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Some Remarks on $\lambda_{p,q}$-Connectedness

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Abstract

A connected graph is $\lambda_{p,q}$-connected if there is a set of edges whose deletion leaves two components of order at least p and q, respectively. In this paper we present some sufficient conditions for graphs to be $\lambda_{p,q}$-connected. Furthermore, we study $\lambda_{2,q}$-connected graphs in more detail.

Keywords: Edge-connectivity; restricted edge-connectivity

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1 Introduction

We consider finite graphs without loops or multiple edges and use standard terminology as in [1] or [3].

It is an easy exercise to show that every connected graph has a vertex whose deletion results in a connected graph. The main problem we consider in this paper concerns similar results for pairs of adjacent vertices. Clearly, a general connected graph need not contain an edge such that the deletion of the two incident vertices results in a connected graph. In fact, deleting the vertices incident with any edge can result in a graph not having any large component.

This motivates the study of so-called $\lambda_{2,q}$-connected graphs for integers $q \geq 2$. A connected graph is $\lambda_{2,q}$-connected if it contains an edge such that the deletion of the two incident vertices results in a graph one component of which has order at least q. Equivalently, a connected graph is $\lambda_{2,q}$-connected if there is an edge cut whose removal results in two components containing at least 2 and at least q vertices, respectively.

This second definition makes it apparent that the problem we consider is closely related to the so-called restricted edge-connectivity first proposed by Harary [10]. In general, for integers $p, q \geq 1$, a connected graph is called $\lambda_{p,q}$-connected if it contains an edge cut whose removal results in two components containing at least p and at least q vertices, respectively. The smallest size of such an edge cut has been proposed by Esfahanian and Hakimi [6, 7] as a natural measure of fault-tolerance and was studied for various special network topologies [4, 5, 6, 16, 17].

For general graphs, the research mainly focused on the case $p = q$ [2, 11, 13, 14, 15, 18, 19, 20]. Next to explicit characterizations for small values of p and q, sufficient conditions for $\lambda_{p,q}$-connectedness and bounds on the sizes of the corresponding edge cuts were studied. We mention just some results and refer to the abundant cited literature for more.

Esfahanian and Hakimi [6] showed that a connected graph of order $n \geq 4$ is $\lambda_{2,2}$-connected if and only if it is not a star. Later Bonsma, Ueffing and Volkmann [2] characterized all $\lambda_{3,3}$-connected of order $n \geq 6$. Hellwig, Rautenbach and Volkmann [11] studied sufficient conditions for arbitrary values of p and q and Ou [14] characterized $\lambda_{p,p}$-connected graphs of order $n \geq 3p-2$. Recently, Zhao Zhang and Jinjiang Yuan [19] characterized the graphs which are $\lambda_{p,p}$-connected for some p which is at most the minimum degree of the graph plus one.

In the next section we will first show that the last mentioned result is an immediate consequence of a theorem due to Győri [9] and Lovász [12]. In the third section, we will then consider $\lambda_{2,q}$-connected graphs in detail.

2 $\lambda_{p,q}$-connected graphs

The main tool of this section is the following beautiful result which was first conjectured by Frank [8] in 1976. A subgraph of some graph $G = (V, E)$ induced by a set $X \subseteq V$ is denoted by $G[X]$.

Theorem 2.1 (Győri [9] 1978, Lovász [12] 1977) *For every k-connected graph $G = (V, E)$ of order n, k vertices $v_1, v_2, \ldots, v_k \in V$, and k positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \ldots + n_k = n$ there exists a partition $\{V_1, V_2, \ldots, V_k\}$ of V such that $v_i \in V_i$, $|V_i| = n_i$ and $G[V_i]$ is connected for $1 \leq i \leq k$.*

Corollary 2.2 *Let p and q be integers with $q \geq p \geq 1$. A connected graph G of order $n \geq p+q$ and minimum degree δ is $\lambda_{p,q}$-connected provided one of the following conditions is satisfied.*

(i) G is 2-connected.

(ii) G has a block of order at least $p+1$ containing at most one cut vertex.

(iii) $p = q \leq \delta + 1$ and G contains a block with at least two cut vertices.

(iv) $n \geq 2q-1$ and G contains a cut vertex u such that all components of $G[V \setminus \{u\}]$ are of order at least p.

Proof: (i) follows immediately from Theorem 2.1.

(ii) In view of (i), we may assume that G is not 2-connected. Let $G' = (V', E')$ denote a block of order at least $p+1$ containing exactly one cut vertex v_1. Let $v_2 \in V' \setminus \{v_1\}$. Applying Theorem 2.1 to G' with $n_1 = |V'| - p$ and $n_2 = p$ yields a partition $V_1 \cup V_2$ of V' such that $G[V_2]$ is a connected graph of order (exactly) p and $G[V \setminus V_2]$ is a connected graph of order (exactly) $n - p \geq q$.

(iii) Let G' arise from G by deleting all edges in a block of G with at least two cut vertices v_1 and v_2. If G_i denotes the component of G' containing v_i, then G_i obviously has order at least $\delta + 1 \geq p = q$ for $i = 1, 2$ which implies the desired result.

(iv) If $G' = (V', E')$ is a smallest component of $G[V \setminus \{u\}]$, then $|V'| \geq p$, $|V \setminus V'| \geq n - \frac{n-1}{2} = \frac{n+1}{2} \geq q$ and $G[V \setminus V']$ is connected. $\square$

Corollary 2.3 (Zhao Zhang and Jinjiang Yuan [19]) *If $p \in \mathbb{N}$ and G is a connected graph of order at least $2p$ and minimum degree at least $p - 1$, then G is $\lambda_{p,p}$-connected if and only if G does not arise by identifying one vertex from each of at least three disjoint cliques of order p.*

Proof: This follows immediately from Corollary 2.2 (i)-(iii). $\square$

Note that Zhao Zhang and Jinjiang Yuan [19] also estimate the size of the corresponding edge cut.

Before we proceed to the next section we phrase the following immediate observation for further reference.

Observation 2.4 *For integers $p, q \geq 1$ a connected graph is $\lambda_{p.q}$-connected if and only if it has a $\lambda_{p,q}$-connected spanning tree.*

Proof: The 'if'-part is immediate. The 'only if'-part follows for a graph $G = (V, E)$ with $\lambda_{p,q}$-edge cut S by joining spanning trees of the two components of $(V, E \setminus S)$ by an edge in S. $\square$

3 $\lambda_{2,q}$-connected graphs

In the following let a, b, c be non-negative integers with $a, c \geq 2$. Let $S(a, b, c)$ denote the tree of order $a + b + c$ that arises by joining the centers of two stars $K_{1,a-1}$ and $K_{1,c-1}$ by a path containing b internal vertices (cf. Figure 3). Furthermore, let $R(a, b, c)$ denote the graph of order $a + b + c$ that arises by joining the centers of two stars $K_{1,a-1}$ and $K_{1,c-1}$ with a new edge and adding b vertices which are adjacent to the centers of the two stars (cf. Figure 1).

We begin with a result about $\lambda_{2,q}$-connected trees.

Theorem 3.1 *Let n, q be two integers such that $n \geq q + 2 \geq 4$.*

(i) A tree T of order n is $\lambda_{2,q}$-connected if and only if it contains a non-endvertex u of degree at most $n - q$ which is adjacent to at most one non-endvertex.

(ii) A tree T of order $n \geq \frac{3(q-1)}{2}$ is not $\lambda_{2,q}$-connected if and only if

$$T \in \{K_{1,n-1}\} \cup \{S(a, b, c) \mid a, c \geq n - q + 1; a + b + c = n\}.$$

Proof: (i) The 'if'-part is immediate. For the 'only if'-part consider a $\lambda_{2,q}$-cut edge xy in a $\lambda_{2,q}$-connected tree T. Let the components of $T - xy$ containing x and y have at least 2 and q vertices, respectively. Let u denote a non-endvertex of T in the component of $T - xy$ containing x at maximum possible distance from y. Clearly, u has degree at most $n - q$ in T and is adjacent to at most one non-endvertex.

(ii) The 'if'-part is immediate. For the 'only if'-part consider a tree T of order $n \geq \frac{3(q-1)}{2}$ which is not $\lambda_{2,q}$-connected. By (i), every non-endvertex which is adjacent to at most one non-endvertex has degree at least $n - q + 1$, i.e. such a vertex together with the adjacent endvertices constitute already at least $n - q + 1$ vertices. Note that every tree with at least three such vertices necessarily also contains a non-endvertex which is adjacent to more than one non-endvertex. Since $3(n - q + 1) + 1 > n$, T has at most two such vertices which immediately implies that T is either a star or $T = S(a, b, c)$ with $a, c \geq n - q + 1$ and $a + b + c = n$. $\square$

The next result characterizes the $\lambda_{2,q}$-connected graphs for orders at least $2q - 3$. Note that (i) implies the result due to Esfahanian and Hakimi [6] mentioned in the introduction.

Theorem 3.2 *Let n, q be two integers such that $n \geq q + 2 \geq 4$.*

(i) A connected graph G of order $n \geq 2q - 1$ is not $\lambda_{2,q}$-connected if and only if $G = K_{1,n-1}$.

(ii) A connected graph G of order $n = 2q - 2$ is not $\lambda_{2,q}$-connected if and only if

$$G \in \{K_{1,n-1}, S(q-1, 0, q-1)\}.$$

(iii) A connected graph G of order $n = 2q - 3$ is not $\lambda_{2,q}$-connected if and only if (cf. Figure 1)

$$G \in \{K_{1,n-1}, S(q-1, 0, q-2), S(q-2, 1, q-2), R(q-2, 1, q-2)\}.$$

Proof: (i) The 'if'-part is immediate. For the 'only if'-part we consider a connected graph G of order $n \geq 2q - 1$ which is not $\lambda_{2,q}$-connected. By Observation 2.4 and (ii) of Theorem 3.1, the only spanning tree of G is $K_{1,n-1}$. This implies the desired result that $G = K_{1,n-1}$.

(ii) The 'if'-part is immediate. For the 'only if'-part we consider a connected graph G of order $n = 2q - 2$ which is not $\lambda_{2,q}$-connected. By (i) of the present result and (ii) of Theorem 3.1, all spanning trees of G belong to $\{K_{1,n-1}, S(q-1, 0, q-1)\}$. Since adding any further edge to one of these trees results in a $\lambda_{2,q}$-connected graph, it follows that $G \in \{K_{1,n-1}, S(q-1, 0, q-1)\}$.

(iii) The 'if'-part is immediate. For the 'only if'-part we consider a connected graph G of order $n = 2q - 3$ which is not $\lambda_{2,q}$-connected. By (i) of the present result and (ii) of Theorem 3.1, all spanning trees of G belong to $\{K_{1,n-1}, S(q-1, 0, q-2), S(q-2, 1, q-2)\}$. To $S(q-1, 0, q-2)$ one can only add one further edge, for example vy_1 (cf. Figure 1), and also to $S(q-1, 1, q-2)$ one can only add the edge vw (cf. Figure 1) resulting in a graph, namely $R(q-2, 1, q-2)$, which is not $\lambda_{2,q}$-connected. This leads to the desired result that

$$G \in \{K_{1,n-1}, S(q-1, 0, q-2), S(q-2, 1, q-2), R(q-2, 1, q-2)\}. \quad \square$$

Remark 3.3 Analogously to the proof of Theorem 3.2 (iii), one can show the following supplement to Theorem 3.2.

(iv) A connected graph G of order $n = 2q - 4 \geq q + 2$ is not $\lambda_{2,q}$-connected if and only if it is isomorhic to a star or to one of the ten graphs in Figure 2. (Note that $n \geq q + 2$ implies $q \geq 6$.)

The next result characterizes $\lambda_{2,q}$-connected graphs in terms of a specified spanning tree and properties of the additional edges.

Theorem 3.4 *Let n, q be two integers such that $n \geq q + 2 \geq 4$. If $G = (V, E)$ is a graph of order n which has a spanning tree that is isomorphic to $S(a, b, c)$ with $a, c \geq n - q + 1$ and $a + b + c = n$, then G is $\lambda_{2,q}$-connected if and only if one of the following conditions is satisfied. We denote the vertices of G as specified in Figure 3. (Note that the conditions on a, b and c imply $n \leq 2q - 2$.)*

(i) $x_i x_j \in E$ *for indices* i *and* j *with*

either $1 \leq i < j \leq a - 1$,
or $a + b + 2 \leq i < j \leq n$,
or $1 \leq i \leq a - 1$ *and* $a + 3 \leq j \leq n$,
or $1 \leq i \leq a + b - 2$ *and* $a + b + 2 \leq j \leq n$,
or $a \leq i < j \leq a + b + 1$ *and* $j - i \geq 3$.

(ii) $x_i x_j, x_k x_l \in E$ *for indices* i, j, k *and* l *with*

either $1 \leq i < k \leq a - 1$, $j = a + 1 \leq a + b$ *and* $l = a + 2$,
or $a + b + 2 \leq i < k \leq n$, $j = a + b - 1$ *and* $l = a + b \geq a + 1$,
or $1 \leq i \leq a - 1$, $j = a + 1 \leq a + b$, $k = a$ *and* $l = a + 2$,
or $a + b + 2 \leq i \leq n$, $j = a + b \geq a + 1$, $k = a + b - 1$ *and* $l = a + b + 1$,
or $1 \leq i \leq a - 1$, $j = a + 2 \leq a + b$, $k = a + 1$ *and* $l = a + 3$,
or $a + b + 2 \leq i \leq n$, $j = a + b - 1 \geq a + 1$, $k = a + b - 2$ *and* $l = a + b$.

(iii) $x_i x_{i+2}, x_{i+1} x_{i+3}, x_{i+2} x_{i+4} \in E$ *for* $a \leq i \leq a + b - 3$.

(iv) $x_{i_1} x_j, x_{i_2} x_j, \ldots, x_{i_r} x_j \in E$ *with*

either $2 \leq i_1 < i_2 < \ldots < i_r \leq a - 1$, $r \geq q + a - n$ *and* $j \in \{a + 1, a + 2\}$,
or $a + b + 3 \leq i_1 < i_2 < \ldots < i_r \leq n$, $r \geq q + b - n$ *and* $j \in \{a + b, a + b + 1\}$.

Proof: For the 'if'-part we have to check that every edge configuration as specified in one of the above conditions would result in a $\lambda_{2,q}$-connected graph. This is equivalent to showing the existence of an edge e^* such that the deletion of the two vertices incident with e^* results in a graph that has one component with at least q vertices. We will specify such an edge for the various conditions. Since it is trivial to check the existence of a large component, we leave this task to the reader.

If two endvertices x_i and x_j are adjacent (cf. the first four conditions in (i)), then $e^* = x_i x_j$. If $x_i x_j \in E$ with $1 \leq i \leq a-1$ and $a+3 \leq j \leq a+b+1$, then $e^* = x_{a+1}x_{a+2}$. If $x_i x_j \in E$ with $a \leq i \leq a+b-2$ and $a+b+2 \leq j \leq n$, then $e^* = x_{a+b-1}x_{a+b}$. If $x_i x_j \in E$ with $a \leq i < j \leq a+b+1$ and $j-i \geq 3$, then $e^* = x_{i+1}x_{i+2}$.

If $x_i x_j, x_k x_l \in E$ satisfy any of the conditions given in (ii), then $e^* = x_i x_j$.

If $x_i x_{i+2}, x_{i+1}x_{i+3}, x_{i+2}x_{i+4} \in E$ for $a \leq i \leq a+b-3$, then $e^* = x_{i+1}x_{i+3}$.

If $x_{i_1}x_j, x_{i_2}x_j, \ldots, x_{i_r}x_j \in E$ with $2 \leq i_1 < i_2 < \ldots < i_r \leq a-1$, $r \geq q+a-n$ and $j \in \{a+1, a+2\}$, then $e^* = x_1 x_a$. If $x_{i_1}x_j, x_{i_2}x_j, \ldots, x_{i_r}x_j \in E$ with $a+b+3 \leq i_1 < i_2 < \ldots < i_r \leq n$, $r = q+b-n$ and $j \in \{a+b, a+b+1\}$, then $e^* = x_{a+b+1}x_{a+b+2}$.

For the 'only if'-part one has to check that a graph G as in the theorem for which none of the above conditions holds is not $\lambda_{2,q}$-connected. Equivalently, one can argue that the deletion of the endpoints of any edge $\tilde{e} = \tilde{u}\tilde{v}$ from G results in a graph all components of which have order at most $q-1$. Note that $a+b, b+c \leq q-1$. Let $\tilde{G} = G[V \setminus \{\tilde{u}, \tilde{v}\}]$.

First, we consider the case that $\tilde{e}$ is an edge of $S(a,b,c)$. If $\tilde{e}$ is incident to an endvertex of $S(a,b,c)$, then at least $\min\{(a-1)-(q+a-n-1)-1, (b-1)-(q+b-n-1)-1\} = n-q-1$ of the endvertices of $S(a,b,c)$ will be isolated in $\tilde{G}$, because condition (iv) does not hold. Therefore, there are at most $n-2-(n-q-1) = q-1$ vertices in any component of $\tilde{G}$. If $\tilde{e}$ is $x_a x_{a+1}$ or $x_{a+b}x_{a+b+1}$, then a very similar argument applies. If $\tilde{e}$ is any other edge of $S(a,b,c)$, then no component of $\tilde{G}$ contains at least q vertices, because the last three conditions in (i) do not hold.

Next, we consider the case that $\tilde{e}$ is not an edge of $S(a,b,c)$. Similarly, as above one can consider the cases that $\tilde{e}$ is or is not incident to an endvertex of $S(a,b,c)$. In each case it is very simple to see which of the conditions not being satisfied implies that there is no large component. We leave the details to the reader. $\Box$

Theorem 3.1 together with the last result allow a characterization of $\lambda_{2,q}$-connected graphs for orders between $\max\left\{q+2, \frac{3q-3}{2}\right\}$ and $2q-2$.

Corollary 3.5 *Let n, q be two positive integers with $\max\left\{q+2, \frac{3q-3}{2}\right\} \leq n \leq 2q-2$. If $G = (V,E)$ is a connected graph of order n and T is a spanning tree of G, then G is $\lambda_{2,q}$-connected if and only if*

(i) either $T \notin \{K_{1,n-1}\} \cup \{S(a,b,c) \mid a, c \geq n-q+1; a+b+c = n\}$,

(ii) or $T = K_{1,n-1}$ and G has at least n edges,

(iii) or $T = S(a,b,c)$ for some positive integers a,b,c with $a, c \geq n-q+1$ and $a+b+c = n$ such that one of the conditions specified in Theorem 3.4 are satisfied.

Proof: The 'if'-part is immediate. For the 'only if'-part let G be a connected and $\lambda_{2,q}$-connected graph of order n with spanning tree T. If $T \notin \{K_{1,n-1}\} \cup \{S(a,b,c) \mid a, c \geq$

$n-q+1; a+b+c=n\}$, then (i) holds. Hence, we may assume that either $T=K_{1,n-1}$ or $T \in \{S(a,b,c) \mid a,c \geq n-q+1, a+b+c=n\}$. In the first case, G needs at least one edge more than T in order to be $\lambda_{2,q}$-connected. In the second case, the result follows directly from Theorem 3.4. Hence (ii) or (iii) hold which completes the proof. $\square$

From the last result one can deduce the following extremal result about $\lambda_{2,q}$-connected graphs for orders between $\max\left\{q+2, \frac{3q-3}{2}\right\}$ and $2q-2$.

Corollary 3.6 *Let n, q be two positive integers with $\max\left\{q+2, \frac{3q-3}{2}\right\} \leq n \leq 2q-2$. If $G=(V,E)$ is a connected graph of order n and size at least $2q-2$, then G is $\lambda_{2,q}$-connected.*

The graph $R(n-q+1, 2p-n-2, n-q+1)$ of size exactly $2q-3$ shows that the above bound on the size is best possible.

Proof: The proof relies on Theorem 3.4. By considering the maximum possible number of edges to add to $S(a,b,c)$ which do not result in a $\lambda_{2,q}$-connected graph, one easily sees that the largest such number of edges can be added to $S\left(\left\lfloor \frac{n}{2} \right\rfloor, 0, \left\lceil \frac{n}{2} \right\rceil\right)$. In fact, one can add $2q-n-2$ edges to this graph resulting in $R(n-q+1, 2p-n-2, n-q+1)$ which implies the desired result. $\square$

For orders below $\frac{3q-3}{2}$ a number of edges linear in q does no longer suffice in order to ensure $\lambda_{2,q}$-connectedness. A simple example for this effect is obtained by attaching $l \geq 2$ new envertices to all but one vertex of a clique of order k. Clearly, this graph has order $n=(k-1)(l+1)+1$, is not $\lambda_{2,n-l}$-connected and has $\binom{k}{2}+(k-1)l$ edges which is quadratic in n for fixed l.

The next result analyses the effect of long cycles on $\lambda_{2,q}$-connectedness.

Theorem 3.7 *For an integer $t \geq 1$ and $r \in \{0,1,2,3\}$ let $q=4t+r$. If G is a connected graph of order n with $n \geq q+2$ that contains a cycle C of length l with $l \geq 2t+r+1$ for $r \in \{0,1,2\}$ and $l \geq 2t+r$ for $r=3$, then G is $\lambda_{2,q}$-connected.*

Proof: Let $C: v_0v_1v_2...v_{l-1}v_0$. It is easy to see that G has a spanning tree T which contains all but one edge from C. Deleting from T the remaining $l-1$ edges of C yields l components with vertex sets $V_0, V_1, V_2, ..., V_{l-1}$ where $v_i \in V_i$ for $0 \leq i \leq l-1$. Let $n_i = |V_i|$ for $0 \leq i \leq l-1$.

Note that for every $0 \leq i \leq l-1$ and $1 \leq j \leq l-1$ the two graphs $G[V_i \cup V_{i+1} \cup ... \cup V_{i+j-1}]$ and $G[V_{i+j} \cup V_{i+j+1} \cup ... \cup V_{i-1}]$ are connected (all indices are taken modulo l).

Therefore, if $n_i = 2$ or $n_i = n_{i+1} = 1$ for some $0 \leq i \leq l-1$, then G is clearly $\lambda_{2,q}$-connected. Hence we may assume that $n_i \neq 2$ and that no two consecutive n_i's are equal to 1. This implies that

$$n_i + n_{i+1} \geq 1+3=4 \tag{1}$$

for $0 \leq i \leq l-1$. This implies

$$n = \sum_{i=0}^{l-1} n_i = (n_0+n_1)+(n_2+n_3)+... \geq \begin{cases} 2l & , l \text{ even}, \\ 2(l-1)+3=2l+1 & , l \text{ odd}. \end{cases} \tag{2}$$

If one of the l connected graphs $G[V_i \cup V_{i+1} \cup ... \cup V_{i+l-3}]$ for $0 \le i \le l-1$ has order at least q, then G is $\lambda_{2,q}$-connected because $G[V_{i+l-2} \cup V_{i+l-1}]$ has order at least 2. Hence, we may assume that

$$\sum_{j=0}^{l-3} n_{i+j} \le q-1$$

for $0 \le i \le l-1$. This implies

$$(l-2)n = (l-2)\sum_{i=0}^{l-1} n_i = \sum_{i=0}^{l-1}\sum_{j=0}^{l-3} n_{i+j} \le (q-1)l = (4t+r-1)l. \tag{3}$$

The second equation follows from a double-counting argument - each n_i is counted exactly $l-2$ times in the inner sums of the right term.

First we assume that l is even. Now (2) and (3) imply $(l-2)2l \le (4t+r-1)l$, i.e. $l \le 2t + \frac{3}{2} + \frac{r}{2}$. Since l is even, this implies $l \le 2t$ for $r = 0$ and $l \le 2t+2$ for $r \in \{1,2,3\}$. For $r \in \{0,2,3\}$ this contradicts the assumption on l.

Hence it remains the case that $r = 1$ and $l = 2t+2$. If $n = 2l = 4t+4 = q+3$, then we may assume that $n_0 = 3$ which clearly implies that G is $\lambda_{2,q}$-connected. If $n \ge 2l+1$, then (3) implies $(l-2)(2l+1) \le 4tl$, i.e. $l \le 2t + \frac{3}{2} + \frac{1}{l}$. Since $l \ge 3$ is even, this implies the contradiction $l \le 2t$.

Next we assume that l is odd. Now (2) and (3) imply $(l-2)(2l+1) \le (4t+r-1)l$, i.e. $l \le 2t+1+\frac{r}{2}+\frac{1}{l}$. Since $l \ge 3$ is odd, this implies $l \le 2t+1$. For $r \in \{1,2,3\}$ this contradicts the assumption on l.

Hence it remains the case that $r = 0$ and $l = 2t+1$. If $n = 2l+1 = 4t+3 = q+3$, then we may assume that $n_0 = 3$ which clearly implies that G is $\lambda_{2,q}$-connected. If $n \ge 2l+2$, then (3) implies $(l-2)(2l+2) \le (4t-1)l$, i.e. $l \le 2t + \frac{1}{2} + \frac{2}{l}$. If $l \ge 5$, then this implies the contradiction $l \le 2t-1$. If $l = 3$, then $q = 4$ and we may assume that $n_0 + n_1 \ge 4$ and $n_2 \ge 2$ which clearly implies that G is $\lambda_{2,q}$-connected. This completes the proof. □

The next example shows that Theorem 3.7 is best-possible in the case that $r = 2$. Similar examples exist for all parities $r \in \{0,1,3\}$.

Example 3.8 Let $q = 4t+2$ for an integer $t \ge 1$, and let $C = v_1v_2\dots v_{q-2t} = v_1v_2\dots v_{2t+2}$ be a cycle of length $q-2t = 2t+2$. In addition, let $u_1, u_2, \dots, u_{2t+2}$ be $2t+2$ further vertices such that v_{2i-1} is adjacent to u_{2i-1} and u_{2i} for $1 \le i \le t+1$. The resulting graph H (cf. Figure 4) is of order $q+2$ with a cycle of length $q-2t$, however, it is a simple matter to verify that H is not $\lambda_{2,q}$-connected.

We want to close with the following observation which establishes $\lambda_{2,q}$-connectedness using a structural property.

Observation 3.9 *Let $q \ge 2$ be an integer. If G is a connected and claw-free graph of order $n \ge q+2$, then G is $\lambda_{2,q}$-connected.*

Proof: Let $u_1 u_2 \ldots u_t$ be a longest path in G. Since G is claw-free, it is a simple matter to verify that $G - \{u_1, u_2\}$ is connected, and the proof is complete. $\Box$

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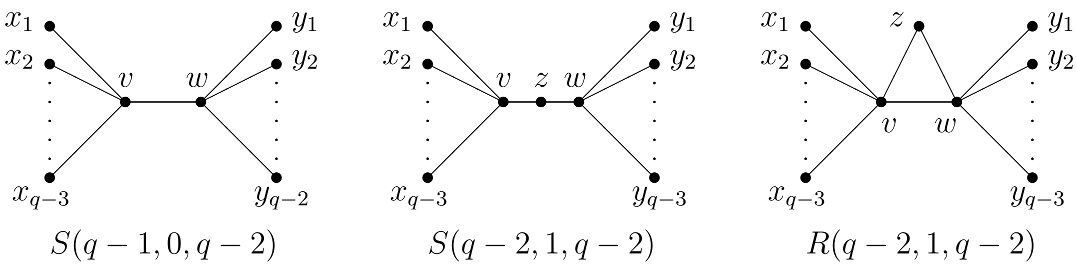

Figure 1: $S(q-1,0,q-2)$, $S(q-2,1,q-2)$ and $R(q-2,1,q-2)$.

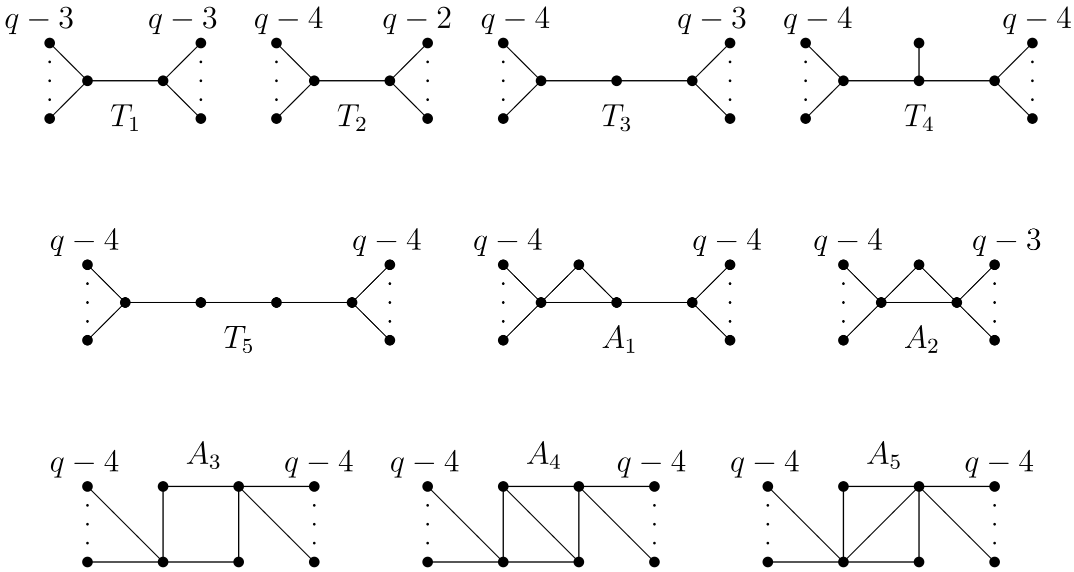

Figure 2: Graphs of order $2q-4$ which are not $\lambda_{2,q}$-connected.

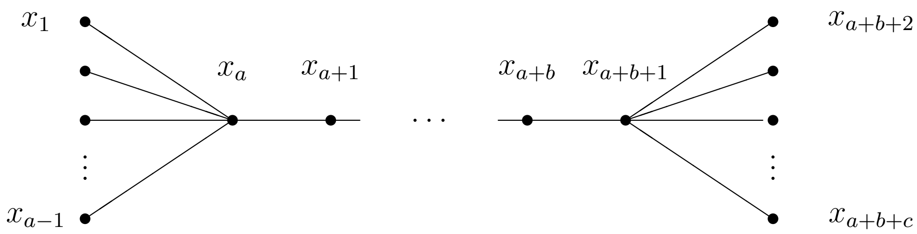

Figure 3: $S(a,b,c)$.

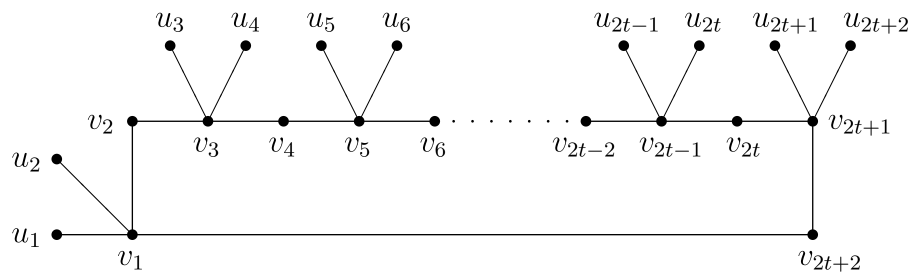

Figure 4: Graph showing that Theorem 3.7 is best-possible.

List of figure captions