Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **08/20**

Methods for finding fixed points of nonexpansive operators in a Hilbert space

Cegielski, Andrzej

Jul 2008

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

ilmedia

Methods for finding fixed points of nonexpansive operators in a Hilbert space

Andrzej Cegielski
Faculty of Mathematics,
Computer Sciences and Econometrics
University of Zielona Góra

A series of lectures
presented at the Institute of Mathematics
Technical University Ilmenau,
July 2008

Contents

1 Introduction

1.1 Notation and basic facts

- The elements of $\mathbb{R}^n$ are column vectors, e.g., $x = (\xi_1, ..., \xi_n)^\top$, $y = (\eta_1, ...\eta_n)^\top$.

For vectors $x, y \in \mathbb{R}^n$:

- $x \geq 0$ denotes that all coordinates of x are nonnegative,
- $f = \max_{i \in I} f_i$ denotes that for all x there holds the equality $f(x) = \max\{f_i(x) : i \in I\}$
- $x_+ = \max\{x, 0\} = (\max\{\xi_1, 0\}, ..., \max\{\xi_n, 0\})^\top$,
- $e_j = (0, ..., 0, 1, 0, ..., 0)^\top \in \mathbb{R}^m$ with 1 at the j-th coordinate,
- $e = (1, ..., 1)^\top$
- $\mathbb{R}^m_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ – nonnegative orthant,
- $\Delta_m = \{u \in \mathbb{R}^m : e^\top u = 1, u \geq 0\}$ – the standard simplex
- $B(x, \alpha) = \{y \in \mathcal{H} : \|y - x\| \leq \alpha\}$ – the ball with centre $x \in \mathcal{H}$ and radius $\alpha \geq 0$,
- $\bar{D}$ – closure of $D \subset \mathcal{H}$,
- $\operatorname{Fix} T = \{z \in \mathcal{H} : Tz = z\}$ – the subset of fixed points of an operator $T : D \to D$, where $D \subset \mathcal{H}$,
- $\operatorname{Argmin}_{x \in D} f(x) = \{z \in D : f(z) \leq f(x)$ for all $x \in D\}$, where $D \subset \mathcal{H}$ and $f : D \to \mathbb{R}$ – a subset on which the function f attains its minimum on D,
- $\operatorname{argmin}_{x \in D} f(x)$ – a minimizer of a function $f : D \to \mathbb{R}$, i.e., an element of $\operatorname{Argmin}_{x \in D} f(x)$,
- a function $f : \mathcal{H} \to \mathbb{R}$ is said to be coercive if $\lim_{\|x\| \to \infty} f(x) = +\infty$,
- a continuous and coercive function $f : \mathcal{H} \to \mathbb{R}$ attains its minimum,
- $N_D(x) = \{y \in \mathcal{H} : \langle y, z - x\rangle \leq 0$ for all $z \in D\}$ – the normal cone to a convex subset $D \subset \mathcal{H}$ in the point $x \in D$,
- $H(a, \beta) = \{x \in \mathcal{H} : \langle a, x\rangle = \beta\}$, where $a \in \mathcal{H}$ and $\beta \in \mathbb{R}$ – a hyperplane in $\mathcal{H}$,
- $H^+(a, \beta) = \{x \in \mathcal{H} : \langle a, x\rangle \geq \beta\}$ and $H^-(a, \beta) = \{x \in \mathcal{H} : \langle a, x\rangle \leq \beta\}$ half-spaces in $\mathcal{H}$,
- $S(f, \alpha) = \{x \in \mathcal{H} : f(x) \leq \alpha\}$ – the sublevel set of a function $f : \mathcal{H} \to \mathbb{R}$ at a level $\alpha \in \mathbb{R}$.
- $\mathrm{d}(x, D) = \inf_{y \in D} \|x - y\|$ – the distance of $x \in \mathcal{H}$ to a subset $D \subset \mathcal{H}$,
- $T^1 = T$, $T^m = T^{m-1} \circ T$, $m = 2, 3, ...$, where $T : D \to D$ for $D \subset \mathcal{H}$,
- $\operatorname{diag} v$ – a diagonal matrix, with a vector $v \in \mathbb{R}$ on the main diagonal.

Let $x, y \in \mathcal{H}$

- **The Schwarz inequality**
$$\langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2,$$
The equality holds if and only if the vectors x and y are linear dependent

- **Parallelogram law**
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

- **Strict convexity**
$$\|x+y\| = \|x\| + \|y\| \Longleftrightarrow \|x\|y = \|y\|x$$

1.2 Convex subsets and convex functions

Let X, Y be linear spaces.

- The intersection$\bigcap_{\alpha\in\Lambda} C_\alpha$ of a family $\{C_\alpha\}_{\alpha\in\Lambda}$ of convex subsets of X is a convex subset.
- Any norm $\|\cdot\|$ in X is a convex function
- If $f : X \to \mathbb{R}$ is a convex function and $g : \mathbb{R} \to \mathbb{R}$ is a convex and nondecreasing function, then $g \circ f$ is a convex function.
- For any norm $\|\cdot\|$ in X the function $\|\cdot\|^2$ is convex
- If $f_i : X \to \mathbb{R}$, $i = 1, ..., m$, are convex functions and $F : \mathbb{R}^m \to \mathbb{R}$ is a convex function which is nondecreasing with respect to any coordinate, then $f = F(f_1, ..., f_m)$ is a convex function.
- If $f : Y \to \mathbb{R}$ is a convex function and $A : X \to Y$ is a linear operator, then $f \circ A$ is a convex function.
- The sublevel set $S(f, \alpha)$ of a convex function $f : X \to \mathbb{R}$ is a convex subset.

Let $\mathcal{H}$ be a Hilbert space and let $D \subset \mathcal{H}$ be closed and convex.

- The distance function $d(\cdot, D) : \mathcal{H} \to \mathbb{R}$, $d(x, D) = \inf_{z\in D} \| x - z \|$, is convex.
- The function $d^2(\cdot, D)$ is convex as a composition of a convex function $d(\cdot, D)$ and a convex and a nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}$, $g(u) = u^2$.
- The function $\frac{1}{2}d^2(\cdot, D)$ is differentiable and its derivative has the form
$$D(\frac{1}{2}d^2(x, D)) = x - P_D x.$$

1.3 Operators

In methods for the CFP the important role play nonexpansive operators, firmly nonexpansive operators and the Fejér monotone operators.

Let $D \subset \mathcal{H}$ be closed and convex subset.

Definition 1 We say that an operator $T : D \to \mathcal{H}$ is *nonexpansive* (NE) if for all $x, y \in D$
$$\|Tx - Ty\| \le \|x - y\|.$$

If the inequality is strict for $x \neq y$ then T is said to be *strictly nonexpansive.*

Definition 2 We say that an operator $T : D \to \mathcal{H}$ is *a contraction* if for some $\alpha \in (0, 1)$ and for all $x, y \in D$
$$\|Tx - Ty\| \le \alpha\|x - y\|.$$

Definition 3 We say that an operator $T : D \to \mathcal{H}$ is monotone if for all $x, y \in D$

$$\langle Tx - Ty, x - y\rangle \geq 0$$

Definition 4 We say that an operator $T : D \to \mathcal{H}$ is *Fejér monotone* (FM) with respect to a subset $M \subset D$ if for all $x \in D$ and for all $z \in M$

$$\|Tx - z\| \leq \|x - z\|.$$

If T is Fejér monotone with respect to $\operatorname{Fix} T$ then T is called *quasi-nonexpansive* (QNE) If the inequality is strict for $x \notin M$ then we say that T is *strictly Fejér monotone* (or *strictly quasi-nonexpansive if* $M = \operatorname{Fix} T$) with respect to a subset $M \subset D$.

Definition 5 We say that a sequence (x_k) is *Fejér monotone* (FM) with respect to a subset M if for all k and for all $z \in M$

$$\|x_{k+1} - z\| \leq \|x_k - z\|.$$

Definition 6 Let $\alpha > 0$. We say that an operator $T : D \to \mathcal{H}$ is *α-strongly Fejér monotone* – α-SFM with respect to a subset $M \subset D$, or *strongly Fejér monotone* – SFM, if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|Tx - x\|^2$$

for all $x \in D$ and for all $z \in M$. If T is α-SFM with respect to $\operatorname{Fix} T$ then T is called *α-strongly quasi-nonexpansive* or *strongly quasi-nonexpansive* – SQNE.

Remark 7 A nonexpansive operator T with $\operatorname{Fix} T \neq \varnothing$ is quasi-nonexpansive.

Definition 8 Let $\lambda \in [0, 2]$. The operator $T_\lambda = I + \lambda(T - I)$ is called a *relaxation* of an operator $T : D \to \mathcal{H}$. The parameter λ is called a relaxation parameter. If $\lambda \in (0, 2)$ then T_λ is called a *strict relaxation* of T.

Definition 9 We say that an operator $T : D \to \mathcal{H}$ is *firmly nonexpansive* (FNE) if for all $x, y \in D$

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2.$$

Definition 10 We say that an operator $T : D \to \mathcal{H}$ is *relaxed firmly nonexpansive* (RFNE) if T is a relaxation of a firmly nonexpansive operator.

Definition 11 We say that an operator $T : D \to \mathcal{H}$ with $\operatorname{Fix} T \neq \varnothing$ is *separating* if

$$\langle z - Tx, x - Tx\rangle \leq 0$$

for all $x \in D$ and for all $z \in \operatorname{Fix} T$.

Definition 12 We say that an operator $T : D \to \mathcal{H}$ is *strongly nonexpansive* (SNE) if T is nonexpansive and for all sequences $(x_k), (y_k) \subset D$ there holds the implication:

If $(x_k - y_k)$ is bounded and $\| x_k - y_k \| - \| Tx_k - Ty_k \| \to 0$ then $(x_k - y_k) - (Tx_k - Ty_k) \to 0$.

Definition 13 We say that an operator $T : D \to \mathcal{H}$ is *averaged* (AV) if

$$T = (1 - \alpha)I + \alpha U$$

for a nonexpansive operator $U : D \to \mathcal{H}$ and for a constant $\alpha \in (0, 1)$.

Definition 14 We say that an operator $T : D \to D$ is *idempotent* if $T^2 = T$

Definition 15 We say that an operator $T : D \to D$ is *asymptotically regular* (AR) if for all $x \in D$

$$\lim_{k\to\infty} \|T^{k+1}x - T^k x\| = 0.$$

1.4 Metric projection

Definition 16 Let $D \subset \mathcal{H}$ be a nonempty subset and let $x \in \mathcal{H}$. The point $y \in D$ is called the *metric projection* of a point x onto a subset D, if for any $z \in D$ there holds the inequality

$$\|y - x\| \leq \|z - x\|.$$

The metric projection of a point x onto D is denoted by $P_D x$.

In one of the next sections, we show a fact which is more general than the result below.

Theorem 17 *Let $D \subset \mathcal{H}$ be a nonempty, convex and closed subset. Then for any $x \in \mathcal{H}$ there exists the metric projection $P_D x$ and is defined uniquely.*

1.4.1 Characterization and basic properties of the metric projection

The theorem below is used in many applications.

Theorem 18 *Let $x \in \mathcal{H}$, $D \subset \mathcal{H}$ be a nonempty, convex and closed subset and let $y \in D$. The following conditions are equivalent:*

(i) $y = P_D x$,

(ii) $\langle x - y, z - y \rangle \leq 0$ *for all* $z \in D$.

Proof. (i)⇒(ii). Let $y = P_D(x)$, $z \in D$ and let $z_\lambda = y + \lambda(z - y)$ for $\lambda \in (0, 1)$. Obviously, $z_\lambda \in D$, since D is convex. We have by the properties of the scalar product

$$\|x - y\|^2 \leq \|x - z_\lambda\|^2 = \|x - y - \lambda(z - y)\|^2$$
$$= \|x - y\|^2 - 2\lambda\langle x - y, z - y \rangle + \lambda^2 \|z - y\|^2.$$

Since $\lambda > 0$, we have

$$\langle x - y, z - y \rangle \leq \frac{\lambda}{2} \|z - y\|^2$$

If we let $\lambda \to 0$ in the last inequality, we obtain (ii) in the limit.

(ii)⇒(i). By the properties of the scalar product and by (ii) we obtain for any $z \in D$

$$\begin{aligned} \|z - x\|^2 &= \|z - y + y - x\|^2 \\ &= \|z - y\|^2 + \|y - x\|^2 + 2\langle z - y, y - x \rangle \\ &\geq \|y - x\|^2, \end{aligned}$$

which, by the definition of the metric projection, gives (i). ■

The following Lemma can be easily proved.

Lemma 19 *Let $x, y, z \in \mathcal{H}$. The following conditions are equivalent:*

(i) $\langle x - y, z - y \rangle \leq 0$,

(ii) $\langle z - x, y - x \rangle \geq \|y - x\|^2$,

(iii) $\|z - y\|^2 \leq \|z - x\|^2 - \|y - x\|^2$,

(iv) $\langle z - x, z - y \rangle \geq 0$.

Corollary 20 *Let $D \subset \mathcal{H}$ be nonempty, convex and closed. Then* $\operatorname{Fix} P_D = D$*. Consequently, the metric projection P_D is an idempotent operator.*

Proof. If $x \in D$, then it follows from the definition of the metric projection that $x = P_D x$. If $x \notin D$, then $x \neq P_D x$ since $P_D x \in D$. ■

Corollary 21 *Let $D \subset \mathcal{H}$ be nonempty, convex and closed. Then for all $x \in \mathcal{H}$ and $z \in D$ there holds the inequality*

$$\|P_D x - z\|^2 \leq \|x - z\|^2 - \|P_D x - x\|^2,$$

consequently, the metric projection P_D is strongly Fejér monotone with respect to D.

Proof. Let $x \in \mathcal{H}$ and let $z \in D$. The inequality follows from the characterization of the metric projection (Theorem 18) and from the equivalence (i)⇔(iii) in Lemma 19 for $y = P_D x$ and $z \in D$. ■

Corollary 22 *Let $D \subset \mathcal{H}$ be nonempty, convex and closed, $x \notin D$ and $y \in D$. Then*

$$y = P_D x \Longleftrightarrow x - y \in N_D(y).$$

Proof. The right side of the above equivalence can be written by the definition of the normal cone in the form $\langle x - y, z - y \rangle \leq 0$ for all $z \in D$. Now we see that the equivalence follows directly from the characterization of the metric projection (Theorem 18). ■

1.5 Fixed points theorems

The theorem below, called *Banach fixed point theorem* or *Banach theorem on contractions*, is widely applied in various areas of mathematics. The theorem holds for any metric complete space, in particular for a closed subset of a Hilbert space.

Theorem 23 *Let $U : X \to X$ be a contraction. Then U has exactly one fixed point $x^* \in X$. Furthermore, for any $x \in X$ the sequence of iterations $(U^k x)$ converges to x^* with a rate of geometric progression.*

Proof. See [GK90, Theorem 2.1], where three various proofs are given. ■

The Banach fixed point theorem is a good tool for iterative approximation of fixed points. Nevertheless, its application is restricted to contractions. We will need, however, appropriate tools for iterative approximation of fixed points of nonexpansive operators.

Below, we present few classical fixed points theorems.

Theorem 24 (Brouwer, 1912) *Let $X \subset \mathbb{R}^n$ be nonempty, compact and convex and let $U : X \to X$ be continuous. Then U has a fixed point.*

Proof. See, e.g. [Bro12], [GD03, Chapter II, §5, Theorem 7.2] or [Goe02, Theorem 7.6]. ■

Theorem 25 (Schauder, 1930) *Let X be nonempty, compact and convex subset of a Banach space and let $U : X \to X$ be continuous. Then U has a fixed point.*

Proof. See, e.g. [Sch30], [GD03, Chapter II, §6, Theorem 3.2] or [Goe02, Theorem 8.1]. ■

Theorem 26 (Browder, 1965) *Let $X \subset \mathcal{H}$ be a nonempty closed, convex and bounded subset of a Hilbert space and let $U : X \to X$ be nonexpansive. Then U has a fixed point.*

Proof. See, e.g. [Bro65, Theorem 1], [GD03, chapter I, §4, Theorem 1.3] or [Goe02, Theorem 4.1]. ■

In one of the next sections we present theorems which can be applied in iterative methods for finding fixed points of nonexpansive operators.

Below, we present some properties of the subset of fixed points of a nonexpansive operator.

Lemma 27 *The subset of fixed points of a nonexpansive operator $T : X \to \mathcal{H}$ is closed and convex.*

Proof. Let $x_k \in \operatorname{Fix} T$ and let $x_k \to x$. We have $x \in X$ since X is closed. Since a nonexpansive operator is continuous, we have

$$x = \lim x_k = \lim_k T x_k = T x,$$

i.e. $\operatorname{Fix} T$ is a closed subset. Now we show the convexity of $\operatorname{Fix} T$. Let $x, y \in \operatorname{Fix} T$ and let $z = (1-\lambda)x + \lambda y$ for $\lambda \in [0,1]$. By the nonexpansivity of T and by the positive homogeneity of the norm we have

$$\|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = \lambda \|x - y\|$$

and

$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = (1-\lambda)\|x - y\|.$$

Now, the triangle inequality yields

$$\|x - y\| \leq \|x - Tz\| + \|Tz - y\| \leq \|x - y\|.$$

Consequently,

$$\|x - y\| = \|x - Tz\| + \|Tz - y\|$$

and the strict convexity yields $Tz = (1-\alpha)x + \alpha y$. It follows easily from the nonexpansivity of T that $\alpha = \lambda$, consequently $Tz = z$. The details are left to the reader. ■

The closedness and the convexity of the subset of fixed points of a nonexpansive operator follows also from a property which will be presented in Corollary **??**.

Let $U_i : X \to X$, $i = 1, ..., m$. Denote $U = U_m U_{m-1} ... U_1$ and $Q_i = U_i U_{i-1} ... U_1 U_m ... U_{i+1}$, $i = 1, 2, ..., m$. We have $Q_m = U$. Let $z_0 \in X$ and let $z_i = U_i z_{i-1}$, $i = 1, 2, ..., m$. There exists a correspondence between fixed points of operators Q_i expressed by the following Lemma.

Lemma 28 *A point $z_0 \in X$ is a fixed point of the operator U if and only if z_i is a fixed point of the operator Q_i, $i = 1, 2, ..., m-1$. Furthermore,* $\operatorname{Fix} Q_1 = U_1(\operatorname{Fix} U)$ *and* $\operatorname{Fix} Q_i = U_i(\operatorname{Fix} Q_{i-1})$, *$i = 2, ..., m$.*

Proof. Suppose that $U z_0 = z_0$. By the equalities $z_j = U_j z_{j-1}$, $j = 1, 2, ..., m$, we have

$$\begin{aligned} Q_i z_i &= U_i U_{i-1} ... U_1 U_m ... U_{i+1} z_i \\ &= U_i U_{i-1} ... U_1 U_m ... U_{i+1} U_i ... U_1 z_0 \\ &= U_i U_{i-1} ... U_1 z_0 \\ &= z_i. \end{aligned}$$

The proof of the converse implication is similar. We leave the proof of the second part of the Lemma to the reader. ■

Theorem 29 *Let $U_i : X \to X$ and let $U = U_m ... U_1$. If U_i are nonexpansive and $U_j(X)$ is bounded for at least one $j \in I$ then* $\operatorname{Fix} U \neq \varnothing$.

Proof. Since U_i are nonexpansive, $i \in I$, the boundedness of $U_j(X)$ yields the boundedness of $U(X)$. Therefore, $Y = \overline{\operatorname{conv} U(X)}$ is closed and convex and bounded (see, e.g. [HUL93, Chapter III, Theorem 1.4.3]). Since $U(X) \subset X$ and X is closed and convex, we have $Y \subset X$. The operator $U\mid_Y$ maps a closed, convex and bounded subset Y into itself. By the Browder theorem, the operator $U\mid_Y$ has a fixed point $z \in Y$. Of course, $z \in Y$ and $Uz = U\mid_Y(z) = z$. ■

Theorem 30 *Let $U_i : X \to X$, $i \in I$, be nonexpansive operators with a common fixed point, let $w \in \Delta_m$, $\omega_i > 0$ for $i \in I$ and let $U = \sum_{i\in I} \omega_i U_i$. Then*

$$\operatorname{Fix} U = \bigcap_{i\in I} \operatorname{Fix} U_i.$$

Proof. The inclusion $\bigcap_{i\in I} \operatorname{Fix} U_i \subset \operatorname{Fix} U$ is obvious. We show that the converse inclusion holds if U_i are nonexpansive operators, $i \in I$. Let $z \in \operatorname{Fix} U$ and $u \in \bigcap_{i\in I} \operatorname{Fix} U_i$. We have by the triangle inequality

$$\begin{aligned}\|z-u\| &= \|Uz-u\| \\ &= \|\sum_{i\in I}\omega_i U_i z - u\| = \|\sum_{i\in I}\omega_i(U_i z - u)\| \\ &\le \sum_{i\in I}\omega_i\|U_i z - u\| = \sum_{i\in I}\omega_i\|U_i z - U_i u\| \\ &\le \sum_{i\in I}\omega_i\|z-u\| = \|z-u\|.\end{aligned}$$

Now we see that

$$\|\sum_{i\in I}\omega_i(U_i z - u)\| = \sum_{i\in I}\omega_i\|U_i z - u\| = \sum_{i\in I}\omega_i\|z-u\|.$$

Since $\omega_i > 0$ for $i \in I$, the first of the equalities above yields positive linear dependence of all pair of vectors $U_i z - u$ and $U_j z - u$, $i, j \in I$, $i \neq j$, i.e.,

$$\|U_i z - u\|(U_j z - u) = \|U_j z - u\|(U_i z - u). \tag{1}$$

The second equality, together with the inequality

$$\|U_i z - u\| \le \|z-u\|,$$

$i \in I$, and with the assumption $\omega_i > 0$ yields

$$\|U_i z - u\| = \|z-u\| \tag{2}$$

for all $i \in I$. Now, it follows from (1) and (2) that $U_i z - u = z - u$, $i \in I$, i.e. $U_i z = z$ for all $i \in I$. Consequently, $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$. ■

Lemma 31 *Let $C \subset \mathcal{H}$ be nonempty, closed and convex, let $T : \mathcal{H} \to \mathcal{H}$ and let $\lambda > 0$. Then*

$$\operatorname{Fix}(P_C T_\lambda) = \operatorname{Fix}(P_C T).$$

In particular, $\operatorname{Fix} T_\lambda = \operatorname{Fix} T$.

Proof. We have by Theorem 22

$$\begin{aligned} x &\in \operatorname{Fix}(P_C T_\lambda) \Longleftrightarrow P_C(x + \lambda(Tx - x)) = x \\ &\Longleftrightarrow \lambda(Tx - x) \in N_C(x) \\ &\Longleftrightarrow Tx - x \in N_C(x) \\ &\Longleftrightarrow P_C T x = x \\ &\Longleftrightarrow x \in \operatorname{Fix}(P_C T).\end{aligned}$$

■

Note that for $\lambda > 0$ and for an arbitrary operator T we have $\operatorname{Fix} T = \operatorname{Fix} T_\lambda$.

2 Problems

$\mathcal{H}$ – a real Hilbert space, $\langle \cdot, \cdot \rangle$ – a scalar product in $\mathcal{H}$, $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ – the norm induced by $\langle \cdot, \cdot \rangle$.

$D \subset \mathcal{H}$ – closed and convex subset.

$T : D \to \mathcal{H}$ a nonexpansive operator with $\operatorname{Fix} T \neq \varnothing$

$$\text{find } x^* \in \operatorname{Fix} T.$$

2.1 Convex minimization problem

Let $f : \mathcal{H} \to \mathbb{R}$ be convex and let $D \subset \mathcal{H}$ be closed and convex. The *constrained minimization problem* expressed in the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{with respect to} & x \in D \end{array} \tag{3}$$

is to find an $x^* \in D$ such that $f(x^*) \leq f(x)$ for all $x \in D$, if such a point exists. The point x^* is called *minimizer* of the function f on D or *optimal solution* of problem (3). The value $f^* = f(x^*)$ is called the minimum of the function f on the subset D. If f is strongly convex then problem (3) has unique solution. If F is differentiable then problem (3) is equivalent to finding a fixed point of the operator $T : D \to D$,

$$T(x) = P_D(x - \gamma \nabla f(x)),$$

where $\gamma > 0$. One can prove that if ∇f is L-Lipschitz continuous then T is RFNE for $\gamma \in [0, \frac{2}{L}]$ (see, e.g. [Byr08, Theorem 17.12]).

2.2 Variational inequality

Let $D \subset \mathcal{H}$ be closed and convex and let $F : D \to \mathcal{H}$ be a monotone operator. The *variational inequality* (VI) problem is to find $x \in D$ such that

$$\langle F(x), y - x \rangle \geq 0 \tag{4}$$

for all $y \in D$. If F is a strongly monotone and Lipschitz continuous operator then VI has a unique solution $x^* \in D$. The convex minimization problem (3) is a special case of the variational inequality. VI (4) is equivalent to finding a fixed point of the operator $T : D \to D$,

$$T(x) = P_D(x - \gamma F(x)),$$

where $\gamma > 0$.

2.3 Convex feasibility problem

Let $C_i \subset \mathcal{H}$, $i \in I = \{1, ..., m\}$ be nonempty, closed and convex. The *convex feasibility problem* (CFP) has the form:

$$\text{Find} \quad x \in C = \textstyle\bigcap_{i \in I} C_i \quad \text{if such a point exists}$$

CFP can be also formulated as minimization of the convex proximity function $f : \mathcal{H} \to \mathbb{R}$

$$f(x) = \frac{1}{2} \sum_{i \in I} \omega_i d^2(x, C_i)$$

where $w = (\omega_1, ..., \omega_m)^\top \in \mathbb{R}^m$ is a vector of weights, i.e., $\omega_i \geq 0$, $i \in I$, $\sum_{i \in I} \omega_i = 1$. By the necessary and sufficient optimality conditions for the unconstrained convex minimization problem the minimization of f is equivalent to the following problem

$$\text{Find a fixed point (if exists) of the operator } T : \mathcal{H} \to \mathcal{H}$$

defined as follows

$$T(x) = \sum_{i\in I} \omega_i P_{C_i}(x).$$

2.3.1 Linear feasibility problem

Let A be a real matrix of type $m \times n$ and let $b \in \mathbb{R}^m$. The *linear feasibility problem* (LFP) has the form:

$$\text{Find } x \in \mathbb{R}^n \text{ with } Ax \le b, \text{ if such } x \text{ exists.}$$

Of course, LFP is a special case of CFP.

2.4 Split feasibility problem

Let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be closed and convex subsets of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The *split feasibility problem* (SFP) has the form:

$$\text{Find } x \in C \text{ with } Ax \in Q, \text{ if such } x \text{ exists}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

SFP can be also formulated as minimization of the convex proximity function $f : C \to \mathbb{R}$

$$f(x) = \frac{1}{2}\|P_Q(Ax) - Ax\|^2$$

By the necessary and sufficient optimality conditions for the constrained convex minimization problem the minimization of f is equivalent to the following problem:

$$\text{Find a fixed point (if exists) of the operator } T : C \to C,$$

defined as follows

$$T(x) = P_C(x + \gamma A^*(P_Q Ax - Ax))$$

for $\gamma > 0$.

2.4.1 Linear split feasibility problem

Let $C \subset \mathbb{R}^n$ be closed and convex, A be a real matrix of type $m \times n$ and bet $b \in \mathbb{R}^m$. The *linear split feasibility problem* (LSFP) has the form:

$$\text{Find } x \in C \text{ with } Ax \le b,$$

if such x exists. Of course, LSFP is a special case of SFP.

Let $r(x) = (\rho_1(x), ..., \rho_m(x))^\top = Ax - b$ be the residual vector and let $r_+(x) = \max\{0, r(x)\}$ be the nonnegative part of $r(x)$. LSFP can be also formulated as minimization of a convex proximity function $f : C \to \mathbb{R}$ defined as follows

$$f(x) = \frac{1}{2}\sum_{i=1}^{m} \nu_i(a_i^\top x - \beta_i)_+^2 = \frac{1}{2}\sum_{i=1}^{m} \nu_i(\rho_i)_+$$

where $v = (\nu_1, ..., \nu_m)^\top \in \mathbb{R}^m_{++}$, $a_i = (a_{i1}, ..., a_{in})^\top$ is the i-th row of $A\, i = 1, 2, ..., m$, and $b = (\beta_1, ..., \beta_m)^\top$. If $\mathbb{R}^m$ is equipped with the scalar product $\langle\cdot,\cdot\rangle_V$ defined by $\langle x, y\rangle_V = x^\top V y$, where $V = \operatorname{diag} v$, then one can prove that $P_Q Ax - Ax = -V r_+(x)$. Consequently, the minimization of f is equivalent to the following problem:

$$\text{Find a fixed point (if exists) of the operator } T : C \to C$$

defined by

$$T(x) = P_C(x - \gamma A^\top V r_+(x))$$

with $\gamma > 0$ for all x and $V = \operatorname{diag} v$.

3 Convergence theorems

3.1 Weak convergence in a Hilbert space

Definition 32 We say that a sequence (x_k) of elements of a Hilbert space $\mathcal{H}$ *converges weakly* to $x \in \mathcal{H}$ if for any $y \in \mathcal{H}$ the sequence $(\langle y, x_k \rangle)$ converges to $\langle y, x \rangle$. We call the point x the *weak limit* of the sequence (x_k) and we write $x_k \rightharpoonup x$. If a point $x \in \mathcal{H}$ is a weak limit of a subsequence $(x_{k'}) \subset (x_k)$, then say that x is a *weak cluster point* of the sequence (x_k).

3.1.1 Properties of the weak convergence

Following properties of weak convergent sequences can be found in handbooks of functional analysis.

(w1) A weakly convergent sequence $(x_k) \subset \mathcal{H}$ has exactly one weak limit.

(w2) A weakly convergent sequence $(x_k) \subset \mathcal{H}$ is bounded.

(w3) A bounded sequence $(x_k) \subset \mathcal{H}$ includes a weakly convergent subsequence.

(w4) If a sequence $(x_k) \subset \mathcal{H}$ is bounded and has exactly one weak cluster point $x \in \mathcal{H}$, then $x_k \rightharpoonup x$.

(w5) If a sequence (x_k) converges to $x \in \mathcal{H}$, then it converges weakly to $x \in \mathcal{H}$.

(w6) A weakly convergent sequence (x_k) of a finite-dimensional Hilbert space $\mathcal{H}$ is convergent.

Remark 33 A bounded sequence (x_k) of a Hilbert space does not need contain a convergent subsequence.

Example 34 Let $\mathcal{H} = l^2$ and let $x_k = (\xi_{k1}, \xi_{k2}, ...)$, where

$$\xi_{ki} = \delta_{ki} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k. \end{cases}$$

Then $\|x_k\| = 1$, although (x_k) does not contain a convergent subsequence since $\|x_k - x_l\| = \sqrt{2}$ for all k, l, $k \neq l$. Note that $x_k \rightharpoonup 0$.

Lemma 35 *If $x_k \rightharpoonup x \in \mathcal{H}$, then $\liminf_k \|x_k\| \geq \|x\|$.*

Proof. Let $x_k \rightharpoonup x \in \mathcal{H}$. If $x = 0$ the Lemma is obvious. Suppose now that $x \neq 0$. We have by the Schwarz inequality

$$\liminf_k \|x\| \cdot \|x_k\| \geq \liminf_k \langle x, x_k \rangle = \|x\|^2.$$

Consequently $\liminf_k \|x_k\| \geq \|x\|$. ∎

Lemma 36 *If $x_k \rightharpoonup x \in \mathcal{H}$ and $\|x_k\| \to \|x\|$, then $x_k \to x$.*

Proof. Let $x_k \rightharpoonup x \in \mathcal{H}$ and $\|x_k\| \to \|x\|$. Then it follows from the parallelogram law

$$\begin{aligned}\|x - x_k\|^2 &= 2(\|x\|^2 + \|x_k\|^2) - \|x + x_k\|^2 \\ &= 2(\|x\|^2 + \|x_k\|^2) - (\|x\|^2 + \|x_k\|^2 + 2\langle x, x_k\rangle) \\ &= \|x\|^2 + \|x_k\|^2 - 2\langle x, x_k\rangle \to 0.\end{aligned}$$

■

In 1967 Zdzisław Opial has proved the following property of a Hilbert space, known also under the name *Opial's property.*

Lemma 37 (Opial, 1967) *If $x_k \rightharpoonup y \in \mathcal{H}$, then for any $y' \in \mathcal{H}$, $y' \neq y$ there holds the inequality*

$$\liminf_k \|x_k - y'\| > \liminf_k \|x_k - y\| \tag{5}$$

Proof. Let $x_k \rightharpoonup y \in \mathcal{H}$, $y' \in \mathcal{H}$ be different from y and let $\delta = \|y - y'\|^2 > 0$. Since a weakly convergent sequence is bounded both limits in (5) are finite. Further, we have by the properties of the scalar product

$$\begin{aligned}\|x_k - y'\|^2 &= \|x_k - y + y - y'\|^2 \\ &= \|x_k - y\|^2 + \|y - y'\|^2 + 2\langle x_k - y, y - y'\rangle \\ &= \|x_k - y\|^2 + \delta + 2\langle x_k - y, y - y'\rangle\end{aligned}$$

Since $\langle x_k - y, y - y'\rangle \to 0$

$$\liminf_k \|x_k - y'\|^2 = \liminf_k \|x_k - y\|^2 + \delta > \liminf_k \|x_k - y\|^2,$$

i.e., $\liminf_k \|x_k - y'\| > \liminf_k \|x_k - y\|$. ■

Lemma 38 (Opial, 1967) *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator, y be a weak cluster point of a sequence (x_k) and let $\|Tx_k - x_k\| \to 0$. Then $y \in \operatorname{Fix} T$.*

Proof. Let $x_{n_k} \rightharpoonup y$ for a subsequence $(x_{n_k}) \subset (x_k)$. Suppose that $Ty \neq y$. Then we have by Lemma 37

$$\begin{aligned}\liminf_{k\to\infty} \|x_{n_k} - y\| &\geq \liminf_{k\to\infty} \|Tx_{n_k} - Ty\| \\ &= \liminf_{k\to\infty} \|Tx_{n_k} - x_{n_k} + x_{n_k} - Ty\| \\ &\geq \liminf_{k\to\infty} (\|x_{n_k} - Ty\| - \|Tx_{n_k} - x_{n_k}\|) \\ &= \liminf_{k\to\infty} \|x_{n_k} - Ty\| \\ &> \liminf_{k\to\infty} \|x_{n_k} - y\|.\end{aligned}$$

We have obtained a contradiction, what shows that the Lemma is true. ■

Lemma 38 is also known under the name *Opial's demi-closedness principle.*

Lemma 39 *Let $C \subset \mathcal{H}$ be a convex and closed subset and let a sequence (x_k) be Fejér monotone with respect to C. Then there exists the unique element $y^* \in C$ such that*

$$\lim_k \|x_k - y^*\| = \inf_{y\in C} \lim_k \|x_k - y\| \tag{6}$$

Proof. Let $d(y) = \lim_k \|x_k - y\|$ for $y \in C$, $\delta = \inf\{d(y) : y \in C\}$ and let the sequence $(y_m) \subset C$ be such that $d(y_m) \to \delta$. First we show that the sequence (y_m) is a Cauchy sequence. By the parallelogram law we obtain for all $k, m, l \in \mathbb{N}$

$$\begin{aligned} \|y_m - y_l\|^2 &= \|(x_k - y_m) - (x_k - y_l)\|^2 \\ &= 2\|x_k - y_m\|^2 + 2\|x_k - y_l\|^2 - \|(x_k - y_m) - (x_k - y_l)\|^2 \\ &= 2\|x_k - y_m\|^2 + 2\|x_k - y_l\|^2 - 4\|x_k - \frac{y_m + y_l}{2}\|^2 \end{aligned}$$

Obviously $\frac{y_m+y_l}{2} \in C$ since C is convex. Hence, we obtain by the Fejér monotonicity of (x_k) with respect to C that

$$\|x_k - \frac{y_m + y_l}{2}\| \geq \lim_k \|x_k - \frac{y_m + y_l}{2}\| = d(\frac{y_m + y_l}{2}) \geq \delta.$$

Now we see that

$$\|y_m - y_l\|^2 \leq 2\|x_k - y_m\|^2 + 2\|x_k - y_l\|^2 - 4\delta^2.$$

If we set $k \to \infty$ in the above inequality, we obtain in the limit

$$\|y_m - y_l\|^2 \leq 2d(y_m)^2 + 2d(y_l)^2 - 4\delta^2$$

Consequently,

$$\lim_{l,m\to\infty} \|y_m - y_l\|^2 = 0,$$

i.e., $(y_m) \subset \mathcal{H}$ is a Cauchy sequence. Since the Hilbert space is complete the sequence (y_m) converges to a point $y^* \in \mathcal{H}$. Since C is a closed subset $y^* \in C$. It is clear that $d(y^*) = \delta$ since by the triangle inequality we have for all $m \in \mathbb{N}$

$$\delta \leq d(y^*) = \lim_k \|x_k - y^*\| \leq \lim_k(\|x_k - y_m\| + \|y_m - y^*\|) = d(y_m) + \|y_m - y^*\| \underset{m\to\infty}{\to} \delta.$$

Now we show the uniqueness of $y^* \in C$ with the property $d(y^*) = \delta$. Suppose that $d(y') = \delta$ for some $y' \in C$. We have $\frac{y^*+y'}{2} \in C$, since C is a convex subset. Furthermore $d(\frac{y^*+y'}{2}) \geq \delta$ by the definition of δ. By the parallelogram law we have

$$\begin{aligned} \|y^* - y'\|^2 &= \|(x_k - y^*) - (x_k - y')\|^2 \\ &= 2\|x_k - y^*\|^2 + 2\|x_k - y'\|^2 - \|(x_k - y^*) + (x_k - y')\|^2 \\ &= 2\|x_k - y^*\|^2 + 2\|x_k - y'\|^2 - 4\|x_k - \frac{y^* + y'}{2}\|^2 \end{aligned}$$

and we obtain in the limit for $k \to +\infty$

$$\|y^* - y'\|^2 = 2d(y^*)^2 + 2d(y')^2 - 4d(\frac{y^* + y'}{2})^2 \leq 0$$

since $d(\frac{y^*+y'}{2}) \geq \delta$. Therefore $y' = y^*$, which proves the uniqueness of y^*. ■

Remark 40 Observe that the theorem of the existence of the metric projection of a point $x \in \mathcal{H}$ onto a convex and closed subset $C \subset \mathcal{H}$ follows directly from Lemma 39. To see it, it is enough to take $x_k = x$, $k = 1, 2, \ldots$.

3.2 Opial's Theorem

Recall that an operator $U : \mathcal{H} \to \mathcal{H}$ is *asymptotically regular* if for any $x \in \mathcal{H}$ $\lim_{k\to\infty} \|U^{k+1}x - U^k x\| = 0$ (see Definition 15).

Theorem 41 (Opial, 1967) *Let $U : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator with $\operatorname{Fix} U \neq \varnothing$. Furthermore, let U be asymptotically regular. If a sequence (x_k) is generated by an iterative procedure $x_k = U^k x$ then x_k converges weakly to an element $x_* \in \operatorname{Fix} U$ for any $x \in \mathcal{H}$.*

Zdzisław Opial (1930-1974)

Proof. Let $x \in \mathcal{H}$, $x_k = U^k x$ and let $z \in \operatorname{Fix} U$. Since U is a nonexpansive operator

$$\|x_{k+1} - z\| = \|U^{k+1}x - z\| = \|U^{k+1}x - Uz\| \le \|U^k x - z\| = \|x_k - z\|$$

i.e., (x_k) is Fejér monotone with respect to $\operatorname{Fix} U$, consequently, (x_k) is bounded. Let $y \in \mathcal{H}$ be a weak cluster point of (x_k). Let $(x_{n_k}) \subset (x_k)$ be a subsequence which is weakly convergent to the point y. Since U is an asymptotically regular operator

$$\|U^{k+1}x - U^k x\| = \|Ux_k - x_k\| \to 0$$

It follows from Lemma 38 that $y \in \operatorname{Fix} U$. Since U is nonexpansive, $\operatorname{Fix} U$ is closed and convex (see Corollary **??**). Let $y^* \in \operatorname{Fix} U$ be such that

$$\lim_k \|x_k - y^*\| = \inf_{y \in \operatorname{Fix} U} \lim_k \|x_k - y\|.$$

The existence and uniqueness of y^* follows from Lemma 39. We show that $x_{n_k} \rightharpoonup y^*$. Suppose that $y^* \neq y$. Then we obtain by Lemma 37 that

$$\lim_k \|x_k - y^*\| = \lim_k \|x_{n_k} - y^*\| > \lim_k \|x_{n_k} - y\| \ge \lim_k \|x_k - y^*\|$$

The contradiction shows that $y = y^*$. We have shown that y^* is the unique weak cluster point of any subsequence of (x_k). Consequently, $x_k \rightharpoonup y^*$ by the property (w2). ∎

3.3 Krasnoselskii–Mann Theorem

In one of the next sections we prove that for $\lambda \in (0, 2)$ a λ-RFNE operator U is asymptotically regular. Therefore, Theorem below is an immediate consequence of the Opial Theorem.

Theorem 42 (Krasnoselskii–Mann, 1953) *Let $T : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator with $\operatorname{Fix} T \neq \varnothing$ and let $x \in \mathcal{H}$. If $x_k = T_\lambda^k x$ for $\lambda \in (0, 2)$ then x_k converges weakly to an element $x_* \in \operatorname{Fix} T$.*

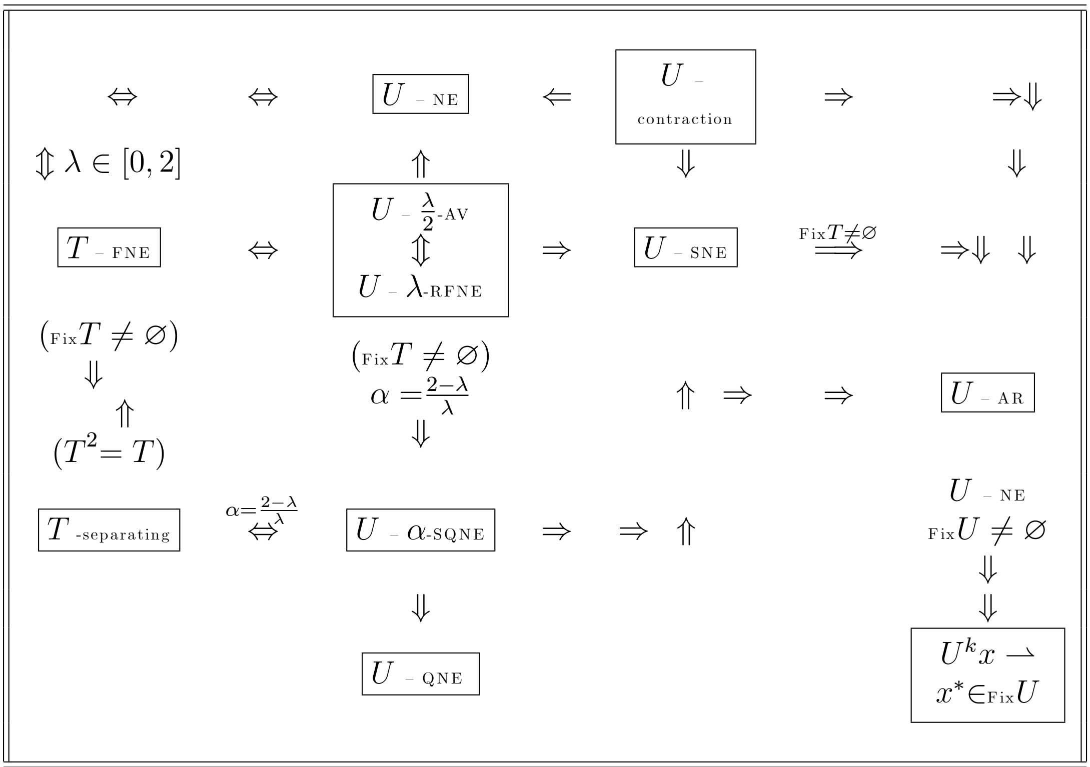
4 Algorithmic operators

In the next Section we present several methods for solving convex optimization problems. We focus our study of *iterative methods* (we also call them *iterative procedures* or *algorithms*) which are given in the form of recurrence $x_{k+1} = Tx_k$ that is defined on a closed and convex subset $X \subset \mathcal{H}$, where $T : X \to X$. We suppose that the starting point x_0 is an element of a starting subset $X_0 \subset X$. Usually, one supposes that $X_0 = X$. A sequence generated by an iterative method is called *approximating sequence*. Any iterative method for solving a convex optimization problem will be constructed in such a way, that the approximating sequences (x_k) generated by this method converge (at least weakly) to an optimal solution of the optimization problem. As we will see, the optimal solution is a fixed point of the operator $T : X \to \mathcal{H}$. The form of this operator depends on the considered optimization problem.

In this Chapter we deal with general properties of operators which define algorithms for solving convex optimization problem. In one iteration of the algorithm an operator $U : X \to X$ defines an actualization x^+ of the current approximation x of a solution of the convex optimization problem. Usually, this actualization has the form $x^+ = Ux$. We call this operator *algorithmic operator*. One can also consider algorithms, where an actualization has the form $x^+ \in Ux$ for a mapping (multifunction) $U : X \rightrightarrows X$. In this case the mapping U is called *algorithmic mapping*.

An operator defining the iteration of an algorithm can depend on some parameters which are constant or vary during the iteration process. The properties of approximation sequences depend on the properties of algorithmic operators defining the iterative method.

Let $T : D \to \mathcal{H}$ i $U = I + \lambda(T - I)$, $\lambda \in (0, 2)$. In the sequel we will prove that there hold the following correspondences between the defined algorithmic operators



4.1 Properties of a firmly nonexpansive operator

4.1.1 Correspondences between FNE and NE and FM operators

Theorem 43 *A firmly nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$ is monotone and nonexpansive.*

Proof. Let T be firmly nonexpansive. We have by the Schwarz inequality

$$\|Tx - Ty\| \cdot \|x - y\| \geq \langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \geq 0.$$

The Theorem follows now easily from the above inequalities. ■

Theorem 44 *Let $T : X \to \mathcal{H}$ be an operator which has a fixed point. If T is firmly nonexpansive, then T is a separator, i.e.*

$$\langle z - Tx, x - Tx\rangle \leq 0 \tag{7}$$

for all $x \in X$ and $z \in \operatorname{Fix} T$. If, furthermore, T is an idempotent operator, then the converse implication also holds.

Proof. Let $x \in \mathcal{H}$ and let $z \in \operatorname{Fix} T$ for a firmly nonexpansive operator T. We have

$$\begin{aligned}\langle x - Tx, z - Tx\rangle &= \langle x - z + z - Tx, Tz - Tx\rangle \\ &= -\langle z - x, Tz - Tx\rangle + \langle Tz - Tx, Tz - Tx\rangle \\ &\leq -\|Tz - Tx\|^2 + \|Tz - Tx\|^2 \\ &= 0,\end{aligned}$$

i.e. T is separating.

Now, suppose that T is an idempotent operator and that inequality (7) holds for all $x \in X$ and $z \in \operatorname{Fix} T$. Let $u, v \in X$. Taking $x = u$ and $z = Tv$ in (7) we get

$$\langle Tv - Tu, u - Tu\rangle \leq 0$$

and taking $x = v$ and $z = Tu$ in (7) we get

$$\langle Tu - Tv, v - Tv\rangle \leq 0,$$

since in both cases $z \in \operatorname{Fix} T$. After adding the inequalities above we get

$$\langle Tu - Tv, (Tu - Tv) - (u - v)\rangle \leq 0,$$

i.e.

$$\|Tu - Tv\|^2 \leq \langle Tu - Tv, u - v\rangle$$

and we see that T is firmly nonexpansive. ■

Corollary 45 *Let $D \subset \mathcal{H}$ be nonempty, convex and closed. Then the metric projection P_D is a firmly nonexpansive operator. Consequently, the metric projection P_D is monotone and nonexpansive.*

Proof. Since the metric projection is an idempotent operator we obtain, by the characterization of the metric projection (Theorem 18), that for all $x \in \mathcal{H}$ and for all $z \in D = \operatorname{Fix} P_D$

$$\langle x - P_D x, z - P_D x\rangle \leq 0$$

i.e., there are satisfied the conditions of the second part of Theorem 44. Consequently, P_D is a firmly nonexpansive operator. The second part of the Corollary follows now from Theorem 43. ■

Lemma 46 *Let $T : \mathcal{H} \to \mathcal{H}$ and let $T_\lambda = \lambda T + (1 - \lambda)I$, where $\lambda > 0$, be a relaxation of the operator T. The following conditions are equivalent:*

(i) *T is a firmly nonexpansive operator,*

(ii) *T_λ is a nonexpansive operator for any $\lambda \in [0, 2]$,*

(iii) *T has the form $T = \frac{1}{2}(S + I)$, where $S : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator,*

(iv) *$I - T$ is a firmly nonexpansive operator.*

Proof.

(i)⇒(ii) Suppose that T is a firmly nonexpansive operator. Then for any $\lambda \in [0, 2]$ and for the operator $T_\lambda = \lambda T + (1 - \lambda)I$ we obtain by the definition of a firmly nonexpansive operator, by the Schwarz inequality and by the monotonicity of T (see Theorem 43)

$$\begin{aligned}
\|T_\lambda x - T_\lambda y\|^2 &= \|\lambda Tx + (1-\lambda)x - \lambda Ty - (1-\lambda)y\|^2 \\
&= \|\lambda(Tx - Ty) + (1-\lambda)(x - y)\|^2 \\
&= \lambda^2(\|Tx - Ty\|^2 - \langle Tx - Ty, x - y\rangle) \\
&\quad +(2\lambda - \lambda^2)\langle Tx - Ty, x - y\rangle + (1-\lambda)^2\|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\langle Tx - Ty, x - y\rangle + (1-\lambda)^2\|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\|Tx - Ty\|\|x - y\| + (1-\lambda)^2\|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\|x - y\|^2 + (1-\lambda)^2\|x - y\|^2 \\
&= \|x - y\|^2.
\end{aligned}$$

We have obtained that T_λ is a nonexpansive operator.

(ii)⇒(iii) By the assumption, the operator $T_2 = 2T - I$ is nonexpansive. Since $T = \frac{1}{2}[(2T - I) + I]$ the implication is obvious.

(iii)⇒(iv) Let S be a nonexpansive operator, $T = \frac{1}{2}(S + I)$ and let $G = I - T$. Then we have $G = \frac{1}{2}(I - S)$ and

$$\begin{aligned}
\|Gx - Gy\|^2 &= \langle Gx - Gy, x - y\rangle + \langle Gx - Gy, (Gx - Gy) - (x - y)\rangle \\
&= \langle Gx - Gy, x - y\rangle \\
&\quad +\frac{1}{4}\langle (Sx - Sy) - (x - y), (Sx - Sy) + (x - y)\rangle \\
&= \langle Gx - Gy, x - y\rangle + \frac{1}{4}(\|Sx - Sy\|^2 - \|x - y\|^2) \\
&\le \langle Gx - Gy, x - y\rangle.
\end{aligned}$$

(iv)⇒(i) Let $G = I - T$ be a firmly nonexpansive operator, i.e.,

$$\langle Gx - Gy, x - y\rangle \ge \|Gx - Gy\|^2.$$

The above inequality is equivalent to

$$\langle Gx - Gy, (x - Gx) - (y - Gy)\rangle \ge 0$$

or to

$$\langle (x - Tx) - (y - Ty), Tx - Ty\rangle \ge 0$$

which is equivalent to

$$\langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2,$$

i.e., T is a firmly nonexpansive operator. ■

Remark 47 One can also prove that any condition (i)-(iv) in Lemma 46 is equivalent to any of the following conditions

(v) for all $x, y \in X$

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \tag{8}$$

(vi) for all $x, y \in X$ and for any $\alpha \geq 0$

$$\|Tx - Ty\| \leq \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|.$$

Some authors define the FNE-operators as operators satisfying (v) or (vi).

Corollary 48 *Let $S : \mathcal{H} \to \mathcal{H}$. The following conditions are equivalent:*

(i) *S is a nonexpansive operator,*

(ii) *$S = 2F - I$, where $F : \mathcal{H} \to \mathcal{H}$ is a firmly nonexpansive operator.*

Proof.
(ii)⇒(i) Let $S = 2F - I$ for a firmly nonexpansive operator F. It follows from the implication (i)⇒(ii) in Lemma 46 that S is a nonexpansive operator.
(i)⇒(ii) Let $S = 2F - I$ be a nonexpansive operator. We have $F = \frac{1}{2}(S + I)$. By the implication (iii)⇒(i) in Lemma 46 the operator F firmly nonexpansive. ■

Corollary 49 *An operator $U : \mathcal{H} \to \mathcal{H}$ is averaged if and only if U is a strict relaxation of a firmly nonexpansive operator.*

Proof. Let U be averaged. Then we have by Corollary 48, for a nonexpansive operator S and for a constant $\alpha \in (0, 1)$,

$$U = (1 - \alpha)I + \alpha S = (1 - \alpha)I + \alpha(2F - I) = (1 - \lambda)I + \lambda F$$

where F is FNE and $\lambda = 2\alpha \in (0, 2)$. Let now $U = (1 - \lambda)I + \lambda F$ for a FNE operator F and for $\lambda \in (0, 2)$. Then, by the implication (i)⇒(iii) in Lemma 46

$$U = (1 - \lambda)I + \frac{1}{2}\lambda(I + S) = (1 - \frac{\lambda}{2}I) + \frac{\lambda}{2}S$$

for a nonexpansive operator S, i.e. U is averaged. ■

Corollary 50 *A convex combination of nonexpansive operators is nonexpansive. A convex combination of firmly nonexpansive operators is firmly nonexpansive.*

Proof. Let $w = (\omega_1, ..., \omega_m)^\top \in \Delta_m$ be a vector of weights. If S_i, $i \in I$, are nonexpansive operators and $S = \sum_{i \in I} \omega_i S_i$ then, by the convexity of the norm $\| \cdot \|$, we have for any $x, y \in \mathcal{H}$

$$\begin{aligned} \|Sx - Sy\| &= \|\sum_{i \in I} \omega_i(S_i x - S_i y)\| \\ &\leq \sum_{i \in I} \omega_i \|S_i x - S_i y\| \\ &\leq \sum_{i \in I} \omega_i \|x - y\| = \|x - y\|, \end{aligned}$$

i.e., S is a nonexpansive operator. Let now T_i, $i \in I$, be firmly nonexpansive and let $T = \sum_{i \in I} \omega_i T_i$. By the implication (i)⇒(iii) in Lemma 46 we have $T_i = \frac{1}{2}(S_i + I)$, for a nonexpansive operator S_i, $i \in I$. The Corollary follows now from the equality $T = \frac{1}{2}(S + I)$ and from the implication (iii)⇒(i) in Lemma 46. ■

Corollary 51 *A convex combination of relaxed firmly nonexpansive operators is relaxed firmly nonexpansive.*

Proof. Let $T_i = I + \lambda_i(U_i - I)$, where the operators U_i are FNE, $\lambda_i \in [0,2]$, $i \in I$, and let $w = (\omega_1, ..., \omega_m) \in \Delta_m$. It is clear that $\lambda = \sum_{j=1}^m \omega_j\lambda_j \in [0,2]$ and $\sum_{i=1}^m \frac{\omega_i\lambda_i}{\sum_{j=1}^m \omega_j\lambda_j} = 1$. Therefore, $U = \sum_{i=1}^m \frac{\omega_i\lambda_i}{\sum_{j=1}^m \omega_j\lambda_j} U_i$ is FNE as a convex combination of FNE operators U_i (see Corollary 50). Let $T = \sum_{i=1}^m \omega_i T_i$. We have

$$\begin{aligned} T &= \sum_{i=1}^m \omega_i[I + \lambda_i(U_i - I)] = I + \sum_{i=1}^m \omega_i\lambda_i(U_i - I) \\ &= I + (\sum_{j=1}^m \omega_j\lambda_j)(\sum_{i=1}^m \frac{\omega_i\lambda_i}{\sum_{j=1}^m \omega_j\lambda_j} U_i - \sum_{i=1}^m \frac{\omega_i\lambda_i}{\sum_{j=1}^m \omega_j\lambda_j} I) = I + \lambda(U - I) \blacksquare \end{aligned}$$

■

Theorem 52 *Let $T : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator with $\operatorname{Fix} T \neq \varnothing$, $\lambda \in (0,2)$ and let $T_\lambda = (1-\lambda)I + \lambda T$ be a relaxation of T. Then T_λ is $\frac{2-\lambda}{\lambda}$-SQNE, i.e.*

$$\|T_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2$$

for all $x \in \mathcal{H}$ and for all $z \in \operatorname{Fix} T$.

Proof. Since a FNE-operator having a fixed point is separating (see Theorem 44), by the properties of the scalar product and by the obvious equality $Tx - x = \frac{1}{\lambda}(T_\lambda x - x)$, we obtain for all $x \in \mathcal{H}$ and for all $z \in \operatorname{Fix} T$

$$\begin{aligned} \|T_\lambda x - z\|^2 &= \|(x - z) + \lambda(Tx - x)\|^2 \\ &= \|x - z\|^2 + \lambda^2\|Tx - x\|^2 - 2\lambda\langle z - x, Tx - x\rangle \\ &= \|x - z\|^2 + \lambda^2\|Tx - x\|^2 - 2\lambda\|Tx - x\|^2 + 2\lambda\langle z - Tx, x - Tx\rangle \\ &\leq \|x - z\|^2 - \lambda(2-\lambda)\|Tx - x\|^2 \\ &= \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2 \end{aligned}$$

■

Theorem 53 *A composition of relaxed firmly nonexpansive operators is relaxed firmly nonexpansive.*

Proof. Let $T, U : \mathcal{H} \to \mathcal{H}$ be FNE and let $\lambda, \mu \in (0,2)$. It follows from Lemma 46 and from Corollary 50 that

$$U \circ T_\lambda = (1 - \frac{2+\lambda}{2})I + \frac{2+\lambda}{2} \cdot \frac{1}{2}[\overbrace{\frac{\lambda}{2+\lambda}\overbrace{(2T - I)}^{NE} + \frac{2}{2+\lambda}\overbrace{\overbrace{(2U - I)}^{NE}\circ \overbrace{T_\lambda}^{NE}}^{NE}}^{NE} + I]$$

i.e., $U \circ T_\lambda$ is a γ-relaxation of the FNE-operator

$$S = \frac{1}{2}[\frac{\lambda}{2+\lambda}(2T - I) + \frac{2}{2+\lambda}(2U - I) \circ T_\lambda + I]$$

with$\gamma = 1 + \frac{\lambda}{2}$. Now we see that

$$U_\mu \circ T_\lambda = [(1-\mu)I + \mu U] \circ T_\lambda = (1-\mu)T_\lambda + \mu U \circ T_\lambda,$$

i.e., $U_\mu \circ T_\lambda$ is RFNE as a convex combination of RFNE operators T_λ and $U \circ T_\lambda$ (see Corollary 51). Furthermore, it follows from the proof of Corollary 51 that $U_\mu \circ T_\lambda$ is ν-RFNE, where $\nu = \mu + \lambda - \frac{\mu\lambda}{2}$.
■

Remark 54 A composition of firmly nonexpansive operators needs not to be firmly nonexpansive.

Example 55 Let $A = \{x \in \mathbb{R}^2 : \xi_2 = 0\}$ and $B = \{x \in \mathbb{R}^2 : \xi_1 = \xi_2\}$. For $x = (2, 2)^\top$ and $y = (4, 0)^\top$ we have for a composition $T = P_B P_A$ of FNE operators P_A and P_B: $Tx = (1, 1)^\top$ and $Ty = (2, 2)^\top$. Therefore, $\langle Tx - Ty, x - y\rangle = 0 < 2 = \|Tx - Ty\|^2$ and we see that T is not FNE.

Theorem 56 *Let $T : \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator, $C \subset \mathcal{H}$ be convex and closed and let $\lambda \in (0, 2)$. Then*

(i) *$P_C T_\lambda$ is nonexpansive.*

(ii) *If $\operatorname{Fix}(P_C T) \neq \varnothing$ then the operator $P_C T_\lambda$ is $\frac{2-\lambda}{2+\lambda}$-SQNE, i.e.*

$$\|P_C T_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2 - \lambda}{2 + \lambda}\|P_C T_\lambda x - x\|^2 \tag{9}$$

Proof.

(i) By Lemma 46 the operator $T_\lambda = (1 - \lambda)I + \lambda T$ is nonexpansive. The nonexpansivity of $P_C T_\lambda$ follows now from the nonexpansivity of the metric projection P_C (Corollary 45) and from the obvious fact that the composition of nonexpansive operators is nonexpansive.

(ii) It follows from the proof of Theorem 53 that $P_C T_\lambda$ is ν-RFNE operator with $\nu = 1+\frac{\lambda}{2} \in (1, 2)$. Theorem 52 yields now

$$\begin{aligned}\|P_C T_\lambda x - z\|^2 &\leq \|x - z\|^2 - \frac{2 - \nu}{\nu}\|P_C T_\lambda x - x\|^2 \\ &= \|x - z\|^2 - \frac{2 - \lambda}{2 + \lambda}\|P_C T_\lambda x - x\|^2.\end{aligned}$$

■

4.2 Asymptotically regular operators

Definition 57 An operator $U : X \to X$ is called *asymptotically regular* (AR) if

$$\lim_{k\to\infty} \|U^{k+1}x - U^k x\| = 0.$$

for all $x \in X$.

Remark 58 It is clear, that any idempotent operator is asymptotically regular. In particular, the metric projection onto a nonempty, closed and convex subset $C \subset \mathcal{H}$ is asymptotically regular.

Since the notion of an asymptotically regular operators plays an important role in iterative methods for finding fixed points of operators, we give below some theorems which are useful in the construction of asymptotically regular operators.

Theorem 59 *Let $U : X \to \mathcal{H}$ be an operator with nonempty* $\operatorname{Fix} U$*. If U is strongly quasi-nonexpansive, then U is asymptotically regular.*

Proof. Let U be strongly quasi-nonexpansive, let $x \in X$ and let $z \in \operatorname{Fix} U$. For $x_k = U^k x$ and for some constant $\alpha > 0$, we have

$$\|x_{k+1} - z\|^2 = \|Ux_k - z\|^2 \leq \|x_k - z\|^2 - \alpha\|Ux_k - x_k\|^2.$$

Consequently, the sequence $(\|x_k - z\|)$ is monotone and therefore, it converges. By setting $k \to \infty$ in the above inequality, we obtain in the limit

$$\|U^{k+1}x - U^k x\|^2 = \|Ux_k - x_k\| \to 0,$$

i.e. U is asymptotically regular. ■

Corollary 60 *Let $C \subset \mathcal{H}$ be nonempty, closed and convex and let $T : X \to \mathcal{H}$ be a firmly nonexpansive operator with* $\operatorname{Fix}(P_C T) \neq \varnothing$*. Then, for any $\lambda \in (0, 2)$, the projected relaxation $R_\lambda = P_C T_\lambda$ of the operator T is asymptotically regular.*

Proof. Let $\lambda \in (0, 2)$, let $x \in X$ and let $x_k = R_\lambda^k x$, $k = 1, 2, \ldots$. By Theorem 56(iii) we have

$$\|x_{k+1} - z\|^2 = \|R_\lambda x_k - z\|^2 \leq \|x_k - z\|^2 - \frac{2-\lambda}{2+\lambda}\|R_\lambda x_k - x_k\|^2$$

for all $z \in \operatorname{Fix}(P_C T)$. Since $\operatorname{Fix}(P_C T) = \operatorname{Fix} R_\lambda$ (see, Theorem 56(ii)), the inequality above says that the operator R_λ is strongly quasi-nonexpansive. The asymptotic regularity of R_λ follows now from Theorem 59. ■

Corollary 61 *Let $T : X \to \mathcal{H}$ be a firmly nonexpansive operator with* $\operatorname{Fix} T \neq \varnothing$*. Then, for any $\lambda \in (0, 2)$, the relaxation T_λ of the operator T is asymptotically regular.*

Corollary 62 *Let $T_i : X \to \mathcal{H}$, $i = 1, \ldots, m$, be relaxed firmly nonexpansive and let the composition $T = T_1 \ldots T_m$ have a fixed point. Then T is asymptotically regular.*

Proof. It follows from Theorem 53 that T is relaxed firmly nonexpansive. Therefore, the asymptotic regularity of T follows from Theorem 59. ■

5 Projection methods

5.1 Cyclic projection method (ART) for CFP

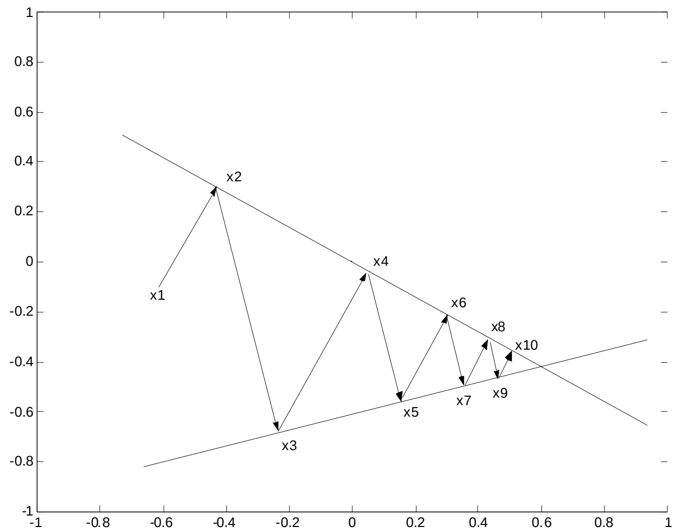
Definition 63 Let $C_i \subset \mathcal{H}$ be nonempty, convex and closed subset, $i \in I$. The operator

$$T = P_{C_m} \circ P_{C_{m-1}} \circ ... \circ P_{C_1}$$

is said to be the *cyclic projection operator*.

The cyclic projection method (CPM) has the form $x_k = T^k x$ for a cyclic projection operator T and for $x \in \mathcal{H}$. If C_i are hyperplanes, the cyclic projection method is also known also under the name the Kaczmarz method or the algebraic reconstruction technique (ART).

- Stefan Kaczmarz (1937) – the cyclic projections method for a nonsingular system of linear equations



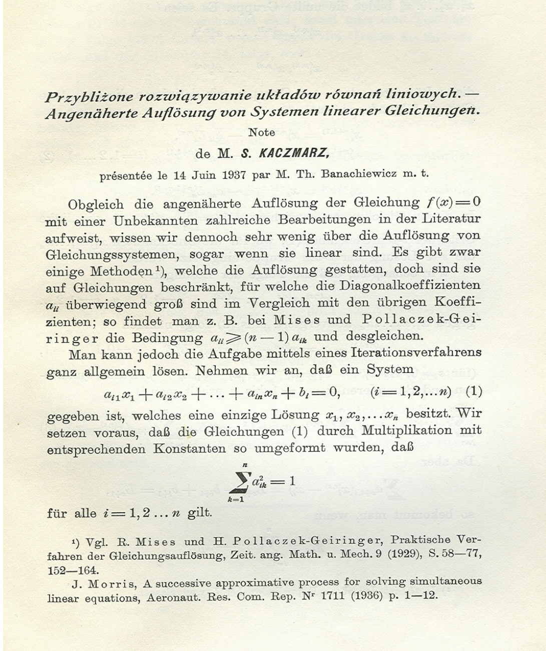
The Kaczmarz method

Stefan Kaczmarz (1895 - 1939)
polish mathematician, associated professor of the Technical University in Lwów, where he has collaborated with Stefan Banach and with Hugo Steinhaus. The area of his research was algebra, theory of real functions, Fourier series, orthogonal series. Author of 35 books and articles, member of the Lwów School of Mathematics Died in a battle in the World War II. His Kaczmarz Method provided the basis for many modern imaging technologies, including the computerized tomography.

Photo from the article:
Stefan Kaczmarz (1895-1939) by L. Maligranda

Mathematicians from Lwów (1930)
(2 – S. Banach, 4 – K. Kuratowski, 5 – S. Kaczmarz, 6 – J. Schauder, 10 – S. Ulam)

1937 **Stefan Kaczmarz**
has published the paper:
Angenährte Auflösung
von Systemen linearer Gleichungen
Bull. Intern. Acad. Polonaise Sci. Lett., Cl. Sci. Math. Nat. A, **35** (1937) 355-357.
translated into English in1993:
S. Kaczmarz, Approximate solution of systems of linear equations, *International Journal of Control* **57** (1993) 1269-1271



Przybliżone rozwiązywanie układów równań liniowych. — Angenäherte Auflösung von Systemen linearer Gleichungen.

Note

de M. **S. KACZMARZ,**

présentée le 14 Juin 1937 par M. Th. Banachiewicz m. t.

Obgleich die angenäherte Auflösung der Gleichung $f(x)=0$ mit einer Unbekannten zahlreiche Bearbeitungen in der Literatur aufweist, wissen wir dennoch sehr wenig über die Auflösung von Gleichungssystemen, sogar wenn sie linear sind. Es gibt zwar einige Methoden[1]), welche die Auflösung gestatten, doch sind sie auf Gleichungen beschränkt, für welche die Diagonalkoeffizienten a_{ii} überwiegend groß sind im Vergleich mit den übrigen Koeffizienten; so findet man z. B. bei Mises und Pollaczek-Geiringer die Bedingung $a_{ii} \geqslant (n-1) a_{ik}$ und desgleichen.

Man kann jedoch die Aufgabe mittels eines Iterationsverfahrens ganz allgemein lösen. Nehmen wir an, daß ein System

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + b_i = 0, \qquad (i=1,2,\ldots n) \quad (1)$$

gegeben ist, welches eine einzige Lösung $x_1, x_2, \ldots x_n$ besitzt. Wir setzen voraus, daß die Gleichungen (1) durch Multiplikation mit entsprechenden Konstanten so umgeformt wurden, daß

$$\sum_{k=1}^{n} a_{ik}^2 = 1$$

für alle $i = 1, 2 \ldots n$ gilt.

[1]) Vgl. R. Mises und H. Pollaczek-Geiringer, Praktische Verfahren der Gleichungsauflösung, Zeit. ang. Math. u. Mech. 9 (1929), S. 58—77, 152—164.
J. Morris, A successive approximative process for solving simultaneous linear equations, Aeronaut. Res. Com. Rep. Nr 1711 (1936) p. 1—12.

The first page of the Kaczmarz paper

- Bregman (1965) – CPM for CFP ($\lambda = 1$)
- Gurin–Polyak–Raik (1967) – CPM for CFP ($\lambda \in (0, 2)$)
- Gordon–Bender–Herman (1970) – application of the Kaczmarz method in radiology
- Tanabe (1971) – a generalization of the Kaczmarz method for arbitrary (also inconsistent) system of linear equations
- McCormick (1977) – the Kaczmarz method in a Hilbert space
- Censor (1981) – almost cyclic control
- Bauschke–Borwein (1996) – general model for projection methods
- Popa–Zdunek (2004) – extended Kaczmarz method
- Haller–Szwarc (2005) – the Kaczmarz method in a Hilbert space
- Herman (since 1970) – application of the Kaczmarz method in the computerized tomography and in the intensity-modulated radiation therapy

Theorem 64 *Let $C_i \subset \mathcal{H}$ be nonempty, convex and closed subset, $i \in I$, and let $T = P_{C_m} P_{C_{m-1}} ... P_{C_1} : C_m \to C_m$ be the cyclic projection operator. If $\bigcap_{i \in I} C_i \neq \varnothing$, then*

$$\operatorname{Fix} T = \bigcap_{i \in I} C_i.$$

Proof. The inclusion $\bigcap_{i \in I} C_i \subset \operatorname{Fix} T$ is obvious. We show the converse inclusion. Since $\operatorname{Fix} P_{C_i} = C_i$ we have for $z \in \bigcap_{i \in I} C_i$

$$Tz = P_{C_m} P_{C_{m-1}} ... P_{C_1} z = z,$$

i.e., $z \in \operatorname{Fix} T$. Suppose now that $\bigcap_{i \in I} C_i \neq \varnothing$, $x \in \operatorname{Fix} T$ and that $x \notin \bigcap_{i \in I} C_i$. Let $j = \min\{i \in I : x \notin C_i\}$. Then we have $P_{C_i} ... P_{C_1} x \in C_i$ for $i < j$ and $P_{C_j} ... P_{C_1} x \notin C_j$. Since the metric projection P_{C_i} is strictly Fejér monotone with respect to C_i, $i \in I$, we have for any $z \in \bigcap_{i \in I} C_i$

$$\begin{aligned} \|x - z\| &= \|Tx - z\| \\ &= \|P_{C_m} P_{C_{m-1}} ... P_{C_1} x - z\| \\ &\leq ... \leq \|P_{C_j} P_{C_{j-1}} ... P_{C_1} x - z\| \\ &< \|P_{C_{j-1}} ... P_{C_1} x - z\| \\ &= \|P_{C_{j-2}} ... P_{C_1} x - z\| \\ &= ... = \|x - z\| \end{aligned}$$

The contradiction shows that $\operatorname{Fix} T = \bigcap_{i \in I} C_i$. ■

Corollary 65 *Let $C_i \subset \mathcal{H}$ be nonempty, convex and closed subset, $i \in I$. The cyclic projection operator $T = P_{C_m} P_{C_{m-1}} ... P_{C_1} : C_m \to C_m$ is nonexpansive.*

Proof. T is nonexpansive as a composition of nonexpansive operators P_{C_i}, $i \in I$. ■

Theorem 66 *Let $C_i \subset \mathcal{H}$ be nonempty, convex and closed subset, $i \in I$ and let $T = P_{C_m} P_{C_{m-1}} ... P_{C_1} : C_m \to C_m$ be the cyclic projection operator.*

(i) *T is a relaxation of a firmly nonexpansive operator.*

(ii) *If at least one of C_i, $i \in I$, is bounded, then has a fixed point $z \in C_m$.*

Proof. (i) The property follows from Theorem 53.

(ii) We can suppose for the simplicity that C_m is a bounded subset. We see that T maps a closed, convex and bounded subset into itself. Furthermore, T is a nonexpansive operator (see Corollary 65). It follows from the Browder fixed point theorem that T has a fixed point $z \in C_m$. ■

Corollary 67 *Let $C_i \subset \mathcal{H}$ be nonempty, convex and closed subset, $i \in I$, and let $T = P_{C_m} P_{C_{m-1}} ... P_{C_1} : C_m \to C_m$ be the cyclic projection operator with $\operatorname{Fix} T \neq \varnothing$. Then for any $x \in \mathcal{H}$ the sequence $x_k = T^k x$ converges weakly to an element $z \in \operatorname{Fix} T$*

Proof. By Corollary 62 T is asymptotically regular. The Corollary follows now from the Opial Theorem. ■

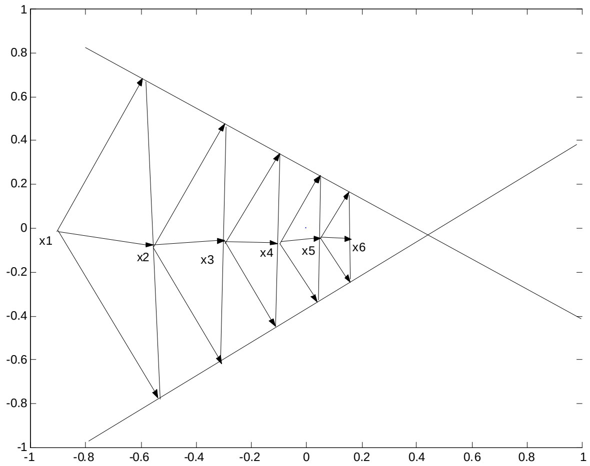
5.2 Simultaneous projection method (SPM) for CFP

The simultaneous projection method (SPM) has the form

$$x_{k+1} = x_k + \lambda_k \Big(\sum_{i \in I} \omega_i P_{C_i} x_k - x_k\Big), \tag{10}$$

where $\lambda_k \in (0, 2)$ is a relaxation parameter and $w = (\omega_1, ..., \omega_m)^\top \in \Delta_m$ is a vector of weights

- Cimmino, (1938) – SPM for a system of linear equations,



- Auslender, (1976) – SPM for CFP ($\lambda = 1$ and equal weights),
- Censor–Elfving (1982)
- Pierra (1988) – SPM as the alternating projection method in a product Hilbert space
- Iusem–de Pierro (1987) – SPM for CFP ($\lambda \in (0, 2)$ and $w > 0$,
- Aharoni–Censor (1989),
- Butnariu–Censor, (1990) – $\lambda \in (0, 2)$ and weights w_k depending on iteration, $C \neq \varnothing$
- Flåm–Zowe (1990) – $\lambda \in (0, 2)$ and weights w_k depending on iteration, $C \neq \varnothing$
- Combettes (1994) – $\lambda_k \in [\varepsilon, 2 - \varepsilon]$, $\varepsilon > 0$
- Bauschke–Borwein (1996) – a general model for SPM, $C \neq \varnothing$

Define the simultaneous projection operator $T : \mathcal{H} \to \mathcal{H}$ by equality

$$T = \sum_{i \in I} \omega_i P_{C_i}$$

and its relaxation $T_\lambda = I + \lambda(T - I)$. The SPM can be written in the form $x_{k+1} = T_\lambda x_k$.

- The operator T is firmly nonexpansive as a convex combination of firmly nonexpansive operators P_{C_i}, $i \in I$ (Corollary 50)

Define the proximity function $f : \mathcal{H} \to \mathbb{R}$ by equality

$$f(x) = \frac{1}{2} \sum_{i \in I} \omega_i \|P_{C_i} x - x\|^2.$$

Lemma 68 *There holds the equality*

$$\operatorname{Fix} T = \operatorname*{Argmin}_{x \in \mathcal{H}} f(x)$$

and $\operatorname{Fix} T \neq \varnothing$ *if at least one* C_i *is bounded and the corresponding weight* $\omega_i > 0$.

Proof. Since f is convex and differentiable we have

$$\begin{aligned} x &\in \operatorname*{Argmin}_{x\in\mathcal{H}} f(x) \Longleftrightarrow Df(x)=0 \\ &\Longleftrightarrow \sum_{i\in I}\omega_i(x-P_{C_i}x)=0 \\ &\Longleftrightarrow x=\sum_{i\in I}\omega_i P_{C_i}x \\ &\Longleftrightarrow x\in \operatorname{Fix} T \\ &\Longleftrightarrow x\in \operatorname{Fix} T_\lambda. \end{aligned}$$

Furthermore, if at least one C_i is bounded and the corresponding weight $\omega_i > 0$ then f is coercive, consequently $\operatorname{Argmin}_{x\in\mathcal{H}} f(x) \neq \varnothing$. ■

There holds the inclusion

$$C \subset \operatorname{Fix} T.$$

Let $w > 0$. If $C = \bigcap_{i\in I} C_i \neq \varnothing$ then

$$C = \operatorname{Fix} T \text{ and } \min_{x\in\mathcal{H}} f(x) = 0$$

Theorem 69 *Let* $\operatorname{Fix} T \neq \varnothing$*. Then for any* $z \in \operatorname{Fix} T$ *there holds the inequality*

$$\|x_{k+1}-z\|^2 \le \|x_k - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x_k - x_k\|^2. \tag{11}$$

Consequently $x_k \rightharpoonup x_* \in \operatorname{Fix} T$

Proof. Since T is firmly nonexpansive inequality (11) follows from Corollary 52. Consequently, $\|T_\lambda x_k - x_k\| \to 0$, i.e., (x_k) is asymptotically regular. Furthermore, T_λ is nonexpansive by Lemma 46. Therefore, the convergence follows from the Opial theorem. ■

5.3 Surrogate constraints method (SCM) for LFP

Consider the linear feasibility problem (LFP) in the form

$$\text{Find } x \in \mathbb{R}^n \text{ with } Ax \le b,$$

where A is a real matrix of type $m \times n$ with rows $a_i = (a_{i1}, ..., a_{in})^\top$ and $b = (\beta_1, ..., \beta_m)^\top \in \mathbb{R}^m$, and suppose that $M = \{x \in \mathbb{R}^n : A^\top x \le b\} \neq \varnothing$. Of course the LFP is a particular case of the CFP with

$$C_i = H_i = \{x \in \mathbb{R}^n : a_i^\top x \le \beta_i\}.$$

Multiply the particular inequalities by weights $\omega_i \ge 0$ and add the formed inequalities. We obtain so called *surrogate inequality*

$$(Aw)^\top x \le w^\top b,$$

where $w = (\omega_1, ..., \omega_m)^\top \in \Delta_m$. Of course

$$M \subset H_w = \{x \in \mathbb{R}^n : (Aw)^\top x \le w^\top b\}.$$

Let $\bar{x} \notin M$ be the current approximation of a solution $x_* \in M$. Let $r = (\rho_1, ..., \rho_m)^\top = A^\top \bar{x} - b$ be the residual vector (for $\bar{x}$). Suppose that $r^\top w > 0$ (e.g., the weights for nonviolated constraints vanish: $\omega_i = 0$ for $\rho_i \le 0$, $i \in I$). In this case $\bar{x} \notin H_w$. One iteration of the *surrogate constraints method* (SCM) has the form

$$x_{k+1} = x_k + \lambda_k(P_{H_{w_k}} x_k - x_k),$$

where $\lambda \in (0, 2)$ and the weights for nonviolated constraints vanish.

Lemma 70 *There holds the following inequality*

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 - \lambda(2-\lambda)\|P_{H_{w_k}} x_k - x_k\|^2,$$

i.e., the SCM is strongly Fejér monotone with respect to M.

Theorem 71 (Yang–Murty, 1992) *If all positive weights are greater than some constant $\gamma > 0$ then for any starting point x_1 the sequence (x_k) generated by the SCM converges to a solution $x_* \in M$.*

One can prove that the SPM can be described as a "short step" version of the SCM (AC, 2005). Consequently,

- the SCM produces longer steps than the SPM and
- behaves numerically better than the SPM if $M \neq \varnothing$.

If $M = \varnothing$ then the SPM converges to $\operatorname{Fix} T$, for $T = \sum_{i \in I} \omega_i P_{H_i}$ but the SCM diverges.

5.4 CQ-method for the SFP

Consider the split feasibility problem in the form:

$$\text{Find } x \in C \text{ with } Ax \in Q$$

if such x exists, where $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are closed and convex and A is a real matrix of type $m \times n$. Define an operator $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$T(x) = x + \frac{1}{\lambda_{\max}(A^\top A)} A^\top (P_Q - I)Ax$$

and its projected relaxation $R_\lambda = P_C T_\lambda$ for $\lambda \in (0, 2)$, i.e.

$$R_\lambda x = P_C\Big(x + \frac{\lambda}{\lambda_{\max}(A^\top A)} A^\top (P_Q - I)Ax\Big), \tag{12}$$

Furthermore, define the proximity function $f : C \to \mathbb{R}$,

$$f(x) = \frac{1}{2}\|P_Q(Ax) - Ax\|^2.$$

Lemma 72 (Byrne, 2002) *There holds the equality*

$$\operatorname{Fix} R_\lambda = \operatorname*{Argmin}_C f.$$

Proof. (Compare [Byr02, Proposition 2.1]) Observe that f is differentiable and that

$$\nabla f(x) = A^\top (P_Q(Ax) - Ax)$$

There holds the following equivalences

$$\begin{aligned}
x &\in \operatorname*{Argmin}_C f \iff -\nabla f(x) \in N_C(x) \\
&\iff -\gamma \nabla f(x) \in N_C(x) \\
&\iff x = P_C(x - \gamma \nabla f(x)) \\
&\iff x = P_C(x + \gamma(A^\top (P_Q(Ax) - Ax))) \\
&\iff x = P_C\Big(x + \frac{\lambda}{\lambda_{\max}(A^\top A)} A^\top (P_Q - I)Ax\Big) = R_\lambda x \\
&\iff x \in \operatorname{Fix} R_\lambda
\end{aligned}$$

■

The iterative step in the CQ-method for the SFP has the form

$$x_{k+1} = R_\lambda(x_k) \tag{13}$$

Lemma 73 *Let $Q \subset \mathbb{R}^n$ be convex and closed and let A be an $m \times n$ matrix. The operator $G : \mathbb{R}^n \to \mathbb{R}^n$,*

$$G(x) = \frac{1}{\lambda_{\max}(A^\top A)} A^\top (I - P_Q)Ax)$$

is firmly nonexpansive. Consequently, the operator $T = I - G$ is firmly nonexpansive.

Proof. Since P_Q is firmly nonexpansive (see Corollary 45) then, by Lemma 46, the operator $I - P_Q$ is also firmly nonexpansive, i.e.,

$$\langle (u - P_Q u) - (v - P_Q v), u - v \rangle \geq \|(u - P_Q u) - (v - P_Q v)\|^2$$

for all $u, v \in \mathbb{R}^n$. If we take $u = A^\top x$ and $v = A^\top y$ for $x, y \in \mathbb{R}^n$ in the above inequality and apply the property $\|A^\top z\|^2 \leq \lambda_{\max}(A^\top A)\|z\|^2$ then we obtain

$$\begin{aligned}
\langle G(x) - G(y), x - y \rangle &= \frac{1}{\lambda_{\max}(A^\top A)} \langle A^\top (I - P_Q)Ax - A^\top (I - P_Q)Ay, x - y \rangle \\
&= \frac{1}{\lambda_{\max}(A^\top A)} \langle (I - P_Q)Ax - (I - P_Q)Ay, Ax - Ay \rangle \\
&\geq \frac{1}{\lambda_{\max}(A^\top A)} \|(I - P_Q)Ax - (I - P_Q)Ay\|^2 \\
&\geq \frac{1}{\lambda_{\max}^2(A^\top A)} \|A^\top [(I - P_Q)Ax - (I - P_Q)Ay]\|^2 \\
&= \|\frac{1}{\lambda_{\max}(A^\top A)} A^\top (I - P_Q)Ax - \frac{1}{\lambda_{\max}(A^\top A)} A^\top (I - P_Q)Ay\|^2 \\
&= \|G(x) - G(y)\|^2,
\end{aligned}$$

i.e., the operator G is firmly nonexpansive. Again, by Lemma 46, the operator $T = I - G$ is also firmly nonexpansive. ■

Theorem 74 (Byrne, 2002) *Let the sequence (x_k) generated by the CQ-method* (13) *where the operator R_λ is given by* (12). *If $\operatorname{Fix} R_\lambda \neq \varnothing$ then for any starting point x_0 the sequence (x_k) converges to an element $z \in \operatorname{Fix} F_\lambda$.*

Proof. (Compare [Byr02, Theorem 2.1]) Observe that the CQ-method has the form

$$x_{k+1} = P_C((1 - \lambda)x_k + \lambda T(x_k))$$

where the operator T has the form

$$T(x) = \frac{1}{\lambda_{\max}(A^\top A)} A^\top (P_Q - I)Ax - x.$$

The operator T is firmly nonexpansive by Lemma 73, consequently, for $\lambda \in (0, 2)$ the operator R_λ is NE and SQNE by Theorem 56 and the sequence (x_k) generated by the CQ-method is asymptotically regular by Theorem 59. Now we see that all conditions of the Opial Theorem are satisfied. Consequently, the sequence (x_k) converges to a point $z \in \operatorname{Fix} F_\lambda$. ■

References

[BB96] H. H.Bauschke, J. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review*, 38 (1996) 367-426.

[Ber07] V. Berinde, *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin, 2007

[Bro12] L. E. J. Brouwer, Über Abbildungen von Mannigfaltigkeiten, *Math. Ann.* 71 (1912) 97-115.

[Bro65] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. USA*, 54 (1965) 1041-1044.

[Byr02] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem, *Inverse Problems*, 18 (2002) 441-453.

[Byr04] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems*, 20 (2004) 103-120.

[Byr08] C. L. Byrne, *Applied Iterative Methods*, A K Peters, Wellesley, MA, 2008.

[Ceg07] A. Cegielski, A generalization of the Opial's Theorem, *Control and Cybernetics* **36** (2007) 601-610.

[CS08] A. Cegielski, A. Suchocka, Relaxed alternating projection methods, *SIAM Journal on Optimization* (in print).

[CZ97] Y. Censor, S. A. Zenios, *Parallel Optimization, Theory, Algorithms and Applications*, Oxford University Press, New York 1997.

[Cim38] G. Cimmino, Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari, *La Ricerca Scientifica*, II, 9 (1938) 326-333.

[Com94] P. L. Combettes, Inconsistent signal feasibility problems: Least-square solutions in a product space, *IEEE Trans. Signal Processing* **42** (1994) 2955-2866.

[G04] A. Galántai, *Projectors and Projection Methods*, Kluwer Academic Publishers, Boston, 2004.

[Goe02] K. Goebel, *Concise Course on Fixed Points Theorems*, Yokohama Publishing, 2002.

[GK90] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge 1990.

[GD03] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[HUL93] J.-B. Hiriart-Urruty, C. Lemaréchal, *Convex Analysis and Minimization Algorithms, Vol I, Vol II*, Springer-Verlag, Berlin 1993.

[Kac37] S. Kaczmarz, Angenäherte Auflösung von Systemen linearer Gleichungen, *Bulletin International de l'Académie Polonaise des Sciences et des Lettres*, A35 (1937) 355-357.

[Opi67] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc., 73 (1967) 591-597.

[Sch30] J. Schauder, Der Fixpunktsatz in Funktionalräumen, *Studia Math.* 2 (1930) 171-180.

[SY98] H. Stark, Y. Yang, *Vector Space Projections. A Numerical Approach to Signal and Image Processing, Neural Nets and Optics*, John Wiley&Sons, Inc, New York, 1998.

[YM92] K. Yang, K. G. Murty, New iterative methods for linear inequalities, *JOTA*, 72 (1992) 163-185.