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Boßecker, Anett ; Rautenbach, Dieter

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Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
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Interpolating between Bounds on the Independence Number

Anett Boßecker and **Dieter Rautenbach**

Institut für Mathematik
TU Ilmenau, Postfach 100565
D-98684 Ilmenau, Germany
emails: {anett.bossecker,dieter.rautenbach}@tu-ilmenau.de

Abstract

For a non-negative integer T, we prove that the independence number of a graph $G = (V, E)$ in which every vertex belongs to at most T triangles is at least $\sum_{u \in V} f(d(u), T)$ where $d(u)$ denotes the degree of a vertex $u \in V$, $f(d, T) = \frac{1}{d+1}$ for $T \geq \binom{d}{2}$ and $f(d, T) = (1 + (d^2 - d - 2T)f(d - 1, T))/(d^2 + 1 - 2T)$ for $T < \binom{d}{2}$. This is a common generalization of the lower bounds for the independence number due to Caro, Wei, and Shearer. We discuss further possible strengthenings of our result and pose a corresponding conjecture.

Keywords: Independence; triangle-free graph
AMS subject classification: 05C69

1 Introduction

We consider finite, simple, and undirected graphs $G = (V, E)$ with vertex set V and edge set E. The degree of a vertex u in G is denoted by $d_G(u)$. A set of vertices $I \subseteq V$ of G is called independent, if no two vertices in I are adjacent. The independence number $\alpha(G)$ is the maximum cardinality of an independent set.

The independence number is among the most fundamental and well-studied graph-theoretical concepts. In view of its computational hardness [7] bounds on the independence number received a lot of attention. The following classical lower bound on the independence number of a graph G was obtained independently by Caro [4] and Wei [13]

$$\alpha(G) \geq \sum_{u \in V} \frac{1}{d_G(u) + 1}. \qquad (1)$$

This bound is best-possible in view of cliques. A simple proof of (1) is based on the observation that the deletion of a vertex of maximum degree at least 1 from G does not decrease the right-hand side of (1). Therefore, iteratively deleting such vertices results in an independent set of at least the desired cardinality.

For triangle-free graphs G, Shearer [11] (cf. also [10]) proved

$$\alpha(G) \quad \geq \quad \sum_{u \in V} f(d_G(u)) \tag{2}$$

where $f(0) = 1$ and $f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$ for $d \in \mathbb{N}$. The bound (2) improved on earlier results [2, 3, 6] which gave bounds of the form $\alpha(G) \geq \Omega\left(\frac{n\ln(d)}{d}\right)$ for triangle-free graph G of order n and average degree d. For related results concerning k-clique-free graphs, we refer to [1, 9, 12].

Shearer's bound (2) is similar to Caro and Wei's bound (1) in the sense that every vertex contributes a suitable degree-dependent weight to the value of the bound. Its inductive proof is considerably harder than the proof for (1). In [11] Shearer exploited his approach further to establish lower bounds on the independence number of graphs of large girth. For d-regular graphs G of order n and girth g, he proved $\alpha(G) \geq (1-o(g))nf(d)$ where $f(3) = \frac{125}{302}$ and $f(d) = \frac{1+(d^2-d)f(d-1)}{d^2+1}$ for $d \geq 4$. The strength of his approach is illustrated by the fact that this last bound was only improved very recently [5, 8].

The goal of the research reported here was to prove a common generalization of (1) and (2). For a graph G and a vertex u of G, let $t_G(u)$ denote the number of triangles of G containing u. Note that $t_G(u)$ equals the number of edges among neighbours of u in G. For a suitable function $f : \mathbb{N}_0^2 \to \mathbb{R}_{\geq 0}$, we wanted to prove a bound of the form

$$\alpha(G) \quad \geq \quad \sum_{u \in V} f(d_G(u), t_G(u))$$

which coincides with (2) for triangle-free graphs and is always at least as good as (1).

In Section 2 we discuss Shearer's approach and the possibility to extend it to graphs which may contain triangles. This leads to a number of properties the function f should possess. In Section 3 we propose a candidate for f and establish most of the desired properties. While we eventually succeed in proving a common generalization of (1) and (2), we found our result not yet totally satisfactory and pose a conjecture concerning a possible strengthening.

2 Extending Shearer's Approach

In this section we discuss how to extend Shearer's approach from [11] to graphs which may contain triangles. Consider a graph G. For a vertex u in G, let $d_u = d_G(u)$ and $t_u = t_G(u)$. Our goal is a lower bound for the independence number of G of the form

$$\alpha(G) \quad \geq \quad w(G) := \sum_{v \in V} f(d_v, t_v) \tag{3}$$

where $f : \mathbb{N}_0^2 \to \mathbb{R}_{\geq 0}$ is a suitable function. In order for Shearer's inductive approach to work, the function f has to possess several properties. For $d, t \in \mathbb{N}_0$, we assume

(P_1) $f(0,0) = 1$,

(P_2) $f(d,t) \geq f(d,t+1)$,

(P_3) $f(d,t) - f(d+1,t) \geq f(d+1,t) - f(d+2,t)$, and

(P_4) $1-(d+2)f(d+1,t)+\left((d+1)^2-(d+1)-2t\right)(f(d,t)-f(d+1,t))\geq 0$ for $t\leq\binom{d+1}{2}$.

Property (P_1) implies (3) for $|V|=1$, i.e. the base case of the induction. Furthermore, by (P_1), we may assume that G has no vertex of degree 0.

For two distinct vertices u and v in G, let $d_{\{u,v\}}$ denote the number of common neighbours of u and v. For a vertex u in G, let N_u denote the set of neighbours of u and let N_u^2 denote the set of vertices at distance exactly two from u, respectively.

If there is a vertex u in G such that the deletion of all vertices in $\{u\}\cup N_u$ results in a graph G_u with $1-w(G)+w(G_u)\geq 0$, then adding u to a maximum independent set of G_u results in an independent set of G of order at least $1+w(G_u)\geq w(G)$. If $w\in N_u^2$, then $d_{G_u}(w)=d_w-d_{\{u,w\}}$ and $t_{G_u}(w)\leq t_w$. Therefore, by the monotonicity property (P_2), it suffices to prove the existence of a vertex u in G with

$$1-f(d_u,t_u)-\sum_{v\in N_u}f(d_v,t_v)+\sum_{w\in N_u^2}\left(f\left(d_w-d_{\{u,w\}},t_w\right)-f(d_w,t_w)\right)\geq 0. \tag{4}$$

In [11] Shearer shows the existence of such a vertex by proving that (4) holds on average. Therefore, let

$$A = \sum_{u\in V}\left(1-f(d_u,t_u)-\sum_{v\in N_u}f(d_v,t_v)+\sum_{w\in N_u^2}\left(f(d_w-d_{\{u,w\}},t_w)-f\left(d_w,t_w\right)\right)\right).$$

Since $\sum\limits_{u\in V}\sum\limits_{v\in N_u}f(d_v,t_v)=\sum\limits_{u\in V}d_uf(d_u,t_u)$ and $w\in N_u^2\Leftrightarrow u\in N_w^2$, we have

$$\begin{aligned}A &= \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\sum_{w\in N_u^2}\left(f(d_w-d_{\{u,w\}},t_w)-f\left(d_w,t_w\right)\right)\right)\\ &= \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\sum_{w\in N_u^2}\left(f(d_u-d_{\{u,w\}},t_u)-f\left(d_u,t_u\right)\right)\right).\end{aligned} \tag{5}$$

By (P_3),

$$f(d_u-d_{\{u,w\}},t_u)-f\left(d_u,t_u\right)\geq d_{\{u,w\}}(f(d_u-1,t_u)-f(d_u,t_u)).$$

Furthermore, simple double-counting yields

$$\sum_{w\in N_u^2}d_{\{u,w\}} = \left(\sum_{v\in N_u}(d_v-1)\right)-2t_u.$$

Together with (5) we obtain

$$\begin{aligned}A &\geq \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\sum_{w\in N_u^2}d_{\{u,w\}}(f(d_u-1,t_u)-f(d_u,t_u))\right)\\ &= \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\left(\left(\sum_{v\in N_u}(d_v-1)\right)-2t_u\right)(f(d_u-1,t_u)-f(d_u,t_u))\right).\end{aligned} \tag{6}$$

A crucial property of f — or of the pair (G, f) — needed at this point to continue along Shearer's argument is that

$$\sum_{u\in V}\sum_{v\in N_u}(d_v-1)(f(d_u-1,t_u)-f(d_u,t_u)) \;\;\geq\;\; \sum_{u\in V}\sum_{v\in N_u}(d_u-1)(f(d_u-1,t_u)-f(d_u,t_u)). \tag{7}$$

If the values of f are independent of the second parameter, i.e. $f(d,t) = f(d,t+1)$ for all $d,t \in \mathbb{N}_0$, then (7) follows from property (P_3) as follows

$$\begin{aligned}
& \sum_{u\in V}\sum_{v\in N_u}(d_v-1)(f(d_u-1,t_u)-f(d_u,t_u)) \\
= & \sum_{uv\in E}((d_v-1)(f(d_u-1,t_u)-f(d_u,t_u))+(d_u-1)(f(d_v-1,t_v)-f(d_v,t_v))) \\
\overset{(P_3)}{\geq} & \sum_{uv\in E}((d_u-1)(f(d_u-1,t_u)-f(d_u,t_u))+(d_v-1)(f(d_v-1,t_v)-f(d_v,t_v))) \\
= & \sum_{u\in V}\sum_{v\in N_u}(d_u-1)(f(d_u-1,t_u)-f(d_u,t_u)).
\end{aligned}$$

Assuming (7) we would obtain from (6) that

$$\begin{aligned}
A & \geq \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\left(\left(\sum_{v\in N_u}(d_u-1)\right)-2t_u\right)(f(d_u-1,t_u)-f(d_u,t_u))\right) \\
& = \sum_{u\in V}\left(1-(d_u+1)f(d_u,t_u)+\left(d_u^2-d_u-2t_u\right)(f(d_u-1,t_u)-f(d_u,t_u))\right).
\end{aligned}$$

Since $t_u \leq \binom{d_u}{2}$ for every vertex u in G, property (P_4) would imply $A \geq 0$ which would complete the inductive proof. In order to turn the sketched approach into a result we need to describe a function f which possesses the desired properties. In fact, apart from a version of (7) in full generality our proposal for f will possess all these properties.

3 A Reasonable Proposal for f

In this section we propose a choice for f which has properties (P_1) through (P_4) and which appears reasonable in the sense that it allows to prove a common generalization of Caro and Wei's bound (1) and Shearer's bound (2).

For non-negative integers d and t let

$$r(d,t,f) \;=\; \frac{1+(d^2-d-2t)f}{d^2+1-2t}. \tag{8}$$

Furthermore, let

$$f(d,t) \;=\; \begin{cases} \frac{1}{(d+1)} & , t \geq \binom{d}{2}, \\ r(d,t,f(d-1,t)) & , t < \binom{d}{2}. \end{cases} \tag{9}$$

Clearly, the function $f(\cdot,0)$ coincides with the function $f(\cdot)$ from (2). Furthermore, we will show $f(d,t) \geq \frac{1}{d+1}$ for $d,t \in \mathbb{N}_0$. In view of Section 2 it makes sense to define $f(d,t)$ also for values of d

and t with $t > \binom{d}{2}$ which are graph-theoretically meaningless. Table 1 shows some specific values of f. The bold entries correspond to vertices whose neighbourhoods induce complete graphs. As soon as the neighbourhood of a vertex is not complete the Shearer-like recursion (8) sets in.

$f(d,t)$	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
$t=0$	**1**	**1/2**	2/5	17/50	127/425	593/2210
$t=1$	1	1/2	**1/3**	7/24	47/180	19/80
$t=2$	1	1/2	1/3	5/18	29/117	581/2574
$t=3$	1	1/2	1/3	**1/4**	5/22	23/110
$t=4$	1	1/2	1/3	1/4	2/9	11/54
$t=5$	1	1/2	1/3	1/4	3/14	11/56
$t=6$	1	1/2	1/3	1/4	**1/5**	13/70
$t=7$	1	1/2	1/3	1/4	1/5	11/60
$t=8$	1	1/2	1/3	1/4	1/5	9/50
$t=9$	1	1/2	1/3	1/4	1/5	7/40
$t=10$	1	1/2	1/3	1/4	1/5	**1/6**

Table 1 $f(d,t)$ for $0 \leq d \leq 5$ and $0 \leq t \leq 10$.

The next lemma collects properties of f. For $t \in \mathbb{N}_0$, let $d_t = \max\left\{d \in \mathbb{N}_0 \mid t \geq \binom{d}{2}\right\}$. Note that (9) is equivalent with $f(d,t) = \frac{1}{d+1}$ for $d \leq d_t$ and $f(d,t) = r(d,t,f(d-1,t))$ for $d > d_t$.

Lemma 1 *Let $d, t \in \mathbb{N}_0$.*

(i) $f(d,t) \geq \frac{3(d+2)}{2(d^2+5d+5+t)}$ *for* $d \geq d_t$.

(ii) $f(d,t) \geq \frac{1}{d+1}$.

(iii) $f(d,t) \geq f(d+1,t)$.

(iv) $f(d,t) \geq f(d,t+1)$.

(v) $f(d,t) - f(d+1,t) \geq f(d+1,t) - f(d+2,t)$.

(vi) $1 - (d+2)f(d+1,t) + \left((d+1)^2 - (d+1) - 2t\right)(f(d,t) - f(d+1,t)) \geq 0$ *for* $t \leq \binom{d+1}{2}$.

Proof: (i) We prove this statement by induction on $d \geq d_t$. By (9), $f(d_t,t) = \frac{1}{d_t+1}$. Since $t \geq \binom{d_t}{2} = \frac{1}{2}d_t(d_t-1)$, we obtain

$$\begin{aligned}\frac{3(d_t+2)}{2(d_t^2+5d_t+5+t)} &\leq \frac{3(d_t+2)}{2\left(d_t^2+5d_t+5+\frac{1}{2}d_t(d_t-1)\right)} \\ &= \frac{3(d_t+2)}{3(d_t+2)(d_t+1)+4} < \frac{1}{d_t+1} = f(d_t,t)\end{aligned}$$

which proves the base case of the induction.

If $d > d_t$, then $2t < d(d-1)$ and, by (9), $f(d,t) = r(d,t,f(d-1,t))$. Since $r(d,t,x)$ is monotonously decreasing as a function of x, we obtain, by induction,

$$\begin{aligned} f(d,t) - \frac{3(d+2)}{2(d^2+5d+5+t)} &= r(d,t,f(d-1,t)) - \frac{3(d+2)}{2(d^2+5d+5+t)} \\ &\geq r\left(d,t,\frac{3((d-1)+2)}{2((d-1)^2+5(d-1)+5+t)}\right) - \frac{3(d+2)}{2(d^2+5d+5+t)} \\ &= \frac{d^2+2d+2-2t}{(d^2+5d+5+t)(d^3+3d+1+t)} \\ &\geq \frac{d^2+2d+2-d(d-1)}{(d^2+5d+5+t)(d^3+3d+1+t)} \\ &= \frac{3d+2}{(d^2+5d+5+t)(d^3+3d+1+t)} > 0 \end{aligned}$$

which completes the proof of (i).

(ii) We prove this statement by induction on d. If $d \leq d_t$, then, by (9), $f(d,t) = \frac{1}{d+1}$.

If $d > d_t$, then $t < \binom{d}{2}$ and, by induction,

$$\begin{aligned} f(d,t) - \frac{1}{d+1} &\overset{(9)}{=} r(d,t,f(d-1,t)) - \frac{1}{d+1} = \frac{1+(d^2-d-2t)f(d-1,t)}{d^2+1-2t} - \frac{1}{d+1} \\ &\geq \frac{1+(d^2-d-2t)\frac{1}{d}}{d^2+1-2t} - \frac{1}{d+1} = \frac{d^2-d-2t}{(d^2+1-2t)d(d+1)} \geq 0 \end{aligned}$$

which completes the proof of (ii).

(iii) If $d \leq d_t - 1$, then $f(d,t) = \frac{1}{d+1} > \frac{1}{d+2} = f(d+1,t)$.

If $d \geq d_t$, then $t < \binom{d+1}{2}$ and

$$f(d,t) - f(d+1,t) \overset{(9)}{=} f(d,t) - r(d+1,t,f(d,t)) \overset{(8)}{=} \frac{(d+2)f(d,t)-1}{(d+1)^2+1-2t} \overset{\text{(ii)}}{\geq} 0$$

which completes the proof of (iii).

(iv) We prove this statement by induction on d. If $d \leq d_t$, then $f(d,t) \overset{(9)}{=} f(d,t+1) \overset{(9)}{=} \frac{1}{d+1}$.

Hence, we may assume that $d > d_t$ which implies $t < \binom{d}{2}$ and $f(d,t) = r(d,t,f(d-1,t))$.

If $t+1 < \binom{d}{2}$, then, by induction,

$$\begin{aligned} f(d,t) - f(d,t+1) &\overset{(9)}{=} r(d,t,f(d-1,t)) - r(d,t+1,f(d-1,t+1)) \\ &= \frac{1+(d^2-d-2t)f(d-1,t)}{d^2+1-2t} - \frac{1+(d^2-d-2t-2)f(d-1,t+1)}{d^2+1-2t-2} \\ &\geq \frac{1+(d^2-d-2t)f(d-1,t)}{d^2+1-2t} - \frac{1+(d^2-d-2t-2)f(d-1,t)}{d^2+1-2t-2} \\ &= \frac{(2d+2)f(d-1,t)-2}{(d^2+1-2t)(d^2+1-2t-2)} \overset{\text{(ii)}}{\geq} 0. \end{aligned}$$

Hence, we may assume that $t+1=\binom{d}{2}$. This implies

$$\begin{aligned} f(d,t)-f(d,t+1) &\stackrel{(9)}{=} r\left(d,t,f(d-1,t)\right)-\frac{1}{d+1}=r\left(d,t,\frac{1}{d}\right)-\frac{1}{d+1}\\ &= \frac{1+(d^2-d-2t)\frac{1}{d}}{d^2+1-2t}-\frac{1}{d+1}=\frac{d^2-d-2t}{(d^2+1-2t)d(d+1)}\geq 0 \end{aligned}$$

which completes the proof of (iv).

(v) If $d\leq d_t-1$, then

$$\begin{aligned} &(f(d,t)-f(d+1,t))-(f(d+1,t)-f(d+2,t))\\ \stackrel{(9)}{=}& \left(\frac{1}{d+1}-\frac{1}{d+2}\right)-\left(\frac{1}{d+2}-f(d+2,t)\right)=f(d+2,t)-\frac{d}{(d+1)(d+2)}\\ \stackrel{(ii)}{\geq}& \frac{1}{d+3}-\frac{d}{(d+1)(d+2)}=\frac{2}{(d+1)(d+2)(d+3)}>0. \end{aligned}$$

If $d\geq d_t$, then

$$\begin{aligned} &(f(d,t)-f(d+1,t))-(f(d+1,t)-f(d+2,t))\\ \stackrel{(9)}{=}& f(d,t)-2r(d+1,t,f(d,t))+r(d+2,t,r(d+1,t,f(d,t))). \end{aligned}$$

It is straightforward to verify that the last expression is non-negative if and only if $f(d,t)\geq\frac{3(d+2)}{2(d^2+5d+5+t)}$ which holds by (i) which completes the proof of (v).

(vi) It is straightforward to verify that the desired statement is equivalent to

$$f(d+1,t)\leq r(d+1,t,f(d,t))$$

for $t\leq\binom{d+1}{2}$.

If $t<\binom{d+1}{2}$, this follows immediately from (9). Hence, we may assume that $t=\binom{d+1}{2}$. This implies

$$\begin{aligned} &r(d+1,t,f(d,t))-f(d+1,t)\\ =& r\left(d+1,\binom{d+1}{2},f\left(d,\binom{d+1}{2}\right)\right)-f\left(d+1,\binom{d+1}{2}\right)\\ =& \frac{1+\left((d+1)^2-(d+1)-2\binom{d+1}{2}\right)\frac{1}{d+1}}{(d+1)^2+1-2\binom{d+1}{2}}-\frac{1}{d+2}=0 \end{aligned}$$

which completes the proof of (vi). □

Having collected numerous properties of f we can now state a joint generalization of (1) and (2).

Theorem 2 *Let $T\in\mathbb{N}_0$. If G is a graph such that every vertex of G belongs to at most T triangles, then*

$$\alpha(G)\geq\sum_{u\in V}f(d_G(u),T).$$

Proof: We proceed by induction on the order of G as in Section 2. By Lemma 1, the function $g : \mathbb{N}_0^2 \to \mathbb{R}_{\geq 0}$ with $g(d,t) = f(d,T)$ has properties (P_1), (P_2), and (P_3). Therefore, we can argue exactly as in Section 2 until the point when (6) is established (with $g(d,t) = f(d,T)$ instead of $f(d,t)$). Also as shown in Section 2, (P_3) for g implies

$$\sum_{u\in V}\sum_{v\in N_u}(d_v-1)(f(d_u-1,T)-f(d_u-1,T)) \geq \sum_{u\in V}\sum_{v\in N_u}(d_u-1)(f(d_u-1,T)-f(d_u-1,T)).\quad(10)$$

Starting with (6) we obtain

$$\begin{aligned} A &\geq \sum_{u\in V}\left(1-(d_u+1)f(d_u,T)+\left(\left(\sum_{v\in N_u}(d_v-1)\right)-2t_u\right)(f(d_u-1,T)-f(d_u,T))\right)\\ &\overset{(10)}{\geq} \sum_{u\in V}\left(1-(d_u+1)f(d_u,T)+\left(\left(\sum_{v\in N_u}(d_u-1)\right)-2t_u\right)(f(d_u-1,T)-f(d_u,T))\right)\\ &= \sum_{u\in V}\left(1-(d_u+1)f(d_u,T)+\left(d_u^2-d_u-2t_u\right)(f(d_u-1,T)-f(d_u,T))\right).\end{aligned}$$

If $T > \binom{d_u}{2}$, then $t_u \leq \binom{d_u}{2}$ and, by Lemma 1,

$$1-(d_u+1)f(d_u,T)+\left(d_u^2-d_u-2t_u\right)(f(d_u-1,T)-f(d_u,T)) \geq 1-(d_u+1)f(d_u,T) \overset{(9)}{=} 0.$$

If $T \leq \binom{d_u}{2}$, then $t_u \leq T$ and, by Lemma 1,

$$\begin{aligned} &1-(d_u+1)f(d_u,T)+\left(d_u^2-d_u-2t_u\right)(f(d_u-1,T)-f(d_u,T))\\ \geq\ &1-(d_u+1)f(d_u,T)+\left(d_u^2-d_u-2T\right)(f(d_u-1,T)-f(d_u,T)) \geq 0.\end{aligned}$$

Altogether, we obtain $A \geq 0$ which completes the proof of the theorem. $\square$

Lemma 1 collected more properties than we actually needed for the proof of Theorem 2. We hope that these are helpful to prove — rather than to disprove — the following conjecture.

Conjecture 3 *If G is a graph, then $\alpha(G) \geq \sum\limits_{u\in V} f(d_G(u), t_G(u))$.*

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