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Precoloring extension for K_4-minor-free graphs

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Abstract

Let $G = (V, E)$ be a graph where every vertex $v \in V$ is assigned a list of available colors $L(v)$. We say that G is list colorable for a given list assignment if we can color every vertex using its list such that adjacent vertices get different colors. If $L(v) = \{1, \ldots, k\}$ for all $v \in V$ then a corresponding list coloring is nothing other than an ordinary k-coloring of G. Assume that $W \subseteq V$ is a subset of V such that $G[W]$ is bipartite and each component of $G[W]$ is precolored with two colors. The minimum distance between the components of $G[W]$ is denoted by $d(W)$. We will show that if G is K_4-minor-free and $d(W) \geq 7$, then such a precoloring of W can be extended to a 4-coloring of all of V. This result clarifies a question posed in [10]. Moreover, we will show that such a precoloring is extendable to a list coloring of G for outerplanar graphs, provided that $|L(v)| = 4$ for all $v \in V \setminus W$ and $d(W) \geq 7$. In both cases the bound for $d(W)$ is best possible.

Keywords: precoloring extension, list coloring, minor-free graphs

1 Introduction

Precoloring problems for special graph classes and different types of precolored subsets are studied by many authors, including [1]-[8], [10],[11],[13],[14]. Let $G = (V, E)$ be a simple graph and $W \subseteq V$ a precolored subset of the vertex set. The shortest distance between components of W in G is denoted by $d(W)$. Obviously, $d(W)$ influences the extendability of the precoloring of W to all of G. Thus we ask for bounds for $d(W)$ such that a precoloring of W can be extended to a proper coloring of V using a given number of colors. An analogous problem for list colorings, introduced independently by Vizing [12] and Erdős, Rubin and Taylor [9] at the end of the 1970s, can also be investigated. In a graph with list coloring, every vertex is assigned a list of available colors $L(v)$ and the graph's vertices are colored in such a way that each vertex is colored with a color from its list and adjacent vertices are colored differently.
Assume that G is a simple graph with chromatic number $\chi(G) = k$. The first result on precoloring extension that takes $d(W)$ into consideration was publicized by Albertson [1]. If $W \subseteq V$ is an independent set and $d(W) \geq 4$, then every $(k+1)$-coloring of W can be extended to a proper $(k+1)$-coloring of V. This result can be generalized. If $G[W]$ is an s-colorable graph where every component is s-colored and $d(W) \geq 4$, then every $(k+s)$-coloring of W can be extended to a proper $(k+s)$-coloring of V [3]. Clearly, the bound for $d(W)$ will grow if we use fewer colors. If W is the union of complete graphs K_s and $d(W) \geq 4s$, then any $(k+1)$-coloring of $G[W]$ is extendable to a $(k+1)$-coloring of V by a result of Kostochka proved in [6].
Consider the bounds for $d(W)$ if we use $k+s-1$ colors. Hutchinson and Moore [10] showed that no distance can insure a color extension with $k+s-1$ colors without topological constraints. They found several results for special graph classes in [10]. Let G be a simple graph with $\chi(G) = k$ and without a minor K_{k+1} and let $G[W]$ be s-colorable with every component of $G[W]$ precolored with s colors.

$k \backslash s$	2	3	4	5
2	3	–	–	–
3	7, 8	5	–	–
4	7, 8	7	6	–
5	7, 8	7, 8	7	6

The above table shows bounds and exact values, respectively, for a smallest

d such that $d(W) \geq d$ ensures the required coloring extension. Albertson and Moore [7] found several results for $s = 1$.
For the list coloring version of this problem, some results in the style of Brook's theorem are known. Axenovich [8] and Albertson, Kostochka and West [5] independently showed that for $|L(v)| = \Delta \geq 3\ \forall v \in V$ and $d(W) \geq 8$ every list coloring of an independent set W extends to a list coloring of V. If G is 2-connected, then $d(W) \geq 4$ for $\Delta(G) \geq 4$ [13] and $d(W) \geq 6$ for $\Delta(G) = 3$ [14] ensure the extendability of such colorings.
In this paper, we will take a closer look at the case with $k = 3$ and $s = 2$ from the above table. This means that G is a K_4-minor-free graph which is always 3-colorable and every component of $G[W]$ is bipartite and precolored with two colors. We can see in the table that $d(W) \geq 8$ ensures that the precoloring is extendable to a $3+2-1 = 4$-coloring of all of G. On the other hand, $d(W) \geq 6$ is not sufficient for such an extension, which is shown by an example of Kostochka [11] (see Fig. 1).

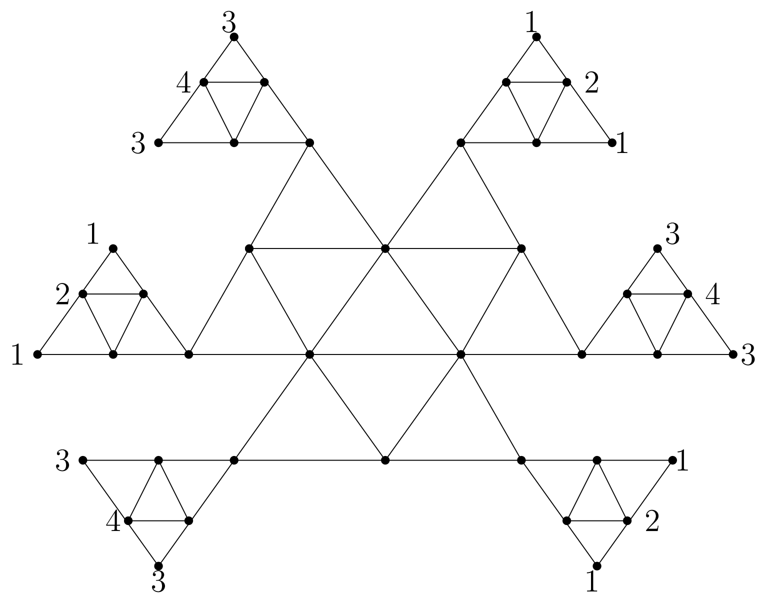

Figure 1: $d(W) = 6$; an extension to a 4-coloring is not possible

To see that the coloring is not extendable, color the central triangle arbitrarily and try to extend the coloring in a proper way.
We will prove in this paper that $d(W) \geq 7$ is sufficient to ensure the extension of the coloring in our case.

Moreover, we investigate the list coloring version of this problem for outerplanar graphs (K_4- and K_{23}-minor-free graphs). For technical reasons, we start with the results of the list coloring version of our problem.

Theorem 1 *Let G be an outerplanar graph, $W \subseteq V(G)$ and $G[W]$ bipartite. Furthermore, assume that the shortest distance $d(W)$ between components of $G[W]$ is at least 7. If every vertex of $G \setminus W$ has a list of at least 4 colors, then any precoloring of $G[W]$ which colors every component of $G[W]$ using two colors can be extended to a proper list coloring of all of G.*

Theorem 2 *Let G be a K_4 minor-free graph, $W \subseteq V(G)$, and $G[W]$ bipartite. Furthermore, assume that the shortest distance $d(W)$ between components of $G[W]$ is at least 7. Then any proper 4-coloring of $G[W]$ which colors every component of $G[W]$ using two colors can be extended to a 4-coloring of all of G.*

Given the example above, it is clear that the bound for $d(W)$ is best possible in both theorems.
The idea behind the proofs of these theorems is taken from the previous example, where the second neighborhood of every precolored component is a single vertex. We start by coloring the first neighborhood of the precolored components and examine the remaining graph G^* after removing these vertices along with the precolored vertices.
In the second section, we will prove a lemma containing a coloring algorithm which will be used for the proofs of both theorems. In both cases, this algorithm realizes the coloring of G^*. Theorem 1 is proved in the third section, showing that for this case the remaining graph G^* fulfils the assumption of Lemma 3. For the proof of Theorem 2 in Section 4, we have to construct a special coloring of the first neighborhood to apply Lemma 3 again.

2 A Coloring Algorithm

Let G be a simple graph where every vertex v has a list $L(v)$ of available colors. A vertex with a list of cardinality i is called an Li vertex. We consider the following types of vertex subsets:

- Type 1: an $L1$ vertex z_1 or a pair z_1, z_2 of adjacent $L1$ vertices with different colors,

- Type 2: a single $L2$ vertex,
- Type 3: a pair x, y of adjacent vertices, where x is an $L2$ vertex and y is an $L3$ vertex,
- Type 4: a tree $T = (V_T, E_T)$, where exactly one of the vertices is an $L2$ vertex and the other vertices are $L3$ vertices and there is no path between two nonadjacent vertices of T in G outside of T.

The union of all subsets of Type 1, 2, 3, and 4 is denoted by S. A vertex belonging to a subset of Type i is called a t_i-vertex.

Lemma 3 *Let $G = (V, E)$ with $|V| = n$ be a connected K_4-minor-free graph where every vertex is assigned a list $L(v)$ of available colors. Let S be a subset of V containing subsets of Type 1, 2, 3 and 4 and $|L(v)| \in \{3, 4\}$ for all $v \in V \setminus S$. Furthermore we assume that*

1. *there is at most one subset of Type 1 in S,*
2. *the distance between two different subsets of S in G is at least 3, and*
3. *$|L(w)| = 4$ for all $w \in V \setminus S$ adjacent to a vertex of S.*

Then G is list colorable.

Proof.
We will prove this lemma by induction on the number of subsets of Type 4. If there is no subset of Type 4 then do the following
Algorithm:
Since G and all of its subgraphs are K_4-minor-free, each of them contains at least two non-adjacent vertices of degree at most 2 [15].

1. Determine an order $v_1, \ldots, v_n$ for the vertices of V such that v_i has degree at most 2 in $G[v_{i+1}, \ldots, v_n]$.

   If there is a t_1-vertex z_1 in G, then choose $v_n = z_1$. If there is a second t_1-vertex z_2 in G (which is adjacent to z_1 by definition), then choose $v_{n-1} = z_2$.

2. If $v_i = x$ is a t_2-vertex and there are two neighbors, say $v_j, v_s \in \{v_{i+1}, \ldots, v_n\}$ with $i < j < s$, then reorder the vertices by moving x behind v_j: $v_k = v_{k+1}$ for $k = i, \ldots, j-1$ and $v_j = x$.

   This reordering is shown in Figure 2. Note that only two edges of the graph (xv_j and xv_s in the original order) are given explicitly. A dashed line means that we do not know whether the vertices are adjacent or not.

   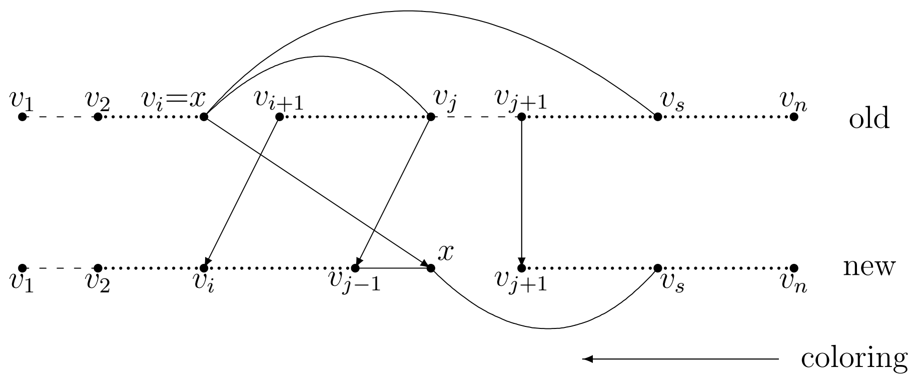

   Figure 2: Reordering if $|L(x)| = 2$ and $|L(v_j)| = 4$

   The new vertex v_{j-1} is an $L4$ vertex and has at most three neighbors in $v_j, \ldots, v_n$ regarding the new order. For all other vertices in the ordering, the number of neighbors in the graph induced by the vertices of higher index in the ordering does not change.

3. Let $v_i = x$ be a t_3-$L2$-vertex and $v_j, v_s \in \{v_{i+1}, \ldots, v_n\}$ two neighbors with $i < j < s$.

   If v_j is an $L4$ vertex, reorder analogously to 2.

   If $v_j = y$ is the corresponding t_3-$L3$-vertex, then move y to the end of the ordering: $v_k = v_{k+1}$ for $k = j, \ldots, n-1$ and $v_n = y$. This reordering is shown in Figure 3.

   If $v_\ell \neq x$ is a neighbor of y, then v_ℓ is an $L4$-vertex. If $\ell > j$ then the number of v_ℓ's neighbors of higher index increases by one to at most three after moving y.

Observation after the reordering:

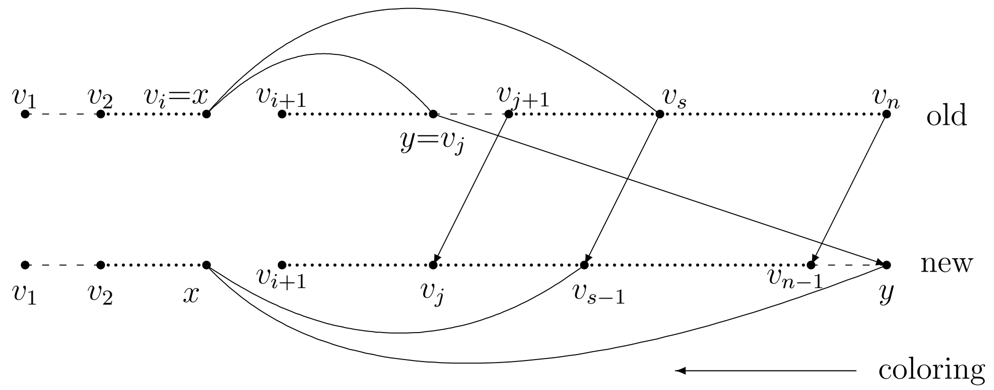

Figure 3: Reordering if $|L(x)| = 2$ and $|L(v_j)| = 3$

- Every $L4$ vertex has at most three neighbors in the graph induced by the vertices of higher index since it has at most one "reordered neighbor" according to the lemma's second assumption.

- Every $L3$ vertex has at most two neighbors in the graph induced by the vertices of higher index.

- Every $L2$ vertex x has either one neighbor in the graph induced by the vertices of higher index, or two neighbors such that one of them is the $L3$ vertex y belonging to x. In the last case, y is one of the last vertices of the ordering and may be followed by some other reordered $L3$ vertices.

- z_1 has no neighbors in the graph induced by the vertices of higher index. If the set of vertices of higher index is not empty, then its elements are reordered $L3$ vertices which are not adjacent to z_1, due to the lemma's assumption.
  z_2 has only one neighbor in the graph induced by the vertices of higher index, namely z_1.

4. Color the vertices in the reverse order $v_n, \dots, v_1$:

   - Let $v_n, v_{n-1}, \dots, v_i$ be reordered t_3-$L3$-vertices. Note that these vertices build an independent set in G because of the lemma's assumption. Color each of these vertices with a color that does not belong to the list of its corresponding t_3-$L2$-vertex.

- Color the remaining vertices with an arbitrary available color which has not been used to color its previously colored neighbors.

Now assume that there are subsets of Type 4.
Let $T = (V_T, E_T)$ be a Type 4 subset. If $V(G) = V_T$, then we are done coloring the tree.
Otherwise, note that every component of $G - T$ is adjacent to at most two vertices of T since there are no paths between two non-adjacent vertices of T outside of the tree. If such a component is adjacent to two vertices of T, then the vertices are adjacent in T for the same reason. At most one of these components has a t_1 vertex or a t_1 pair. This component, together with the adjacent vertex/vertices of T, is denoted by G'. If there is no component with t_1-vertex/vertices, choose an arbitrary component and add the corresponding vertices of T to obtain G'.
Color G' first. This is possible because G' has fewer Type 4 subsets than G and fulfils the lemma's assumption. Next, color the remaining vertices of T. Note that each of the remaining components, together with the corresponding adjacent (and already colored) vertex/vertices of T, fulfils the lemma's assumption (having one t_1 vertex or one t_1 pair belonging to T), and again has fewer subsets of Type 4 than G. Therefore, we can color these components as well. □

3 List Coloring for Outerplanar Graphs

We will use the following notations throughout the rest of the paper.

- The components of $G[W]$ are denoted by $C_1, \ldots, C_r$, the first neighborhoods by $F_1, \ldots, F_r$, and the corresponding second neighborhoods by $S_1, \ldots, S_r$.
- Let G^* be the graph that remains if we delete $C_1, \ldots, C_r$ and $F_1, \ldots, F_r$ from G.
- Let G_j, $j = 1, \ldots, s$ be the components of G^*.
- $S_{ij} = S_i \cap V(G_j)$.
- If $x \in S_{ij}$ has at least k neighbors in F_i ,then x is called a k^+-vertex. If $x \in S_{ij}$ has exactly k neighbors in F_i, then x is called a k-vertex.

Proof of Theorem 1.
Note that an outerplanar graph is K_4- and K_{23}-minor-free. Thus, such a graph has no subdivision of K_4 or K_{23}.
First we shall prove the following property:
If G is an outerplanar graph, then every S_{ij} is a collection of vertices where either all vertices are 1-vertices or there is exactly one 2-vertex x (the others being 1-vertices). In this case, the component of $G[S_{ij}]$ containing x is a tree $T = (V_T, E_T)$ and there is no path in G between nonadjacent $v_i, v_j \in V_T$ outside of T.

- If S_{ij} has a 3^+ vertex, then there is a subdivision of K_{23} which is forbidden in an outerplanar graph.
- If there are at least two 2 vertices, then we can always find a subdivision of K_4 or K_{23}, which contradicts the graph's outerplanarity. Three of these configurations are given in Figure 4.

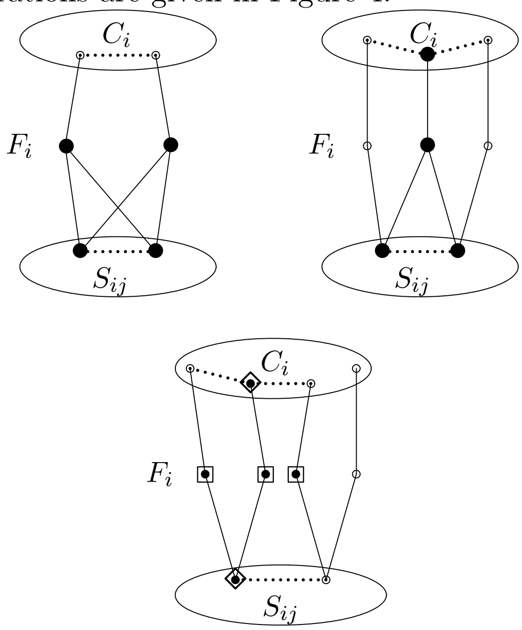

Figure 4: configurations with subdivisions of K_4 or $K_{2,3}$

- If there is no 2-vertex we are done.

We may assume that there is exactly one 2-vertex x and the rest are 1-vertices. Moreover, the component of $G^*[S_{ij}]$ containing x cannot contain a

cycle since there would otherwise be a subdivision of K_4 in G. Therefore, such a component is a tree.
If such a tree $T = (V_T, E_T)$ has more than two vertices, then there are no paths between nonadjacent $v_i, v_j \in V_T$ in G^* because there would otherwise be a subdivision of K_4 in G, since every vertex of V_T has a path to C_i.
We are now ready to extend the coloring. Note that for all i the subgraph $G[F_i]$ does not contain a cycle since there would otherwise be a subdivision of K_4 in G. Thus, every $G[F_i]$ is a forest and every vertex has at least two available colors if we delete the colors used for its neighbors in C_i. We are therefore able to color the vertices of F_i properly with respect to C_i for all i and delete the used colors from the lists of the corresponding neighbors in S_i. It follows that a 2-vertex of S_i now has at least two available colors and a 1-vertex of S_i now has at least three available colors.
Every component G_j of G^* now fulfills the assumption of Lemma 3. The distance constraints of the assumption are also satisfied because of $d(W) \geq 7$. Thus, according to Lemma 3, we can extend the coloring. □

4 Ordinary Coloring for K_4-minor-free Graphs

Proof of Theorem 2.
We will first find a coloring for the vertices of F_i for all i and forbid the used colors for the coloring of the neighbors. Then every vertex in S_{ij} has a set (list) of some available colors.
We will prove the following claim:
Claim: *We can color the F_i's in such a way that every S_{ij} contains at most one vertex with two available colors and all other vertices have at least three available colors.*
Proof.

- *F_i is a forest for every i.*

  Assume the contrary. Then there is a cycle in F_i which implies, together with a suitable vertex in C_i, a subdivision of K_4 (and therefore a K_4-minor), which leads to a contradiction.

- *Every vertex in S_{ij} has at least two available colors.*

  Assume w.l.o.g. that C_i is colored by $1, 2$. We use the colors 3 and 4 for the coloring of F_i. Thus, every vertex in S_i still has at least two available colors, namely 1 and 2.

If S_{ij} contains only one vertex with at least two neighbors in F_i, we are done. Thus, we may assume that there is an S_{ij} with at least two 2^+-vertices, say v_1, v_2.

- *v_1 and v_2 do not have a common neighbor in F_i.*

  Assume the contrary. Let x_1, x_2 be two neighbors of v_1 and x_3, x_4 be two neighbors of v_2 in F_i. If v_1 and v_2 have two common neighbors in F_i, that is $x_1 = x_3$ and $x_2 = x_4$, then we immediately find a subdivision of K_4. If v_1 and v_2 have exactly one common neighbor in F_i, say $x_2 = x_3$, we can also find a subdivision of K_4 (v_1, v_2, x_2 and a suitable vertex of C_i).

Let x_1, x_2 be two neighbors of v_1 and x_3, x_4 be two neighbors of v_2 in F_i.

- *Every path from x_1 to x_2 in G contains vertices of G_j or C_i.*

  If there is a path without vertices of G_j and C_i, then we can find a subdivision of K_4.

- *Every path from x_2 to x_3 contains vertices of G_j or C_i.*

  If there is a path without vertices of G_j and C_i, then we can find a subdivision of K_4.

If an S_{ij} has at least two 2^+-vertices, then the set of all neighbors of 2^+-vertices in F_i is denoted by X_{ij}. Note that every X_{ij} is an independent set due to the above remarks.
Now, for every i consider the subgraph $H_i := G[F_i]$ of G induced by F_i. Construct a new graph T_i from H_i identifying the vertices of each set X_{ij} to a vertex y_j.

- *T_i is a forest for every i and does not contain double edges.*

  Assume the contrary. Then there is a cycle (possibly of length 2) in T_i. Thus, there is a cycle in $G[V(G^*) \cup F_i]$ in the original graph which contains at least three vertices of F_i, since any two vertices of a fixed X_{ij} are joined by a path which uses vertices from the corresponding G_j. But these three vertices, together with a suitable vertex of C_i, imply a subdivision of K_4 in G.

Coloring: We are now ready for the coloring of F_i. First, color the vertices of T_i properly with 3 and 4, assuming that C_i is colored with 1 and 2. Color each of vertices of X_{ij} with the color which is used for the corresponding vertex $y_j \in V(T_i)$. Such a coloring is possible since every X_{ij} is independent. Thus, all vertices of X_{ij} have the same color for a fixed i and j . Consequently, if there are multiple 2^+-vertices in S_{ij}, then all of their neighbors in F_i belong to X_{ij} and have the same color. Thus, the coloring of F_i forbids only one color for each of these vertices. This completes the proof of the claim.
Note that every component of $G^*[S_{ij}]$ is a tree $T = (V_T, E_T)$, because we can otherwise find a subdivision of K_4 in G, which is forbidden. For the same reason, there is no path between nonadjacent vertices of T outside of T in G^*. Now every component G_j of G^* fulfills the assumption of Lemma 3. The distance constraints of the assumption are also satisfied because of $d(W) \geq 7$. Thus, by Lemma 3, we can extend the coloring. □

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