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On positivity and stability of linear time-varying Volterra equations

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Abstract. Linear time-varying Volterra integro-differential equations of non convolution type are considered. Positivity is characterized and a sufficient condition for exponential stability is presented.

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Keywords. Linear Volterra equations, positive systems, exponential stability.

Nomenclature

$\mathbb{R}_+^{\ell\times q}$		the set of all nonnegative matrices $M=(m_{ij})\in\mathbb{R}^{\ell\times q}$ with $m_{ij}\geq 0$ for all $i=1,\dots,\ell$, $j=1,\dots,q$
$A\gg B$	iff	$a_{ij}>b_{ij}$ for all entries of $A=(a_{ij})$, $B=(b_{ij})\in\mathbb{R}^{l\times q}$
$\phi\geq 0$	iff	the function $\phi\in C(J,\mathbb{R}^{l\times q})$ is *nonnegative*, i.e. $\phi(t)\in\mathbb{R}_+^{l\times q}$ for all $t\in J$
$\lvert\cdot\rvert$	:	a vector norm or induced matrix norm
$C(D,\mathbb{R}^{\ell\times q})$		the vector space of continuous functions $y:D\to\mathbb{R}^{\ell\times q}$, $D\subset\mathbb{R}^p$
$\lVert\cdot\rVert$	:	the supremum norm of continuous functions $y:[0,T]\to\mathbb{R}^n$, with respect to a given norm $\lvert\cdot\rvert$ on $\mathbb{R}^n$
$C_0([0,T],\mathbb{R}^n)$	:=	$\{\phi\in C([0,T],\mathbb{R}^n)\mid \phi(T)=0\}$, $T>0$
$NBV([0,T],\mathbb{R}^{n\times n})$	:=	$\{\eta\in BV([0,T],\mathbb{R}^{n\times n})\mid \eta$ is cont. f. left$\}$
Δ	:=	$\{(t,s)\in\mathbb{R}^2\mid t\geq s\geq 0\}$

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1. Introduction

We consider linear time-varying Volterra integro-differential equations of non-convolution type

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\mathrm{d}s, \quad t \geq t_0 \geq 0, \tag{1.1}$$

where

$$A(\cdot) \in C([0,\infty), \mathbb{R}^{n\times n}), \qquad B(\cdot,\cdot) \in C(\Delta, \mathbb{R}^{n\times n}). \tag{1.2}$$

In Section 2, we characterize positivity of the system (1.1); positivity means, roughly speaking, that for any nonnegative initial condition the corresponding (unique) solution is also nonnegative. The theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius, as references we mention [1] and [4]. In recent time, problems of positive systems have attracted a lot of attention from researchers, see [7]-[10] and the references therein.
The characterization of positivity, which we present in Theorem 2.2, generalizes a recent result by [10] where A in (1.1) is constant and the equation is of convolution type.

In Section 3, we exploit positivity and present a sufficient condition for exponential stability of (1.1) provided $A(\cdot)$ is a diagonal matrix. Theorem 3.2 generalizes the result by [6] and [3], both of them allow only for scalar equations and the kernel B must be of convolution type.

2. Positivity

Before we state our main theorem, we recall some basic facts on linear time-varying Volterra equations of non-convolution type (1.1) provided, for example, in [2]. For $A(\cdot), B(\cdot,\cdot)$ as in (1.2) and initial data $T \geq 0$ and $\phi \in C([0,T], \mathbb{R}^n)$, the initial value problem

$$\dot{x}(t) = A(t)x(t) + \int_{0}^{t} B(t,s)x(s)\mathrm{d}s, \ t \geq T, \quad x(\cdot)\big|_{[0,T]} = \phi(\cdot). \tag{2.1}$$

has a unique *solution* $x(\,\cdot\,; T, \phi)\colon \mathbb{R}_+ \to \mathbb{R}^n$, i.e. $x(\cdot; T, \phi)$ is continuously differentiable on $[T, \infty)$ and satisfies (2.1) for all $t \geq T$.

Definition 2.1. Equation (1.1) is said to be *positive* if, and only if, for any $T \geq 0$ and any nonnegative $\phi \in C([0,T], \mathbb{R}^n)$ the solution $x(\,\cdot\,; T, \phi)$ of the initial value problem (2.1) is nonnegative.

We are now ready to state our main result on positivity of (1.1) in terms of the system matrices $A(\cdot)$ and $B(\cdot,\cdot)$.

Theorem 2.2. *Equation* (1.1) *is positive if, and only if,* $A(t)$ *is a Metzler matrix for every* $t \geq 0$ *and* $B(t,s) \geq 0$ *for all* $(t,s) \in \Delta$.

Theorem 2.2 generalizes a recent result by [10] who consider (1.1) with constant $A(\cdot) = A$ and non-convolutional kernel $B(t,s) = B_1(t-s)$; moreover they assume that $B_1(\cdot) \in L^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$.

We first collect some well-known facts, see e.g. [2], on time-varying Volterra equations of non-convolution type (1.1) and prove some lemmata. For $A(\cdot), B(\cdot,\cdot)$ as in (1.2), the matrix initial value problem

$$\frac{\partial R(t,s)}{\partial t} = A(t)R(t,s) + \int_s^t B(t,u)R(u,s)\mathrm{d}u, \qquad R(s,s) = I_n \tag{2.2}$$

has a unique solution $R(\cdot,\cdot)\colon \Delta \to \mathbb{R}^{n\times n}$, called the *resolvent* of equation (1.1); this solution is continuously differentiable.

Moreover, for $f \in C(\mathbb{R}_+, \mathbb{R}^n)$ and fixed $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the initial value problem

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^t B(t,s)x(s)\mathrm{d}s + f(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{2.3}$$

has a unique solution which may be represented by the variation of constants formula

$$x(t) = R(t,t_0)x_0 + \int_{t_0}^t R(t,s)f(s)\mathrm{d}s, \quad t \geq t_0. \tag{2.4}$$

Lemma 2.3. *Let $t_0 \geq 0$ and consider, for* (2.1)*, a sequence of initial functions $(\phi_k)_{k\in\mathbb{N}}$ in $C([0,t_0], \mathbb{R}^n)$ satisfying*

(i) *$(\phi_k)_{k\in\mathbb{N}}$ is bounded,*

(ii) *there exists a measurable function $\phi\colon [0,t_0] \to \mathbb{R}^n$ such that $\lim_{k\to\infty} \phi_k(t_0) = \phi(t_0)$ and $\lim_{k\to\infty} \phi_k(t) = \phi(t)$ for almost all $t \in [0,t_0]$.*

Then

$$\lim_{k\to\infty} x(t;t_0,\phi_k) = R(t,t_0)\phi(t_0) + \int_{t_0}^t R(t,s)\left(\int_0^{t_0} B(s,u)\phi(u)\mathrm{d}u\right)\mathrm{d}s, \quad \forall\ t \geq t_0. \tag{2.5}$$

Proof. Setting

$$x_0 := \phi(t_0) \quad \text{and} \quad f(t) := \int_0^{t_0} B(t,u)x(u)\mathrm{d}u \ \forall\ t \geq t_0$$

in (2.3) yields, by invoking (2.4), that

$$x(t;t_0,\phi_k) = R(t,t_0)\phi_k(t_0) + \int_{t_0}^t R(t,s)\left(\int_0^{t_0} B(s,u)\phi_k(u)\mathrm{d}u\right)\mathrm{d}s, \quad \forall\ t \geq t_0.$$

Since R and B are continuous and the assertions (i) and (ii) hold, we may apply the Lebesgue dominated theorem to arrive at (2.5). □

Corollary 2.4. *If* (1.1) *is positive, then the resolvent satisfies $R(t,s) \geq 0$ for all $(t,s) \in \Delta$.*

Proof. Fix $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ and consider, for $k \in \mathbb{N}$, the continuous function

$$\phi_k\colon [0,t_0] \to \mathbb{R}^n, \quad t \mapsto \phi_k(t) = \begin{cases} 0 & \text{if } t \in [0, t_0 - 1/k] \\ ktx_0 + (1-kt_0)x_0 & \text{if } t \in [t_0 - 1/k, t_0]. \end{cases}$$

Then we have, for all $t \in [0, t_0]$,

$$\lim_{k\to\infty} \phi_k(t) = \phi(t) := \begin{cases} 0 & \text{if } t \in [0, t_0) \\ x_0 & \text{if } t = t_0. \end{cases}$$

Since ϕ so defined is measurable, Lemma 2.3 yields, for all $t \geq t_0$,

$$\begin{aligned} \lim_{k\to\infty} x(t; t_0, \phi_k) &= \lim_{k\to\infty} \left[R(t,t_0)\phi_k(t_0) + \int_{t_0}^t R(t,s) \left(\int_0^{t_0} B(s,u)\phi_k(u)\mathrm{d}u \right) \mathrm{d}s \right] \\ &= R(t,t_0)x_0\,. \end{aligned}$$

Finally, by positivity of (1.1), we may conclude $\lim_{k\to\infty} x(t; t_0, \phi_k) = R(t,t_0)x_0 \geq 0$, and since $x_0 \in \mathbb{R}^n_+$ is arbitrary, it follows that $R(t,s) \geq 0$ for all $(t,s) \in \Delta$. □

Lemma 2.5. *Let $T > 0$ and $\eta \in NBV([0,T], \mathbb{R}^{n\times n})$. Then the linear operator*

$$L \ : \ C_0([0,T], \mathbb{R}^n) \to \mathbb{R}^n, \qquad \phi \mapsto L\phi = \int_0^T \mathrm{d}[\eta(\theta)]\phi(\theta)$$

is positive (i.e. $L\phi \geq 0$ for every $\phi \in C_0([0,T]$, $\phi \geq 0)$ if, and only if, η is increasing (i.e. every entry function η_{ij} satisfies, for all t_1, t_2 with $0 \leq t_1 \leq t_2 \leq T$, $\eta_{ij}(t_2) \geq \eta_{ij}(t_1)$).

Proof. Let $\eta \in NBV([0,T], \mathbb{R}^{n\times n})$ be increasing. By definition of the Riemann-Stieltjes integrals,

$$L\phi = \lim_{d(P)\to 0} \sum_{k=1}^{p} (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k) \geq 0, \qquad \forall\ \phi \in C_0([0,T], \mathbb{R}^n), \phi \geq 0.$$

Therefore, L is positive.

Conversely, assume that L is positive on $C_0([0,T], \mathbb{R}^n)$. We show that $\eta(\cdot) = (\eta_{ij}(\cdot)) \in NBV([0,T], \mathbb{R})$ is an increasing scalar function for every $i, j \in \{1, 2, ..., n\}$. Since L is positive, it is easy to see that the operator

$$L_{ij} : C_0([0,T], \mathbb{R}) \to \mathbb{R}, \quad \phi \mapsto L_{ij}\phi := \int_0^T \phi(\theta)\mathrm{d}[\eta_{ij}(\theta)]$$

is also positive for every $i, j \in \{1, 2, ..., n\}$. Fix

$$\theta_1, \theta_2 \in (0,T) \text{ with } \theta_1 < \theta_2 \quad \text{and} \quad k \in \mathbb{N} \ \text{ with } \ k > \max\left\{ \frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_2 - \theta_1} \right\}$$

and consider the continuous function

$$\phi_k : [0,T] \to \mathbb{R}, \quad \theta \mapsto \phi_k(\theta) := \begin{cases} 0 & \text{if } \theta \in [0, \theta_1 - \frac{1}{k}] \\ k\theta + 1 - k\theta_1 & \text{if } \theta \in (\theta_1 - \frac{1}{k}, \theta_1] \\ 1 & \text{if } \theta \in (\theta_1, \theta_2 - \frac{1}{k}] \\ -k\theta + k\theta_2 & \text{if } \theta \in (\theta_2 - \frac{1}{k}, \theta_2] \\ 0 & \text{if } \theta \in (\theta_2, T]. \end{cases}$$

Since ϕ_k is continuous on $[0,T]$, it follows from a standard property of the Riemann-Stieltjes integral, see e.g. [11, p.109], that

$$\int_0^T \phi_k(\theta)\mathrm{d}[\eta_{ij}(\theta)] = \left(\int_0^{\theta_1 - \frac{1}{k}} + \int_{\theta_1 - \frac{1}{k}}^{\theta_1} + \int_{\theta_1}^{\theta_2 - \frac{1}{k}} + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} + \int_{\theta_2}^{T} \right) \phi_k(\theta)\mathrm{d}[\eta_{ij}(\theta)].$$

This gives, for all $k \in \mathbb{N}$ sufficiently large,

$$\int_{\theta_1 - \frac{1}{k}}^{\theta_1} \phi_k(\theta) d[\eta_{ij}(\theta)] + \eta_{ij}(\theta_2 - \frac{1}{k}) - \eta_{ij}(\theta_1) + \int_{\theta_2 - \frac{1}{k}}^{\theta_2} \phi_k(\theta)\mathrm{d}[\eta_{\mathrm{ij}}(\theta)] \geq 0.$$

Taking into account that η_{ij} is continuous from the left at θ_1, θ_2 and letting $k \to \infty$, we have $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$ for every $\theta_1, \theta_2 \in (0,T), \theta_2 \geq \theta_1$. In case of $\theta_1 = 0 < \theta_2 < T$, a similar argument gives $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$. Finally, since η_{ij} is continuous from the left at T, we have $\eta_{ij}(T) \geq \eta_{ij}(\theta)$ for all $\theta \in [0,T]$. This completes the proof. □

We are now in the position to prove Theorem 2.2.
Proof of Theorem 2.2**:** Assume that (1.1) is positive. Fix $s \geq 0$. It follows from (2.2) that

$$A(s) = \frac{\partial R(t,s)}{\partial t}\Big|_{t=s} = \lim_{t \downarrow s} (R(t,s) - I_n)/(t-s).$$

Since by Corollary 2.4, $R(t,s) \geq 0$ for all $(t,s) \in \Delta$, we conclude that $A(s)$ is a Metzler matrix.

We now show that $B(t,s) \geq 0$ for all $(t,s) \in \Delta$. Fix $T \geq 0$ and $\phi \in C_0([0,T], \mathbb{R}^n), \phi \geq 0$. Then $x(\cdot) := x(\cdot; T, \phi)$ satisfies

$$\begin{aligned} 0 \leq \lim_{t \downarrow T} (x(t) - x(T))/(t-T) &= \lim_{t \downarrow T} x(t)/(t-T) \\ = \dot{x}(T) &= A(T)\phi(T) + \int_0^T B(T,s)\phi(s)\mathrm{d}s = \int_0^T B(T,s)\phi(s)\mathrm{d}s. \end{aligned}$$

Thus, for $\eta(s) := \int_0^s B(T,\theta)\mathrm{d}\theta$, $s \in [0,T]$,

$$\int_0^T B(T,s)\phi(s)\mathrm{d}s = \int_0^T \mathrm{d}[\eta(\mathrm{s})]\phi(s) \geq 0, \quad \forall \phi \in C_0([0,T], \mathbb{R}^n),\ \phi \geq 0.$$

By Lemma 2.5, $\eta(\cdot)$ is increasing on $[0,T]$. This implies that $B(T,s) \geq 0$ for $s \in [0,T]$. Since $T \geq 0$ is arbitrary, we have $B(t,s) \geq 0$ for all $(t,s) \in \Delta$.

Conversely, assume $A(t)$ is a Metzler matrix for every $t \geq 0$ and $B(t,s) \geq 0$ for all $(t,s) \in \Delta$. Fix $T \geq 0$ and $\phi \in C([0,T],\mathbb{R}^n)$ with $\phi \geq 0$. We prove that $x(t;T,\phi) \geq 0$ for all $t \geq T$. Fix $T_1 > T$, since $A(\cdot)$ is continuous on $[0,T_1]$ and $A(t)$ is a Metzler matrix for every $t \geq 0$, we may choose $r > 0$ such that $rI_n + A(t) \geq 0$ for all $t \in [0,T_1]$. Consider

$$z\colon [0,T_1] \to \mathbb{R}^n, \quad t \mapsto z(t) := e^{rt}x(t;T,\phi).$$

Then z satisfies

$$\dot{z}(t) = (A(t) + rI_n)z(t) + \int_0^t e^{r(t-s)}B(t,s)z(s)\mathrm{d}s, \quad \forall t \in [T,T_1]. \tag{2.6}$$

It remains to consider two cases:
(i) Assume $\phi(T) \gg 0$. We show that $z(t) \geq 0$ for all $t \in [T,T_1]$. Seeking a contradiction, suppose

$$T_0 = \inf\{t \in [T,T_1] |\ z(t) \not\geq 0\} \in [T,T_1].$$

Then by continuity $z(T_0) \geq 0$ and so (2.6) yields

$$\begin{aligned} z(T_0) &= z(T) + \int_T^{T_0} \dot{z}(\tau)\mathrm{d}\tau \\ &= \phi(T) + \int_T^{T_0} \left((A(\tau) + rI_n)z(\tau) + \int_0^\tau e^{r(\tau-s)}B(\tau,s)z(s)\mathrm{d}s \right) \mathrm{d}\tau \geq \phi(T) \gg 0. \end{aligned}$$

By continuity, there exists $\epsilon > 0$ such that $z(t) \gg 0$ for all $t \in [T_0, T_0 + \epsilon)$. However, this contradicts the definition of T_0; whence $z(t) \geq 0$ for all $t \in [T,T_1]$. Since $T_1 \geq T$ is arbitrary, we have $z(t) \geq 0$ for all $t \geq T$ and therefore, $x(t) \geq 0$ for all $t \geq T$.
(ii) Assume $\phi(T) \geq 0$. Then $\phi_k := \phi + (1/k)e$, where $e = (1,1,...,1)^T \in \mathbb{R}^n$ and $k \in \mathbb{N}$, yields $\phi_k(T) \gg 0$. Now (2.5) together with Part (i) gives

$$\lim_{k\to\infty} x(t;T,\phi_k) = x(t;T,\phi) \geq 0, \qquad \forall\ t \geq T.$$

This completes the proof of the theorem. □

We finalize this section with a remark showing that positivity of (1.1) holds uniformly in time t.

Remark 2.6. Let $t_0, T \geq 0$ and $\psi \in C([t_0, t_0+T],\mathbb{R}^n)$ be given. Then it is easy to see that there exists a unique solution $y(\ \cdot\ ; t_0, t_0+T, \psi) : [t_0,\infty) \to \mathbb{R}^n$ of the initial value problem

$$\dot{y}(t) = A(t)y(t) + \int_{t_0}^t B(t,s)y(s)\mathrm{d}s,\ t \geq t_0 + T, \quad y(\cdot)\Big|_{[t_0,t_0+T]} = \psi(\cdot). \tag{2.7}$$

Furthermore, $y(\cdot) := y(\,\cdot\,; t_0, t_0+T, \psi)$ is the solution of (2.7) if, and only if, $x(\cdot) := y(\cdot + t_0)$ is the solution of the initial value problem

$$\dot{x}(t) = A(t+t_0)x(t) + \int_0^t B(t+t_0, s+t_0)x(s)\mathrm{d}s,\ t \geq T, \quad x(\cdot)\big|_{[0,T]} = \psi(\cdot+t_0). \quad (2.8)$$

In view of Definition 2.1, it is not more general to define positivity of (1.1) as: *For any* $t_0, T \geq 0$ *and any nonnegative* $\psi \in C([t_0, t_0+T], \mathbb{R}^n)$ *the solution* $y(\,\cdot\,; t_0, t_0+T, \psi)$ *of* (2.7) *is nonnegative.*

3. Exponential stability

In this section, we give a sufficient condition for exponential stability of (1.1) which is defined as follows.

Definition 3.1. Equation (1.1) is said to be *exponentially stable* if, and only if,

$$\exists\, M, \alpha > 0\ \forall\,(t, t_0) \in \Delta\ \forall\, T \in [0, t_0]\ \forall\, \phi \in C([t_0 - T, t_0], \mathbb{R}^n)\ :$$
$$|x(t; t_0 - T, t_0, \phi)| \ \leq\ M e^{-\alpha(t-t_0)}\|\phi\| \qquad \forall\, t \geq t_0 - T,$$

where $x(\,\cdot\,; t_0 - T, t_0, \phi)$ denotes the unique solution of (1.1) satisfying the initial condition $x(t) = \phi(t), t \in [t_0 - T, t_0]$.

Opposed to [3, 6], the above definition of exponential stability is uniform in time t.

Theorem 3.2. *Suppose* $A(\cdot) \in C([0,\infty), \mathbb{R}^{n\times n})$ *and* $B(\cdot,\cdot) \in C(\Delta, \mathbb{R}^{n\times n})$ *satisfy*

(i) $A(\cdot) = diag\{a_1(\cdot), \ldots, a_n(\cdot)\}$,

(ii) $\exists\, \alpha > 0\ \forall\,(t, t_0) \in \Delta\ :\ A(t) + \int_{t_0}^t B(t,s)\, e^{-\int_s^t A(u)\mathrm{d}u}\mathrm{d}s\ \leq -\alpha I_n$,

(iii) $\sup_{s\geq 0} \int_0^\infty e^{\alpha\tau}|B(\tau+s, s)|\,\mathrm{d}\tau < \infty$.

Then positivity of (1.1) *implies exponentially stability of* (1.1)*.*

Theorem 3.2 present a sufficient condition for a diagonal matrix $A(\cdot)$ only. However, this generalizes a result by [6] and a recent contribution by [3]; both of them allow only for scalar equations and the kernel B must be of convolution type.

In what follows the matrix norms are assumed to be induced by monotonic vector norms; this implies that

$$\forall P, Q \in \mathbb{R}^{l\times q}, \quad 0 \leq P \leq Q \implies |P| \leq |Q|, \quad (3.1)$$

see e.g. [10].

Proof of Theorem 3.2**:** Fix $t_0 \geq 0$ and let R denote the resolvent of (2.2).
Step 1: We show

$$R(s, t_0) \leq e^{-\int_s^t A(\mu)\mathrm{d}\mu} R(t, t_0) \ =:\ G(t), \quad \forall s \in [t_0, t]\ \forall t \geq t_0. \quad (3.2)$$

An elementary calculation yields, for all $\tau \in [s,t]$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau) &= \frac{\partial}{\partial\tau}e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \cdot R(\tau,t_0) + e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \cdot \frac{\partial}{\partial\tau}R(\tau,t_0) \\
&= -e^{-\int_s^\tau A(\mu)\mathrm{d}\mu}\ A(\tau)\ R(\tau,t_0) \\
&\quad + e^{-\int_s^\tau A(\mu)\mathrm{d}\mu}\ \left[A(\tau)R(\tau,t_0)\int_{t_0}^\tau B(\tau,u)R(u,t_0)\mathrm{d}u\right] \\
&= e^{-\int_s^\tau A(\mu)\mathrm{d}\mu}\ \int_{t_0}^\tau B(\tau,u)R(u,t_0)\mathrm{d}u. \qquad (3.3)
\end{aligned}
$$

By assertion (i), Theorem 2.2 and Corollary 2.4 we may conclude that

$$\forall\ t_0 \le s \le \tau \le t \quad \forall\ t_0 \le u \le \tau\ :\ e^{-\int_s^\tau A(\mu)\mathrm{d}\mu} \ge 0, \quad B(\tau,u) \ge 0, \quad R(u,t_0) \ge 0,$$

and (3.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau) \ge 0, \quad \forall \tau \in [s,t].$$

Therefore,

$$\forall\ t_0 \le s \le t\ :\ R(s,t_0) = G(s) \le G(t) = e^{-\int_s^t A(u)\mathrm{d}u}R(t,t_0).$$

Step 2: We show that

$$0 \le R(t,t_0) \le e^{-\alpha(t-t_0)}I_n \quad \forall (t,t_0) \in \Delta. \qquad (3.4)$$

The first inequality of (3.4) follows from Corollary 2.4. An elementary calculation yields, for all $t \ge t_0$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\alpha(t-t_0)}R(t,t_0)\right) &\overset{(2.2)}{=} \alpha e^{-\alpha(t-t_0)}R(t,t_0) \\
&\quad + e^{-\alpha(t-t_0)}\left[A(t)R(t,t_0) + \int_{t_0}^t B(t,u)R(u,t_0)\mathrm{d}u\right] \\
&\overset{(3.2)}{\le} e^{-\alpha(t-t_0)}\left[\alpha I_n + A(t) + \int_{t_0}^t B(t,u)e^{-\int_u^t A(\mu)\mathrm{d}\mu}\mathrm{d}u\right]R(t,t_0) \\
&\overset{(ii)}{\le} 0.
\end{aligned}
$$

Step 3: We finally have for any solution $x(\,\cdot\,; t_0-T, t_0, \phi)$ of (2.1) and for all $t \geq t_0$,

$$\begin{aligned}|x(t;t_0-T,t_0,\phi)| &\overset{(2.4)}{\leq} |R(t,t_0)|\ |\phi(t_0)| + \int_{t_0}^{t} |R(t,\tau)| \int_{t_0-T}^{t_0} |B(\tau,s)|\ |\phi(s)| \mathrm{d}s\ \mathrm{d}\tau \\ &\overset{(3.1)-(3.4)}{\leq} e^{-\alpha(t-t_0)}\ \|\phi\| + \int_{t_0}^{t} e^{-\alpha(t-\tau)} \int_{t_0-T}^{t_0} |B(\tau,s)| \mathrm{d}s\ \mathrm{d}\tau\ \|\phi\| \\ &= e^{-\alpha(t-t_0)}\ \|\phi\| + \int_{t_0-T}^{t_0} e^{-\alpha(t-s)} \int_{t_0}^{t} e^{\alpha(\tau-s)} |B(\tau,s)| \mathrm{d}\tau\ \mathrm{d}s\ \|\phi\| \\ &\overset{(iii)}{\leq} \left[e^{-\alpha(t-t_0)} + K \int_{0}^{t_0} e^{\alpha(s-t)} \mathrm{d}s \right] \|\phi\|,\end{aligned}$$

where

$$K := \sup_{s\geq 0} \int_0^\infty e^{\alpha\tau} |B(\tau+s,s)| \mathrm{d}\tau,$$

and therefore

$$\begin{aligned}|x(t;t_0-T,t_0,\phi)| &\leq \left(e^{-\alpha(t-t_0)} + \frac{K}{\alpha}(e^{-\alpha(t-t_0)} - e^{-\alpha t}) \right) \|\phi\| \\ &= \left(1 + \frac{K}{\alpha}(1-e^{-\alpha t_0})\right) e^{-\alpha(t-t_0)}\ \|\phi\| \\ &\leq \left(1+\frac{K}{\alpha}\right) e^{-\alpha(t-t_0)}\ \|\phi\|, \qquad t \geq t_0 \geq 0.\end{aligned}$$

This completes the proof. □

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