Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 07/18

Perturbation theory for self-adjoint operators in Krein spaces

Trunk, Carsten

2007

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

Perturbation Theory for Self-Adjoint Operators in Krein spaces

Carsten Trunk

Institut für Mathematik, Technische Universität Ilmenau,
Postfach 10 05 65, 98684 Ilmenau, Germany

E-mail: `carsten.trunk@tu-ilmenau.de`

Abstract. We show that the spectrum of negative type and the spectrum of positive type of self-adjoint operators in Krein spaces are stable under perturbations small in the gap metric. Moreover, we show how these notions can be used to decide whether a given operator has real spectrum only. We apply this to a $\mathcal{PT}$-symmetric multiplication operator.

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

It is well-known that an isolated real eigenvalue of definite type of a self-adjoint oprerator A in a Krein space remain real under sufficiently small self-adjoint perturbations.

It is the aim of this paper to extend these results to points from the continuous spectrum and for non-isolated eigenvalues. In order to do this we recall the definition of different kind of spectra: The spectral points of positve and of negative type and the spectral points of type π_+ and of π_-. A real point λ of the spectrum $\sigma(A)$ is called a spectral point of positive (negative) type, if for every normed approximative eigensequence (x_n) corresponding to λ all accumulation points of the sequence $([x_n, x_n])$ are positive (resp. negative). These spectral points were introduced by P. Lancaster, A. Markus and V. Matsaev in [19]. In [21] the existence of a local spectral function was proved for intervals containing only spectral points of positive (negative) type or points of the resolvent set $\rho(A)$. Moreover it was shown that, if A is perturbed by a compact selfadjoint operator, a spectral point of positive type of A becomes either an inner point of the spectrum of the perturbed operator or it becomes an eigenvalue of type π_+. A point from the approximative point spectrum of A is of type π_+ if the abovementioned property of approximative eigensequences (x_n) holds only for sequences (x_n) belonging to some linear manifold of finite codimension (see Definition 4 below). Every spectral point of a selfadjoint operator in a Pontryagin space with finite index of negativity is of type π_+. For a detailed study of the properties of the spectrum of type π_+ we refer to [2].

In this paper we show how these notions can be used to decide whether a given operator has real spectrum only. Moreover, as the main result of this paper, we show that the spectrum of negative type and the spectrum of positive type of self-adjoint operators in Krein spaces are stable under perturbations small in the gap metric.

Sign type spectrum is used in the classification of eigenvalues, e.g. [6, 7, 9, 12, 18, 22] and it can be applied to $\mathcal{PT}$-symmetric problems. We will give an example with a $\mathcal{PT}$-symmetric multiplication operator in Section 3. Moreover, it is used in the theory of indefinite Sturm-Liouville operators, e.g. [3, 5, 10, 16], and in the mathematical system theory, see e.g. [14, 20].

2. Sign type spectrum of self-adjoint operators in Krein spaces

2.1. Self-adjoint operators in Krein spaces

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. We briefly recall that a complex linear space $\mathcal{H}$ with a hermitian nondegenerate sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a so called *fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot,\cdot]$ such that $(\mathcal{H}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces. In the following all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent. An element $x \in \mathcal{H}$ is called *positive* (*negative, neutral*, respectively) if $[x,x] > 0$ ($[x,x] < 0$, $[x,x] = 0$, respectively). For the basic theory of Krein space and operators acting therein we refer to [8], [1] and, in the context of $\mathcal{P}T$ symmetry, we refer to [22].

Let A be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. The resolvent set of A is denoted by $\rho(A)$. The operator A is called *Fredholm* if the dimension of the kernel of A and the codimension of the range of A are finite. The set

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of A. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* A, denoted by $\sigma_{ap}(A)$, if there exists a sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$$

(see e.g. [11, 25]). Obviously, the continuous and the point spectrum of a closed operator are subsets of the approximate point spectrum. Moreover, we have the following.

Remark 1 *The boundary points of $\sigma(A)$ in $\mathbb{C}$ belong to $\sigma_{ap}(A)$.*

Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. Then all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. Corollary VI.6.2 in [8]).

2.2. Spectral points of positive and negative type of A

The indefiniteness of the scalar product $[\cdot,\cdot]$ on $\mathcal{H}$ induces a natural classification of isolated real eigenvalues: A real isolated eigenvalue λ_0 of A is called of *positive* (*negative*) *type* if all the corresponding eigenelements (i.e. all elements of all Jordan chains corresponding to λ_0) are positive (negative, respectively). Observe that there is no Jordan chain of length greater than one which corresponds to a eigenvalue of A of positive type (or of negative type). This classification of real isolated eigenvalues is used frequently, we mention here only [6, 7, 9, 12, 18, 22].

There is a corresponding notion for points from the approximate point spectrum. The following definition was given in [19] and [21] for bounded self-adjoint operators.

Definition 2 *For a self-adjoint operator A in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* positive (negative) type of A *if $\lambda_0 \in \sigma_{ap}(A)$ and for every sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0 I)x_n\| \to 0$ as $n \to \infty$ we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp.} \limsup_{n\to\infty} [x_n, x_n] < 0).$$

The point ∞ *is said to be of* positive (negative) type of A *if* A *is unbounded and for every sequence* $(x_n) \subset \mathrm{dom}(A)$ *with* $\lim_{n\to\infty} \|x_n\| = 0$ *and* $\|Ax_n\| = 1$ *we have*

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0).$$

We denote the set of all points of $\sigma_e(A)$ *of positive (negative) type by* $\sigma_{++}(A)$ *(resp.* $\sigma_{--}(A)$*).*

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. Indeed, for $\lambda \in \sigma_{++}(A)\setminus\{\infty\}$ and (x_n) as in the first part of Definition 2 we have $-(\mathrm{Im}\,\lambda)[x_n, x_n] = \mathrm{Im}\,[(A-\lambda)x_n, x_n] \to 0$ for $n \to \infty$ which implies $\mathrm{Im}\,\lambda = 0$. Here $\overline{\mathbb{R}}$ denotes the set $\mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{C}}$ the set $\mathbb{C} \cup \{\infty\}$, each equipped with the usual topology. In the following proposition we collect some properties. For a proof we refer to [2].

Proposition 3 *Let* λ_0 *be a point of* $\sigma_{++}(A)$ $(\sigma_{--}(A)$, *respectively*$)$. *Then there exists an open neighbourhood* $\mathcal{U}$ *in* $\overline{\mathbb{C}}$ *of* λ_0 *such that the following holds.*

(i) *We have*

$$\mathcal{U} \setminus \overline{\mathbb{R}} \subset \rho(A),$$

this is, the non-real spectrum of A *cannot accumulate to* $\sigma_{++}(A) \cup \sigma_{--}(A)$.

(ii) $\mathcal{U} \cap \sigma_e(A) \cap \overline{\mathbb{R}} \subset \sigma_{++}(A)$ *(resp.* $\mathcal{U} \cap \sigma_e(A) \cap \overline{\mathbb{R}} \subset \sigma_{--}(A)$*).*

(iii) *There exists a number* $M > 0$ *such that*

$$\|(A-\lambda)^{-1}\| \leq \frac{M}{|\mathrm{Im}\,\lambda|} \quad \textit{for all } \lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}.$$

2.3. Spectral points of type π_+ *and of type* π_- *of* A

In a similar way as above we define subsets $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$ of $\sigma_e(A)$ containing $\sigma_{++}(A)$ and $\sigma_{--}(A)$, respectively (cf. Definition 5 in [2]). They will play an important role in the following.

Definition 4 *For a self-adjoint operator* A *in* $\mathcal{H}$ *a point* $\lambda_0 \in \sigma(A)$ *is called a spectral point of* type π_+ (type π_-) of A *if* $\lambda_0 \in \sigma_{ap}(A)$ *and if there exists a linear submanifold* $\mathcal{H}_0 \subset \mathcal{H}$ *with* $\mathrm{codim}\,\mathcal{H}_0 < \infty$ *such that for every sequence* $(x_n) \subset \mathcal{H}_0 \cap \mathrm{dom}(A)$ *with* $\|x_n\| = 1$ *and* $\|(A - \lambda_0 I)x_n\| \to 0$ *as* $n \to \infty$ *we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0).$$

The point ∞ *is said to be of* type π_+ (type π_-) of A *if* A *is unbounded and if there exists a linear submanifold* $\mathcal{H}_0 \subset \mathcal{H}$ *with* $\mathrm{codim}\,\mathcal{H}_0 < \infty$ *such that for every sequence* $(x_n) \subset \mathcal{H}_0 \cap \mathrm{dom}(A)$ *with* $\lim_{n\to\infty} \|x_n\| = 0$ *and* $\|Ax_n\| = 1$ *we have*

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0).$$

We denote the set of all points of $\sigma_e(A)$ *of type* π_+ *(type* π_-*) of* A *by* $\sigma_{\pi_+}(A)$ *(resp.* $\sigma_{\pi_-}(A)$*).*

Spectral point of type π_+ and type π_- of A have properties comparable to those mentioned in Proposition 3. We will collect them in the following proposition (for a proof see [2] and [4]).

Proposition 5 *Let λ_0 be a point of $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$, respectively). Then there exists an open neighbourhood $\mathcal{U}$ in $\overline{\mathbb{C}}$ of λ_0 such that the following holds.*

(i) *We have*

$$\mathcal{U} \setminus \overline{\mathbb{R}} \subset \sigma_p(A).$$

Moreover, if $\lambda_0 \in \mathbb{C}$ is non-real then the operator $A - \lambda_0$ has a closed range and $\dim \ker (A - \lambda_0) < \infty$.

(ii) $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_+}(A)$ *(resp.* $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_-}(A)$*).*

(iii) *If $\lambda_0 = \infty$ then $\infty \in \sigma_{++}(A)$. If $\infty \in \sigma_{\pi_-}(A)$ then $\infty \in \sigma_{--}(A)$.*

(iv *Assume, in addition, that $\lambda_0 \in \mathbb{R}$ and that there is $[a, b] \subset \mathcal{U}$, $\lambda_0 \in [a, b]$, such that each point of $[a, b]$ is an accumulation point of $\rho(A)$. Then there exists an open neighbourhood $\mathcal{V}$ in $\mathbb{C}$ of $[a, b]$ such that $\mathcal{V} \setminus \mathbb{R} \subset \rho(A)$ and either $\mathcal{V} \cap \sigma(A) \cap \mathbb{R} \subset \sigma_{++}(A)$ or there exists a finite number of points $\lambda_1, \ldots, \lambda_n \in \sigma_{\pi_+}(A) \cap \sigma_p(A)$ such that*

$$(\mathcal{V} \cap \sigma(A) \cap \mathbb{R}) \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{++}(A).$$

Moreover, in this case there exist numbers $m \geq 1$ and $M > 0$ such that

$$\|(A - \lambda)^{-1}\| \leq \frac{M}{|\operatorname{Im} \lambda|^m} \quad \text{for all } \lambda \in \mathcal{V} \setminus \overline{\mathbb{R}}.$$

3. Statements of the results

3.1. Location of the spectrum

Recall that a Krein space $(\mathcal{H}, [\cdot, \cdot])$ is called a *Pontryagin space* if one of the spaces $\mathcal{H}_+, \mathcal{H}_-$ in (1) is finite dimensional. Moreover, we will call a Krein space $(\mathcal{H}, [\cdot, \cdot])$ an *anti Hilbert space* if $(\mathcal{H}, -[\cdot, \cdot])$ is a Hilbert space.

It follows from Proposition 3 that the resolvent near a spectral point of positive type of a self-adjoint operator in a Krein space grows like the resolvent of a self-adjoint operator in a Hilbert space (up to a multiplicative constant). If, for some reason, the spectrum of a self-adjoint operator in a Krein space consists entirely out of spectrum of positive type, then the underlying Krein space turns out to be a Hilbert space, see Theorem 6 below.

Theorem 6 *Let A be a self-adjoint operator in $(\mathcal{H}, [\cdot, \cdot])$. Let A satisfy*

$$\sigma_e(A) = \sigma_{++}(A) \quad (\text{resp. } \sigma_e(A) = \sigma_{--}(A)). \tag{2}$$

Then $(\mathcal{H}, [\cdot, \cdot])$ is a Hilbert space (anti-Hilbert space, respectively).

If A satisfies instead of (2) the following condition

$$\sigma_e(A) = \sigma_{++}(A) \cup \sigma_{--}(A),$$

then A is similar to a self-adjoint operator in a Hilbert space.

Observe, that in the case of an unbounded operator A condition (2) implies also $\infty \in \sigma_{++}(A)$. There are comparable results for the spectrum of type π_+.

Theorem 7 *Let A be a self-adjoint operator in $(\mathcal{H}, [\cdot,\cdot])$ with $\rho(A) \neq \emptyset$ satisfying*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \textit{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \quad (\textit{resp. } \sigma_e(A) = \sigma_{\pi_-}(A)). \tag{3}$$

Then $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space and the space $\mathcal{H}_-$ in the fundamental decomposition (1) *is of finite dimension. Moreover, the set $\sigma(A) \setminus \mathbb{R}$ consists of at most finitely many eigenvalues with finite dimensional root subspaces, i.e.*

$$\sigma(A) \setminus \mathbb{R} \subset \sigma_p(A) \setminus \sigma_{ess}(A)$$

If A with $\rho(A) \neq \emptyset$ satisfies instead of (3) *the following condition*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \textit{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A). \tag{4}$$

Then the non-real spectrum of A consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$.

3.2. Stability properties of sign type spectrum under compact perturbations and under perturbations small in gap

Let A be a self-adjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. We assume that A is fundamentally reducible, that is, the operator A admits a matrix representation

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \tag{5}$$

with respect to a fundamental decomposition (1) of $(\mathcal{H}, [\cdot,\cdot])$ such that A_+ and A_- are self-adjoint operators in the Hilbert spaces $(\mathcal{H}_+, (\cdot,\cdot))$ and $(\mathcal{H}_-, -(\cdot,\cdot))$, respectively. In the case of a bounded A and a perturbed operator B of the form

$$B = \begin{pmatrix} A_+ & C \\ -C^* & A_- \end{pmatrix}$$

with some bounded operator C acting from $\mathcal{H}_-$ to $\mathcal{H}_+$, it was shown in [21] that

$$\operatorname{dist}(\lambda, \sigma(A_-)) > \|C\| \Longrightarrow \lambda \in \rho(B) \cup \sigma_{++}(B),$$
$$\operatorname{dist}(\lambda, \sigma(A_+)) > \|C\| \Longrightarrow \lambda \in \rho(B) \cup \sigma_{--}(B).$$

Theorem 8 below can be viewed as a generalization of this result.

Recall that the gap between two subspaces M and N of a Hilbert space is defined by

$$\hat{\delta}(M, N) := \max\{\sup_{u \in M, \|u\|=1} \operatorname{dist}(u, N), \sup_{v \in N, \|v\|=1} \operatorname{dist}(v, M)\}$$

(cf. [17]). If P_M and P_N denote the orthogonal projections on M and N, respectively, it follows

$$\hat{\delta}(M, N) = \|P_M - P_N\|.$$

Theorem 8 *Let A and B be self-adjoint operators in $(\mathcal{H},[\cdot,\cdot])$. Let A be a fundamentally reducible operator. If A_+ and A_- are given by the matrix representation* (5) *and if there exists a real $\gamma > 0$ such that for $\lambda \in \mathbb{R}$*

$$\hat{\delta}(\mathrm{graph}\,(A-\lambda), \mathrm{graph}\,(B-\lambda)) < \gamma \quad \textit{and} \quad \gamma^2\left(1 + (\mathrm{dist}\,(\lambda, \sigma(A_-)))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{++}(B).$$

If there exists a real $\gamma > 0$ such that for $\lambda \in \mathbb{R}$

$$\hat{\delta}(\mathrm{graph}\,(A-\lambda), \mathrm{graph}\,(B-\lambda)) < \gamma \quad \textit{and} \quad \gamma^2\left(1 + (\mathrm{dist}\,(\lambda, \sigma(A_+)))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{--}(B).$$

Theorem 8 can be considered as a generalization of Corollary 3.4 in [23].

Finally, we mention a perturbation result for spectral points of type π_+ (type π_-) which was already proved in [2].

Theorem 9 *Let A and B be self-adjoint operators in the Krein space $(\mathcal{H},[\cdot,\cdot])$. Assume that $\rho(A) \cap \rho(B) \neq \emptyset$ and that for some (and hence for all) $\mu \in \rho(A) \cap \rho(B)$*

$$(A-\mu)^{-1} - (B-\mu)^{-1} \quad \textit{is a compact operator.} \tag{6}$$

Then

$$(\sigma_{\pi_+}(A) \cup \rho(A)) \cap \mathbb{R} = (\sigma_{\pi_+}(B) \cup \rho(B)) \cap \mathbb{R}, \tag{7}$$

$$(\sigma_{\pi_-}(A) \cup \rho(A)) \cap \mathbb{R} = (\sigma_{\pi_-}(B) \cup \rho(B)) \cap \mathbb{R}. \tag{8}$$

Moreover, $\infty \in \sigma_{++}(A)$ ($\infty \in \sigma_{--}(A)$) if and only if $\infty \in \sigma_{++}(B)$ (resp. $\infty \in \sigma_{--}(B)$).

We mention that a similar statement as in Theorem 9 for spectral points of positive or negative type is in general not true.

3.3. Example

Denote by I the closed interval $[-1,1]$. Suppose V and W are functions from $L^\infty(I)$, that is, V and W are essentially bounded. Moreover, we assume that V is real-valued and even, that is

$$V(-x) = V(x) = \overline{V(x)}, \quad x \in I,$$

and that W is $\mathcal{PT}$-symmetric, that is

$$W(-x) = \overline{W(x)}, \quad x \in I.$$

Let $\mathfrak{D}$ be the set of all $f \in L^2(I)$ such that f and f' are absolutely continuous, $f(-1) = f(1) = 0$ with $f'' \in L^2(I)$. In the Hilbert space $L^2(I)$, equipped with the usual inner product

$$(f,g) = \int_I f(x)\overline{g(x)}dx, \quad f,g \in L^2(I),$$

we consider the operators A and B defined on $\mathfrak{D}$,

$$A := -\frac{d^2}{dx^2} + V \quad \text{and} \quad B := -\frac{d^2}{dx^2} + W, \quad \operatorname{dom} A = \operatorname{dom} B = \mathfrak{D}.$$

It is easily seen that A is a self-adjoint operator in the Hilbert space $(L^2(I), (\cdot, \cdot))$. In general the potential W is not real-valued and the operator B is not a self-adjoint operator in the Hilbert space $(L^2(I), (\cdot, \cdot))$. Therefore, we consider the inner product

$$[f, g] = \int_I f(x)\overline{g(-x)}dx, \quad f, g \in L^2(I).$$

Then $(L^2(I), [\cdot, \cdot])$ is a Krein space and W is a a self-adjoint operator in the Krein space $(L^2(I), [\cdot, \cdot])$, see [22]. The operator A is also self-adjoint in the Krein space $(L^2(I), [\cdot, \cdot])$ and A is fundamental reducible. If $\|V\|_{L^\infty} < \frac{3\pi^2}{8}$, then the spectrum consists of eigenvalues which are alternating between positive and negative type (see Theorem 4.1 in [22]), that is

$$\sigma(A) = \sigma_{++}(A) \cup \sigma_{--}(A).$$

Moreover, we have $\infty \notin \sigma_{++}(A) \cup \sigma_{--}(A)$.

Observe that

$$\hat{\delta}(\operatorname{graph}(A - \lambda), \operatorname{graph}(B - \lambda)) \leq \|V - W\|_{L^\infty}.$$

Now Theorem 8 implies the following.

Theorem 10 *Let $\lambda \in \mathbb{R}$. If*

$$\|V - W\|_{L^\infty}^2 \left(1 + (\operatorname{dist}(\lambda, \sigma_{--}(A)))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{++}(B).$$

If

$$\|V - W\|_{L^\infty}^2 \left(1 + (\operatorname{dist}(\lambda, \sigma_{++}(A)))^{-2}\right) < \frac{1}{4},$$

then

$$\lambda \in \rho(B) \cup \sigma_{--}(B).$$

4. Proofs

In this section we will prove Theorems 6-8 from Section 3.

4.1. Proof of Theorems 6 and 7

Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ with $\rho(A) \neq \emptyset$ satisfying (4). The resolvent set of a self-adjoint operator in a Krein space is symmetric with respect to the real axis (cf. [8]), hence there are points from $\rho(A)$ in the upper and in the lower half-plane. This and $\sigma_{ess}(A) \subset \mathbb{R}$ imply that $\sigma(A) \setminus \mathbb{R}$ consists only of isolated eigenvalues with finite algebraic multiplicity (see §5.6 in [17]). In particular, each point in $\overline{\mathbb{R}}$ is an accumulation point of $\rho(A)$ and Proposition 5 implies that the spectrum of A cannot accumulate to a real point. Moreover, from (4) and Proposition 5 (iii) we conclude $\infty \in \sigma_{++}(A) \cup \sigma_{--}(A)$. Therefore the non-real spectrum of A is bounded and the second part of Theorem 7 is proved. In order to show the first part of Theorem 7 we assume without loss of generality

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A). \tag{9}$$

It remains to show that $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space. Relation (9), Theorem 23 in [2] and Theorem 4.7 in [15] imply that A is a definitizable operator. Recall that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *definitizable* if $\rho(A) \neq \emptyset$ and if there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the spectrum of A is real or its non-real part consists of a finite number of points. Moreover, A has a spectral function $E(\cdot)$ defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not belong to some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [24]). Now Corollary 28 and Theorem 26 of [2] show that Theorem 7 holds true.

Theorem 6 is now a consequence of Theorem 7: Assume without loss of generality

$$\sigma_e(A) = \sigma_{++}(A).$$

Then Theorem 7 implies that $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space and the space $\mathcal{H}_-$ in the fundamental decomposition (1) is of finite dimension. If $\mathcal{H}_- \neq 0$, then there exists at least one non-positive eigenvector of A (see §12 in [13]) for some eigenvalue λ_0. This implies $\lambda_0 \notin \sigma_{++}(A)$, hence $\mathcal{H}_- = 0$ and $(\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space. The second part of Theorem 6 follows from Proposition 25 and Corollary 28 in [2].

4.2. Proof of Theorem 8

We will only prove the first part of Theorem 8. The second one follows then by a similar reasoning. Let λ be a real number in $\sigma(B)$ and assume that there exists a $\gamma > 0$ such that

$$\hat{\delta}(\operatorname{graph}(A-\lambda), \operatorname{graph}(B-\lambda)) < \gamma \quad \text{and} \quad \gamma^2\left(1 + (\operatorname{dist}(\lambda, \sigma(A_-)))^{-2}\right) < \frac{1}{4}.$$

Then $\lambda \in \sigma_{ap}(B)$. Let $(x_n^+ + x_n^-) \in \operatorname{dom} B$, $n = 1, 2\ldots$, $x_n^+ \in \mathcal{H}_+$, $x_n^- \in \mathcal{H}_-$, be a sequence with

$$\|x_n^+\|^2 + \|x_n^-\|^2 = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(B-\lambda)(x_n^+ + x_n^-)\| = 0 \tag{10}$$

We have

$$\operatorname{dist}\left(\begin{pmatrix} x_n^+ + x_n^- \\ (B-\lambda)(x_n^+ + x_n^-) \end{pmatrix}, \operatorname{graph}(A-\lambda)\right) < \gamma \left\| \begin{pmatrix} x_n^+ + x_n^- \\ (B-\lambda)(x_n^+ + x_n^-) \end{pmatrix} \right\|.$$

Hence, there exists $y_n^+ \in \operatorname{dom} A_+$, $y_n^- \in \operatorname{dom} A_-$ with

$$\|x_n^+ - y_n^+\|^2 + \|x_n^- - y_n^-\|^2 + \|(B-\lambda)(x_n^+ + x_n^-) - (A_+ - \lambda)y_n^+ - (A_- - \lambda)y_n^-\|^2$$

is less than

$$\gamma^2 \left\| \begin{pmatrix} x_n^+ + x_n^- \\ (B-\lambda)(x_n^+ + x_n^-) \end{pmatrix} \right\|^2.$$

In view of (10), we have

$$\limsup_{n\to\infty} \|x_n^- - y_n^-\|^2 + \|A_- y_n^- - \lambda y_n^-\|^2 < \gamma^2. \tag{11}$$

With (11), (10) and $\|A_- y_n^- - \lambda y_n^-\| \geq \operatorname{dist}(\lambda, \sigma(A_-))\|y_n^-\|$ we obtain

$$\begin{aligned}
\liminf_{n\to\infty} & \left[x_n^+ + x_n^-, x_n^+ + x_n^-\right] = \\
&= \liminf_{n\to\infty} \|x_n^+\|^2 - \|x_n^-\|^2 = \liminf_{n\to\infty} 1 - 2\|x_n^-\|^2 \\
&= 1 - 2\limsup_{n\to\infty} \|x_n^- - y_n^- + y_n^-\|^2 \\
&\geq 1 - 2\limsup_{n\to\infty} \left(2\|x_n^- - y_n^-\|^2 + 2\|y_n^-\|^2\right) \\
&\geq 1 - 4\limsup_{n\to\infty} \left(1 + \operatorname{dist}(\lambda, \sigma(A_-))^{-2}\right)\left(\|x_n^- - y_n^-\|^2 + \|A_- y_n^- - \lambda y_n^-\|^2\right) \\
&\geq 1 - 4\gamma^2\left(1 + \operatorname{dist}(\lambda, \sigma(A_-))^{-2}\right) > 0
\end{aligned}$$

and Theorem 8 is proved.

References

[1] Azizov T Ya and Iokhvidov I S 1989 *Linear Operators in Spaces with an Indefinite Metric* (John Wiley & Sons)

[2] Azizov T Ya, Jonas P and Trunk C 2005 *J. Funct. Anal.* **226** 114

[3] Behrndt J 2007 *J. Math. Anal. Appl.* **334** 1439

[4] Behrndt J, Philipp F and Trunk C 2006 *Methods Funct. Anal. Topology* **12** 326

[5] Behrndt J and Trunk C 2007 *J. Differential Equations* **238** 491

[6] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 270401

[7] Bender C M, Brody D C and Jones H F 2003 *Am. J. Phys.* **71** 1095

[8] Bognár J 1974 *Indefinite Inner Product Spaces* (Springer)

[9] Caliceti E, Graffi S and Sjöstrand J 2005 *J. Phys. A* **38** 185

[10] Ćurgus B and Langer H 1989 *J. Differential Equations* **79** 31

[11] Engel K-J and Nagel R 2000 *One-Parameter Semigroups for Linear Evolution Equations* (Springer)

[12] Günther U, Stefani F and Znojil M 2005 *J. Math. Phys.* **46** 063504

[13] Iohvidov I S, Krein M G and Langer H 1982 *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric* (Akademie-Verlag)

[14] Jacob B and Trunk C 2007 *Operators and Matrices* **1** 45

[15] Jonas P 2003 Ion Colojoara Anniversary Volume (Theta) 95

[16] Karabash I and Trunk C *submitted*
[17] Kato T 1976 *Perturbation Theory for Linear Operators* (Springer)
[18] Kirillov O N and Günther U 2006 *J. Phys. Math. Gen.* **39** 10057
[19] Lancaster P, Markus A and Matsaev V 1995 *J. Funct. Anal.* **131** 1
[20] Lancaster P and Shkalikov A 1994 *Canadian Applied Mathematics Quarterly* **2** 45
[21] Langer H, Markus A and Matsaev V 1997 *Math. Ann.* **308** 405
[22] Langer H and Tretter C 2004 *Czech. J. Phys.* **54** 1113
[23] Langer H and Tretter C 2006 *Czech. J. Phys.* **56** 1063
[24] Langer H 1982 *Functional Analysis* (Springer) **948** 1
[25] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics. IV: Analysis of Operators* (Academic Press)