If $\Delta(G) \geq 2\mathrm{col}(G) - 2$ then $\mathrm{fan}(G) \leq \Delta(G)$.*

Proof: Suppose on the contrary that there is a simple graph G with $\Delta(G) \geq 2\mathrm{col}(G) - 2$ and $\mathrm{fan}(G) > \Delta(G)$. Since, by Theorem 2.10, we have $\mathrm{fan}(G) \leq \Delta(G)+1$, this implies $\mathrm{fan}(G) = \Delta(G) + 1$. Clearly, there is a fan-critical subgraph G' of G with $\mathrm{fan}(G') = \mathrm{fan}(G)$. Since col is a monotone graph parameter, this subgraph G' satisfies $\Delta(G') \geq \mathrm{fan}(G') - 1 = \Delta(G) \geq 2\mathrm{col}(G) - 2 \geq 2\mathrm{col}(G') - 2$. Hence, without loss of generality, we can assume that G itself is fan-critical.

Let $d = \mathrm{col}(G)$. Since $\mathrm{fan}(G) > 0$, the graph G is not edgeless, and we have $d \geq \delta(G)+1 \geq 2$. Hence $\Delta(G) \geq 2d-1 > d-1$. Let $Y = \{y \in V(G) \mid d_G(y) \leq d-1\}$. Since $\delta(G) \leq \mathrm{col}(G) - 1 = d - 1$ and $\Delta(G) > d - 1$, both sets Y and $V(G) \setminus Y$ are non-empty. Furthermore, the subgraph $H = G - Y$ satisfies $\delta(H) + 1 \leq d$ and, therefore, H contains a vertex x with $d_H(x) \leq d - 1$. Since $x \notin Y$, we have $d_G(x) \geq d$. Consequently, there is a vertex $y \in N_G(x) \cap Y$. Now let $Z = \{z \in N_G(x) \mid d_G(z) = \Delta(G)\}$. Then, by Lemma 2.13, we have $|Z| \geq \Delta(G) - d_G(y) + 1 \geq (2d-2) - (d-1) + 1 = d$. Since $\Delta(G) > d-1$, we also have $Z' \subseteq V(G) \setminus Y = V(H)$ and, therefore, $|Z'| \leq d_H(x) \leq d - 1$. This contradiction proves the claim. ■

In 1890 Heawood [12] established an upper bound for the chromatic number of a graph embedded on a surface of Euler genus $g \geq 1$. This upper bound became known as the **Heawood number**

$$\mathbf{H}(g) = \left\lfloor \tfrac{7+\sqrt{24g+1}}{2} \right\rfloor$$

and is in fact an upper bound for the colouring number of a simple graph embedded on a surface of Euler genus $g \geq 1$. Consequently, Theorem 2.14 implies that if a simple graph G can be embedded on a surface of Euler genus $g \geq 1$ such that $\Delta(G) \geq 2\mathrm{H}(g) - 2$, then $\mathrm{fan}(G) \leq \Delta(G)$. For a simple planar graph, the colouring number is at most 6. Hence, every simple planar graph G with $\Delta(G) \geq 10$ implies that $\mathrm{fan}(G) \leq \Delta(G)$. The next result shows that 10 is not the optimum.

Theorem 2.15 *Let G be a planar, simple graph. If $\Delta(G) \geq 8$ then $\mathrm{fan}(G) \leq \Delta(G)$.*

Proof: Suppose on the contrary that there is a planar, simple graph G with $\Delta(G) = \Delta \geq 8$ but $\mathrm{fan}(G) > \Delta$. Since, by Theorem 2.10, we have $\mathrm{fan}(G) \leq \Delta(G) + 1$, this implies $\mathrm{fan}(G) = \Delta(G) + 1$. Clearly, there is a fan-critical subgraph G' of G with $\mathrm{fan}(G') = \mathrm{fan}(G)$. Then this subgraph G' satisfies $\Delta(G') \geq \mathrm{fan}(G') - 1 = \Delta(G) \geq 8$. Hence, without loss of generality, we may assume that G itself is fan-critical.

Let $n = |V(G)|$ and $m = |E(G)|$. Since we have $n \geq 3$, the planarity of G implies $m \leq 3n-6$. For $i \in \mathbb{N}$, let n_i be the number of vertices x of G with degree $d_G(x) = i$. Since G is fan-critical, we have $n_0 = n_1 = 0$. Then, evidently,

$$\sum_{i=2}^{\Delta} i \cdot n_i = \sum_{x \in V(G)} d_G(x) = 2m \leq 6n - 12 = 6 \cdot \sum_{i=2}^{\Delta} n_i - 12$$

and, therefore,

$$\sum_{i=7}^{\Delta} (i-6)n_i \leq n_5 + 2n_4 + 3n_3 + 4n_2 - 12. \tag{2.6}$$

Let $n_\Delta(i_2, i_3, i_4, i_5)$ be the number of vertices x of G with degree $d_G(x) = \Delta$ such that x has i_d neighbours z of degree $d_G(z) = d$, respectively for $d = 2, 3, 4, 5$. If $i_d > 0$ for such a vertex x and a $d \leq 5$ then, by Lemma 2.13, x has at least $\Delta + 1 - d$ neighbours of degree Δ and, therefore, we have $i_2 + i_3 + i_4 + i_5 \leq d - 1$. Hence, there are only 16 types of vertices of degree Δ where $n_\Delta(i_2, i_3, i_4, i_5)$ may be positive.

$$\begin{aligned} n_\Delta = {} & n_\Delta(1,0,0,0) + n_\Delta(0,1,0,0) + n_\Delta(0,1,1,0) + n_\Delta(0,1,0,1) \\ & + n_\Delta(0,2,0,0) + n_\Delta(0,0,1,0) + n_\Delta(0,0,1,1) + n_\Delta(0,0,1,2) \\ & + n_\Delta(0,0,2,0) + n_\Delta(0,0,2,1) + n_\Delta(0,0,3,0) + n_\Delta(0,0,0,1) \\ & + n_\Delta(0,0,0,2) + n_\Delta(0,0,0,3) + n_\Delta(0,0,0,4) + n_\Delta(0,0,0,0). \end{aligned} \tag{2.7}$$

Similarly, let $n_{\Delta-1}(i_3, i_4, i_5)$ be the number of vertices x of G with degree $d_G(x) = \Delta - 1$ such that x has i_d neighbours z of degree $d_G(z) = d$, respectively for $d = 3, 4, 5$. Note that, by Lemma 2.13, a vertex of degree $\Delta - 1$ cannot have a neighbour of degree 2. If $i_d > 0$ for such a vertex x and a $d \leq 5$ then, by Lemma 2.13, x has at least $\Delta + 1 - d$ neighbours of degree Δ and, therefore, we have $i_3 + i_4 + i_5 \leq d - 2$. Hence, there are only 9 types of vertices of degree $\Delta - 1$ where $n_{\Delta-1}(i_3, i_4, i_5)$ may be positive.

$$\begin{aligned} n_{\Delta-1} = {} & n_{\Delta-1}(1,0,0) + n_{\Delta-1}(0,1,0) + n_{\Delta-1}(0,1,1) \\ & + n_{\Delta-1}(0,2,0) + n_{\Delta-1}(0,0,1) + n_{\Delta-1}(0,0,1) \\ & + n_{\Delta-1}(0,0,2) + n_{\Delta-1}(0,0,3) + n_{\Delta-1}(0,0,0). \end{aligned} \tag{2.8}$$

Now we want to derive a contradiction to (2.6). To do this we will count edges between vertices of given degree in two different ways, using the given partitions of n_Δ and $n_{\Delta-1}$.

First we count the edges between vertices of degree 2 and Δ. From (2.7) it follows that there are exactly $n_\Delta(1,0,0,0)$ edges of this type. Moreover, by Lemma 2.13, every vertex with degree 2 has two neighbours with degree Δ. Hence, we have

$$2n_2 = n_\Delta(1,0,0,0). \tag{2.9}$$

Now we count the edges joining a vertex of degree 3 and one of degree $\Delta - 1$ or Δ. By Lemma 2.13, every vertex with degree 3 has only neighbours with degrees Δ or $\Delta - 1$. Hence, we have $3n_3$ edges of this type. Then (2.7) and (2.8) imply that

$$\begin{aligned} 3n_3 = {} & n_\Delta(0,1,0,0) + n_\Delta(0,1,1,0) + n_\Delta(0,1,0,1) \\ & + 2n_\Delta(0,2,0,0) + n_{\Delta-1}(1,0,0). \end{aligned} \tag{2.10}$$

By Lemma 2.13, every vertex with degree 5 has at least two neighbours with degree Δ. Hence, there are at least $2n_5$ edges joining a vertex of degree 5 and one of degree Δ or $\Delta - 1$. Then, by (2.7) and (2.8), we obtain

$$\begin{aligned} 2n_5 \le\ & n_\Delta(0,1,0,1) + n_\Delta(0,0,1,1) + 2n_\Delta(0,0,1,2) \\ & + n_\Delta(0,0,2,1) + n_\Delta(0,0,0,1) + 2n_\Delta(0,0,0,2) \\ & + 3n_\Delta(0,0,0,3) + 4n_\Delta(0,0,0,4). \end{aligned} \tag{2.11}$$

By Lemma 2.13, every vertex with degree 4 either has one neighbour with degree $\Delta - 2$ and 3 neighbours with degree Δ, or has 4 neighbours with degrees Δ or $\Delta - 1$. Let n_4' be the number of vertices of G with degree 4 having a neighbour of degree $\Delta - 2$. Then there are $3n_4' + 4(n_4 - n_4')$ edges joining a vertex of degree 4 and one of degree Δ or $\Delta - 1$, and from (2.7) and (2.8) it follows that

$$\begin{aligned} 3n_4' + 4(n_4 - n_4') =\ & n_\Delta(0,1,1,0) + n_\Delta(0,0,1,0) + n_\Delta(0,0,1,1) \\ & + n_\Delta(0,0,1,2) + 2n_\Delta(0,0,2,0) + 2n_\Delta(0,0,2,1) \\ & + 3n_\Delta(0,0,3,0) + n_{\Delta-1}(0,1,0) + n_{\Delta-1}(0,1,1) \\ & + 2n_{\Delta-1}(0,2,0). \end{aligned} \tag{2.12}$$

Furthermore, by Lemma 2.13, every vertex with degree 4 has at least two neighbours with degree Δ. Consequently, in the case, where a vertex with degree 4 has only neighbours with degrees Δ or $\Delta - 1$, it has at most two neighbours with degree $\Delta - 1$. Hence, there are at most $2(n_4 - n_4')$ edges between vertices of degrees 4 and $\Delta - 1$. Since there are at least $2n_{\Delta-1}(0,2,0)$ edges of this type, we obtain $n_4 - n_4' \ge n_{\Delta-1}(0,2,0)$. From this and (2.12) it then follows that

$$\begin{aligned} 3n_4 \le\ & n_\Delta(0,1,1,0) + n_\Delta(0,0,1,0) + n_\Delta(0,0,1,1) \\ & + n_\Delta(0,0,1,2) + 2n_\Delta(0,0,2,0) + 2n_\Delta(0,0,2,1) \\ & + 3n_\Delta(0,0,3,0) + n_{\Delta-1}(0,1,0) + n_{\Delta-1}(0,1,1) \\ & + n_{\Delta-1}(0,2,0). \end{aligned} \tag{2.13}$$

From the four inequalities (2.9),(2.10),(2.11) and (2.13) we now conclude that

$$\begin{aligned} & n_5 + 2n_4 + 3n_3 + 4n_2 \\ & \quad \le 2n_\Delta(1,0,0,0) + n_\Delta(0,1,0,0) + \tfrac{5}{3}n_\Delta(0,1,1,0) + \tfrac{2}{3}n_\Delta(0,1,0,1) \\ & \qquad + 2n_\Delta(0,2,0,0) + \tfrac{2}{3}n_\Delta(0,0,1,0) + \tfrac{7}{6}n_\Delta(0,0,1,1) + \tfrac{5}{8}n_\Delta(0,0,1,2) \\ & \qquad + \tfrac{4}{3}n_\Delta(0,0,2,0) + \tfrac{11}{6}n_\Delta(0,0,2,1) + 2n_\Delta(0,0,3,0) + \tfrac{1}{2}n_\Delta(0,0,0,1) \\ & \qquad + n_\Delta(0,0,0,2) + \tfrac{3}{2}n_\Delta(0,0,0,3) + 2n_\Delta(0,0,0,4) \\ & \qquad + n_{\Delta-1}(1,0,0) + \tfrac{2}{3}n_{\Delta-1}(0,1,0) + \tfrac{2}{3}n_{\Delta-1}(0,1,1) + \tfrac{2}{3}n_{\Delta-1}(0,2,0). \end{aligned}$$

By (2.7) and (2.8), this implies $n_5 + 2n_4 + 3n_3 + 4n_2 \le 2n_\Delta + n_{\Delta-1}$. Consequently, since $\Delta \ge 8$, we have

$$n_5 + 2n_4 + 3n_3 + 4n_2 \le 2n_\Delta + n_{\Delta-1} \le \sum_{i=7}^{\Delta}(i-6)n_i,$$

a contradiction to (2.6). This completes the prove. ■

That a simple planar graph of maximum degree 7 is of class one, was recently proved by Zhang [41] and, independently, by Sanders and Zhao [26]. However, both proofs use adjacency lemmas that cannot be derived from the fan equation. We do not know whether every simple planar graph G with maximum degree 7 satisfies $\text{fan}(G) \leq \Delta(G)$.

2.5 The Kierstead Path

Kierstead [19] invented a new type of test objects for the edge colouring problem. He used it to give a strengthening of Vizing's result. Kierstead's method can also be used to give an alternative colouring algorithm. This algorithm is based on recolouring the edges of a path instead of recolouring the edges of a fan.

Definition 2.16 *Let G be a graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer k. A* **Kierstead path** *with respect to e and φ is a sequence $(y_0, e_1, y_1, \ldots, e_p, y_p)$ consisting of edges $e_1, \ldots, e_p$ and vertices $y_0, \ldots, y_p$ satisfying the following two conditions:*

(K1) *The vertices $y_0, \ldots, y_p$ are distinct, $e_1 = e$, and $e_i \in E_G(y_{i-1}, y_i)$ for $i = 1, \ldots, p$.*

(K2) *For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $0 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.*

Kierstead [19] proved 1984 that for every graph G with $\chi'(G) = k+1$, the vertex set of any Kierstead path with respect to a critical edge $e \in E(G)$ and a colouring $\varphi \in \mathcal{C}_k(G-e)$ is elementary with respect to φ if $k \geq \Delta(G)+1$. That Kierstead's argument also works in case of $k = \Delta(G)$ provided we add a degree condition seems to be first noticed by Zhang [41]. In the following we formulate an algorithmic version of Kierstead's result.

If $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ is a Kierstead path, then (K1) implies that the corresponding graph $(V(K), E(K))$ is indeed a path in G with endvertices y_0 and y_p. We call K a **maximal Kierstead path** with respect to e and φ if there is no edge-vertex pair (f, v) such that (K, f, v) is a Kierstead path with respect to e and φ. Clearly, Ky_q remains a Kierstead path whenever $1 \leq q \leq p$. Furthermore, if $V(K)$ is not elementary with respect to φ, then there exists an integer $1 \leq q \leq p$ such that $V(Ky_q)$ is not elementary, but $V(Ky_{q-1})$ is elementary with respect to φ.

Theorem 2.17 *Let G be a graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)$. Furthermore, let $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ be a Kierstead path with respect to e and φ such that $d_G(y_j) < k$ for all $j = 2, \ldots, p$. If $V(K)$ is not elementary but $V(Ky_{p-1})$ is elementary with respect to φ, then a colouring $\varphi' \in \mathcal{C}_k(G)$ can be derived from φ by at most $\frac{p(p+1)}{2} - 1$ Kempe changes plus colouring the edge e.*

Proof: The proof will be based on induction with respect to p. If $p = 1$ then K consists only of the edge $e_1 = e$ and the two endvertices y_0 and y_1. Clearly,

$V(Ky_0) = \{y_0\}$ is elementary with respect to φ. Since $V(K)$ is not elementary with respect to φ, there is a colour $\alpha \in \bar{\varphi}(y_0) \cap \bar{\varphi}(y_1)$. Hence we can get φ' simply by colouring e with α.

Now let $p > 1$. Since $V(K)$ is not elementary, but $V(Ky_{p-1})$ is elementary with respect to φ, there is a maximal index $i \in \{0, \ldots, p-1\}$ such that $\bar{\varphi}(y_i) \cap \bar{\varphi}(y_p) \neq \emptyset$. Let $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$. Now we claim that there is a colouring $\varphi'' \in \mathcal{C}_k(G - e)$ and an index $r < p$ such that Ky_r is a Kierstead path with respect to e and φ'', $V(Ky_r)$ is not elementary, but $V(Ky_{r-1})$ is elementary with respect to φ''. Moreover, the colouring φ'' can be derived from φ by at most $p - i$ Kempe changes.

If this claim is proved then, by induction, we can derive the desired colouring φ' from φ'' by at most $\frac{r(r+1)}{2} - 1$ Kempe changes plus colouring the edge e. From $r \leq p - 1$ and $p - i \leq p$ we then conclude that in total we need at most $\frac{p(p+1)}{2} - 1$ Kempe changes plus the colouring of e to derive φ' from φ.

To complete the proof we only have to prove the claim. To do this we use again an induction, this time with respect to i and secondarily to the induction with respect to p.

If $i = p - 1$ then let $\beta = \varphi(e_p)$. By definition, there is an index $j < p$ with $\beta \in \bar{\varphi}(y_j)$. Since $e_p \in E_G(y_{p-1}, y_p)$, we have $j < p - 1$. Since $\alpha \in \bar{\varphi}(y_{p-1}) \cap \bar{\varphi}(y_p)$, we can recolour e_p with α, resulting in a colouring $\varphi'' \in \mathcal{C}_k(G - e)$ such that Ky_{p-1} is a Kierstead path with respect to e and φ'' satisfying $\beta \in \bar{\varphi}''(y_j) \cap \bar{\varphi}''(y_{p-1})$. Clearly, since $e_p \in E_G(y_{p-1}, y_p)$, the set $V(Ky_{p-2})$ remains elementary with respect to φ''. We used exactly one Kempe change, namely the recolouring of e_p, because $P_{y_p}(\alpha, \beta, \varphi) = P(y_p, e_p, y_{p-1})$. This settles the case.

Now let $i < p - 1$. Since the edge e_1 is uncoloured with respect to φ, we have $|\bar{\varphi}(y_j)| = k - d_G(y_j) + 1$ for $j = 1, 2$ and $|\bar{\varphi}(y_j)| = k - d_G(y_j)$ otherwise. From $d_G(y_j) < k$ for $j = 2, \ldots, p$ and $k \geq \Delta(G)$ we conclude that $\bar{\varphi}(y_j) \neq \emptyset$ for $j = 0, \ldots, p$. In particular, this implies that there is a colour $\beta \in \bar{\varphi}(y_{i+1})$. Since $y_{i+1} \in V(Ky_{p-1})$ and $V(Ky_{p-1})$ is elementary with respect to φ, the colours α and β are distinct and, moreover, $\alpha \in \varphi(y_{i+1})$, $\beta \in \varphi(y_i)$, and $\alpha, \beta \in \varphi(y_h)$ for $h = 0, \ldots, i - 1$. Consequently, by definition, we have $\varphi(e_h) \notin \{\alpha, \beta\}$ for $h = 2, \ldots, i + 1$ and, moreover, the (α, β)-chain $P = P_{y_{i+1}}(\alpha, \beta, \varphi)$ is a path with one endvertex y_{i+1} and another endvertex $z \in V(G) \backslash \{y_0, \ldots, y_{i-1}, y_{i+1}\}$. Hence $\varphi_1 = \varphi/P \in \mathcal{C}_k(G - e)$, and we distinguish two cases.

If $z = y_i$, then K is a Kierstead path with respect to e and φ_1, where $\alpha \in \bar{\varphi}_1(y_{i+1}) \cap \bar{\varphi}_1(y_p)$. Clearly, the set $V(Ky_{p-1})$ remains elementary with respect to φ_1. Hence, by induction, we can derive the colouring φ'' from φ_1 by at most $p - i - 1$ Kempe changes. Therefore, to derive φ'' from φ we need $p - i$ Kempe changes in total. This settles the case $z = y_i$.

If $z \neq y_i$ then Ky_{i+1} is a Kierstead path with respect to e and φ_1, where $\alpha \in \bar{\varphi}_1(y_i) \cap \bar{\varphi}_1(y_{i+1})$. Since the set $V(Ky_i)$ remains elementary with respect to φ_1, then we already have our desired colouring $\varphi'' = \varphi_1$. There was just one Kempe change needed to derive $\varphi'' = \varphi_1$ from φ. Since $1 \leq p - i$, this settles the case $z \neq y_i$. This completes the induction and proves the claim. ■

This result gives us a way to construct an alternative kernel **KierExt** for our colouring algorithm.

<u>**KierExt**(G, e, x, y, k, φ):</u>

1) $p \leftarrow 1$, $e_p \leftarrow e$, $y_p \leftarrow y$, $y_0 \leftarrow x$, $K \leftarrow (y_0, e_p, y_p)$.

2) `If` $\bar{\varphi}(V(Ky_{p-1})) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  2a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.17.`

  2b) `Return` (k, φ').

3) `If` $\exists e_{p+1} \in E_G(y_p, V(G) \setminus V(K)) : \varphi(e_{p+1}) \in \bar{\varphi}(V(K))$ `then`

  3a) `Let` y_{p+1} `be the endvertex of` e_{p+1} `that is not` y_p.

  3b) $K \leftarrow (K, e_{p+1}, y_{p+1})$, $p \leftarrow p + 1$.

  3c) `Goto 2.`

4) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k + 1$.

5) `Return` $(k + 1, \varphi')$.

Correctness of KierExt: Let G be a graph, let $e \in E_G(x, y)$, and let $\varphi \in \mathcal{C}_k(G - e)$ for an integer $k \geq \Delta(G) + 1$. Then, on the input (G, e, x, y, k, φ), the algorithm KierExt subsequently builds a Kierstead path $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with respect to e and φ until either $V(K)$ is not elementary with respect to φ, or K is a maximal Kierstead path with respect to e and φ. In the first case, $V(Ky_{p-1})$ is elementary with respect to φ, otherwise the algorithm had detected it before. Further, $k \geq \Delta(G) + 1$ implies that $d(y) < k$ for all $y \in V(K)$. Hence, the requirements of Theorem 2.17 are fulfilled, and KierExt can compute the colouring $\varphi' \in \mathcal{C}_k(G)$. Consequently, KierExt works correctly in this case. In the other case, clearly, $V(K)$ is maximal and elementary with respect to φ, and KierExt simply assigns a new colour to e. In any case, KierExt computes a colouring $\varphi' \in \mathcal{C}_{k'}(G)$ where $k' \in \{k, k + 1\}$.

Note that, in order to fulfill the requirements of Theorem 2.17, we had to require $k \geq \Delta(G) + 1$. So from now on require this for an input (G, e, x, y, k, φ) to be valid.

Time Complexity of KierExt: Now we have to analyse the running time of the algorithm KierExt on a valid input, that is, an input (G, e, x, y, k, φ) with $|V(G)| = n$ and $\Delta(G) = \Delta$. The first initializing step only needs constant time. Due to the goto-statement in step 3c there is a loop starting in step 2. In every round the value of p is increased, the number of rounds is bounded by the order of a maximal Kierstead path with respect to e and φ. Since the vertices of a Kierstead path are distinct, we have $p \leq n$. Moreover, in the loop $V(Ky_{p-1})$ is always an elementary set with respect to φ and, since $k \geq \Delta(G) + 1$, we also have $p \leq |\bar{\varphi}(V(Ky_{p-1}))| \leq k$. Hence, the loop is repeated at most $\min\{n, k\}$ times.

In every sweep of the loop the algorithm has to check the condition of step 2. This can be done in time $O(\Delta)$ by maintaining, for every colour α, an entry $\mathrm{IND}[\alpha]$ that is the index i of the vertex y_i where α is missing. During step 2 it can easily be updated. Moreover, during this step we can identify the maximal index i such that there is a colour $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$ which is needed for the following recolouring procedure, see the proof of Theorem 2.17. The recolouring step 2a does not count to the loop, because the algorithm will terminate right after this step. The next step inside the loop is step 3. This check can be done in time $O(\Delta)$ by simply scanning the adjacency list of y_p and using the IND-entries. Note, that to decide whether a vertex belongs to $V(K)$ or not, we simply mark every vertex when adding it to $V(K)$. Updating K clearly needs constant time, then the loop starts over again. Consequently, every round in the loop needs time $\mathrm{O}(\Delta)$. For the whole loop, this gives a running time of $\mathrm{O}(n\Delta)$ or $\mathrm{O}(k\Delta)$, respectively.

The step 2a is computed at most once during the algorithm. During the check in step 2, the necessary index and colour (which are needed to start the recolouring procedure) are already computed, and the time costs for this have already been taken into account. Hence, the running time of step 2a contains only the real recolouring costs, which are determined by the number of the performed Kempe changes, see Section 2.2. By Theorem 2.17, at most $\mathrm{O}(p^2)$ Kempe changes are needed and, as already stated, we have $p \leq n$ as well as $p \leq k$. Consequently, step 2a has a running time of $\mathrm{O}(n^2(n+\Delta))$ or $\mathrm{O}(k^2(n+\Delta))$, respectively.

If the algorithm uses a new colour in step 4, this needs only constant time. Since we can assume $k \in \mathrm{O}(\Delta)$, see Section 2.2, KierExt has a total running time of $\mathrm{O}\big(n^2(n+\Delta)\big)$ or $\mathrm{O}\big(\Delta^2(n+\Delta)\big)$, respectively. Since we may have $n \in \mathrm{o}(\Delta)$ or $\Delta \in \mathrm{o}(n)$, neither of these two bounds is generally better than the other one. Hence, we have a running time of $\mathrm{O}\big((n+\Delta)\min\{n^2,\Delta^2\}\big)$ for the algorithm KierExt.

Kierstead's colouring algorithm: For the Kierstead path, there is a inequality similar to the fan equation. To see this, let G be a graph, let $e \in E_G(x,y)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. If KierExt increases the number of colours for the input (G,e,x,y,k,φ), then there is a maximal Kierstead path $K = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with respect to e and φ such that $V(K)$ is elementary with respect to φ. Hence, we have $p \geq 2$ and, for the vertex set $X = V(Ky_{p-1})$ we have $\bar{\varphi}(X) \subseteq \varphi(y_p)$. From this and the fact that K is a maximal Kierstead path with respect to e and φ, we infer that for every colour $\alpha \in \bar{\varphi}(X)$ there is an edge $f \in E_G(y_p, X)$ with $\bar{\varphi}(f) = \alpha$. This implies $|\bar{\varphi}(X)| \leq |E_G(y_p, X)| = \sum_{z\in X} \mu(z, y_p)$. On the other hand, since $x, y \in X$ and X is elementary with respect to φ, we have $|\bar{\varphi}(X)| \geq \sum_{z\in X}(k - d_{G-e}(z)) = \sum_{z\in X}(k - d_G(z)) + 2$. Consequently, this gives the inequality

$$\sum_{z\in X}(d_G(z) + \mu_G(z, y_p) - k) \geq 2. \tag{2.14}$$

Let **Kierstead** be a colouring algorithm that uses an arbitrary edge order, KierExt as kernel, and that starts with $k = \Delta(G)+1$ for any input graph with $\Delta(G) \geq 2$. Note that in the case $\Delta(G) \leq 1$ the algorithm can simply colour the graph optimal with $\Delta(G)$ colours. Clearly, Kierstead only increases the number of colours if $\Delta(G) \geq 2$ and (2.14) holds for a subgraph of G. From (2.14) then follows that Kierstead attains Vizing's and Shannon's bound. This can be seen as follows. If G is a graph and $k \geq \Delta(G) + \mu(G)$ then (2.14) fails for G as well as for any subgraph of G. Since Kierstead starts with $k = \Delta(G)+1 \leq \Delta(G)+\mu(G)$, it follows that the value of k is never increased above $\Delta(G)+\mu(G)$ by the subroutine KierExt. Consequently, Kierstead uses not more than $\Delta(G)+\mu(G)$ colours, which proves Vizing's bound. Shannon's bound can be proved the same way, because for $\Delta(G) \geq 2$ the algorithm starts with $k \leq \frac{3}{2}\Delta(G)$, and (2.14) also fails for every $k > \frac{3}{2}\Delta(G) - 1$. This can be seen as follows. Using $\sum_{z\in X} \mu_G(z, y_p) \leq d_G(y_p)$, (2.14) implies $(p+1)\Delta(G) - kp \geq 2$ which fails for $k > \Delta(G) + \frac{\Delta(G)-2}{p}$. Since $\Delta(G) \geq 2$ and $p \geq 2$, it fails for every $k > \frac{3}{2}\Delta(G) - 1$. Since KIERSTEAD starts with $k \leq \frac{3}{2}\Delta(G)$, it follows that the value of k is never increased above $\frac{3}{2}\Delta(G)$ by the subroutine KIEREXT. Consequently, KIERSTEAD uses not more than $\frac{3}{2}\Delta(G)$ colours, which proves Shannon's bound.

2.6 The Tashkinov Tree

Tashkinov [38] obtained a common generalization, Tashkinov trees, of both the Vizing fans and the Kierstead paths. Tashkinov trees form a very useful type of test objects for the edge colouring problem.

Definition 2.18 *Let G be a graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer k. A* **Tashkinov tree** *with respect to e and φ is a sequence $(y_0, e_1, y_1, \ldots, e_p, y_p)$ consisting of edges $e_1, \ldots, e_p$ and vertices $y_0, \ldots, y_p$ satisfying the following two conditions:*

(T1) *The vertices $y_0, \ldots, y_p$ are distinct, $e_1 = e$ and, for $i = 1, \ldots, p$, we have $e_i \in E_G(y_i, y_j)$ where $0 \leq j < i$.*

(T2) *For every edge e_i with $2 \leq i \leq p$, there is a vertex y_j with $0 \leq j < i$ such that $\varphi(e_i) \in \bar{\varphi}(y_j)$.*

If T is a Tashkinov tree, then condition (T1) implies that the corresponding graph $(V(T), E(T))$ is indeed a tree in G. If $F = (e_1, y_1, \ldots, e_p, y_p)$ is a multi-fan at a vertex x with respect to a colouring $\varphi \in \mathcal{C}_k(G-e)$, then $T = (x, e_1, y_1, \ldots, e_p, y_p)$ is a Tashkinov tree with respect to e and φ, provided that the vertices $y_1, \ldots, y_p$ are distinct. Furthermore, every Kierstead path with respect to e and φ is also a Tashkinov tree with respect to e and φ.

Let $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ be a Tashkinov tree with respect to e and a colouring $\varphi \in \mathcal{C}_k(G-e)$. Clearly, Ty_r with $1 \leq r \leq p$ is a Tashkinov tree with respect to e and φ, too. Furthermore, $y_pT = (y_p)$ is a path in G of length 0. Hence there is a smallest integer $i \in \{0, \ldots, p\}$ such that the sequence $y_iT = (y_i, e_{i+1}, \ldots, e_p, y_p)$ corresponds to a path in G, that is, $e_j \in E_G(y_{j-1}, y_j)$ for $j = i+1, \ldots, p$. We refer to this number i as the **path number** of T and write $p(T) = i$. Clearly, if $p(T) = 0$ then T is a Kierstead path with respect to e and φ.

We call T a **maximal Tashkinov tree** with respect to e and φ if there is no edge-vertex pair (f, z) such (T, f, z) is a Tashkinov tree with respect to e and φ. This means that there is no edge $f \in E_G(V(T), V(G) \setminus V(T))$ with $\varphi(f) \in \bar{\varphi}(V(T))$. Consequently, T is a maximal Tashkinov tree with respect to e and φ iff $V(T)$ is closed with respect to φ. Furthermore, we say that a colour α is **used** on T with respect to φ if $\varphi(f) = \alpha$ for some edge $f \in E(T)$. Otherwise, we say that α is **unused** on T with respect to φ.

In 2000 Tashkinov [38] proved that for every graph G with $\chi'(G) = k+1$, the vertex set of any Tashkinov tree with respect to a critical edge $e \in E(G)$ and a colouring $\varphi \in \mathcal{C}_k(G-e)$ is elementary with respect to φ if $k \geq \Delta(G) + 1$. We will give an algorithmic version of this result, that allows us to construct a colouring algorithm using Tashkinov trees as test objects.

Lemma 2.19 *Let G be a graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. Furthermore, let $T = (y_0, e_1, y_1, \dots, e_p, y_p)$ be a Tashkinov tree with respect to e and φ such that $V(Ty_r)$ is elementary with respect to φ for some integer $1 \leq r \leq p$.*

(a) *Then there are at least 4 colours in $\bar{\varphi}(V(Ty_r))$ which are unused on Ty_r with respect to φ.*

Let $0 \leq i < j \leq r$, let (y_i, y_j) be a (γ, δ)-pair with respect to φ, and let γ be unused on Ty_j with respect to φ. Furthermore, let $P = P_{y_j}(\gamma, \delta, \varphi)$ and $\varphi' = \varphi/P$. Then the following statements hold:

(b) *If y_i is not an endvertex of P then $T' = Ty_j$ is a Tashkinov tree with respect to e and φ', and $V(T')$ is not elementary, but $V(T'y_{j-1})$ is elementary with respect to φ'.*

(c) *If y_i is an endvertex of P, then T is a Tashkinov tree with respect to e and φ', and $V(Ty_r)$ is elementary with respect to φ'.*

Proof: For the set $\bar{\varphi}(v)$ of missing colours at $v \in V(T)$, we have $|\bar{\varphi}(v)| = k - d_G(v) + 1 \geq 2$ if $v \in \{y_0, y_1\}$ and $|\bar{\varphi}(v)| = k - d_G(v) \geq 1$ otherwise. Since $V(Ty_r)$ is elementary with respect to φ, and since $y_0, y_1 \in V(Ty_r)$, it then follows that $|\bar{\varphi}(V(Ty_r))| \geq |V(Ty_r)| + 2 = r + 3$. Since $|E(Ty_r)| = r$, $e_1 = e \in E(Ty_r)$, and e is uncoloured, at least 4 colours in $\bar{\varphi}(V(Ty_r))$ are unused on Ty_r with respect to φ. This proves (a).

Since $V(Ty_r)$ is elementary with respect to φ, no vertex $v \in V(Ty_r) \backslash \{y_i, y_j\}$ can be an endvertex of the (γ, δ)-chain P and, therefore, we have $\bar{\varphi}'(v) = \bar{\varphi}(v)$ for every such vertex v. Since $\delta \in \bar{\varphi}(y_j)$, $j \leq r$, and $V(Ty_r)$ is elementary with respect to φ, the colour δ is unused on Ty_j with respect to φ. Since the colour γ also is unused on Ty_j, this implies $E(Ty_j) \cap E(P) = \emptyset$. Consequently, Ty_j remains a Tashkinov tree with respect to e and φ'.

If y_i is not an endvertex of P then, since $\gamma \in \bar{\varphi}(y_i)$, y_i does not belong to P at all. Consequently, we have $\bar{\varphi}'(y_i) = \bar{\varphi}(y_i)$. Hence, we have $\gamma \in \bar{\varphi}'(y_i) \cap \bar{\varphi}'(y_j)$ and, therefore, $V(T')$ is not elementary with respect to φ'. Since $\bar{\varphi}'(v) = \bar{\varphi}(v)$ for every $v \in V(Ty_{j-1})$, and since $V(Ty_{j-1}) \subseteq V(Ty_r)$ is elementary with respect to φ, we infer that $V(T'y_{j-1}) = V(Ty_{j-1})$ is elementary with respect to φ'. This proves (b).

If y_i is an endvertex of P then $\bar{\varphi}'(y_i) = (\bar{\varphi}(y_i) \backslash \{\gamma\}) \cup \{\delta\}$ and $\bar{\varphi}'(y_j) = (\bar{\varphi}(y_i) \backslash \{\delta\}) \cup \{\gamma\}$. Since $\bar{\varphi}'(v) = \bar{\varphi}(v)$ for all vertices $v \in V(G) \backslash \{y_i, y_j\}$, we conclude that $V(Ty_r)$ still is elementary with respect to φ' and, moreover, that $\bar{\varphi}'(V(Ty_j)) = \bar{\varphi}(V(Ty_j))$. Then, for $h \in \{i, \dots, p\}$, we clearly have $\varphi'(e_h) \in \{\varphi(e_h), \gamma, \delta\} \subseteq \bar{\varphi}(V(Ty_{h-1})) = \bar{\varphi}'(V(Ty_{h-1}))$. Hence, T remains a Tashkinov tree with respect to e and φ'. This proves (c). ■

Theorem 2.20 *Let G be a graph with $|V(G)| = n$, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. Furthermore, let $T = (y_0, e_1, y_1, \dots, e_p, y_p)$ be a Tashkinov tree with respect to e and φ such that $V(T)$ is not elementary, but $V(Ty_{p-1})$ is elementary with respect to φ. Then a colouring $\varphi^* \in \mathcal{C}_k(G)$ can be derived from φ by performing at most $O(\min\{pn^2, pk^2\})$ Kempe changes plus colouring the edge e.*

Proof: The proof is by induction on the path number $q = p(T)$. If $q \leq 2$ then, since $e_1 \in E_G(y_0, y_1)$, we have either $q = 0$ or $q = 2$. In the first case T is a Kierstead path with respect to e and φ. In the second case $e_2 \in E_G(y_2, y_0)$ and $T' = (y_1, e_1, y_0, e_2, y_2, \ldots, e_p, y_p)$ is a Kierstead path with respect to e and φ. Hence, in both cases, Theorem 2.17 implies that the desired colouring φ^* can be derived from φ using at most $\mathrm{O}(p^2)$ Kempe changes.

For the induction step, let $q \geq 3$. Clearly, we have $p \geq q$. Since $V(Ty_{p-1})$ is elementary, but $V(T)$ is not elementary with respect to φ, there is at least one index $i \in \{0, 1, \ldots, p-1\}$ such that $\bar{\varphi}(y_i) \cap \bar{\varphi}(y_p) \neq \emptyset$. We denote the set of all these indices by I_φ. Now, for the proof of the induction step, we use induction again, this time with respect to $p = |V(T)| - 1$.

Case 1: (the basic case) $p = q$. This implies that $e_p \in E_G(y_h, y_p)$ for some integer $h \leq p-2$. Let $\beta = \varphi(e_p)$. Now we have to distinguish between some cases.

Case 1a: $\beta \in \varphi(y_{p-1})$ *and* $\min I_\varphi \leq p-2$. Then $\beta \in \bar{\varphi}(V(Ty_{p-2}))$, and there is an index $i \leq p-2$ satisfying $\bar{\varphi}(y_i) \cap \bar{\varphi}(y_p) \neq \emptyset$. Then, evidently, $T' = (y_0, e_1, y_1, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$ is a Tashkinov tree with respect to e and φ satisfying $p(T') \leq |V(T')| - 1 < |V(T)| - 1 = p(T)$ and, moreover, $V(T')$ is not elementary, but $V(T'y_{p-2})$ is elementary with respect to φ. Consequently, by induction, we can derive the desired colouring φ^* from φ.

Case 1b: $\beta \in \varphi(y_{p-1})$ *and* $I_\varphi = \{p-1\}$. Then there is a colour $\alpha \in \bar{\varphi}(y_{p-1}) \cap \bar{\varphi}(y_p)$. By Lemma 2.19(a), there are at least 4 colours in $\bar{\varphi}(V(Ty_{p-2}))$ that are unused on Ty_{p-2} with respect to φ. Consequently, there is a colour $\gamma \in \bar{\varphi}(V(Ty_{p-2}))$, say $\gamma \in \bar{\varphi}(y_j)$ with $j \leq p-2$, such that γ is unused on T with respect to φ. Since $V(Ty_{p-1})$ is elementary with respect to φ, it follows that γ is distinct from α. Hence, (y_j, y_{p-1}) is a (γ, α)-pair with respect to φ. Now let $P = P_{y_{p-1}}(\alpha, \gamma, \varphi)$ and $\varphi' = \varphi/P$.

If y_j is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Moreover, we have $\alpha \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_p)$ and $\beta = \varphi'(e_p) \in \varphi'(y_{p-1})$. Hence, we can derive the desired colouring φ^* from φ' as in Case 1a.

If y_j is not an endvertex of P then, by Lemma 2.19(b), $T' = Ty_{p-1}$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_{p-2})$ is elementary with respect to φ'. Moreover, we have $p(T') \leq |V(T')| - 1 = p - 1 < p = q = p(T)$. Consequently, by induction, we can derive the desired colouring φ^* from φ'.

Case 1c: $\beta \in \bar{\varphi}(y_{p-1})$ *and there is an index* $i \leq p-2$ *and a colour* $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$ *that is not used on* T *with respect to* φ. Since $V(Ty_{p-1})$ is elementary with respect to φ, we have $\alpha \neq \beta$, implying that (y_i, y_{p-1}) is an (α, β)-pair with respect to φ. Now let $P = P_{y_{p-1}}(\alpha, \beta, \varphi)$ and $\varphi' = \varphi/P$.

If y_i is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Since $\beta = \varphi(e_p)$ and $\alpha \in \bar{\varphi}(y_p)$, we conclude that y_p does not belong to P. This implies that $\beta = \varphi'(e_p)$ and $\alpha \in \bar{\varphi}'(y_p)$. Furthermore, we have $\beta \in \bar{\varphi}'(y_i)$ and $\alpha \in \bar{\varphi}'(y_{p-1})$. Hence, we can derive the desired colouring φ^* from φ' as in Case 1a or Case 1b.

If otherwise y_i is not an endvertex of P then, by Lemma 2.19(c), $T' = Ty_{p-1}$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_{p-2})$ is elementary with respect to φ'. Moreover, we have $p(T') \leq |V(T')|-1 = p-1 < p = q = p(T)$. Consequently, by induction, we can derive the desired colouring φ^* from φ'.

Case 1d: $\beta \in \bar{\varphi}(y_{p-1})$ *and* $\min I_\varphi \leq p-2$. Then there is an index $i \leq p-2$ and a colour $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$. We may assume that α is used on T with respect to φ, otherwise we are in Case 1c. By Lemma 2.19(a), there are at least 4 colours in $\bar{\varphi}(V(Ty_{p-2}))$ that are unused on Ty_{p-2} with respect to φ. Consequently, there is a colour $\gamma \in \bar{\varphi}(V(Ty_{p-2}))$, say $\gamma \in \bar{\varphi}(y_j)$ with $j \leq p-2$, such that γ is unused on T with respect to φ. We may assume that $\gamma \in \varphi(y_p)$, otherwise we are again in Case 1c (with γ instead of α). Since $V(Ty_{p-1})$ is elementary with respect to φ, it follows that γ is distinct from α. Now let $P = P_{y_p}(\alpha, \gamma, \varphi)$. Clearly, P is a path, and y_p is an endvertex of P.

If $V(P) \cap V(Ty_{p-1}) = \emptyset$ then $E(P) \cap E(T) = \emptyset$ and, therefore, T is a Tashkinov tree with respect to e and $\varphi' = \varphi/P$. Moreover, we have $\bar{\varphi}(y) = \bar{\varphi}(y)$ for all $y \in V(Ty_{p-1})$ and, therefore, $V(Ty_{p-1})$ remains elementary with respect to φ'. Since we now have $\gamma \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_p)$ and γ is not used on T with respect to φ', we can derive the desired colouring φ^* from φ' as in Case 1c.

If $V(P) \cap V(Ty_{p-1}) \neq \emptyset$ then there is a number $i_0 \in \{0, \ldots, p-1\}$ such that y_{i_0} belongs to P, and the subpath P' of P joining y_p and y_{i_0} does not contain any other vertex of $V(Ty_{p-1})$. Let $P' = (y_{i_0}, f_1, z_1, \ldots, f_m, z_m)$ where $z_m = y_p$. If $i_0 < p-1$ then, since $\alpha \in \bar{\varphi}(y_i)$ and $\gamma \in \bar{\varphi}(y_j)$, where $i, j \leq p-2$,

$$T' = (y_0, e_1, y_1, \ldots, e_{p-2}, y_{p-2}, f_1, z_1, \ldots, f_m, z_m)$$

is a Tashkinov tree with respect to e and φ satisfying $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(z_m)$. Consequently, since $V(T'y_{p-2})$ is elementary with respect to φ, there is a smallest index $m' \in \{1, \ldots, m\}$ such that $T'' = T'z_{m'}$ is not elementary with respect to φ. Moreover, we have $p(T'') \leq p-1 < p = q = p(T)$. Hence, by induction, we can derive the desired colouring φ^* from φ. In the other case we have $i_0 = p-1$, and

$$T' = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1}, f_1, z_1, \ldots, f_m, z_m)$$

is a Tashkinov tree with respect to e and φ satisfying $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(z_m)$. Consequently, since $V(T'y_{p-1})$ is elementary with respect to φ, there is a smallest index $m' \in \{1, \ldots, m\}$ such that $T'' = T'z_{m'}$ is not elementary with respect to φ. Moreover, we have $p(T'') \leq p-1 < p(T)$. Hence, by induction, we can derive the desired colouring φ^* from φ.

Case 1e: $\beta \in \bar{\varphi}(y_{p-1})$ *and* $I_\varphi = \{p-1\}$. Then there is a colour $\alpha \in \bar{\varphi}(y_{p-1}) \cap \bar{\varphi}(y_p)$. Clearly, $\alpha \neq \beta$. By Lemma 2.19(a), there are at least four colours in $\bar{\varphi}(V(Ty_{p-2}))$ that are unused on Ty_{p-2} with respect to φ. Consequently, there is a colour $\gamma \in \bar{\varphi}(V(Ty_{p-2}))$, say $\gamma \in \bar{\varphi}(y_j)$ with $j \leq p-2$, such that γ is unused on T with respect to φ. Since $V(Ty_{p-1})$ is elementary with respect to φ, it follows that γ is distinct from α and β. Hence, (y_j, y_p) is a (γ, α)-pair with respect to φ. Now let $P = P_{y_{p-1}}(\alpha, \gamma, \varphi)$ and $\varphi' = \varphi/P$.

If y_j is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Moreover, we have $\alpha \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_p)$ and $\beta = \varphi'(e_p) \in \bar{\varphi}'(y_{p-1})$. Therefore, we can derive the desired colouring φ^* from φ' as in Case 1d.

If y_j is not an endvertex of P then, by Lemma 2.19(b), $T' = Ty_{p-1}$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_{p-2})$ is elementary with respect to φ'. Moreover, we have $p(T') \leq |V(T')|-1 = p-1 < p(T)$. Consequently, by induction, we can derive the desired colouring φ^* from φ'.

Case 2: (induction step) $p > q$. Again we distinguish between several cases.

Case 2a: $\max I_\varphi \geq q$. Let $i = \max I_\varphi$. Then there is a colour $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$. Now we use a third induction, this time with respect to $i = \max I_\varphi$ and secondarily to the induction with respect to p.

If $i = p-1$ then let $\beta = \varphi(e_p)$. By definition, β is missing at some vertex y_j where $j \leq p-1$. Since $q < p$, we have $e_p \in E_G(y_{p-1}, y_p)$ and, therefore, $j \leq p-2$. Recolour e_p with α. This results in a colouring $\varphi' \in \mathcal{C}_k(G-e)$ such that $T' = Ty_{p-1}$ is a Tashkinov tree with respect to e and φ' satisfying $\beta \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_{p-1})$ and $\bar{\varphi}'(y_h) = \bar{\varphi}(y_h)$ for all $h \leq p-2$. Consequently, $V(T')$ is not elementary, but $V(T'y_{p-2})$ is elementary with respect to φ'. Moreover, we have $|V(T')| < |V(T)|$ and $p(T') = p(T)$, because of $q < p$. Therefore, by induction (with respect to p), we can derive the desired colouring φ^* from φ'. This settles the basic case $i = p-1$.

For the induction step, let $i < p-1$. From $k \geq \Delta(G)+1$ we infer that there is a colour $\beta \in \bar{\varphi}(y_{i+1})$. Since $V(Ty_{p-1})$ is elementary with respect to φ, the colours α and β are distinct and, moreover, the colour α is not used on Ty_i. Since $i \geq q$, we have $e_{i+1} \in E_G(y_i, y_{i+1})$ and, therefore, $\alpha \neq \varphi(e_{i+1})$. Consequently, α is not used on Ty_{i+1} either. Clearly, (y_i, y_{i+1}) is an (α,β)-pair with respect to φ. Now let $P = P_{y_{i+1}}(\alpha,\beta,\varphi)$ and $\varphi' = \varphi/P$.

If y_i is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Moreover, we have $\alpha \in \bar{\varphi}'(y_{i+1}) \cap \bar{\varphi}'(y_p)$ and, therefore, $V(T)$ is not elementary with respect to φ', and $\max I_{\varphi'} > i$. Consequently, by induction (with respect to i), we can derive the desired colouring φ^* from φ'.

If y_i is not an endvertex of P then, by Lemma 2.19(b), $T' = Ty_{i+1}$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_i)$ is elementary with respect to φ'. Moreover, we have $|V(T')| = i+2 < p+1 = |V(T)|$ and, since $i \geq q$, also $p(T') = p(T)$. Consequently, by induction (with respect to p), we can derive the desired colouring φ^* from φ'.

Case 2b: $\max I_\varphi < q$ *and there is an index* $i \in I_\varphi$ *and a colour* $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$ *that is not used on* Ty_q *with respect to* φ. Since $k \geq \Delta(G)+1$, there is a colour $\beta \in \bar{\varphi}(y_q)$. From $\max I_\varphi < q$ we infer that $\beta \neq \alpha$. Hence, (y_i, y_q) is an (α,β)-pair with respect to φ. Now let $P = P_{y_q}(\alpha,\beta,\varphi)$ and $\varphi' = \varphi/P$.

If y_i is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Moreover, we have $\alpha \in \bar{\varphi}'(y_q) \cap \bar{\varphi}'(y_p)$ and, therefore, $V(T)$ is not elementary with respect to φ', and $\max I_{\varphi'} = q$. Hence, we can derive the desired colouring φ^* from φ' as in Case 2a.

If y_i is not an endvertex of P then, by Lemma 2.19(b), $T' = Ty_q$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_{q-1})$ is elementary with respect to φ'. Moreover, we have $p(T') = p(T)$ and $|V(T')| = q+1 < p+1 = |V(T)|$. Consequently, by induction (with respect to p), we can derive the desired colouring φ^* from φ'.

Case 2c: $\max I_\varphi < q$ *and* $\min I_\varphi < q-1$. Then there is a colour $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(y_p)$ where $i = \min I_\varphi$. We may assume that α is used on T with respect to φ, otherwise we are in Case 2b. By Lemma 2.19(a), there are at least 4 colours in $\bar{\varphi}(V(Ty_{q-2}))$ that are unused on Ty_{q-2} with respect to φ. Consequently, there is a colour $\gamma \in \bar{\varphi}(V(Ty_{q-2})) \setminus \{\alpha\}$, say $\gamma \in \bar{\varphi}(y_j)$ with $j \leq q-2$, such that γ is unused on Ty_q with respect to φ. We may assume that $\gamma \in \varphi(y_p)$, otherwise we are again in Case 2b. Now let $P = P_{y_p}(\alpha, \gamma, \varphi)$. Then P is a path where one endvertex is y_p, and the other endvertex is some vertex $z \neq y_p$. Since $V(Ty_{p-1})$ is elementary with respect to φ, we conclude that $z \in \{y_i, y_j\}$ or $z \in V(G) \setminus V(T)$.

If $V(P) \cap V(Ty_{q-1}) = \emptyset$ then $E(P) \cap E(Ty_q) = \emptyset$ and, therefore, Ty_q is a Tashkinov tree with respect to e and $\varphi' = \varphi/P$. Moreover, $z \in V(G) \setminus V(T)$ and, therefore, we have $\bar{\varphi}'(y) = \bar{\varphi}(y)$ for all $y \in V(Ty_{p-1})$, and $\varphi'(f) \in \{\varphi(f), \alpha, \gamma\} \subseteq \bar{\varphi}(V(Ty_q)) = \bar{\varphi}'(V(Ty_q))$ for all $f \in E(T) \setminus E(Ty_q)$. Consequently, even T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Since we have $\gamma \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_p)$ and γ is not used on Ty_q with respect to φ', we can derive the desired colouring φ^* from φ' as in Case 2b.

If $V(P) \cap V(Ty_{q-1}) \neq \emptyset$ then there is a number $i_0 \in \{0, \ldots, q-1\}$ such that y_{i_0} belongs to P, and the subpath P' of P joining y_p and y_{i_0} does not contain any other vertex of $V(Ty_{q-1})$. Let $P' = (y_{i_0}, f_1, z_1, \ldots, f_m, z_m)$ where $z_m = y_p$. If $i_0 < q-1$ then, since $\alpha \in \bar{\varphi}(y_i)$ and $\gamma \in \bar{\varphi}(y_j)$ where $i, j \leq q-2$,

$$T' = (y_0, e_1, y_1, \ldots, e_{q-2}, y_{q-2}, f_1, z_1, \ldots, f_m, z_m)$$

is a Tashkinov tree with respect to e and φ satisfying $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(z_m)$. Consequently, since $V(T'y_{q-2})$ is elementary with respect to φ, there is a smallest index $m' \in \{1, \ldots, m\}$ such that $T'' = T'z_{m'}$ is not elementary with respect to φ. Moreover, we have $p(T'') \leq q-1 < p(T)$. Hence, by induction (with respect to q), we can derive the desired colouring φ^* from φ. In the other case we have $i_0 = q-1$, and

$$T' = (y_0, e_1, y_1, \ldots, e_{q-1}, y_{q-1}, f_1, z_1, \ldots, f_m, z_m)$$

is a Tashkinov tree with respect to e and φ satisfying $\alpha \in \bar{\varphi}(y_i) \cap \bar{\varphi}(z_m)$. Consequently, since $V(T'y_{q-1})$ is elementary with respect to φ, there is a smallest index $m' \in \{1, \ldots, m\}$ such that $T'' = T'z_{m'}$ is not elementary with respect to φ. Moreover, we have $p(T'') \leq q-1 < p(T)$. Hence, by induction (with respect to q), we can derive the desired colouring φ^* from φ.

Case 2d: $I_\varphi = \{q-1\}$. Then there is a colour $\alpha \in \bar{\varphi}(y_{q-1}) \cap \bar{\varphi}(y_p)$. By Lemma 2.19(a), there are at least 4 colours in $\bar{\varphi}(V(Ty_{q-2}))$ that are unused on Ty_{q-2} with respect to φ. Consequently, there is a colour $\gamma \in \bar{\varphi}(V(Ty_{q-2}))$, say $\gamma \in \bar{\varphi}(y_j)$ with $j \leq q-2$, such that γ is not used on Ty_q with respect to φ. Since $V(Ty_{p-1})$ is elementary with respect to φ, it follows that γ is distinct from α. Hence, (y_j, y_{q-1}) is a (γ, α)-pair with respect to φ. Now let $P = P_{y_{q-1}}(\alpha, \gamma, \varphi)$ and $\varphi' = \varphi/P$.

If y_j is an endvertex of P then, by Lemma 2.19(c), T is a Tashkinov tree with respect to e and φ', and $V(Ty_{p-1})$ is elementary with respect to φ'. Moreover, we have $\alpha \in \bar{\varphi}'(y_j) \cap \bar{\varphi}'(y_p)$ and, therefore, $V(T)$ is not elementary with respect to φ'. Hence, we can derive the desired colouring φ^* from φ' as in Case 2c.

If y_j is not an endvertex of P then, by Lemma 2.19(b), $T' = Ty_{q-1}$ is a Tashkinov tree with respect to e and φ' such that $V(T')$ is not elementary, but $V(T'y_{q-2})$ is elementary with respect to φ'. Moreover, we have $p(T') \leq |V(T')| - 1 = q-1 < p(T)$. Consequently, by induction (with respect to q), we can derive the desired colouring φ^* from φ'.

Now Case 2 is settled, which completes the induction with respect to p and also the induction with respect to q. Hence, we can derive the desired colouring φ^* from φ. We just have to check the number of Kempe changes (before colouring the edge e) to do this. As already stated, in the case $q \leq 2$ we need $\mathrm{O}(p^2)$ Kempe changes. In the case $q > 2$ we can inductively reduce the value of q.

For the cost of one such reduction, we have to analyse the different cases from the first part of the proof. In Case 1a we can simply reduce q without recolouring anything. In Case 1b one Kempe change is needed to either reduce q or get to Case 1a. Hence, we need a total of $\mathrm{O}(1)$ Kempe changes to reduce q. In Case 1c one Kempe change is needed to either reduce q or get to Case 1b. Hence, we need a total of $\mathrm{O}(1)$ Kempe changes to reduce q. In Case 1d either we use at most one Kempe change to get to Case 1c, or we can reduce q by just finding but not recolouring a chain. Hence, we need a total of $\mathrm{O}(1)$ Kempe changes to reduce q. In Case 1e one Kempe change is needed to either reduce q or get to Case 1d. Hence, we need a total of $\mathrm{O}(1)$ Kempe changes to reduce q. Consequently, in the case $p = q$ (Case 1) we need $\mathrm{O}(1)$ Kempe changes to reduce the value of q.

In the case $p > q$ (Case 2) we may have to reduce p while leaving q unchanged. So we have to look also at the costs of such a reduction of p. In Case 2a we need at most $p - 1 - \max I_\varphi \in \mathrm{O}(p)$ Kempe changes to increase $\max I_\varphi$ until we can reduce p using one Kempe change. Hence, we need a total of $\mathrm{O}(p)$ Kempe changes to reduce p. In Case 2b we need one Kempe change to either reduce p or get to Case 2a. Hence, we need a total of $\mathrm{O}(p)$ Kempe changes to reduce p. In Case 2c either we use at most one Kempe change to get to Case 2b, or we can reduce q by just finding but not recolouring a chain. Hence, we need a total of $\mathrm{O}(p)$ Kempe changes to reduce p or q. In Case 2d we need one Kempe change to either reduce q or get to Case 2c. Hence, we need a total of $\mathrm{O}(p)$ Kempe changes to reduce p or q. Consequently, in the case $p > q$ (Case 2) we need $\mathrm{O}(p)$ Kempe changes to reduce p or q. Hence, after at most $p - q$ reductions of p we can reduce q using $\mathrm{O}(p)$ or $\mathrm{O}(1)$ (Case 1) Kempe changes, which leads to a total of $\mathrm{O}((p-q)p + p) \subseteq \mathrm{O}(p^2)$ Kempe changes to reduce the value of q.

Note that, during a reducing step for q, the order of the new Tashkinov tree may significantly increase, namely in Case 1d and in Case 2c. Hence, to estimate the costs for the whole reduction of q we cannot use the value p of the original Tashkinov tree, we have to use an upper bound of the order of all constructed Tashkinov trees. Clearly, one possible upper bound is $|V(G)| = n$. Moreover, the vertex set of every constructed Tashkinov T is elementary with respect to the current colouring, except for the last vertex. Since we have $k \geq \Delta + 1$, this implies that the order of every constructed Tashkinov tree is bounded by the number of missing colours of $V(T)$ which itself is at most k. Consequently, the number of Kempe changes needed for one reduction of q is in $O(n^2)$ as well as in $O(k^2)$.

Then we get the total costs as follows. After at most $q-2$ reductions of the path number we get to the basic case $q \leq 2$ where we need at most $O(n^2)$ and also at most $O(k^2)$ Kempe changes using Kierstead's method. Hence, to derive the colouring $\varphi^* \in \mathcal{C}_k(G)$, the total number of Kempe changes needed is in $O((q-2)n^2 + n^2) \subseteq O(pn^2)$ as well as in $O((q-2)k^2 + k^2) \subseteq O(pk^2)$. This completes the proof. ∎

Let G be a graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. Moreover, let T be a Tashkinov tree with respect to e and φ. If $V(T)$ is not elementary with respect to φ, then Theorem 2.20 shows how to get a k-edge colouring of G. If $V(T)$ is elementary with respect to φ, but there is an edge $f \in E_G(V(T), z)$ satisfying $z \in V(G) \backslash V(T)$ and $\varphi(f) \in \bar{\varphi}(V(T))$, then $T' = (T, f, z)$ is a Tashkinov tree with respect to e and φ. This gives us a way to construct an algorithm **TashExt1** that extends a partial colouring $\varphi \in \mathcal{C}_k(G-e)$ and uses Tashkinov trees as test objects.

TashExt1(G, e, x, y, k, φ):

1) $p \leftarrow 1$, $e_p \leftarrow e$, $y_p \leftarrow y$, $y_0 \leftarrow x$, $T \leftarrow (y_0, e_p, y_p)$.

2) `If` $\bar{\varphi}(V(Ty_{p-1})) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  2a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.20.`

  2b) `Return` (k, φ').

3) `If` $\exists e_{p+1} \in E_G(V(T), V(G) \backslash V(T)) : \varphi(e_{p+1}) \in \bar{\varphi}(V(T))$ `then`

  3a) `Let` y_{p+1} `be the endvertex of` e_{p+1} `that is not in` $V(T)$.

  3b) $T \leftarrow (T, e_{p+1}, y_{p+1})$, $p \leftarrow p+1$.

  3c) `Goto 2.`

4) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k+1$.

5) `Return` $(k+1, \varphi')$.

Correctness of TashExt1: If G is a graph, $e \in E_G(x, y)$, and $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$, then TashExt1 subsequently builds a Tashkinov tree $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with respect to e and φ until either $V(T)$ is not elementary with respect to φ, or T is a maximal Tashkinov tree with respect to e and φ. In the first case, $V(Ty_{p-1})$ is elementary with respect to φ, otherwise the algorithm had detected it before. Hence, by Theorem 2.20, TashExt1 can compute the colouring $\varphi' \in \mathcal{C}_k(G)$. Consequently, TashExt1 works correctly in this case. In the other case, $V(T)$ is maximal and elementary with respect to φ, and TashExt1 simply assigns a new colour to e. In any case, TashExt1 computes a colouring $\varphi' \in \mathcal{C}_{k'}(G)$ where $k' \in \{k, k+1\}$.

Note that, in order to use Theorem 2.20, we have to require $k \geq \Delta(G)+1$. So we consider only inputs (G, e, x, y, k, φ) with $k \geq \Delta(G)+1$ as valid.

Time Complexity of TashExt1: Now, let us analyse the running time of the algorithm TashExt1. Let (G, e, x, y, k, φ) be a valid input with $|V(G)| = n$ and $\Delta(G) = \Delta$. We can assume that $k \in O(\Delta)$, see Section 2.2. The first initializing step only needs constant time. Due to the goto-statement in step 3c, there is a loop starting in step 2. In every round the value of p is increased, the number of rounds is bounded by the order of a maximal Tashkinov tree with respect to e and φ. Since the vertices of a Tashkinov tree are distinct, we have $p \leq n$. Moreover, in the loop $V(Ty_{p-1})$ is always an elementary set with respect to φ and, since $k \geq \Delta(G) + 1$, we also have $p \leq |\bar{\varphi}(V(Ky_{p-1}))| \leq k$. Hence, the loop is repeated at most $\min\{n, k\}$ times.

In every sweep of the loop, the algorithm has to check the condition of step 2. This can be done in time $O(\Delta)$, using the same method as described for KierExt. The recolouring step 2a does not count to the loop, because the algorithm will terminate right after this step. The next step inside the loop is step 3. This step can also be implemented in time $O(\Delta)$. We simply maintain, for every colour, the number of edges with this colour that join $V(T)$ and $V(G)\setminus V(T)$. This can easily be updated in time $O(\Delta)$ when increasing T. With this information, in step 3 we can decide in time $O(\Delta)$ whether there is a suitable colour and, if this is the case, find an edge by scanning the corresponding same-colour list. Consequently, for the whole loop this gives a running time of $O(\Delta \cdot \min\{n, \Delta\})$.

The step 2a is computed at most once during the algorithm. Since $k \in O(\Delta)$ and the order of T is in $O(\min\{n, \Delta\})$, Theorem 2.20 implies that there are at most $O(\min\{n^3, \Delta^3\})$ Kempe changes needed to compute the k-edge colouring of G. This gives a running time of $O\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$, see Section 2.2. In most steps of this recolouring the costs for the Kempe changes dominate the other costs. The only exceptions are the cases in which the path number of T is decreased by adding a path to a part of the tree, see Case 1d and Case 2c in the proof of Theorem 2.20. In these cases a part of the Tashkinov tree is extended by several new vertices until the vertex set is not elementary with respect to the current colouring φ. For every new vertex added, it is necessary to check whether the vertex set remains elementary with respect to φ or not. This needs time $O(\Delta)$ for every new vertex. Since vertices are added only as long as the resulting Tashkinov tree remains elementary with respect to φ, the number of new vertices is bounded by k as well as by n. Consequently, this whole step costs time $O(\Delta \cdot \min\{n, \Delta\})$ which may exceed the time cost of the corresponding Kempe change. Since the path number is never increased, this decreasing step occurs at most $\min\{n, k\}$ times during the algorithm. This gives an additional running time of $O(\Delta \cdot \min\{n^2, \Delta^2\})$ for step 2a. Hence, the number of Kempe changes still determines the time cost for this step which is in $O\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$.

If TashExt1 uses a new colour in step 4, this needs only constant time. Hence, the algorithm has a total running time of $O\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$.

Tashkinov's colouring algorithm: Let **Tashkinov1** be a colouring algorithm that uses an arbitrary edge order, TashExt1 as kernel, and that starts with $k = \Delta(G) + 1$ for any input graph with $\Delta(G) \geq 2$. In the case $\Delta(G) \leq 1$ the algorithm simply colours the graph optimal with $\Delta(G)$ colours.

Since every Kierstead path with respect to an edge e and a colouring φ is also a Tashkinov tree with respect to e and φ, the algorithm Tashkinov1 uses not more colours than Kierstead. Moreover, for every multi-fan F at a vertex x with respect to an edge e and a colouring φ, there is a Tashkinov tree T with respect to e and φ satisfying $V(T) = V(F) \cup \{x\}$. Consequently, Tashkinov1 also uses not more colours than VIZING provided that VIZING uses at least $\Delta(G) + 1$ colours. This 'flaw' could be overcome by a hybrid algorithm. This algorithm would start with $k = \Delta(G)$ and use VIZEXT as kernel until the number of colours is increased to $\Delta(G) + 1$, from this point on the algorithm would use TASHEXT1 as kernel. This hybridizing technique is always possible for kernels with such a requirement for the starting value of k, but we will not concentrate further on this. We will rather improve TASHEXT1 in other ways.

If, on the input (G, e, x, y, k, φ), TASHEXT1 increases the number of colours to $k + 1$, then there is a maximal Tashkinov tree T with respect to e and φ such that $V(T)$ is elementary with respect to φ. In [38] Tashkinov developed some methods which may allow us to compute a k-edge colouring of G even in this case. We will use this methods to improve the algorithm TASHEXT1.

2.7 A New Upper Bound for the Chromatic Index

For a graph G, let

$$\boldsymbol{\tau(G)} = \max\{\Delta(G) + \sqrt{\tfrac{\Delta(G)-1}{2}}, \mathcal{W}(G)\}$$

if G contains at least one edge, and $\tau(G) = 0$ otherwise. Further, for every $\epsilon > 0$ let

$$\boldsymbol{\tau_\epsilon(G)} = \max\{\lfloor (1+\epsilon)\Delta(G) + 1 - 3\epsilon \rfloor, \lfloor \Delta(G) - 1 + \tfrac{1}{2\epsilon} \rfloor, \mathcal{W}(G)\}$$

In this section we shall demonstrate how Tashkinov's methods can be used to show that the parameters τ and τ_ϵ are upper bounds of the chromatic index. First we need some further notation.

Let G be a graph, and let $k \geq \Delta(G) + 1$ be an integer. We denote by $\boldsymbol{\mathcal{T}_k(G)}$ the set of all triples (T, e, φ) such that $e \in E(G)$, $\varphi \in \mathcal{C}_k(G - e)$, T is a maximal Tashkinov tree with respect to e and φ, and $V(T)$ is elementary with respect to φ. Evidently, if $\mathcal{T}_k(G) = \emptyset$ then either $\chi'(G - e) > k$ for all edges $e \in E(G)$, or, by Theorem 2.20, there is a k-edge colouring of G.

For a triple $(T, e, \varphi) \in \mathcal{T}_k(G)$, we introduce the following notation. For a colour α, let $\boldsymbol{E_\alpha(e, \varphi)} = \{e' \in E(G) \backslash \{e\} \mid \varphi(e') = \alpha\}$ be the set of all edges of G coloured with α with respect to φ. Further, let

$$\boldsymbol{E_\alpha(T, e, \varphi)} = E_\alpha(e, \varphi) \cap E_G(V(T), V(G) \backslash V(T)).$$

The colour α is said to be **defective** with respect to (T, e, φ), if $|E_\alpha(T, e, \varphi)| \geq 2$. The set of all defective colours with respect to (T, e, φ) is denoted by $\boldsymbol{\Gamma^d(T, e, \varphi)}$. The colour α is said to be **free** with respect to (T, e, φ), if $\alpha \in \bar{\varphi}(V(T))$ and α is unused on T with respect to φ. The set of all free colours with respect to (T, e, φ) is denoted by $\boldsymbol{\Gamma^f(T, e, \varphi)}$.

Proposition 2.21 *Let G be a graph, let $k \geq \Delta(G) + 1$ be an integer, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$. Then the following statements hold:*

(a) *$V(T)$ is elementary and closed with respect to φ.*

(b) *$|V(T)| \geq 3$ is odd.*

(c) *$V(T)$ is strongly closed with respect to φ iff $\Gamma^d(T, e, \varphi) = \emptyset$.*

(d) *If $\gamma \in \bar{\varphi}(V(T))$ then $E_\gamma(T, e, \varphi) = \emptyset$.*

(e) *If $\delta \in \Gamma^d(T, e, \varphi)$ then $|E_\delta(T, e, \varphi)| \geq 3$ is odd.*

(f) *For a vertex $x \in V(T)$, we have $|\bar{\varphi}(x)| = k - d_G(x) + 1 \geq 2$ if $e \in E_G(x)$ and $|\bar{\varphi}(x)| = k - d_G(x) \geq 1$ otherwise. Moreover, $|\Gamma^f(T, e, \varphi)| \geq 4$.*

(g) *Every colour in $\Gamma^d(T, e, \varphi) \cup \Gamma^f(T, e, \varphi)$ is unused on T with respect to φ.*

(h) *Let $u, v \in V(T)$, and let (u, v) be an (α, β)-pair with respect to φ. Then $P = P_u(\alpha, \beta, \varphi)$ is a path with endvertices u and v, where $V(P) \subseteq V(T)$. Moreover, if α and β are unused on Tu and on Tv, then T is a maximal Tashkinov tree with respect to e and $\varphi' = \varphi/P$ such that $V(T)$ is elementary with respect to φ, that is, $(T, e, \varphi') \in \mathcal{T}_k(G)$.*

Proof: By definition of $\mathcal{T}_k(G)$, the vertex set $V(T)$ is elementary with respect to φ. Let $f \in E_G(V(T), V(G) \setminus V(T))$. Since T is a maximal Tashkinov tree with respect to e and φ, it follows that $\varphi(f) \notin \bar{\varphi}(V(T))$. Hence, $V(T)$ is also closed with respect to φ. This proves (a).

From $k \geq \Delta(G) + 1$ we infer that for any vertex $v \in V(T)$ there is a colour $\gamma \in \bar{\varphi}(v)$. Then (a) implies that every other vertex of $V(T)$ is incident to an edge in $E_G(V(T), V(T))$ coloured with γ. In particular, this implies that $|V(T)|$ is odd. Since both endvertices of e belong to $V(T)$, we also obtain $|V(T)| \geq 3$. This proves (b). Evidently, (b) implies both statements (c) and (d).

For the proof of (e), consider a defective colour $\delta \in \Gamma^d(T, e, \varphi)$, and let $E_\delta = E_\delta(T, e, \varphi)$. Then, by definition, $|E_\delta| \geq 2$. By (d) and (b), we know that $\delta \in \varphi(v)$ for every $v \in V(T)$ and that $|V(T)|$ is odd. Since $E_\delta(e, \varphi)$ is a matching of G, this implies that $|E_\delta|$ is odd and, therefore, $|E_\delta| \geq 3$. Thus (e) is proved.

The first part of statement (f) follows simply from the fact that $\varphi \in \mathcal{C}_k(G - e)$ and $k \geq \Delta(G) + 1$. Since, by (a), $V(T)$ is elementary with respect to φ, this implies that $|\bar{\varphi}(V(T))| \geq p + 2$ where $p = |V(T)|$. From $|E(T)| = p - 1$ and $e \in E(T)$ we then conclude that at least 4 colours in $\bar{\varphi}(V(T))$ are unused on T with respect to φ. Hence, we have $|\Gamma^f(T, e, \varphi)| \geq 4$, and (f) is proved.

That every free colour is unused on T with respect to φ follows from the definition. If $\delta \in \Gamma^d(T, e, \varphi)$ is a defective colour, then $E_\delta(T, e, \varphi) \neq \emptyset$. By (d), this implies that $\delta \notin \bar{\varphi}(V(T))$. Consequently, δ is unused on T with respect to φ. This proves (g).

Let $u, v \in V(T)$, let (u, v) be an (α, β)-pair with respect to φ, and let $P = P_u(\alpha, \beta, \varphi)$. Hence, we have $u \neq v$, $\alpha \in \bar{\varphi}(u)$, and $\beta \in \bar{\varphi}(v)$. Since $V(T)$ is elementary with respect to φ, the vertex u respectively v is the unique vertex of T where α respectively β is missing with respect to φ. Therefore, P is a path, where one endvertex is u and the other endvertex is some vertex $z \neq u$. From (d) we know that $E_\alpha(T, e, \varphi) = E_\beta(T, e, \varphi) = \emptyset$. This clearly implies that $V(P) \subseteq V(T)$ and $z = v$. This proves the first part of (h). Now suppose that α and β are unused on both Tu and Tv with respect to φ. Without loss of generality we may assume that u comes before v in T. Then $E(P) \cap E(V(Tv)) = \emptyset$ and, therefore, Tv is a Tashkinov tree with respect to e and $\varphi' = \varphi/P$. Clearly, we have $\bar{\varphi}'(u) = (\bar{\varphi}(u) \setminus \{\alpha\}) \cup \{\beta\}$,

$\bar{\varphi}'(v) = (\bar{\varphi}(v) \setminus \{\beta\}) \cup \{\alpha\}$, and $\bar{\varphi}'(z) = \bar{\varphi}(z)$ for every vertex $z \notin \{u, v\}$. In particular this implies that $V(T)$ remains elementary with respect to φ'. Furthermore, we have $\bar{\varphi}'(V(T)) = \bar{\varphi}(V(T))$ as well as $\bar{\varphi}'(V(Tv)) = \bar{\varphi}(V(Tv))$. Obviously, for all edges $f \in E(T) \setminus E(Tv)$, we have $\varphi'(f) \in \{\varphi(f), \alpha, \beta\}$. Consequently, T is a Tashkinov tree with respect to e and φ'. From $E_\alpha(T, e, \varphi) = E_\beta(T, e, \varphi) = \emptyset$ it follows that $\varphi'(f) = \varphi(f)$ for all edges $f \in E_G(V(T), V(G) \setminus V(T))$. Since $\bar{\varphi}'(V(T)) = \bar{\varphi}(V(T))$, it then follows from (d) that $E_\gamma(T, e, \varphi') = \emptyset$ for all colours $\gamma \in \bar{\varphi}'(V(T))$. Hence, T is a maximal Tashkinov tree with respect to e and $\varphi' \in \mathcal{C}_k(G - e)$ such that $V(T)$ is elementary with respect to φ', that is, $(T, e, \varphi') \in \mathcal{T}_k(G)$. This proves (h). ■

Let G be graph, let $e \in E(G)$, and let $\varphi \in \mathcal{C}_k(G - e)$ for an integer $k \geq \Delta(G) + 1$. Further, let T be a maximal Tashkinov tree with respect to e and φ such that $V(T)$ is elementary with respect to φ. The following results will give some conditions that allow us to recolour $G - e$ and derive from T a new Tashkinov tree T' with respect to e and φ' with $|V(T')| > |V(T)|$. These results are already implicitly given in [38].

Proposition 2.22 *Let G be a graph, let $k \geq \Delta(G) + 1$ be an integer, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$. Furthermore, let $\delta \in \Gamma^d(T, e, \varphi)$ be a defective colour, and let $u \in V(T)$ be a vertex such that $\bar{\varphi}(u)$ contains a free colour $\gamma \in \Gamma^f(T, e, \varphi)$. Then, for the (γ, δ)-chain $P = P_u(\gamma, \delta, \varphi)$ and the colouring $\varphi' = \varphi/P$, the following statements hold:*

(a) *P is a path where one endvertex is u and the other endvertex belongs to $V(G) \setminus V(T)$.*

(b) *$E(P) \cap E_G(V(T), V(G) \setminus V(T)) \subseteq E_\delta(T, e, \varphi)$.*

(c) *T is a Tashkinov tree with respect to e and φ' and, moreover, $V(T)$ is elementary with respect to φ'.*

(d) *If $E_\delta(T, e, \varphi) \nsubseteq E(P)$ then $V(T)$ is not closed with respect to φ'.*

Proof: By definition, $V(T)$ is elementary and closed with respect to φ. Hence, the vertex u is the only vertex in $V(T)$ where the colour γ is missing with respect to φ. Since $\delta \in \Gamma^d(T, e, \varphi)$, we have $\delta \in \varphi(v)$ for all $v \in V(T)$. Consequently, P is a path where one endvertex is u and the other endvertex belongs $V(G) \setminus V(T)$. This proves (a).

The edges of P are coloured either with γ or δ with respect to φ and, therefore, we clearly have $E(P) \cap E_G(V(T), V(G) \setminus V(T)) \subseteq E_\gamma(T, e, \varphi) \cup E_\delta(T, e, \varphi)$. Since, by Proposition 2.21(d), we have $E_\gamma(T, e, \varphi) = \emptyset$, this implies (b).

Since, by Proposition 2.21(g), γ and δ both are unused on T with respect to φ, we have $\varphi'(f) = \varphi(f)$ for all $f \in E(T)$. Moreover, $\bar{\varphi}'(u) = (\bar{\varphi}(u) \setminus \{\gamma\}) \cup \{\delta\}$ and $\bar{\varphi}'(v) = \bar{\varphi}(v)$ for all vertices $v \in V(T) \setminus \{u\}$. Consequently, T remains a Tashkinov tree with respect to e and φ'. Since δ does not belong to $\bar{\varphi}(V(T))$, the vertex set $V(T)$ remains elementary with respect to φ'. Thus (c) is proved.

If $E_\delta(T, e, \varphi) \nsubseteq E(P)$ then there is an edge $f \in E_\delta(T, e, \varphi)$ that does not belong to $E(P)$. Hence, we have $\varphi'(f) = \varphi(f) = \delta$ and, therefore, $E_\delta(T, e, \varphi') \neq \emptyset$. Since $\delta \in \bar{\varphi}'(V(T))$, this implies that $V(T)$ is not closed with respect to φ'. This proves (d). ■

Theorem 2.23 *Let G be a graph, let $k \geq \Delta(G)+1$ be an integer, and let $(T,e,\varphi) \in \mathcal{T}_k(G)$. Furthermore, let $\delta \in \Gamma^d(T,e,\varphi)$, let $u \in V(T)$ be a vertex such that $\bar{\varphi}(u)$ contains a free colour $\gamma \in \Gamma^f(T,e,\varphi)$, and let $P = P_u(\gamma,\delta,\varphi)$. If $E_\delta(T,e,\varphi) \subseteq E(P)$ then the following statements hold:*

(a) *In the linear order $\preceq_{(u,P)}$ there is a last vertex v_0 that belongs to $V(T)$ and a first vertex v_1 that belongs to $V(G)\setminus V(T)$, where $v_1 \preceq_{(u,P)} v_0$. In the same order v_1 has a successor v_2 which belongs to $V(G)\setminus V(T)$, too.*

(b) *If $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ then there is a colouring $\varphi' \in \mathcal{C}_k(G-e)$ such that T is a Tashkinov tree with respect to e and φ' where $V(T)$ is elementary but not closed with respect to φ'. Moreover, φ' can be derived from φ by at most two Kempe changes.*

(c) *If the set $X = V(T) \cup \{v_1, v_2\}$ is not elementary with respect to φ, then there is a colouring $\varphi' \in \mathcal{C}_k(G-e)$ such that T is a Tashkinov tree with respect to e and φ' where $V(T)$ is elementary but not closed with respect to φ'. Moreover, φ' can be derived from φ by at most four Kempe changes.*

(d) *If $X = V(T) \cup \{v_1, v_2\}$ is elementary with respect to φ, then*

$$|V(T)| \leq \frac{\Delta(G)-3}{k-\Delta(G)} - 1.$$

(e) *If $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) = \emptyset$ then at least $k - d_G(v_0) + 1$ colours are used on T with respect to φ.*

Proof: The existence of v_0 and v_1 is a simple consequence of Proposition 2.22(a) and the fact that $u \in V(T)$. Since, by definition, we have $|E_\delta(T,e,\varphi)| \geq 2$, it follows from $E_\delta(T,e,\varphi) \subseteq E(P)$ that $v_1 \preceq_{(u,P)} v_0$. Moreover, v_1 is not an endvertex of P and hence, in the linear order $\preceq_{(u,P)}$, v_1 has a successor v_2. Then there is an edge $f \in E_G(v_1, v_2)$ with $\varphi(f) = \gamma$. From Proposition 2.21(d) it then follows that $v_2 \in V(G)\setminus V(T)$. Thus (a) is proved.

To prove (b), we assume that there is a colour $\alpha \in \bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi)$. By (a), we have $v_0 \neq u$ and, therefore, $\alpha \neq \gamma$. Since α and γ are unused on T with respect to φ, Proposition 2.21(h) implies that $P_1 = P_u(\alpha,\gamma,\varphi)$ is a path joining u and v_0 and, moreover, that $(T,e,\varphi_1) \in \mathcal{T}_k(G)$ where $\varphi_1 = \varphi/P_1$. Further, we infer from Proposition 2.21(d) that $E_\alpha(T,e,\varphi) = E_\gamma(T,e,\varphi) = \emptyset$ and, therefore, $V(P_1) \subseteq V(T)$. Now let z be the endvertex of P that is not u. By Proposition 2.22(a), we have $z \in V(G)\setminus V(T)$. Since v_0 is in the linear $\preceq_{(u,P)}$ the last vertex that belongs to $V(T)$, we infer for $P_2 = v_0 P z$ that $V(P_2) \cap V(T) = \{v_0\}$ and $|E(P_2) \cap E(V(G)\setminus V(T))| = 1$. In particular, this implies $E(P_2) \cap E(P_1) = \emptyset$. Consequently, since $\gamma \in \bar{\varphi}_1(v_0)$, we have $P_2 = P_{v_0}(\gamma,\delta,\varphi_1)$. By Proposition 2.22(c), T is a Tashkinov tree with respect to e and $\varphi' = \varphi_1/P_2$, and $V(T)$ is elementary with respect to φ'. Moreover, since $\delta \in \Gamma^d(T,e,\varphi)$ and $|E(P_2) \cap E(V(T), V(G)\setminus V(T))| = 1$, there is an edge $f' \in E_G(V(T), V(G)\setminus V(T)) \setminus E(P_2)$ with $\varphi(f') = \delta$. Then, clearly, $\varphi'(f') = \varphi_1(f') = \delta$. Since $\delta \in \bar{\varphi}'(v_0)$, this implies that $V(T)$ is not closed with respect to φ'. The colouring φ' was derived from φ by two Kempe changes and hence, (b) is proved.

To prove (c), we assume that $X = V(T) \cup \{v_1, v_2\}$ is not elementary with respect to φ. Since $V(T)$ is elementary with respect to φ, we distinguish two cases.

Case 1: *There is a colour* $\alpha \in \bar{\varphi}(v_1) \cap \bar{\varphi}(V(T) \cup \{v_2\})$. Since $\gamma \notin \bar{\varphi}(v_1)$, we have $\alpha \neq \gamma$. Let $P_1 = P_{v_1}(\alpha, \gamma, \varphi)$. In the case $\alpha \in \bar{\varphi}(v_2)$ we have $P_1 = P(v_1, f, v_2)$. In the other case we have $\alpha, \gamma \in \bar{\varphi}(V(T))$. Since $V(T)$ is closed with respect to φ, this implies that $V(P_1) \cap V(T) = \emptyset$. Hence, in both cases we have $V(P_1) \cap V(T) = \emptyset$ and, therefore, T remains a Tashkinov tree with respect to e and $\varphi_1 = \varphi/P_1$. Furthermore, we have $(T, e, \varphi_1) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi_1)$, and $\gamma \in \bar{\varphi}_1(u)$ remains unused on T with respect to φ_1. Moreover, the chain $P_2 = P_u(\gamma, \delta, \varphi_1)$ satisfies $P_2 = uPv_1$ and, therefore, $E(P_2)$ contains only one, but not all edges from $E_\delta(T, e, \varphi_1)$. Then, by Proposition 2.22(c),(d), T is a Tashkinov tree with respect to e and $\varphi' = \varphi_1/P_2$, and $V(T)$ is elementary, but not closed with respect to φ'. Since we used two Kempe changes to derive φ' from φ, this settles Case 1.

Case 2: *There is a colour* $\alpha \in \bar{\varphi}(V(T)) \cap \bar{\varphi}(v_2)$. By Proposition 2.21(f), there is a free colour $\alpha' \in \Gamma^f(T, e, \varphi) \setminus \{\alpha, \gamma\}$. Clearly, $\alpha' \neq \delta$. Since $V(T)$ is elementary with respect to φ, there is a unique vertex $u' \in V(T)$ such that $\alpha' \in \bar{\varphi}(u')$. Let $P_1 = P_{v_2}(\alpha, \alpha', \varphi)$. Since $\alpha, \alpha' \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we have $E_\alpha(T, e, \varphi) = E_{\alpha'}(T, e, \varphi) = \emptyset$ and hence $V(P_1) \cap V(T) = \emptyset$. Consequently, T remains a Tashkinov tree with respect to e and $\varphi_1 = \varphi/P_1$, and we have $(T, e, \varphi_1) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi_1)$, $\gamma \in \bar{\varphi}_1(u)$, $P = P_u(\gamma, \delta, \varphi_1)$, $\alpha' \in \bar{\varphi}_1(u') \cap \bar{\varphi}_1(v_2)$, and both colours α' and γ remain unused on T with respect to φ_1.

From Proposition 2.21(f) we also infer that there is a colour $\beta \in \bar{\varphi}(v_1)$. We may assume that $V(T) \cup \{v_1\}$ is elementary with respect to φ, since otherwise we are in Case 1. Hence, we have $\beta \notin \bar{\varphi}(V(T)) = \bar{\varphi}_1(V(T))$ and $\beta \in \bar{\varphi}_1(v_1)$. In particular, $\beta \notin \{\alpha, \alpha', \gamma\}$. Since $\delta \notin \bar{\varphi}_1(v_1)$, we also have $\beta \neq \delta$. Now let $P_2 = P_{v_1}(\alpha', \beta, \varphi_1)$. Clearly, P_2 is a path where one endvertex is v_1. Let z be the other endvertex of P_2.

Case 2a: $z \neq u'$. Since $\bar{\varphi}_1(z) \cap \{\alpha', \beta\} \neq \emptyset$ and $V(T)$ is elementary with respect to φ_1, this implies that $z \notin V(T)$. The colours α' and β both are unused on T with respect to φ_1. Consequently, T is a Tashkinov tree with respect to e and $\varphi_2 = \varphi_1/P_2$, and we have $(T, e, \varphi_2) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi_2)$, $\gamma \in \bar{\varphi}_2(u)$, $P = P_u(\gamma, \delta, \varphi_2)$, $\alpha' \in \bar{\varphi}_2(u') \cap \bar{\varphi}_2(v_1)$, and α', γ remain unused on T with respect to φ_2. Now we can continue as in Case 1 with φ_2 instead of φ, and with the colour α'. Hence, we can recolour in the same way, that is, $\varphi_3 = \varphi_2/P_{v_1}(\alpha', \gamma, \varphi_2)$ and $\varphi' = \varphi_3/P_u(\gamma, \delta, \varphi_3)$, where φ' is the desired colouring. Since we used four Kempe changes to derive φ' from φ, this settles Case 2a.

Case 2b: $z = u'$. Then let $P_3 = P_{v_2}(\alpha', \beta, \varphi_1)$. Since $\alpha' \in \bar{\varphi}_1(v_2)$, we obtain that v_2 is an endvertex of P_3, and $P_3 \neq P_2$. Therefore, the second endvertex z' of P_3 is distinct from u'. Since $\bar{\varphi}_1(z') \cap \{\alpha', \beta\} \neq \emptyset$, and $V(T)$ is elementary with respect to φ_1, this implies that $z' \notin V(T)$. The colours α' and β both are unused on T with respect to φ_1. Consequently, T is a Tashkinov tree with respect to e and $\varphi_2 = \varphi_1/P_3$, and we have $(T, e, \varphi_2) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi_2)$, $\gamma \in \bar{\varphi}_2(u)$, $P = P_u(\gamma, \delta, \varphi_2)$, $\beta \in \bar{\varphi}_2(v_1) \cap \bar{\varphi}_2(v_2)$, and γ remains unused on T with respect to φ_2. Now we can continue as in Case 1 with φ_2 instead of φ, and with the colour β. Hence we can recolour in the same way, that is, $\varphi_3 = \varphi_2/P_{v_1}(\beta, \gamma, \varphi_2)$ and $\varphi' = \varphi_3/P_u(\gamma, \delta, \varphi_3)$, where φ' is the desired colouring. Since we used four Kempe changes to derive φ' from φ, this settles Case 2b. Now all cases are done, and (c) is proved.

To prove (d), we assume that $X = V(T) \cup \{v_1, v_2\}$ is elementary with respect to φ. From $\delta \in \Gamma^d(T, e, \varphi)$ it follows, by Proposition 2.21, that $\delta \notin \bar{\varphi}(V(T))$ and $|E_\delta(T, e, \varphi)| \geq 3$. Since $E_\delta(T, e, \varphi) \subseteq E(P)$, this implies in particular that v_2 is not an endvertex of P. Hence, we have $\delta \notin \bar{\varphi}(\{v_1, v_2\}$ and, therefore, $\delta \notin \bar{\varphi}(X)$. This implies that $|\bar{\varphi}(X)| \leq k-1$. Moreover, since X is elementary with respect to φ, we have

$$|\bar{\varphi}(X)| = \sum_{z \in X}(k - d_{G-e}(z)) = 2 + \sum_{z \in X}(k - d_G(z)) \geq 2 + |X|(k - \Delta(G)).$$

From this an $|\bar{\varphi}(X)| \leq k-1$ we then obtain

$$|X| \leq \frac{k-3}{k-\Delta(G)} = \frac{\Delta(G)-3}{k-\Delta(G)} + 1.$$

This proves (d).

To prove (e) we assume that $\bar{\varphi}(v_0) \cap \Gamma^f(T, e, \varphi) = \emptyset$. Then, clearly, all colours of $\bar{\varphi}(v_0)$ are used on T with respect to φ. In the case that v_0 is an endvertex of e, we have $|\bar{\varphi}(v_0)| = k - d_G(v_0) + 1$ and, therefore, at least $k - d_G(v_0) + 1$ colours are used on T with respect to φ. Now consider the other case, where v_0 is not an endvertex of e. If $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$, this means that $v_0 = y_i$ for an index $i \geq 2$. Hence, the colour $\alpha = \varphi(e_i)$ is used on T with respect to φ. Since e_i is incident with $y_i = v_0$, the colour α does not belong to $\bar{\varphi}(v_0)$ and, therefore, at least $\bar{\varphi}(v_0) + 1 = k - d_G(v_0) + 1$ colours are used on T with respect to φ. This completes the proof of (e). ∎

Since Theorem 2.23(b) works if there is a free colour missing at a certain vertex, it might be useful to reduce the number of used colours in a Tashkinov tree. The following simple result shows how to do that. In fact it shows that, for every triple $(T, e, \varphi) \in \mathcal{T}_k(G)$, the vertex set of T is a unique set, completely determined by e and φ. It does not depend on the structure of T at all.

Lemma 2.24 *Let G be a graph, let $k \geq \Delta(G) + 1$ be an integer, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$. Further, let $T' = (y_0, e_1, y_1, \ldots, e_s, y_s)$ be an arbitrary Tashkinov tree with respect to e and φ. Then the following statements hold:*

(a) *$V(T') \subseteq V(T)$.*

(b) *There is a Tashkinov tree T'' with respect to e and φ such that $V(T'') = V(T)$, $(T'', e, \varphi) \in \mathcal{T}_k(G)$, and $T''y_s = T'$.*

(c) *There is a Tashkinov tree $\tilde{T}$ with respect to e and φ such that $V(\tilde{T}) = V(T)$, $(\tilde{T}, e, \varphi) \in \mathcal{T}_k(G)$, and at most $\frac{|V(T)|-1}{2}$ colours are used on $\tilde{T}$ with respect to φ.*

Proof: Suppose that (a) is not true. Then there is a smallest index $i \in \{0, \ldots, s\}$ such that $y_i \notin V(T)$. Since the endvertices y_0, y_1 of $e_1 = e$ belong to $V(T)$, we conclude that $i \geq 2$. Hence, by definition, we have $e_i \in E_G(V(T'y_{i-1})), y_i)$ and

$\varphi(e_i) \in \bar{\varphi}(V(T'y_{i-1}))$. Because of the choice of i, we have $V(T'y_{i-1}) \subseteq V(T)$ and, therefore, $e_i \in E_G(V(T), V(G) - V(T))$ and $\varphi(e_i) \in \bar{\varphi}(V(T))$. Hence, $V(T)$ is not closed with respect to φ, a contradiction to Proposition 2.21(a). This proves (a).

Clearly, T' can simply be extended to a maximal Tashkinov tree T'' with respect to e and φ. Then, by (a), we have $V(T'') \subseteq V(T)$. Since $V(T)$ is elementary with respect to φ, this implies that $V(T'')$ is elementary with respect to φ as well. Consequently, we have $(T'', e, \varphi) \in \mathcal{T}_k(G)$. Then, again from (a) but with exchanged roles of T'' and T, we conclude that $V(T) \subseteq V(T'')$. Hence $V(T'') = V(T)$. This proves (b).

Eventually, we prove (c). From $k \geq \Delta(G) + 1$ we infer that there is a colour $\alpha \in \bar{\varphi}(y_0)$. Since $y_0, y_1 \in V(T)$ and $V(T)$ is elementary with respect to φ, there is an edge $f_2 \in E_G(y_1, z_2)$ with $z_2 \in V(G)\setminus\{y_0, y_1\}$ and $\varphi(f) = \alpha$. Consequently, $T_1 = (y_0, e_1, y_1, f_2, z_2)$ is a Tashkinov tree with respect to e and φ. Moreover, only $\frac{|V(T_1)|-1}{2} = 1$ colours are used on T_1 with respect to φ. Starting from T_1 we can successively extend the Tashkinov tree as follows.

For $j \geq 0$, let $T_j = (y_0, e_1, y_1, f_2, z_2, \ldots, f_{2j}, z_{2j})$ be a Tashkinov tree with respect to e and φ such that at most j colours are used on T_j with respect to φ. By (a), we have $V(T_j) \subseteq V(T)$. If $|V(T_j)| < |V(T)|$ then it follows from (b) that there is an edge $f_{2j+1} \in E_G(V(T_j), z_{2j+1})$ with $z_{2j+1} \in V(T)\setminus V(T_j)$ and $\varphi(f_{2j+1}) \in \bar{\varphi}(V(T_j))$. Hence, $T'_j = (T_j, f_{2j+1}, z_{2j+1})$ is a Tashkinov tree with respect to e and φ, and at most $j+1$ colours are used on T with respect to φ. Let α_j be a colour that is used on T'_j with respect to φ. Then $\alpha_j \in \bar{\varphi}(V(T'_j))$. Since $V(T'_j) \subseteq V(T)$ is elementary with respect to φ, this implies that the colour α_j is missing at exactly one vertex of T'_j. Moreover, $|V(T'_j)|$ is even. Hence, there is an edge $f_{2j+2} \in E_G(V(T'_j), V(G)\setminus V(T'_j))$ with $\varphi(f) = \alpha_j$. Let z_{2j+2} be the endvertex of f_{2j+2} that belongs to $V(G)\setminus V(T'_j)$, and let $T_{j+1} = (T'_j, f_{2j+2}, z_{2j+2})$. Evidently, T_{j+1} is a Tashkinov tree with respect to e and φ. Moreover, since α_j is already used on T'_j, there are at most $j+1$ colours used on T_{j+1} with respect to φ.

If $|V(T)| = |V(T_j)| = 2j + 1$ then (a) implies that $V(T_j) = V(T)$ and, therefore, $(T_j, e, \varphi) \in \mathcal{T}_k(G)$. Since there are at most $\frac{|V(T)|-1}{2}$ colours used on T_j with respect to φ, (c) is proved. ■

Using the methods of Theorem 2.23 and Lemma 2.24, we can improve the algorithm TashExt1. This leads to the following algorithm **TashExt2**.

TashExt2(G, e, x, y, k, φ):

1) $p \leftarrow 1$, $e_p \leftarrow e$, $y_p \leftarrow y$, $y_0 \leftarrow x$, $T \leftarrow (y_0, e_p, y_p)$.

2) `If` $\bar{\varphi}(V(Ty_{p-1})) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  2a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.20.`

  2b) `Return` (k, φ').

3) `If` $\exists e_{p+1} \in E_G(V(T), V(G)\setminus V(T)) : \varphi(e_{p+1}) \in \varphi(E(T))$ `then`

  3a) `Let` y_{p+1} `be the endvertex of` e_{p+1} `that is not in` $V(T)$.

  3b) $T \leftarrow (T, e_{p+1}, y_{p+1})$, $p \leftarrow p + 1$.

  3c) `Goto 2.`

4) `If` $\exists e_{p+1} \in E_G(V(T), V(G)\setminus V(T)) : \varphi(e_{p+1}) \in \bar{\varphi}(V(T))\setminus\varphi(E(T))$ `then`

4a) `Let` y_{p+1} `be the endvertex of` e_{p+1} `that is not in` $V(T)$.

4b) $T \leftarrow (T, e_{p+1}, y_{p+1})$, $p \leftarrow p+1$.

4c) `Goto 2.`

5) `If` $\Gamma^d(T, e, \varphi) = \emptyset$ `then`

5a) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k+1$.

5b) `Return` $(k+1, \varphi')$.

6) `Choose` $\delta \in \Gamma^d(T, e, \varphi)$ `and` $\gamma \in \Gamma^f(T, e, \varphi)$.
`Let` $u \in V(T)$ `with` $\gamma \in \bar{\varphi}(u)$, `and set` $P \leftarrow P_u(\gamma, \delta, \varphi)$.

7) `If` $E_\delta(T, e, \varphi) \nsubseteq E(P)$ `then`

7a) `Compute` $\varphi' = \varphi/P$, `and set` $\varphi \leftarrow \varphi'$.

7b) `Goto 3.`

8) `Set` v_0, v_1, v_2 `according to Theorem 2.23(a).`

9) `If` $\bar{\varphi}(v_0) \cap \Gamma^f(T, e, \varphi) \neq \emptyset$ `then`

9a) `Compute` $\varphi' \in \mathcal{C}_k(G-e)$ `as in Theorem 2.23(b), and set` $\varphi \leftarrow \varphi'$.

9b) `Goto 3.`

10) `If` $V(T) \cup \{v_1, v_2\}$ `is not elementary with respect to` φ `then`

10a) `Compute` $\varphi' \in \mathcal{C}_k(G-e)$ `as in Theorem 2.23(c), and set` $\varphi \leftarrow \varphi'$.

10b) `Goto 3.`

11) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k+1$.

12) `Return` $(k+1, \varphi')$.

Theorem 2.25 *Let G be a graph, let $e \in E_G(x,y)$, and let $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. On the input (G, e, x, y, k, φ), the algorithm* TashExt2 *returns a tuple (k', φ') with $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}_{k'}(G)$. Moreover, if $k' = k+1$ then $\mathcal{W}(G) = k+1$, or there is a triple $(T, e, \varphi^*) \in \mathcal{T}_k(G)$ satisfying:*

$$|V(T)| \leq \frac{\Delta(G)-3}{k-\Delta(G)} - 1 \tag{2.15}$$

and

$$|V(T)| \geq 2(k - \Delta(G)) + 3. \tag{2.16}$$

Proof: We will refer to all variables as they are valued in the current state of the algorithm, not as they are valued in the input. Note that the value of k is never changed during the algorithm. In particular, we always have $k \geq \Delta(G)+1$ and will use this fact without mentioning it every time. Moreover, if TashExt2 terminates then it will always return a value $k' \in \{k, k+1\}$. Hence, this part of the proof is already done. Now we will prove the correctness of the algorithm TashExt2. To do this, we first formulate some conditions that has to be fulfilled when entering the several steps of the algorithm. Then we show that these conditions are always fulfilled during the algorithm and that, under these conditions, the algorithm gives the required output. The entering conditions are the following:

(C1) When entering step 2, we have $p \geq 1$, and $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ is a Tashkinov tree with respect to e and φ such that $V(Ty_{p-1})$ is elementary with respect to φ.

(C2) When entering step 3 or step 4, T is Tashkinov tree with respect to e and φ such that $V(T)$ is elementary with respect to φ.

(C3) When entering step 5, we have $(T, e, \varphi) \in \mathcal{T}_k(G)$.

(C4) When entering step 7, we have $(T, e, \varphi) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi)$, $\gamma \in \bar{\varphi}(u)$, $\gamma \in \Gamma^f(T, e, \varphi)$, and $P = P_u(\gamma, \delta, \varphi)$.

(C5) When entering step 9, we have $(T, e, \varphi) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi)$, $\gamma \in \bar{\varphi}(u)$, $\gamma \in \Gamma^f(T, e, \varphi)$, $P = P_u(\gamma, \delta, \varphi)$, $E_\delta(T, e, \varphi) \subseteq E(P)$, and v_0, v_1, v_2 are the vertices as defined in Theorem 2.23(a).

(C6) When entering step 10, we have $(T, e, \varphi) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T, e, \varphi)$, $\gamma \in \Gamma^f(T, e, \varphi)$, $\gamma \in \bar{\varphi}(u)$, $P = P_u(\gamma, \delta, \varphi)$, $E_\delta(T, e, \varphi) \subseteq E(P)$, v_0, v_1, v_2 are the vertices as defined in Theorem 2.23(a), and $\bar{\varphi}(v_0) \cap \Gamma^f(T, e, \varphi) = \emptyset$.

After step 1 of TashExt2, $T = (x, e, y)$ is the trivial Tashkinov tree with respect to e and φ, and condition (C1) is satisfied. Now let step 2 be entered with fulfilled condition (C1). If $\bar{\varphi}(V(T)) \cap \bar{\varphi}(y_p) \neq \emptyset$ then the requirements of Theorem 2.20 are fulfilled. Then, in step 2a, TashExt2 computes a k-edge colouring φ' of G and returns (k, φ'). In the other case, the check in step 2 fails. Then $V(T)$ is elementary with respect to φ and, therefore, condition (C2) is satisfied.

Let step 3 be entered with satisfied condition (C2). Then, in this step, the algorithm checks whether there is an edge joining $V(T)$ and $V(G) \setminus V(T)$ that is coloured with a colour already used on T with respect to φ. If yes then the order of T is increased, condition (C1) is satisfied again, and the algorithm starts over at step 2. If not then step 4 checks whether there is an edge in $E_G(V(T), V(G) \setminus V(T))$, coloured with a colour in $\bar{\varphi}(V(T))$, but not used on T with respect to φ. If yes then the order of T is increased, condition (C1) is satisfied again, and the algorithm starts over at step 2. If not, then the checks of step 3 and step 4 both failed and, therefore, $V(T)$ is closed with respect to φ. Consequently, we have $(T, e, \varphi) \in \mathcal{T}_k(G)$ and hence condition (C3) is satisfied.

Let step 5 be entered with fulfilled condition (C3). If the check fails then there is no defective colour with respect to (T, e, φ) and, by Proposition 2.21(c), $V(T)$ is strongly closed with respect to φ. Hence TashExt2 terminates with a new colour used for the edge e. In the other case, TashExt2 chooses in step 6 an existing defective colour δ as well as a free colour γ with respect to (T, e, φ). The existence of γ follows from Proposition 2.21(f). Then, after finding u and P, condition (C4) is clearly satisfied.

Now let step 7 be entered with fulfilled condition (C4). If $E_\delta(T, e, \varphi) \nsubseteq E(P)$ then TashExt2 computes a k-edge colouring $\varphi' = \varphi/P$ of $G - e$. By Proposition 2.22, T is a Tashkinov tree with respect to e and φ', and $V(T)$ is elementary, but not closed with respect to φ'. After φ is set to φ', condition (C2) is satisfied, and TashExt2 starts over at step 3. In the other case we have $E_\delta(T,e,\varphi) \subseteq E(P)$, and the algorithm finds v_0, v_1, v_2 in the next step. Hence, condition (C5) is fulfilled.

Let step 9 be entered with satisfied condition (C5). If $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ then the requirements of Theorem 2.23(b) are fulfilled and, therefore, TashExt2 computes a k-edge colouring φ' of $G-e$ such that T is a Tashkinov tree with respect to e and φ', and $V(T)$ is elementary but not closed with respect to φ'. After φ is set to φ', condition (C2) is satisfied, and TashExt2 starts over at step 3. In the other case we have $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) = \emptyset$ and, therefore, condition (C6) is satisfied.

Let step 10 be entered with fulfilled condition (C6). If $V(T) \cup \{v_1, v_2\}$ is not elementary with respect to φ, then the requirements of Theorem 2.23(c) are satisfied and, therefore, TashExt2 computes a k-edge colouring φ' of $G-e$ such that T is a Tashkinov tree with respect to e and φ', and $V(T)$ is elementary, but not closed with respect to φ'. After φ is set to φ', condition (C2) is satisfied, and TashExt2 starts over at step 3. In the other case $V(T) \cup \{v_1, v_2\}$ is elementary with respect to φ. Then the algorithm uses a new colour for the edge e and terminates.

Now we only have to show that the algorithm terminates. Then, since the conditions (C1)-(C6) are always satisfied, the algorithm works correctly. Due to several goto-statements the algorithm may jump to step 2 or step 3 at some points. There are no other loops. If the algorithm jumps to step 2 then it came either from step 3c or step 4c. In both cases the value of p was increased before. Since p is never decreased and $p+1 = |V(T)| \leq |V(G)|$, this implies that there is only a finite number of jumps to step 2. If TashExt2 jumps to step 3 then it comes from step 7b, 9b, or 10b. In any of these cases, see above, the set $V(T)$ is not closed with respect to φ and, therefore, the checks in step 3 and 4 cannot both fail. Then the algorithm has to jump back to step 2 again, but this is possible only a finite number of times. Consequently, the algorithm TashExt2 has to terminate at some point. This, eventually, proves the first part of the theorem.

To prove the second part, we assume that TashExt2 returns (k', φ') with $k' = k+1$. Then the algorithm terminates either in step 5b or in step 12.

In the first case, see above, we have $(T,e,\varphi) \in \mathcal{T}_k(G)$ and, therefore, the set $V(T)$ is elementary and strongly closed with respect to $\varphi \in \mathcal{C}_k(G-e)$. By Proposition 2.21(b), $|V(T)| \geq 3$ is odd. Since $V(T)$ is elementary and closed with respect to φ, for every colour $\alpha \in \bar{\varphi}(V(T))$ there are $\frac{|V(T)|-1}{2}$ edges in $E_G(V(T), V(T))$ coloured with α. Since $V(T)$ is also strongly closed with respect to φ, it follows from Proposition 2.21(c) that $\Gamma^d(T,e,\varphi) = \emptyset$. Then, for every colour $\alpha \notin \bar{\varphi}(V(T))$, we have $|E_\alpha(T,e,\varphi)| = 1$. Since α is present at every vertex of $V(T)$, this implies that there are $\frac{|V(T)|-1}{2}$ edges in $E_G(V(T), V(T))$ coloured with α. Consequently, the induced subgraph $H = G[V(T)]$ of G contains $k\frac{|V(T)|-1}{2}$ edges plus the edge e. From this we infer that $\mathcal{W}(G) \geq \mathcal{W}(H) \geq k+1$. Since $\mathcal{W}(G) \leq \chi'(G)$ and φ' is a $(k+1)$-edge colouring of G, this implies $\mathcal{W}(G) = k+1$.

In the other case TashExt2 terminates in step 12. Then, see above, we have $(T,e,\varphi) \in \mathcal{T}_k(G)$, $\delta \in \Gamma^d(T,e,\varphi)$, $E_\delta(T,e,\varphi) \subseteq E(P)$, $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) = \emptyset$, and $V(T) \cup \{v_1, v_2\}$ is elementary with respect to φ. Then (2.15) follows directly from Theorem 2.23(d). Moreover, in steps 7a, 9a and 10a of the algorithm TashExt2, the number of used colours on T is never increased with respect to the new colouring, see proof of Theorem 2.23. Hence the number of used colours on T could only change while increasing the order of T. Step 3b uses an already used colour to increase T. Only in step 4b a new colour is used, but then the check in step 3 must have failed. Consequently, the algorithm uses new colours only if necessary and hence, see proof of Lemma 2.24(c), at most $\frac{|V(T)|-1}{2}$ colours are used on T with respect to φ. Moreover, since $\bar{\varphi}(v_0) \cap \Gamma^f(T, e, \varphi) = \emptyset$, Theorem 2.23(e) implies that at least $k - d_G(v_0) + 1 \geq k - \Delta(G) + 1$ colours are used on T with respect to φ. Consequently, we have $\frac{|V(T)|-1}{2} \geq k - \Delta(G) + 1$ and, therefore, $|V(T)| \geq 2(k - \Delta(G)) + 3$. This, eventually proves (2.16). ∎

Now we have to analyse the running time of TashExt2 on a valid input, that is, an input (G, e, x, y, k, φ) with $|V(G)| = n$ and $\Delta(G) = \Delta$. For this, we can consider $k \in \mathrm{O}(\Delta)$, see Section 2.2. Since T is always a Tashkinov tree with respect to e and φ and $|V(T)| = p + 1$, it follows that $p \leq n$. Since $k \geq \Delta(G) + 1$ and $V(Ty_{p-1})$ is always elementary with respect to φ, we also infer that $p \leq k$. Consequently, we have $p \in \mathrm{O}(\min\{n, \Delta\})$.

Now we give some thoughts about maintaining some extra information during the algorithm. For every colour, we store the information whether it is missing in $\bar{\varphi}(V(T))$ or not and, in the first case, at which vertex it is missing. This can easily be updated when adding an edge to T or performing a Kempe change. We also store, for every colour, the number of edges of this colour joining $V(T)$ and $V(G) \setminus V(T)$. When increasing the order of T this can be updated in time $\mathrm{O}(\Delta)$ by scanning the adjacency list of the new vertex. When performing a Kempe change it can be updated in time $\mathrm{O}(n)$, less than the time for the Kempe change itself. Clearly, the checks in the steps 3, 4, 5, and 7 then only need time $\mathrm{O}(\Delta)$.

Step 2 needs time $\mathrm{O}(\Delta)$. By Theorem 2.20, step 2a requires $\mathrm{O}(\min\{n^3, \Delta^3\})$ Kempe changes. That gives a running time of $\mathrm{O}\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$. Step 6 needs time $\mathrm{O}(n + \Delta)$. From Theorem 2.23 we infer that the steps 7a, 9a, and 10a require only a constant number of Kempe changes and, therefore, time $\mathrm{O}(n + \Delta)$. The steps 9 and 10 only need time $\mathrm{O}(\Delta)$.

Due to several goto-statements, some of these steps are repeated. When the algorithm returns to step 2, the value of p was increased before. Hence this happen only $\mathrm{O}(\min\{n, \Delta\})$ times. When the algorithm returns to step 3, $V(T)$ is not closed with respect to φ, and the algorithm reaches either step 3c or step 4c and jumps back to step 2; see also the proof of Theorem 2.25. Consequently, every step of the algorithm is repeated at most $\mathrm{O}(\min\{n, \Delta\})$ times. Clearly, step 2a is performed at most once. This leaves the time of $\mathrm{O}\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$ for this step. The rest of the algorithm has a running time of $\mathrm{O}\big((n + \Delta)\min\{n, \Delta\}\big)$. This gives a total running time of $\mathrm{O}\big((n + \Delta)\min\{n^3, \Delta^3\}\big)$.

Using the algorithm TashExt2 as kernel, we can construct a new edge colouring algorithm Tashkinov2 that will attain some new bounds which follow from Theorem 2.25.

Tashkinov2(G):

1) `If` $\Delta(G) \leq 2$ `then compute an optimal colouring` φ `of` G `and return` φ`.`

2) `Let` G' `be the edgeless graph with` $V(G') = V(G)$`,`
`let` φ `be the empty colouring of` G'`, and let` $k = \Delta(G) + 1$`.`

3) `For every edge` $e \in E(G)$ `do`

  3a) `Let` x, y `be the two endvertices of` e`.`

  3b) $E(G') \leftarrow E(G') \cup \{e\}$

  3c) $(k, \varphi) \leftarrow$ TashExt2$(G', e, x, y, k, \varphi)$

4) `Return` φ`.`

Theorem 2.26 *Let G be a graph. Then, on the input G,* Tashkinov2 *returns an edge colouring of G using at most $\tau(G)$ colours. Hence, the parameter τ is an efficiently realizable upper bounds of χ'.*

Proof: If $\Delta(G) \leq 2$ then $\chi'(G) = \mathcal{W}(G) \leq \tau(G)$ and, since Tashkinov2 returns an optimal edge colouring of G, we are done. Now consider the case $\Delta(G) \geq 3$. Then the algorithm starts with k colours, where

$$k = \Delta(G) + 1 \leq \Delta(G) + \sqrt{\frac{\Delta(G) - 1}{2}} \leq \tau(G).$$

Hence, if the algorithm never uses a new colour, we are done. Otherwise there is a last call of the subroutine TashExt2 where a new colour is used. The input is then a tuple (H, e, x, y, k, φ), where H is a subgraph of G and $k \geq \Delta(H) + 1$. The output is a $(k+1)$-edge colouring of H, and we have $\chi'(G) \leq k + 1$. By Theorem 2.25, we have two possible cases. The first case is that $\mathcal{W}(H) = k + 1$. This implies

$$\mathcal{W}(G) \geq k + 1$$

and we are done. The second case is that there is a triple $(T, e, \varphi^*) \in \mathcal{T}_k(H)$ satisfying

$$2(k - \Delta(H)) + 3 \leq |V(T)| \leq \frac{\Delta(H) - 3}{k - \Delta(H)} - 1.$$

This implies

$$(k - \Delta(H))^2 + 2(k - \Delta(H)) \leq \frac{\Delta(H) - 3}{2}$$

and hence $k - \Delta(H) + 1 \leq \sqrt{\frac{\Delta(H)-1}{2}}$. From this and $\Delta(H) \leq \Delta(G)$ we then conclude that

$$\chi'(G) \leq k + 1 \leq \Delta(G) + \sqrt{\frac{\Delta(G) - 1}{2}} \leq \tau(G).$$

Consequently, the computed colouring uses at most $\tau(G)$ colours.

Since the time complexity of Tashkinov2 is bounded by a polynomial in the number of vertices and edges of the input graph, τ is an efficiently realizable upper bound of χ'. This completes the proof. ■

A consequence of Theorem 2.26 is the following result about an asymptotic approximation of the chromatic index for graphs with sufficiently large maximum degree. In particular, it supports Goldberg's conjecture asymptotically and extends a result of Kahn [18], see Theorem 1.3.

Corollary 2.27 *Let $\epsilon > 0$, and let G be a graph with $\Delta(G) \geq \frac{1}{2\epsilon^2}$. Then, on the input G, the algorithm* Tashkinov2 *returns an edge colouring of G using at most $\tilde{\tau}(G)$ colours, where $\tilde{\tau}(G) = \max\{(1+\epsilon)\Delta(G), \mathcal{W}(G)\} \leq (1+\epsilon)\chi'_f(G)$.*

Proof: Suppose that $\epsilon > 0$ and $\Delta(G) \geq \frac{1}{2\epsilon^2}$. Then $\epsilon \geq \sqrt{\frac{1}{2\Delta(G)}}$ and, therefore,

$$(1+\epsilon)\Delta(G) = \Delta(G) + \epsilon\Delta(G) \geq \Delta(G) + \sqrt{\frac{\Delta(G)}{2}}.$$

This implies that

$$\tilde{\tau}(G) = \max\{(1+\epsilon)\Delta(G), \mathcal{W}(G)\} \geq \max\{\Delta(G) + \sqrt{\frac{\Delta(G)}{2}}, \mathcal{W}(G)\} = \tau(G).$$

Hence, by Theorem 2.26, Tashkinov2 returns an edge colouring of G using at most $\tilde{\tau}(G)$ colours.

If $\Delta(G) \leq 1$ then we have $\chi'_f(G) = \mathcal{W}(G) = \Delta(G)$ and, therefore, we obtain $\tilde{\tau}(G) \leq \max\{(1+\epsilon)\Delta(G), \mathcal{W}(G)\} \leq (1+\epsilon)\chi'_f(G)$. If $\Delta(G) \geq 2$ then $\Delta(G) \geq \frac{1}{2\epsilon}$ implies that $\epsilon\Delta(G) \geq 1$. By (1.1), we have $\chi'_f(G) \geq \Delta(G)$ and $\lceil\chi'_f(G)\rceil \geq \mathcal{W}(G)$. Consequently, we obtain $(1+\epsilon)\chi'_f(G) \geq \chi'_f(G) + \epsilon\Delta(G) \geq \chi'_f(G) + 1 \geq \mathcal{W}(G)$ and, therefore, $\tilde{\tau}(G) \leq \max\{(1+\epsilon)\Delta(G), \mathcal{W}(G)\} \leq (1+\epsilon)\chi'_f(G)$. This completes the proof. ■

The next two results are improvements of results due to Sanders and Steurer [25]. The first one is a simple consequence of Theorem 2.26, the second one states that, for every $\epsilon > 0$, the parameter τ_ϵ is realized by the algorithm Tashkinov2.

Corollary 2.28 *The parameter $\chi'_f + \sqrt{\frac{1}{2}\chi'_f}$ is an efficiently realizable upper bound of the chromatic χ'.*

Proof: Let G be a graph. If $\Delta(G) \leq 1$ then, clearly, we have $\chi'_f(G) = \mathcal{W}(G) = \Delta(G)$. In the other case, if $\Delta(G) \geq 2$ then we obtain from $\chi'_f(G) \geq \Delta(G)$ and $\lceil\chi'_f(G)\rceil \geq \mathcal{W}(G)$ that

$$\chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f} \geq \chi'_f(G) + 1 \geq \mathcal{W}(G).$$

Consequently, in any case, we have

$$\chi'_f(G) + \sqrt{\frac{1}{2}\chi'_f} \geq \tau(G)$$

and, therefore, the desired result follows from Theorem 2.26. This completes the proof. ■

Theorem 2.29 *Let $\epsilon > 0$, and let G be a graph. Then, on the input G, the algorithm* Tashkinov2 *returns an edge colouring of G using at most $\tau_\epsilon(G)$ colours. Hence, the parameter τ_ϵ is an efficiently realizable upper bounds of χ'.*

Proof: If $\Delta(G) \leq 2$ then $\chi'(G) = \mathcal{W}(G)$ and, since TASHKINOV2 returns an optimal edge colouring of G, we are done. Let us consider the case $\Delta(G) \geq 3$. Then the algorithm starts with a value of $k = \Delta(G) + 1 \leq (1+\epsilon)\Delta(G) + 1 - 3\epsilon \leq \tau_\epsilon(G)$. Hence, if k is never increased, we are done, too. If it is increased then there is a last time that k is increased to $k+1$ by the subroutine TASHEXT2 for an input (H, e, x, y, k, φ) where H is a subgraph of G. Then, by Theorem 2.25, there are two possible cases. The first one is that we have $\mathcal{W}(H) = k+1$, implying $\mathcal{W}(G) \geq k+1$. This settles the case. In the other case Theorem 2.25 implies that there is a triple $(T, e, \varphi^*) \in \mathcal{T}_k(H)$ satisfying

$$2(k - \Delta(H)) + 3 \leq |V(T)| \leq \frac{\Delta(H) - 3}{k - \Delta(H)} - 1.$$

In particular, this implies $\Delta(H) \geq 3$. Now we distinguish two cases. If $|V(T)| \leq \frac{1}{\epsilon} - 1$ then we infer that

$$k + 1 \leq \Delta(H) + \frac{|V(T)| - 1}{2} \leq \Delta(H) - 1 + \frac{1}{2\epsilon}.$$

Since $\Delta(H) \leq \Delta(G)$ and k is an integer, it follows that $k + 1 \leq \lfloor \Delta(G) - 1 + \frac{1}{2\epsilon} \rfloor \leq \tau_\epsilon(G)$. If otherwise $|V(T)| > \frac{1}{\epsilon} - 1$ then we have

$$\frac{\Delta(H) - 3}{k - \Delta(H)} \geq |V(T)| + 1 > \frac{1}{\epsilon}$$

and, therefore, $k < (1+\epsilon)\Delta(H) - 3\epsilon$. Since $\Delta(H) \leq \Delta(G)$ and k is an integer, this implies $k+1 \leq (1+\epsilon)\Delta(G) + 1 - 3\epsilon \leq \tau_\epsilon(G)$. Consequently, the computed colouring uses at most $\tau_\epsilon(G)$ colours.

Since the time complexity of TASHKINOV2 is bounded polynomially in the number of vertices and edges of the input graph, τ_ϵ is an efficiently realizable upper bounds of χ'. This completes the proof. ∎

3 Goldberg's Conjecture

3.1 On the 15/14 Edge Colouring of Graphs

In this section we will extend Tashkinov's methods to prove a result related to Goldberg's conjecture 1.2. An equivalent formulation of this conjecture can be obtained as follows. For an integer $m \geq 3$, let $\mathcal{J}_m$ denote the class of all graphs G with

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}.$$

Then for every integer $m \geq 3$ we have $\mathcal{J}_m \subseteq \mathcal{J}_{m+1}$. Moreover, the class

$$\mathcal{J} = \bigcup_{m=3}^{\infty} \mathcal{J}_m$$

consists of all graphs G with $\chi'(G) \geq \Delta(G) + 2$. Consequently, Goldberg's conjecture is equivalent to the following conjecture.

Conjecture 3.1 *For every integer $m \geq 3$, every graph G with*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

is elementary.

Up to now, this conjecture is known to be true for $m \leq 13$. It was proved for $m = 5$ by Sørensen (unpublished), Andersen [1] and Goldberg [9], for $m = 7$ by Sørensen (unpublished) and Andersen [1], for $m = 9$ by Goldberg [11], for $m = 11$ by Nishizeki and Kashiwagi [22] and by Tashkinov [38] and, eventually, for $m = 13$ by Favrholdt, Stiebitz and Toft [8]. We will make the next step and show the following result.

Theorem 3.2 *Every graph G with*

$$\chi'(G) > \frac{15}{14}\Delta(G) + \frac{12}{14}$$

is elementary.

Corollary 3.3 *Every graph G satisfies*

$$\chi'(G) \leq \max\left\{ \left\lfloor \frac{15}{14}\Delta(G) + \frac{12}{14} \right\rfloor, \mathcal{W}(G) \right\}.$$

To prove Theorem 3.2, we will further extend Tashkinov's methods by generalizing the ideas of Theorem 2.23. Most of these improvements were already introduced by Favrholdt, Stiebitz and Toft in [8].

Clearly, the statements of Theorem 3.2 and Corollary 3.3 are equivalent. From Proposition 1.1 it then follows that it is sufficient to prove Theorem 3.2 for critical graphs. Let us mention a further consequence of Theorem 3.2.

Corollary 3.4 *Let $3 \leq m \leq 15$ be an odd integer. If G is a critical graph such that*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1},$$

then $|V(G)| \leq m-2$ is odd.

Proof: By Theorem 3.2, G is an elementary graph, that is, $\chi'(G) = \mathcal{W}(G)$. Since G is critical, every proper subgraph H of G satisfies $\mathcal{W}(H) \leq \chi'(H) < \chi'(G) = \mathcal{W}(G)$. This implies that

$$\chi'(G) = \mathcal{W}(G) = \left\lceil \frac{|E(G)|}{\lfloor \frac{1}{2}|V(G)| \rfloor} \right\rceil.$$

Consequently, $|V(G)|$ is odd, since otherwise $\chi'(G) = \lceil 2|E(G)|/|V(G)| \rceil \leq \Delta(G)$, a contradiction to the assumption $G \in \mathcal{J}_m$. From Theorem 1.4 and Proposition 1.5 it then follows that $|V(G)| \leq m-2$. ■

That the above result holds for every odd integer $m \geq 3$ was conjectured in 1974 by Jakobsen [16].

Conjecture 3.5 *Let $m \geq 3$ be an odd integer. Every critical graph $G \in \mathcal{J}_m$ has at most $m-2$ vertices.*

3.2 Tashkinov Trees in Critical Graphs

Although, in the last section, we introduced Tashkinov's methods totally from an algorithmic point of view, we will now change this and concentrate more on the analysis of critical graphs. As explained in Section 2.1, this gives some advantages in formulating and proving results. Since the results deal with optimal colourings in critical graphs, the results are weaker than algorithmic versions. However, it will always be possible to adapt the methods of the proofs to arbitrary colourings and to construct a corresponding algorithm. In principle, this will always work in the following way. A result states that in a critical graph G with $\chi'(G) = k + 1$ some condition is fulfilled. Then the proof will show how a k-edge colouring of G could be constructed otherwise. We will always discuss how this can be translated into a similar recolouring method for a non-critical graph.

First we will reformulate some results of Section 2.6 and Section 2.7 for critical graphs. One simple consequence of Theorem 2.20 is the following.

Theorem 3.6 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. Furthermore, let $e \in E(G)$, let $\varphi \in \mathcal{C}_k(G - e)$, and let T be a Tashkinov tree with respect to e and φ. Then $V(T)$ is elementary with respect to φ.*

Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. Since G is critical, for every edge $e \in E(G)$ and every colouring $\varphi \in \mathcal{C}_k(G - e)$, there is a Tashkinov tree T with respect to e and φ. Hence, there is a largest number p such that $p = |V(T)|$ for such a Tashkinov tree T. We call p the **Tashkinov order** of G and write $\boldsymbol{t(G)} = p$. Furthermore, we denote by $\boldsymbol{\mathcal{T}(G)}$ the set of all triples (T, e, φ) such that $e \in E(G)$, $\varphi \in \mathcal{C}_k(G - e)$, and T is a Tashkinov tree on $t(G)$ vertices with respect to e and φ. Evidently, $\mathcal{T}(G) \neq \emptyset$.

Corollary 3.7 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $(T, e, \varphi) \in \mathcal{T}(G)$ then $V(T)$ is elementary and closed with respect to φ. As a consequence $\mathcal{T}(G) \subseteq \mathcal{T}_k(G)$, and $t(G) \geq 3$ is odd.*

Proof: Clearly, T is a maximal Tashkinov tree with respect to e and φ and, therefore, $V(T)$ is closed with respect to φ. Moreover, by Theorem 3.6, $V(T)$ is also elementary with respect to φ. This implies that $(T, e, \varphi) \in \mathcal{T}_k(G)$. By Proposition 2.21(b), we obtain $t(G) = |V(T)| \geq 3$ is odd. ■

Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. By Corollary 3.7, the vertex set $V(T)$ is elementary and closed with respect to φ and, therefore, we have $\mathcal{T}(G) \subseteq \mathcal{T}_k(G)$. In particular, this implies that the sets $\Gamma^d(T, e, \varphi)$ and $\Gamma^f(T, e, \varphi)$ are well defined. Thus we can use the results of the last section also for $(T, e, \varphi) \in \mathcal{T}(G)$. From now on, we will use the facts of Corollary 3.7 quite often without mentioning it explicitly.

Note that considering Tashkinov trees of maximum order gives another problem for the translation of the upcoming results into algorithms. However, this will not be a problem. Whenever this maximality is used to prove some condition, it would otherwise be possible to construct a Tashkinov tree of larger order. A corresponding algorithm simply checks the condition and, in case it is not fulfilled, increases the order of the Tashkinov tree.

Theorem 3.8 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Furthermore, let $\delta \in \Gamma^d(T, e, \varphi)$, let $u \in V(T)$ be a vertex such that $\bar{\varphi}(u)$ contains a free colour $\gamma \in \Gamma^f(T, e, \varphi)$, and let $P = P_u(\gamma, \delta, \varphi)$. Then the following statements hold:*

(a) $E_\delta(T, e, \varphi) \subseteq E(P)$.

(b) *In the linear order $\preceq_{(u,P)}$ there is a last vertex v_0 that belongs to $V(T)$ and a first vertex v_1 that belongs to $V(G)\setminus V(T)$, where $v_1 \preceq_{(u,P)} v_0$. In the same order v_1 has a successor v_2 which belongs to $V(G)\setminus V(T)$, too.*

(c) $\bar{\varphi}(v_0) \cap \Gamma^f(T, e, \varphi) = \emptyset$.

(d) *The set $X = V(T) \cup \{v_1, v_2\}$ is elementary with respect to φ.*

Proof: To prove (a), assume that $E_\delta(T, e, \varphi) \nsubseteq E(P)$. Then Proposition 2.22 implies that there is a colouring $\varphi' \in \mathcal{C}(G-e)$ such that T is a Tashkinov tree with respect to e and φ', but $V(T)$ is not closed with respect to φ'. Consequently, there is a Tashkinov tree T' with respect to e and φ' satisfying $|V(T')| > |V(T)|$, a contradiction to $|V(T)| = t(G)$. This proves (a).

Statement (b) is an immediate consequence of (a) and Theorem 2.23(a). Statements (c) and (d) follow from (a) and Theorem 2.23(b)-(c) and the fact that $|V(T)| = t(G)$. ■

Theorem 3.8 shows how the vertex set of a maximum Tashkinov tree can be extended to an elementary set by adding two vertices. Now we will generalize these ideas to get elementary sets of potentially larger size.

Let G be a graph with $\chi'(G) = k+1$, where $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. A vertex v in $V(G)\setminus V(T)$ is called **absorbing** with respect to (T, e, φ) if, for every colour $\delta \in \bar{\varphi}(v)$ and every free colour $\gamma \in \Gamma^f(T, e, \varphi)$ with $\gamma \neq \delta$, the (γ, δ)-chain $P_v(\gamma, \delta, \varphi)$ contains a vertex $u \in V(T)$ satisfying $\gamma \in \bar{\varphi}(u)$. Since $V(T)$ is elementary with respect to φ, this vertex u is the unique vertex in T with $\gamma \in \bar{\varphi}(u)$ and, moreover, $P_v(\gamma, \delta, \varphi)$ is a path whose endvertices are u and v. Clearly, u belongs to $P_v(\gamma, \delta, \varphi)$ if and only if v belongs to $P_u(\gamma, \delta, \varphi)$. Let $\boldsymbol{A(T, e, \varphi)}$ denote the set of all vertices in $V(G)\setminus V(T)$ which are absorbing with respect to (T, e, φ).

Proposition 3.9 (Favrholdt, Stiebitz and Toft [8] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Then the vertex set $V(T) \cup A(T, e, \varphi)$ is elementary with respect to φ.*

Proof: Suppose, on the contrary, that $Z = V(T) \cup A(T, e, \varphi)$ is not elementary with respect to φ. Then there are two distinct vertices $v, v' \in Z$, and there is a colour α such that $\alpha \in \bar{\varphi}(v) \cap \bar{\varphi}(v')$. Since, by Corollary 3.7, $V(T)$ is elementary with respect to φ, at least one of the these two vertices belong to $A = A(T, e, \varphi)$, say $v \in A$. From Proposition 2.21(f) it follows that there is a free colour $\gamma \in \Gamma^f(T, e, \varphi)$ that is distinct from α. Then, since $V(T)$ is elementary with respect to φ, there is a unique vertex u in $V(T)$ such that $\gamma \in \bar{\varphi}(u)$. Since $v \in A$, it follows that v belongs to the chain $P_u = P_u(\gamma, \alpha, \varphi)$. Hence, P_u is a path and u, v are the endvertices of P_u where $u \neq v$. Since $\alpha \in \bar{\varphi}(v')$ and $v' \neq v$, we then conclude that $v' \notin V(P_u)$ and, therefore, $v' \neq u$. Clearly, this implies that $v' \notin A$. Consequently, $v' \in V(T) \setminus \{u\}$. It then follows from Proposition 2.21(h) that the vertex v' is an endvertex of P_u, a contradiction. This completes the proof. ■

Note that the proof of Proposition 3.9 does not depend on the criticality of the graph G or on the fact that $(T, e, \varphi) \in \mathcal{T}(G)$. A similar result also holds for any graph G and any $(T, e, \varphi) \in \mathcal{T}_k(G)$ with $k \geq \Delta(G) + 1$. The result is only due to the special definition of absorbing vertices. This definition makes it impractical to find the set of absorbing vertices algorithmically. However, it is a useful tool to prove that special vertex sets are elementary.

Let G be a critical graph with $\chi'(G) = k + 1$, where $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. We call $v \in V(G)$ a **defective vertex** with respect to (T, e, φ) if there are two distinct colours δ and γ such that $\delta \in \Gamma^d(T, e, \varphi)$ is a defective colour, $\gamma \in \Gamma^f(T, e, \varphi)$ is a free colour, and v is the first vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(G) \setminus V(T)$, where u is the unique vertex in T with $\gamma \in \bar{\varphi}(u)$ and $P = P_u(\delta, \gamma, \varphi)$. The set of all defective vertices with respect to (T, e, φ) is denoted by $\boldsymbol{D(T, e, \varphi)}$.

Proposition 3.10 (Favrholdt, Stiebitz and Toft [8] 2006) *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Then $D(T, e, \varphi) \subseteq A(T, e, \varphi)$.*

Proof: Let v be an arbitrary vertex of $D(T, e, \varphi)$. Then there are two distinct colours $\gamma \in \Gamma^f(T, e, \varphi)$ and $\delta \in \Gamma^d(T, e, \varphi)$ such that v is the first vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(G) \setminus V(T)$ where u is the unique vertex in T with $\gamma \in \bar{\varphi}(u)$ and $P = P_u(\gamma, \delta, \varphi)$.

Now consider an arbitrary colour $\alpha \in \bar{\varphi}(v)$ and a free colour $\gamma' \in \Gamma^f(T, e, \varphi)$. By Theorem 3.8(d), the set $V(T) \cup \{v\}$ is elementary with respect to φ. Consequently, there is a unique vertex $u' \in V(T)$ such that $\gamma' \in \bar{\varphi}(u')$ and $\gamma' \neq \alpha$. Since $\delta \in \varphi(v)$ is a defective colour, $\alpha \in \bar{\varphi}(v)$ and γ' is a free colour, we conclude that $\delta \neq \alpha$ and, by Proposition 2.21(d), $\delta \neq \gamma'$.

In order to prove that $v \in A(T, e, \varphi)$, we have to show that u' belongs to the (α, γ')-chain $P' = P_v(\alpha, \gamma')$. Suppose, on the contrary, that $u' \notin V(P')$. Since $V(T) \cup \{v\}$ is elementary with respect to φ, this implies, in particular, that no endvertex of P' belongs to $V(T)$. Moreover, $\alpha \notin \bar{\varphi}(V(T))$ and $\gamma' \in \Gamma^f(T, e, \varphi)$ and hence, both colours α and γ' are unused on T with respect to φ. Consequently, T is a Tashkinov tree with respect to e and $\varphi' = \varphi/P'$ satisfying $\bar{\varphi}'(V(T)) = \bar{\varphi}(V(T))$ and, therefore, $(T, e, \varphi') \in \mathcal{T}(G)$. For the colouring φ', we then obtain that $\gamma' \in \bar{\varphi}'(u') \cap \bar{\varphi}'(v)$, $E_\delta(T, e, \varphi') = E_\delta(T, e, \varphi)$ and, moreover, the chain $P' = P_u(\gamma, \delta, \varphi')$ satisfies $P' = uPv$. Hence, we have $\delta \in \Gamma^d(T, e, \varphi')$, and v is the first vertex in the linear order $\preceq_{(u,P')}$ that belongs to $V(G) \setminus V(T)$. From Theorem 3.8(d) it then follows that $V(T) \cup \{v\}$ is elementary with respect to φ'. Since $\gamma' \in \bar{\varphi}'(u') \cap \bar{\varphi}'(v)$, this is a contradiction. Consequently, $v \in A(T, e, \varphi)$ and we are done. ■

Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. A simple consequence of Proposition 3.9 and Proposition 3.10 is that the set $V(T) \cup D(T, e, \varphi)$ is elementary with respect to φ. This is a first generalization of Theorem 3.8 where v_1 is the only considered defective vertex.

Algorithmic aspect: Let us say something about a realization of Proposition 3.10 in a colouring algorithm. The input of our subroutine is an arbitrary graph G, an edge $e \in E(G)$, and a colouring $\varphi \in \mathcal{C}_k(G-e)$ for an integer $k \geq \Delta(G)+1$. Like in TashExt2, we successively construct a Tashkinov tree and maybe change the colouring φ until we either get a k-edge colouring of G, or we get an elementary and strongly closed set, or we get a triple $(T, e, \varphi) \in \mathcal{T}_k(G)$. At this point we successively compute the defective vertices. This means, for every free colour γ and every defective colour δ, we check the (γ, δ)-chain $P = P_u(\gamma, \delta, \varphi)$ where $u \in V(T)$ and $\gamma \in \bar{\varphi}(u)$. This process stops in three cases. First, the path P contains not all edges of $E_\delta(T, e, \varphi)$. Then, as in TashExt2, we recolour, increase the order of T and start over with this larger Tashkinov tree. Second, we get a set D of defective vertices such that $V(T) \cup D$ is not elementary with respect to φ. If there is a vertex $v \in D$ such that $V(T) \cup \{v\}$ is not elementary then, again as in TashExt2, we recolour, increase the order of T and start over again with this larger T. Otherwise there are two distinct vertices $v_1, v_2 \in D$ and a colour $\alpha \in \bar{\varphi}(v_1) \cap \bar{\varphi}(v_2)$. Then we choose a free colour $\gamma' \neq \alpha$. Since v_1, v_2 are distinct, the vertex u is not an endvertex of at least one of the chains $P_i = P_{v_i}(\alpha, \gamma', \varphi)$, $i \in \{1, 2\}$. The proof of Proposition 3.10 then shows how, by performing one Kempe change, colouring φ' can be derived from φ such that $v_i \in D(T, e, \varphi')$ and $V(T) \cup \{v_i\}$ is not elementary with respect to φ'. Again, this allows us to recolour, to increase the order of T, and to start over with this larger T. In the third case, the process stops when all defective vertices are found and $V(T) \cup D(T, e, \varphi)$ is elementary with respect to φ.

Clearly, the search for the defective vertices can be done in a time that is polynomial in $\Delta(G)$ and $|V(G)|$. This search is only repeated after the order of the Tashkinov tree T was increased. Since the order of T is bounded by $|V(G)|$, the running time of the whole algorithm is bounded by a polynomial in $\Delta(G)$ and $|V(G)|$.

Definition 3.11 *Let G be graph, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$ for an integer $k \geq \Delta(G)+1$. Furthermore, let Z be a vertex set with $V(T) \subseteq Z \subseteq V(G)$. A sequence $F = (e_1, u_1, \ldots, e_p, u_p)$ is called a* **fan** *at Z with respect to φ if the following conditions hold:*

(F1) *The edges $e_1, \ldots, e_p \in E(G)$ as well as the vertices $u_1, \ldots, u_p \in V(G)$ are distinct.*

(F2) *For every $i \in \{1, \ldots, p\}$, there are two vertices $z \in Z$ and $z' \in Z \cup \{u_1, \ldots, u_{i-1}\}$ satisfying $e_i \in E_G(z, u_i)$ and $\varphi(e_i) \in \bar{\varphi}(z')$.*

Theorem 3.12 (Favrholdt, Stiebitz and Toft [8] 2006) *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Furthermore, let $Y \subseteq D(T, e, \varphi)$ and $Z = V(T) \cup Y$. If F is a fan at Z with respect to φ, then $Z \cup V(F)$ is elementary with respect to φ.*

For a proof of Theorem 3.12, see [8]. This result is another generalization of Theorem 3.8. There v_1 is the only considered defective vertex, and there is an edge $f \in E_G(v_1, v_2)$ with $\varphi(f) = \gamma$. Consequently, $F = (f, v_2)$ is a fan at $V(T) \cup \{v_1\}$ with respect to φ. Now, Theorem 3.12 allows to add a whole fan to get an elementary vertex set.

Algorithmic aspect: Let us say something about a realization in a colouring algorithm, where the input of our subroutine is a valid tuple (G, e, x, y, k, φ). As already described, we can algorithmically construct a maximal Tashkinov tree and the set of defective vertices in polynomial time, resulting in either a k-edge colouring of G, or a triple $(T, e, \varphi) \in \mathcal{T}_k(G)$ such that $Z = V(T) \cup D(T, e, \varphi)$ is elementary with respect to φ. Then we can successively build the fan F at Z with respect to φ until either $Z \cup V(F)$ is not elementary with respect to φ, or F is maximal and $Z \cup V(F)$ is elementary with respect to φ. In the first case the proof of Theorem 3.12, see [8], shows how the colouring φ can be changed in such a way that the order of T can be increased. Then we start over with this larger Tashkinov tree. Moreover, the time for the necessary operations is bounded by a polynomial in $\Delta(G)$ and $|V(G)|$ and hence, also the time for the whole algorithm is.

Consider a graph G and a triple $(T, e, \varphi) \in \mathcal{T}_k(G)$ for an integer $k \geq \Delta(G) + 1$. Let $\alpha \in \bar{\varphi}(u)$ for a vertex $u \in V(T)$, and let $\delta \in \Gamma^d(T, e, \varphi)$. Clearly, the (α, δ)-chain $P = P_u(\alpha, \delta, \varphi)$ is a path where u is one endvertex of P and, moreover, exactly one of the two colours α or δ is missing at the second endvertex of P with respect to φ. Since $V(T)$ is elementary and δ is present at every vertex in $V(T)$, both with respect to φ, the second endvertex of P belongs to $V(G) \setminus V(T)$. Hence, in the linear order $\preceq_{(u,P)}$ there is a last vertex v that belongs to $V(T)$. This vertex is said to be an **exit vertex** with respect to (T, e, φ). The set of all exit vertices with respect to (T, e, φ) is denoted by $\boldsymbol{F(T, e, \varphi)}$.

Lemma 3.13 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Then $\bar{\varphi}(F(T, e, \varphi)) \cap \Gamma^f(T, e, \varphi) = \emptyset$.*

Proof: Let $v \in F(T, e, \varphi)$. Then there is a vertex $u \in V(T)$, a colour $\alpha \in \bar{\varphi}(u)$, and a colour $\delta \in \Gamma^d(T, e, \varphi)$ such that v is the last vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(T)$, where $P = P_u(\alpha, \delta, \varphi)$. Clearly, P is a path with one endvertex u and the other endvertex $z \in V(G) \setminus V(T)$.

Suppose that there is a colour $\gamma \in \bar{\varphi}(v) \cap \Gamma^f(T, e, \varphi)$. By Corollary 3.7, $V(T)$ is elementary and closed with respect to φ and, therefore, no edge in $E_G(V(T), V(G) \setminus V(T))$ is coloured with α or γ with respect to φ. Consequently, there is a colouring $\varphi' \in \mathcal{C}_k(G - e)$, obtained from φ by interchanging the colours α and γ on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. Evidently, we then have $(T, e, \varphi') \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi') = \Gamma^f(T, e, \varphi)$, and $\Gamma^d(T, e, \varphi') = \Gamma^d(T, e, \varphi)$. In particular, we obtain $\alpha \in \bar{\varphi}'(v) \cap \Gamma^f(T, e, \varphi')$ and $\delta \in \Gamma^d(T, e, \varphi')$. Moreover, for $P' = P_v(\alpha, \delta, \varphi')$ we have $P' = vPz$. One one hand, this implies $|E(P') \cap E_G(V(T), V(G) \setminus V(T))| = 1$. On the other hand, it follows from Theorem 3.8(a) that $|E(P') \cap E_G(V(T), V(G) \setminus V(T)| = |E_\delta(T, e, \varphi'))| > 1$, a contradiction. Consequently, we have $\bar{\varphi}(v) \cap \Gamma^f(T, e, \varphi) = \emptyset$. This completes the proof. ■

In Theorem 3.8 the vertex v_0 was the only considered exit vertex. In Theorem 2.23, the algorithmic version of Theorem 3.8, the exit vertex v_0 was used to increase the order of the Tashkinov tree and to estimate the number of used colours. In some of the later results the exit vertices will be used in a similar way.

3.3 Balanced Tashkinov Trees

The last results were dealing with generating elementary vertex sets. This is supported by another useful idea from Section 2.7, namely building Tashkinov trees in such a way that preferably few colours are used. Lemma 2.24 dealt with this issue in a general way that still leaves a wide range of possible Tashkinov trees. We will now focus on constructing Tashkinov trees that not only use few colours but have a more definite structure.

Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$ for an integer $k \geq \Delta(G) + 1$. Then T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1}).$$

T is called a **normal Tashkinov tree** with respect to e and φ if there are two colours $\alpha \in \bar{\varphi}(y_0)$ and $\beta \in \bar{\varphi}(y_1)$, an integer $2 \leq q \leq p-1$, and an edge $f \in E_G(y_0, y_{q-1})$ such that the path $P(y_1, e_2, y_2, \ldots, e_{q-1}, y_{q-1}, f, y_0)$ is an (α, β)-chain with respect to φ. In this case Ty_{q-1} is called the **$(\boldsymbol{\alpha}, \boldsymbol{\beta})$-trunk** of T, and the number q is called the **height** of T, denoted by $\boldsymbol{h(T)} = q$. Furthermore, let $\boldsymbol{\mathcal{T}_k^N(G)}$ denote the set of all triples $(T, e, \varphi) \in \mathcal{T}_k(G)$ for which T is a normal Tashkinov tree with respect to e and φ. The following result shows that normal Tashkinov trees can be generated from arbitrary ones.

Lemma 3.14 *Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k(G)$ with $k \geq \Delta(G) + 1$ and $e \in E_G(x, y)$. Furthermore, let $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$ be two colours, and let $P = P_x(\alpha, \beta, \varphi)$. Then there is a triple $(T', e, \varphi) \in \mathcal{T}_k^N(G)$ such that $V(T') = V(T)$ and $h(T') = |V(P)|$.*

Proof: By Proposition 2.21(h), we have $\alpha \neq \beta$, and the (α, β)-chain $P = P_x(\alpha, \beta, \varphi)$ is a path with endvertices x and y. This means that P is a path of the form

$$P = P(v_1, f_2, v_2, \ldots, f_q, v_q)$$

with $v_1 = y$ and $v_q = x$. Evidently, $f_j \in E_G(v_{j-1}, v_j)$ and $\varphi(f_j) \in \{\alpha, \beta\} \subseteq \bar{\varphi}(\{x, y\})$ for all $j \in \{2, \ldots, p\}$. Hence, $T_1 = (x, e, y, f_2, v_2, \ldots, f_{q-1}, v_{q-1})$ is a Tashkinov tree with respect to e and φ. By Lemma 2.24(b), there is a Tashkinov tree T' with respect to e and φ with $V(T') = V(T)$ and $T'v_{q-1} = T_1$. Then T' is a normal Tashkinov tree with respect to e and φ and, therefore, $(T', e, \varphi) \in \mathcal{T}_k^N(G)$. Moreover, T_1 is the (α, β)-trunk of T' and $h(T') = |V(T_1)| = |V(P)|$. This completes the proof. ∎

Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k^N(G)$ for an integer $k \geq \Delta(G) + 1$, where T has the form $T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1})$. Then T is called a **balanced Tashkinov tree** with respect to e and φ if $\varphi(e_{2j}) = \varphi(e_{2j-1})$ for $h(T) < 2j < p$. Let $\boldsymbol{\mathcal{T}_k^B(G)}$ denote the set of all triples $(T, e, \varphi) \in \mathcal{T}_k(G)$ for which T is a balanced Tashkinov tree with respect to e and φ. A triple $(T, e, \varphi) \in \mathcal{T}_k^B(G)$ is also called a **balanced triple**. The following result shows that balanced Tashkinov trees can be generated from normal ones.

Lemma 3.15 *Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k^N(G)$ for an integer $k \geq \Delta(G)+1$. Then there is a balanced triple $(T', e, \varphi) \in \mathcal{T}_k^B(G)$ such that $V(T') = V(T)$, $h(T') = h(T)$, and all colours used on T' are used on T, both with respect to φ.*

Proof: Let $(T, e, \varphi) \in \mathcal{T}_k^N(G)$, and let $q = h(T)$. Then, clearly, T has the form $T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1})$, and Ty_{q-1} is the (α, β)-trunk of T. By definition, we have $\alpha \in \bar{\varphi}(y_0)$, $\beta \in \bar{\varphi}(y_1)$, and there is an edge $f \in E_G(y_0, y_{q-1})$ satisfying $P = P_{y_0}(\alpha, \beta, \varphi) = P(y_1, e_2, y_2, \ldots, e_{q-1}, y_{q-1}, f, y_0)$. Hence, P is a path with two distinct endvertices, and moreover P is alternately coloured with two colours. Since every of these two colours is missing at one of the endvertices, we conclude that $|E(P)| \geq 2$ is even. Hence, $q \geq 3$ is odd.

Let $i \leq p$ be the greatest odd integer for which there exists a Tashkinov tree $T' = (y'_0, e'_1, y'_1, \ldots, e'_{i-1}, y'_{i-1})$ with respect to e and φ satisfying $T'y'_{q-1} = Ty_{q-1}$, $\varphi(E(T')) \subseteq \varphi(E(T))$ and $\varphi(e'_{2j-1}) = \varphi(e'_{2j})$ for $q < 2j < i$. Evidently, we have $i \geq q$, because T fulfils these requirements for $i = q$.

Now suppose that $i < p$. Then there is a smallest integer r satisfying $y_r \in V(T) \setminus V(T')$. Let $y'_i = y_r$ and $e'_i = e_r$. Consequently, $e'_i \in E_G(V(T'), y'_i)$ and $\varphi(e'_i) \in \bar{\varphi}(V(T'))$ and, therefore, $T_1 = (T', e'_i, y'_i)$ is a Tashkinov tree with respect to e and φ. Let $\gamma = \varphi(e'_i)$. Clearly, $|V(T_1)|$ is even and $\gamma \in \bar{\varphi}(V(T_1))$. Since $V(T)$ is elementary with respect to φ, the set $V(T_1)$ is elementary with respect to φ, too. It then follows that there is an edge $e'_{i+1} \in E_G(V(T_1), y'_{i+1})$ satisfying $y'_{i+1} \in V(G) \setminus V(T_1)$ and $\varphi(e'_{i+1}) = \gamma$. Evidently, $T_2 = (T_1, e'_{i+1}, y'_{i+1})$ is a Tashkinov tree with respect to e and φ satisfying $T_2 y'_{q-1} = Ty_{q-1}$, $\varphi(E(T_2)) \subseteq \varphi(E(T))$, and $\varphi(e'_{2j-1}) = \varphi(e'_{2j})$ for $p < 2j < i + 2$. This contradicts the maximality of i. Consequently, we have $i = p$ and, by Lemma 2.24(b), $V(T') = V(T)$. Then $(T', e, \varphi) \in \mathcal{T}_k^B(G)$ with $h(T') = h(T) = q$ and $\varphi(E(T')) \subseteq \varphi(E(T))$. Hence, the proof is finished. ■

Algorithmic aspect: Clearly, the rearranging procedure described in the proof of Lemma 3.15 only needs polynomial time. Note that, when building a Tashkinov tree, we need not to build a normal tree first and rearrange it later. Given a colouring $\varphi \in \mathcal{C}_k(G - e)$, we can simply start with the trivial Tashkinov tree (x, y, e), then extend it to the trunk using the (α, β)-chain joining x and y. Eventually, we build the rest by repeatedly adding two edges of the same colour. As long as the vertex set remains elementary with respect to our colouring φ, these steps can easily be accomplished, resulting in a balanced triple (T, e, φ). Otherwise, that is, if at some point the vertex of the Tashkinov tree is not elementary with respect to φ, Theorem 2.20 shows how to construct a colouring $\varphi' \in \mathcal{C}_k(G)$.

Although we can get balanced Tashkinov trees in the direct way, the rearranging procedure of Lemma 3.15 will be useful after operations that turn a balanced Tashkinov tree into an unbalanced one. For example, there will be cases where we are able to increase the height of a balanced Tashkinov tree by using other colours for the trunk. Using Lemma 3.14 we can easily get a new normal Tashkinov tree with the same order and larger height, but it may not be balanced anymore, so we have to use Lemma 3.15.

Another recolouring and rearranging operation that we will use, is the following one. Consider a graph G and a balanced triple $(T, e, \varphi) \in \mathcal{T}_k^B(G)$ for an integer $k \geq \Delta(G) + 1$. Then T has the form

$$T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1}),$$

and Ty_{q-1} is the (α, β)-trunk of T, where $q = h(T)$, $\alpha \in \bar{\varphi}(y_0)$, and $\beta \in \bar{\varphi}(y_1)$. Moreover, there is an edge $f_q \in E_G(y_0, y_{q-1})$ with $\varphi(f_q) = \beta$. For $i = 1, \ldots, q-1$, let $f_i = e_i$. Clearly, the edges $f_1, \ldots, f_q$ form a cycle in G. Furthermore, the edge $f_1 = e$ is uncoloured, and the edges $f_2, \ldots, f_q$ are coloured alternately with α and β with respect to φ. Now choose a $j \in \{1, \ldots, q-1\}$. Since (y_0, y_1) is an (α, β)-pair with respect to φ, there is a colouring $\varphi' \in \mathcal{C}_k(G - f_{j+1})$ such that $\varphi'(e') = \varphi(e')$ for all edges $e' \in E(G) \setminus \{f_1, \ldots, f_q\}$, and the edges $f_{j+2}, \ldots, f_q, f_1, \ldots, f_j$ are coloured alternately with α and β with respect to φ'. Then

$$T' = (y_j T y_{q-1}, f_q, y_0 T y_{j-1}, e_q, y_q, \ldots, e_{p-1}, y_{p-1})$$

is a balanced Tashkinov tree with respect to f_{j+1} and φ', where $T' y_{j-1}$ is the (α, β)-trunk of T'. Clearly, $(T', f_{j+1}, \varphi') \in \mathcal{T}_k^B(G)$, and we write $(T', f_{j+1}, \varphi') = \boldsymbol{(T, e, \varphi)(y_0 \to y_j)}$.

Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Then let $\boldsymbol{\mathcal{T}^N(G)} = \mathcal{T}(G) \cap \mathcal{T}_k^N(G)$, that is, $\mathcal{T}^N(G)$ is the set of all triples $(T, e, \varphi) \in \mathcal{T}(G)$ for which T is a normal Tashkinov tree. Since $\mathcal{T}(G) \neq \emptyset$, Lemma 3.14 implies that $\mathcal{T}^N(G) \neq \emptyset$. Then, clearly, there is a greatest number q such that there is a triple $(T, e, \varphi) \in \mathcal{T}^N(G)$ with $h(T) = q$. We denote this number by $\boldsymbol{h(G)}$. Furthermore, let $\boldsymbol{\mathcal{T}^B(G)}$ denote the set of all balanced triples of $\mathcal{T}(G)$ with $h(T) = h(G)$.

Lemma 3.16 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Then the following statements hold:*

(a) $h(G) \geq 3$ *is odd.*

(b) $\mathcal{T}^B(G) \neq \emptyset$.

Proof: Let $(T, e, \varphi) \in \mathcal{T}^N(G)$ with $h(T) = h(G) = q$. Then T has the form $T = (y_0, e_1, y_1, \ldots, e_{p-1}, y_{p-1})$, and Ty_{q-1} is the (α, β)-trunk of T. By definition, we have $\alpha \in \bar{\varphi}(y_0)$, $\beta \in \bar{\varphi}(y_1)$, and there is an edge $f \in E_G(y_0, y_{q-1})$ satisfying $P = P_{y_0}(\alpha, \beta, \varphi) = P(y_1, e_2, y_2, \ldots, e_{q-1}, y_{q-1}, f, y_0)$. Hence, P is a path with two distinct endvertices and, moreover, the edges of P are alternately coloured with two colours. Since every of these two colours is missing at one of the endvertices of P, we conclude that $|E(P)| \geq 2$ is even. Hence, $q = h(G) \geq 3$ is odd and (a) is proved.

Since $\mathcal{T}^N(G) \neq \emptyset$, (b) is a direct consequence of Lemma 3.15. This completes the proof. ■

Algorithmic aspect: Some of the following results will depend on handling triples $(T, e, \varphi) \in \mathcal{T}^B(G)$. This seems to make it difficult to transform those results into algorithms, because it is impracticable to search for balanced Tashkinov trees with maximum order and height. However, the proofs of those results will always be in the following way. If for a balanced triple (T, e, φ) some conditions are not fulfilled, then we can recolour and increase either the order or the height of T. We will discuss this in detail for the specific results.

3.4 The Main Lemma

Next we will establish several conditions related to Tashkinov trees, which imply that a critical graph is elementary. Some of these results are generalizations of results implicitly given in [8], others, like the following one, are new. The next lemma analyses Tashkinov trees with a special structure, they will be used to handle some of the cases in the proof of Theorem 3.2.

Lemma 3.17 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Suppose that T has the form*

$$T = (y_0, e_1, y_1, \ldots, e_{r-1}, y_{r-1}, f^1_{\gamma_1}, u^1_{\gamma_1}, f^2_{\gamma_1}, u^2_{\gamma_1}, \ldots, f^1_{\gamma_s}, u^1_{\gamma_s}, f^2_{\gamma_s}, u^2_{\gamma_s})$$

where $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ is a set of s colours, $Y = \{y_0, \ldots, y_{r-1}\}$, and the following conditions hold:

(S1) *$f^j_\gamma \in E_G(Y, u^j_\gamma)$ for every $\gamma \in \Gamma$ and $j \in \{1, 2\}$.*

(S2) *$\varphi(f^1_\gamma) = \varphi(f^2_\gamma) = \gamma \in \bar{\varphi}(Y)$ for every $\gamma \in \Gamma$.*

(S3) *$\varphi(e_j) \notin \Gamma$ for $2 \leq j \leq r - 1$.*

(S4) *For every $v \in F(T, e, \varphi)$, there is a colour $\gamma \in \Gamma$ satisfying $\gamma \in \bar{\varphi}(v)$.*

Then G is an elementary graph.

Proof: Since $(T, e, \varphi) \in \mathcal{T}(G)$ and G is critical, it follows from Corollary 3.7 that $V(T)$ is elementary and closed both with respect to φ, and $(T, e, \varphi) \in \mathcal{T}_k(G)$. If $V(T)$ is also strongly closed with respect to φ, then Theorem 1.4 implies that G is an elementary graph and we are done. Hence, we only need to consider the case that $V(T)$ is not strongly closed with respect to φ. Then we construct a superset X of $V(T)$ such that X is elementary and strongly closed with respect to φ. Again, by Theorem 1.4, this implies that G is elementary and we are done, too.

Since $V(T)$ is not strongly closed with respect to φ, it follows from Proposition 2.21(c) that $\Gamma^d(T, e, \varphi) \neq \emptyset$. This implies $F(T, e, \varphi) \neq \emptyset$ and, therefore, $s' = |F(T, e, \varphi)| \geq 1$. Since $V(T)$ is elementary with respect to φ, we conclude from (S4) that there is a set $\Gamma' \subseteq \Gamma$ of s' colours such that, for every $v \in F(T, e, \varphi)$, there is a unique colour $\gamma \in \Gamma'$ satisfying $\gamma \in \bar{\varphi}(v)$. Hence, there is a one to one correspondence between $F(T, e, \varphi)$ and Γ'. In particular, $|\bar{\varphi}(v) \cap \Gamma'| = 1$ for every $v \in F(T, e, \varphi)$, and $\bar{\varphi}(v) \cap \Gamma' = \emptyset$ for every $v \in V(T) \setminus F(T, e, \varphi)$.

By Proposition 2.21(f), $|\bar{\varphi}(y_0)| \geq 2$. Hence, there is a colour $\alpha_0 \in \bar{\varphi}(y_0) \setminus \Gamma'$. Now consider an arbitrary defective colour $\delta \in \Gamma^d(T, e, \varphi)$. Then let P_δ be defined by

$$P_\delta = P_{y_0}(\alpha_0, \delta, \varphi).$$

By Proposition 2.21(d),(e), the colour δ is present at any vertex of T. Since $\alpha_0 \in \bar{\varphi}(y_0)$ and $V(T)$ is elementary as well as closed with respect to φ, we then conclude that P_δ is a path where one endvertex is y_0 and the other endvertex, denoted by z_δ, belongs to $V(G) \setminus V(T)$. This implies that there is a last vertex in the linear order $\preceq_{(y_0, P_\delta)}$ that belongs to $V(T)$. We denote this vertex by v^0_δ. By definition,

$v_\delta^0 \in F(T,e,\varphi)$ is an exit vertex. Hence, there is a unique colour $\gamma \in \Gamma'$ such that $\gamma \in \bar{\varphi}(v_\delta^0)$. We denote this colour by $\gamma = \gamma(\delta)$. Furthermore, v_δ^0 is incident with an edge, denoted by f_δ^0, such that $\varphi(f_\delta^0) = \delta$. We denote the second endvertex of f_δ^0 by u_δ^0. Clearly, $f_\delta^0 \in E_\delta(T,e,\varphi)$ and $u_\delta^0 \in V(G)\setminus V(T)$. As we shall see later, $u_\delta^0 \neq z_\delta$ and $|E_\delta(T,e,\varphi)| = 3$. Hence, there is a set U_δ of two vertices distinct from u_δ^0 that are incident with edges in $E_\delta(T,e,\varphi)$. Our aim is to show that the set

$$X = V(T) \cup \bigcup_{\delta \in \Gamma^d(T,e,\varphi)} U_\delta$$

is elementary and strongly closed with respect to φ. The proof is long and relies on several statements.

(1) $F(T,e,\varphi) \subseteq Y$ and, as a consequence, $v_\delta^0 \in Y$ for all $\delta \in \Gamma^d(T,e,\varphi)$.

Proof of (1): From (S2) it follows that all colours used on T with respect to φ are contained in $\bar{\varphi}(Y)$. Since $V(T)$ is elementary with respect to φ, and since $\bar{\varphi}(v) \neq \emptyset$ for every $v \in V(G)$, this implies that $\bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ for every vertex $v \in V(T)\setminus Y$. By Lemma 3.13, $\bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi) = \emptyset$ for every exit vertex $v \in F(T,e,\varphi)$. Hence, we obtain that $F(T,e,\varphi) \subseteq Y$. □

For a colour $\gamma \in \Gamma$, let $T - \gamma$ denote the sequence obtained from T by deleting the edges f_γ^1, f_γ^2 as well as the vertices u_γ^1, u_γ^2. By (S1) and (S2), it follows that $T - \gamma$ is a Tashkinov tree with respect to e and φ. In the sequel, let

$$U_\gamma = \{u_\gamma^1, u_\gamma^2\}$$

and

$$Z_\gamma = V(T - \gamma) = V(T)\setminus U_\gamma.$$

(2) $E_G(u_\gamma^1, u_\gamma^2) \cap E_\alpha(e,\varphi) \neq \emptyset$ for every colour $\gamma \in \Gamma'$ and every colour $\alpha \in \bar{\varphi}(Z_\gamma)\setminus\{\gamma\}$.

Proof of (2): Since $V(T)$ is elementary and closed with respect to φ, for every $\gamma \in \Gamma'$ and every $\alpha \in \bar{\varphi}(Z_\gamma)\setminus\{\gamma\}$, there is an edge $f \in E_G(u_\gamma^1, V(T))$ with $\varphi(f) = \alpha$. Now suppose that $f \in E_G(u_\gamma^1, Z_\gamma)$. Then there is a second edge $f' \in E_G(u_\gamma^2, Z_\gamma)$ with $\varphi(f') = \alpha$. Hence, $T' = (T - \gamma, f, u_\gamma^1, f', u_\gamma^2)$ is a Tashkinov tree with respect to e and φ satisfying $(T',e,\varphi) \in \mathcal{T}(G)$, $V(T') = V(T)$ and $\gamma \in \Gamma^f(T',e,\varphi)$. Moreover, $\Gamma^d(T',e,\varphi) = \Gamma^d(T,e,\varphi)$, which implies $F(T',e,\varphi) = F(T,e,\varphi)$ and, therefore, $\gamma \in \bar{\varphi}(F(T',e,\varphi)) \cap \Gamma^f(T',e,\varphi)$, a contradiction to Lemma 3.13. Consequently, $f \notin E_G(u_\gamma^1, Z_\gamma)$, but $f \in E_G(u_\gamma^1, u_\gamma^2)$. □

(3) $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e,\varphi) = \emptyset$ for every $\delta \in \Gamma^d(T,e,\varphi)$.

Proof of (3): Suppose, on the contrary, that there is a colour $\delta \in \Gamma^d(T,e,\varphi)$ and an edge $g_1 \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(g_1) = \delta$, say g_1 is incident to $u_{\gamma(\delta)}^1$. From (2) we know that there is an edge $g_2 \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(g_2) = \alpha_0$.

Clearly, we have $|E_G(U_{\gamma(\delta)}, V(G)\backslash V(T)) \cap E_\delta(e,\varphi)| \leq 1$. Then, evidently, Proposition 2.21(e) implies $|E_G(Z_{\gamma(\delta)}, V(G)\backslash V(T)) \cap E_\delta(e,\varphi)| \geq 2$. Hence, there is an edge $g_3 \in E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \setminus \{f_\delta^0\}$ with $\varphi(g_3) = \delta$. Let u_3 be the endvertex of g_3 that belongs to $V(G) \setminus V(T)$.

Now consider the subpath $P_1 = v_\delta^0 P_\delta z_\delta$. Then, clearly, $V(P_1) \cap V(T) = \{v_\delta^0\}$. Furthermore, since $\alpha_0, \gamma(\delta) \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we can obtain a new colouring $\varphi_1 \in \mathcal{C}_k(G-e)$ from φ by interchanging the colours α_0 and $\gamma(\delta)$ on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. Then we conclude that $P_1 = P_{v_\delta^0}(\gamma(\delta), \delta, \varphi_1)$. For the colouring $\varphi_2 = \varphi_1/P_1$, we then obtain that $\varphi_2 \in \mathcal{C}_k(G-e)$, and $T_1 = T - \gamma(\delta)$ is a Tashkinov tree with respect to e and φ_2 satisfying $V(T_1) = Z_{\gamma(\delta)}$ and $\delta \in \bar{\varphi}_2(v_\delta^0) \subseteq \bar{\varphi}_2(V(T_1))$. Since g_1, g_2, g_3 neither belong to $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$ nor to $E(P_1)$, their colours did not change and, therefore, we have $\varphi_2(g_1) = \varphi_2(g_3) = \delta$ and $\varphi_2(g_2) = \alpha_0$. Then, evidently, $T_2 = (T_1, g_1, u^1_{\gamma(\delta)}, g_2, u^2_{\gamma(\delta)}, g_3, u_3)$ is a Tashkinov tree with respect to e and φ_2 satisfying $|V(T_2)| > |V(T)| = t(G)$, a contradiction. □

For a defective colour $\delta \in \Gamma^d(T, e, \varphi)$, let P'_δ be the chain defined by

$$P'_\delta = P_{v_\delta^0}(\gamma(\delta), \delta, \varphi),$$

and let $\varphi_\delta \in \mathcal{C}_k(G-e)$ be the colouring

$$\varphi_\delta = \varphi/P'_\delta.$$

Evidently, P'_δ is a path where one endvertex is v_δ^0 and the other endvertex, denoted by z'_δ, belongs to $V(G) \setminus V(T)$.

(4) $V(P'_\delta) \cap V(T) = \{v_\delta^0\}$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (4): Suppose, on the contrary, that there is a $\delta \in \Gamma^d(T, e, \varphi)$ with $V(P'_\delta) \cap V(T) \neq \{v_\delta^0\}$. Since $v_\delta^0 \in V(P'_\delta) \cap V(T)$, the last vertex v_1 in the linear order $\preceq_{(v_\delta^0, P'_\delta)}$ belonging to $V(T)$ satisfies $v_1 \neq v_\delta^0$. Obviously, $v_1 \in F(T, e, \varphi)$, and from (1) it then follows that $v_1 \in Y$.

Clearly, there is an edge $f_1 \in E_G(v_1, V(G) \backslash V(T))$ with $\varphi(f_1) = \delta$. Let $u_0 \in V(G) \backslash V(T)$ be the second endvertex of f_δ^0, and let $u_1 \in V(G) \backslash V(T)$ be the second endvertex of f_1. Furthermore, let $P_1 = v_\delta^0 P_\delta z_\delta$, and let $P'_1 = v_1 P'_\delta z'_\delta$. Then $V(P_1) \cap V(T) = \{v_\delta^0\}$ and $V(P'_1) \cap V(T) = \{v_1\}$. Since $v_1 \in F(T, e, \varphi)$, there is an index $j \in \{1, \ldots, s\}$ with $\gamma_j \in \bar{\varphi}(v_1)$. Moreover, $v_1 \neq v_\delta^0$ implies $\gamma(\delta) \neq \gamma_j$. To simplify notation, let $\gamma = \gamma(\delta)$.

Since $V(T)$ is closed with respect to φ, no edge in $E_G(V(T), V(G) \setminus V(T))$ is coloured with α_0, γ, or γ_j with respect to φ. Hence, we can obtain two new colourings from φ, the first one $\varphi_1 \in \mathcal{C}_k(G-e)$ by interchanging the colours α_0 and γ on all edges in $E_G(V(G) \backslash V(T), V(G) \backslash V(T))$, the second one $\varphi'_1 \in \mathcal{C}_k(G-e)$ by interchanging the colours γ and γ_j on all edges in $E_G(V(G) \setminus V(T), V(G) \setminus V(T))$. Clearly $(T, e, \varphi_1) \in \mathcal{T}(G)$ and $(T, e, \varphi'_1) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_1) = \Gamma^f(T, e, \varphi'_1) = \Gamma^f(T, e, \varphi)$, $\Gamma^d(T, e, \varphi_1) = \Gamma^d(T, e, \varphi'_1) = \Gamma^d(T, e, \varphi)$ and, moreover, $P_1 = P_{v_\delta^0}(\gamma, \delta, \varphi_1)$ and $P'_1 = P_{v_1}(\gamma_j, \delta, \varphi'_1)$.

For the colouring $\varphi_2 = \varphi_1/P_1$, we then obtain that $\varphi_2 \in \mathcal{C}_k(G-e)$, and $T_1 = T - \gamma$ is a Tashkinov tree with respect to e and φ_2 satisfying $V(T_1) = Z_\gamma$ and $\delta \in \bar{\varphi}_2(v_\delta^0) \subseteq$

$\bar\varphi(V(T_1))$. Since f_1 belongs neither to $E_G(V(G)\setminus V(T), V(G)\setminus V(T))$ nor to $E(P_1)$, its colour did not change, so we have $\varphi_2(f_1) = \delta$. Moreover, $v_1 \in Y \subseteq V(T_1)$ and, therefore, $T_2 = (T_1, f_1, u_1)$ is a Tashkinov tree with respect to e and φ_2.

Analogously, for the colouring $\varphi_2' = \varphi_1'/P_1'$, we obtain that $\varphi_2' \in \mathcal{C}_k(G-e)$, and $T_1' = T - \gamma_j$ is a Tashkinov tree with respect to e and φ_2' satisfying $V(T_1') = Z_{\gamma_j}$ and $\delta \in \bar\varphi_2'(v_1) \subseteq \bar\varphi(v(T_1'))$. Since f_δ^0 belongs neither to $E_G(V(G)\backslash V(T), V(G)\backslash V(T))$ nor to P_1', its colour did not change, so we have $\varphi_2'(f_\delta^0) = \delta$. Moreover, $v_\delta^0 \in Y \subseteq V(T_1')$ and, therefore, $T_2' = (T_1', f_\delta^0, u_0)$ is a Tashkinov tree with respect to e and φ_2'.

Let $Z = V(T_1) \cap V(T_1') = V(T) \setminus U_\gamma \setminus U_{\gamma_j}$. Since $\delta \notin \bar\varphi(Z)$ and $|Z|$ is odd, also $|E_G(Z, V(G)\setminus Z) \cap E_\delta(e,\varphi)|$ is odd. So besides f_δ^0 and f_1 there is another edge $f_2 \in E_G(Z, V(G)\setminus Z)$ with $\varphi(f_2) = \delta$. Since f_2 has an endvertex in $Z \subseteq V(T)$, but is distinct from f_δ^0 or f_1, it neither belongs to $E(P_1)$, to $E(P_1')$, or to $E_G(V(G)\setminus V(T), V(G)\setminus V(T))$. So none of the recolourings have an effect on f_2, which leads to $\varphi_2(f_2) = \varphi_2'(f_2) = \delta$.

Let u_2 be the endvertex of f_2 that belongs to $V(G)\backslash Z$. We claim that $u_2 \notin U_{\gamma_j}$. Suppose, on the contrary, that $u_2 \in U_{\gamma_j}$, say $u_2 = u_{\gamma_j}^1$. From (2) we then conclude that there is an edge $f' \in E_G(u_{\gamma_j}^1, u_{\gamma_j}^2)$ with $\varphi(f') = \alpha_0$. Obviously, we have $\varphi_2'(f') = \alpha_0 \in \bar\varphi_2'(V(T_2'))$ and, therefore, $T' = (T_2', f_2, u_{\gamma_j}^1, f', u_{\gamma_j}^2)$ is a Tashkinov tree with respect to e and φ_2' satisfying $|V(T')| > |V(T)| = t(G)$, a contradiction. This proves the claim, thus we have $u_2 \notin U_{\gamma_j}$. Moreover, from (3) we conclude that $u_2 \notin U_\gamma$ and, therefore, $u_2 \in V(G)\backslash V(T)$. Hence, $T_3 = (T_2, f_2, u_2)$ is a Tashkinov tree with respect to e and φ_2, and $T_3' = (T_2', f_2, u_2)$ is a Tashkinov tree with respect to e and φ_2'. Since $|V(T_3)| = |V(T_3')| = |V(T)| = t(G)$, this implies $(T_3, e, \varphi_2), (T_3', e, \varphi_2') \in \mathcal{T}(G)$.

From Proposition 2.21(f) it follows that $|\bar\varphi(\{y_0, y_1\})| \geq 4$. So there is a colour $\beta \in \bar\varphi(\{y_0, y_1\})$ with $\beta \notin \{\alpha_0, \gamma, \gamma_j\}$. Obviously, we also have $\beta \neq \delta$ and, therefore, the colour β does not matter in any of the mentioned recolourings This leads to $E_\beta(e,\varphi) = E_\beta(e,\varphi_2) = E_\beta(e,\varphi_2')$. Then, evidently, $\beta \in \bar\varphi_2(V(T_3))$. By Corollary 3.7, $V(T_3)$ is elementary and closed both with respect to φ_2. Hence, there is an edge $f_3 \in E_G(u_2, V(T_3))$ with $\varphi_2(f_3) = \beta$. Clearly, we also have $\varphi(f_3) = \beta$, but since $V(T)$ is closed with respect to φ, the edge f_3 cannot have an endvertex in $V(T)$. Therefore, we conclude $f_3 \in E_G(u_2, u_1)$. Moreover, we have $\varphi_2'(f_3) = \beta \in \bar\varphi_2'(V(T_3'))$. Hence $T_4' = (T_3, f_3, u_1)$ is a Tashkinov tree with respect to e and φ_2' satisfying $|V(T_4')| > |V(T)| = t(G)$, a contradiction. This proves (4). □

For a defective colour $\delta \in \Gamma^d(T, e, \varphi)$, the colouring φ_δ satisfies the following conditions. It is a simple consequence of (4).

(5) For every $\delta \in \Gamma^d(T, e, \varphi)$, the colouring $\varphi_\delta \in \mathcal{C}_k(G-e)$ satisfies:

- $\varphi_\delta(f_\delta^0) = \gamma(\delta)$,
- $\varphi_\delta(f) = \varphi(f)$ for every edge $f \in E_{G-e}(V(T), V(G))\backslash\{f_\delta^0\}$,
- $\varphi_\delta(f) = \varphi(f)$ for every edge $f \in E(G-e)\backslash E(P_\delta')$,
- $\bar\varphi_\delta(v_\delta^0) = \bar\varphi(v_\delta^0)\backslash\{\gamma(\delta)\} \cup \{\delta\}$,
- $\bar\varphi_\delta(v) = \bar\varphi(v)$ for every vertex $v \in V(T)\backslash\{v_\delta^0\}$,
- $\bar\varphi_\delta(v) = \bar\varphi(v)$ for every vertex $v \in V(G)\backslash V(P_\delta')$.

Next we claim that

(6) $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| = 3$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (6): Suppose, on the contrary, that there is a $\delta \in \Gamma^d(T, e, \varphi)$ satisfying $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| \neq 3$.

Consider the case $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| > 3$. Then besides f_δ^0 there are another three edges $g_1, g_2, g_3 \in E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T))$ with endvertices $z_1, z_2, z_3 \in V(G) \setminus V(T)$ and $\varphi(g_1) = \varphi(g_2) = \varphi(g_3) = \delta$. From (4) and (5) it then follows that $\varphi_\delta(g_1) = \varphi_\delta(g_2) = \varphi_\delta(g_3) = \delta$. Hence, $T_1 = (T - \gamma(\delta), g_1, z_1, g_2, z_2, g_3, z_3)$ is a Tashkinov tree with respect to e and φ_δ satisfying $|V(T_1)| > |V(T)| = t(G)$, a contradiction. Consequently, we have $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| < 3$.

From (3) we know that $|E_G(Z_{\gamma(\delta)}, U_{\gamma(\delta)}) \cap E_\delta(e, \varphi)| = 0$. Consequently, we obtain $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi)| = |E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| < 3$. Since $\delta \notin \bar{\varphi}(Z_{\gamma(\delta)})$ and $|Z_{\gamma(\delta)}|$ is odd, this implies that $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi)|$ is odd, too, which leads to $|E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi)| = 1$. Therefore, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi) = \{f_\delta^0\}$.

By (2), we have $E_G(Z_{\gamma(\delta)}, U_{\gamma(\delta)}) \cap E_{\alpha_0}(e, \varphi) = \emptyset$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, this implies $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_{\alpha_0}(e, \varphi) = \emptyset$. Hence, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E(P_\delta) = \{f_\delta^0\}$. For the subpath $P_1 = y_0 P_\delta v_\delta^0$, this means $V(P_1) \subseteq Z_{\gamma(\delta)}$. Since v_δ^0 is the last vertex in the linear order $\preceq_{(y_0, P_\delta)}$ that belongs to $V(T)$, we conclude $V(P_\delta) \cap V(T) \subseteq Z_{\gamma(\delta)}$.

Then especially the vertices $u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2$ do not belong to P_δ. Hence, the chain $P_2 = P_{u_{\gamma(\delta)}^1}(\alpha_0, \delta, \varphi)$ is vertex disjoint to P_δ and, moreover, $V(P_2) \cap V(T) \subseteq U_{\gamma(\delta)}$. Then, evidently, $E(P_2) \cap E(T) = \emptyset$ and hence T is a Tashkinov tree with respect to e and the colouring $\varphi_2 = \varphi / P_2$. Since $|E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi)| = 1$, Proposition 2.21(e) implies that there are two edges $g_4 \in E_G(u_\delta^1, V(G) \setminus V(T))$ and $g_5 \in E_G(u_\delta^2, V(G) \setminus V(T))$ with $\varphi(g_4) = \varphi(g_5) = \delta$. Evidently, $g_4 \in E(P_2)$ and $\varphi_2(g_4) = \alpha_0 \in \bar{\varphi}_2(y_0)$. If u_4 is the endvertex of g_4 belonging to $V(G) \setminus V(T)$, then $T_2 = (T, g_4, u_4)$ is a Tashkinov tree with respect to e and φ_2 satisfying $|V(T_2)| > |V(T)| = t(G)$, a contradiction. This proves the claim. □

For every $\delta \in \Gamma^d(T, e, \varphi)$, we know from (6) that beside f_δ^0 there are two other edges $f_\delta^1, f_\delta^2 \in E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T))$ with $\varphi(f_\delta^1) = \varphi(f_\delta^2) = \delta$. For $j = 1, 2$, let $f_\delta^j \in E_G(v_\delta^j, u_\delta^j)$ where $v_\delta^1, v_\delta^2 \in Z_{\gamma(\delta)}$ and $u_\delta^1, u_\delta^2 \in V(G) \setminus V(T)$. Furthermore, let

$$U_\delta = \{u_\delta^1, u_\delta^2\}.$$

By (4), we have $f_\delta^1, f_\delta^2 \notin E(P'_\delta)$ and, therefore, $u_\delta^1, u_\delta^2 \notin V(P'_\delta)$. Then (5) implies:

(7) $\varphi_\delta(f) = \varphi(f)$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $f \in E_G(U_\delta, V(G))$.

In particular, for every $\delta \in \Gamma^d(T, e, \varphi)$, this leads to $\varphi_\delta(f_\delta^1) = \varphi_\delta(f_\delta^2) = \delta$. From $\delta \in \bar{\varphi}_\delta(v_\delta^0)$ then follows that T_δ, defined by

$$T_\delta = (T - \gamma(\delta), f_\delta^1, u_\delta^1, f_\delta^2, u_\delta^2),$$

is a Tashkinov tree with respect to e and φ_δ satisfying $|V(T_\delta)| = |V(T)|$. Therefore, we obtain that

(8) $(T_\delta, e, \varphi_\delta) \in \mathcal{T}(G)$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Since $V(T)$ is closed with respect to φ, for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(Z_{\gamma(\delta)}) \setminus \{\gamma(\delta)\}$, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \emptyset$. This implies $E_G(Z_{\gamma(\delta)}, U_\delta) \cap E_\alpha(e, \varphi) = \emptyset$ and, moreover, by (7), $E_G(Z_{\gamma(\delta)}, U_\delta) \cap E_\alpha(e, \varphi_\delta) = \emptyset$. Since $\alpha \in \bar{\varphi}_\delta(Z_{\gamma(\delta)}) \subseteq \bar{\varphi}_\delta(V(T_\delta))$, and since, by Corollary 3.7, $V(T_\delta)$ is elementary and closed with respect to φ_δ, there must be an edge between u^1_δ and u^2_δ coloured with α with respect to φ_δ as well as, by (7), with respect to φ. Therefore, we have

(9) $E_G(u^1_\delta, u^2_\delta) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(Z_{\gamma(\delta)}) \setminus \{\gamma(\delta)\}$.

Further, we claim that the following two statements are true.

(10) $E_G(u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)}) \cap E_\delta(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

(11) $E_G(u^1_\delta, u^2_\delta) \cap E_{\gamma(\delta)}(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (10): Let $\delta \in \Gamma^d(T, e, \varphi)$. We have $|Z_{\gamma(\delta)}| = r + 2s - 2$ and hence, by Proposition 2.21(f), $|\bar{\varphi}(Z_{\gamma(\delta)})| \geq r + 2s$. Since there are at most $r - 2 + s$ colours used on T with respect to φ, there is a colour $\beta \in \bar{\varphi}(Z_{\gamma(\delta)}) \cap \Gamma^f(T, e, \varphi)$.

Let $v \in Z_{\gamma(\delta)}$ be the unique vertex with $\beta \in \bar{\varphi}(v)$, and let $P = P_v(\beta, \delta, \varphi)$. From (6) and Proposition 2.22 we conclude that P is a path having one endvertex v and another endvertex $z \in V(G) \setminus V(T)$ satisfying $E(P) \cap E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T)) = \{f^0_\delta, f^1_\delta, f^2_\delta\}$. By (1), we have $F(T, e, \varphi) \subseteq Y \subseteq Z_{\gamma(\delta)}$, so for the last vertex v' in the linear order $\preceq_{(v,P)}$ belonging to $V(T)$, we conclude $v' \in \{v^0_\delta, v^1_\delta, v^2_\delta\}$.

From (9) it follows that there is an edge $g \in E_G(u^1_\delta, u^2_\delta)$ with $\varphi(g) = \beta$. Hence, the path $P_1 = P(v^1_\delta, f^1_\delta, u^1_\delta, g, u^2_\delta, f^2_\delta, v^2_\delta)$ is a subpath of P and, therefore, $v' \notin \{v^1_\delta, v^2_\delta\}$, but $v' = v^0_\delta$.

By (2), we have $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi) = \emptyset$. Moreover, it follows from (3) that $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi) = \emptyset$. Hence, for the subpath $P_2 = vPv^0_\delta$, we conclude $V(P_2) \subseteq Z_{\gamma(\delta)} \cup U_\delta$. Further, for $P_3 = v^0_\delta P z$, we clearly have $V(P_3) \cap V(T) = \{v^0_\delta\}$ and hence $V(P) \cap U_{\gamma(\delta)} = \emptyset$. Then from Theorem 3.8(a) it follows that $E_G(U_{\gamma(\delta)}, V(G) \setminus V(T)) \cap E_\delta(e, \varphi) = \emptyset$. Since $E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)}) \cap E_\delta(e, \varphi) = \emptyset$ and $\delta \notin \bar{\varphi}(V(T))$, we conclude $E_G(u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)}) \cap E_\delta(e, \varphi) \neq \emptyset$. This proves the claim. □

Proof of (11): Let $\delta \in \Gamma^d(T, e, \varphi)$. We have $|Z_{\gamma(\delta)}| = r + 2s - 2$ and hence, by Proposition 2.21(f), $|\bar{\varphi}_\delta(Z_{\gamma(\delta)})| \geq r + 2s$. Since there are at most $r - 2 + s$ colours used on T_δ with respect to φ_δ, there is a colour $\beta \in \bar{\varphi}_\delta(Z_{\gamma(\delta)}) \cap \Gamma^f(T, e, \varphi_\delta)$. Let $v \in Z_{\gamma(\delta)}$ be the unique vertex with $\beta \in \bar{\varphi}_\delta(v)$.

By (5), we have $\varphi_\delta(f^0_\delta) = \varphi_\delta(f^1_{\gamma(\delta)}) = \varphi_\delta(f^2_{\gamma(\delta)}) = \gamma(\delta)$ and, therefore, we have $\gamma(\delta) \in \Gamma^d(T_\delta, e, \varphi_\delta)$. Moreover, beside this three edges there can be no further edge in $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T_\delta))$ coloured with $\gamma(\delta)$ with respect to φ_δ. Otherwise such an edge f would, by (5), satisfy $\varphi(f) = \gamma(\delta)$ and it would belong to $E_G(V(T), V(G) \setminus V(T))$. This would contradict the fact that $V(T)$ is closed with respect to φ. Hence, we have $E_G(Z_{\gamma(\delta)}, V(G) \setminus V(T_\delta)) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \{f^0_\delta, f^1_{\gamma(\delta)}, f^2_{\gamma(\delta)}\}$. From this and Proposition 2.22 we conclude that $P = P_v(\beta, \gamma(\delta), \varphi_\delta)$ is a path having one endvertex

v and another endvertex $z \in V(G)\setminus V(T_\delta)$ satisfying $E(P) \cap E_G(Z_{\gamma(\delta)}, V(G)\setminus V(T_\delta)) = \{f_\delta^0, f_{\gamma(\delta)}^1, f_{\gamma(\delta)}^2\}$.

By Proposition 2.21(f), we have $\bar{\varphi}_\delta(u_\delta^1) \neq \emptyset$ and $\bar{\varphi}_\delta(u_\delta^2) \neq \emptyset$. Since no colour in $\bar{\varphi}_\delta(U_\delta)$ is used on T_δ with respect to φ_δ, Lemma 3.13 implies that $F(T_\delta, e, \varphi_\delta) \subseteq Z_{\gamma(\delta)}$. Hence, for the last vertex v' in the linear order $\preceq_{(v,P)}$ belonging to $V(T)$, we conclude $v' \in \{v_\delta^0, v_1, v_2\}$, where v_1, v_2 are the two endvertices of $f_{\gamma(\delta)}^1, f_{\gamma(\delta)}^2$ belonging to $Z_{\gamma(\delta)}$.

From (2) it follows that there is an edge $g \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(g) = \beta$ and, by (5), also $\varphi_\delta(g) = \beta$. Hence, $P_1 = P(v_1, f_{\gamma(\delta)}^1, u_{\gamma(\delta)}^1, g, u_{\gamma(\delta)}^2, f_{\gamma(\delta)}^2, v_2)$ is a subpath of P and, therefore, $v' \notin \{v_1, v_2\}$, but $v' = v_\delta^0$.

Since $V(T)$ is closed with respect to φ, we clearly have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi) = \emptyset$ and $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi) = \emptyset$. Therefore, by (5), we have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi_\delta) = \emptyset$ and $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$. Hence, for the subpath $P_2 = vPv_\delta^0$, we conclude $V(P_2) \subseteq Z_{\gamma(\delta)} \cup U_{\gamma(\delta)}$. Furthermore, for $P_3 = v_\delta^0 Pz$, we clearly have $V(P_3) \cap V(T_\delta) = \{v_\delta^0\}$ and, therefore, $V(P) \cap U_\delta = \emptyset$. Then from Theorem 3.8(a) it follows that $E_G(U_\delta, V(G)\setminus V(T_\delta)) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$. Since we have $E_G(U_\delta, Z_{\gamma(\delta)}) \cap E_{\gamma(\delta)}(e, \varphi_\delta) = \emptyset$ and $\gamma(\delta) \notin \bar{\varphi}_\delta(V(T_\delta))$, we conclude $E_G(u_\delta^1, u_\delta^2) \cap E_{\gamma(\delta)}(e, \varphi_\delta) \neq \emptyset$. From (7) it then follows that $E_G(u_\delta^1, u_\delta^2) \cap E_{\gamma(\delta)}(e, \varphi) \neq \emptyset$. This proves the claim. □

Further, we claim that the following two statements are true:

(12) $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(U_\delta)$.

(13) $E_G(u_\delta^1, u_\delta^2) \cap E_\alpha(e, \varphi) \neq \emptyset$ for every $\delta \in \Gamma^d(T, e, \varphi)$ and every $\alpha \in \bar{\varphi}(U_{\gamma(\delta)})$.

Proof of (12): Let $\delta \in \Gamma^d(T, e, \varphi)$, and let $\alpha \in \bar{\varphi}(U_\delta)$. Then, clearly, we have $\alpha \neq \delta$ and, by (11), also $\alpha \neq \gamma(\delta)$. Consequently, $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$ and, therefore, $\alpha \in \bar{\varphi}_\delta(U_\delta)$. Moreover $\alpha_0 \in \bar{\varphi}_\delta(y_0)$, and $V(T_\delta)$ is elementary with respect to φ_δ. Hence, we have $\alpha \neq \alpha_0$. Since $V(T_\delta)$ is also closed with respect to φ_δ, for $P = P_{u_{\gamma(\delta)}^1}(\alpha_0, \alpha, \varphi) = P_{u_{\gamma(\delta)}^1}(\alpha_0, \alpha, \varphi_\delta)$, we have $V(P) \subseteq V(G) \setminus V(T_\delta)$. This implies $V(P) \cap V(T) \subseteq U_{\gamma(\delta)}$ and hence $E(P) \cap E(T) = \emptyset$. Then T is a Tashkinov tree with respect to e and the colouring $\varphi' = \varphi/P$, and from $y_0 \notin V(P)$ we conclude $\alpha_0 \in \bar{\varphi}'(V(T))$.

From (2) we know that $E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2) \cap E_{\alpha_0}(e, \varphi) \neq \emptyset$ and, therefore, we have $u_{\gamma(\delta)}^2 \in V(P)$. If there is an edge $g \in E_G(U_{\gamma(\delta)}, z)$ for a vertex $z \in V(G) \setminus V(T)$ and $\varphi(g) = \alpha$, then we have $\varphi'(g) = \alpha_0 \in \bar{\varphi}'(V(T))$, and $T' = (T, g, z)$ is a Tashkinov tree with respect to e and φ', satisfying $|V(T')| > |V(T)| = t(G)$, a contradiction. If there is an edge $g \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(g) = \alpha$, then we have $\varphi_\delta(g) = \alpha \in \bar{\varphi}_\delta(V(T_\delta))$, a contradiction, too, because $V(T_\delta)$ is closed with respect to φ_δ. Consequently, we have $E_G(U_{\gamma(\delta)}, V(G) \setminus U_{\gamma(\delta)}) \cap E_\alpha(e, \varphi) = \emptyset$. Since $V(T)$ is elementary with respect to φ, we conclude that there is an edge $g \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(g) = \alpha$. This proves the claim. □

Proof of (13): Let $\delta \in \Gamma^d(T, e, \varphi)$, and let $\alpha \in \bar{\varphi}(U_{\gamma(\delta)})$. Then, clearly, we have $\alpha \neq \gamma(\delta)$ and, by (10), also $\alpha \neq \delta$. Consequently, $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$. Moreover, $\alpha_0 \in \bar{\varphi}_\delta(y_0)$, and $V(T)$ is elementary with respect to φ and hence $\alpha \neq \alpha_0$. Since $V(T)$ is also closed with respect to φ, for $P = P_{u_\delta^1}(\alpha_0, \alpha, \varphi) = P_{u_\delta^1}(\alpha_0, \alpha, \varphi_\delta)$, we have

$V(P) \subseteq V(G) \backslash V(T)$. This implies $V(P) \cap V(T_\delta) \subseteq U_\delta$ and, therefore, $E(P) \cap E(T_\delta) = \emptyset$. Then T_δ is a Tashkinov tree with respect to e and the colouring $\varphi' = \varphi_\delta/P$, and from $y_0 \notin V(P)$ we conclude $\alpha_0 \in \bar{\varphi}'(V(T_\delta))$.

From (9) we know $E_G(u_\delta^1, u_\delta^2) \cap E_{\alpha_0}(e, \varphi) \neq \emptyset$ and, therefore, $u_\delta^2 \in V(P)$. If there was an edge $g \in E_G(U_\delta, z)$ with $z \in V(G) \backslash V(T_\delta)$ and $\varphi_\delta(g) = \alpha$, then we would have $\varphi'(g) = \alpha_0 \in \bar{\varphi}'(V(T_\delta))$, and $T' = (T_\delta, g, z)$ would be a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. If there was an edge $g \in E_G(U_\delta, Z_{\gamma(\delta)})$ with $\varphi_\delta(g) = \alpha$, then we would have $\varphi(g) = \alpha \in \bar{\varphi}(V(T))$, a contradiction, too, because $V(T)$ is closed with respect to φ. Consequently, we have $E_G(U_\delta, V(G) \backslash U_\delta) \cap E_\alpha(e, \varphi_\delta) = \emptyset$. Since $V(T_\delta)$ is elementary with respect to φ_δ, we conclude that there is an edge $g \in E_G(u_\delta^1, u_\delta^2)$ with $\varphi_\delta(g) = \varphi(g) = \alpha$. This proves the claim. □

Next we claim

(14) $U_\delta \subseteq A(T, e, \varphi)$ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (14): Let $\delta \in \Gamma^d(T, e, \varphi)$, let $\alpha \in \bar{\varphi}(U_\delta)$, and let $\beta \in \Gamma^f(T, e, \varphi) \setminus \{\alpha\}$. Clearly, we have $\alpha \neq \delta$ and, by (11), also $\alpha \neq \gamma(\delta)$. Since neither δ nor $\gamma(\delta)$ is a free colour with respect to (T, e, φ), we also have $\beta \notin \{\delta, \gamma(\delta)\}$. Hence, $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$ and $E_\beta(e, \varphi) = E_\beta(e, \varphi_\delta)$.

Let $u \in U_\delta$ be the unique vertex with $\alpha \in \bar{\varphi}(u) = \bar{\varphi}_\delta(u)$, and let $v \in V(T)$ be the unique vertex with $\beta \in \bar{\varphi}(v) = \bar{\varphi}_\delta(v)$. Moreover, let $P = P_u(\alpha, \beta, \varphi) = P_u(\alpha, \beta, \varphi_\delta)$. Obviously, P is a path having u as an endvertex. We now have to show that v is the second endvertex of P, this then implies $u \in A(T, e, \varphi)$.

In the case $v \in Z_{\gamma(\delta)}$ we have $v \in V(T_\delta)$ and, therefore, $\beta \in \bar{\varphi}_\delta(V(T_\delta))$. Since also $\alpha \in \bar{\varphi}_\delta(V(T_\delta))$, Proposition 2.21(h) implies that v is the second endvertex of P.

In the other case, we have $v \in U_{\gamma(\delta)}$. By (13), we have $E_G(u_\delta^1, u_\delta^2) \cap E_\beta(e, \varphi) \neq \emptyset$ and, therefore, $U_\delta \subseteq V(P)$. Since $\alpha \in \bar{\varphi}_\delta(u)$ and $V(T_\delta)$ is elementary and closed with respect to φ_δ, there is an edge $f_2 \in E_G(U_\delta, Z_{\gamma(\delta)})$ with $\varphi_\delta(f_2) = \alpha$ and, moreover, $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\alpha(e, \varphi_\delta) = \{f_2\}$. Therefore, $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\alpha(e, \varphi) = \{f_2\}$ and $f_2 \in E(P)$. Since $\beta \in \bar{\varphi}(U_{\gamma(\delta)})$ and $V(T)$ is elementary and closed with respect to φ, there is an edge $f_3 \in E_G(U_{\gamma(\delta)}, Z_{\gamma(\delta)})$ with $\varphi(f_3) = \beta$ and, moreover, $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)}) \cap E_\beta(e, \varphi) = \{f_3\}$. Since we have $f_2 \in E(P)$ and $\alpha, \beta \notin \bar{\varphi}(Z_{\gamma(\delta)})$, and since f_2, f_3 are the only two edges in $E_G(Z_{\gamma(\delta)}, V(G) \setminus Z_{\gamma(\delta)})$ coloured with α or β with respect to φ, we conclude that $f_3 \in E(P)$. By (12), there is an edge $f_4 \in E_G(u_{\gamma(\delta)}^1, u_{\gamma(\delta)}^2)$ with $\varphi(f_4) = \alpha$. This implies $f_4 \in E(P)$ and, therefore, $U_{\gamma(\delta)} \subseteq V(P)$. Hence, we have $v \in V(P)$, so v must be the second endvertex of P.

In both cases P is a path with endvertices u and v. Hence, by definition, we have $u \in A(T, e, \varphi)$ and the claim is proved. □

Now let

$$X = V(T) \cup \bigcup_{\delta \in \Gamma^d(T, e, \varphi)} U_\delta.$$

Then, by (14), we have $X \subseteq V(T) \cup A(T, e, \varphi)$. Hence, Proposition 3.9 implies

(15) X is elementary with respect to φ.

The aim is to show that X is also closed with respect to φ. To do this, we first claim

(16) $X_\delta = V(T) \cup U_\delta$ is closed with respect to φ for every $\delta \in \Gamma^d(T, e, \varphi)$.

Proof of (16): Let $\delta \in \Gamma^d(T, e, \varphi)$, and let $\alpha \in \bar{\varphi}(X_\delta)$. We have to show that $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

If $\alpha \in \bar{\varphi}(V(T))$ we conclude from (9), (11), and (13) that $E_G(u^1_\delta, u^2_\delta) \cap E_\alpha(e, \varphi) \neq \emptyset$ and, therefore, $E_G(U_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Moreover, since $V(T)$ is closed with respect to φ, we also have $E_G(V(T), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Hence, we conclude $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

If $\alpha \in \bar{\varphi}(U_\delta)$ then, clearly, we have $\alpha \neq \delta$ and, by (11), also $\alpha \neq \gamma(\delta)$. Hence, we have $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$. Consequently, $\alpha \in \bar{\varphi}_\delta(V(T_\delta))$ and, since $V(T_\delta)$ is closed with respect to φ_δ, we obtain $E_G(V(T_\delta), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = E_G(V(T_\delta), V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi_\delta) = \emptyset$. Moreover, from (12) we know that $E_G(u^1_{\gamma(\delta)}, u^2_{\gamma(\delta)}) \cap E_\alpha(e, \varphi) \neq \emptyset$ and, therefore, $E_G(U_{\gamma(\delta)}, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. Hence, we obtain $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$.

In any case, we have $E_G(X_\delta, V(G) \setminus X_\delta) \cap E_\alpha(e, \varphi) = \emptyset$. This proves the claim. □

Since, by (9), we have $E_G(u^1_\delta, u^2_\delta) \cap E_{\alpha_0}(e, \varphi) \neq \emptyset$ for all $\delta \in \Gamma^d(T, e, \varphi)$, we easily conclude the following.

(17) For any $\delta, \delta' \in \Gamma^d(T, e, \varphi)$, the sets U_δ and $U_{\delta'}$ are either equal or disjoint.

Now we can show that

(18) X is closed with respect to φ.

Proof of (18): Suppose, on the contrary, that X is not closed with respect to φ, that is, there exists a colour $\alpha \in \bar{\varphi}(X)$ and an edge $f \in E_G(X, V(G) \setminus X)$ with $\varphi(f) = \alpha$. Then, clearly, there is a colour $\delta \in \Gamma^d(T, e, \varphi)$ satisfying $f \in E_G(V(T) \cup U_\delta, V(G) \setminus X)$. Since, by (16), $V(T) \cup U_\delta$ is closed with respect to φ, we conclude that $\alpha \in \bar{\varphi}(X \setminus V(T) \setminus U_\delta)$. Consequently, by (17), $\alpha \in \bar{\varphi}(U_{\delta'})$ for a colour $\delta' \in \Gamma^d(T, e, \varphi)$ with $U_\delta \cap U_{\delta'} = \emptyset$.

Since, by (16), also $V(T) \cup U_{\delta'}$ is closed with respect to φ, we have $f \notin E_G(V(T) \cup U_{\delta'}, V(G) \setminus X)$. In particular, this means $f \notin E_G(V(T), V(G) \setminus X)$ and, therefore, we conclude $f \in E_G(u, v)$ for two vertices $u \in U_\delta$ and $v \in V(G) \setminus X$. Now let $P = P_u(\alpha_0, \alpha, \varphi)$. Since $\alpha_0, \alpha \in \bar{\varphi}(V(T) \cup U_{\delta'})$ and $V(T) \cup U_{\delta'}$ is closed with respect to φ, this implies $V(P) \cap V(T) = \emptyset$. Hence, we have $E(P) \cap E(T_\delta) = \emptyset$.

By (15), X is elementary with respect to φ. Since $\alpha \in \bar{\varphi}(U_{\delta'})$ and $\Gamma \subseteq \bar{\varphi}(V(T))$, we conclude that $\alpha \notin \Gamma$. Moreover, since, by (16), $V(T) \cup U_{\delta'}$ is closed with respect to φ, and since $f^1_\delta \in E_G(V(T), V(G) \setminus V(T) \setminus U_{\delta'})$, we also conclude that $\alpha \neq \delta$. Moreover, we also have $\alpha_0 \notin \Gamma$ and $\alpha_0 \neq \delta$. Evidently, we conclude $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$ and $E_{\alpha_0}(e, \varphi) = E_{\alpha_0}(e, \varphi_\delta)$, which especially implies $P = P_u(\alpha_0, \alpha, \varphi_\delta)$. From $E(P) \cap E(T_\delta) = \emptyset$ it then follows that T_δ is a Tashkinov tree with respect to e and $\varphi' = \varphi_\delta / P$. Since $f \in E(P)$, we have $\varphi'(f) = \alpha_0 \in \bar{\varphi}'(V(T_\delta))$. Hence, $T' = (T_\delta, f, v)$ is a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. This proves the claim. □

Next, we claim the following:

(19) If $\alpha \notin \bar{\varphi}(X)$ and $P = P_{y_0}(\alpha_0, \alpha, \varphi)$, then $|E(P) \cap E_G(X, V(G) \setminus X)| = 1$.

Proof of (19): By (15), X is elementary with respect to φ and, since $\alpha_0 \in \bar{\varphi}(y_0)$, we know that P is a path with one endvertex y_0 and another endvertex $z \in V(G) \setminus X$. Evidently, there is a last vertex v in the linear order $\preceq_{(y_0,P)}$ that belongs to X, and there is an edge in $g \in E_G(v, V(G) \setminus X)$ with $\varphi(g) = \alpha$. For the subpath $P_1 = y_0 P v$ of P, we have to show that $V(P_1) \subseteq X$, this would complete the proof of (19). To do this, we distinguish the following cases.

Case 1: *$v \in V(T)$ and $\alpha \notin \Gamma^d(T, e, \varphi)$.* Then we have $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \{g\}$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we conclude that $E(P) \cap E_G(V(T), V(G) \setminus V(T)) = \{g\}$ and, therefore, $V(P_1) \subseteq V(T) \subseteq X$.

Case 2: *$v \in V(T)$ and $\alpha \in \Gamma^d(T, e, \varphi)$.* Then from (6) and (10) we conclude that $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \{f^0_\alpha, f^1_\alpha, f^2_\alpha\}$. Hence, we have $E_G(X_\alpha, V(G) \setminus X_\alpha) = \{f^0_\alpha\}$, which implies $g = f^0_\alpha$. Since $\alpha_0 \in \bar{\varphi}(X_\alpha)$ and, by (16), X_δ is closed with respect to φ, it follows that $V(P_1) \subseteq X_\alpha \subseteq X$.

Case 3: *$v \notin V(T)$.* Then, evidently, $v \in U_\delta$ for some $\delta \in \Gamma^d(T, e, \varphi)$. Since $\varphi(g) = \alpha$, we conclude that $\alpha \neq \delta$. Clearly, we also have $\alpha \neq \gamma(\delta)$ and, therefore, we infer that $E_\alpha(e, \varphi) = E_\alpha(e, \varphi_\delta)$. Moreover, we also have $\alpha_0 \notin \{\delta, \gamma(\delta)\}$ and, therefore, it follows that $P = P_{y_0}(\alpha_0, \alpha, \varphi_\delta)$. Clearly, v is the last vertex in $\preceq_{(y_0,P)}$ that belongs to $V(T_\delta)$. Since no colour from $\bar{\varphi}_\delta(v)$ is used on T_δ with respect to φ_δ, we infer that $\bar{\varphi}_\delta(v) \cap \Gamma^f(T_\delta, e, \varphi_\delta) \neq \emptyset$. From Lemma 3.13 it then follows that $v \notin F(T_\delta, e, \varphi_\delta)$ and, therefore, $\alpha \notin \Gamma^d(T_\delta, e, \varphi_\delta)$. Hence, we conclude that $E(P) \cap E_G(V(T_\delta), V(G) \setminus V(T_\delta)) = \{g\}$, which implies $V(P_1) \subseteq V(T_\delta) \subseteq X$. This settles the case.

In any of the three cases we have $V(P_1) \subseteq X$, which implies $E(P) \cap E_G(X, V(G) \setminus X) = \{g\}$. This completes the proof. □

Eventually, we can show that

(20) X is strongly closed with respect to φ.

Proof of (20): Suppose, on the contrary, that X is not strongly closed with respect to φ. Since, by (18), X is closed with respect to φ, this implies that there is a colour α satisfying $\alpha \notin \bar{\varphi}(X)$ and $|E_G(X, V(G) \setminus X) \cap E_\alpha(e, \varphi)| \geq 2$. Obviously, this implies $|E_G(X, V(G) \setminus X) \cap E_\alpha(e, \varphi)| \geq 3$, because $|X|$ is odd.

If $|E_G(V(T), V(G) \setminus X) \cap E_\alpha(e, \varphi)| \geq 2$ then we would have $\alpha \in \Gamma^d(T, e, \varphi)$, but then (6) and (10) would imply $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \{f^0_\alpha, f^1_\alpha, f^2_\alpha\}$ and, therefore, $E_G(V(T), V(G) \setminus X) \cap E_\alpha(e, \varphi) = \{f^0_\alpha\}$, a contradiction. Consequently, we have $|E_G(V(T), V(G) \setminus X) \cap E_\alpha(e, \varphi)| \leq 1$, which leads to $|E_G(X \setminus V(T), V(G) \setminus X) \cap E_\alpha(e, \varphi)| \geq 2$.

For the path $P = P_{y_0}(\alpha_0, \alpha, \varphi)$, it follows from (19) that $|E(P) \cap E_G(X, V(G) \setminus X)| = 1$. Consequently, there is a colour $\delta \in \Gamma^d(T, e, \varphi)$, and there is an edge $f \in E_G(U_\delta, V(G) \setminus X)$ satisfying $\varphi(f) = \alpha$ and $f \notin E(P)$. Let u be the endvertex of f that belongs to $V(G) \setminus X$, and let $P' = P_u(\alpha_0, \alpha, \varphi)$. Since $f \in E(P')$ but $f \notin E(P)$, we infer that P and P' are vertex disjoint. Further, we claim that $V(P') \cap V(T) = \emptyset$. To prove this, we have two consider two cases.

Case 1: $\alpha \in \Gamma^d(T, e, \varphi)$. From (6) and (10) we then conclude $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi) = \{f_\alpha^0, f_\alpha^1, f_\alpha^2\}$ and, therefore, we have $|E_G(X_\alpha, V(G) \setminus X_\alpha) \cap E_\alpha(e, \varphi)| = 1$. Since $\alpha_0 \in \bar{\varphi}(X_\alpha)$ and, by (16), X_α is closed with respect to φ, we also have $E_G(X_\alpha, V(G) \setminus X_\alpha) \cap E_{\alpha_0}(e, \varphi) = \emptyset$. Since the only edge in $E_G(X_\alpha, V(G) \setminus X_\alpha) \cap E_\alpha(e, \varphi)$ must belong to $E(P)$, we conclude that $V(P') \cap X_\alpha = \emptyset$ and, therefore, also $V(P') \cap V(T) = \emptyset$.

Case 2: $\alpha \notin \Gamma^d(T, e, \varphi)$. Then the only edge in $E_G(V(T), V(G) \setminus V(T)) \cap E_\alpha(e, \varphi)$ belongs to $E(P)$. Since $\alpha_0 \in \bar{\varphi}(V(T))$ and $V(T)$ is closed with respect to φ, we conclude that $V(P') \cap V(T) = \emptyset$. This settles the case.

In any case we have $V(P') \cap V(T) = \emptyset$, which implies $E(P') \cap E(T_\delta) = \emptyset$. Moreover, from $\alpha, \alpha_0 \notin \{\delta, \gamma(\delta)\}$ we conclude that $P' = P_u(\alpha_0, \alpha, \varphi_\delta)$. Then, evidently, T_δ is a Tashkinov tree with respect to e and $\varphi' = \varphi_\delta / P'$. From $\varphi'(f) = \alpha_0 \in \bar{\varphi}'(V(T_\delta))$ it then follows that $T' = (T_\delta, f, u)$ is also a Tashkinov tree with respect to e and φ' satisfying $|V(T')| > |V(T_\delta)| = t(G)$, a contradiction. This proves (20). □

Now, by (15) and (20), X is elementary and strongly closed with respect to φ. Then Theorem 1.4 implies that G is an elementary graph, which completes the proof. ■

Algorithmic aspect: An algorithmic version of the last result would be of the following kind. Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k(G, e, \varphi)$ for an integer $k \geq \Delta(G) + 1$ such that T is of the described structure. Then G is an elementary graph, or there is a k-edge colouring of G, or there is a triple $(T', e, \varphi') \in \mathcal{T}_k(G)$ with $|V(T')| > |V(T)|$. The proof shows, if one of the many claims is not fulfilled, how this colouring of G or the new triple can be computed. Note that, since we are not dealing with a critical graph, many of the necessary recolouring and rearranging operations need additionally steps for checking the usual properties like elementarity and so on. If any of these checks fails, we can use the methods from the last section to compute the k-edge colouring of G or the triple (T', e, φ'). Evidently, a detailed algorithmic proof would be very nasty, but it is clear how an algorithm had to be constructed. It is also very difficult to get, from this non-algorithmic proof, an exact running time of such an algorithm. However, the running time is bounded polynomial in $\Delta(G)|$ and $|V(G)|$, because all algorithms from the last section have such a running time and have to be performed only a polynomially bounded number of times.

3.5 Proof of Theorem 3.2

Proposition 3.18 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $h(G) > t(G) - 4$ then G is an elementary graph.*

Proof: Let $h(G) > t(G) - 4$. By Corollary 3.7 and Lemma 3.16(a), both $t(G)$ and $h(G)$ are odd. Hence, we have either $h(G) = t(G)$ or $h(G) = t(G) - 2$.

By Lemma 3.16(b), we have $\mathcal{T}^B(G) \neq \emptyset$. Consequently, there is an edge $e \in E_G(x, y)$, a colouring $\varphi \in \mathcal{C}_k(G - e)$ and a Tashkinov tree T with respect to e and φ satisfying $(T, e, \varphi) \in \mathcal{T}^B(G)$. Hence, $h(T) = h(G)$.

If $h(G) = t(G)$ then T consists only of its trunk, and only two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$ are used on T with respect to φ. Therefore, Proposition 2.21(f) implies that $\bar{\varphi}(v) \cap \Gamma^f(T, e, \varphi) \neq \emptyset$ for every vertex $v \in V(T)$. From Lemma 3.13 we then conclude that $F(T,e,\varphi) = \emptyset$ and hence, also $\Gamma^d(T,e,\varphi) = \emptyset$. From Proposition 2.21(a),(c) it then follows that $V(T)$ is elementary and strongly closed with respect to φ. Consequently, by Theorem 1.4, G is an elementary graph.

In the other case we have $h(G) = t(G) - 2$ and, therefore, T has the form

$$T = (y_0, e_1, y_1, \dots, e_{q-1}, y_{q-1}, f_1, u_1, f_2, u_2)$$

where $x = y_0$, $y = y_1$ and $q = h(T)$. Clearly, exactly two colours $\alpha \in \bar{\varphi}(x)$ and $\beta \in \bar{\varphi}(y)$ are used on Ty_{q-1} with respect to φ. Moreover $\varphi(f_1) = \varphi(f_2) = \gamma \in \bar{\varphi}(\{y_0, \dots, y_{q-1}\}) \setminus \{\alpha, \beta\}$, and for $j = 1, 2$ we have $f_j \in E_G(\{y_0, \dots, y_{q-1}\}, u_j)$. Since $V(T)$ is elementary with respect to φ, there is a unique vertex $y_r \in \{y_0, \dots, y_{q-1}\}$ with $\gamma \in \bar{\varphi}(y_r)$. Since there are exactly three colours $\alpha \in \bar{\varphi}(x)$, $\beta \in \bar{\varphi}(y)$ and $\gamma \in \bar{\varphi}(y_r)$ used on T with respect to φ, we conclude from Proposition 2.21(f) that $\bar{\varphi}(v) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ for every vertex $v \in V(T) \setminus \{y_r\}$. Then Lemma 3.13 implies $F(T,e,\varphi) \subseteq \{y_r\}$. Hence, (T,e,φ) fulfils the conditions of Lemma 3.17 and, therefore, G is an elementary graph. This completes the proof. ■

Algorithmic aspect: An algorithmic version of this result would deal with an arbitrary graph G and a balanced triple $(T,e,\varphi) \in \mathcal{T}_k^B(G,e,\varphi)$ for an integer $k \geq \Delta(G) + 1$. If $h(T) > |V(T)| - 4$ then G is an elementary graph, or there is a k-edge colouring of G, or there is a triple $(T', e, \varphi') \in \mathcal{T}_k(G)$ with $|V(T')| > |V(T)|$. Similar to the proof, the algorithm works as follows. If T consists only of its trunk then either $V(T)$ is elementary and strongly closed with respect to φ, or there is an exit vertex where a free colour is missing. In the first case, G is elementary; in the second case, we can use Theorem 2.23 to construct a larger Tashkinov tree. Now consider the case $h(T) = |V(T)| - 2$. We can assume that there is a defective colour, otherwise G is elementary. Then y_r is the unique exit vertex and we can use the algorithm corresponding to Lemma 3.17. We may use a specialized version of this algorithm, because we have only one exit vertex. Since we are not interested in an exact running time, it does not matter. In any case the running time of our algorithm is bounded by a polynomial in $\Delta(G)$ and $|V(G)|$.

Proposition 3.19 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If $h(G) < 5$ then G is an elementary graph.*

Proof: Let $q = h(G) < 5$. Then, by Lemma 3.16(a), we have $q = 3$ and, by Lemma 3.16(b), there is a triple $(T,e,\varphi) \in \mathcal{T}^B(G)$. Hence, T has the form

$$T = (y_0, e_1, y_1, \dots, e_{p-1}, y_{p-1})$$

where $p = t(G)$, and $T_1 = (y_0, e_1, y_1, e_2, y_2)$ is the (α, β)-trunk of T where $\alpha \in \bar{\varphi}(y_0)$ and $\beta \in \bar{\varphi}(y_1)$. Moreover, there is an edge $f \in E_G(y_2, y_0)$ with $\varphi(f) = \beta$.

If $p = 3$ then $t(G) = h(G)$, and Proposition 3.18 implies that G is an elementary graph.

Now consider the case $p > 3$. Since (T,e,φ) is a balanced triple and $q = 3$, we have $\varphi(e_3) = \varphi(e_4) = \gamma$ and $\gamma \in \bar{\varphi}(y_j)$ for some $j \in \{0, 1, 2\}$. Without loss of generality we may assume that $j = 0$, otherwise we could replace the balanced triple (T,e,φ) by the balanced triple $(T,e,\varphi)(y_0 \to y_j)$. Therefore, we have $e_3, e_4 \in$

$E_G(\{y_1, y_2\}, \{y_3, y_4\})$ and, moreover, (y_0, y_1) is a (γ, β)-pair with respect to φ. From Proposition 2.21(h) we then conclude that there is a (γ, β)-chain P with respect to φ having endvertices y_1 and y_0. Evidently, $q' = |V(P)|$ is odd, $f, e_3, e_4 \in E(P)$ and $y_0, y_1, y_2, y_3, y_4 \in V(P)$. Therefore, we have $q' \geq 5$ and, by Lemma 3.14, there is a Tashkinov tree T' with respect to e and φ satisfying $(T', e, \varphi) \in \mathcal{T}^N(G)$ and $h(T') = q' \geq 5 > h(G)$, a contradiction. Hence, G is an elementary graph. This completes the proof. ■

Algorithmic aspect: The corresponding algorithm to this result is very simple. Let G be a graph, and let $(T, e, \varphi) \in \mathcal{T}_k^B(G)$ for an integer $k \geq \Delta(G) + 1$. If $h(T) < 5$ then $h(T) = 3$, and then the algorithm works as follows. If there is no defective colour then G is elementary. So we assume that there is a defective colour. If $|V(T)| = 3$ then there is an exit vertex where a free colour is missing and hence, we can use Theorem 2.23 to construct a larger Tashkinov tree. If $|V(T)| > 3$ then we can easily construct a normal triple $(T', e', \varphi') \in \mathcal{T}_k^N(G)$ with $V(T') = V(T)$ and $h(T') \geq 5$.

Lemma 3.20 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$ such that $e \in E_G(x, y)$. Furthermore, let (x, y) be an (α, β)-pair with respect to φ, let $P = P_x(\alpha, \beta, \varphi)$, and let $\varphi' = \varphi/P$. Then P is a path having endvertices x and y, $(T, e, \varphi') \in \mathcal{T}(G)$, and there is a triple $(T', e, \varphi) \in \mathcal{T}^N(G)$ such that $h(T) = |V(P)|$.*

Proof: By Corollary 3.7, $(T, e, \varphi) \in \mathcal{T}_k(G)$. Hence, Proposition 2.21(h) implies that P is a path with endvertices x and y, and $(T, e, \varphi') \in \mathcal{T}_k(G)$. Since $|V(T)| = t(G)$, we then have $(T, e, \varphi') \in \mathcal{T}(G)$. Since (x, y) is an (α, β)-pair with respect to φ, it follows from Lemma 3.14 that there is a triple $(T', e, \varphi) \in \mathcal{T}_k^N(G)$ such that $V(T') = V(T)$ and $h(T') = |V(P)|$. Then $|V(T')| = |V(T)| = t(G)$ and, therefore, $(T', e, \varphi) \in \mathcal{T}^N(G)$. This completes the proof. ■

Lemma 3.21 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$ such that $h(G) = 5$. Furthermore, let $(T, e, \varphi) \in \mathcal{T}^N(G)$, and let $T' = (y_0, e_1, y_1, e_2, y_2, e_3, y_3, e_4, y_4)$ be the (α, β)-trunk of T. If $\gamma \in \bar{\varphi}(y_0)$ is a colour satisfying $E_G(V(T'), V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) \neq \emptyset$, then the following statements hold:*

(a) *There are three edges $f_1 \in E_G(y_1, V(T) \setminus V(T'))$, $f_2 \in E_G(y_4, V(T) \setminus V(T'))$ and $f_3 \in E_G(y_2, y_3)$ with $\varphi(f_1) = \varphi(f_2) = \varphi(f_3) = \gamma$.*

(b) *For the two endvertices $v_1, v_2 \in V(T) \setminus V(T')$ of the two edges f_1, f_2, we have $E_G(v_1, v_2) \cap E_\alpha(e, \varphi) \neq \emptyset$ and $E_G(v_1, v_2) \cap E_\beta(e, \varphi) \neq \emptyset$.*

Proof: Since T is a normal Tashkinov tree with respect to e and φ and $T' = Ty_4$ is the (α, β)-trunk of T, we conclude that T' is a path satisfying $e_1 = e$, $\varphi(e_2) = \varphi(e_4) = \alpha \in \bar{\varphi}(y_0)$, and $\varphi(e_3) = \beta \in \bar{\varphi}(y_1)$. Furthermore, there is an edge $e_0 \in E_G(y_4, y_0)$ with $\varphi(e_0) = \beta$.

Let $P_1 = P_{y_0}(\gamma, \beta, \varphi)$. Clearly, (y_0, y_1) is a (γ, β)-pair with respect to φ. Since $(T, e, \varphi) \in \mathcal{T}^N(G)$, it then follows from Lemma 3.20 that P_1 is a path having endvertices y_0 and y_1, and there is a triple $(T_1, e, \varphi) \in \mathcal{T}^N(G)$ such that $h(T_1) = |V(P_1)|$. Since $h(G) = 5$, we then obtain $|V(P_1)| \leq 5$. Since y_0, y_1 are the endvertices of P_1, $e_0 \in E_G(y_4, y_0)$, and $\varphi(e_0) = \beta$, we conclude that $e_0 \in E(P_1)$ and $y_0, y_1, y_4 \in V(P_1)$.

Now we claim that $E_G(\{y_2, y_3\}, V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$. Suppose this is not true. Then there is an edge $g \in E_G(\{y_2, y_3\}, V(T) \setminus V(T'))$ with $\varphi(g) = \gamma$. Let v be the endvertex of g that belongs to $V(T) \setminus V(T')$. Since $e_3 \in E_G(y_2, y_3)$ and $\varphi(e_3) = \beta$, we conclude that either none of the vertices y_2, y_3, v belong to P_1, or all three vertices belong to P_1. In the latter case, however, we would obtain $|V(P_1)| \geq |V(T') \cup \{v\}| \geq 6$, a contradiction. Hence, none of the vertices y_2, y_3, v belong to P_1. Then, for the colouring $\varphi_1 = \varphi/P_1$, we have $\alpha, \beta \in \bar{\varphi}_1(y_0)$, $\gamma \in \bar{\varphi}_1(y_1)$, $\varphi_1(e_0) = \varphi_1(g) = \gamma$, and $\varphi_1(e_2) = \varphi_1(e_4) = \alpha$. Consequently, the chain $P_2 = P_{y_0}(\alpha, \gamma, \varphi_1)$ contains the two subpaths $P(y_0, e_0, y_4, e_4, y_3)$ and $P(y_1, e_2, y_2)$, implying $g \in E(P_2)$ and $v \in V(P_2)$. Hence, we have $|V(P_2)| \geq 6$. Moreover, by Lemma 3.20, we have $(T, e, \varphi_1) \in \mathcal{T}(G)$. Since (y_0, y_1) is an (α, γ)-pair with respect to φ_1, it then follows from Lemma 3.20 that there is a triple $(T_2, e, \varphi_1) \in \mathcal{T}^N(G)$ such that $h(T_2) = |V(P_2)| \geq 6$, a contradiction to $h(G) = 5$. This proves the claim that $E_G(\{y_2, y_3\}, V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$.

Since $(T, e, \varphi) \in \mathcal{T}(G)$, Corollary 3.7 implies that $V(T)$ is elementary and closed with respect to φ. Since $\gamma \in \bar{\varphi}(y_0)$, we then conclude that $E_\gamma(T, e, \varphi) = \emptyset$ and, for every vertex $y \in \{y_1, y_2, y_3, y_4\}$, there is an edge $f_y \in E_G(y, v_y)$ such that $v_y \in V(T)$ and $\varphi(f_y) = \gamma$.

Now, we claim that $f_{y_2} = f_{y_3} = f_3$. Otherwise, $E_G(\{y_2, y_3\}, V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$ would imply that $f_{y_2}, f_{y_3} \in E_G(\{y_2, y_3\}, \{y_1, y_4\})$ and, therefore, $E_G(V(T'), V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$, contradicting the assumption. This proves the claim. Clearly, $f_3 \in E_G(y_2, y_3)$ and, therefore, this part of (a) is proved.

Then the edges $f_1 = f_{y_1}$ and $f_2 = f_{y_4}$ are distinct, since otherwise we would again have $E_G(V(T'), V(T) \setminus V(T')) \cap E_\gamma(e, \varphi) = \emptyset$, a contradiction to the assumption. Since $f_1 \neq f_2$, the vertices $v_1 = v_{y_1}$ and $v_2 = v_{y_4}$ are distinct, too, and both vertices are contained in $V(T) \setminus V(T')$. This completes the proof of (a).

Then, the chain $P_1 = P_{y_0}(\gamma, \beta, \varphi)$ contains the two subpaths $P(y_0, e_0, y_4, f_2, v_2)$ and $P(y_1, f_1, v_1)$. Since $|V(P_1)| \leq 5$, this implies that there is an edge $g_1 \in E_G(v_1, v_2)$ such that $\varphi(g_1) = \beta$. This proves the first part of (b).

Eventually, consider the chain $P_3 = P_{y_0}(\alpha, \beta, \varphi)$ and the colouring $\varphi_2 = \varphi/P_3$. Clearly, $P_3 = P(y_1, e_2, y_2, e_3, y_3, e_4, y_4, e_0, y_0)$. Since $(T, e, \varphi) \in \mathcal{T}^N(G)$ and (y_0, y_1) is an (α, β)-pair with respect to φ_2, Lemma 3.20 implies that $(T, e, \varphi_2) \in \mathcal{T}(G)$. Moreover, we have $\gamma, \beta \in \bar{\varphi}_2(y_0)$, $\alpha \in \bar{\varphi}_2(y_1)$, $\varphi_2(e_0) = \alpha$, and $\varphi_2(f_1) = \varphi_2(f_2) = \gamma$. Hence, the chain $P_4 = P_{y_0}(\gamma, \alpha, \varphi_2)$ contains the two subpaths $P(y_0, e_0, y_4, f_2, v_2)$ and $P(y_1, f_1, v_1)$, implying $|V(P_4)| \geq 5$. Since $(T, e, \varphi_2) \in \mathcal{T}(G)$ and (y_0, y_1) is a (γ, α)-pair with respect to φ_2, Lemma 3.20 implies that there is a triple $(T', e, \varphi_2) \in \mathcal{T}^N(G)$ such that $h(T') = |V(P_4)|$. Since $h(G) = 5$, we conclude that $|V(P_4)| = 5$. Hence, there is an edge $g_2 \in E_G(v_1, v_2)$ with $\varphi_2(g_2) = \alpha$. Since $g_2 \notin E(P_3)$, we also have $\varphi(g_2) = \alpha$. This completes the proof of (b). ■

This result gives some structural information about Tashkinov trees of height 5, which is used in the proof of the next result. An algorithmic implementation of the last result is quite simple. If an algorithm does not find the expected structure of a Tashkinov tree T, then it can increase either the order or the height of T.

Proposition 3.22 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. If $t(G) < 11$ then G is an elementary graph.*

Proof: By Corollary 3.7, $t(G) \geq 3$ is odd and, therefore, we have $t(G) \leq 9$. If $h(G) > t(G) - 4$ or $h(G) < 5$, then it follows from Proposition 3.18 respectively from Proposition 3.19 that G is elementary. So we only have to consider the case where $h(G) \leq t(G) - 4$ and $h(G) \geq 5$. Since $t(G) \leq 9$, this implies that $h(G) = 5$ and $t(G) = 9$. Our aim is to show that there is a balanced triple that fulfils the conditions of Lemma 3.17. Clearly, this would imply that G is elementary.

By Lemma 3.16 and Corollary 3.7, we have $\emptyset \neq \mathcal{T}^B(G) \subseteq \mathcal{T}_k(G)$. Hence, we can choose a balanced triple $(T, e, \varphi) \in \mathcal{T}^B(G)$. Since $h(G) = 5$ and $t(G) = 9$, T has the form

$$T = (y_0, e_1, y_1, e_2, y_2, e_3, y_3, e_4, y_4, e_5, y_5, e_6, y_6, e_7, y_7, e_8, y_8)$$

where $e_1 = e$, $\varphi(e_2) = \varphi(e_4) = \alpha \in \bar{\varphi}(y_0)$, $\varphi(e_3) = \beta \in \bar{\varphi}(y_1)$, $\varphi(e_5) = \varphi(e_6) = \gamma_1 \in \bar{\varphi}(\{y_0, \ldots, y_4\})$ and $\varphi(e_7) = \varphi(e_8) = \gamma_2 \in \bar{\varphi}(\{y_0, \ldots, y_6\})$. Moreover, $T_1 = Ty_4$ is the (α, β)-trunk of T and there is an edge $e_0 \in E_G(y_4, y_0)$ with $\varphi(e_0) = \beta$.

Clearly, $\gamma_1 \in \bar{\varphi}(y_i)$ for some $i \in \{0, \ldots, 4\}$. We may assume that $i = 0$, since otherwise we could replace the triple (T, e, φ) by the balanced triple $(T, e, \varphi)(y_0 \to y_i)$. Since $e_5 \in E_G(\{y_0, \ldots, y_4\}, y_5)$, we conclude from Lemma 3.21 that there are five edges f_1, f_2, f_3, g_1, g_2 satisfying $f_1 \in E_G(y_1, v_1)$ for a vertex $v_1 \in \{y_5, \ldots, y_8\}$, $f_2 \in E_G(y_4, v_2)$ for a vertex $v_2 \in \{y_5, \ldots, y_8\}$, $\varphi(f_1) = \varphi(f_2) = \gamma_1$, $f_3 \in E_G(y_2, y_3)$, $\varphi(f_3) = \gamma_1$, $g_1, g_2 \in E_G(v_1, v_2)$, $\varphi(g_1) = \alpha$ and $\varphi(g_2) = \beta$. In particular, this implies $\{e_5, e_6\} = \{f_1, f_2\}$ and $\{y_5, y_6\} = \{v_1, v_2\}$. By symmetry, we may assume that $e_5 = f_1$ and $e_6 = f_2$, implying $y_5 = v_1$ and $y_6 = v_2$.

Now we have $\varphi(e_2) = \varphi(e_4) = \varphi(g_1) = \alpha \in \bar{\varphi}(y_0)$, $\varphi(e_0) = \varphi(e_3) = \varphi(g_2) = \beta \in \bar{\varphi}(y_1)$, $\varphi(e_5) = \varphi(e_6) = \varphi(f_3) = \gamma_1 \in \bar{\varphi}(y_0)$, and $e_7, e_8 \in E_G(\{y_0, \ldots, y_6\}, \{y_7, y_8\})$. Since $\varphi(e_7) = \varphi(e_8) = \gamma_2$, we then conclude that $\gamma_2 \notin \{\alpha, \beta, \gamma_1\}$. Since, by Corollary 3.7, $V(T)$ is elementary and closed with respect to φ, there are three edges $f_4, g_3, g_4 \in E_G(y_7, y_8)$ satisfying $\varphi(f_4) = \gamma_1$, $\varphi(g_3) = \alpha$ and $\varphi(g_4) = \beta$.

We may assume that $\gamma_2 \in \bar{\varphi}(\{y_0, \ldots, y_4\})$, since otherwise we could replace T by $T_1 = (y_0, e_1, y_1, e_5, y_5, g_2, y_6, e_6, y_4, e_2, y_2, e_4, y_3, e_7, y_7, e_8, y_8)$. Obviously, $(T_1, e, \varphi) \in \mathcal{T}^B(G)$ is a balanced triple, $T_1 y_4$ is the (γ_1, β)-trunk of T_1, and $\alpha \in \bar{\varphi}(y_0)$. Hence, T_1 has the same structure as T, just the two colours α and γ_1 changed their role.

Now we claim that $E_G(\{y_0, \ldots, y_4\}, \{y_7, y_8\}) \cap E_{\gamma_2}(e, \varphi) \neq \emptyset$. Suppose this is not true. Then we have $e_7, e_8 \in E_G(\{y_5, y_6\}, \{y_7, y_8\})$ and, by symmetry, we may assume that $e_7 \in E_G(y_5, y_7)$ and $e_8 \in E_G(y_6, y_8)$. Evidently, the chain $P_1 = P_{y_7}(\gamma_2, \beta, \varphi) = P(y_7, e_7, y_5, g_2, y_6, e_8, y_8, g_4, y_7)$ is a cycle. Consequently, T is a Tashkinov tree with respect to e and $\varphi_1 = \varphi / P_1$ and, therefore, we have $(T, e, \varphi_1) \in \mathcal{T}(G)$. Then (y_0, y_1) is a (γ, β)-pair with respect to φ_1, and the chain $P_2 = P_{y_0}(\gamma_1, \beta, \varphi_1)$ satisfies $P_2 = P(y_1, f_1, v_1, e_7, y_7, f_4, y_8, e_8, v_2, f_2, y_4, e_0, y_0)$. By Lemma 3.20, it then follows that there is a triple $(T_2, e, \varphi_1) \in \mathcal{T}^N(G)$ such that $h(T_2) = |V(P_2)| = 7 > h(G)$, a contradiction. This proves the claim.

Since we have $\gamma_2 \in \bar{\varphi}(y_j)$ for some $j \in \{0, \dots, 4\}$, we can construct a new balanced triple as follows. In the case $j = 0$ let $(T', e', \varphi') = (T, e, \varphi)$, otherwise let $(T', e', \varphi') = (T, e, \varphi)(y_0 \to y_j)$. In any case we have $(T', e', \varphi') \in \mathcal{T}^B(G)$, and T' has the form

$$T' = (y'_0, e'_0, y'_1, e'_1, y'_2, e'_3, y'_3, e'_4, y'_4, e_5, y_5, e_6, y_6, e_7, y_7, e_8, y_8)$$

where $\{y'_0, \dots, y'_4\} = \{y_0, \dots, y_4\}$, Ty'_4 is the (α, β)-trunk of T', and $\varphi'(e_7) = \varphi'(e_8) = \gamma_2 \in \bar{\varphi}'(y'_0)$. By Lemma 3.21, there are two vertices $v'_1, v'_2 \in \{y_5, \dots, y_8\}$ and four edges f'_1, f'_2, f'_3, g'_1 satisfying $f'_1 \in E_G(y'_1, v'_1)$, $f'_2 \in E_G(y'_4, v'_2)$, $\varphi'(f_1) = \varphi'(f_2) = \gamma_2$, $f'_3 \in E_G(y'_2, y'_3)$, $\varphi'(f'_3) = \gamma_2$, $g'_1 \in E_G(v'_1, v'_2)$, and $\varphi'(g'_1) = \alpha$. Consequently, we have $f'_1, f'_2 \in E_G(\{y_0, \dots, y_4\}, \{y_5, \dots, y_8\})$, $f'_3 \in E_G(\{y_0, \dots, y_4\}, \{y_0, \dots, y_4\})$, $\varphi(f'_1) = \varphi(f'_2) = \varphi(f'_3) = \gamma_2$, and $\varphi(g'_1) = \alpha$. Evidently, it then follows that $|E_G(\{y_0, \dots, y_4\}, \{y_5, \dots, y_8\}) \cap E_{\gamma_2}(e, \varphi)| = 2$. Since $E_G(\{y_0, \dots, y_4\}, \{y_7, y_8\}) \cap E_{\gamma_2}(e, \varphi) \neq \emptyset$, we conclude that $\{y_7, y_8\} \cap \{v'_1, v'_2\} \neq \emptyset$. From $\varphi(g'_1) = \varphi(g_3) = \alpha$ it then follows that $\{y_7, y_8\} = \{v'_1, v'_2\}$.

Now we have $\alpha, \gamma_1 \in \bar{\varphi}(y_0)$, $\beta \in \bar{\varphi}(y_1)$, and $\gamma_2 \in \bar{\varphi}(y_j)$ for some $j \in \{0, \dots, 4\}$. Since $(T, e, \varphi) \in \mathcal{T}_k(G)$, Proposition 2.21(f) then implies that $|\bar{\varphi}(v) \setminus \{\alpha, \beta\}| \geq 1$ for every $v \in V(T)$. Since no colours beside α, β, γ_1 and γ_2 are used on T with respect to φ, we conclude that, for every vertex $v \in V(T) \setminus \{y_0, y_j\}$, the set $\bar{\varphi}(v)$ contains at least one free colour with respect to (T, e, φ). From Lemma 3.13 it then follows that $F(T, e, \varphi) \subseteq \{y_0, y_j\}$. Since $\gamma_1 \neq \gamma_2$ and $e_5, e_6, e_7, e_8 \in E_G(\{y_0, \dots, y_4\}, \{y_5, \dots, y_8\})$, the triple (T, e, φ) fulfils the conditions from Lemma 3.17 and, therefore, G is an elementary graph. This completes the proof. ■

Algorithmic aspect: This result handles the small Tashkinov trees up to a order less than 11. It uses the previous results for most of the cases. The only new case is a balanced Tashkinov tree of order 9 and height 5. Again, the proof can be translated into an algorithm that, in the case when G is not elementary, either computes a k-edge colouring of the graph G, or increases the order of the Tashkinov until it is at least 11. Clearly, the running time again is bounded by a polynomial in $\Delta(G)$ and $|V(G)|$.

Lemma 3.23 *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \varphi) \in \mathcal{T}(G)$. Moreover, let $\alpha, \beta \in \{1, \dots, k\}$, and let P be an (α, β)-chain with respect to φ satisfying $V(P) \cap V(T) = \emptyset$. Then, for the colouring $\varphi' = \varphi/P$, the following statements hold:*

(a) *$\varphi'(f) = \varphi(f)$ for every $f \in E_{G-e}(V(T), V(G))$, and $\bar{\varphi}'(v) = \bar{\varphi}(v)$ for every $v \in V(T)$.*

(b) *$(T, e, \varphi') \in \mathcal{T}(G)$ and $\Gamma^f(T, e, \varphi') = \Gamma^f(T, e, \varphi)$.*

(c) *$\Gamma^d(T, e, \varphi') = \Gamma^d(T, e, \varphi)$ and $E_\delta(T, e, \varphi') = E_\delta(T, e, \varphi)$ for every $\delta \in \Gamma^d(T, e, \varphi)$.*

(d) *$D(T, e, \varphi') = D(T, e, \varphi)$.*

Proof: From $V(P) \cap V(T) = \emptyset$ we conclude that $\varphi'(f) = \varphi(f)$ for every edge $f \in E_{G-e}(V(T), V(G))$. Clearly, this implies (a). Since $(T, e, \varphi) \in \mathcal{T}(G)$, statement (a) implies (b) as well as (c).

Now let $v \in D(T, e, \varphi)$. Then there are two colours $\gamma \in \Gamma^f(T, e, \varphi)$ and $\delta \in \Gamma^d(T, e, \varphi)$ such that v is the first vertex in the linear order $\preceq_{(u,P_1)}$ that belongs to $V(G) \setminus V(T)$, where $u \in V(T)$ is the unique vertex with $\gamma \in \bar{\varphi}(u)$ and $P_1 = P_u(\gamma, \delta, \varphi)$. Consequently, for $P_2 = uP_1v$ we have $E(P_2) \subseteq E_{G-e}(V(T), V(G))$. Since $\gamma \in \Gamma^f(T, e, \varphi')$, $\delta \in \Gamma^d(T, e, \varphi')$ and $\gamma \in \bar{\varphi}'(u)$, we conclude that $\varphi'(f) = \varphi(f)$ for every edge $f \in E(P_2)$. Hence, we have $P_2 = uP'_1v$ where $P'_1 = P_u(\gamma, \delta, \varphi')$. This leads to $v \in D(T, e, \varphi')$ and hence, we have $D(T, e, \varphi) \subseteq D(T, e, \varphi')$.

Since P is also an (α, β)-chain with respect to φ' and since we not only have $\varphi' = \varphi/P$, but also $\varphi = \varphi'/P$, we conclude $D(T, e, \varphi') \subseteq D(T, e, \varphi)$ in an analogous way. Consequently, we obtain $D(T, e, \varphi') = D(T, e, \varphi)$, and the proof is finished. ■

This result just gives some invariants of a recolouring operation that will be used in the next proof. In the next result, the parameter m from the parameterized version of Goldberg's conjecture (Conjecture 3.1) is used for the first time.

Proposition 3.24 *Let G be a critical graph with*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

for an odd integer $m \geq 3$. Moreover, let $(T, e, \varphi) \in \mathcal{T}(G)$ and $Z = V(T) \cup D(T, e, \varphi)$. Then the following statements hold:

(a) $|Z| \leq m - 2$.

(b) *If $|Z| = m - 2$ then G is elementary.*

Proof: From the assumption follows that $\chi'(G) > \Delta(G)$ and hence $\Delta(G) \geq 2$. Then we obtain $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} \geq \Delta(G) + 1$ and, therefore, $\chi'(G) \geq \Delta(G) + 2$. Hence, for $k = \chi'(G) - 1$, we have $k \geq \Delta(G) + 1$ and $\varphi \in \mathcal{C}_k(G - e)$.

From Proposition 3.9 and Proposition 3.10 we conclude that Z is elementary with respect to φ. Then Proposition 1.5(c) implies $|Z| \leq m - 1$.

Now suppose $|Z| = m - 1$. Since $k \geq \Delta(G) + 1$, there is a colour $\alpha \in \bar{\varphi}(Z)$. Moreover, Z is elementary with respect to φ, and $|Z|$ is even, so there is an edge $g \in E_G(Z, V(G) \setminus Z)$ having an endvertex $z \in V(G) \setminus Z$ and satisfying $\varphi(g) = \alpha$. Then $F = (g, z)$ is a fan at Z with respect to φ, and Theorem 3.12 implies that $Z \cup \{z\}$ is elementary with respect to φ. Since $|Z \cup \{z\}| = m$, this contradicts Proposition 1.5(c). Consequently, we have $|Z| \leq m - 2$ and (a) is proved.

For the proof of (b), suppose that $|Z| = m - 2$. In the sequel, we will use the following abbreviation. For a colouring $\varphi' \in \mathcal{C}_k(G - e)$, a colour $\alpha \in \{1, \ldots, k\}$, and a set $X \subseteq V(G)$, let

$$E_\alpha(X, e, \varphi') = E_G(X, V(G) \setminus X) \cap E_\alpha(e, \varphi').$$

First, we claim that Z is closed with respect to φ. Suppose this is not true. Then there is a colour $\alpha \in \bar{\varphi}(Z)$ satisfying $E_\alpha(Z, e, \varphi) \neq \emptyset$. Since Z is elementary with respect to φ, and since $|Z|$ is odd, we then conclude that $|E_\alpha(Z,e,\varphi)| \geq 2$. Then there are two distinct edges $g_1, g_2 \in E_\alpha(Z,e,\varphi)$. For $j = 1,2$, let z_j denote the endvertex of g_j that belongs to $V(G)\setminus Z$. Clearly, $z_1 \neq z_2$ and, therefore, $F' = (g_1, z_1, g_2, z_2)$ is a fan at Z with respect to φ. Hence, by Theorem 3.12, $Z \cup \{z_1, z_2\}$ is elementary with respect to φ, but then $|Z \cup \{z_1, z_2\}| = m$ contradicts Proposition 1.5(c). This proves the claim that Z is closed with respect to φ.

Now we want to show that Z is also strongly closed with respect to φ. Suppose this is not true. Then there is a colour $\delta \in \{1, \ldots, k\}$ satisfying $\delta \notin \bar{\varphi}(Z)$ and $|E_\delta(Z,e,\varphi)| \geq 2$. Since $|Z|$ is odd, we then conclude that $|E_\delta(Z,e,\varphi)| \geq 3$. Moreover, by Proposition 2.21(f), there is a colour $\gamma \in \Gamma^f(T,e,\varphi)$, and there is a unique vertex $v \in V(T)$ satisfying $\gamma \in \bar{\varphi}(v)$. Let $P = P_v(\gamma,\delta,\varphi)$. Then P is a path and v is an endvertex of P. Since $\delta \notin \bar{\varphi}(Z)$ and Z is elementary with respect to φ, the other endvertex of P belongs to $V(G)\setminus Z$. Hence, in the linear order $\preceq_{(v,P)}$ there is a first vertex u that belongs to $V(G)\setminus Z$. Let Φ denote the set of all colourings $\varphi' \in \mathcal{C}_k(G-e)$ such that $(T,e,\varphi') \in \mathcal{T}(G)$, $\Gamma^f(T,e,\varphi') = \Gamma^f(T,e,\varphi)$, $D(T,e,\varphi') = D$, $E_\delta(e,\varphi') = E_\delta(e,\varphi)$, and $E_G(Z,Z) \cap E_\gamma(e,\varphi') = E_G(Z,Z) \cap E_\gamma(e,\varphi')$. Clearly, $\varphi \in \Phi$.

Consider an arbitrary colouring $\varphi' \in \Phi$. Then $Z = V(T) \cup D = V(T) \cup D(T,e,\varphi')$. By Proposition 3.9 and Proposition 3.10, it follows that Z is elementary with respect to φ'. Since $|Z| = m-2$, it follows from the above proof that Z is closed with respect to φ'. Since $E_\delta(e,\varphi') = E_\delta(e,\varphi)$, it follows that $\delta \notin \bar{\varphi}'(Z)$ and $E_\delta(Z,e,\varphi') = E_\delta(Z,e,\varphi)$. Hence, Z is not strongly closed with respect to φ'. Furthermore, $\gamma \in \Gamma^f(T,e,\varphi) = \Gamma^f(T,e,\varphi')$ and, since $E_G(Z,Z) \cap E_\gamma(e,\varphi') = E_G(Z,Z) \cap E_\gamma(e,\varphi)$, we have $\gamma \in \bar{\varphi}'(v)$. Moreover, since $E(vPu) \subseteq E_\delta(e,\varphi) \cup \big(E_G(Z,Z) \cap E_\gamma(e,\varphi)\big)$, it follows that $\varphi'(f') = \varphi(f')$ for all edges $f' \in E(vPu)$. Hence, we have $vP'u = vPu$, where $P' = P_v(\gamma,\delta,\varphi')$. In particular, this implies that u is the first vertex in the linear order $\preceq_{(v,P')}$ that belongs to $V(G)\setminus Z$.

We claim that there is a colouring $\varphi' \in \Phi$ such that $\gamma \in \bar{\varphi}'(u)$. Clearly, this implies that $P_v(\gamma,\delta,\varphi') = vPu$. For the proof of this claim, we consider the following two cases:

Case 1: $\bar{\varphi}(u) \cap \bar{\varphi}(Z) \neq \emptyset$. Then there is a colour $\beta \in \bar{\varphi}(u) \cap \bar{\varphi}(Z)$. If $\beta = \gamma$ then, since $\varphi \in \Phi$, we are done. If $\beta \neq \gamma$ then let $P_1 = P_u(\gamma,\beta,\varphi)$ and $\varphi' = \varphi/P$. Since Z is closed with respect to φ, we obtain $E_\gamma(Z,e,\varphi) = E_\beta(Z,e,\varphi) = \emptyset$. Hence, we have $V(P_1) \cap Z = \emptyset$. By Lemma 3.23, it follows that $(T,e,\varphi') \in \mathcal{T}(G)$, $\Gamma^f(T,e,\varphi') = \Gamma^f(T,e,\varphi)$, and $D(T,e,\varphi') = D$. Since the recolouring involves neither edges of $E_G(Z,Z)$ nor edges coloured with δ, this implies that $\varphi' \in \Phi$. Moreover, $\gamma \in \bar{\varphi}'(u)$ and, therefore, φ' has the desired properties.

Case 2: $\bar{\varphi}(u) \cap \bar{\varphi}(Z) = \emptyset$. Since Z is elementary with respect to φ, this implies that $Z \cup \{u\}$ is elementary with respect to φ, too. Since $\gamma \in \bar{\varphi}(Z)$, there is a vertex $u' \in V(G)$ and an edge $f \in E_G(u,u')$ with $\varphi(f) = \gamma$. Since Z is closed with respect to φ, it follows that $u' \in V(G)\setminus Z$. Then $|Z \cup \{u,u'\}| = m$ and hence, by Proposition 1.5(c), the set $Z \cup \{u,u'\}$ is not elementary with respect to φ. Since $Z \cup \{u\}$ is elementary with respect to φ, this implies that $\bar{\varphi}(u') \cap \bar{\varphi}(Z \cup \{u\}) \neq \emptyset$. We consider three subcases.

Case 2a: *There is a colour* $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(u)$. Then we can simply obtain the desired colouring $\varphi' \in \Phi$ from φ by recolouring the edge f with the colour γ_1.

Case 2b: *There is a colour* $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(Z)$ *satisfying* $\gamma_1 \in \Gamma^f(T, e, \varphi)$. Then there is a unique vertex $v' \in V(T)$ with $\gamma_1 \in \bar{\varphi}(v')$. Moreover, by Proposition 2.21(f), there is a colour $\gamma_2 \in \bar{\varphi}(u)$. Since $Z \cup \{u\}$ is elementary with respect to φ, we clearly have $\gamma_1 \neq \gamma_2$ and $\gamma_2 \notin \bar{\varphi}(Z)$. Furthermore, from $\varphi(f) = \gamma$ it follows that $\gamma \notin \{\gamma_1, \gamma_2\}$. Since $\delta \notin \bar{\varphi}(Z)$ and $\delta \in \varphi(u)$, we also obtain $\delta \notin \{\gamma_1, \gamma_2\}$.

Now let $P_2 = P_{v'}(\gamma_1, \gamma_2, \varphi)$. Then P_2 is a path, where one endvertex is v' and the other endvertex is some vertex $w' \in V(G) \setminus Z$. Since $\gamma_1 \in \bar{\varphi}(V(T))$, Proposition 2.21(d) implies $E_{\gamma_1}(T, e, \varphi) = \emptyset$. Hence, we obtain $\emptyset \neq E_G(V(T), V(G) \setminus V(T)) \cap E(P_2) \subseteq E_{\gamma_2}(T, e, \varphi)$. Then we conclude that $E_{\gamma_2}(T, e, \varphi) \subseteq E(P_2)$. If $|E_{\gamma_2}(T, e, \varphi)| = 1$, this is evident, otherwise it follows from Theorem 3.8(a), because $\gamma_1 \in \Gamma^f(T, e, \varphi)$.

If $w' = u$ then, evidently, u' is not an endvertex of P_2 and, therefore, u' does not belong to $V(P_2)$ at all. Let $P_3 = P_{u'}(\gamma_1, \gamma_2, \varphi)$, and let $\varphi_3 = \varphi/P_3$. Clearly, P_3 and P_2 are disjoint. Since $E_{\gamma_1}(T, e, \varphi) = \emptyset$ and $E_{\gamma_2}(T, e, \varphi) \subseteq E(P_2)$, this implies that $V(P_3) \cap V(T) = \emptyset$. By Lemma 3.23, it follows that $(T, e, \varphi_3) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_3) = \Gamma^f(T, e, \varphi)$, and $D(T, e, \varphi_3) = D$. Since the recolouring does not involve edges coloured with γ or δ, this implies that $\varphi_3 \in \Phi$. Moreover, since $u \notin V(P_3)$, we have $\gamma_2 \in \bar{\varphi}_3(u) \cap \bar{\varphi}_3(u')$ and $\varphi_3(f) = \gamma$. Hence, we can obtain the desired colouring $\varphi' \in \Phi$ from φ_3 by recolouring the edge f with the colour γ_2.

If otherwise $w' \neq u$ then, evidently, u does not belong to $V(P_2)$ at all. Let $P_4 = P_u(\gamma_1, \gamma_2, \varphi)$ and $\varphi_4 = \varphi/P_4$. Clearly, P_4 and P_2 are disjoint. Since $E_{\gamma_1}(T, e, \varphi) = \emptyset$ and $E_{\gamma_2}(T, e, \varphi) \subseteq E(P_2)$, this implies that $V(P_4) \cap V(T) = \emptyset$. By Lemma 3.23, it follows that $(T, e, \varphi_4) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_4) = \Gamma^f(T, e, \varphi)$, and $D(T, e, \varphi_4) = D$. Since the recolouring does not involve edges coloured with γ or δ, this implies that $\varphi_4 \in \Phi$. Moreover, we have $\gamma_1 \in \bar{\varphi}_4(u) \cap \bar{\varphi}_4(Z)$ and $\varphi_4(f) = \gamma$. Hence, we are in the same situation as in Case 1, just with the colouring φ_4 instead of φ, and the colour γ_1 instead of β. Then we can obtain the desired colouring $\varphi' \in \Phi$ from φ_4 analogously to Case 1, that is, $\varphi' = \varphi_4/P_u(\gamma, \gamma_1, \varphi_4)$.

Case 2c: *There is a colour* $\gamma_1 \in \bar{\varphi}(u') \cap \bar{\varphi}(Z)$ *satisfying* $\gamma_1 \notin \Gamma^f(T, e, \varphi)$. By Proposition 2.21(f), there is a colour $\gamma_3 \in \Gamma^f(T, e, \varphi) \setminus \{\gamma\}$. Evidently, $\gamma_1 \neq \gamma_3$. Let $P_5 = P_{u'}(\gamma_1, \gamma_3, \varphi)$, and let $\varphi_5 = \varphi/P_5$. Since $\gamma_1, \gamma_3 \in \bar{\varphi}(Z)$ and Z is closed with respect to φ, we obtain $E_{\gamma_1}(Z, e, \varphi) = E_{\gamma_3}(Z, e, \varphi) = \emptyset$. Consequently, P_5 is a path satisfying $V(P_5) \cap Z = \emptyset$. From Lemma 3.23 it then follows that $(T, e, \varphi_5) \in \mathcal{T}(G)$, $\Gamma^f(T, e, \varphi_5) = \Gamma^f(T, e, \varphi)$, and $D(T, e, \varphi_5) = D$. Since the recolouring does not involve edges coloured with γ or δ, this implies that $\varphi_5 \in \Phi$. Moreover, we have $\gamma_3 \in \bar{\varphi}_5(u') \cap \Gamma^f(T, e, \varphi_5)$. Since $\gamma_3 \neq \gamma$, we also have $\varphi_5(f) = \gamma$. Hence, we are in the same situation as in Case 2b, just with the colouring φ_5 instead of φ, and the colour γ_3 instead of γ_1. Then we can obtain the desired colouring $\varphi' \in \Phi$ from φ_4 analogously to Case 2b.

Thus the claim is proved, and there is a colouring $\varphi' \in \Phi$ such that $\gamma \in \bar{\varphi}'(u)$ and, therefore, $P' = P_v(\gamma, \delta, \varphi') = vPu$. Evidently, we have $|E(P') \cap E_G(Z, V(G) \setminus Z)| = 1$. Since, by assumption, we also have $|E_\delta(Z, e, \varphi)| \geq 3$, there must be two edges $f_1, f_2 \in E_G(Z, V(G) \setminus Z) \setminus E(P')$ with $\varphi(f_1) = \varphi(f_2) = \delta$. Since $\varphi' \in \Phi$, this implies $\varphi'(f_1) = \varphi'(f_2) = \delta$. For $j = 1, 2$, let $v_j \in Z$ and $u_j \in V(G) \setminus Z$ denote the endvertices of f_j.

Now let $P'_1 = P_{u_1}(\gamma, \delta, \varphi')$ and $P'_2 = P_{u_2}(\gamma, \delta, \varphi')$. Note that P'_1 and P'_2 may be equal. Since $E(P') \cap \{f_1, f_2\} = \emptyset$, both chains P'_1 and P'_2 are vertex disjoint to P'. Since $\gamma \in \bar{\varphi}'(V(T))$, Proposition 2.21(d) implies $E_\gamma(T, e, \varphi') = \emptyset$. Moreover, by Theorem 3.8(a), $E_\delta(T, e, \varphi) \subseteq P'$. Consequently, we obtain $V(P'_1) \cap V(T) = V(P'_2) \cap V(T) = \emptyset$. If $P'_1 = P'_2$ then let $\varphi'_2 = \varphi'/P'_1$, otherwise let $\varphi'_2 = (\varphi'/P'_1)/P'_2$. From Lemma 3.23 we then conclude that $(T, e, \varphi'_2) \in \mathcal{T}(G)$, and $D(T, e, \varphi'_2) = D$. Moreover, we have $\varphi'_2(f_1) = \varphi'_2(f_2) = \gamma \in \bar{\varphi}'_2(V(T))$ and, therefore, $F = (f_1, u_1, f_2, u_2)$ is a fan at Z with respect to φ'_2. From Theorem 3.12 it then follows that $Z \cup \{u_1, u_2\}$ is elementary with respect to φ'_2. Since $|Z \cup \{u_1, u_2\}| = m$, this contradicts Proposition 1.5(c). This proves the claim that Z is strongly closed with respect to φ. Then, by Theorem 1.4, G is an elementary graph. This completes the proof of statement (b). ∎

Algorithmic aspect: Now we will discuss the algorithmic interpretation of this result. Let G be a graph and $(T, e, \varphi) \in \mathcal{T}_k(G)$ for an integer $k \geq \Delta(G) + 1$. We assume that $Z = V(T) \cup D(T, e, \varphi)$ is elementary with respect to φ, otherwise our colouring algorithm had computed a k-edge colouring of G or a larger Tashkinov tree before.

If $|Z| > m - 2$ then we distinguish two cases. First, the case $|Z| > m - 1$. This means that we have an elementary set of size at least m, containing the two endvertices x, y of e. Moreover, at every vertex at least $k - \Delta(G)$ colours are missing, and at x, y at least $k - \Delta(G) + 1$ colours are missing. This implies $k \geq m(k - \Delta(G)) + 2$. From this we obtain

$$k + 1 \leq \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}.$$

Consequently, we can use a new colour for e without exceeding our desired bound. In the second case we have $|Z| = m - 1$. Then $|Z|$ is even and we find a fan (g, z) at Z with respect to φ. If $Z \cup \{z\}$ is not elementary with respect to φ, the proof of Theorem 3.12 shows how we can recolour and increase the order of T. If $Z \cup \{z\}$ is elementary with respect to φ. Again, we have an elementary set of size m containing x, y, allowing to use a new colour for e.

If $|Z| = m - 2$ then we first check whether T is closed with respect to φ. If not then we find a fan (g_1, z_1, g_2, z_2) at Z with respect to φ. On one hand, if $Z \cup \{g_1, g_2\}$ is not elementary with respect to φ then, by Theorem 3.12, we can recolour and increase the order of T. On the other hand, if $Z \cup \{g_1, g_2\}$ is elementary with respect to φ then we can again use a new colour for e without exceeding our desired bound. So we only have to consider the case that Z is closed with respect to φ. We can further assume that Z is not strongly closed with respect φ, otherwise G is elementary.

Again we have to distinguish some cases, we skip the details here, they can be found in the proof. If we cannot, at some point, use Theorem 3.8 to increase the order of T then, eventually, we get again an elementary set of size m, allowing to use a new colour for the edge e. Clearly, the algorithm only has a running time polynomial in $\Delta(G)$ and $|V(G)|$.

Proposition 3.25 *Let G be a critical graph with*

$$\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$$

for an odd integer $m \geq 3$. Then the following statements hold:

(a) *If $t(G) > m-4$ then G is elementary.*

(b) *If $t(G) = m-4$ and $h(G) > t(G) - 8$ then G is elementary.*

Proof: Let $t(G) > m-4$. Since m is odd and, by Corollary 3.7, $t(G)$ is odd, too, this implies $t(G) \geq m-2$. Evidently, for any $(T,e,\varphi) \in \mathcal{T}(G)$ we have $|V(T) \cup D(T,e,\varphi)| \geq m-2$. Then Proposition 3.24 implies that $|V(T) \cup D(T,e,\varphi)| = m-2$, and that G is elementary. This proves (a).

Now let $t(G) = m-4$ and $h(T) > |V(T)| - 8$. By Lemma 3.16, there is a triple $(T,e,\varphi) \in \mathcal{T}^B(G)$. Then we have $e \in E_G(x,y)$ for two vertices $x, y \in V(T)$, and $\varphi \in \mathcal{C}_k(G-e)$ with $k = \chi'(G) - 1$. Since $\chi'(G) > \Delta(G)$, we conclude that $\Delta(G) \geq 2$. Then $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} \geq \Delta(G) + 1$ and, therefore, $\chi'(G) \geq \Delta(G) + 2$ and $k \geq \Delta(G) + 1$.

By Corollary 3.7, $V(T)$ is elementary and closed with respect to φ, and $(T,e,\varphi) \in \mathcal{T}_k(G)$. If $V(T)$ is strongly closed with respect to φ, then Theorem 1.4 implies that G is elementary, and we are done. So for the rest of the proof, assume that $V(T)$ is not strongly closed with respect to φ. Then, by Proposition 2.21(c), we have $\Gamma^d(T,e,\varphi) \neq \emptyset$ and, therefore, $D = D(T,e,\varphi) \neq \emptyset$. If $|D| \geq 2$ then $|V(T) \cup D| \geq t(G) + 2 = m-2$, and from Proposition 3.24 it then follows that $|V(T) \cup D| = m-2$ and that G is elementary. So from now on we assume that $|D| = 1$.

Let $\alpha_1 \in \bar{\varphi}(x)$ and $\alpha_2 \in \bar{\varphi}(y)$ be the two colours used on the trunk of T with respect to φ. By Proposition 2.21(f), we have $|\bar{\varphi}(v)| \geq k - \Delta(G) + 1$ for $v \in \{x,y\}$, and $|\bar{\varphi}(v)| \geq k - \Delta(G)$ for $v \in V(T) \setminus \{x,y\}$. Since $V(T)$ is elementary with respect to φ, this implies that $|\bar{\varphi}(v) \setminus \{\alpha_1, \alpha_2\}| \geq k - \Delta(G)$ for every $v \in V(T)$. Since (T,e,φ) is a balanced triple and $h(T) > |V(T)| - 8$, we conclude that beside α_1, α_2 there are at most 3 other colours used on T with respect to φ. This leads to $(k - \Delta(G))s \leq 3$, where s is the number of vertices $v \in V(T)$, such that $\bar{\varphi}(v)$ contains no free colour with respect to (T,e,φ).

Let $\delta \in \Gamma^d(T,e,\varphi)$, and let $E' = E_\delta(T,e,\varphi)$. By Proposition 2.21(e), we have $|E'| \geq 3$. Let $E' = \{f_j \mid 1 \leq j \leq |E'|\}$, and for $j = 1, \ldots, |E'|$ let $v_j \in V(T)$ and $u_j \in V(G) \setminus V(T)$ be the endvertices of f_j. By Proposition 2.21(f), there is a free colour $\gamma_0 \in \Gamma^f(T,e,\varphi)$. Hence, there is a vertex $v_0 \in V(T)$ with $\gamma_0 \in \bar{\varphi}(v_0)$. Let $P_0 = P_{v_0}(\gamma_0, \delta, \varphi)$. Then Proposition 2.22(a),(b) implies that, in the linear order $\preceq_{(v_0,P_0)}$, one of the vertices $u_1, \ldots, u_{|E'|}$, say u_1, is the first one that belongs to $V(G) \setminus V(T)$. Consequently, $u_1 \in D$. Since $|D| = 1$, we then obtain $D = \{u_1\}$.

Now let $j \in \{2, \ldots, |E'|\}$. Suppose that the set $\bar{\varphi}(v_j)$ contains a free colour γ_j. Then (γ_j, δ)-chain $P_j = P_{v_j}(\gamma_j, \delta, \varphi)$ is a path, and v_j is an endvertex of P_j. Moreover, on the linear order $\preceq_{(v_j,P_j)}$, the vertex u_j is the first vertex that belongs to $V(G) \setminus V(T)$. Hence, we have $u_j \in D$, a contradiction. Consequently, $\bar{\varphi}(v_j) \cap \Gamma^f(T,e,\varphi) = \emptyset$ for $j = 2, \ldots, |E'|$. Clearly, this implies $s \geq |E'| - 1 \geq 2$. Since we also have $(k - \Delta(G))s \leq 3$, it follows that $2 \leq s \leq \frac{3}{k - \Delta(G)}$. Since $k \geq \Delta(G) + 1$, this implies $k = \Delta(G) + 1$ and, therefore, $2 \leq |E'| - 1 \leq 3$. Since, by Proposition 2.21(e), $|E'|$ is odd, we then conclude that $|E'| = 3$, that is, $E' = \{f_1, f_2, f_3\}$. From Proposition 3.9 and Proposition 3.10 we conclude that $Z = V(T) \cup D = V(T) \cup \{u_1\}$ is elementary with respect to φ. Since $V(T)$ is closed with respect to φ, this implies that, for every colour $\gamma' \in \bar{\varphi}(V(T))$, there is a vertex $u_{\gamma'} \in V(G) \setminus Z$ and an edge $f_{\gamma'}$ such that $f_{\gamma'} \in E_G(u_1, u_{\gamma'})$ and $\varphi(f_{\gamma'}) = \gamma'$. Now let $u = u_{\gamma'}$ and $f = f_{\gamma'}$ for some $\gamma' \in \bar{\varphi}(V(T))$. Then (f, u) is a fan at Z with respect to φ. Hence, by Theorem 3.12, $X = Z \cup \{u\}$ is elementary with respect to φ.

Now we claim that $\delta \notin \bar{\varphi}(X)$. Suppose this is not true. Since $\delta \notin \bar{\varphi}(V(T))$ and $\delta \notin \bar{\varphi}(u_1)$, this implies that $\delta \in \bar{\varphi}(u)$. Then $(f, u, f_2, u_2, f_3, u_3)$ is a fan at Z with respect to φ and hence, by Theorem 3.12, $X_1 = X \cup \{u_2, u_3\}$ is elementary with respect to φ. Since $|X_1| = m$, this contradicts Proposition 1.5(c). This proves the claim.

This clearly implies $k \geq |\bar{\varphi}(X)| + 1$. Since $k = \Delta(G) + 1$, Proposition 2.21(f) implies that $|\bar{\varphi}(v)| \geq 2$ for $v \in \{x, y\}$, and $|\bar{\varphi}(v)| \geq 1$ for $v \in V(T) \setminus \{x, y\}$. Since X is elementary with respect to φ, we then obtain $|\bar{\varphi}(X)| \geq |X| + 2 = m$. Hence, on the one hand, we have $k \geq m + 1$. On the other hand, since $k = \Delta(G) + 1$, we have

$$k + 1 = \chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1} = \Delta(G) + 1 + \frac{\Delta(G) - 2}{m-1} = k + \frac{k-3}{m-1}$$

and, therefore, $k < m + 2$. Since both k and m are integers, we then conclude that $k = m + 1 = |\bar{\varphi}(X)| + 1$.

Now we claim that X is closed with respect to φ. Suppose this is not true. Then there is a colour $\alpha \in \bar{\varphi}(X)$ satisfying $E_1 = E_G(X, V(G) \setminus X) \cap E_\alpha(e, \varphi) \neq \emptyset$. Since X is elementary with respect to φ, there is a unique vertex in X where the colour α is missing with respect to φ. Moreover, $|X| = m - 2$ is odd and, therefore, $|E_1|$ is even and $|E_1| \geq 2$. Hence, there is at least one edge $f' \in E_1$ having an endvertex in Z. Let u' be the endvertex of f' that belongs to $V(G) \setminus X$. Then (f, u, f', u') is a fan at Z with respect to φ and, by Theorem 3.12, $X_2 = X \cup \{u'\}$ is elementary with respect to φ. From $k = m + 1$ and $|X_2| = m - 1$ we then conclude that $k = |X_2| + 2 \leq |\bar{\varphi}(X_2)| \leq k$ and, therefore, $|\bar{\varphi}(X_2)| = k$. This implies $\delta \in \bar{\varphi}(X_2)$, and from $\delta \notin \bar{\varphi}(X)$ it then follows that $\delta \in \bar{\varphi}(u')$. Consequently, we have $u' \notin \{u_2, u_3\}$ and, therefore, at least one of the vertices u_2, u_3 does not belong to X_2, say $u_2 \notin X_2$. Then, evidently, (f, u, f', u', f_2, u_2) is a fan at Z with respect to φ and hence, by Theorem 3.12, $X_3 = X_2 \cup \{u_2\}$ is elementary with respect to φ. Since $|X_3| = m$, this contradicts Proposition 1.5(c). This proves the claim.

Let $E'' = E_G(X, V(G) \setminus X) \cap E_\delta(e, \varphi)$. Since $\delta \notin \bar{\varphi}(X)$ and $|X|$ is odd, we conclude that $|E''| \geq 1$ is odd, too. We claim that $|E''| = 1$. Suppose on the contrary that $|E''| > 1$. Since $|E''|$ is odd, this implies $|E''| \geq 3$. From $E' = \{f_1, f_2, f_3\}$ and $f_1 \in E_G(X, X)$ it then follows that $|E''| = 3$ and $E'' = \{f_2, f_3, g\}$, where $g \in E_G(u, v)$ and $v \in V(G) \setminus X$. Clearly, there is a colour $\beta \in \bar{\varphi}(v)$. Evidently, we have $\beta \neq \delta$, and since $k = |\bar{\varphi}(X)| + 1$ and $\delta \notin \bar{\varphi}(X)$, we have $\beta \in \bar{\varphi}(X)$. Moreover, by Proposition 2.21(f), there is a colour $\gamma \in \Gamma^f(T, e, \varphi) \setminus \{\beta\}$. Now let $P = P_v(\beta, \gamma, \varphi)$ and $\varphi' = \varphi/P$. Since $\beta, \gamma \in \bar{\varphi}(X)$ and X is closed with respect to φ, we conclude that $V(P) \cap X = \emptyset$. By Lemma 3.23, we then have $(T, e, \varphi') \in \mathcal{T}(G)$, $\gamma \in \Gamma^f(T, e, \varphi')$, $\delta \in \Gamma^d(T, e, \varphi')$ and $D(T, e, \varphi') = \{u_1\}$. Moreover, we have $\varphi'(f_1) = \varphi'(f_2) = \varphi'(f_3) = \varphi'(g) = \delta$ and $\gamma \in \bar{\varphi}'(v)$. Since both $V(T)$ and X are closed with respect to φ, there is an edge $g' \in E_G(u_1, u)$ satisfying $\varphi'(g') = \varphi(g') = \gamma$. Let $v' \in V(T)$ be the unique vertex with $\gamma \in \bar{\varphi}'(v)$, and let $P' = P_{v'}(\gamma, \delta, \varphi')$. Since $D(T, e, \varphi') = \{u_1\}$, we conclude that u_1 is the first vertex in the linear order $\preceq_{(P', v')}$ that belongs to $V(G) \setminus V(T)$. Then, evidently, u, v are the next vertices in the linear order $\preceq_{(P',v')}$ and, moreover, v is the second endvertex of P'. Consequently, we have $f_2, f_3 \notin E(P')$, a contradiction to Theorem 3.8(a). Hence, the claim is proved.

Since $k = |\bar{\varphi}(X)| + 1$ and $\delta \notin \bar{\varphi}(X)$, and since X is closed with respect to φ, we then conclude that every edge in $E_G(X, V(G) \setminus X)$ is coloured with δ with respect to φ. Then $|E''| = 1$ implies that X is strongly closed with respect to φ. Since X is also elementary with respect to φ, we infer from Theorem 1.4 that G is elementary. This, eventually, proves (b). ■

Algorithmic aspect: The last result handles large Tashkinov trees with a order near to the parameter m. A corresponding algorithm works as follows. Let G be a graph, and $(T, e, \varphi) \in \mathcal{T}_k^B(G)$ for an integer $k \geq \Delta(G) + 1$ such that either $|V(T)| > m - 4$, or $|V(T)| = m - 4$ and $h(T) > |V(T)| - 8$. If $|V(T) \cup D(T, e, \varphi)| \geq m - 2$ then we use the algorithm from the preceding result. This leaves only the case $|V(T)| = m - 4$, $h(T) > |V(T)| - 8$ and $|D(T, e, \varphi)| = 1$. Now we have to distinguish several cases, leading to one of three outcomes. First, we get a larger Tashkinov tree due to Theorem 3.8. Second, we get an elementary set of size m containing the two endvertices of e. This allows us to use a new colour for e without exceeding our desired bound. And last, we get an elementary and strongly closed set, implying that G is elementary. This algorithm also has a polynomial (in $\Delta(G)$ and $|V(G)|$) running time like the others.

Proof of Theorem 3.2: Let G be a graph with $\chi'(G) > \frac{15}{14}\Delta(G) + \frac{12}{14}$. Moreover, let H be a critical subgraph of G satisfying $\chi'(H) = \chi'(G)$. Clearly, we have $\Delta(H) \leq \Delta(G)$ and, therefore,

$$\chi'(H) > \frac{15}{14}\Delta(H) + \frac{12}{14}.$$

In the following we show that H is elementary. From this we then conclude that also G is elementary, because $\mathcal{W}(G) \leq \chi'(G) = \chi'(H) = \mathcal{W}(H) \leq \mathcal{W}(G)$.
To show that H is elementary, we distinguish four cases.

Case 1: $t(H) < 11$. Then, by Proposition 3.22, H is elementary.

Case 2: $t(H) > 11$. Then, by Proposition 3.25(a), H is elementary.

Case 3: $t(H) = 11$ *and* $h(H) \leq 3$. Then from Proposition 3.19 it follows that H is elementary.

Case 4: $t(H) = 11$ *and* $h(H) > 3$. Then from Proposition 3.25(b) it follows that H is elementary.

In any case H is an elementary graph, this completes the proof. ■

Algorithmic aspect: Now it is easy to construct an algorithm that colours a given graph G with at most $\max\{\frac{15}{14}\Delta(G) + \frac{12}{14}, \mathcal{W}(G)\}$ colours. The kernel algorithm that extends a partial colouring to a new edge, is a combination of the algorithms corresponding the previous results. Given a graph H, an edge $e \in E(H)$, and a colouring $\varphi \in \mathcal{C}_k(H - e)$ for an integer $k \geq \Delta(H) + 1$, this algorithm works as follows. It successively builds a Tashkinov tree T with respect to e and φ until either the vertex set $V(T)$ is not elementary with respect to φ, or T is a maximal Tashkinov tree. In the first case, due to Theorem 2.20, we can colour H with k colours. In the second case, due to Lemma 3.15, we may assume that (T, e, φ) is a balanced triple. Depending on the order and the height of T, we use one of algorithms of Proposition 3.22, Proposition 3.25, or Proposition 3.19. This may have different outcomes. First, we get a k-edge colouring of H. Second, we can recolour and increase the order of T. Then we start over again. Third, we get an elementary set of size 15. Then we can use a new colour for e and still use at most $\frac{15}{14}\Delta(G) + \frac{12}{14}$ colours. And last, we get an elementary and strongly closed set, implying that H is elementary. Then we use a new colour for e and still use at most $\mathcal{W}(G)$ colours.

An colouring algorithm that uses this described kernel and, on the input G, starts with $\Delta(G)$ colours, will not use more than $\max\{\frac{15}{14}\Delta(G) + \frac{12}{14}, \mathcal{W}(G)\}$ for the whole graph G. Since all described subroutines have a running time polynomial in $\Delta(G)$ and $|V(G)|$, the running time of the whole algorithm is bounded polynomial in $|V(G)|$ and $|E(G)|$.

Corollary 3.26 *The parameter*

$$\max\left\{\left\lfloor\frac{15}{14}\Delta + \frac{12}{14}\right\rfloor, \mathcal{W}\right\}$$

is an efficiently realizable upper bound of the chromatic index χ'.

3.6 Some Conclusions

As a consequence of Theorem 3.2, Goldberg's conjecture holds for graphs of small order or small maximum degree.

Theorem 3.27

(a) *Every graph with $\Delta(G) \leq 15$ satisfies $\chi'(G) \leq \max\{\Delta(G) + 1, \mathcal{W}(G)\}$.*

(b) *Every graph with $|V(G)| \leq 15$ satisfies $\chi'(G) \leq \max\{\Delta(G) + 1, \mathcal{W}(G)\}$.*

Proof: If G is a graph with $\Delta(G) \leq 15$ then

$$\left\lfloor\frac{15}{14}\Delta(G) + \frac{12}{14}\right\rfloor = \left\lfloor\Delta(G) + 1 + \frac{\Delta(G) - 2}{14}\right\rfloor \leq \Delta(G) + 1.$$

Hence, Corollary 3.3 implies that $\chi'(G) \leq \max\{\Delta(G) + 1, \mathcal{W}(G)\}$. This proves (a).

To prove (b), let G be a graph with $|V(G)| \leq 15$. Suppose that $\chi'(G) \geq \Delta(G)+2$. We have to show that $\chi'(G) = \mathcal{W}(G)$, that is, G is elementary with respect to φ. By Proposition 1.1(a), G contains a critical subgraph H satisfying $\chi'(H) = \chi'(G) \geq \Delta(G) + 2 \geq \Delta(H) + 2$. Let $k = \chi'(H) - 1 \geq \Delta(H) + 1$. By Lemma 3.16(b), there is a triple $(T, e, \varphi) \in \mathcal{T}^B(H)$. Then T is a Tashkinov tree with respect to e and φ satisfying $|V(T)| = t(H)$ and $h(T) = h(H)$.

If $t(H) < 11$ then, by Proposition 3.22, H is elementary, and we are done. So consider the case that $t(H) \geq 11$. By Corollary 3.7, $V(T)$ is elementary and closed with respect to φ, and $(T, e, \varphi) \in \mathcal{T}_k(G)$. If $V(T)$ is also strongly closed with respect to φ, then Theorem 1.4 implies that H is elementary, and we are done, too. So consider the case that $V(T)$ is not strongly closed with respect to φ. Then, by Proposition 2.21(c), we have $\Gamma^d(T, e, \varphi) \neq \emptyset$ and, therefore, $D = D(T, e, \varphi) \neq \emptyset$.

Let $u \in D(T, e, \varphi)$, and let $Z = V(T) \cup \{u\}$. Then, by Proposition 3.9 and Proposition 3.10, Z is elementary with respect to φ. Since $\bar{\varphi}(Z) \neq \emptyset$ and, by Proposition 2.21 (b), $|Z| = |V(T)| + 1$ is even, this implies that there is a colour $\gamma \in \bar{\varphi}(Z)$ and an edge $f_1 \in E_H(Z, V(H) \setminus Z)$ with $\varphi(f_1) = \gamma$. Let u_1 be the endvertex of f_1 that belongs to $V(H) \setminus Z$. Then (f_1, u_1) is a fan at Z with respect to φ and, by Theorem 3.12, $X_1 = Z \cup \{u_1\}$ is elementary with respect to φ. Since $|V(T)| = t(H) \geq 11$, we have $|X_1| \geq 13$. Since $|V(H)| \leq |V(G)| \leq 15$, this implies that $|V(H) \setminus X_1| \leq 2$.

For every colour $\beta \notin \bar{\varphi}(X_1)$, since $|X_1|$ is odd, $|E_H(X_1, V(H) \setminus X_1) \cap E_\beta(e, \varphi)|$ is odd, too. Since $|V(H) \setminus X_1| \leq 2$, this implies that $|E_H(X_1, V(H) \setminus X_1) \cap E_\beta(e, \varphi)| = 1$. Consequently, if X_1 is closed with respect to φ then X_1 is also strongly closed with respect to φ. From Theorem 1.4 it then follows that G is elementary, and we are done. So consider the case that X_1 is not closed with respect to φ. Then there is a colour $\gamma' \in \bar{\varphi}(X_1)$ and, since $|X_1|$ is odd, there are two distinct edges $f_2, f_3 \in E_H(X_1, V(H) \setminus X_1)$ with $\varphi(f_2) = \varphi(f_3) = \gamma'$. For $j = 2, 3$, let u_j be the endvertex of f_j that belongs to $V(H) \setminus X_1$. At least one of these two edges, say f_2, has an endvertex in Z. Hence, (f_1, u_1, f_2, u_2) is a fan at Z with respect to φ. Hence, by Theorem 3.12, the set $X_2 = Z \cup \{u_1, u_2\}$ is elementary with respect to φ. Moreover, we have $15 = |X_2 \cup \{u_3\}| \leq |V(H)| \leq |V(G)| \leq 15$ and, therefore, $V(H) \setminus X_2 = \{u_3\}$. Since $|X_2|$ is even, this implies that, for every colour $\alpha \in \bar{\varphi}(X_2)$, there is an edge $g_\alpha \in E_H(X_2, u_3)$ satisfying $\varphi(g_\alpha) = \alpha$, and hence, $\alpha \in \varphi(u_3)$. Consequently, $V(H) = X_2 \cup \{u_3\}$ is elementary with respect to φ. Then from Theorem 1.4 we conclude that G is elementary. This completes the proof of (b). ■

Theorem 3.2 can also be used to improvement a result about the existence of **extreme graphs**, i.e., graphs for which the chromatic index attains Vizing's bound. In [27],[30] it was shown that, for any positive Δ, μ, p satisfying

$$2p(\mu - 1) + 2 \leq \Delta \leq 2p\mu,$$

there exists an extreme graph G with $\Delta(G) = \Delta$ and $\mu(G) = \mu$. It is also proved in [27],[30] that this characterization is complete provided that Goldberg's conjecture holds. That means, if Goldberg's conjecture is true, then every graph G with $\Delta(G) = \Delta$, $\mu(G) = \mu$ and

$$2p\mu + 1 \leq \Delta \leq 2(p + 1)\mu - (2p + 1),$$

for an integer $p \geq 1$, satisfies $\chi'(G) \leq \Delta + \mu - 1$. In [27],[30] is it proved, without needing Goldberg's conjecture, that this holds for $p \leq 5$. With Theorem 3.2 we now can slightly improve this result.

Lemma 3.28 (Scheide [27] 2007) *Let Δ, μ be positive integers, and let G be a graph with $\Delta(G) = \Delta$ and $\mu(G) = \mu$ such that*

$$2p\mu + 1 \leq \Delta \leq 2(p + 1)\mu - (2p + 1)$$

holds for some integer $p \geq 1$. If G is critical and elementary, then $\chi'(G) \leq \Delta + \mu - 1$.

Theorem 3.29 *Let Δ, μ be positive integers, and let G be a graph with $\Delta(G) = \Delta$ and $\mu(G) = \mu$ such that*

$$2p\mu + 1 \leq \Delta \leq 2(p+1)\mu - (2p+1)$$

holds for some integer $p \geq 1$. If $p \leq 6$ then $\chi'(G) \leq \Delta + \mu - 1$.

Proof: Suppose, on the contrary, that $\chi'(G) \geq \Delta + \mu$. By Vizing's bound, this implies that $\chi'(G) = \Delta + \mu$, that is, G is an extreme graph. Let H be a critical subgraph of G satisfying $\chi'(H) = \chi'(G) = \Delta + \mu$. Clearly, we have $\Delta(H) \leq \Delta$ and $\mu(H) \leq \mu$. Since, by Vizing's bound, we also have $\chi'(H) \leq \Delta(H) + \mu(H)$, this implies that $\Delta(H) = \Delta$ and $\mu(H) = \mu$. Since $2p\mu + 1 \leq \Delta \leq 2(p+1)\mu - (2p+1)$ and $1 \leq p \leq 6$, we conclude that

$$\mu \geq \frac{\Delta + (2p+1)}{2p+2} = \frac{\Delta - 1}{2p+2} + 1 \geq \frac{\Delta - 1}{14} + 1 > \frac{1}{14}\Delta + \frac{12}{14}.$$

Consequently, we have

$$\chi'(H) = \Delta + \mu > \frac{15}{14}\Delta + \frac{12}{14}.$$

From Theorem 3.2 it then follows that H is an elementary graph. Since H is critical, Lemma 3.28 then implies that $\chi'(H) \leq \Delta + \mu - 1$, a contradiction. ■

4 Polynomial-Time Algorithms

Usually, a graph G can be described by its vertex set V, its edge set E, and its incidence function i assigning to each edge a 2-subset of V. Instead of $i(e) = \{x, y\}$ we always wrote $e \in E_G(x, y)$. However, up to isomorphisms the graph G is completely determined by its vertex set V and the function μ_G assigning to each vertex pair (x, y) with $x \neq y$ its multiplicity $\mu_G(x, y) = |E_G(x, y)|$. Since our graphs are undirected, we need to know the multiplicity only for each 2-subset of V. For a set X, let $X^{(2)}$ denote the set of all 2-subsets of X. A graph $\boldsymbol{G = (V, \mu)}$ is then a pair consisting of the vertex set V and the multiplicity function $\mu : V^{(2)} \to \mathbb{N}$. Then the representation of a graph G has length bounded, from above as well as from below, by polynomials in $|V(G)|$ and $\log \mu(G)$. So far, all considered edge colouring algorithms had a running time polynomial in the number of edges and the number of vertices of the input graph. Hence, these algorithms are not polynomial in the length of the input. The running time of a real polynomial-time edge colouring algorithm has also to be bounded, for an input graph G, by a polynomial in $|V(G)|$ and $\log \mu(G)$. Using two concepts from Sanders and Steurer [25], we will develop an scheme that allows us to transform all our former algorithms into polynomial-time algorithms which achieve the same approximation guaranties. The first concept is an elegant data storing during the algorithm such that the standard recolouring operations have a running time that is polynomial only in the number of vertices, provided that the size of the stored data is sufficiently small. The second concept is a divide-and-conquer-strategy that produces a partial edge colouring of a graph G, that uses not more than $\max\{\Delta(G), \mathcal{W}(G)\}$ colours, needs sufficiently small storage space, and leaves only a sufficiently small number of edges uncoloured. Then this partial edge colouring is completed by simply colouring the remaining edges one by one, using any of our kernel routines from the last chapters.

Let $G = (V, \mu)$ be a graph. A subgraph of G is then a graph $H = (V', \mu')$ such that $V' \subseteq V$ and $\mu'(e) \leq \mu(e)$ for every $e \in V'^{(2)}$. We say that the subgraph H is obtained from G by deleting an edge e from $E_G(x, y)$ if $V' = V$, $\mu'(\{x, y\}) = \mu(\{x, y\}) - 1$, and $\mu'(f) = \mu(f)$ for all $f \in V^{(2)} \setminus \{\{x, y\}\}$. In this case we write, as usual, $H = G - \{e\}$ or $H = G - e$. Note that e denotes an edge from $E_G(x, y)$ rather than a 2-subset. However, we do not mean a specific edge just one edge joining x and y in G. Hence it would be more natural to replace e by the 2-set $e' = \{x, y\}$, and to say that H is obtained from G by deleting an edge from the 2-set e', written $H = \boldsymbol{G - \{e'\} = G - \{\{x, y\}\}}$.

Since we do not distinguish between parallel edges, we consider a **k-edge colouring** φ of G as a function that assigns to each 2-subset of V a subset of the colour set $\{1, \ldots, k\}$ such that $|\varphi(e)| = \mu(e)$ for every $e \in V^{(2)}$ and $\varphi(e) \cap \varphi(e') = \emptyset$ whenever $e, e' \in V^{(2)}$ are distinct, but not disjoint.

Now let $\varphi \in \mathcal{C}_k(G)$ be an k-edge colouring of G. A **colour class** for a given colour $\alpha \in \{1, \ldots, k\}$ is then defined as the set $\boldsymbol{E_\alpha(\varphi)} = \{e \in V^{(2)} \mid \alpha \in \varphi(e)\}$. An **$(\alpha, \beta)$-chain** of G with respect to φ is a component of the subgraph $\boldsymbol{G_{\alpha,\beta}}$ consisting of the vertex set V, and having the multiplicity function μ' defined by $\mu'(e) = |\varphi(e) \cap \{\alpha, \beta\}|$ for every $e \in V^{(2)}$. Let P be an (α, β)-chain of G with respect to φ, having the multiplicity function μ_P. Clearly, P is a path or a cycle, and if we interchange the two colours on P, we get a new k-edge colouring φ' of G satisfying

$$\varphi'(e) = \begin{cases} (\varphi(e) \setminus \{\alpha\}) \cup \{\beta\} & \text{if } \mu_P(e) = 1 \text{ and } \varphi(e) \cap \{\alpha, \beta\} = \{\alpha\}, \\ (\varphi(e) \setminus \{\beta\}) \cup \{\alpha\} & \text{if } \mu_P(e) = 1 \text{ and } \varphi(e) \cap \{\alpha, \beta\} = \{\beta\}, \\ \varphi(e) & \text{otherwise.} \end{cases}$$

For a vertex v of G, the set of colours **present** at v consists of all colours $\alpha \in \{1, \ldots, k\}$ such that there is a 2-set $e \in V^{(2)}$ with $v \in e$ and $\alpha \in \varphi(e)$. Clearly, $\bar{\varphi}(v) = \{1, \ldots, k\} \setminus \varphi(v)$ is then the set of colours **missing** at v.

Let $G = (V, \mu)$ be a graph, let e be an edge of $E_G(x, y)$ (or let $e = \{x, y\}$), and let $\varphi \in \mathcal{C}_k(G - \{e\})$ be a colouring of the graph obtained from G by deleting an edge from $\{x, y\}$. We can now easily adapt all concepts, like multi-fans, Kierstead paths, or Tashkinov trees to the new graph model. Clearly, we may still think of a Tashkinov tree T with respect to e and φ as sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ of vertices $y_0, \ldots, y_p$ and edges $e_1, \ldots, e_p$ satisfying the conditions (T1) and (T2) from Section 2.6. However, formally we have to replace the edges by 2-sets and hence, the conditions (T1) and (T2) by the following two conditions.

(T1′) The vertices $y_0, \ldots, y_p$ are distinct, $e_1 = \{y_0, y_1\} = \{x, y\}$ and, for $i = 1, \ldots, p$, there exists $0 \leq j < i$ such that $e_i = \{y_i, y_j\}$ and $\mu(e_i) \geq 1$.

(T2′) For every e_i with $2 \leq i \leq p$, there is a vertex y_h with $0 \leq h < i$ such that $\varphi(e_i) \cap \bar{\varphi}(y_h) \neq \emptyset$.

To obtain a better performance for our colouring algorithms it is not enough to consider a graph with multiple edges as a pair consisting of a vertex set and a multiplicity function. This graph model, however, gives us the possibility to introduce more efficient data structures.

4.1 Implementation Details

Let $G = (V, \mu)$ be a graph with $|V| = n$ and $\Delta = \Delta(G)$. To obtain a k-edge colouring of G, we start with the edgeless subgraph and the empty colouring, and extend the colouring in several steps using any of our subroutines EXT. In each step, we have a **partial k-edge colouring** of G, that means a k-edge colouring of a subgraph of G. Then φ is a function from $V^{(2)}$ in the power set of $\{1, \dots, k\}$ such that $|\varphi(e)| \leq \mu(e)$ for every $e \in V^{(2)}$ and $\varphi(e) \cap \varphi(e') = \emptyset$ for every $e, e' \in V^{(2)}$ such that $|e \cap e'| = 1$. Clearly, φ may be considered as a k-edge colouring of the subgraph $G' = (V, \mu')$, where $\mu'(e) = |\varphi(e)|$ for all $e \in V^{(2)}$.

Now consider a partial k-edge colouring φ of G. For each colour $\alpha \in \{1, \dots, k\}$, we have the colour class $E_\alpha(\varphi) = \{e \in V^{(2)} \mid \alpha \in \varphi(e)\}$. Using an idea of Sanders and Steurer [25], we can contract consecutive colours with the same colour class to **colour intervals**. If I is such a colour interval, then let $\boldsymbol{E_I(\varphi)} = E_\alpha(\varphi)$ for some $\alpha \in I$. For every colour interval I, we store the corresponding colour class $E_I(\varphi)$ as a doubly-linked list, we call it the **cc-list** of I. Every element of a cc-list also contains a pointer to the colour interval the cc-list belongs to. Since a colour class corresponds to a matching of G, we have $|E_I(\varphi)| \leq \frac{n}{2}$. Since the colour intervals are pairwise disjoint, we can order them (in the natural way) and store them, along with their cc-lists, as a sorted, doubly-linked list. Let ℓ be the number of colour intervals. Then the space needed to store the partial edge colouring φ of G is $O(n \cdot \ell)$.

Note that the cc-lists of the colour intervals correspond to the same-colour lists described in Section 2.2. Only that we now work with 2-subsets of vertices instead of edges, and that such a list is stored for every colour interval instead of every colour. The graph structure itself is stored by adjacency lists that contain, for every vertex, its neighbours along with the corresponding multiplicities. Every 2-subset $e = \{x, y\}$ of vertices with $\mu(x, y) \geq 1$ can now appear in the cc-list of several colour intervals. For every such 2-set e, we store a list of pointers to the corresponding elements in the cc-lists of these colour intervals. It is similar to the representation described in Section 2.2, only that in this former representation an edge appeared only in one same colour list.

Since all colours of a colour interval have the same colour class, the algorithms will usually work on the colour intervals, not on special colours. Such a colour interval will always be represented as a pointer to an element of the list of all colour intervals. This way, the algorithm has direct access to the colour class of a colour interval without having to scan the list of colour intervals for a particular interval. If an algorithm has to handle a set of colours (e.g., missing colours, free colours, defective colours), this set will always be represented as an ordered list of colour intervals. Of course, since we already have an ordered list of all colour intervals, we avoid copying them and store the set of colours simply as a list of pointers to elements of the list of all colour intervals, maintaining its ordering. If an algorithm has to perform a recolouring operation, for example colouring an edge or performing a Kempe change, then special colours from given intervals are needed. In this case the algorithm can simply choose an arbitrary colour from a given colour interval. Clearly, to perform such a simple recolouring operation, the involved colour intervals usually have to be split, because such an operation changes the colour class of only one colour per colour interval. Hence, it would be clever to choose the maximum or the minimum colour of the interval. This way, the interval is split into at most two parts instead of at most three. However, it is always split into a constant number of parts, so it does not affect the time and space complexities of such an algorithmic step. Of course, an algorithm can significantly increase the number of colour intervals, affecting the time complexities of subsequent steps.

When formulating algorithms, we will still use colours instead of colour intervals. Otherwise we would have to change much of our notation. For example, an (α, β)-chain $P_x(\alpha, \beta, \varphi)$ would have to be a (I, J)-chain $P_x(I, J, \varphi)$ for two colour intervals I and J. Clearly, this is not of much benefit, so we simply use the old notation. Just keep in mind, what this means for the algorithm and its time complexity.

Our aim is to construct an edge colouring of G in such a way, that in each step the number of colour intervals remains bounded by a polynomial in $n = |V(G)|$. Then the space of the colouring representation is polynomial in the length of the graph representation, a simple requirement for a polynomial-time edge colouring algorithm.

Before we construct such an algorithm, we first need to analyse how the new colouring representation affects the time complexity of our colouring operations for a partial k-edge colouring φ of G that is contracted to ℓ colour intervals. Since the representation resembles the one described in Section 2.2, we can basically perform the operations in a similar way, using colour intervals and cc-lists instead of particular colours and same colour lists.

Apart from determining the time complexity of an operation, we also have to analyse how this operation affects the number of colour intervals. This number has to remain sufficiently small, in order to get a polynomial-time edge colouring algorithm in the end.

Since colour intervals are represented as pointers to elements of the sorted list of all colour intervals, we have direct access to the colour class of a given colour interval. We need not to find an interval by scanning the whole interval list. So there is no disadvantage compared to the old representation, where colours was only integers.

To determine the colour set $\varphi(e)$ for a given 2-set e of vertices, we need time $\mathrm{O}(\ell)$. This is because we stored pointers to the elements of the cc-lists corresponding to e, and these elements are linked to colour interval of their cc-list. Keep in mind that we handle the set $\varphi(e)$ always as an ordered list of pointers to colour intervals. To maintain the ordering of the list that represents the set $\varphi(e)$, we first mark the intervals in our colour interval list, and then scan the whole list (which is ordered) to find the marked intervals. Clearly, we unmark them along the way. To determine the colour set $\varphi(x)$ for a given vertex x, we also need time $\mathrm{O}(\ell)$. We simply determine, for all neighbours y of x, the set $\varphi(\{x, y\})$. Since, for different neighbours y of x, the colour sets $\varphi(\{x, y\})$ are disjoint, every colour interval appears in at most one of the lists of such a 2-set $\{x, y\}$ and, therefore, we have to handle at most ℓ colour intervals. This gives the mentioned time complexity. Note, that we can maintain the ordering of the list representing $\varphi(x)$ the same way as described above. Clearly, we can also determine the set $\bar{\varphi}(x)$ in time $O(\ell)$ by constructing $\varphi(x)$ and comparing to the whole colour interval list. Consequently, for two vertices x, y, we can determine the colour set $\bar{\varphi}(x) \cap \bar{\varphi}(y)$ in time $O(\ell)$. Note that, for all these operations, the ordering of the lists is essential to obtain these time complexities. However, using unordered representations, they still would need only polynomial time.

Another important operation is, for two distinct colour intervals I, J containing colours α respectively β, to find an (α, β)-chain P containing a given vertex x. We can find P in the following way. First we construct the graph $H = G_{\alpha,\beta}$. This graph is defined by the two colour classes $E_I(\varphi)$ and $E_J(\varphi)$. To construct H, we start with the edgeless graph containing the n vertices of G, and then, for every 2-set $\{u, v\}$ in one of the cc-lists of I and J, we add to H an edge between u and v. Note that we only need a standard adjacency list to represent H. Since all components of H are paths or cycles, H contains at most n edges and can be constructed in time $O(n)$. For the same reason, we can find P in time $O(n)$ by simply starting at x and following the edges of H. Clearly, we need time $O(n)$ in total to construct P.

If $|\varphi(e)| = \mu(e)$ for every $e \in V^{(2)}$, then all edges of G are coloured and $\varphi \in \mathcal{C}_k(G)$. Otherwise, there is a 2-subset $e = \{x, y\}$ such that $|\varphi(e)| < \mu(e)$. This means that at least one edge of e (or of $E_G(x, y)$) is uncoloured. Colouring one more edge of $e = \{x, y\}$ with a colour from a given colour interval I, or with the new colour $\alpha = k + 1$, needs time $O(n)$ and increases the number of colour intervals by only a constant. This can be done in the following way. If we use a new colour $\alpha = k + 1$ then $I' = \{\alpha\}$ forms a new interval, and the cc-list of I' only contains the 2-set e. Note that, for e, we also have to store a list of pointers to the cc-list elements of the corresponding colour intervals. This list can easily be updated by adding the pointer to the element e in the cc-list of I'. If we use a colour from a colour interval I then we choose an arbitrary $\alpha \in I$, and split I into at most three intervals, one containing the colours smaller than α, one containing the colours higher than α, and one interval $I' = \{\alpha\}$. Clearly, we can choose α as the minimum or maximum of I, and then we have to split I into at most two parts, but we will consider the general case. Further, note that, in order to maintain an ordered colour interval list, we have to include the new intervals at the right position. Since we use a doubly-linked list, we can include them directly before or after the position of I in constant time. We also have to make copies of the cc-list of I for the new intervals. This includes linking the copied cc-list elements to the new intervals they now belong to. Since the cc-list contains at most $\frac{n}{2}$ elements, this is done in time $O(n)$. Afterwards we add the 2-set e to the cc-list of I'. Note that, for all 2-sets in the cc-list of the former interval I, we have to update the list of pointers to the cc-lists elements of the corresponding colour intervals. Since the interval I just became one of the new intervals, the old pointers are still valid. For each 2-set in the cc-list of the former interval I, we just have to add pointers to the copied cc-list elements in the cc-lists of the new intervals. Clearly, this also needs time $O(n)$. Additionally, for the 2-set e, we add a similar pointer to the colour interval I'. Then we are done. Consequently, the time needed for the whole colouring operation is $O(n)$. Moreover, the number of colour intervals is increased by only a constant.

Performing a Kempe change of a (α, β)-chain P, where α, β belong to the colour intervals I respectively J, also needs time $O(n)$, and increases the number of colour intervals by only a constant. It can be done in the following way. If P consists only of a multiple edge then we have nothing to do. Otherwise I and J are distinct. First we need to split I and J into at most three intervals, respectively. We do this in the same way as described above, this includes copying the cc-list and, for all 2-sets in the cc-list, updating the list of pointers to the cc-list elements. As already described, this needs time $O(n)$. Now we have to interchange some elements between the two cc-lists of I and J. We can do this in the following way. First we build the graph $H = G_{\alpha,\beta}$ in the same way as we did for finding the chain P. We even may still have this graph from this step. If not, we need time $O(n)$ to do it. In H, we then mark the edges of the component corresponding to P in time $O(n)$. These edges, respectively the 2-sets in G, have to be recoloured. For every element e in the cc-lists of I and J, we can decide in constant time whether it is marked in H or not. If it is marked, we move this element to the cc-list of the other colour interval, and change the pointer from the former interval to the new one. This also needs only constant time. Note that we have to move this element instead of copying it and deleting the original, otherwise we would have to find and update the pointer from the 2-set e to the corresponding element e in the cc-list, and this could not be done in constant time. Since any cc-list contains at most $\frac{n}{2}$ elements, this whole interchanging is done in time $O(n)$. That gives a total time of $O(n)$ to perform a Kempe change. Moreover, the number of colour intervals is increased by only a constant.

Note that all these time complexities correspond to the time complexities of the same operations described in Section 2.2. The time for finding (α, β)-chains is in both cases linear in the number of vertices. Only that in the new system two colour intervals are given instead of the two colours α and β. In the new system we can simply choose these colours from the given intervals. Constructing and comparing colour sets like $\bar{\varphi}(x)$ needed time linear in the number k of colours and, hence, linear in the maximum degree $\Delta(G)$. Now we have a time linear in the number ℓ of colour intervals. We have this correspondence between $\Delta(G)$ and ℓ in general. If an algorithm using the old data structures has to scan all edges incident to a vertex x, then the corresponding algorithm using the new data structures has to scan all neighbours of x and, for every neighbour, the list of pointers to the cc-lists elements of the corresponding colour intervals. Since, for different neighbours y of x, the colour sets $\varphi(\{x, y\})$ are disjoint, every colour interval appears in at most one of these lists and, therefore, there are at most ℓ colour intervals to handle. Consequently, for any graph operation that works only in a neighbourhood of a vertex, a time complexity polynomial in $|V(G)|$ and $\Delta(G)$ when using the old data structures of Section 2.2 can be implemented with the new data structures of this chapter with a time complexity polynomial in $|V(G)|$ and the resulting number ℓ of colour intervals. Clearly, the same holds for operations that can be divided into a polynomial (in $|V(G)|$) number of such operations. Since the old and the new data structures and techniques correspond in such a strong way, we can even expect that in many cases the time complexities of the two approaches are nearly exactly the same, only that $\Delta(G)$ and ℓ are interchanged.

Let Ext be some algorithm that extends a given partial edge colouring of a graph

G to an additional uncoloured edge. Further, suppose that EXT uses the old data structures of Section 2.2, and that EXT only uses simple recolouring techniques such as Kempe changes. Clearly, this algorithm EXT can transformed into a version EXT′ that uses the new data structures an techniques from this chapter. If the running time of EXT is polynomially bounded in $|V(G)|$ and $\Delta(G)$, then the running time of EXT′ is polynomially bounded in $|V(G)|$ and in the number of colour intervals ℓ the resulting colouring is contracted to. Moreover, in many cases we can even expect that EXT′ has basically the same running time as EXT, only that $\Delta(G)$ is replaced by ℓ.

All the algorithms of Chapter 2 or Chapter 3 that extend a given colouring to an additional edge, can be transformed into similar algorithms using the new data structures. Then these new versions have running times bounded by a polynomial in $|V(G)|$ and ℓ. Moreover, the number of recolouring operations (usually Kempe changes) in all these algorithms, is bounded by a polynomial in $|V(G)|$. Since a Kempe change increases the number of colour intervals by only a constant, this implies that these algorithms increase the number of colour intervals by at most a polynomial in $|V(G)|$. We will formalize this and define a class of algorithms that will be useful.

For an integer $r \in \mathbb{N}$, let $\boldsymbol{\mathcal{CE}_r}$ be the class of algorithms ALG that satisfy the following conditions:

(C1) The input of ALG is a tuple (G, x, y, k, φ), where $G = (V, \mu)$ is a graph, $x, y \in V$ are two adjacent vertices, and $\varphi \in \mathcal{C}_k(G - \{x, y\})$ is a k-edge colouring, that is, $|\varphi(\{x, y\})| = \mu(x, y) - 1$ and $|\varphi(e)| = \mu(e)$ for every $e \in V^{(2)} \setminus \{\{x, y\}\}$.

(C2) If $k \geq \Delta(G) + r$ then ALG returns a tuple (k', φ'), where $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}_{k'}(G)$.

(C3) The running time of ALG is bounded by a polynomial in $|V|$ and ℓ', where ℓ' is the number of colour intervals φ' is contracted to. Moreover, if φ is contracted to ℓ colour intervals then $\ell' - \ell$ is bounded by a polynomial in $|V|$.

Let **VIZEXT′**, **KIEREXT′**, **TASHEXT1′** respectively **TASHEXT2′** be the versions of VIZEXT, KIEREXT, TASHEXT1 respectively TASHEXT2 using our new data structures and techniques. Evidently, we then have

$$\text{VIZEXT}' \in \mathcal{CE}_0 \tag{4.1}$$

$$\text{KIEREXT}', \text{TASHEXT1}', \text{TASHEXT2}' \in \mathcal{CE}_1 \tag{4.2}$$

In Chapter 3 we, bit by bit, described another kernel algorithm that extends a partial edge colouring by additionally colouring one edge. Although this algorithm is somewhat more complex, constructing elementary and strongly closed sets in many different ways, it mainly works on Tashkinov trees and fans, and uses Kempe changes as recolouring operations. Hence, this algorithm also has a polynomial-time implementation **TASHKINOV3′** using our new data structures. Therefore, we have

$$\text{TASHKINOV3}' \in \mathcal{CE}_1 \tag{4.3}$$

4.2 Divide and Conquer

To get a polynomial-time edge colouring algorithm, we cannot simply colour all the edges one by one, since the number of edges is the critical parameter which may be too large. An obvious idea is to use a divide-and-conquer strategy to split the graph into halves, colour one half, and double the colouring. Clearly, it is not this easy. For one thing, a graph usually cannot split exactly into halves. If some edge multiplicities are odd, we have to additionally colour the remaining edges. A much greater problem is that the bounds, we want to achieve with our colouring algorithm, usually are not homogeneous in the sense that the parameter value for an edge-doubled graph is the double of the parameter value for the original graph. A way out of this situation is given by an idea from Sanders and Steurer [25]. We first compute a partial edge colouring with the divide-and-conquer strategy, and the colour the remaining edges one by one. Note that this restricts the number of uncoloured edges in the partial colouring that we compute in the first step. On one hand we can leave only so many edges uncoloured, that we can colour them one by one in polynomial time. On the other hand, to ensure that the divide-and-conquer algorithm works properly, we have to leave at least so many edges uncoloured, that the maximal number of used colours is homogeneous with respect to doubling the edges of a graph.

Let $G = (V, \mu)$ be graph, and let $G' = (V, \mu')$ be the graph satisfying $\mu'(e) = 2\mu(e)$ for all $e \in V^{(2)}$. What graph parameters ρ satisfy $\rho(G') = 2\rho(G)$? Clearly, this equality holds for $\rho = \Delta$, the maximum degree. It also holds for $\rho = \mathcal{W}$, the density. That $\mathcal{W}(G') \geq 2\mathcal{W}(G)$ simply follows from the fact that every subgraph H of G corresponds to a subgraph H' of G' that results from H by doubling all edges. Moreover, every subgraph H'' of G' is also a subgraph of another subgraph H' of G' with only even edge multiplicities. Since H' corresponds to a subgraph H of G, that results from H' by dividing all edges multiplicities by 2, we obtain $\mathcal{W}(G') \leq 2\mathcal{W}(G)$.

The aim is to use a divide-and-conquer strategy to compute, for a graph G, a partial edge colouring that uses not more than $\max\{\Delta(G), \mathcal{W}(G)\}$ colours and leaves only few edges (bounded by a polynomial in the number of vertices) uncoloured. The remaining edges can be coloured one by one. Since both parameters Δ and $\mathcal{W}$ are lower bounds of the chromatic index χ', the partial edge colouring uses only as many colours as we need at least to colour the whole graph G. This gives a wide choice of colouring algorithms for the remaining edges to achieve several approximation guarantees.

To construct the partial edge colouring, we also need some kernel algorithm that extends an edge colouring by colouring an additional edge, which was uncoloured before. At this point we do not give a special kernel, but we define some conditions for such an algorithm, that will be sufficient to achieve our goal. For a function $d : \mathbb{N} \to \mathbb{N}$, let $\boldsymbol{\mathcal{CE}^*_d}$ be the class of algorithms Alg that satisfy the following conditions:

(C1*) The input of Alg is a tuple (G, x, y, k, φ), where $G = (V, \mu)$ is a graph, $x, y \in V$ are two adjacent vertices, and $\varphi \in \mathcal{C}_k(G - \{x, y\})$ is a k-edge colouring, that is, $|\varphi(\{x, y\})| = \mu(x, y) - 1$ and $|\varphi(e)| = \mu(e)$ for every $e \in V^{(2)} \setminus \{\{x, y\}\}$.

(C2*) If $k \geq \Delta(G) + d(|V|)$ then Alg returns a tuple (k', φ'), where $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}_{k'}(G)$. Moreover, if $k' = k + 1$ then $\mathcal{W}(G) = k' = k + 1$.

(C3*) If $k \geq \Delta(G)+d(|V|)$ then the running time of ALG is bounded by a polynomial in $|V|$ and ℓ', where ℓ' is the number of colour intervals φ' is contracted to. Moreover, if φ is contracted to ℓ colour intervals then $\ell' - \ell$ is bounded by a polynomial in $|V|$.

Let $d : \mathbb{N} \to \mathbb{N}$ be a function that is polynomially bounded, and let EXT $\in \mathcal{CE}_d^*$. Then we can construct the following algorithm PARTIALCOL[EXT, d] that computes a partial edge colouring of a given graph $G = (V, \mu)$, that is, a tuple (G', k, φ) such that $G' = (V, \mu')$ is a subgraph of G, and $\varphi \in \mathcal{C}_k(G')$ is a k-edge colouring of G'.

PARTIALCOL[EXT, d]($G = (V, \mu)$):

1) $\Delta \leftarrow \Delta(G)$, $D \leftarrow d(|V|)$
`If` $\Delta \leq D$ `then Return` $((V, \emptyset), 0, \emptyset)$.

2) $\mu' \leftarrow \mu$, $G' \leftarrow (V, \mu')$.

3) `While` $\Delta(G') > \Delta - D$ `do`

  3a) `Find` $x, y \in V$ `such that` $d_{G'}(x) > \Delta - D$ `and` $\mu'(\{x, y\}) \geq 1$.

  3b) $\mu'(\{x, y\}) \leftarrow \max\{0, \mu'(\{x, y\}) - d_{G'}(x) + \Delta - D\}$.
  $G' \leftarrow (V, \mu')$.

4) $\forall e \in V^{(2)}$: $\mu_0(e) \leftarrow \left\lfloor \frac{\mu'(e)}{2} \right\rfloor$.

5) $((V, \mu_0'), k, \varphi) \leftarrow$ `PARTIALCOL`[`EXT`, d]$((V, \mu_0))$.

6) $\forall e \in V^{(2)}$: $\mu_1'(e) \leftarrow 2\mu_0'(e)$.

7) $k \leftarrow 2k$.
$I \leftarrow [2a - 1, 2b]$ `for every colour interval` $I = [a, b]$ `of` φ.

8) `If` $k < \Delta(G)$ `then` $k \leftarrow \Delta(G)$.

9) `While` $\mu_1' \neq \mu'$ `do`

  9a) `Let` $x, y \in V$ `such that` $\mu_1'(\{x, y\}) < \mu'(\{x, y\})$.

  9b) $\mu_1'(\{x, y\}) \leftarrow \mu_1'(\{x, y\}) + 1$.

  9c) $(k, \varphi) \leftarrow$ `EXT`$((V, \mu_1'), x, y, k, \varphi)$.

10) `Return` (G', k, φ).

Proposition 4.1 *Let $d : \mathbb{N} \to \mathbb{N}$ be a function that is polynomially bounded, and let* EXT $\in \mathcal{CE}_d^*$. *Furthermore, let $G = (V, \mu)$ be a graph. Then, on the input G, the algorithm* PARTIALCOL[EXT, d] *returns a tuple (G', k, φ) satisfying the following conditions:*

(a) *$G' = (V', \mu')$ is a subgraph of G, that is, $V' = V$ and $\mu'(e) \leq \mu(e)$ for all $e \in V^{(2)}$.*

(b) *$|E(G')| \geq |E(G)| - d(|V|)|V|$.*

(c) *$\Delta(G') = \max\{0, \Delta(G) - d(|V|)\}$.*

(d) *$\varphi \in \mathcal{C}_k(G')$.*

(e) $k \le \max\{\Delta(G), \mathcal{W}(G)\}$.

(f) *If φ is contracted to ℓ colour intervals, then $\ell \le p(|V|) \cdot \log \mu(G)$, where p is a polynomial.*

(g) *On the input $G = (V, \mu)$, the algorithm* PartialCol[Ext, d] *has a running time bounded by a polynomial in $|V|$ and $\log \mu(G)$.*

Proof: First we proof (a)-(e), that is, the correctness of the algorithm. This proof is based on induction over the maximum degree $\Delta(G)$. If $\Delta(G) \le d(|V|)$ then, in step 1, the algorithm returns the edgeless graph on $n = |V|$ vertices, and the empty colouring. Then the statements (a)-(e) are clearly true. This settles the basic case.

For the inductive step, let $\Delta(G) > d(|V|) = D$. In step 2 the graph G' is initialized with G. The loop starting in step 3 deletes edges of $G' = (V, \mu')$ as long as $\Delta(G') > \Delta - D$. The algorithm chooses only 2-sets $\{x, y\}$ with $d_{G'}(x) > \Delta - D$ and $\mu'(\{x, y\}) \ge 1$ (step 3a), and then deletes at least one and at most $\min\{d_{G'}(x) - \Delta + D, \mu'(\{x, y\})\}$ edges from $E_{G'}(x, y)$ (step 3b). Hence, the loop ends after at most $D|V|$ iterations, and then we have $\Delta(G') = \Delta - D > 0$ and $|E(G')| \ge |E(G)| - D|V|$. Further, note that the graph G' does not change during the rest of the algorithm. Consequently, if the algorithm terminates, this proves (a)-(c).

Step 4 computes the graph $G_0 = (V, \mu_0)$ by dividing the edge multiplicities of G' into halves. Then step 5 recursively calls the algorithm PartialCol[Ext, d] with the input G_0. Clearly, we have $\Delta(G_0) \le \left\lfloor \frac{1}{2}\Delta(G') \right\rfloor \le \left\lfloor \frac{1}{2}\Delta(G) \right\rfloor$. Since $\Delta(G) \ge D + 1 \ge 1$, this implies that $\Delta(G_0) < \Delta(G)$ and, therefore, we can use the induction. Consequently, after step 5, the vertex set V is the same as before, $G'_0 = (V, \mu'_0)$ is a subgraph of G_0, the graph G'_0 has at least $|E(G_0)| - D|V|$ edges, $\Delta(G'_0) = \max\{0, \Delta(G_0) - D\}$, $\varphi \in \mathcal{C}_k(G'_0)$, $k \le \max\{\Delta(G_0), \mathcal{W}(G_0)\}$.

In step 6 the graph $G'_1 = (V, \mu'_1)$ is computed by doubling the edges of G'_0. Clearly, G'_1 is a subgraph of G'. Moreover, we have $\Delta(G'_1) = 2\Delta(G'_0)$ and $\mathcal{W}(G'_1) = 2\mathcal{W}(G'_0)$. Then in step 7, the value of k and the length of the colour intervals of φ is doubled. Hence, after this step, φ is a k-edge colouring of G'_1. Moreover, we have $k \le 2\max\{\Delta(G_0), \mathcal{W}(G_0)\} \le \max\{\Delta(G'), \mathcal{W}(G')\}$. If $k \le \Delta(G)$ then, in the next step 8, k is increased to $\Delta(G)$. Consequently, since G'_1 is a subgraph of G', we then have $\Delta(G) \le k \le \max\{\Delta(G), \mathcal{W}(G')\}$. Moreover, since k is not decreased, we still have $\varphi \in \mathcal{C}_k(G'_1)$.

The loop starting in step 9 increases the graph $G'_1 = (V, \mu'_1)$ and updates the colouring $\varphi \in \mathcal{C}_k(G'_1)$. The loop ends when $G'_1 = G'$. In every sweep of the loop, an edge is added to G'_1, such that G'_1 remains a subgraph of G' (step 9a-9b). Then in step 9c the colouring φ is extended to this edge, using the algorithm Ext. Since Ext $\in \mathcal{CE}^*_d$ and $k \ge \Delta(G) = \Delta(G') + D \ge \Delta(G'_1) + D$, the value of k is increased only if $k < \mathcal{W}(G'_1) \le \mathcal{W}(G')$. Since we had $k \le \max\{\Delta(G), \mathcal{W}(G')\}$ before the loop, this implies that, after step 9c, we always have $k \le \max\{\Delta(G), \mathcal{W}(G')\}$. Since every sweep of the loop increases G'_1 by one edge, at some point we have $G'_1 = G'$, and the loop ends. Then we have $\varphi \in \mathcal{C}_k(G')$ where $k \le \max\{\Delta(G), \mathcal{W}(G')\} \le \max\{\Delta(G), \mathcal{W}(G)\}$. This proves (d)-(e).

Before we prove (f) and (g), we first analyse the recursion depth of the algorithm. If $\Delta(G) \leq d(|V|)$ then there is no recursion. Otherwise, there is one recursive call of PartialCol[Ext, d] with the input graph G_0. Since $\Delta(G_0) \leq \frac{1}{2}\Delta(G)$, we have a recursion depth of at most $\log_2(\Delta(G)) \leq \log_2(|V|\mu(G))$.

Besides the recursion step, the number of colour intervals of the colouring φ is increased only in step 9c, inside the loop starting in step 9. From Ext $\in \mathcal{CE}_d^*$ we then conclude that the step itself increases the number of colour intervals by at most a polynomial in $|V|$. We have to count, how often this step is repeated. The loop runs exactly $|E(G')| - 2|E(G_0')|$ times. Since we have $|E(G_0')| \geq |E(G_0)| - D|V|$, the loop runs at most $|E(G')| - 2|E(G_0)| + 2D|V|$ times. Moreover, we also have $\mu'(e) - 2\mu_0(e) \leq 1$ for every $e \in V^{(2)}$ and, therefore, $|E(G')| - 2|E(G_0)| \leq |V|^2$. Then it follows that the loop runs at most $|V|^2 + 2D|V|$ times. Since $D = d(|V|)$ and d is polynomially bounded, it follows the number of executing step 9c is bounded by a polynomial in $|V|$. Consequently, besides the recursion, the number of colour intervals is increased by at most $p'(|V|)$, where p' is a polynomial. Since the recursion depth is at most $\log_2(|V|\mu(G))$, we conclude that, at the end of the algorithm, the number ℓ of colour intervals is at most $p(|V|) \cdot \log \mu(G)$, where p is a polynomial. This proves (f).

Since the number of colour intervals is never decreased during the algorithm, ℓ is an upper bound of the number of colour intervals at earlier points in the algorithm. When estimating the time complexity of the algorithm, we can always use this highest value ℓ instead of the possibly smaller number in the actual analysed step. Clearly, the steps 1-2, 3a, 3a-3b, 4, 6, 8, 9, and 9a-9b need only polynomial time in $|V|$. The loop starting in step 3 runs at most $D|V| = |V|d(|V|)$ times. Since d is polynomially bounded, this implies that the whole loop only needs polynomial time in $|V|$. Step 7 clearly needs time $O(\ell)$ which is bounded by a polynomial in $|V|$ and $\log \mu(G)$. Since Ext $\in \mathcal{CE}_d^*$, step 9c needs time bounded by a polynomial in $|V|$ and ℓ and hence, by a polynomial in $|V|$ and $\log \mu(G)$. The loop starting in step 9 is repeated at most $|V|^2 + 2D|V| = |V|^2 + 2|V|d(|V|)$. Since d is polynomially bounded, this implies that the whole loop only needs polynomial time in $|V|$ and $\log \mu(G)$. Consequently, besides the recursion step, the algorithm has a running time bounded by a polynomial in $|V|$ and $\log \mu(G)$. Since the recursion depth is at most $\log_2(|V|\mu(G))$, this proves (g). ■

This result shows, that we can compute a partial edge colouring with our desired properties, as long as we find a kernel algorithm Ext$'$ $\in \mathcal{CE}_d^*$ for some function $d : \mathbb{N} \to \mathbb{N}$ that is polynomially bounded. As we will see, the algorithm TashExt2, or rather the version TashExt2$'$ that uses our new data structures, will suffice. Although the algorithm TashExt2$'$ basically works in the same way as TashExt2, we will give it here using the new notation. Keep in mind that even in this new formulation we use several colours, but the algorithm uses only the colour intervals and chooses a colour if necessary, see Section 4.1 for details.

TashExt2′$(G=(V,\mu),x,y,k,\varphi)$:

1) `p` $\leftarrow 1$, $e_p \leftarrow \{x,y\}$, $y_p \leftarrow y$, $y_0 \leftarrow x$, $T \leftarrow (y_0, e_p, y_p)$.

2) `If` $\bar{\varphi}(V(Ty_{p-1})) \cap \bar{\varphi}(y_p) \neq \emptyset$ `then`

  2a) `Compute` $\varphi' \in \mathcal{C}_k(G)$ `as in Theorem 2.20.`

  2b) `Return` (k, φ').

3) `If` $\exists x_{p+1} \in V(T), y_{p+1} \in V(G) \setminus V(T), e_{p+1} = \{x_{p+1}, y_{p+1}\} : \varphi(e_{p+1}) \cap \varphi(E(T)) \neq \emptyset$ `then`

  3a) $T \leftarrow (T, e_{p+1}, y_{p+1})$, $p \leftarrow p+1$.

  3b) `Goto 2.`

4) `If` $\exists x_{p+1} \in V(T), y_{p+1} \in V(G) \setminus V(T), e_{p+1} = \{x_{p+1}, y_{p+1}\} : \varphi(e_{p+1}) \cap (\bar{\varphi}(V(T)) \setminus \varphi(E(T))) \neq \emptyset$ `then`

  4a) $T \leftarrow (T, e_{p+1}, y_{p+1})$, $p \leftarrow p+1$.

  4b) `Goto 2.`

5) `If` $\Gamma^d(T,e,\varphi) = \emptyset$ `then`

  5a) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow \varphi'(e) \cup \{k+1\}$.

  5b) `Return` $(k+1, \varphi')$.

6) `Choose` $\delta \in \Gamma^d(T,e,\varphi)$ `and` $\gamma \in \Gamma^f(T,e,\varphi)$.
`Let` $u \in V(T)$ `with` $\gamma \in \bar{\varphi}(u)$ `and` $P \leftarrow P_u(\gamma,\delta,\varphi)$.

7) `If` $E_\delta(T,e,\varphi) \nsubseteq E(P)$ `then`

  7a) `Compute` $\varphi' = \varphi/P$ `and set` $\varphi \leftarrow \varphi'$.

  7b) `Goto 3.`

8) `Set` v_0, v_1, v_2 `according to Theorem 2.23(a).`

9) `If` $\bar{\varphi}(v_0) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ `then`

  9a) `Compute` $\varphi' \in \mathcal{C}_k(G-e)$ `as in Theorem 2.23(b), and set` $\varphi \leftarrow \varphi'$.

  9b) `Goto 3.`

10) `If` $V(T) \cup \{v_1, v_2\}$ `is not elementary with respect to` φ `then`

  10a) `Compute` $\varphi' \in \mathcal{C}_k(G-e)$ `as in Theorem 2.23(c), and set` $\varphi \leftarrow \varphi'$.

  10b) `Goto 3.`

11) $\varphi' \leftarrow \varphi$, $\varphi'(e) \leftarrow k+1$.

12) `Return` $(k+1, \varphi')$.

Proposition 4.2 *Let $d : \mathbb{N} \to \mathbb{N}$ a function defined by $d(n) = \max\{0, \lfloor \frac{n-1}{2} \rfloor\}$. Then* TashExt2′ $\in \mathcal{CE}^*_d$.

Proof: Since TashExt2′ $\in \mathcal{CE}_1$, condition (C1*) is fulfilled. Let (G,x,y,k,φ) an input of TashExt2′, where $G=(V,\mu)$ and $k \geq \Delta(G) + d(|V|)$. Since G contains an edge, we have $|V| \geq 2$.

If $|V| = 2$ then G consists only of a multiple edge. Obviously, we then have $\bar{\varphi}(x) = \bar{\varphi}(y)$. Moreover, from $k \geq \Delta$ is it follows that $\bar{\varphi}(x) \neq \emptyset$. Consequently, the condition of step 2 in TashExt2′ is fulfilled, and TashExt2′ computes a colouring $\varphi' \in \mathcal{C}_k(G)$. Consequently, the conditions (C2*) and (C3*) are fulfilled in this case.

If $|V| \geq 3$ then $d(|V|) \geq 1$. Then it follows from TashExt2′ $\in \mathcal{CE}_1$ that condition (C3*) is fulfilled in this case. Since TashExt2′ works in the same way as TashExt2, Theorem 2.25 applies to TashExt2′. Consequently, TashExt2′ computes a colouring $\varphi' \in \mathcal{C}_{k'}(G)$ satisfying $k' \in \{k, k+1\}$. Moreover, if $k' = k+1$ then $\mathcal{W}(G) = k+1$ or, as a consequence of (2.16), $|V| \geq 2(k - \Delta(G)) + 3$. Since $2(k - \Delta(G)) + 3 \geq 2d(|V|) + 3 = 2\left\lfloor \frac{1}{2}(|V| - 1)\right\rfloor + 3 \geq |V| + 1$, we conclude that $\mathcal{W}(G) = k+1$ if $k' = k+1$. Hence, condition (C2*) is also fulfilled in the case $|V| \geq 3$. This completes the proof. ■

4.3 A Polynomial Time Colouring Scheme

Let $d : \mathbb{N} \to \mathbb{N}$ be a function that is polynomially bounded, and let Ext $\in \mathcal{CE}^*_d$. Furthermore, let $r \in \mathbb{N}$, and let Ext′ $\in \mathcal{CE}_r$. Then we can formulate the following algorithm PolyCol[Ext, d, Ext′, r].

PolyCol[Ext, d, Ext′, r]($G = (V, \mu)$):

1) $((V, \mu'), k, \varphi) \leftarrow$ `PartialCol`[`Ext`, d](G).
2) `If` $k < \Delta(G) + r$ `then` $k \leftarrow \Delta(G) + r$.
3) `While` $\mu' \neq \mu$ `do`
   - 3a) `Let` $x, y \in V$ `such that` $\mu'(\{x,y\}) < \mu(\{x,y\})$.
   - 3b) $\mu'(\{x,y\}) \leftarrow \mu'(\{x,y\}) + 1$.
   - 3c) $(k, \varphi) \leftarrow$ `Ext`′$((V, \mu'), x, y, k, \varphi)$.
4) `Return` (k, φ).

Theorem 4.3 *Let $d : \mathbb{N} \to \mathbb{N}$ be a function that is polynomially bounded, let* Ext $\in \mathcal{CE}^*_d$*, let $r \in \mathbb{N}$, and let* Ext′ $\in \mathcal{CE}_r$*. Furthermore, let $G = (V, \mu)$ be a graph. Then the algorithm* PolyCol[Ext, d, Ext′, r] *is a polynomial-time algorithm that, on the input G, returns a tuple (k, φ) such that $\varphi \in \mathcal{C}_k(G)$.*

Proof: The algorithm PolyCol[Ext, d, Ext′, r] calls PartialCol[Ext, d](G) in step 1. Since the function d is polynomially bounded and Ext $\in \mathcal{CE}^*_d$, it follows from Proposition 4.1 that, after step 1, the graph $G' = (V, \mu')$ is a subgraph of G, $k \leq \max\{\Delta(G), \mathcal{W}(G)\}$, φ is a k-edge colouring of G' (or a partial k-edge colouring of G), and at most $d(|V|)|V|$ edges of G are uncoloured. Moreover, the colouring φ is contracted to at most $p(|V|) \log \mu(G)$ colour intervals, and the running time of step 1 is bounded by a polynomial in $|V|$ and $\log \mu(G)$.

Step 2 simply increases the value of k if necessary. This is only to ensure that the inputs of Ext′ in the later steps are valid. The loop starting in 3 extends the colouring φ step by step to an edge colouring of the whole graph G. In every sweep, the graph $G' = (V, \mu')$ is extended by an edge of G that is uncoloured (steps 3a and 3b), and then this edge is coloured using the algorithm Ext′ (step 3c). Since the value of k is at least $\Delta(G) + r$ before the loop, and it is never decreased during the loop, it follows from $\text{Ext}' \in \mathcal{CE}_r$ that the input of Ext' is always valid, and that step 3c works correctly, increases the number of colour intervals by just a number that is polynomially bounded in $|V|$, and needs only time polynomially bounded in $|V|$ and in the resulting number of colour intervals. Consequently, since the loop ends after at most $d(|V|)|V|$ iterations, in step 4 φ is a k-edge colouring of G, and the number ℓ of colour intervals is bounded by a polynomial in $|V|$ and $\log \mu(G)$. Moreover, the running time of the whole loop is bounded by a polynomial in $|V|$ and ℓ, and hence by a polynomial in $|V|$ and $\log \mu(G)$. Consequently, $\text{PolyCol}[\text{Ext}, d, \text{Ext}', r]$ works correctly and is a polynomial-time algorithm. This completes the proof. ■

This result shows how a polynomial-time edge colouring algorithm can be constructed, and still leaving a wide choice for the subroutines Ext and Ext′. We already have suitable candidates at hand. Since we have (4.1), (4.2), and (4.3), Proposition 4.2 and Theorem 4.3 imply the following result.

Corollary 4.4 *Let* $d : \mathbb{N} \to \mathbb{N}$ *a function defined by* $d(n) = \max\{0, \lfloor \frac{n-1}{2} \rfloor\}$. *Then the following algorithms are polynomial-time edge colouring algorithms:*

(a) $\text{PolyCol}[\text{TashExt2}', d, \text{VizExt}', 0]$

(b) $\text{PolyCol}[\text{TashExt2}', d, \text{KierExt}', 1]$

(c) $\text{PolyCol}[\text{TashExt2}', d, \text{TashExt1}', 1]$

(d) $\text{PolyCol}[\text{TashExt2}', d, \text{TashExt2}', 1]$

(e) $\text{PolyCol}[\text{TashExt2}', d, \text{Tashkinov3}', 1]$

From Proposition 4.1(e) it follows that, for any suitable algorithms Ext and Ext′, and for any input graph G, the first part of $\text{PolyCol}[\text{Ext}, d, \text{Ext}', r]$ produces a partial edge colouring of G using at most $\max\{\Delta(G), \mathcal{W}(G)\}$ colours. Since this is a lower bound for the chromatic index, $\text{PolyCol}[\text{Ext}, d, \text{Ext}', r]$ has the same approximation guarantees as a an algorithm that, for the input G, starts with $\Delta(G) + r$ colours and uses Ext′ to colour G edge by edge. For example, if $d(n) = \max\{0, \lfloor \frac{n-1}{2} \rfloor\}$ then $\text{PolyCol}[\text{TashExt2}', d, \text{VizExt}', 0]$ uses, on the input G, not more than $\Delta(G) + \mu(G)$ or $\frac{3}{2}\Delta(G)$ colours. Further, Theorem 2.26, Corollary 2.28 and Theorem 2.29 imply that $\text{PolyCol}[\text{TashExt2}', d, \text{TashExt2}', 1]$ uses not more than $\tau(G)$, and, for every $\epsilon > 0$, not more than $\tau_\epsilon(G)$ colours. Moreover, from Corollary 3.26, or rather the algorithmic versions of the preceding results of Chapter 3, it follows that the algorithm $\text{PolyCol}[\text{TashExt2}', d, \text{Tashkinov3}', 1]$ uses at most $15/14\Delta(G) + 12/14$ colours. In general, one can expect that for all colouring algorithms, that use an arbitrary edge order and colour the graph edge by edge, there exists a polynomial-time edge colouring algorithm that has the same approximation guarantees.

Corollary 4.5 *The graph parameters* $\Delta + \mu$, $\frac{3}{2}\Delta$, τ, $\chi'_f + \sqrt{\frac{1}{2}\chi'_f}$, τ_ϵ *(for every* $\epsilon > 0$*), and* $\frac{15}{14}\Delta + \frac{12}{14}$ *are upper bounds for* χ' *that can be realized by a polynomial-time edge colouring algorithm.*

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Symbols