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Robustness of funnel control in the gap metric*

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Abstract

For m-input, m-output, finite-dimensional, linear systems satisfying the classical assumptions of adaptive control (i.e., (i) minimum phase, (ii) relative degree one and (iii) positive high-frequency gain), the well known funnel controller $k(t) = \frac{\varphi(t)}{1-\varphi(t)\|e(t)\|}$, $u(t) = -k(t)e(t)$ achieves output regulation in the following sense: all states of the closed-loop system are bounded and, most importantly, transient behaviour of the tracking error $e = y - y_{\text{ref}}$ is ensured such that the evolution of $e(t)$ remains in a performance funnel with prespecified boundary $\varphi(t)^{-1}$, where y_{ref} denotes a reference signal bounded with essentially bounded derivative. As opposed to classical adaptive high-gain output feedback, system identification or internal model is not invoked and the gain $k(\cdot)$ is not monotone.
We show that the funnel controller is robust by invoking the conceptual framework of the nonlinear gap metric: the funnel controller copes with bounded input and output disturbances and, more importantly, it may even be applied to a system not satisfying any of the classical conditions (i)–(iii) as long as the initial conditions and the disturbances are "small" and the system is "close" (in terms of a "small" gap) to a system satisfying (i)–(iii).

1 Introduction

In the early 1980s, a novel feature in classical adaptive control was introduced: adaptive control without identifying the entries of the system being controlled. Pioneering contributions to the area include [1, 13, 14, 16, 19] (see, also, the survey [9] and references therein). The classical assumptions on such a system class – rather than a single system – of linear m-input, m-output systems are: (i) minimum phase, (ii) strict relative degree one and (iii) positive-definite high-frequency gain matrix. Then the simple output feedback $u(t) = -k(t)\, y(t)$ stabilizes each system belonging to the above class and $k(\cdot)$ adapted by $\dot{k}(t) = \|y(t)\|^2$ and variations thereof. Two major drawbacks of the latter strategy (and its variations) are first, the gain $k(t)$ is, albeit bounded, monotonically increasing which might finally become too large whence amplifying measurement noise, and secondly, whilst asymptotic performance is guaranteed, transient behaviour is not taken into account (apart from [15], where the issue of prescribed transient behaviour is successfully addressed).

A fundamentally different approach, the so-called "funnel controller", was introduced in [8] in the context of the following output regulation problem: this controller ensures prespecified transient behaviour of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (actually it is not adaptive in so far the gain is not dynamically generated) and does not invoke any internal model. Funnel control has been applied to a large class of systems described by functional differential equations including nonlinear or/and infinite dimensional systems and systems with higher relative degree [10], it has been successfully applied

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in experiments controlling the speed of electric devices [11] (see [9] for further applications and a survey), and recently it has be shown that funnel control copes with input constraints if a certain feasibility inequality holds [6].

The contribution of the present paper is to show that the funnel controller is *robust* in the sense that the control objectives (bounded signals and tracking within a prespecified performance funnel) are still met if the funnel controller is applied to any system "close" (in terms of the gap metric) to a system satisfying the classical assumptions (i)–(iii). This will be achieved by exploiting the concept of (nonlinear) gap metric and graph topology from [5, 2]. The results are analogous in structure to its precursors: robustness of the common adaptive controller [4] and of the λ-tracker [7]. However, some care must be exercised in finding the appropriate signal spaces, mainly in proving the existence of a gain function, and applying the known robustness results from [5, 2].

1.1 System class

We consider the class of linear n-dimensional, m-input m-output systems ($n, m \in \mathbb{N}$ with $n \geq m$)

$$\left.\begin{array}{ll} \dot{x}(t) = A\,x(t) + B\,u_1(t)\,, & x(0) = x^0 \in \mathbb{R}^n, \\ y_1(t) = C\,x(t)\,, & \end{array}\right\} \tag{1.1}$$

which satisfy the classical assumptions in high-gain adaptive control, that is minimum phase with relative degree one and positive definite high-frequency gain matrix, i.e. they belong to

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} CB + (CB)^T > 0\,, \\ \forall\, s \in \overline{\mathbb{C}}_+ \,:\, \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \right\}.$$

The state space dimension $n \in \mathbb{N}$ needs not to be known but only the dimension $m \in \mathbb{N}$ of the input/output space. Most importantly, only structural assumptions are required but the system entries may be completely unknown.

Note that for any $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ with $\det CB \neq 0$ we may choose $V \in \mathbb{R}^{n\times(n-m)}$ with $\operatorname{rk} V = n - m$ and $\operatorname{im} V = \ker C$; then $T := [B(CB)^{-1}, V]$ is invertible and

$$T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} CB \\ 0 \end{bmatrix} =, \quad CT = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}.$$

Moreover, if (A,B,C) is minimum-phase, then A_4 has spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we replace $\widetilde{\mathcal{M}}_{n,m}$ by

$$\mathcal{M}_{n,m} := \left\{ (A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \;\middle|\; \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\; B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},\; C = \begin{bmatrix} I & 0 \end{bmatrix}, \\ B_1, A_1 \in \mathbb{R}^{m\times m},\; \operatorname{spec}(A_4) \subset \mathbb{C}_-\,, \\ B_1 + B_1^T > 0 \end{array} \right\},$$

and restrict our attention to systems $(A,B,C) \in \mathcal{M}_{n,m}$ in Byrnes-Isidori normal form, see for example [12, Sec. 4], i.e.

$$\left.\begin{array}{rclrcl} \dot{y}_1 & = & A_1 y_1 + A_2 z + CB\,u_1\,, & y_1(0) & = & y_1^0 \in \mathbb{R}^m\,, \\ \dot{z} & = & A_3 y_1 + A_4 z\,, & z(0) & = & z^0 \in \mathbb{R}^{n-m}\,. \end{array}\right\} \tag{1.2}$$

We will study the initial value problem (1.1) or (1.2) as *plant* P mapping the interior input signal u_1 to the interior output signal y_1, in conjunction with the *controller* C (the funnel controller (1.4) in our setup), mapping the interior output-signal y_2 to the interior input signal u_2, and in the presence of additive input/output disturbances u_0, y_0 so that

$$u_0 \;=\; u_1 + u_2, \qquad y_0 \;=\; y_1 + y_2\,, \tag{1.3}$$

as depicted in Figure 1.

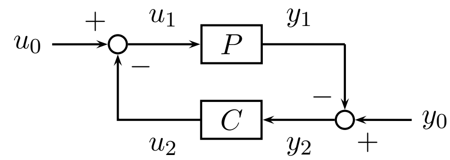


Figure 1: The closed-loop system $[P, C]$.

1.2 Performance funnel and funnel control

The control objective, defined in the following sub-section, will be captured in terms of the *performance funnel*

$$\mathcal{F}_\varphi \;:=\; \{(t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1\}\,,$$

determined by $\varphi(\cdot)$ belonging to

$$\Phi := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}) \;\middle|\; \begin{array}{l} \varphi(0) = 0\,,\ \forall\, t > 0 \,:\, \varphi(t) > 0\,,\ \liminf_{t\to\infty} \varphi(t) > 0\,, \\ \forall\, \varepsilon > 0 : \varphi|_{[\varepsilon,\infty)}(\cdot)^{-1} \text{ is globally Lipschitz continuous} \end{array} \right\}.$$

Note that the funnel boundary is given by $\varphi(t)^{-1}$, $t > 0$; see Figure 3. The concept of performance funnel had been introduced by [8]. There it is not assumed that $\varphi(\cdot)$ has the Lipschitz condition as given in Φ; we incorporate this mild assumption for technical reasons. The assumption $\varphi(0) = 0$ allows to start with arbitrarily large initial conditions x_0 and output disturbances y_0. If for special applications the initial value and y_0 are known, then $\varphi(0) = 0$ may be relaxed by $\varphi(0)\|y_0(0) - Cx^0\| < 1$, see also the simulations in Example 4.6.

The funnel controller, for prespecified $\varphi(\cdot) \in \Phi$, is given by

$$u_2(t) = -k(t)y_2(t)\,, \qquad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \tag{1.4}$$

and will be applied to (1.1) or (1.2). Note that the funnel controller (1.4) is actually not an adaptive controller in the sense that it is not dynamic. The gain $k(t)$ is the reciprocal of the distance between $y_2 = y_0 - y_1$ (i.e. the difference of a reference signal y_0 and the output of (1.1)) and the funnel boundary $\varphi(t)^{-1}$; and, loosely speaking, if the error approaches the funnel boundary, then $k(t)$ becomes large, thereby exploiting the high-gain properties of the system and precluding boundary contact.

We will study properties of the closed-loop system generated by the application of the funnel controller (1.4) to systems (1.1) of class $\mathcal{M}_{n,m}$ or of class $\mathcal{P}_{n,m}$ (see below) in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (1.3). The closed-loop system (1.2), (1.4), (1.3) is depicted in Figure 2.

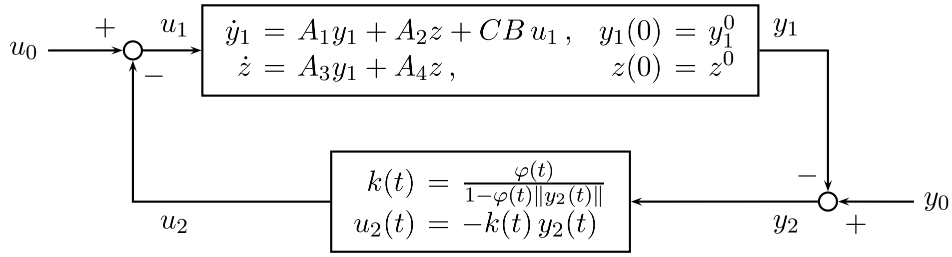


Figure 2: The "funnel controlled" closed-loop system.

1.3 Control objectives

We are ready to formulate the control objectives. If the funnel controller (1.4), for prespecified $\varphi \in \Phi$ determining the funnel boundary, is applied to any system (1.1), belonging to the class $\mathcal{M}_{n,m}$, in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (1.3), then the closed-loop system (1.2), (1.4), (1.3), as depicted in Figure 2, is supposed to meet the following control objectives:

- all signals are bounded;
- the output error $y_2(t) = y_0(t) - y_1(t)$ of the output disturbance and the output of the linear system evolves in the funnel, in other words

$$\forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,.$$

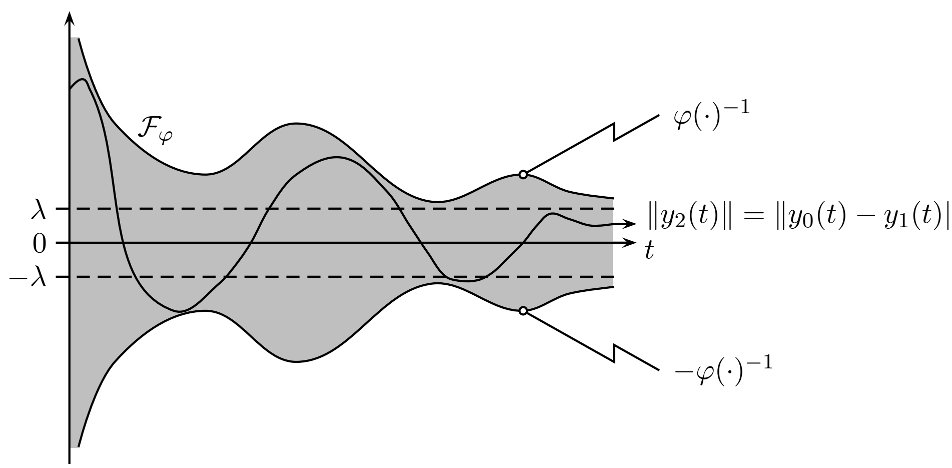


Figure 3: Funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ and $\inf_{t>0} \varphi(t)^{-1} \geq \lambda > 0$.

1.4 Main result: robustness

The main result of the present paper is to show robustness of the funnel controller in the following sense: The control objectives should still be met if $(A, B, C) \in \mathcal{M}_{n,m}$ is replaced by some system $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ belonging to the system class

$$\mathcal{P}_{q,m} := \big\{(A,B,C) \in \mathbb{R}^{q\times q} \times \mathbb{R}^{q\times m} \times \mathbb{R}^{m\times q} \mid (A,B,C) \text{ is stabilizable and detectable}\big\} \supsetneq \mathcal{M}_{q,m}$$

where $q, m \in \mathbb{N}$ with $q \geq m$, and $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is close (in terms of the gap metric) to a system belonging to $\mathcal{M}_{n,m}$ and the initial conditions and the disturbances are "small".

For the purpose of illustration, we will further show that a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of the transfer function

$$s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}, \qquad \alpha, N, M > 0, \tag{1.5}$$

(which obviously does not satisfy any of the classical assumptions since it is not minimum phase, has relative degree 2 and negative high-frequency gain) is the closer to a system in $\mathcal{M}_{n,m}$ the larger N and M.

The paper is organized as follows. In Section 2 we show that the funnel controller achieves all control objectives if applied to a linear system (1.1) belonging to class $\mathcal{M}_{n,m}$ in the presence of $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ input/output disturbances, see Figure 2. In Section 3, we collect the basics of the framework of gap metric and graph topology from [5, 2, 4] necessary for our setup. The final Section 4 contains the main result, i.e. robustness of funnel control.

Nomenclature

$\mathbb{C}_+$, $\mathbb{C}_-$	$=$	$\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$, $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, respectively
$M > 0$		if, and only if, $x^T M x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, where $M \in \mathbb{R}^{n \times n}$
$\|x\|$	$=$	$\sqrt{x^T x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\|M\|$	$=$	$\max\left\{\|M x\| \,\middle|\, x \in \mathbb{R}^m,\ \|x\| = 1\right\}$, induced matrix norm of $M \in \mathbb{R}^{n \times m}$
$\|v\|_{\mathcal{V}}$		the norm of $v \in \mathcal{V}$ for any normed vector space $\mathcal{V}$
$L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of p-integrable functions $y \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$, $1 \leq p < \infty$ with norm
$\|y\|_{L^p(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	$=$	$\left(\int_0^\infty \|y(t)\|^p \, \mathrm{d}t\right)^{\frac{1}{p}}$
$L^p_{\text{loc}}(I \to \mathbb{R}^\ell)$		the space of locally p-integrable functions $y \colon I \to \mathbb{R}^\ell$, with $\int_K \|y(t)\|^p \, \mathrm{d}t < \infty$ for all compact $K \subset I$, where $1 \leq p < \infty$ and $I \subset \mathbb{R}_{\geq 0}$ is an interval
$L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the space of essentially bounded functions $y \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with norm
$\|y\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	$=$	$\operatorname{ess\,sup}_{t \geq 0} \|y(t)\|$
$L^\infty_{\text{loc}}(I \to \mathbb{R}^\ell)$		the space of locally bounded functions $y \colon I \to \mathbb{R}^\ell$, with $\operatorname{ess\,sup}_{t \in K} \|y(t)\| < \infty$ for all compact $K \subset I$, where $I \subset \mathbb{R}_{\geq 0}$ is an interval
$W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$		the Sobolev space of absolutely continuous functions $y \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^\ell$ with $y, \dot{y} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$ and norm
$\|y\|_{W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$	$=$	$\|y\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)} + \|\dot{y}\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)}$

2 Funnel control

In this section we show that the funnel controller (1.4) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves, in presence of input/output disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the control objectives: y_2 is forced to evolve within a performance funnel $\mathcal{F}_\varphi$ for prespecified $\varphi \in \Phi$ and all signals and states of the closed-loop (1.2), (1.3), (1.4), as depicted in Figure 2, remain essentially bounded. Moreover, it is shown that the derivatives of the output signals y_1, y_2 and the state $\left(\begin{smallmatrix} y_1 \\ \eta \end{smallmatrix}\right)$ are essentially bounded, too.

Write, for notational convenience,

$$\mathcal{D}_{n,m} := \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \Phi \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \qquad n, m \in \mathbb{N},\ n \geq m,$$

the set of all tuples of systems, initial values y_1^0, η^0 of the linear system, functions φ describing the funnel $\mathcal{F}_\varphi$ and input/output disturbances (u_0, y_0).

Proposition 2.1 *Let $n, m \in \mathbb{N}$ with $n \geq m$ and $\varphi \in \Phi$. Then there exists a continuous map $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all tuples $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$, the associated closed-loop initial value problem* (1.2), (1.3), (1.4) *satisfies*

$$\|(k, u_2, y_2, \eta)\|_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^{1+m}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n-m})} \leq \nu(d)\,, \tag{2.1}$$

and

$$\forall\, t \geq 0 \ :\ (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\ . \tag{2.2}$$

Note that Proposition 2.1 also yields that all control objectives are met if the funnel controller (1.4) is applied to $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$. This had already been shown, for $u_0 = 0$, in [8]; the essential difference to [8] is that here we prove the result by the construction of a continuous function ν so that (2.1) holds. The latter is crucial for the robustness analysis of funnel control in Section 4. The proof of Proposition 2.1 uses ideas from [4] and from [6].

Proof of Proposition 2.1. Let $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right) \in \mathcal{D}_{n,m}$. Then the closed-loop initial value problem (1.2), (1.3), (1.4) may be written as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_2 \\ \eta \end{pmatrix} = f(t, y_2, \eta)\,, \qquad \begin{pmatrix} 0 \\ y_2(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_0(0) - y_1^0 \\ \eta^0 \end{pmatrix} \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}\,, \tag{2.3}$$

where the right hand side is given by

$$\begin{aligned} &f\colon \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \\ &\quad (t, y_2, \eta) \mapsto \begin{pmatrix} A_1\, y_2 - A_2\, \eta - CB \frac{\varphi(t)}{1-\varphi(t)\|y_2\|} y_2 + \dot{y}_0(t) - A_1\, y_0(t) - CB\, u_0(t) \\ -A_3\, y_2 + A_4\, \eta + A_3\, y_0(t) \end{pmatrix}. \end{aligned}$$

We proceed in several steps.

Step 1: We show that the initial value problem (2.3) has an absolutely continuous solution $(y_2, \eta)\colon [0, \omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ for maximal $\omega \in (0, \infty]$; this solution satisfies $(t, y_2(t), \eta(t)) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ for all $t \in [0, \omega)$, is unique and maximality of ω means that the solution is extended up to the boundary of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$: the closure of graph $\left((y_2, \eta)|_{[0,\omega)}\right)$ is not a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$, i.e. for every compact $\mathcal{K} \subset \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ there exists $t \in [0, \omega)$ such that $(t, y_2(t), \eta(t)) \notin \mathcal{K}$.

Since $\varphi|_{[\varepsilon,\infty)}(\cdot)^{-1}$, is globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$, it follows that f is locally Lipschitz on the relatively open set $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ in the sense that, for all $(\tau, \xi, \zeta) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m}$, there exists an open neighbourhood O of (τ, ξ, ζ) and a constant $L > 0$ such that

$$\forall\, (t, y, \eta) \in O \ :\ \|f(t, y, \eta) - f(t, \xi, \zeta)\| \leq L(\|y - \xi\| + \|\eta - \zeta\|)\,.$$

Now by the standard theory of ordinary differential equations, see, for example, [18, Th. III.11.III], the initial value problem (2.3) has the desired properties.

Step 2: We collect some definition and technicalities.

By Step 1 and the properties of φ it follows that

$$\exists\, \delta = \delta(d) > 0 \;\forall\, t \in [0,\delta] \;:\; \|y_2(t)\| \le \|y_2(0)\| + 1 \;\wedge\; 1 - \varphi(t)\|y_2(t)\| \ge \max\{1/2, \varphi(t)\}\,. \tag{2.4}$$

Let $L_\delta > 0$ denote a global Lipschitz constant of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$ (which exists by definition of Φ) and set $\lambda := \inf\left\{\varphi(t)^{-1} \,|\, t > 0\right\}$. Note that $(t, y_2(t)) \in \mathcal{F}_\varphi$ for all $t \in [0,\omega)$ yields

$$\forall\, t \in [0,\omega) \;:\; \|y_2(t)\| \le \max\left\{\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty}, \|y_0(0) - y_1^0\| + 1\right\}. \tag{2.5}$$

By the minimum phase property of (1.2), i.e. $\operatorname{spec} A_4 \subset \mathbb{C}_-$,

$$\exists\, \alpha, \beta > 0 \;\forall\, t \ge 0 \;:\; \|e^{A_4}t\| \le \beta e^{-\alpha t}\,. \tag{2.6}$$

In view of positive definiteness of CB, let $\gamma_{CB} > 0$ denote the smallest singular value of $CB + (CB)^T$, and thus

$$\forall\, v \in \mathbb{R}^m \setminus \{0\} \;:\; \langle v, CBv\rangle \ge \gamma_{CB}\|v\|^2\,.$$

Step 3: We show:

$$\forall\, t \in [\delta,\omega) \;:\; \varphi(t)^{-1} - \|y_2(t)\| \ge \varepsilon\,, \tag{2.7}$$

where $\delta > 0$ is defined by (2.4) and, for $\gamma_{CB}, \lambda, L_\delta, \alpha$ and β defined in Step 2,

$$\varepsilon := \min\left\{\frac{1}{2}, \frac{\lambda}{2}, \frac{\gamma_{CB}\,\lambda}{2}, \left[L_\delta + \left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right)\cdot\left(\|y_0\|_{L^\infty} + \|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\|_{L^\infty}\right)\right.\right.$$
$$\left.\left. + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\right]^{-1}\right\}. \tag{2.8}$$

Seeking a contradiction, suppose that

$$\exists\, t_1 \in [\delta,\omega) \;:\; \varphi(t_1)^{-1} - \|y_2(t_1)\| < \varepsilon\,. \tag{2.9}$$

Since $t \mapsto \varphi(t)\|y_2(t)\|$ is continuous on $[0,\omega)$ and in view of (2.4) it follows that

$$\exists\, t_0 \ge \delta \;:\; t_0 = \max\left\{t \in [\delta, t_1) \,\middle|\, \varphi(t)^{-1} - \|y_2(t)\| = \varepsilon\right\}.$$

Thus, by definition of Φ,

$$\forall\, t \in [t_0, t_1] \;:\; \varphi(t)^{-1} - \|y_2(t)\| \le \varepsilon \;\wedge\; \|y_2(t)\| \ge \varphi(t)^{-1} - \varepsilon \ge \lambda - \lambda/2 \tag{2.10}$$

and hence

$$\forall\, t \in [t_0, t_1] \;:\; \frac{\|y_2(t)\|}{\varphi(t)^{-1} - \|y_2(t)\|} \ge \frac{\lambda}{2\,\varepsilon}\,. \tag{2.11}$$

By Variation of Constants, the second line of the differential equation (2.3) yields

$$\forall\, t \ge 0 \;:\; \eta(t) = e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3\left(y_0(s) - y_2(s)\right) \mathrm{d}s\,, \tag{2.12}$$

thus the first line of the differential equation (2.3) writes, for almost all $t \ge 0$,

$$\dot y_2(t) = -A_1(y_0(t) - y_2(t)) + A_2 \int_0^t e^{A_4(t-s)} A_3\left(y_0(s) - y_2(s)\right) \mathrm{d}s$$
$$- A_2 e^{A_4 t}\eta^0 + \dot y_0(t) - CBu_0(t) + CB\frac{-\varphi(t)}{1 - \varphi(t)\|y_2(t)\|}y_2(t)\,.$$

Hence, by (2.5), (2.6), (2.11) and (2.8), we conclude, for almost all $t \in [t_0, t_1]$,

$$\begin{aligned}
\langle y_2(t), \dot y_2(t)\rangle &\le \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right) \left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\
&\qquad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\varphi(t)\,\langle y_2(t), CBy_2(t)\rangle}{1-\varphi(t)\|y_2(t)\|} \\
&\le \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right) \left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\
&\qquad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\varphi(t)\gamma_{CB}\|y_2(t)\|}{\varphi(t)\left(\varphi(t)^{-1} - \|y_2(t)\|\right)}\|y_2(t)\| \\
&\le \|y_2(t)\| \Bigg[\left(\|A_1\| + \|A_2\|\,\|A_3\|\frac{\beta}{\alpha}\right) \left[\|y_0\|_{L^\infty} + \left\|\varphi|_{[\delta,\infty)}(\cdot)^{-1}\right\|_{L^\infty}\right] \\
&\qquad + \|A_2\|\beta\|\eta^0\| + \|\dot y_0\|_{L^\infty} + \|CB\|\,\|u_0\|_{L^\infty}\Bigg] - \frac{\gamma_{CB}\lambda}{2\varepsilon}\|y_2(t)\| \\
&\le -L_\delta\|y_2(t)\|\,. \qquad (2.13)
\end{aligned}$$

Thus

$$\begin{aligned}
\|y_2(t_1)\| - \|y_2(t_0)\| &= \int_{t_0}^{t_1} \frac{\langle y_2(\tau), \dot y_2(\tau)\rangle}{\|y_2(\tau)\|}\,\mathrm{d}\tau \\
&\le -L_\delta(t_1 - t_0) \le -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \le \varphi(t_1)^{-1} - \varphi(t_0)^{-1}\,,
\end{aligned}$$

whence the contradiction $\varepsilon = \varphi(t_0)^{-1} - \|y_2(t_0)\| \le \varphi(t_1)^{-1} - \|y_2(t_1)\| < \varepsilon$. This proves (2.7).

Step 4: We show that $\omega = \infty$.

Let $\sigma := \min\left\{1, \inf_{t\in[\delta,\omega)}\varphi(t)\right\} > 0$. By (2.7) it follows, for $\varepsilon > 0$ as defined in (2.8), that

$$\forall\, t \in [\delta, \omega)\ :\ 1 - \varphi(t)\|y_2(t)\| \ge \varepsilon\varphi(t) \ge \varepsilon\sigma\,,$$

and so, in view of (2.4),

$$\forall\, t \in [0, \omega)\ :\ 1 - \varphi(t)\|y_2(t)\| \ge \varepsilon\sigma\,.$$

Seeking a contradiction, suppose that $\omega < \infty$. By (2.5) and (2.12) follows that $\eta \in L^\infty([0,\omega) \to \mathbb{R}^{n-m})$ with $\left\|\eta|_{[0,\omega)}\right\|_{L^\infty} \le c$ for some $c > 0$. Then

$$\mathcal{K} := \left\{(t, y, z) \in \mathcal{F}_\varphi \times \mathbb{R}^{n-m} \,\middle|\, t \in [0, \omega]\,,\ 1 - \varphi(t)\|y\| \ge \varepsilon\sigma\,,\ \|z\| \le c\right\}$$

is a compact subset of $\mathcal{F}_\varphi \times \mathbb{R}^{n-m}$ with $(t, y_2(t), \eta(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, which contradicts the fact that the closure of graph $\left((y_2, \eta)|_{[0,\omega)}\right)$ is not a compact set, see Step 1. Therefore, $\omega = \infty$.

Step 5: We show (2.1).

Step 4 yields $\omega = \infty$. Then Step 3 and (2.4) guarantee that $(t, y_2(t)) \in \mathcal{F}_\varphi$ for all $t \ge 0$. Moreover, for some $\delta > 0$ as in (2.4), $\|y_2(t)\| \le \varphi(t)^{-1} - \varepsilon$ for all $t \ge \delta$, and, in view of (2.4), we have $\|y_2(t)\| \le \|y_2(0)\| + 1 \le \|y_0(0)\| + \|y_1^0\| + 1$ for all $t \in [0, \delta]$. Thus $y_2 \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0\right)$. Moreover, (2.7) and (2.4) yield

$$\forall\, t \ge 0\ :\ 1 - \varphi(t)\|y_2(t)\| \ge \varepsilon\varphi(t)$$

and so

$$\forall\, t \ge 0\ :\ k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \le \varepsilon^{-1}$$

which gives $k \in L^\infty(R_{\geq 0} \to \mathbb{R})$ and, in view of (2.4), $\|k\|_{L^\infty} \leq \frac{1}{\varepsilon}$, thus k is uniformly bounded in terms of d. Hence, $u_2 = -k\, y_2 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is also uniformly bounded in terms of d. By (2.12) we have, for all $t \geq 0$,

$$
\begin{aligned}
\|\eta(t)\| &= \left\| e^{A_4 t}\eta^0 + \int_0^t e^{A_4(t-s)} A_3 \left(y_0(s) - y_2(s)\right) \mathrm{d}s \right\| \\
&\leq \beta e^{-\alpha t}\|\eta^0\| + \int_0^t \beta \|A_3\| e^{-\alpha(t-s)} \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right) \mathrm{d}s \\
&\leq \beta \|\eta^0\| e^{-\alpha t} + \frac{\beta}{\alpha}\|A_3\| \left(\|y_0\|_{L^\infty} - \|y_2\|_{L^\infty}\right) \left(1 + e^{-\alpha t}\right) ,
\end{aligned}
$$

hence $\eta \in L^\infty\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^{n-m}\right)$ and moreover, η is uniformly bounded in terms of the system matrices and the L^∞-norms of y_0 and y_2 which yields that η is uniformly bounded in terms of $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0 \right)$.

Finally, in view of (2.3), it follows that the derivatives of y_2 and η are also uniformly bounded in terms of d which yields that $(y_2, \eta) \in W^{1,\infty}\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^m \times \mathbb{R}^{n-m}\right)$. Moreover, this proves the existence of a continuous function $\nu\colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that (2.1) holds true.

Step 6: Finally, we show (2.2).

By Step 5 we have $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Thus, and since y_2 is continuous, it follows that, for all $t \geq 0$, $1 - \varphi(t)\|y_2(t)\| > 0$, which shows (2.2) and completes the proof. □

3 The concept of gap metric

The material in this section is based on [5, Sec. II], [4, Sec. 2], [2, Sec. 2] and mainly [3, Sec. 2]. Definitions for extended and ambient spaces, well posedness and the nonlinear gap can be found in [3, Sec. 2]; however, gain-functions and gain-function stability, which are required for the robust stability results in Section 4, is not defined in [3]. A section about the basic concepts of the gap metric needed in the setup of robustness is in [7]; however, the latter contains a technical flaw: extended and ambient spaces are defined there as in [4, Sec. 2] and [2, Sec. 2] and are not applicable to function spaces of continuous functions. Therefore, in the following we correct this flaw when defining extended and ambient spaces and well posedness more carefully. The results in [7] hold true if this minor correction is applied; only the proof of [7, Prop. 4.4] is effected: one has to apply [17, Th. 6.5.3 and Th. 6.5.4] which are revisions of [2, Th. 5.2 and Th. 5.3], see also Sub-section 4.3 for more details.

3.1 Generalized signal spaces

Let $\mathcal{X}$ be a nonempty set. For $0 < \omega \leq \infty$, let $\mathcal{S}_\omega$ denote the set of all locally integrable maps in $\mathrm{map}([0,\omega) \to \mathcal{X})$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \leq \infty$, define the *truncation operator* T_τ and the *restriction* of maps as follows:

$$
\begin{aligned}
T_\tau : \mathcal{S}_\omega \to \mathcal{S}, \qquad & v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau) \\ 0, & t \in [\tau,\infty) \end{cases} \right), \\
(\cdot)\big|_{[0,\tau)} : \mathcal{S}_\omega \to \mathcal{S}_\tau, \qquad & v \mapsto v\big|_{[0,\tau)} := \left(t \mapsto v(t), \quad t \in [0,\tau)\right).
\end{aligned}
$$

Consider next a space $\mathcal{V} \subset \mathcal{S}$ of maps defined on $[0,\infty)$ with norm $\|\cdot\|_{\mathcal{V}}\colon \mathcal{V} \to \mathbb{R}_{\geq 0}$. Note that $T_\tau v$ may not belong to $\mathcal{V}$, for example if $\mathcal{V}$ contains continuous functions. Therefore, we introduce the norm $\|\cdot\|_{\mathcal{V}|_{[0,\tau)}} : \{v|_{[0,\tau)} \mid v \in \mathcal{V}\} \to \mathbb{R}_{\geq 0}$ where $\|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ denotes the norm on the restriction $[0,\tau) \subset \mathbb{R}_{\geq 0}$, and write, for ease of notation, $\|T_\tau v\|_{\mathcal{V}} = \|v|_{[0,\tau)}\|_{\mathcal{V}|_{[0,\tau)}}$ for $v \in \mathcal{V}$.

We associate with $\mathcal{V}$ spaces as follows:

$$\begin{aligned}
\mathcal{V}[0,\tau) &= \left\{ v \in \mathcal{S}_\tau \,\middle|\, \exists\, w \in \mathcal{V} \text{ with } \|T_\tau w\|_{\mathcal{V}} < \infty \;:\; v = w|_{[0,\tau)} \right\}, \text{ for } \tau > 0\,; \\
\mathcal{V}_e &= \left\{ v \in \mathcal{S} \,\middle|\, \forall\, \tau > 0 \,:\, v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \textit{ the extended space}\,; \\
\mathcal{V}_\omega &= \left\{ v \in \mathcal{S}_\omega \,\middle|\, \forall\, \tau \in (0,\omega) \,:\, v|_{[0,\tau)} \in \mathcal{V}[0,\tau) \right\}, \text{ for } 0 < \omega \leq \infty\,; \\
\mathcal{V}_a &= \textstyle\bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega\,, \textit{ the ambient space}\,.
\end{aligned}$$

If $v, w \in \mathcal{V}_a$ with $v|_I = w|_I$ on $I = \operatorname{dom}(v) \cap \operatorname{dom}(w)$, then write $v = w$. For $(u,y) \in \mathcal{V}_a \times \mathcal{V}_a$, the domains of u and y may be different; adopt the convention

$$\operatorname{dom}(u,y) \;:=\; \operatorname{dom}(u) \cap \operatorname{dom}(y)\,.$$

The set $\mathcal{V} \subset \mathcal{S}$ is a said to be a *signal space* if, and only if, it is a) a normed vector space and b) $\sup_{\tau \geq 0} \|T_\tau v\|_{\mathcal{V}} < \infty$ implies $v \in \mathcal{V}$.

For the purpose of illustration, consider $\mathcal{V} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, which obviously satisfies the aforementioned assumptions a) and b): $L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is a normed space and, if $\sup_{\tau \geq 0} \|T_\tau v\|_{L^\infty} < \infty$, then $v \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Note that this also holds for the Sobolev space $W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. For $\mathcal{V} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $\mathcal{V}_e = L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{V}_\omega = L^\infty_{\text{loc}}([0,\omega) \to \mathbb{R}^m)$ for $\omega \in (0,\infty]$, and $\mathcal{V}_a = \cup_{0<\omega\leq\infty} L^\infty_{\text{loc}}([0,\omega) \to \mathbb{R}^m)$. It is important to note that $\mathcal{V}_\omega \supsetneq L^\infty([0,\omega) \to \mathbb{R}^m)$.

For a normed signal space $\mathcal{U}$ and the Euclidean space $\mathbb{R}^l$, $l \in \mathbb{N}$, also subsets of $\mathcal{V} = \mathbb{R}^l \times \mathcal{U}$ will be considered, which, on identifying each $\theta \in \mathbb{R}^l$ with the constant signal $t \mapsto \theta$, can be thought of as a normed signal space with norm given by $\|(\theta, x)\|_{\mathcal{V}} = \sqrt{|\theta|^2 + \|x\|^2_{\mathcal{U}}}$.

3.2 Well posedness

A mapping $Q \colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if, and only if,

$$\forall\, x, y \in \mathcal{U}_a \;\forall\, \tau \in \operatorname{dom}(x,y) \cap \operatorname{dom}(Qx, Qy) \;:\; \Big[\; x|_{[0,\tau)} = y|_{[0,\tau)} \;\Rightarrow\; (Qx)|_{[0,\tau)} = (Qy)|_{[0,\tau)} \;\Big]\,.$$

Consider $P \colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C \colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$ being causal mappings representing the plant and the controller, respectively, and satisfying the closed-loop equations:

$$[P, C] \;:\quad y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2\,, \tag{3.1}$$

corresponding to the closed-loop shown in Figure 1.

For $w_0 = (u_0, y_0) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$, a pair $(w_1, w_2) = ((u_1, y_1), (u_2, y_2)) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a *solution* if, and only if, (3.1) holds on $\operatorname{dom}(w_1, w_2)$. The (possibly empty) set of solutions is denoted by

$$\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves } (3.1)\}$$

The closed-loop system $[P, C]$, given by (3.1), is said to have:

- the *existence property* if, and only if, $\mathcal{X}_{w_0} \neq \emptyset$;

- the *uniqueness property* if, and only if,

$$\forall\, w_0 \in \mathcal{W} \;:\; \Big[(\hat w_1, \hat w_2), (\tilde w_1, \tilde w_2) \in \mathcal{X}_{w_0} \Longrightarrow \\ (\hat w_1, \hat w_2) = (\tilde w_1, \tilde w_2) \quad \text{on} \quad \operatorname{dom}(\hat w_1, \hat w_2) \cap \operatorname{dom}(\tilde w_1, \tilde w_2)\Big].$$

Assume that $[P, C]$ has the existence and uniqueness property. For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in (0, \infty]$, by the property

$$[0, \omega_{w_0}) := \bigcup_{(\hat w_1, \hat w_2) \in \mathcal{X}_{w_0}} \operatorname{dom}(\hat w_1, \hat w_2)$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\operatorname{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This construction induces the closed-loop operator

$$H_{P,C} \colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2)\,.$$

The closed-loop system $[P, C]$, given by (3.1), is said to be:

- *locally well posed* if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C} \colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a\,,\ w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e\,$;
- *$\mathcal{W}$-stable* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$;
- *regularly well posed* if, and only if, it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \;:\; \Big[\, \omega_{w_0} < \infty \;\Rightarrow\; \big\|(H_{P,C} w_0)|_{[0,\tau)}\big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0} \,\Big]\,. \tag{3.2}$$

If $[P, C]$ is globally well posed, then for each $w_0 \in \mathcal{W}$ the solution $H_{P,C}(w_0)$ exists on the half line $\mathbb{R}_{\geq 0}$. Regular well posedness means that if the closed-loop system has a finite escape time $\omega_{w_0} > 0$ for some disturbance $w_0 \in \mathcal{W}$, then at least one of the components u_1, u_2 or y_1, y_2 is not a restriction to $[0, \omega_{w_0})$ of a function in $\mathcal{U}$ or $\mathcal{Y}$, respectively. If $[P, C]$ is regularly well posed and satisfies

$$\forall\, w_0 \in \mathcal{W} \;:\; \Big[\, \omega_{w_0} < \infty \;\Rightarrow\; H_{P,C}(w_0)|_{[0,\omega_{w_0})} \in \mathcal{W}[0, \omega_{w_0}) \times \mathcal{W}[0, \omega_{w_0})\Big]\,,$$

there does not exist a solution of $[P, C]$ with a finite escape time, and therefore $[P, C]$ is globally well posed. However, global well posedness does not guarantee that each solution belongs to $\mathcal{W} \times \mathcal{W}$; the latter is ensured by $\mathcal{W}$-stability of $[P, C]$. Note also that neither regular nor global well posedness implies the other.

3.3 Graphs, the nonlinear gap metric and gain-function stability

To measure the distance between two plants P and P_1 it is necessary to find sets associated with the plant operators within some space where one may define a map which identifies the gap. These set are the *graphs* of the operators: for the plant operator $P \colon \mathcal{U}_a \to\ \mathcal{Y}_a$ and the controller operator $C \colon \mathcal{Y}_a \to\ \mathcal{U}_a$ define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \;\middle|\; u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}\,, \qquad \mathcal{G}_C \quad := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \;\middle|\; Cy \in \mathcal{U},\, y \in \mathcal{Y} \right\} \subset \mathcal{W}\,.$$

Note that $\mathcal{G}_P$ and $\mathcal{G}_C$ are, strictly speaking, not subsets of $\mathcal{W}$; however, abusing the notation one may identify $\mathcal{G}_P \ni \left(\begin{smallmatrix} u \\ Pu \end{smallmatrix}\right) = (u, Pu) \in \mathcal{W}$ and $\mathcal{G}_C \ni \left(\begin{smallmatrix} Cy \\ y \end{smallmatrix}\right) = (Cy, y) \in \mathcal{W}$.

The essence of Section 4 is the study of robust stability of funnel control in a specific control context. Robust stability is the property that the stability properties of a globally well posed closed-loop system $[P, C]$ persists under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$, when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the present context, plants P and P_1 are deemed to be close if, and only if, their respective graphs are *close* in the gap sense of [5]. The nonlinear gap is defined as follows:

Let, for signal spaces $\mathcal{U}$ and $\mathcal{Y}$,

$$\Gamma(\mathcal{U}, \mathcal{Y}) := \left\{ P : \mathcal{U}_a \to \mathcal{Y}_a \;\middle|\; P \text{ is causal} \right\}$$

and, for $P_1, P_2 \in \Gamma$, define the (possibly empty) set

$$\mathcal{O}_{P_1,P_2} := \left\{ \Phi : \mathcal{G}_{P_1} \to \mathcal{G}_{P_2} \;\middle|\; \Phi \text{ is causal, surjective, and } \Phi(0) = 0 \right\}.$$

The *directed nonlinear gap* is given by

$$\vec{\delta}\colon \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty],\ (P_1, P_2) \mapsto \inf_{\Phi \in \mathcal{O}_{P_1,P_2}} \sup_{x \in \mathcal{G}_{P_1} \setminus \{0\},\, \tau > 0} \left(\frac{\|T_\tau (\Phi - I)|_{\mathcal{G}_{P_1}}(x)\|_{\mathcal{U}\times\mathcal{Y}}}{\|T_\tau x\|_{\mathcal{U}\times\mathcal{Y}}} \right),$$

with the convention that $\vec{\delta}(P_1, P_2) := \infty$ if $\mathcal{O}_{P_1,P_2} = \emptyset$, and the *nonlinear gap* δ is

$$\delta\colon \Gamma(\mathcal{U}, \mathcal{Y}) \times \Gamma(\mathcal{U}, \mathcal{Y}) \to [0, \infty], (P_1, P_2) \mapsto \max\{\vec{\delta}(P_1, P_2), \vec{\delta}(P_2, P_1)\}.$$

The following definition of gain-function stability goes back to [5]: A causal operator $F\colon \mathcal{X} \to \mathcal{V}_a$, where $\mathcal{X}, \mathcal{V}$ are subsets of normed signal spaces, is said to be *gain-function stable* if, and only if, $F(\mathcal{X}) \subset \mathcal{V}$ and the following nonlinear so-called *gain-function* is well defined:

$$g[F] : (r_0, \infty) \to \mathbb{R}_{\geq 0},\ r \mapsto g[F](r) = \sup\left\{ \|T_\tau F x\|_{\mathcal{V}} \;\middle|\; x \in \mathcal{X},\ \|T_\tau x\|_{\mathcal{X}} \in (r_0, r],\ \tau > 0 \right\}, \tag{3.3}$$

where $r_0 := \inf_{x\in\mathcal{X}} \|x\|_{\mathcal{X}} < \infty$.

A closed-loop system $[P, C]$ is said to be *gain-function stable* if, and only if, it is globally well posed and $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e$ is gain-function stable.

Observe that $\|T_\tau F x\|_{\mathcal{V}} \leq g[F](\|T_\tau x\|_{\mathcal{X}})$ and note the following facts:

(i) global well posedness of $[P, C]$ implies that $\operatorname{im} H_{P,C} \subset \mathcal{W}_e \times \mathcal{W}_e$;

(ii) gain function stability of $[P, C]$ implies $\mathcal{W}$-stability of $[P, C]$;

(iii) if $[P, C]$ is $\mathcal{W}$-stable, then $H_{P,C} : \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is a bijective operator with inverse $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

To see (iii), note that $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$ implies that $H_{P,C}(\mathcal{W}) \subset \mathcal{G}_P \times \mathcal{G}_C$, and since, for any $w_1 \in \mathcal{G}_P \subset \mathcal{W}$, $w_2 \in \mathcal{G}_C \subset \mathcal{W}$ one has $w_1 + w_2 \in \mathcal{W}$, it follows that $H_{P,C}(\mathcal{W}) \supset \mathcal{G}_P \times \mathcal{G}_C$. Therefore, think of a gain-function stable $H_{P,C}$ as a surjective operator $H_{P,C}\colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$. The inverse of $H_{P,C}\colon \mathcal{W} \to \mathcal{G}_P \times \mathcal{G}_C$ is obviously $H_{P,C}^{-1} : (w_1, w_2) \mapsto w_1 + w_2$.

Next, we associate with the closed-loop system $[P, C]$ given by (3.1) the following two parallel projection operators:

$$\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a,\ w_0 \mapsto w_1 \qquad \text{and} \qquad \Pi_{C//P}\colon \mathcal{W} \to \mathcal{W}_a,\ w_0 \mapsto w_2.$$

Clearly, $H_{P,C} = \big(\Pi_{P//C}\,,\, \Pi_{C//P}\big)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Note that gain stability of either $\Pi_{P//C}$ and $\Pi_{C//P}$ implies $\mathcal{W}$-gain stability of the closed-loop system $[P,C]$ and that $\|\Pi_{P//C}\|_{\mathcal{W},\mathcal{W}}, \|\Pi_{C//P}\|_{\mathcal{W},\mathcal{W}} \geq 1$ since $\Pi_{P//C} = \Pi_{P//C}^2$, $\Pi_{C//P} = \Pi_{C//P}^2$.

Finally, we associate with the closed-loop system $[P,C]$ given by (3.1) the following two parallel projection operators:

$$\Pi_{P//C} \colon \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_1 \qquad \text{and} \qquad \Pi_{C//P} \colon \mathcal{W} \to \mathcal{W}_a\,,\ w_0 \mapsto w_2\,.$$

Clearly, $H_{P,C} = \big(\Pi_{P//C}\,,\, \Pi_{C//P}\big)$ and $\Pi_{P//C} + \Pi_{C//P} = I$. Therefore, gain-function stability of one of the operators $\Pi_{P//C}$ and $\Pi_{C//P}$ implies the gain-function stability of the other, and so gain-function stability of either operator implies gain-function stability of the closed-loop system $[P,C]$.

We close this section with an example. Define, for $\alpha > 0$, $x^0 \in \mathbb{R}$ and $N, M > 0$, the plant operator

$$P_\alpha \colon L_e^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to W_e^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,,\quad u_1 \mapsto y_1 = x,\ \dot{x} = \alpha x + u_1\,, x(0) = x^0, \tag{3.4}$$

and, for

$$\widetilde{A} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha NM & -NM + \alpha N + \alpha M & \alpha - N - M \end{bmatrix},\ \widetilde{b} := \begin{bmatrix} 0 \\ 0 \\ N \end{bmatrix},\ \widetilde{c} := \begin{bmatrix} M \\ -1 \\ 0 \end{bmatrix}^T,\ \widetilde{x}^0 \in \mathbb{R}^3\,, \tag{3.5}$$

the plant operator

$$P_{N,M,\alpha} \colon L_e^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to W_e^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})\,,\ u_1 \mapsto y_1 = \widetilde{c}\,x,\ \dot{x} = \widetilde{A}\,x + \widetilde{b}\,u_1\,, x(0) = \widetilde{x}^0\,. \tag{3.6}$$

In [7, Sec. 3] it is shown that, for sufficiently large $M > 0$ and $N = 2M$, P_α is close to $P_{N,M,\alpha}$ in the sense

$$\limsup_{M \to \infty}\ \vec{\delta}(P_\alpha, P_{2M,M,\alpha}) = 0\,. \tag{3.7}$$

4 Robustness of the funnel controller

4.1 Well posedness of the nominal closed-loop system

For $n, m \in \mathbb{N}$ with $n \geq m$, consider $\mathcal{P}_{n,m}$ as a subspace of the Euclidean space $\mathbb{R}^{n^2+2mn}$ by identifying a plant $\theta = (A, B, C)$ with a vector θ consisting of the elements of the plant matrices, ordered lexicographically. With normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where $x^0 \in \mathbb{R}^n$ is the initial value of a linear system (1.1), we associate the causal plant operator

$$P(\theta, x^0)\ :\ \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto P(\theta, x^0)(u_1) := y_1\,, \tag{4.1}$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0, \omega)$, we have $y_1 = cx$, x being the unique solution of (1.1) on $[0, \omega)$. Note that P is a map from $\bigcup_{n \geq m}(\mathcal{P}_{n,m} \times \mathbb{R}^n)$ to the space of maps $\mathcal{U}_a \to \mathcal{Y}_a$. Consider, for $\varphi \in \Phi$, the control strategy (1.4) and associate the causal control operator, parameterized by φ, i.e.

$$C(\varphi)\ :\ \mathcal{Y}_a \to \mathcal{U}_a, \qquad y_2 \mapsto C(\varphi)(y_2) := u_2\,. \tag{4.2}$$

Note that C is a map from the set of inverse funnel boundary functions Φ to the space of causal maps $\mathcal{Y}_a \to \mathcal{U}_a$.

In this sub-section we show that, for $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the closed-loop system $[P(\theta, x^0), C(\varphi)]$ of any plant of the form (1.1) (with associated operator $P(\theta, x^0)$) and controller (1.4) (with associated operator $C(\varphi)$), where $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $\varphi \in \Phi$, is regularly well posed. Furthermore we show that, for $\theta \in \mathcal{M}_{n,m}$, the closed-loop system $[P(\theta, x^0), C(\varphi)]$ is globally well posed and $\big(\mathcal{U} \times \mathcal{Y}\big)$-stable.

Proposition 4.1 *Let $n, m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and funnel control operator $C(\varphi)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by* (1.2), (1.3), (1.4)*, is globally well posed and moreover $[P(\theta, x^0), C(\varphi)]$ is $\big(L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stable.*

Proof. The proposition is a direct consequence of Proposition 2.1. □

In the following sub-section we show that an application of the funnel controller to any stabilizable and detectable linear system (A, B, C) yields a closed-loop system which is regularly well posed. This is required for the robustness analysis in Sub-section 4.3, namely the application of [17, Th. 6.5.3 and Th. 6.5.4].

4.2 Well posedness of the general closed-loop system

Note that, for $(A, B, C) \in \mathcal{P}_{n,m}$, $x^0 \in \mathbb{R}^n$ and $\varphi \in \Phi$, the closed-loop initial value problem (1.1), (1.3), (1.4) may be written as

$$\left.\begin{array}{rcll} \dot{x}(t) &=& Ax(t) + B[u_0(t) - u_2(t)]\,, & x(0) = x^0 \in \mathbb{R}^n\,, \\ k(t) &=& \dfrac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|}\,, & \\ y_2(t) &=& y_0(t) - Cx(t)\,, & \\ u_2(t) &=& -k(t)y_2(t)\,. & \end{array}\right\} \tag{4.3}$$

Proposition 4.2 *Let $n \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and funnel control operator $C(\varphi)$, given by* (4.1) *and* (4.2)*, respectively, the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by* (4.3)*, has the following properties:*

(i) *there exists a unique solution $x\colon [0, \omega) \to \mathbb{R}^n$, for some $\omega \in (0, \infty]$, and the solution is maximal in the sense that for every compact $\mathcal{K} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ exists $t \in [0, \omega)$ such that $(t, x(t)) \notin \mathcal{K}$;*

(ii) *if $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$, then $\omega = \infty$, $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and y_2 is uniformly bounded away from the funnel boundary $\varphi(\cdot)^{-1}$;*

(iii) *$[P(\theta, x^0), C(\varphi)]$ is regularly well posed.*

Proof. Set, for $\varphi \in \Phi$ and $y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$,

$$\mathcal{H}_{\varphi, y_0} := \left\{ (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \,\middle|\, \varphi(t)\|y_0(t) - C\,x\| < 1 \right\}.$$

(i): The initial value problem (4.3) may be written as

$$\dot{x} = g(t, x)\,, \qquad x(0) = x^0\,, \quad (0, y_0(0) - C\,x_0) \in \mathcal{H}_{\varphi, y_0}\,, \tag{4.4}$$

where

$$g\colon \mathcal{H}_{\varphi, y_0} \to \mathbb{R}^n\,, \quad (t, x) \mapsto Ax + Bu_0(t) + \frac{\varphi(t)}{1 - \varphi(t)\|y_0(t) - C\,x\|} B(y_0(t) - C\,x)\,,$$

satisfies, in view of $\varphi|_{[\varepsilon,\infty)}(\cdot)^{-1}$ being globally Lipschitz for every $\varepsilon > 0$ and $\varphi(0) = 0$, see the definition of Φ in Section 1, a local Lipschitz condition on the relatively open set $\mathcal{H}_{\varphi,y_0}$ in the sense that, for all $(\tau,\xi) \in \mathcal{H}_{\varphi,y_0}$, there exists an open neighbourhood O of (τ,ξ) and a constant $L > 0$ such that

$$\forall\,(t,x) \in O \;:\; \|g(t,x) - g(t,\xi)\| \le L\|x - \xi\|\,.$$

Therefore, standard theory of ordinary differential equations, see, for example, [18, Th. III.11.III], yields that (4.3) has an absolutely continuous solution $x\colon [0,\omega) \to \mathbb{R}^n$ for some $\omega \in (0,\infty]$, which satisfies $(t,x) \in \mathcal{H}_{\varphi,y_0}$. Moreover, the solution is unique and the solution can be extended up to the boundary of $\mathcal{H}_{\varphi,y_0}$. In other words: for every compact $\mathcal{K} \subset \mathcal{H}_{\varphi,y_0}$ exists $t \in [0,\omega)$ such that $(t,x(t)) \notin \mathcal{K}$, as required.

(ii): Suppose $(u_2,y_2) \in L^\infty([0,\omega) \to \mathbb{R}^m) \times W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega < \infty$. By boundedness of φ, see the definition of Φ, it follows that there exists $\lambda > 0$ such that $\varphi(t) \le 1/\lambda$ for all $t \in [0,\omega)$. Thus

$$\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \le \frac{1}{2} \quad\Rightarrow\quad \frac{1}{2} \le \varphi(t)\|y_2(t)\| \le \frac{\|y_2(t)\|}{\lambda} \quad\Rightarrow\quad \|y_2(t)\| \ge \frac{\lambda}{2}$$

which yields, in view of $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $\frac{-\varphi}{1-\varphi\|y_2\|}y_2 = u_2 \in L^\infty([0,\omega) \to \mathbb{R})$, that

$$\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \le \frac{1}{2} \quad\Rightarrow\quad \|u_2\|_{L^\infty} \ge \frac{\varphi(t)\,\|y_2(t)\|}{1 - \varphi(t)\|y_2(t)\|} \ge \frac{\lambda\varphi(t)}{2(1 - \varphi(t)\|y_2(t)\|)}\,,$$

thus $\frac{\varphi}{1-\varphi\|y_2\|}$ is bounded on $\{t \in [0,\omega) \,|\, 1 - \varphi(t)\|y_2(t)\| \le 1/2\}$. Moreover, for all $t \in [0,\omega)$ such that $1 - \varphi(t)\|y_2(t)\| > 1/2$, $\left(\frac{\varphi(t)}{1-\varphi(t)\|y_2(t)\|}\right) \le 2/\lambda$. Thus $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty([0,\omega) \to \mathbb{R})$. Hence, by continuity of the solution

$$\exists\,\varepsilon > 0\;\forall\,t \in [0,\omega) \;:\; 1 - \varphi(t)\|y_2(t)\| \ge \varepsilon\,. \tag{4.5}$$

Then, Variation of Constants applied to (4.3) yields the existence of constants $c_0 = c_0(B,\lambda,\varepsilon)$, $c_1 = c_1(A) > 0$ such that

$$\forall\;t \in [0,\omega) \;:\; \|x(t)\| \le c_0\left(e^{c_1\omega} + \int_0^\omega e^{c_1(\omega-s)}\left(\|u_0(s)\| + \|y_2(s)\|\right)\,\mathrm{d}s\right). \tag{4.6}$$

Since $y_2 \in L^\infty([0,\omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, it follows from the convolution in (4.6) that the right hand side of (4.6) is bounded by $c_3 = c_0\big(e^{c_1\omega} + (e^{c_1\omega} + 1)(\|u_0\|_{L^\infty([0,\omega)\to\mathbb{R}^m)} + \|y_2\|_{L^\infty([0,\omega)\to\mathbb{R}^m)})/c_1\big) > 0$ on $[0,\omega)$ which gives that

$$\mathcal{K} := \left\{(t,x) \in \mathcal{H}_{\varphi,y_0} \,\middle|\, t \in [0,\omega]\,,\; \|x\| \le c_3\right\}$$

is a compact subset of $\mathcal{H}_{\varphi,y_0}$ with $(t,x(t)) \in \mathcal{K}$ for all $t \in [0,\omega)$, which contradicts the fact that the closure of graph $\left(x|_{[0,\omega)}\right)$ is not a compact set, see (i). Therefore, $\omega = \infty$ and in view of (4.5) we have k bounded and y_2 is uniformly bounded away from the funnel boundary $\varphi(\cdot)^{-1}$.

(iii): By (i), the closed-loop initial value problem is $[P(\theta,x^0),C(\varphi)]$ is locally well posed. To prove that $[P(\theta,x^0),C(\varphi)]$ is regularly well posed, it suffices to show that (3.2) holds. For arbitrary $w_0 = (u_0,y_0) \in \mathcal{W}$ consider $(w_1,w_2) = H_{P(\theta,x^0),C(\varphi)}(w_0)$ where $\mathrm{dom}(w_1,w_2) = [0,\omega)$ is maximal. Suppose, contrary to the right hand side of (3.2), $\left\|(w_1,w_2)|_{[0,\omega)}\right\|_{\mathcal{W}_\omega\times\mathcal{W}_\omega} < \infty$. Then $(u_2,y_2) \in L^\infty([0,\omega) \to \mathbb{R}^m) \times W^{1,\infty}([0,\omega) \to \mathbb{R}^m)$, which, in view of (ii), yields $\omega = \infty$, i.e. the contrary of the left hand side of (3.2). Hence the closed-loop system is regularly well posed and the proof is complete. $\square$

4.3 Robustness of funnel control

In Proposition 4.1 we have established that, for $(\theta, x^0, \varphi) \in \mathcal{M}_{n,m} \times \mathbb{R}^n \times \Phi$ and $n, m \in \mathbb{N}$ with $n \geq m$, $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the closed-loop system $[P(\theta, x^0), C(\varphi)]$ is globally well posed and has certain stability properties.

The purpose of this sub-section is to determine conditions under which these properties are maintained when the plant $P(\theta, x^0)$ is perturbed to a plant $P\big(\widetilde{\theta}, \widetilde{x}^0\big)$ where $\big(\widetilde{\theta}, \widetilde{x}^0\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ for some $q \in \mathbb{N}$, $q \geq m$, in particular when $\widetilde{\theta} \notin \mathcal{M}_{q,m}$. Proposition 4.2 shows that the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)]$ is regularly well posed. This provides the basis for our main result: Theorem 4.5 shows that stability properties of the funnel controller persist if (a) the plant $P\big(\widetilde{\theta}, 0\big)$ and $P(\theta, 0)$ is sufficiently close (in the gap sense) and (b) the initial data $\widetilde{x}^0$ and disturbance $w_0 = (u_0, y_0)$ are sufficiently small.

To establish gap margin results, we will need to construct the augmented plant and controller operators as in [7] and [4]. Note that $0 \notin \mathcal{M}_{n,m}$. Define $\widetilde{\mathcal{U}} := \mathbb{R}^{n^2+2mn} \times \mathcal{U} = \mathbb{R}^{n^2+2mn} \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and let $\widetilde{\mathcal{W}} := \widetilde{\mathcal{U}} \times \mathcal{Y} = \widetilde{\mathcal{U}} \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, which can be considered as signal spaces by identifying $\theta \in \mathbb{R}^{n^2+2mn}$ with the constant function $t \mapsto \theta$ and endowing $\widetilde{\mathcal{U}}$ with the norm $\|(\theta, u)\|_{\widetilde{\mathcal{U}}} := \sqrt{\|\theta\|^2 + \|u\|^2_{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)}}$. For given $P(\theta, 0)$ as in (4.1), we define the (augmented) plant operator as

$$\widetilde{P} \;:\; \widetilde{\mathcal{U}}_a \to W^{1,\infty}_a(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\,, \quad (\theta, u_1) = \widetilde{u}_1 \mapsto y_1 = \widetilde{P}(\widetilde{u}_1) := P(\theta, 0)(u_1)\,. \tag{4.7}$$

Fix $\varphi \in \Phi$ and define, for $C(\varphi)$ as in (4.2), the (augmented) controller operator as

$$\widetilde{C} \;:\; W^{1,\infty}_a(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \to \widetilde{\mathcal{U}}_a, \quad y_2 \mapsto \widetilde{u}_2 = \widetilde{C}(y_2) := \big(0, C(\varphi)(y_2)\big) = (0, u_2)\;. \tag{4.8}$$

For each non-empty $\Omega \subset \mathcal{M}_{n,m}$, define

$$\mathcal{W}^\Omega := (\Omega \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \quad \text{and} \quad H^\Omega_{\widetilde{P},\widetilde{C}} := H_{\widetilde{P},\widetilde{C}}|_{\mathcal{W}^\Omega}\,. \tag{4.9}$$

It follows from Proposition 4.1 that $H^\Omega_{\widetilde{P},\widetilde{C}} : \mathcal{W}^\Omega \to \widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}}$ is a causal operator for any $\Omega \subset \mathcal{M}_{n,m}$. In the following Proposition 4.3 we show gain-function stability of $H^\Omega_{\widetilde{P},\widetilde{C}}$. This is a supposition of Theorem 5.2 in [2], the latter being used to show Proposition 4.4 and thus the main result Theorem 4.5.

Proposition 4.3 *Let $n, m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$ and assume $\Omega \subset \mathcal{M}_{n,m}$ is closed. Then, for the closed-loop system $[\widetilde{P}, \widetilde{C}]$ given by* (3.1), (4.7) *and* (4.8)*, the operator $H^\Omega_{\widetilde{P},\widetilde{C}}$ given by* (4.9) *is gain-function stable.*

The proof for Proposition 4.3 is equivalent to the proof of [7, Prop. 4.3], when applying Proposition 2.1 instead of [7, Prop. 2.1], and therefore omitted.

The following proposition establishes $\big(L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stability of the closed-loop system $[P(\widetilde{\theta}, \widetilde{x}^0), C(\varphi)]$ for a system $\widetilde{\theta}$ belonging to the system class $\mathcal{P}_{q,m}$ if, for a system θ belonging to $\mathcal{M}_{n,m}$, the gap between $P(\widetilde{\theta}, 0)$ and $P(\theta, 0)$, the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ and the input/output disturbances $w_0 = (u_0, y_0)$ are sufficiently small. The proof uses the robustness results [17, Th. 6.5.3 and Th. 6.5.4].

Proposition 4.4 *Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0, \varphi) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \Phi$, consider $P(\widetilde{\theta}, \widetilde{x}^0) \colon \mathcal{U}_a \to \mathcal{Y}_a$, and*

$C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by (4.1) and (4.2), respectively. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ such that the following holds:

$$\forall \left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \quad \left.\begin{array}{l} \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r) \end{array}\right\} \implies H_{P(\widetilde{\theta},\widetilde{x}^0), C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}. \tag{4.10}$$

The proof of Proposition 4.4 is equivalent to the proof of [7, Prop. 4.4] if the gain-function stability result Proposition 4.3 for funnel control is applied instead of the corresponding result [7, Prop. 4.3]. Moreover, one has to choose signal spaces as in Section 2, namely $\mathcal{U} = L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ and $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ instead of $\mathcal{U} = \mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, and apply [17, Th. 6.5.3 and Th. 6.5.4] instead of [2, Th. 5.2 and Th. 5.3].

Finally, we are in the position to state and prove the main result of the present paper. Loosely speaking, we show that funnel control achieves the control objectives if applied to a system $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ as long as this system is sufficiently close – in the terms of the gap metric – to a system $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ and the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ for $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ and the input/output disturbances (u_0, y_0) are sufficiently small. As a consequence $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ may not even satisfy any of the classical assumptions: minimum phase, relative degree one and positive high-frequency gain.

Theorem 4.5 *Let $n,q,m \in \mathbb{N}$ with $n,q \ge m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, $\varphi \in \Phi$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ consider the associated operators $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by* (4.1) *and* (4.2)*, respectively, and the closed-loop initial value problem* (1.1)*,* (1.3)*,* (1.4)*. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ such that the following holds:*

$$\forall \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \quad \left.\begin{array}{l} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \le r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \eta(r) \end{array}\right\} \implies \left\{\begin{array}{l} \forall\, t \ge 0 \ : \ (t, y_2(t)) \in \mathcal{F}_\varphi \\ k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}) \\ x \in W^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)\,, \end{array}\right. \tag{4.11}$$

where (x,k) and y_2 satisfy (4.3)*.*

Proof. *Step 1*: We show

$$\big((u_1,y_1),(u_2,y_2)\big) = H_{P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}. \tag{4.12}$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Proposition 4.4. Let

$$\big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \ \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \le r \ \wedge \ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \le \ \eta(r)\,.$$

Then Proposition 4.4 gives (4.12).

Step 2: By Proposition 4.2 it follows that (4.3) has a unique solution $x\colon [0,\omega) \to \mathbb{R}^q$ on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. Proposition 4.2(iii) yields $\omega = \infty$ and $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$, the second assertion of (4.11).

Step 3: By Step 2 we have $k \in L^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R})$ which, in view of continuity of $1 - \varphi\|y_2\|$ on $(0,\infty)$, yields $1 - \varphi(t)\|y_2(t)\| \ge \|\varphi\|_{L^\infty}\|k\|_{L^\infty}^{-1} > 0$. Thus, for all $t \ge 0$, $\varphi(t)\|y_2(t)\| < 1$, which yields the first assertion of (4.11).

Step 4: It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$.
Let $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ associated with (1.1). Detectability of $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ yields the existence of $F \in \mathbb{R}^q$ such that $\operatorname{spec}(\widetilde{A} + F\widetilde{C}) \subset \mathbb{C}_-$. Setting $g := -\left[F + k\,\widetilde{B}\right](y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0$ gives

$$\dot{x} = \left[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\right]x + \widetilde{B}\,u_0 + \widetilde{B}\,ky_0 = \left[\widetilde{A} + F\widetilde{C}\right]x + g\,. \tag{4.13}$$

By Proposition 4.4 and Step 3 we have $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and since $w_0 = (u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $g \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (4.13) and Variation of Constants we obtain $x \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. The first equation in (4.3) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ which shows the third assertion in (4.11) and the proof is complete. □

Example 4.6 Finally, we revisit the example systems (3.4) and (3.6).

We have already shown that for zero initial conditions the gap between the system $\big(\widetilde{A}, \widetilde{b}, \widetilde{c}\big) \in \mathcal{P}_{3,1} \backslash \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$ tends to zero as $N = 2M$ and M tends to infinity, see (3.7). Now, in view of Theorem 4.5, there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{3,1} \to (0,\infty)$ such that, for all $(\widetilde{x}^0, w_0, r) \in \mathbb{R}^3 \times \mathcal{W} \times (0,\infty)$, we have

$$\left.\begin{array}{l} \psi\big((\widetilde{A}, \widetilde{b}, \widetilde{c})\big)\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \leq r \\ \vec{\delta}\left(\widetilde{P}_1\big((\alpha,1,1),0\big), \widetilde{P}_2\big((\widetilde{A}, \widetilde{b}, \widetilde{c}),0\big)\right) \leq \eta(r) \end{array}\right\} \implies \left\{\begin{array}{l} \forall\, t \geq 0 \;:\; (t, y_0(t) - y_1(t)) \in \mathcal{F}_\varphi\,, \\ k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})\,, \\ x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^3)\,, \end{array}\right.$$

where $\mathcal{W} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R})$. Note that Theorem 4.5 shows only existence of two continuous functions $\psi\colon \mathcal{P}_{n,m} \to (0,\infty)$ and $\eta\colon (0,\infty) \to (0,\infty)$ in (4.11); however, it could be hard to find these functions for a given system.

The above theoretical result is visualized by MATLAB simulations. System (3.6) has a state space realization

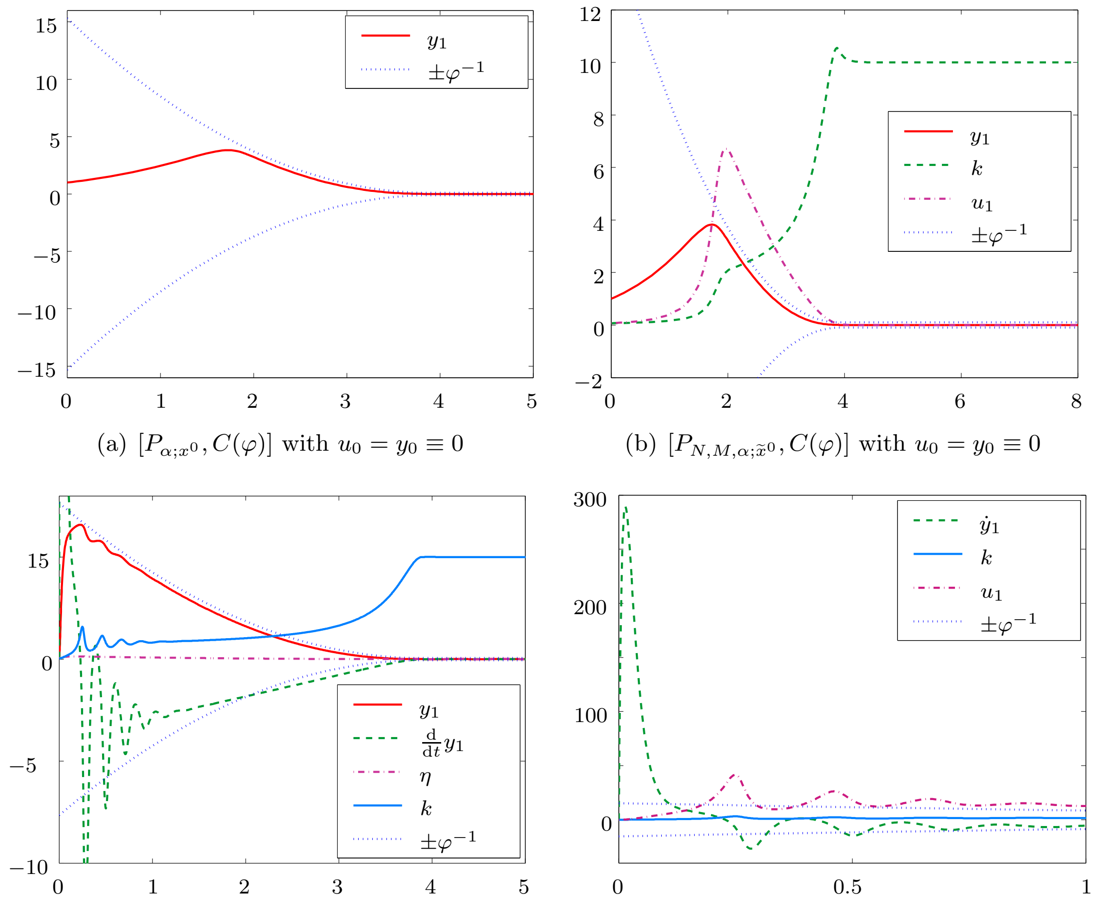
$$\left.\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \alpha N + 2M(\alpha - M - N)\,, & \alpha - M - N\,, & 2M(NM + M^2 - \alpha M - \alpha N) \\ -1 & 0 & -M \end{bmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} + \begin{pmatrix}0\\ -N\\ 0\end{pmatrix}u_1 \\ \quad y_1 = \xi_1\,. \end{array}\right\} \tag{4.14}$$

Let $\alpha = 1$ and $N = 2M = 100$. In [7, Sec. 3] it is shown that $\vec{\delta}(P_\alpha, P_{N,M,\alpha}) \leq 8/51$. Let the funnel boundary be specified, for $\lambda = 0.1$, by

$$\varphi(\cdot)^{-1}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}\,, \qquad t \mapsto \begin{cases} 15.31 - 7.8\,t + t^2\,, & \text{if } t \in [0, 3.9) \\ \lambda\,, & \text{if } t \geq 3.9. \end{cases}$$

Then, for initial values $x^0 = 1$ for system (3.4) and $\widetilde{x}^0 = (0.1, 0.1, 0.08)^T$ for system (3.6) and input/output disturbances $u_0 = y_0 \equiv 0$, Figures 4(a) and 4(b) show the solution $t \mapsto y_1(t)$, k and the input u_1 of the closed-loop system (3.4), (1.4), (1.3) with $u_0 = y_0 \equiv 0$. Moreover, Figures 4(c) and 4(d) show all components of the solution $t \mapsto \begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} = \begin{pmatrix}y_1(t)\\ \dot{y}_1(t)\\ \eta(t)\end{pmatrix}$, k and u_1 of the closed-loop system (4.14), (1.4), (1.3) with $u_0 = y_0 \equiv 0$, where Figures 4(d) indicates that all states (in particular $\xi_2 = \dot{y}_1$) are bounded.

Figure 4 illustrates that the funnel controller (1.4) work for linear systems, which do not satisfy the classical assumptions for funnel control, but are close in terms of the gap metric to minimum phase systems with relative degree one and positive high–frequency gain.



(a) $[P_{\alpha;x^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$

(b) $[P_{N,M,\alpha;\widetilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$

(c) $[P_{N,M,\alpha;\widetilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$

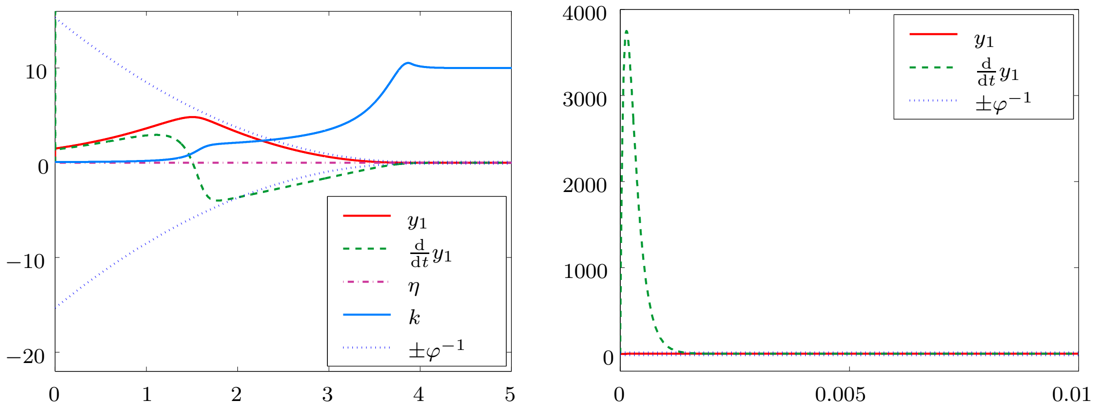
(d) $[P_{N,M,\alpha;\widetilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$

Figure 4: Funnel control simulations

A shortcoming of the main result is that it shows sheer existence of functions ψ and η in (4.11), compare also with the result for λ-tracking. For a given systems $\widetilde{\theta}$ it is maybe hard to calculate the value $\psi(\widetilde{\theta})$. It could be also possible that this functions counteract in some ways. For example: given small $r > 0$ and $\widetilde{\theta} \in \mathcal{P}_{q,m}$ such that $\vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \leq \eta(r)$ it could be possible that $\psi(\widetilde{\theta})$ is very large which requires then a very small initial value $\widetilde{x}_0 \in \mathbb{R}^q$ so that the left hand side of (4.11) holds. However, in view of (4.11) given that the second inequality holds for r and $\widetilde{\theta}$ it is always possible to choose a sufficiently small initial value. This is shown with the simulation in Figure 5: choose $P_{N,M,\alpha;\widetilde{x}^0}$ with $\alpha = 1$, $N = 2M = 10000$ and the initial value $\widetilde{x}^0 = (0.0001, 0.0001, 0.0001)$.

Figure 5(a) shows that the output y_2 is within the funnel and k is bounded. Figure 5(b) shows that all states are bounded, although though $\dot{y}_1$ is very large.

This shows in particular that funnel control works for system (3.6) despite the fact that it has unstable zero dynamics, relative degree two and negative high-frequency gain. The only restrictions are that the zero is "far" in the right half complex plane, the initial condition $\widetilde{x}^0$ is "small" and the $L^\infty \times W^{1,\infty}$ input/output disturbances u_0 and y_0 are "small", too.



(a) $[P_{N,M,\alpha;\widetilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$ (b) $[P_{N,M,\alpha;\widetilde{x}^0}, C(\varphi)]$ with $u_0 = y_0 \equiv 0$

Figure 5: Funnel control simulations for $P_{N,M,\alpha;\widetilde{x}^0}$ with "huge" $N = 2M = 10000$

5 Conclusions

We have shown robustness of the funnel controller (1.4) for a class of linear systems which are close in the gap metric to minimum phase systems with (strict) relative degree one; moreover, funnel control copes with certain bounded input/output disturbances. The only shortcoming of the present approach is that the main result shows sheer existence of continuous functions ψ and η in (4.11). For a given systems $\widetilde{\theta}$ it maybe hard to calculate the value $\psi(\widetilde{\theta})$. It could be also possible that this functions counteract in some ways. For example: given small $r > 0$ and $\widetilde{\theta} \in \mathcal{P}_{q,m}$ such that $\vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r)$ it could be possible that $\psi(\widetilde{\theta})$ is very large which requires then a very small initial value $\widetilde{x}_0 \in \mathbb{R}^q$ so that the left hand side of (4.11) holds. However, in view of (4.11) given that the second inequality holds for r and $\widetilde{\theta}$ it is always possible to choose a sufficiently small initial value.

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