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An Independent Dominating Set in the Complement of a Minimum Dominating Set of a Tree

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Abstract

We prove that for every tree T of order at least 2 and every minimum dominating set D of T which contains at most one endvertex of T, there is an independent dominating set I of T which is disjoint from D. This confirms a recent conjecture of Johnson, Prier, and Walsh.

Keywords: domination; independence; inverse domination
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1 Introduction

We consider finite, undirected and simple graphs and use standard terminology as in [3]. A *dominating set* of a graph G is a set D of vertices of G such that every vertex of G which

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does not lie in D has a neighbour in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called *minimum*. An *independent set* in a graph G is a set of pairwise non-adjacent vertices. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set in G.

Johnson, Prier, and Walsh recently posed the following conjecture.

Conjecture 1 (Johnson et al. [4]) *If T is a tree of order at least 2 and D is a minimum dominating set of T containing at most one endvertex of T, then there is an independent dominating set I of T which is disjoint from D.*

As pointed out in [4], Conjecture 1, if true, is best-possible. This may be seen by considering a path $P : v_1v_2v_3 \dots v_{3k+1}$ on $3k + 1 \geq 4$ vertices and the dominating set $D = \{v_1, v_4, \dots, v_{3k+1}\}$ of P. Note that D is minimum and that P has no independent dominating which is disjoint from D.

The motivation due to Johnson, Prier, and Walsh [4] for posing their conjecture is based on a related conjecture concerning the so-called inverse domination in graphs. A classical observation in domination theory is that, if D is a minimum dominating set of a graph $G = (V, E)$, then $V \setminus D$ is also a dominating set of G. A set D' is an *inverse dominating set* of G if D' is a dominating set of G and $D' \subseteq V \setminus D$ for some minimum dominating set D of G. The *inverse domination number* $\gamma'(G)$ of G is the minimum cardinality of an inverse dominating set of G. Inverse domination in graphs was introduced by Kulli and Sigarkant [5]. In their original paper in 1991, they include a proof that for all graphs with no isolated vertex, the inverse domination number is at most the independence number. However, this proof contained an error and in 2004, Domke, Dunbar, and Markus [1] formally posed this "result" of Kulli and Sigarkant as a conjecture. This conjecture still remains open and has been proved for many special families of graphs, including claw-free graphs, bipartite graphs, split graphs, very well covered graphs, chordal graphs and cactus graphs (see [2]).

Our result is the proof of Conjecture 1.

2 Result

In this section we prove Conjecture 1.

Theorem 2 *Conjecture 1 is true.*

Before we proceed to the proof, we explain our general strategy. Given T and D as in the statement of the conjecture, it suffices to determine an independent set J of vertices which is disjoint from D and contains a neighbour of every vertex in D, because a maximal independent set I which contains J but is disjoint from D is clearly a dominating set of T. A simple strategy to select the elements of J is to root T in some vertex r in D and to select a child of every vertex in D which itself is not contained in D. Since T has order at least 2 and D contains at most one endvertex of T, choosing the root r of T as an endvertex, if possible, every vertex in D has at least one child. If this strategy succeeds, then the selected vertices will clearly form an independent set. Nevertheless, this strategy fails in the presence of vertices u in D all children of which are also in D. For such a vertex, we necessarily have to choose its parent. Since J has to be independent, this choice affects the choosability of the children of ancestors of u in D. Working out the consequences of this reasoning, leads to the algorithm SELECT (cf. Algorithm 1 below).

Input: A tree T of order at least 2 and a minimum dominating set D of T containing at most one endvertex of T

Output: An independent dominating set I of T which is disjoint from D

1 **begin**
2   Choose a vertex $r \in D$ of minimum degree $d_T(r) = \min\{d_T(u) \mid u \in D\}$;
3   Root T in r;
4     $J \leftarrow \emptyset$;
5   **while** $\exists\ u \in D$ *such that* $u \notin N_T(J)$ *and all children of* u *lie in* $D \cup N_T(J)$ **do**
6   Let v be the parent of u;
7         $J \leftarrow J \cup \{v\}$;
8   partner$(u) \leftarrow v$;
9   **end**
10   **while** $\exists\ u \in D$ *such that* $u \notin N_T(J)$ **do**
11   Choose a child v of u such that $v \notin D \cup N_T(J)$;
12         $J \leftarrow J \cup \{v\}$;
13   **end**
14   Let I be a maximal independent set of T with $J \subseteq I$ and $D \cap I = \emptyset$;
15 **end**

Algorithm 1: SELECT

We proceed to the

Proof of Theorem 2: In view of the above remarks it suffices to argue that SELECT successfully determines an independent set J of T such that $D \cap J = \emptyset$ and $D \subseteq N_T(J)$. Note that, since D contains at most one endvertex and by the choice of r in line 3, every vertex in D has at least one child.

Claim *The vertex u in line 5 has a parent which does not belong to D.*

Proof: For contradiction, we consider the first execution of the **while**-loop in line 5 for which the vertex u has no parent which does not belong to D, i.e. either u is the root r of T or the parent of u belongs to D.

Let D' denote the set of vertices u' from D which can be reached from u on a path P of the form

$$P : u_0 w_1 v_1 u_1 w_2 v_2 u_2 \ldots w_l v_l u_l \tag{1}$$

with $u_0 = u$, $u_l = u'$, $l \in \mathbb{N}$, $w_i \notin D$, and partner$(u_i) = v_i$ for $1 \le i \le l$. Note that w_1 is a child of u. Let the set D'' contain the parent of the parent of u' — the grandparent of u' — for every vertex u' in D'. Let $\tilde{D} = (D \setminus D' \cup \{u\}) \cup D''$. Note that $|\tilde{D}| < |D|$.

Let w'' be a child of u. Clearly, $w'' \notin J$. If $w'' \in D$, then $w'' \in \tilde{D}$. If $w'' \notin D$, then w'' has a child v'' which belongs to J, and v'' has a child u'' which belongs to D such that partner$(u'') = v''$. Since $uw''v''u''$ is a path as in (1), we obtain, by the definition of D', that $u'' \in D'$. This implies $w'' \in D''$, and hence $w'' \in \tilde{D}$. Therefore, in both cases, $u, w'' \in N_T[\tilde{D}]$ and all vertices which were dominated by u in D are still dominated by vertices in $\tilde{D}$.

Let $u' \in D'$. Let P be as in (1) with $u' = u_l$. Since $w_l \in \tilde{D}$, we have $v_l \in N_T[\tilde{D}]$. If w'' is a child of u', then exactly the same argument as above implies that $w'' \in \tilde{D}$. Hence again all vertices which were dominated by u' in D are still dominated by vertices in $\tilde{D}$.

Altogether, we obtain that $\tilde{D}$ is a dominating set of T which contradicts the assumption that D is a minimum dominating set. $\square$

By the claim, the **while**-loop in line 5 successfully adds to the set J the parents of vertices in D which do not belong to D. By the condition for the **while**-loop in line 5, just before the first execution of the **while**-loop in line 10, the set J is independent and every vertex $u \in D$ with $u \notin N_T(J)$ has at least one child which does not belong to D and is non-adjacent to the vertices in J. Since during the executions of the **while**-loop in line 10 only children of vertices in D are added to J, this property is maintained throughout the remaining execution of SELECT. Hence the **while**-loop in line 10 successfully adds to the set J the children of vertices in D which do not belong to D such that after the last execution of the **while**-loop in line 10, the set J is independent, disjoint from D and $D \subseteq N_T(J)$.

By the above remarks, the set I defined in line 14 is an independent dominating set of T which completes the proof. $\square$

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