Bifurcations from Homoclinic Orbits to a Saddle-Centre in Reversible Systems

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Contents

1 Introduction

Within this chapter we give an overview of the historical background of this thesis. In particular we point out the papers, which influenced our work. Further, we describe how this thesis is organised and state the underlying scenario.

1.1 Prologue

Already in the late 19th century the French mathematician and physicist Poincaré discovered the possibility of complicated, nearly irregular behaviour in deterministic model systems, [Po1890]. His investigations can be seen as the beginning of the qualitative analysis of dynamical systems. Qualitative analysis aims at understanding a system with respect to its asymptotic behaviour or the existence of special types of solutions, thereby using geometric, statistical or analytical techniques. Particular relevance has the study of how external parameters influence a system; corresponding research has established *bifurcation theory* as one of the main branches of modern applied analysis. In the last years in particular homoclinic orbits and their bifurcation behaviour have attracted much attention, since they are an "organising centre" for the nearby dynamics of the system. Under certain conditions complicated or even chaotic dynamics near these homoclinic orbits can occur. For historical notes of homoclinic bifurcations in general systems we refer to [Kuz98]. Champneys, [Cha98], presents a detailed overview of homoclinic bifurcations in reversible systems.
A second aspect for the importance of homoclinic orbits is their occurrence as solutions of dynamical systems arising as a travelling wave equation for a partial differential equation by an appropriate travelling wave ansatz. Then homoclinic solutions describe solitary waves (or solitons). We refer to [Rem96] for a detailed introduction and to [Cha99, CMYK01a, CMYK01b].

Many applications lead to dynamical systems with symmetries or systems that conserve quantities. For example the equations of motion of a mechanical system without friction are Hamiltonian, i.e., they preserve energy. Very frequently those systems are also *reversible*. Roughly speaking this means that they behave the same when considered in forward or in backward time. Reversibility has also been found in many systems, which are not Hamiltonian. Indeed, there are examples from nonlinear optics, where a spatial symmetry in the governing partial differential equation leads to reversibility of a corresponding travelling wave ordinary differential equation, without this equation being Hamiltonian, see [Cha99]. Considerations regarding reversible or Hamiltonian systems show the remarkable fact that those systems have many interesting dynamical features in common, see [Cha98, LR98] and the references therein. This concerns in particular the occurrence of certain orbits such as homoclinic or periodic ones. However, recently Homburg and Knobloch [HK06] could prove essential differences regarding the existence of more complicated dynamics such as shift dynamics. So, it is of interest to work out differences and similarities of reversible and Hamiltonian systems.

While earlier studies of homoclinic bifurcations were bound to homoclinic orbits to hyperbolic equilibria, in recent years many authors turned to systems with non-hyperbolic equilibria. In general in this case one expects bifurcations of the equilibrium, for example saddle-node bifurcations considered by Schecter, Hale and Lin in [Sch87, Sch93, HL86, Lin96]. We also refer to the monograph [IL99]. But under certain conditions non-hyperbolic equilibria can be robust, i.e. no bifurcations of the equilibrium occur under perturbation. For instance an equilibrium of saddle-centre type (there is a pair of purely imaginary eigenvalues; the rest of the spectrum consists of eigenvalues with non-zero real part) in a Hamiltonian or reversible system is robust. In both Hamiltonian and reversible systems the centre manifold of a saddle-centre equilibrium is filled with a family of periodic orbits, called Liapunov family, see [AM67, Dev76].
Within this thesis we consider bifurcations of homoclinic orbits to a saddle-centre equilibrium in reversible systems. Concerning this investigations the papers of Mielke, Holmes and O'Reilly, [MHO92], and Koltsova and Lerman, [Ler91, KL95, KL96] are of particular interest. Mielke, Holmes and O'Reilly studied reversible Hamiltonian systems in $\mathbb{R}^4$ having a codimension-two homoclinic orbit to a saddle-centre equilibrium (i.e., it unfolds in a two-parameter family). There they focussed on k-homoclinic orbits to the equilibrium and shift dynamics. The k-homoclinic orbits are orbits which intersect a cross-section to the primary homoclinic orbit in a tubular neighbourhood k times. Koltsova and Lerman made similar considerations in purely Hamiltonian systems. Besides they considered homoclinic orbits asymptotic to the periodic orbits lying in the centre manifold. However, in each case the underlying Hamiltonian structure was heavily exploited. So, it is a natural question to ask for a complete analysis for purely reversible systems with homoclinic orbits to a saddle-centre, [Cha98]. Champneys and Härterich, [CH00], gave first answers to the posed question for vector fields in $\mathbb{R}^4$. Thereby they focussed on bifurcating two-homoclinic orbits to the equilibrium. For that concern it is sufficient to confine the studies to one-parameter families of vector fields; the parameter controls the splitting of the (one-dimensional) stable and unstable manifolds.
In all mentioned papers [MHO92, Ler91, KL95, KL96, CH00] the analysis is based on the construction of a return map. This method was originally developed by Poincaré, and is nowadays a standard tool for the analysis of the dynamics near periodic orbits. Shilnikov adapted this method for homoclinic bifurcation analysis in flows, [Shi65, Shi67]; we also refer to Deng, [Den88, Den89], for the modern treatment of this technique.

In this thesis we address the above mentioned issue of [Cha98]. To study a similar scenario as Mielke, Lerman and their co-workers, we consider a codimension-two

homoclinic orbit Γ to a saddle-centre equilibrium in a purely reversible system in $\mathbb{R}^{2n+2}$. In the following Section 1.2 we explain the considered scenario in detail. We focus on bifurcating one-homoclinic orbits to the centre manifold and symmetric one-periodic orbits. (One-periodic orbits are orbits which intersect a cross-section to Γ in a tubular neighbourhood once.)
In Chapter 2 we give a survey of the main results concerning the dynamics. There we also outline the method which we use. In contrast to [MHO92, Ler91, KL95, KL96, CH00] we use Lin's method, [Lin90], which originally was developed for the investigation of the dynamics near orbits connecting hyperbolic equilibria. Indeed, in recent years this method has been advanced by other authors, see for instance [VF92, San93, Kno97], and [Kno04] for a detailed survey of Lin's method and its applications. However, the improvements and extensions do not touch the restriction to systems with hyperbolic equilibria. For that reason one important issue of this thesis is the corresponding extension of Lin's method. But this thesis does not provide a general theory of Lin's method for problems with non-hyperbolic equilibria. In fact, we adapt Lin's ideas to our problem only as far as necessary. However, our approach can be seen as a first step towards an aspired general theory.
The bifurcating one-homoclinic orbits are discussed in Chapter 3. Due to the orbit structure of the centre manifold (Liapunov family of periodic orbits) we distinguish one-homoclinic orbits to the equilibrium, one-homoclinic orbits to a periodic orbit and heteroclinic orbits connecting different orbits of the centre manifold. Our investigations are based on a modification of the derivation of Lin's method, [San93]. We prove the existence of special solutions $\gamma^{s(u)}$ within the (un)stable manifold of the equilibrium and search for solutions $\gamma^{+(-)}$ in the centre-(un)stable manifold as perturbations of $\gamma^{s(u)}$. Solving the bifurcation equation $\gamma^+(0) - \gamma^-(0) = 0$ leads to one-homoclinic orbits to the centre manifold. Thereby we have to distinguish two different cases regarding the relative position of the centre-stable manifold and the fixed space of the involution R (which is associated with the reversibility). Later these cases will be specified as elementary and non-elementary case, respectively. Our procedure allows to differentiate between homoclinic orbits to the equilibrium and orbits connecting periodic orbits of the centre manifold.
Bifurcating symmetric one-periodic orbits are studied in Chapter 4. As a generalisation of the method of Sandstede the solutions $\gamma^\pm$ serve as a basis for the search of these orbits. For technical reasons we restrict our investigations to vector fields in $\mathbb{R}^4$. The analysis in higher dimensions would be more complex. Furthermore, we restrict our considerations to the non-elementary case. Our analysis yields that each one-homoclinic orbit to the centre manifold is accompanied by a family of symmetric one-periodic orbits.
In Chapter 5 we present a detailed discussion of problems arising during our analysis. Further, we relate this thesis to the previous considerations of bifurcations of homoclinic orbits to a saddle-centre equilibrium in [CH00, MHO92, Ler91, KL95, KL96]. To keep this thesis self-contained, in Appendix A.1 we give a survey of some results about reversible systems. Our analysis exploits that the variational equation along a solution in the stable or unstable manifold has an exponential trichotomy

(see [HL86]). Therefore, in Appendix A.2 we introduce the idea of exponential trichotomy.

For standard notions and assertions from the theory of dynamical systems and functional analysis we refer for instance to [Rob95] and [Zei93].

1.2 Main Scenario

We consider a smooth system

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^{2n+2}, \lambda \in \mathbb{R}^2 . \tag{1.1}$$

The adjective smooth means, that f is in C^r for a sufficiently large number r which we do not specify here. Later within this section we will explain why we assume the parameter λ to be two-dimensional.
Further we assume that the system under consideration is reversible, i.e., there exists a linear involution R ($R^2 = id$) with

(H 1.1) $\quad Rf(x, \lambda) = -f(Rx, \lambda).$

A summary of fundamental facts concerning reversible systems can be found in Appendix A.1. We will use these essential properties of reversible systems without referring to them in detail.

For $\lambda = 0$ system (1.1) is assumed to possess a saddle-centre $\mathring{x}$:

(H 1.2) $\quad f(\mathring{x}, 0) = 0$ with $\sigma(D_1 f(\mathring{x}, 0)) = \{\pm i\} \cup \{\pm\mu\} \cup \sigma^{ss} \cup \sigma^{uu}$,

where $\mu \in \mathbb{R}^+$ and $|\Re(\tilde{\mu})| > \mu \;\; \forall \tilde{\mu} \in \sigma^{ss} \cup \sigma^{uu}$. Here σ^{ss} denotes the strong stable and σ^{uu} denotes the strong unstable spectrum of $D_1 f(\mathring{x}, 0)$. Because of the reversibility we have $\sigma^{uu} = -\sigma^{ss}$. Hypotheses (H 1.2) implies that n is the dimension of the stable and the unstable manifold of the equilibrium, respectively, the dimension of its centre manifold is two.

By our assumptions the local dynamics around $\mathring{x}$ is completely determined: First observe that $D_1 f(\mathring{x}, 0)$ is non-singular. Therefore we have for all (sufficiently small) λ a unique equilibrium point x_λ nearby $\mathring{x}$. Thus, we may assume that $x_\lambda \equiv 0$ (in particular $\mathring{x} \equiv 0$), i.e.,

$$f(0, \lambda) \equiv 0 \quad \forall \;\text{ small } \lambda \, ,$$

because we find a linear transformation generating this situation. Furthermore, the reversibility prevents that simple eigenvalues can move off the imaginary axis. Thus the spectrum of $D_1 f(0, \lambda)$ contains exactly one pair of purely imaginary eigenvalues as well, and for each λ we have a two-dimensional (local) centre manifold W^c_λ. By the Liapunov Centre Theorem for reversible systems, see [Dev76], the centre manifold is filled with symmetric periodic orbits surrounding the equilibrium, hence the local centre manifold is uniquely determined. Altogether, there are no local bifurcations (around the equilibrium $\mathring{x}$), neither of equilibria nor of periodic orbits.

All orbits in the local centre manifold are bounded. Thus we can define the centre-stable manifold as union of the stable manifolds of the periodic orbits filling the local centre manifold. This ensures the uniqueness of W^{cs}_λ. Analogously the centre-unstable manifold W^{cu}_λ is uniquely determined. Notice, that the uniqueness of both centre and centre-(un)stable manifold is a particular feature of the present situation. In general systems those manifolds are not unique, see [SSTC98] and [Van89].

Further, we assume the existence of a symmetric homoclinic orbit.

(H 1.3) For $\lambda = 0$ there exists a symmetric homoclinic orbit $\Gamma := \{\gamma(t) : t \in \mathbb{R}\}$ to the saddle-centre $\mathring{x}$ with $R\gamma(0) = \gamma(0)$.

The homoclinic orbit Γ has exactly one intersection point with the fixed space $\operatorname{Fix} R$ of the involution R. So, it makes sense to assume $R\gamma(0) = \gamma(0)$ which is equivalent to $\gamma(0) \in \operatorname{Fix} R$.

Although the equilibrium $\mathring{x}$ is non-hyperbolic, all solutions approaching the equilibrium for $t \to \infty$ are contained in its stable manifold W^s_λ. All solutions approaching the equilibrium for $t \to -\infty$ lie in the unstable manifold W^u_λ. (For $\lambda = 0$ we omit the index λ and just write W^s, for instance.) Hence

$$\Gamma \subset W^s \cap W^u \,.$$

Both manifolds are n-dimensional. To exclude degeneracies between W^s and W^{cu} we will suppose that

(H 1.4) $\dim \left(T_{\gamma(0)} W^s \cap T_{\gamma(0)} W^{cu}\right) = 1.$

Here, T_pW denotes the tangent space of a manifold W at a point p. By reversibility we also have $\dim \left(T_{\gamma(0)} W^u \cap T_{\gamma(0)} W^{cs}\right) = 1$. So (H 1.4) can be read as a non-degeneracy condition as it is usual for homoclinic orbits to hyperbolic equilibria. The assumption (H 1.4) does not imply that the homoclinic orbit Γ appears stably because the n-dimensional manifold W^s cannot intersect the $(n+1)$-dimensional fixed point space $\operatorname{Fix} R$ of R transversally. To be sure to consider a typical family we will assume

(H 1.5) $\{W^s_\lambda \,,\, \lambda \in U(0)\} \pitchfork \operatorname{Fix} R\,,$

where $U(0) \subset \mathbb{R}^2$ is a certain neighbourhood of zero in the parameter space. Recall that $\dim W^s_\lambda = n$ and $\dim \operatorname{Fix} R = n + 1$. Thus, to fulfil Hypothesis (H 1.5) a scalar parameter would be sufficient. Consequently there is a curve in the parameter plane corresponding to homoclinic orbits to the equilibrium.
Since $\dim W^{cu} = \dim W^{cs} = n{+}2$ the manifolds W^{cs} and W^{cu} can intersect transversally along Γ. Here we assume, however,

(H 1.6) W^{cs} and W^{cu} do not intersect transversally in $\gamma(0)$.

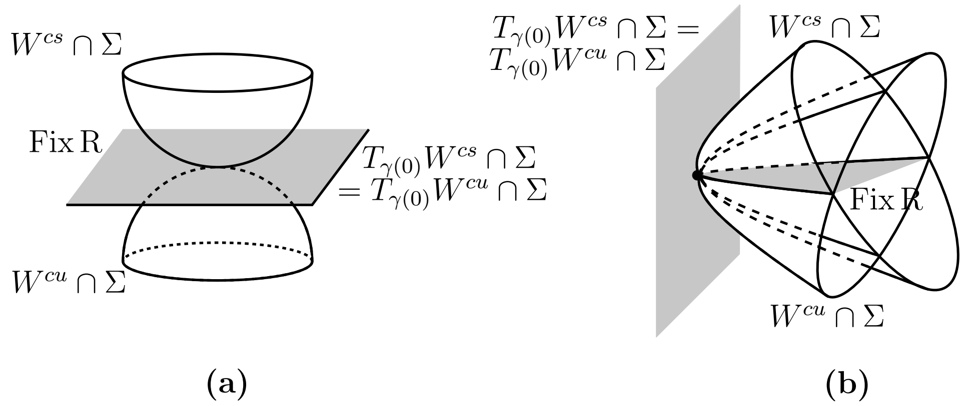

Figure 1.1: Possible intersection of centre-stable and centre-unstable manifold with Σ in case of **(a)** a non-elementary and **(b)** an elementary homoclinic orbit in the $\mathbb{R}^4$-case ($\dim\Sigma = 3$)

In our investigations it turns out that, similarly to the situations encountered in the case of hyperbolic equilibria, the relative position of W^{cs} and $\mathrm{Fix}\, R$ are of importance in the analysis. We call the symmetric homoclinic orbit Γ of Equation (1.1) ***non-elementary*** if W^{cs} intersects $\mathrm{Fix}\, R$ non-transversally. Otherwise we speak of an ***elementary*** homoclinic orbit. Mind that here, in contrast to the case of a hyperbolic equilibrium, an elementary homoclinic orbit (in general) does not persist under perturbations. The pictures depicted in Figure 1.1 should give an impression of the relative position of the manifolds that are involved. In drawing the pictures we restricted ourselves to vector fields in $\mathbb{R}^4$ and have only drawn the traces of the manifolds in a cross section $\Sigma \cong \mathbb{R}^3$ to the primary homoclinic orbit Γ. However, our analysis shows that these pictures also reflect the essential geometry in $\mathbb{R}^{2n+2}$ for arbitrary n.

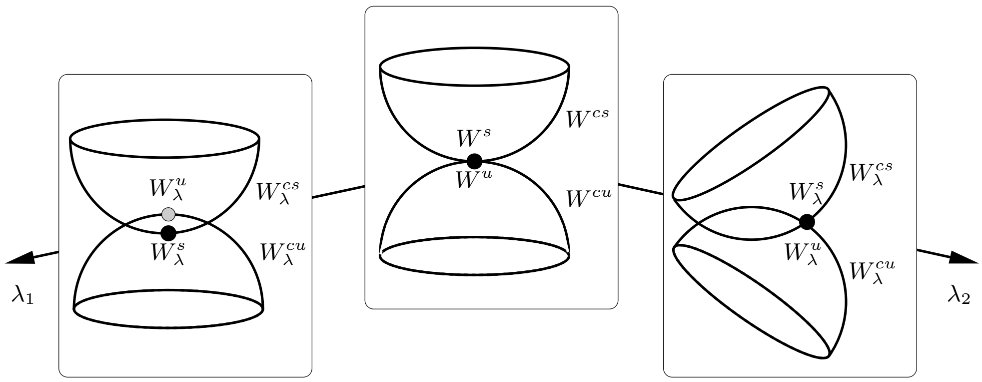

Figure 1.2: The geometrical meaning of the parameters λ_1 and λ_2 in case of a non-elementary homoclinic orbit Γ

Symmetric homoclinic orbits fulfilling the assumptions above occur generically in two-parameter families. The geometrical meaning of these two parameters can be seen as follows: One parameter, here λ_1, describes the drift of the stable and unstable manifold of the equilibrium. Now, keeping this parameter equal to zero the other parameter λ_2 can be used to unfold the non-transversal intersection of W^{cs} and W^{cu}. So it makes sense to consider the two-parameter system (1.1). We refer to Figure 1.2 and Figure 1.3 for a visualisation of the meaning of the two parameters.

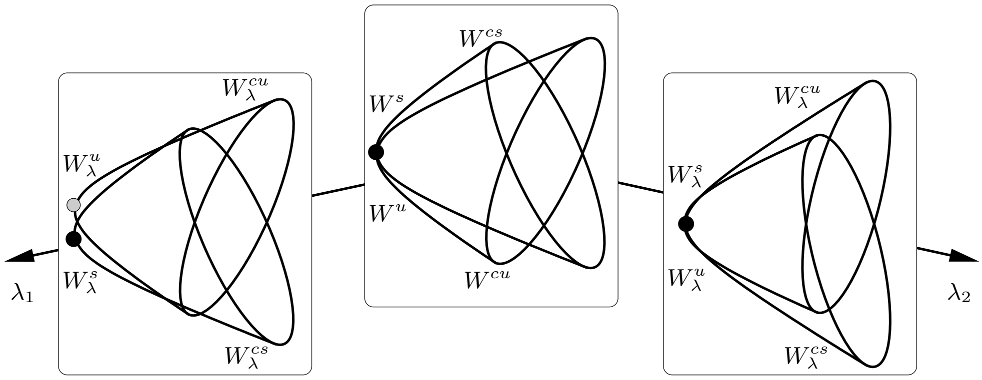

Figure 1.3: The geometrical meaning of the parameters λ_1 and λ_2 in case of an elementary homoclinic orbit Γ

2 Main Ideas and Results

In this chapter we present the main ideas and results of this thesis without giving proofs. We divide the exposition in two parts. In the first part we explain the method we use for our investigations. The dynamical results we describe in the second part.

2.1 Adaptation of Lin's method

The method we want to use was originally introduced by Lin [Lin90]. For that reason it is commonly referred to as Lin's method. In Lin's work a heteroclinic chain connecting hyperbolic equilibria is considered. The main idea is to construct "discontinuous orbits" near Γ, which we will call Lin orbits. Their main feature is that the only "discontinuities" which are allowed are jumps in prescribed directions. Making these jumps to zero we obtain actual orbits.

Our first objective is to adapt Lin's method to our purpose. In our explanations we will confine to homoclinic orbits to a saddle-centre as introduced. (In order to align this scenario with the general setting in Lin's method, consider a heteroclinic chain where both all equilibria and all connecting orbits coincide.)

We consider a homoclinic orbit Γ to a saddle-centre $\mathring{x}$ with properties (H 1.1)–(H 1.6). Let Σ be a cross section to Γ in $\gamma(0) \in \Gamma$. Furthermore, let Z, with $\gamma(0) + Z \subset \Sigma$, be a subspace transversally to $T_{\gamma(0)}W^s + T_{\gamma(0)}W^u$:

$$\mathbb{R}^{2n+2} = Z \oplus (T_{\gamma(0)}W^s + T_{\gamma(0)}W^u) .$$

By Hypothesis (H 1.4) $\dim Z = 3$.

In the following we give a precise definition of a Lin orbit. In preparation for that we introduce the notions partial solution and partial orbit.

Definition 2.1.1 *Let $\mathcal{U}$ be a neighbourhood of $\Gamma \cup \{\mathring{x}\}$ and let $x(\cdot)$ be a solution of Equation (1.1) on an interval $[0, \tau]$, $\tau \in \mathbb{R}^+$, with values in $\mathcal{U}$ satisfying*

(i) $x(0), x(\tau) \in \Sigma$;

(ii) $x(t) \notin \Sigma \quad \forall t \in (0, \tau)$.

*We call $x(\cdot)$ a **partial solution** connecting Σ; the corresponding orbit $X := \{x(t) : t \in [0, \tau]\}$ is referred to as **partial orbit**.*

Notice, that the partial solution is defined with respect to a given neighbourhood $\mathcal{U}$.

Definition 2.1.2 *A family* $\mathcal{X} := \{x_i\}_{i\in\mathbb{Z}}$ *of partial solutions* $x_i(\cdot)$ *connecting* Σ, *where* $x_i(\cdot)$ *is defined on* $[0,\tau_i]$, *is called* ***Lin solution*** *if*

$$\xi_i := x_i(\tau_i) - x_{i+1}(0) \in Z\ , i \in \mathbb{Z}\ .$$

The corresponding family $\mathcal{L} := \{X_i\}_{i\in\mathbb{Z}}$ *of partial orbits* X_i *is called* ***Lin orbit***.

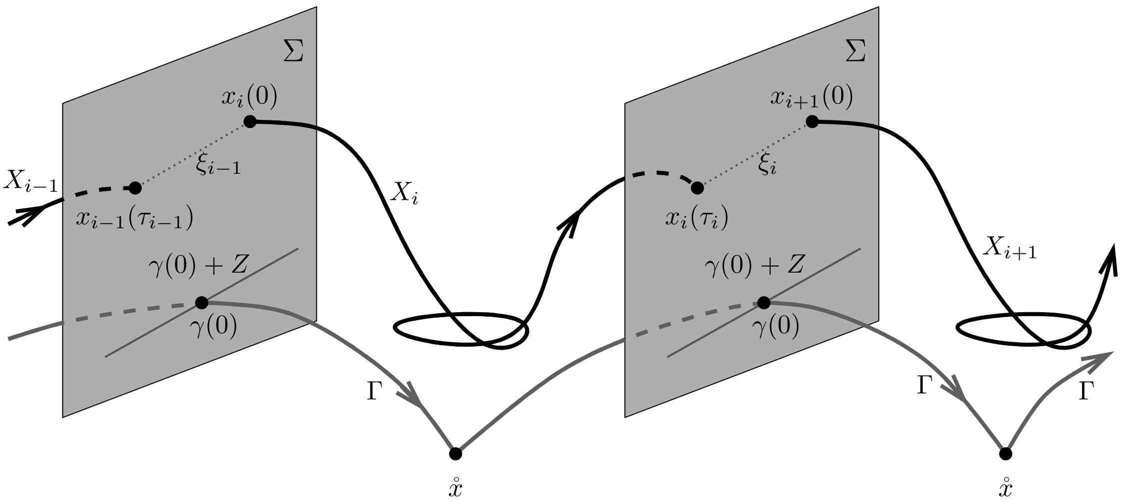

Figure 2.1: Lin orbit $\{X_i\}_{i\in\mathbb{Z}}$ near the homoclinic orbit Γ where Γ is considered as a heteroclinic chain

In Figure 2.1 a Lin orbit near the homoclinic orbit Γ is drawn. In order to depict the situation clearly, Γ is considered as a heteroclinic chain where both all equilibria and all heteroclinic connections are identified.

We want to remark that the notions *Lin solution* and *Lin orbit* are not bound to the particular configuration under consideration. In the same way one can define Lin orbits near an arbitrary heteroclinic chain (by Figure 2.1 one can get an idea of that). Further, we want to mention that the reversibility of the underlying equation does not play any role in the definition.

Notice, that a Lin orbit with corresponding Lin solution $\mathcal{X} := \{x_i\}_{i\in\mathbb{Z}}$ fulfilling

$$\xi_i = x_i(\tau_i) - x_{i+1}(0) = 0\ , i \in \mathbb{Z}\ ,$$

is a real orbit staying for all time near the primary homoclinic orbit Γ.

In order to construct partial orbits we decompose, as depicted in Figure 2.2, such an orbit into three parts: $X_i = (X_i^+, X_i^{loc}, X_i^-)$. The corresponding solutions $x_i^+(\cdot)$, $x_i^{loc}(\cdot)$ and $x_i^-(\cdot)$ are defined on $[0,\omega_i^+]$, $[0,\omega_i^{loc}]$ and $[-\omega_i^-,0]$. Further, they satisfy the conditions

$$x_i^+(0),\ x_i^-(0) \in \Sigma$$

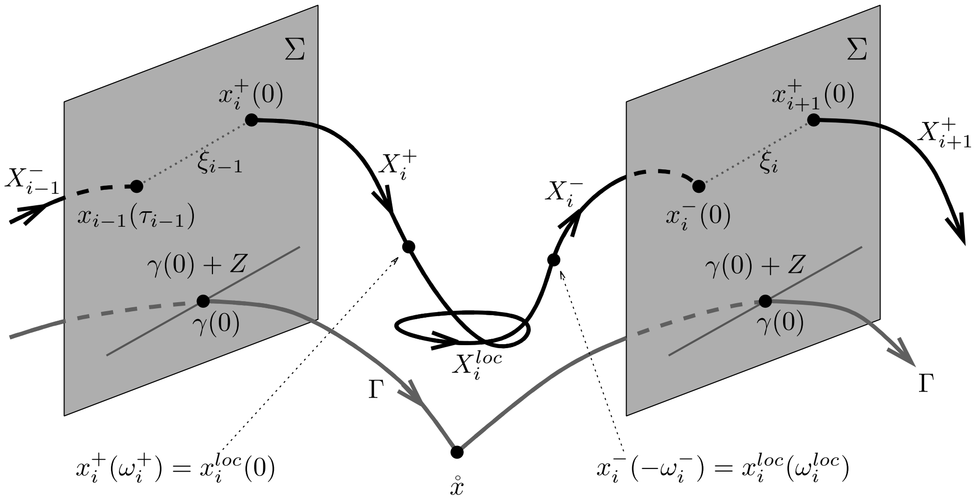

Figure 2.2: Lin orbit $\{X_i\}_{i\in\mathbb{Z}}$ near the homoclinic orbit Γ where the partial orbits X_i are decomposed into three parts: $X_i = (X_i^+, X_i^{loc}, X_i^-)$

and

$$x_i^+(\omega_i^+) = x_i^{loc}(0)\ , \quad x_i^-(-\omega_i^-) = x_i^{loc}(\omega_i^{loc})\ , \quad i \in \mathbb{Z}\ . \tag{2.1}$$

Roughly speaking the orbits X_i^+ and X_i^- follow the primary homoclinic orbit Γ between Σ and a small neighbourhood of the equilibrium $\mathring{x}$. The orbit X_i^{loc} on the other hand follows the flow in the centre manifold. The conditions (2.1) are referred to as coupling conditions. Note, that alternatively the local solutions can be characterised by the number N_i of "circulations around the equilibrium".

Remark 2.1.3 In the case of a homoclinic orbit asymptotic to a hyperbolic equilibrium the partial orbits X_i are decomposed only into the two orbits X_i^+ and X_i^- which are directly coupled near the equilibrium. □

For the construction of the local part X^{loc} of a partial orbit $X = (X^+, X^{loc}, X^-)$ we describe the local flow near the centre manifold by means of a Poincaré map $\Pi(\cdot,\lambda)$ with respect to a Poincaré section Σ_{loc} containing $\mathring{x}$. This Poincaré map is defined by means of the flow $\varphi(t,\cdot,\lambda)$ of the underlying differential equation (1.1). To be able to do that we have to assume the existence of a leaf which contains $\mathring{x}$ and is locally invariant with respect to φ, see (H 4.2).
For each small $\lambda \in \mathbb{R}^2$ the section Σ_{loc} is smoothly foliated into $\Pi(\cdot,\lambda)$-invariant leaves $\mathcal{M}_{p,\lambda}$ with base points p, which mark the intersections of the periodic orbits within the local centre manifold with Σ_{loc}. We assume this foliation to be smooth, see (H 4.3). Restricted to a leaf $\mathcal{M}_{p,\lambda}$ the Poincaré map $\Pi(\cdot,\lambda)$ possesses the hyperbolic fixed point p. For given (positive, sufficiently large) ω^+ and ω^- the traces $\mathcal{C}^+(\omega^+,p,\lambda)$ and $\mathcal{C}^-(\omega^-,p,\lambda)$ of $\varphi(\omega^+,\Sigma,\lambda)$ and $\varphi(-\omega^-,\Sigma,\lambda)$ in $\mathcal{M}_{p,\lambda}$ intersect the (local) stable and unstable manifolds of $\Pi(\cdot,\lambda)|_{\mathcal{M}_{p,\lambda}}$, respectively, transversally. An argument based on the λ-lemma allows for each sufficiently large $N \in \mathbb{N}$

and $\omega := (\omega^+, \omega^-)$ to connect $\mathcal{C}^+(\omega^+, p, \lambda)$ and $\mathcal{C}^-(\omega^-, p, \lambda)$ by an orbit segment $Z^{loc} = \{z^{loc}(0), \ldots, z^{loc}(N)\}$ of $\Pi(\cdot, \lambda)|_{\mathcal{M}_{p,\lambda}}$. This corresponds to a local orbit X^{loc} of (1.1) connecting $z^{loc}(0)$ and $z^{loc}(N)$.
With that we get the orbits X^+ and X^- of (1.1) as orbits connecting $z^{loc}(0)$ and $z^{loc}(N)$, respectively, with Σ. The detailed construction of $X^\pm$ and X^{loc} is carried out in Chapter 4. Altogether for each ω, p, λ and N this construction yields a partial orbit $X(\omega, p, \lambda, N)$ of (1.1) connecting Σ, see Lemma 4.1.12. Note, that N counts the circulations of this orbit around $\mathring{x}$ near the centre manifold and correlates directly with ω_i^{loc}, just like the flows of Π and φ.

If we consider (1.1) in $\mathbb{R}^4$ (see (H 4.1)) the difference of the starting point and the final point of $X(\omega, p, \lambda, N)$ in Σ lies automatically in Z. Moreover, the jump between the final point of some partial orbit and the final point of another one is parallel to Z. Therefore, for any λ and any sequences (ω_i), (p_i) and (N_i) there is a unique Lin orbit $\mathcal{L} = \{X_i\}$, $X_i = X(\omega_i, p_i, \lambda, N_i)$ satisfying

(i) $x_i(\omega_i^+)$, $x_i(\tau_i - \omega_i^-) \in \mathcal{M}_{p_i,\lambda}$,
(ii) $\Pi^{N_i}(x_i(\omega_i^+), \lambda) = x_i(\tau_i - \omega_i^-)$.

Here $\mathcal{X} = \{x_i\}_{i\in\mathbb{Z}}$ is the corresponding Lin solution, where x_i is defined on $[0, \tau_i]$ and $x_i(\cdot) = x(\omega_i, p_i, \lambda, N_i)(\cdot)$.

Therefore $\mathcal{L}((\omega_i), (p_i), \lambda, (N_i)) := \{X(\omega_i, p_i, \lambda, N_i)\}_{i\in\mathbb{Z}}$ is an actual orbit, if for all $i \in \mathbb{Z}$

$$\xi_i((\omega_i), (p_i), \lambda, (N_i)) = x(\omega_i, p_i, \lambda, N_i)(\tau_i) - x(\omega_{i+1}, p_{i+1}, \lambda, N_{i+1})(0) = 0 \,. \qquad (2.2)$$

In what follows we will address these equations as bifurcation equations. Note, that here (i.e. for $n=1$), the jump ξ_i does not depend on the entire sequences (p_i), (ω_i) and (N_i) but only on (ω_i, p_i, N_i) and $(\omega_{i+1}, p_{i+1}, N_{i+1})$. This is due to the above mentioned fact that the jump between the final point and the starting point of two consecutive partial orbits is automatically parallel to Z. This is a remarkable difference to the general situation, i.e. $n>1$, where $\gamma(0) + Z$ is a proper subset of Σ. In that general case the adaptation of Lin's method is much more complex and has not been carried out in the course of this thesis.

In particular we are interested in k-periodic orbits near the primary homoclinic orbit Γ. These orbits are defined as follows.

Definition 2.1.4 *A periodic orbit in the neighbourhood $\mathcal{U}$ of $\Gamma \cup \{\mathring{x}\}$ is called **k-periodic** if it intersects Σ k times.*

Note, that k counts the number of circulations along Γ and does not denote the period of the periodic orbit.
A k-periodic orbit corresponds to a k-periodic sequence (ω_i, p_i, N_i), i.e. $(\omega_i, p_i, N_i) = (\omega_{i+k}, p_{i+k}, N_{i+k})$, that solves the bifurcation equation (2.2). So, in order to find k-periodic orbits we can confine to search for k-periodic sequences solving (2.2). Our above considerations show that a Lin orbit $\mathcal{L} = \mathcal{L}((\omega_i), (p_i), \lambda, (N_i))$ corresponding

to k-periodic sequences (ω_i), (p_i) and (N_i) is itself k-periodic, i.e. $X_{i+k} = X_i$. We denote such a Lin orbit a ***k-periodic Lin orbit*** and the corresponding solution ***k-periodic Lin solution***. Because we find $\xi_{i+k} = \xi_i$, the bifurcation equation for detecting those orbits shrinks down to

$$\xi_i((\omega_i),(p_i),\lambda,(N_i)) = 0\ , \quad i = 1,\dots,k\ . \tag{2.3}$$

In the following we show how we arrive at a bifurcation equation for k-homoclinic orbits to the centre manifold. In the same way as we defined k-periodic orbits we define k-homoclinic orbits by

Definition 2.1.5 *An orbit is called homoclinic orbit to the centre manifold W^c_{loc} if both its α-limit set and its ω-limit set are subsets of W^c_{loc}.*
*A homoclinic orbit to the centre manifold W^c_{loc} lying in the neighbourhood $\mathcal{U}$ of $\Gamma \cup \{\mathring{x}\}$ is called **k-homoclinic orbit** if it intersects Σ k times.*

Depending on its limit sets a homoclinic orbit to W^c_{loc} can be homoclinic to the equilibrium $\mathring{x}$, or it can be homoclinic to a periodic orbit lying in the centre manifold, or it can be a heteroclinic orbit connecting two different orbits of the centre manifold.

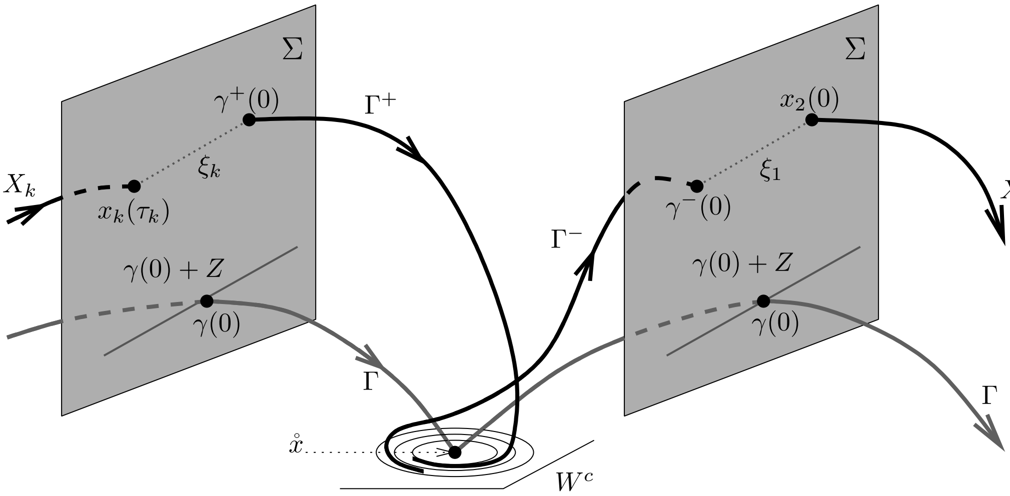

Figure 2.3: A k-homoclinic Lin orbit $\{(\Gamma^+,\Gamma^-), X_2,\dots,X_k\}$ asymptotic to the centre manifold W^c of the non-hyperbolic equilibrium $\mathring{x}$

The principle approach for the detection of k-periodic orbits is also applicable to the search for k-homoclinic orbits (see Figure 2.3). The main observation in this respect is that the τ_i have not necessarily to be finite, so we allow $\omega_1^+ = \omega_1^- = \infty$. This leads to the following definition of a k-homoclinic Lin orbit:

Definition 2.1.6 *Let $\mathcal{U}$ be a neighbourhood of $\Gamma \cup \{\mathring{x}\}$ and let $\gamma^-(\cdot)$ and $\gamma^+(\cdot)$ be solutions of Equation (1.1) on the intervals $(-\infty, 0]$ and $[0,\infty)$, respectively, with values in $\mathcal{U}$. We call γ^- and γ^+ **partial solution** connecting W^c_{loc} and Σ, respectively, if* *(i)* $\gamma^-(0), \gamma^+(0) \in \Sigma$ *;*

(ii) $\alpha(\Gamma^-), \omega(\Gamma^+) \subset W^c_{loc}$*;*

(iii) $\gamma^+(t) \notin \Sigma \quad \forall t \in (0,\infty)$*,* $\gamma^-(t) \notin \Sigma \quad \forall t \in (-\infty, 0)$*.*

Here Γ^- and Γ^+ are the corresponding orbits and are referred to as ***partial orbit*** *connecting W^c_{loc} with Σ.*

Let X_i, $i \in \{2,\dots,k\}$, be partial orbits connecting Σ, and let Γ^+ and Γ^- be partial orbits connecting W^c_{loc} with Σ. If the jumps between Γ^- and X_2, between two consecutive partial orbits X_i and X_{i+1}, $i = 2,\dots,k-1$, and between X_k and Γ^+ are parallel to Z, then we call $\mathcal{L}_{hom} := \{(\Gamma^+,\Gamma^-), X_2,\dots,X_k\}$ a ***k-homoclinic Lin orbit*** *and the corresponding solution $\mathcal{X}_{hom} := \{(\gamma^+,\gamma^-), x_2,\dots,x_k\}$ a* ***k-homoclinic Lin solution****.*

Again we denote the jumps addressed in the definition by ξ_i. Let again $[0,\tau_i]$ be the domains of the corresponding partial solutions $x_i(\cdot)$, $i = 2,\dots,k$, then ξ_i reads more detailed:

$$\begin{aligned} \xi_1 &:= \gamma^-(0) - x_2(0), \\ \xi_i &:= x_i(\tau_i) - x_{i+1}(0), \quad i = 2,\dots,k-1\ , \\ \xi_k &:= x_k(\tau_k) - \gamma^+(0). \end{aligned}$$

Note, that a k-homoclinic Lin orbit connects W^c_{loc} with itself, more precisely it connects the α-limit set of Γ^- and the ω-limit set of Γ^+. If the jumps ξ_i, $i = 1,\dots,k$, are equal to zero the k-homoclinic Lin orbit is an actual k-homoclinic orbit connecting the α-limit set of Γ^- and the ω-limit set of Γ^+.

Bifurcation equation for one-homoclinic orbits to the centre manifold

Within this thesis we are concerned with one-homoclinic orbits to the centre manifold. In a first step we focus on one-homoclinic orbits to the equilibrium $\mathring{x}$. Here the procedure resembles the first step of Lin's method for hyperbolic equilibria (which can be found in [San93] or [Kno04]). We prove that for each λ there is a unique one-homoclinic Lin solution $\mathcal{X}^0_{hom} = \{(\gamma^s(\lambda), \gamma^u(\lambda))\}$ to $\mathring{x}$, see Lemma 3.1.4. The corresponding orbits $\Gamma^s(\lambda)$ and $\Gamma^u(\lambda)$ are subsets of W^s_λ and W^u_λ, respectively, see Figure 2.4. We have to take into account that, in contrast to the case of a hyperbolic equilibrium, solutions in the (un)stable manifold of the equilibrium have to be characterised as solutions which are *exponentially* bounded for (negative) positive time.
Solutions of the bifurcation equation

$$\xi(\lambda) := \gamma^s(\lambda)(0) - \gamma^u(\lambda)(0) = 0 \tag{2.4}$$

correspond to one-homoclinic orbits to the equilibrium.

The other one-homoclinic orbits to the centre manifold we find by means of one-homoclinic Lin orbits to the centre manifold. Let Y^c be a certain two-dimensional

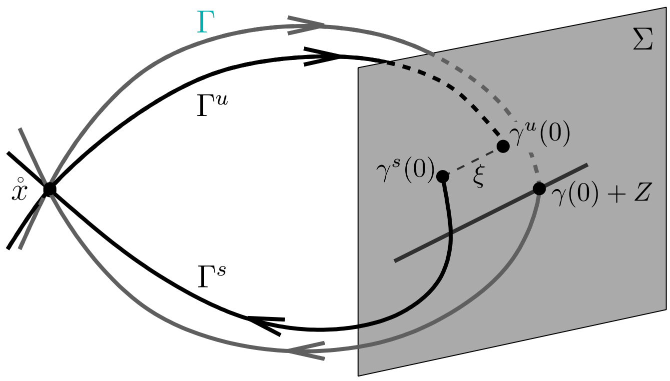

Figure 2.4: One-homoclinic Lin orbit $\mathcal{L}^0_{hom} = \{(\Gamma^s, \Gamma^u)\}$ asymptotic to the equilibrium $\mathring{x}$

subspace of Z. Then, for each $y_c^+, y_c^- \in Y^c$ and $\lambda \in \mathbb{R}^2$ we prove the existence of one-homoclinic Lin solutions $\mathcal{X}_{hom} = \{(\gamma^+(y_c^+, y_c^-, \lambda), \gamma^-(y_c^+, y_c^-, \lambda))\}$, see Lemma 3.2.3. For that we consider the corresponding Lin orbit $\mathcal{L}_{hom}$ as perturbation of the one-homoclinic Lin orbit $\mathcal{L}^0_{hom} = \{(\Gamma^s(\lambda), \Gamma^u(\lambda))\}$. This is illustrated in Figure 2.5. It is worth to remark that this part has no classical pendant.

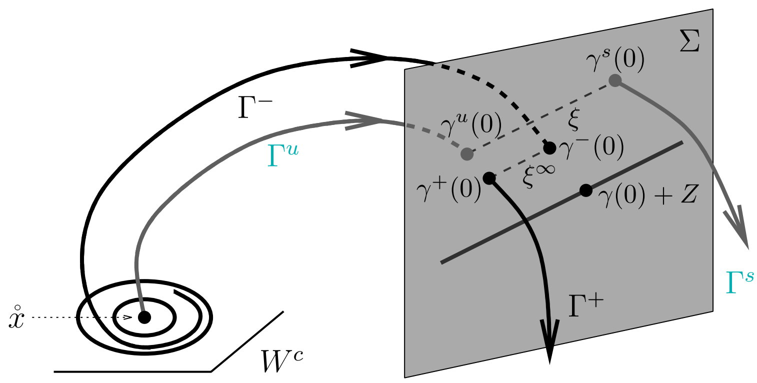

Figure 2.5: One-homoclinic Lin orbit $\mathcal{L}_{hom} = \{(\Gamma^+, \Gamma^-)\}$ asymptotic to the centre manifold, where $\mathcal{L}_{hom}$ is considered as a perturbation of $\mathcal{L}^0_{hom} = \{(\Gamma^s, \Gamma^u)\}$

Solutions of the bifurcation equation

$$\xi^\infty(y_c^+, y_c^-, \lambda) := \gamma^+(y_c^+, y_c^-, \lambda)(0) - \gamma^-(y_c^+, y_c^-, \lambda)(0) = 0$$

correspond to one-homoclinic orbits to the centre manifold. By distinguishing bases ξ^∞ can be read as a mapping $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3$. In our analysis we reduce $\xi^\infty = 0$ to a real valued equation. In the non-elementary case this reduction is due to the setting $y_c^+ = y_c^- =: y$, which is necessary for $\xi^\infty = 0$. We define (for $\lambda = (\lambda_1, \lambda_2)$)

$$\hat{\xi}^\infty(y, \lambda_1, \lambda_2) := \xi^\infty(y, y, (\lambda_1, \lambda_2)) ,$$

where, with a distinguishing bases, $\hat{\xi}^\infty$ can be seen as a mapping

$$\hat{\xi}^\infty : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \;, \quad (y, \lambda_1, \lambda_2) \mapsto \hat{\xi}^\infty(y, \lambda_1, \lambda_2)$$

and get the following bifurcation equation

$$\hat{\xi}^\infty(y, \lambda_1, \lambda_2) = \gamma^+(y, y, (\lambda_1, \lambda_2))(0) - \gamma^-(y, y, (\lambda_1, \lambda_2))(0) = 0 \;. \tag{2.5}$$

Similar, in the elementary case after a λ-dependent transformation we get the jump in the form

$$\hat{\xi}^\infty : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \;, \quad (y_R, y_{-R}, \lambda) \mapsto \hat{\xi}^\infty(y_R, y_{-R}, \lambda) \;,$$

where $y_R \in Y^c \cap \operatorname{Fix} R$ and $y_{-R} \in Y^c \cap \operatorname{Fix}(-R)$.

Bifurcation equation for symmetric one-periodic orbits

In the final part of this thesis we detect symmetric one-periodic orbits near a *non-elementary* homoclinic orbit Γ. For technical reasons we consider only vector fields in $\mathbb{R}^4$.

We ensure the existence of symmetric one-periodic Lin orbits. In $\mathbb{R}^4$ we have that $\Sigma = \gamma(0) + Z$, hence each partial orbit is a one-periodic Lin orbit. Thus, our analysis reduces to the search of partial orbits. In difference to the case of a hyperbolic equilibrium, we compose a partial orbit of three parts X^+, X^{loc} and X^-, that depend on $\omega = (\omega^+, \omega^-)$, p, λ and N. The symmetry requires $\omega^+ = \omega^- =: \Omega$; hence $\omega = (\Omega, \Omega)$. A corresponding symmetric one-periodic Lin solution $\{(x^+(\Omega, p, \lambda, N), x^{loc}(\Omega, p, \lambda, N), x^-(\Omega, p, \lambda, N))\}$ is given by Lemma 4.1.12. So, we get a bifurcation equation

$$\xi_{per}(\Omega, p, \lambda, N) := x^+(\Omega, p, \lambda, N)(0) - x^-(\Omega, p, \lambda, N)(0) = 0.$$

Further, the symmetry implies

$$\xi_{per}(\Omega, p, \lambda, N) \in \operatorname{Fix}(-R) \cap Z \;.$$

Because $\dim Z = 3$, $\dim \operatorname{Fix}(-R) = 2$ and hence $\dim(\operatorname{Fix}(-R) \cap Z) = 1$ the jump ξ_{per} can be read as a mapping

$$\xi_{per} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{N} \to \mathbb{R}, \quad (\Omega, p, \lambda, N) \mapsto \xi_{per}(\Omega, p, \lambda, N) \;.$$

In order to solve $\xi_{per} = 0$ we describe this equation as a "small perturbation" of (2.5), see Figure 2.6, in the following form

$$\xi_{per}(\Omega, p, \lambda, N) = \hat{\xi}^\infty(y, \lambda_1, \lambda_2) + \xi_r(y, \Omega, p, \lambda, N) = 0 \;,$$

where $y = y^*(\Omega, p, \lambda)$ such that the norm of $\xi_r(y, \Omega, p, \lambda, N)$ becomes "exponentially small" for increasing N.

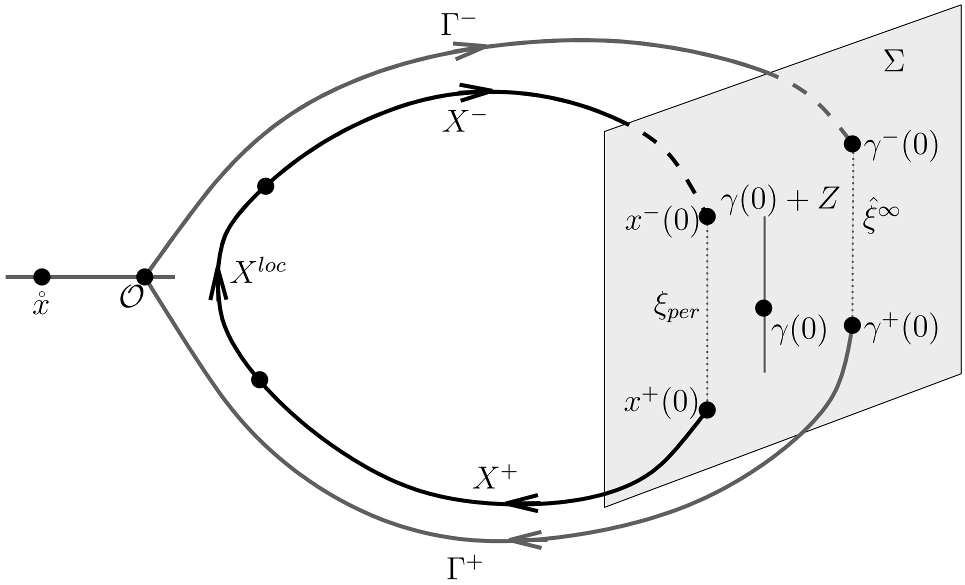

Figure 2.6: Symmetric one-periodic Lin orbit $\mathcal{L} = \{(X^+, X^{loc}, X^-)\}$ where $\mathcal{L}$ is considered as a perturbation of $\mathcal{L}_{hom} = \{(\Gamma^+, \Gamma^-)\}$; $\mathcal{O}$ is a periodic orbit in the centre manifold

We discuss $\hat{\xi}^\infty = 0$ in terms of y and λ, thus it would be favourable to express Ω and p as functions of y and λ. Each point w on the trace of the centre-stable manifold in Σ is uniquely determined by $y \in Y^c$ and λ: $w(y, \lambda)$. Then, for $y \neq 0$, i.e., $w(y, \lambda) \notin W^s$, we define $\Omega(y, \lambda)$ as the "first-hit-time" of $w(y, \lambda)$ in Σ_{loc} under the flow. Note, that $\Omega(y, \lambda)$ is determined modulo the time of first return to Σ_{loc}. However, if $w(y, \lambda) \in W^s$ then $\varphi(t, w(y, \lambda), \lambda)$ stays in Σ_{loc} for all (sufficiently large) t. In order to extend $\Omega(\cdot, \lambda)$ to $y = 0$ we introduce polar coordinates (ϱ, ϑ) in Y^c. With that we define a "first-hit-time" $\Omega^*(\varrho, \vartheta, \lambda)$ locally around $(\varrho, \vartheta, \lambda) = (0, 0, 0)$, see Lemma 4.2.4. Further, $p = p^*(\varrho, \vartheta, \lambda)$ is determined by the leaf $\mathcal{M}_{p,\lambda}$ in which $w(y, \lambda)$ is carried under the flow.

In our analysis we assume the jump $\hat{\xi}^\infty$ to be in the form

$$\hat{\xi}^\infty((y_1, y_2), \lambda_1, \lambda_2) = \lambda_1 + \mathfrak{c}(\lambda_2) - y_1^2 - y_2^2$$

for $y = (y_1, y_2)$ and some appropriate function $\mathfrak{c}$. In the new coordinates this finally gives

$$\lambda_1 + \mathfrak{c}(\lambda_2) - \varrho^2 + \hat{\xi}_r(\varrho, \vartheta, \lambda, N) = 0$$

as the bifurcation equation for symmetric one-periodic orbits. Note, that $\hat{\xi}_r$ is defined only for small ϑ.

2.2 Dynamical issues

In this section we present our main results concerning the dynamics in a neighbourhood of the primary homoclinic orbit Γ of Equation (1.1). Throughout we assume the Hypotheses (H 1.1) – (H 1.6) to be satisfied. We put emphasise on the dynamical issues. The exact formulations of all statements and the genericity conditions, on which our analysis relies, are given in Chapters 3 and 4.

We discuss in detail the one-homoclinic orbits to the centre manifold near Γ. Thereby we distinguish the cases of an elementary and a non-elementary primary homoclinic orbit Γ . Beyond it we discuss nearby symmetric one-periodic orbits. Here, however, we confine to consider non-elementary primary homoclinic orbits Γ.

2.2.1 One-homoclinic orbits to the centre manifold

Our first result concerns one-homoclinic orbits to the equilibrium.

Theorem 1 *(compare with Theorem 3.1.6) There is a curve crossing $\lambda = 0$ in the parameter plane corresponding to symmetric one-homoclinic orbits to the equilibrium.*

First we assume that Γ is non-elementary. In this case all bifurcating one-homoclinic orbits to the centre manifold are symmetric, see Lemma 3.3.1. Therefore these orbits are homoclinic to periodic orbits or to the equilibrium.
For further considerations we have to distinguish two cases. For a more geometrical explanation of these cases we confine ourselves to $\mathbb{R}^4$, $n = 1$. However, all results are true also in higher dimensions. First, we want to remark that $W^{cs} \cap \Sigma$ and $W^{cu} \cap \Sigma$ are R-images of each other. So it suffices to consider $W^{cs} \cap \Sigma$. This manifold can be seen as graph of a function $h^{cs} : \mathrm{Fix}\, R \to \mathbb{R}$. Moreover, (for $\lambda = 0$) the tangent space of this manifold at $\gamma(0)$ coincides with $\mathrm{Fix}\, R$. Hence the higher order terms of h^{cs} determine the local shape of $W^{cs} \cap \Sigma$ (around $\gamma(0)$). Generically the Hessian $D^2 h^{cs}(\gamma(0))$ of h^{cs} is non-degenerate, therefore it is either definite (if all its eigenvalues have the same sign) or indefinite (if the eigenvalues have different signs).

Theorem 2 *(compare with Theorem 3.3.4) Let Γ be a non-elementary homoclinic orbit and let $D^2 h^{cs}(\gamma(0))$ be (positive) definite. Then, near $\lambda = 0$, there is a curve $\mathfrak{C}$ in the parameter plane (as depicted in Figure 2.7) such that for parameter values above $\mathfrak{C}$ there exists a one-parameter family of one-homoclinic orbits to the centre manifold. For parameter values on $\mathfrak{C}$ there exists one one-homoclinic orbit. Below $\mathfrak{C}$ no one-homoclinic orbit does exist.*
All one-homoclinic orbits to the centre manifold are symmetric.

The bifurcation diagram presented in Figure 2.7 explains the situation more precisely. We describe this bifurcation diagram for $n = 1$. For further discussion we refer to Section 3.3.1.1.

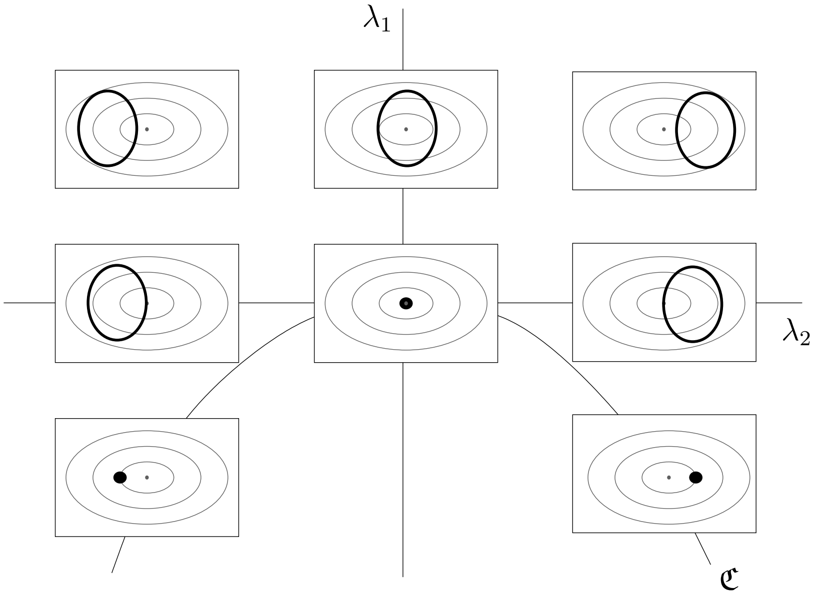

Figure 2.7: Bifurcation diagram corresponding to Theorem 2

The rectangles reflect the dynamics of one-homoclinic orbits at the parameter value which is beneath the midpoint of the rectangle. The fine curves (within each rectangle) display the periodic orbits within the centre manifold. The boldfaced curves (or points) κ^c_λ are the projections (along stable fibres) of the intersection of $W^{cs}_\lambda \cap W^{cu}_\lambda \cap \Sigma$. Each intersection of κ^c_λ with an periodic orbit $\mathcal{O}$ represents a one-homoclinic orbit asymptotic to $\mathcal{O}$. In the case under consideration κ^c_λ is a closed curve. While approaching $\mathfrak{C}$ from above κ^c_λ shrinks down, degenerates into a point (for $\lambda \in \mathfrak{C}$), and disappears for λ below $\mathfrak{C}$.
Generically we may expect that κ^c_λ intersects a periodic orbit transversally, if it intersects a periodic orbit at all. However, there are periodic orbits which have (also) a non-transversal intersection with κ^c_λ. This indicates a bifurcation of one-homoclinic orbits to such a distinguished periodic orbit in the centre manifold.

Next we assume that $D^2h^{cs}(\gamma(0))$ is indefinite.

Theorem 3 *(compare with Theorem 3.3.6) Let Γ be a non-elementary homoclinic orbit and let $D^2h^{cs}(\gamma(0))$ be indefinite. Then, near $\lambda = 0$, there exist infinitely many one-homoclinic orbits to the centre manifold.*
All one-homoclinic orbits to the centre manifold are symmetric.

The corresponding bifurcation diagram is depicted in Figure 2.8. It should be read in the same way as the bifurcation diagram corresponding to Theorem 2.
However, we want to mention that here the curves κ^c_λ are no longer closed but hyperbola-like. This implies the remarkable difference to the foregoing case that here for each λ there are (infinitely many) one-homoclinic orbits to the centre manifold. (This is a consequence of the transversal intersection of W^{cs} and $\operatorname{Fix} R$.)

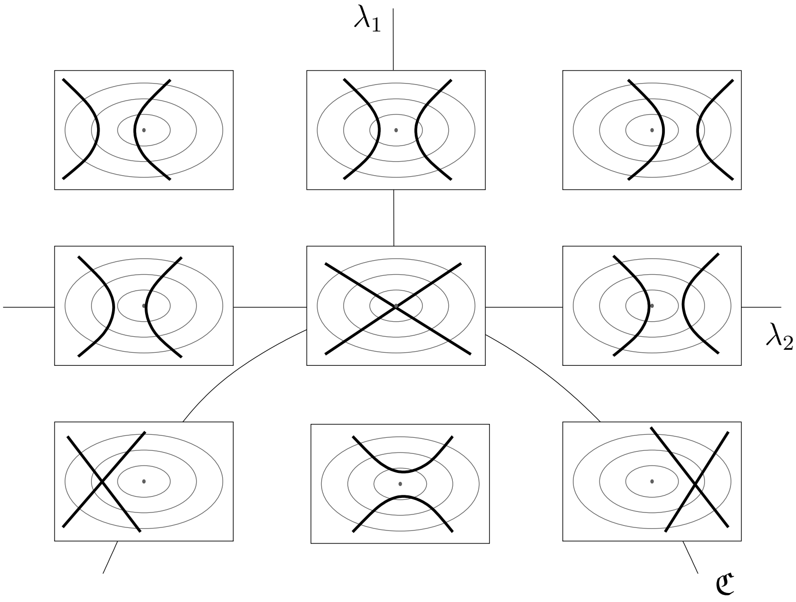

Figure 2.8: Bifurcation diagram corresponding to Theorem 3

The curve $\mathfrak{C}$ represents all those λ for which W^{cs}_λ and W^{cu}_λ intersect non-transversally within Σ.

Now, we assume that Γ is elementary, that means that within Σ the centre-stable manifold (and hence by symmetry also the centre-unstable manifold) intersects $\operatorname{Fix} R$ transversally. Here we find both symmetric homoclinic orbits and heteroclinic orbits bifurcating from Γ.

Theorem 4 *(compare with Theorem 3.3.10) Let Γ be an elementary homoclinic orbit. Then, near $\lambda = 0$, there exist a one-parameter family of symmetric and two one-parameter families of non-symmetric one-homoclinic orbits to the centre manifold. The two families of non-symmetric homoclinic orbits are R-images of each other.*

The corresponding bifurcation diagram is depicted in Figure 2.9. This diagram should be read in the same way as the bifurcation diagram corresponding to Theorem 2. Here, the boldfaced black lines are associated to symmetric homoclinic orbits to the centre manifold. For $\lambda_1 = 0$, there exist connections of the equilibrium. The grey lines (both, the dashed one as well as the solid one) correspond to the two families of non-symmetric one-homoclinic orbits to the centre manifold. For parameter values $\lambda = (\lambda_1, \lambda_2^*(\lambda_1))$ the grey lines intersect in the equilibrium. For those λ there exists a heteroclinic cycle involving the equilibrium. For more details we refer to Section 3.3.2.

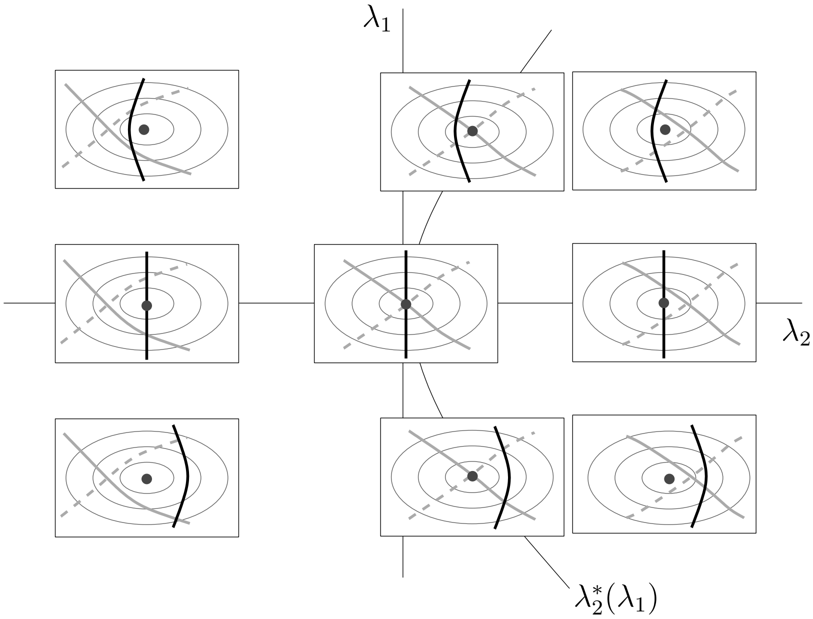

Figure 2.9: Bifurcation diagram corresponding to Theorem 4

2.2.2 Symmetric one-periodic orbits

We are looking for symmetric one-periodic orbits near a non-elementary homoclinic orbit in $\mathbb{R}^4$. With $U_\vartheta(0)$ as a neighbourhood of 0 in $\mathbb{R}$ we get the following result.

Theorem 5 *(compare with Theorem 4.2.9) Let Γ be a non-elementary homoclinic orbit and let $D^2h^{cs}(\gamma(0))$ be (positive) definite. Then, near $\lambda = 0$, there is a curve $\mathfrak{C}$ in the parameter plane (as depicted in Figure 2.10) such that for all parameter values above $\mathfrak{C}$ there exists $N_\lambda \in \mathbb{N}$, such that there is a two-parameter family $\{\mathcal{O}_{\vartheta,N}(\lambda) : \vartheta \in U_\vartheta(0),\, N \in \mathbb{N},\, N > N_\lambda\}$ of symmetric one-periodic orbits. The difference of the periods of $\mathcal{O}_{\vartheta,N}(\lambda)$ and $\mathcal{O}_{\vartheta,N+1}(\lambda)$ is approximately 2π. If λ tends to $\mathfrak{C}$ then the period of the symmetric one-periodic orbits converges to infinity.*

For parameter values on $\mathfrak{C}$ as well as below symmetric one-periodic orbits does not exist.

Above $\mathfrak{C}$ there exists a family of infinitely many one-homoclinic orbits to the centre manifold (bold dotted lines), see Theorem 2. To each $\vartheta \in U(0)$ one of the one-homoclinic orbits to the centre manifold is assigned. So, as depicted in Figure 2.10, these one-homoclinic orbits are accompanied by a one-parameter family of symmetric one-periodic orbits (bold solid lines) $\{\mathcal{O}_{\vartheta,N}(\lambda) : N > N_\lambda\}$.

For parameter values on $\mathfrak{C}$ there are only one-homoclinic orbits while below $\mathfrak{C}$ neither one-homoclinic nor symmetric one-periodic orbits exist.

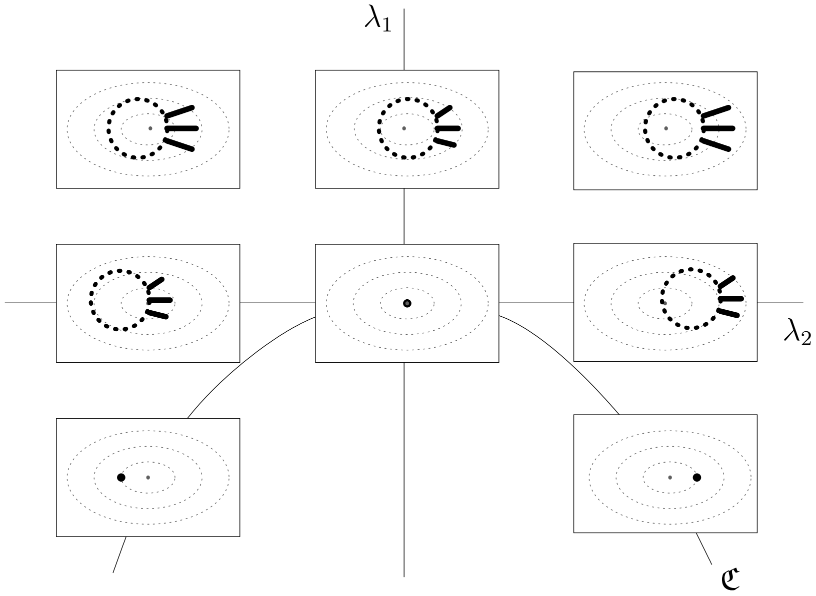

Figure 2.10: Bifurcation diagram corresponding to Theorem 5

3 The Existence of One-Homoclinic Orbits to the Centre Manifold

Our aim is to clarify the occurrence of one-homoclinic orbits to the centre manifold. Our considerations are based on the direct sum decomposition

$$\mathbb{R}^{2n+2} = \operatorname{span}\{f(\gamma(0),0)\} \oplus Y^s \oplus Y^u \oplus Z\,, \tag{3.1}$$

where $\operatorname{span}\{f(\gamma(0),0)\} \oplus Y^{s(u)} = T_{\gamma(0)}W^{s(u)}$ and Z is complementary to the sum of the corresponding tangent spaces of the stable and unstable manifold $T_{\gamma(0)}W^s + T_{\gamma(0)}W^u$. Note that $\dim Y^s = \dim Y^u = n-1$ and $\dim Z = 3$ (see Hypotheses (H 1.2) and (H 1.4)). Using this decomposition we define a cross section Σ by

$$\Sigma := \gamma(0) + \{Y^s \oplus Y^u \oplus Z\}\,. \tag{3.2}$$

As a first step in the investigation of one-homoclinic orbits, in Section 3.1 we prove the existence of homoclinic Lin solutions $\{(\gamma^s,\gamma^u)\}$ tending to the equilibrium. There, we obtain a first result concerning the occurrence of one-homoclinic orbits to the centre manifold, more precisely, we find one-homoclinic orbits being asymptotic to the saddle-centre equilibrium. In Section 3.2, we search one-homoclinic Lin solutions $\{(\gamma^+,\gamma^-)\}$ tending to orbits within the centre manifold as perturbations of $\{(\gamma^s,\gamma^u)\}$. Solving the bifurcation equation $\gamma^+(0)-\gamma^-(0)=0$ we detect one-homoclinic orbits to the centre manifold within Section 3.3 . Thereby we have to distinguish the case of a non-elementary and an elementary primary homoclinic orbit Γ.
Note, that the procedure allows us to distinguish between orbits approaching the equilibrium (in forward or backward time) and orbits connecting the periodic orbits of the centre manifold.

3.1 One-homoclinic orbits to the equilibrium

The main objective of this section is the detection of one-homoclinic Lin orbits to the equilibrium. This prepares the proof of the existence of one-homoclinic orbits to the centre manifold.
Although we are looking for orbits homoclinic to a non-hyperbolic equilibrium we can proceed in principle as in the case of a hyperbolic equilibrium. The main point which makes life easy at this stage is that all orbits approaching the equilibrium are contained in its (un)stable manifold. So we can restrict our considerations to one-homoclinic Lin solutions $\mathcal{X}^0 = \{(\gamma^s,\gamma^u)\}$ of (1.1) where the corresponding orbits $\Gamma^{s(u)}$ are subsets of $W^{s(u)}$.

As in the case of a hyperbolic equilibrium variational equations along solutions in the (un)stable manifold will play an essential role. Here such an equation has no longer an exponential dichotomy but an exponential trichotomy (see Section A.2). Using this fact we will show (see Lemma 3.1.4) that for each small λ there is a unique homoclinic Lin orbit $\{(\Gamma^s(\lambda), \Gamma^u(\lambda))\}$ to the equilibrium. With the corresponding solutions $\gamma^s(\lambda)(\cdot)$ and $\gamma^u(\lambda)(\cdot)$ we derive a bifurcation equation $\xi(\lambda) = \gamma^s(\lambda)(0) - \gamma^u(\lambda)(0) = 0$ for the detection of one-homoclinic orbits to the equilibrium. The subspace Z is 3-dimensional, hence the values of $\xi(\lambda)$ are 3-dimensional. Using the reversibility of the system this equation can be reduced to a one-dimensional equation. This is obvious in the 4-dimensional case where both the stable and unstable manifold are one-dimensional. The traces of these manifolds in Σ are the points $\gamma^{s(u)}(\lambda)(0)$. Due to the reversibility these points are R-images of each other. Hence their difference is parallel to the one-dimensional space $\mathrm{Fix}\,(-R) \cap \Sigma$.

The further considerations are devoted to the precise analysis leading to the Lin solutions $\mathcal{X}^0$ and to solutions of the bifurcation equation $\xi(\lambda) = 0$.

In addition to (3.1) we demand

$$Y^{s(u)} \perp \mathrm{span}\{f(\gamma(0), 0)\}\,, \tag{3.3}$$

$$Z \perp \mathrm{span}\{f(\gamma(0), 0)\} \oplus Y^s \oplus Y^u \tag{3.4}$$

with respect to an R-invariant scalar product $\langle \cdot, \cdot \rangle_R$ in $\mathbb{R}^{2n+2}$. Such a scalar product exists because $\{id, R\}$ forms a finite group.

Due to the reversibility of the vector field and the symmetry of the equilibrium we have $RW^s = W^u$. Since $\gamma(0) \in \mathrm{Fix}\,R$ this implies $RT_{\gamma(0)}W^s = T_{\gamma(0)}W^u$ and from (3.3) it finally follows

$$RY^s = Y^u. \tag{3.5}$$

Obviously, if $\eta^+ \in Y^s$ then $\eta^+ + R\eta^+ \in \mathrm{Fix}\,R \cap (Y^s \oplus Y^u)$ and $\eta^+ - R\eta^+ \in \mathrm{Fix}\,(-R) \cap (Y^s \oplus Y^u)$. Hence:

Lemma 3.1.1 *There are* $(n-1)$*-dimensional subspaces of both* $\mathrm{Fix}\,R$ *and* $\mathrm{Fix}\,(-R)$ *contained in* $Y^s \oplus Y^u$*.* ■

Because of (H 1.1) and (H 1.3) we have $\mathrm{span}\{f(\gamma(0), 0)\} \subset \mathrm{Fix}\,(-R)$. Then a consequence of (3.4) and (3.5) is

$$RZ = Z\,. \tag{3.6}$$

Further, counting dimensions, Lemma 3.1.1 and (3.6) give the next assertion.

Lemma 3.1.2 *The space* Z *is a direct sum of a one-dimensional subspace of* $\mathrm{Fix}\,(-R)$ *and a two-dimensional subspace of* $\mathrm{Fix}\,R$*.* ■

As an immediate consequence of the latter two lemmas, $R\gamma(0) = \gamma(0)$ and (3.2) we obtain:

Corollary 3.1.3 *The cross section* Σ *contains* $\mathrm{Fix}\,R$*.* ■

Now we are prepared to construct the Lin solutions $\{(\gamma^s, \gamma^u)\}$, which are solutions of (1.1) on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, satisfying

(P 3.1) (i) The orbits of $\gamma^{s(u)}$ are near Γ;

(ii) $\gamma^s(0)\,,\ \gamma^u(0) \in \Sigma\,;$

(iii) $\gamma^s(0) \in W^s\,,\ \gamma^u(0) \in W^u\,;$

(iv) $\gamma^s(0) - \gamma^u(0) \in Z\,.$

We consider $\gamma^{s(u)}$ as perturbations of γ. For that we define functions $v^\pm(\cdot)$ on $\mathbb{R}^\pm$ by

$$\gamma^s(t) = \gamma(t) + v^+(t)\,, \quad t \in \mathbb{R}^+ \quad \text{and} \quad \gamma^u(t) = \gamma(t) + v^-(t)\,, \quad t \in \mathbb{R}^-. \tag{3.7}$$

In order to formulate an equivalent problem for $v^\pm$ we first observe that on $\mathbb{R}^\pm$ the perturbations $v^\pm$ have to satisfy

$$\dot{v} = D_1 f(\gamma(t), 0)v + h(t, v, \lambda)\,, \tag{3.8}$$

where $h(t, v, \lambda) = f(\gamma(t) + v, \lambda) - f(\gamma(t), 0) - D_1 f(\gamma(t), 0)v$. We know

$$h(t, 0, 0) \equiv 0 \quad \text{and} \quad D_2 h(t, 0, 0) \equiv 0\,. \tag{3.9}$$

Moreover the reversibility of f and the symmetry of the homoclinic orbit Γ imply

$$\begin{aligned} RD_1 f(\gamma(t), 0) &= -D_1 f(\gamma(-t), 0)R \quad \text{and} \\ Rh(t, x, \lambda) &= -h(-t, Rx, \lambda)\,. \end{aligned} \tag{3.10}$$

This means Equation (3.8) is reversible.

In the case of a hyperbolic equilibrium the usual way to solve (3.8) is to rewrite it as a fixed point equation in the space of continuous bounded functions (see for instance [Van92], [Kno97]). Here, in principle, we tread the same path. But due to the centre part of the considered system stable and unstable manifold are no longer characterised by solutions staying bounded as t or $-t$, respectively, tends to infinity, but by solutions which are exponentially bounded. This forms the reason for solving (3.8) in spaces $V^\pm_{\bar{\alpha}}$ of exponentially bounded functions. To be able to define these spaces we consider the variational equation along γ

$$\dot{v} = D_1 f(\gamma(t), 0)v\,. \tag{3.11}$$

Let $\Phi(\cdot,\cdot)$ be the corresponding transition matrix. This variational equation is a reversible non-autonomous linear equation, that means,

$$R\Phi(t, s) = \Phi(-t, -s)R\,. \tag{3.12}$$

Proof of (3.12): Let $\psi(t) := R\Phi(t,s)\xi$ and $\phi(t) := \Phi(-t,-s)R\xi$ for $\xi \in \mathbb{R}^{2n+2}$. Equation (3.11) is solved by $\Phi(t,s)$, thus by (3.10) both $\psi(\cdot)$ and $\phi(\cdot)$ solve the initial value problem $\dot{Y}(t) = -D_1 f(\gamma(-t),0)Y(t)$, $Y(s) = R\xi$. ∎

From the deliberations in Section A.2, Lemma A.2.11, we know that (3.11) has exponential trichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. That is, there are continuous projections $P_u^\pm(t)$, $P_s^\pm(t)$ and $P_c^\pm(t)$ satisfying $id = P_u^\pm(t) + P_s^\pm(t) + P_c^\pm(t)\,,\; t \in \mathbb{R}^\pm$ and commuting with the transition matrix $\Phi(\cdot,\cdot)$, i.e.,

$$\Phi(t,s)P_i^\pm(s) = P_i^\pm(t)\Phi(t,s)\,,\; i = s,u,c\,. \tag{3.13}$$

Let μ be the leading unstable eigenvalue as introduced in (H 1.2). For any α and α_c with $\mu > \alpha > \alpha_c > 0$ and $t \geq s \geq 0$ it holds

$$\begin{aligned} &\|\Phi(t,s)P_s^+(s)\| \leq Ke^{-\alpha(t-s)}, \quad \|\Phi(s,t)P_u^+(t)\| \leq Ke^{-\alpha(t-s)},\\ &\|\Phi(t,s)P_c^+(s)\| \leq Ke^{\alpha_c(t-s)}, \quad \|\Phi(s,t)P_c^+(t)\| \leq Ke^{\alpha_c(t-s)}. \end{aligned} \tag{3.14}$$

Notice here, that due to the reversibility of f the constants α_s and α_u in Definition A.2.8 satisfy $-\alpha_s = \alpha_u$, and thus we obtain the above estimates by the setting $\alpha_u =: \alpha$.
Similar expressions hold for $P_i^-(t)$, $i = s,u,c$. Due to the reversibility the projections corresponding to the exponential trichotomy on $\mathbb{R}^-$ can be defined by

$$P_u^-(t) = RP_s^+(-t)R\,, \quad P_s^-(t) = RP_u^+(-t)R\,, \quad P_c^-(t) = RP_c^+(-t)R\,. \tag{3.15}$$

In accordance with Lemma A.2.11

$$\operatorname{im} P_s^+(0) = T_{\gamma(0)}W^s\,.$$

We are free to choose

$$\ker P_s^+(0) = Z \oplus Y^u\,.$$

Indeed $T_{\gamma(0)}W^s$ and $Z \oplus Y^u$ are complementary. Because of (3.5), (3.6) and (3.15)

$$\operatorname{im} P_u^-(0) = T_{\gamma(0)}W^u \quad \text{and} \quad \ker P_u^-(0) = Z \oplus Y^s\,.$$

Due to (3.13) $P_s^+(t)$ and $P_u^-(-t)$ are well-defined for all $t \in \mathbb{R}^+$.

For $\bar\alpha \in (\alpha_c, \alpha)$ we define

$$\begin{aligned} V_{\bar\alpha}^+ &:= \{v \in C^0([0,\infty),\mathbb{R}^{2n+2}) :\ \sup\nolimits_{t\geq 0} e^{\bar\alpha t}\|v(t)\| =: \|v\|_{\bar\alpha}^+ < \infty\}\,,\\ V_{\bar\alpha}^- &:= \{v \in C^0((-\infty,0],\mathbb{R}^{2n+2}) :\ \sup\nolimits_{t\leq 0} e^{-\bar\alpha t}\|v(t)\| =: \|v\|_{\bar\alpha}^- < \infty\} \end{aligned} \tag{3.16}$$

and rewrite (3.8) into a fixed point equation in $V_{\bar\alpha}^\pm$ (see (3.18) below). Further, by (3.7), Problem (P 3.1) reads as follows: The functions $v^+(\cdot)$ and $v^-(\cdot)$ have to satisfy

(P 3.2) (i) $\|v^{\pm}(t)\|_{\mathbb{R}^{2n+2}}$ are small for all $t \in \mathbb{R}^{\pm}$;

(ii) $v^{+}(0)\,,\, v^{-}(0) \in Y^s \oplus Y^u \oplus Z\,;$

(iii) $v^{+}(t) \in V_{\bar{\alpha}}^{+}\,, v^{-}(t) \in V_{\bar{\alpha}}^{-}\,;$

(iv) $v^{+}(0) - v^{-}(0) \in Z\,.$

The equivalence of (P 3.2)(iii) and (P 3.1)(iii) becomes clear by the following considerations: Obviously, $\gamma(\cdot)$ restricted on $[0,\infty)$ is in $V_{\bar{\alpha}}^{+}$, and restricted on $(-\infty, 0]$ it is in $V_{\bar{\alpha}}^{-}$. If $v^{\pm} \in V_{\bar{\alpha}}^{\pm}$, too, then by $\gamma^s = \gamma + v^{+}$ and $\gamma^u = \gamma + v^{-}$ we can conclude $\gamma^s \in V_{\bar{\alpha}}^{+}$ and $\gamma^u \in V_{\bar{\alpha}}^{-}$. That means for some constants $C > 0$

$$\| \gamma^s(t) \| \leq Ce^{-\bar{\alpha}t} \leq Ce^{-\mu t}\,, \quad t \in \mathbb{R}^{+}\,;$$
$$\| \gamma^u(t) \| \leq Ce^{\bar{\alpha}t} \leq Ce^{\mu t}\,, \quad t \in \mathbb{R}^{-}$$

and hence $\Gamma^{s(u)}$ lie within the (un)stable manifold.

Consequently, the original task of finding solutions of the system (1.1) fulfilling (P 3.1) has been turned into the problem of determining solutions of the "non-linear" variational equation (3.8) satisfying (P 3.2).

The procedure in attacking this problem is parallel to the hyperbolic case (see [Van92], [San93] or [Kno97]). So we restrict to sketch the course of action and concentrate on working out the differences caused by the centre part. Usually one starts with a "linearised" equation (see (3.17) below). By means of the knowledge of exponentially bounded solutions of this equation the Problem ((3.8),(P 3.2)) can be rewritten into an operator equation (see (3.18) below). Application of the Implicit Function Theorem and a further reduction process (see (3.21) - (3.23)) eventually provide the wanted $v^{+(-)}$ and hence $\gamma^{s(u)}$ (see Lemma 3.1.4).

Now, as described, we consider first the "linearised equation"

$$\dot{v} = D_1 f(\gamma(t), 0)v + g(t)\,, \tag{3.17}$$

where $g(t) \in V_{\bar{\alpha}}^{\pm}$. The functions $L^{\pm}(\cdot, g)$ defined by

$$L^{+}(t,g) := \textstyle\int_0^t \Phi(t,s)P_s^{+}(s)g(s)ds - \int_t^{\infty} \Phi(t,s)(id - P_s^{+}(s))g(s)ds\,,$$

$$L^{-}(t,g) := -\textstyle\int_t^0 \Phi(t,s)P_u^{-}(s)g(s)ds + \int_{-\infty}^{t} \Phi(t,s)(id - P_u^{-}(s))g(s)ds$$

are solutions of (3.17) on $\mathbb{R}^{\pm}$. Moreover we have $L^{\pm}(\cdot, g) \in V_{\bar{a}}^{\pm}$. This can be seen as follows (we will execute it exemplarily for "+"): The norm of $L^{+}(t, g)$ can be estimated by

$$\|L^{+}(t,g)\| \leq \| \int_0^t \Phi(t,s)P_s^{+}(s)g(s)ds\| \\ + \| \int_t^{\infty} \Phi(t,s)P_u^{+}(s)g(s)ds\| + \| \int_t^{\infty} \Phi(t,s)P_c^{+}(s)g(s)ds\|.$$

We show that the right-hand side of the forgoing term is exponentially bounded, for that we consider first $\|\int_t^\infty \Phi(t,s)P_c^+(s)g(s)ds\|$. We use (3.14) for $s > t > 0$, $\alpha_c < \bar\alpha < \alpha$ and further $g \in V_{\bar\alpha}^+$, which gives $\|g(s)\| \le e^{-\bar\alpha s}\|g\|_{\bar\alpha}^+$. Then

$$\|\int_t^\infty \Phi(t,s)P_c^+(s)g(s)ds\| \le Ke^{-\alpha_c t}\|g\|_{\bar\alpha}^+ \int_t^\infty e^{(\alpha_c-\bar\alpha)s}ds \le K_c\, e^{-\bar\alpha t}\,,$$

where $K_c = K\|g\|_{\bar\alpha}^+(\bar\alpha-\alpha_c)^{-1} > 0$. For that reason, in contrast to the procedure in the case of homoclinic orbits to hyperbolic equilibria, we are looking for solutions of (3.17) within $V_{\bar\alpha}^\pm$.
Similar to the above estimation we find

$$\|\int_0^t \Phi(t,s)P_s^+(s)g(s)ds\| \le K_s\, e^{-\bar\alpha}t\,,$$

$$\|\int_t^\infty \Phi(t,s)P_u^+(s)g(s)ds\| \le K_u\, e^{-\bar\alpha}t\,.$$

Thus, with $g(t) \in V_{\bar\alpha}^\pm$, we get the exponential bounded solutions of (3.17) in the form $\Phi(t,0)\eta^\pm + L^\pm(t,g)$. Here $\eta^+ \in T_{\gamma(0)}W^s$ and $\eta^- \in T_{\gamma(0)}W^u$, respectively.

Now we return to Equation (3.8). Replacing in (3.17) the inhomogeneous part $g(t)$ by $h(t,v,\lambda)$ provides that solutions $v^\pm \in V_{\bar\alpha}^\pm$ of (3.8) are exactly the solutions of

$$v^\pm(t) = \Phi(t,0)\eta^\pm + L^\pm(t, h(t,v^\pm,\lambda)) \tag{3.18}$$

if only $h(\cdot, v^\pm(\cdot),\lambda) \in V_{\bar\alpha}^\pm$. Indeed, using the definition of h and Taylor expansion of f we find

$$\|h(t,v,\lambda)\| \le K_1\|v\|^2 + K_2\|\lambda\|(\|\gamma(t)\| + \|v\|)\,. \tag{3.19}$$

Hence $v^\pm \in V_{\bar\alpha}^\pm$ implies $h(\cdot, v^\pm(\cdot),\lambda) \in V_{\bar\alpha}^\pm$.

As a consequence of (3.19) the right-hand side of (3.18) (considered exemplarily for "+") is a map

$$T_{\gamma(0)}W^s(0) \times \mathbb{R}^2 \times V_{\bar\alpha}^+ \to V_{\bar\alpha}^+\,.$$

We know that $(\eta^+,\lambda,v^+) = (0,0,0)$ is a solution of (3.18) and $D_2h(t,0,0) \equiv 0$. Therefore we can solve (3.18) for $v^+ = v^+(\eta^+,\lambda)$ by the Implicit Function Theorem (for η^+, $|\lambda|$ sufficiently small) and get $\|v^+(\eta^+,\lambda)\|_{\bar\alpha}^+ \to 0$ as $(\eta^+,\lambda) \to (0,0)$.

Thus, Problem ((3.8),(P 3.2)(i),(iii)) has been solved. It remains to consider (P 3.2)(ii),(iv). By (3.18) we obtain

$$\begin{aligned} v^+(\eta^+,\lambda)(0) &= \eta^+ - (id - P_s^+(0)) \int_0^\infty \Phi(0,s)h(s, v^+(\eta^+,\lambda)(s),\lambda)ds\,,\\ v^-(\eta^-,\lambda)(0) &= \eta^- + (id - P_u^-(0)) \int_{-\infty}^0 \Phi(0,s)h(s, v^-(\eta^-,\lambda)(s),\lambda)ds\,, \end{aligned} \tag{3.20}$$

with $\eta^+ \in T_{\gamma(0)}W^s = \operatorname{im} P_s^+(0)$ and $\eta^- \in T_{\gamma(0)}W^u = \operatorname{im} P_u^-(0)$. For solutions

$v^{\pm}(\eta^{\pm},\lambda)(\cdot)$ additionally satisfying (P 3.2)(ii) we get

$$\begin{aligned} v^{+}(\eta^{+},\lambda)(0) &= \underbrace{\eta^{+}}_{\in Y^s} + \underbrace{y_u(\eta^{+},\lambda)}_{\in Y^u} + \underbrace{z^{+}(\eta^{+},\lambda)}_{\in Z}\,,\\ v^{-}(\eta^{-},\lambda)(0) &= \underbrace{\eta^{-}}_{\in Y^u} + \underbrace{y_s(\eta^{-},\lambda)}_{\in Y^s} + \underbrace{z^{-}(\eta^{-},\lambda)}_{\in Z}\,. \end{aligned} \tag{3.21}$$

Further, condition (P 3.2)(iv) requires

$$\eta^{+} = y_s(\eta^{-},\lambda), \quad \eta^{-} = y_u(\eta^{+},\lambda)\,. \tag{3.22}$$

From (3.20) and (3.21) we get

$$y_u(\eta^{+},\lambda) + z^{+}(\eta^{+},\lambda) = -(id - P_s^{+}(0)) \int_0^{\infty} \Phi(0,s)h(s,v^{+}(\eta^{+},\lambda)(s),\lambda)ds\,,$$

$$y_s(\eta^{-},\lambda) + z^{-}(\eta^{-},\lambda) = (id - P_u^{-}(0)) \int_{-\infty}^{0} \Phi(0,s)h(s,v^{-}(\eta^{-},\lambda)(s),\lambda)ds\,.$$

So, with (3.9) follows:

$$\begin{array}{llll} y_s(0,0)=0\,, & y_u(0,0)=0\,, & z^{+}(0,0)=0\,, & z^{-}(0,0)=0\,,\\ D_1y_s(0,0)=0\,, & D_1y_u(0,0)=0\,, & D_1z^{+}(0,0)=0\,, & D_1z^{-}(0,0)=0\,. \end{array}$$

Hence, by the Implicit Function Theorem we can solve (3.22) for

$$\eta^{+} = \eta^{+}(\lambda) \quad \text{and} \quad \eta^{-} = \eta^{-}(\lambda)\,. \tag{3.23}$$

Thus, we get functions $v^{\pm}(\eta^{\pm}(\lambda),\lambda)$ solving Equation (3.8) and satisfying Problem (P 3.2). By (3.7) we can formulate

Lemma 3.1.4 *For each sufficiently small $\lambda \in \mathbb{R}^2$ there is a unique one-homoclinic Lin solution $\{(\gamma^s(\lambda),\gamma^u(\lambda))\}$ tending to the equilibrium.*
Moreover, the mappings $\gamma^s(\cdot):\mathbb{R}^2 \to C(\mathbb{R}^+,\mathbb{R}^n)$ and $\gamma^u(\cdot):\mathbb{R}^2 \to C(\mathbb{R}^-,\mathbb{R}^n)$ are C^r smooth.

Proof In accordance with (3.7) we define

$$\begin{aligned} \gamma^s(\lambda)(t) &:= \gamma(t) + v^{+}(\eta^{+}(\lambda),\lambda)(t)\,, \quad t \in \mathbb{R}^+\,;\\ \gamma^u(\lambda)(t) &:= \gamma(t) + v^{-}(\eta^{-}(\lambda),\lambda)(t)\,, \quad t \in \mathbb{R}^-\,. \end{aligned}$$

Obviously, the solutions $\gamma^s(\lambda)(\cdot)$ and $\gamma^u(\lambda)(\cdot)$ are as stated in the above Lemma. ■

As a first result concerning the existence of one-homoclinic orbits to the centre manifold we find special one-homoclinic orbits, which are asymptotic to the equilibrium $\mathring{x} = 0$. For that we solve the bifurcation equation (2.4) which now reads

$$\xi(\lambda) = \gamma^s(\lambda)(0) - \gamma^u(\lambda)(0) = 0\,. \tag{3.24}$$

Lemma 3.1.4 implies $\gamma^s(0)(\cdot) = \gamma(\cdot)$, $t \in \mathbb{R}^+$ and $\gamma^u(0)(\cdot) = \gamma(\cdot)$, $t \in \mathbb{R}^-$. Hence

$$\xi(0) = 0\,. \tag{3.25}$$

As an immediate consequence of the reversibility (see (3.10), (3.15) and (3.12)) and the uniqueness of the solution of the operator equation (3.18) we get

Lemma 3.1.5 *The solutions $v^\pm$ of the fixed point Equation (3.18) satisfy*

$$Rv^+(\eta,\lambda)(t) = v^-(R\eta,\lambda)(-t) \quad \text{and} \quad Rv^-(\eta,\lambda)(t) = v^+(R\eta,\lambda)(-t).$$

This lemma and (3.21) give

$$Ry_u(\eta^+,\lambda) \;=\; y_s(R\eta^+,\lambda)\,, \tag{3.26}$$
$$Rz^+(\eta^+,\lambda) \;=\; z^-(R\eta^+,\lambda)\,. \tag{3.27}$$

Solving system (3.22) by means of the Implicit Function Theorem, the property (3.26) will be transferred to the solving functions and thus

$$\eta^+(\lambda) = R\,\eta^-(\lambda)\,. \tag{3.28}$$

For the details of this conclusion we refer to [Kno97]. Finally, putting things together we get

$$R\gamma^s(\lambda)(0) = \gamma^u(\lambda)(0) \tag{3.29}$$

and eventually

$$R\xi(\lambda) = -\xi(\lambda)\,. \tag{3.30}$$

The decomposition of Z (see Lemma 3.1.2) and $\xi(\lambda) \subset \operatorname{Fix} R$ yield that we can consider ξ as a mapping $\mathbb{R}^2 \to \mathbb{R}$ after introducing a basis in $Z \cap \operatorname{Fix}(-R)$.

Our assumption (H 1.5) translates into

(H 3.1) $\quad D\xi(0) \neq 0$,

which we will use henceforth. Then the following theorem holds true:

Theorem 3.1.6 *Assume (H 3.1). Then, locally around $\lambda = 0$ there is a smooth curve $\mathfrak{C}$ in the parameter plane such that exactly for $\lambda \in \mathfrak{C}$ there exist one-homoclinic orbits to the equilibrium. Moreover, these homoclinic orbits are symmetric.*

Proof The statement of the theorem is an immediate consequence of applying the Implicit Function Theorem to $\xi(\lambda) = 0$ (see (3.25) and Hypothesis (H 3.1)). The symmetry follows from Equation (3.29). ∎

Assume (H 3.1). Let $\lambda = (\lambda_1, \lambda_2)$ such that $D_{\lambda_1}\xi(0) \neq 0$. Then the curve $\mathfrak{C}$ can be understood as the graph of a function $\lambda_1^*(\lambda_2)$. Thus there is a transformation such that $\mathfrak{C} = \{(0, \lambda_2)\}$. This transformation can be chosen such that

$$\xi(\lambda) \equiv \lambda_1 \,. \tag{3.31}$$

3.2 One-homoclinic Lin orbits to the centre manifold

Now we will compute all one-homoclinic Lin solutions $\mathcal{X} = \{(\gamma^+, \gamma^-)\}$ to the centre manifold. The procedure for the detection of these Lin solutions is in principle the same as in Section 3.1. Here we have to adapt (P 3.1)(iii) to the fact that a homoclinic orbit to the centre manifold lies simultaneously in W^{cs}_λ and W^{cu}_λ. This is because the centre-(un)stable manifold of the equilibrium coincides with the (un)stable manifold of the centre manifold. Further, we consider the one-homoclinic Lin orbits to the centre manifold as perturbations of $\{(\Gamma^s(\lambda), \Gamma^u(\lambda)\}$ and discuss the arising variational equations along the solutions in the (un)stable manifold of the centre manifold. These equations again have an exponential trichotomy. We find uniquely determined one-homoclinic Lin solutions $\mathcal{X} = \{(\gamma^+(y_c^+, y_c^-, \lambda), \gamma^-(y_c^+, y_c^-, \lambda))\}$ for small λ and small $y_c^\pm$. The $y_c^\pm$ are located in a certain two-dimensional subspace of Z.

The next considerations are devoted to the precise analysis leading to the Lin solutions $\mathcal{X}$. To facilitate our analysis we assume

(H 3.2) $\quad W^{cs(cu)}_{loc,\lambda} \subset X^{cs(cu)}_\lambda$.

Here $X^{cs(cu)}_\lambda$ denote the centre-(un)stable eigenspaces of $D_1 f(0, \lambda)$. This hypothesis is not a restriction because, for each small $\lambda \in \mathbb{R}^2$, there is a transformation which flattens $W^{cs}_{loc,\lambda}$ and $W^{cu}_{loc,\lambda}$ simultaneously in a ball B_δ around the equilibrium. This transformation is described in Section 3.4.

We begin our discussion with considering the relative position of both centre stable manifold W^{cs} and centre unstable manifold W^{cu} to each other and centre stable manifold and $\operatorname{Fix} R$. First, we observe that the tangent spaces of W^{cs} and W^{cu} intersect at least in a 2-dimensional space because both tangent spaces are $(n+2)$-dimensional. On the other hand, (H 1.4) averts that the dimension of this intersection can be greater than three. We can make similar considerations for the intersection of $T_{\gamma(0)}W^{cs}$ with $\operatorname{Fix} R$: The dimension of the spaces require that the dimension of their intersection is at least one. The reversibility effects that the dimension of their intersection cannot be greater than two: $RW^{cs} = W^{cu}$ and $\gamma(0) \in \operatorname{Fix} R$ provide $T_{\gamma(0)}W^{cs} \cap \operatorname{Fix} R \subset T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu}$. Because the vector field at $\gamma(0)$ is contained in $T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu} \cap \operatorname{Fix}(-R)$ it follows

$$\dim(T_{\gamma(0)}W^{cs} \cap \operatorname{Fix} R) + 1 \leq \dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu}) \leq 3 \,. \tag{3.32}$$

Taking into account (H 1.6) we arrive at

$$\dim(T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu}) = 3 \, . \tag{3.33}$$

The last equation implies that the intersection of $T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu}$ and $\gamma(0) - \Sigma$ is two-dimensional.
For our further considerations we refine the direct sum decomposition (3.1). In particular we decompose Z into two subspaces: $Z = Y^c \oplus \hat{Z}$. For that we construct a particular R-invariant scalar product in such a way that the new decomposition of $\mathbb{R}^{2n+2}$ is still orthogonal. Note, that this construction is based on statements within Appendix A.1. However, we want to emphasise that our previous results are true for any R-invariant scalar product. So all previous results remain untouched in the course of the following construction.
Let Y^c be the complement of $\text{span}\{f(\gamma(0), 0)\}$ within $T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu}$ with respect to an R-invariant scalar product:

$$\text{span}\{f(\gamma(0), 0)\} \oplus Y^c = T_{\gamma(0)}W^{cs} \cap T_{\gamma(0)}W^{cu} \, , \tag{3.34}$$

which implies

$$RY^c = Y^c \, . \tag{3.35}$$

Assumption (H 1.4) implies $Y^c \cap (\text{span}\{f(\gamma(0), 0)\} \oplus Y^s \oplus Y^u) = \{0\}$. This justifies the direct sum $\text{span}\{f(\gamma(0), 0)\} \oplus Y^s \oplus Y^u \oplus Y^c$. Finally, let $\hat{Z}$ be any R-invariant complement of the latter direct sum in $\mathbb{R}^{2n+2}$:

$$\mathbb{R}^{2n+2} = \text{span}\{f(\gamma(0), 0)\} \oplus Y^s \oplus Y^u \oplus Y^c \oplus \hat{Z} \, , \tag{3.36}$$

$$R\hat{Z} = \hat{Z} \, . \tag{3.37}$$

Note, that due to (3.33)

$$\dim Y^c = 2 \quad \text{and} \quad \dim \hat{Z} = 1 \, .$$

Let $< \cdot, \cdot >$ be an R-invariant scalar product such that the R-invariant subspaces $\text{span}\{f(\gamma(0), 0)\}$, $Y^s \oplus Y^u$, Y^c and $\hat{Z}$ are in pairs perpendicular. Then in accordance with (3.1) we define

$$Z = Y^c \oplus \hat{Z} \, . \tag{3.38}$$

From (3.32) we get that there are two possibilities concerning the relative position of $T_{\gamma(0)}W^{cs}$ and $\text{Fix}\, R$:

$$\dim(T_{\gamma(0)}W^{cs} \cap \text{Fix}\, R) = 2 \, ; \tag{3.39}$$

$$\dim(T_{\gamma(0)}W^{cs} \cap \text{Fix}\, R) = 1 \, . \tag{3.40}$$

Note, that (3.39) corresponds to the case of a non-elementary primary homoclinic orbit Γ and (3.40) to the case of an elementary one.

Now, we turn to the detection of one-homoclinic Lin solutions $\mathcal{X} = \{(\gamma^+, \gamma^-)\}$ tending to the centre manifold. Let $B_\delta(0)$ be a ball around $x = 0$ with radius δ in which the centre-(un)stable manifolds are flat, and let $T > 0$ be as large such that $\gamma(T) \in B_{\delta/2}(0)$. With that notation the demands on γ^+ and γ^- read:

(P 3.3)
(i) The orbits of $\gamma^\pm$ are near Γ;
(ii) $\gamma^+(0)\,,\, \gamma^-(0) \in \Sigma\,;$
(iii) $\gamma^+(T) \in W^{cs}_\lambda \cap B_\delta(0)$ and $\gamma^-(-T) \in W^{cu}_\lambda \cap B_\delta(0);$
(iv) $\gamma^+(0) - \gamma^-(0) \in Z\,.$

Note that we can choose $\varepsilon > 0$ such that for all λ, $|\lambda| < \varepsilon$ and all x, $\|x - \gamma(0)\| < \varepsilon$, it holds $\varphi(T, x, \lambda) \in B_{\delta/2}(\gamma(T))$. Here $\varphi(\cdot, \cdot, \lambda)$ is the flow of the vector field $f(\cdot, \lambda)$. This guarantees that $\varphi(T, x, \lambda) \in B_\delta(0)$.

Actually we look for the solutions $\gamma^+(\cdot)$ and $\gamma^-(\cdot)$ as perturbations of $\gamma^s(\lambda)(\cdot)$ and $\gamma^u(\lambda)(\cdot)$, respectively:

$$
\begin{aligned}
\gamma^+(t) &= \gamma^s(\lambda)(t) + v^+(t)\,, \quad t \in \mathbb{R}^+\,,\\
\gamma^-(t) &= \gamma^u(\lambda)(t) + v^-(t)\,, \quad t \in \mathbb{R}^-\,.
\end{aligned}
\tag{3.41}
$$

Therefore the functions $v^\pm$ are solutions of

$$
\begin{aligned}
\dot v &= D_1 f(\gamma^s(\lambda)(t), \lambda)v + h(t, v, \lambda)\,, \quad t \in I^+_T := [0, T]\,,\\
\dot v &= D_1 f(\gamma^u(\lambda)(t), \lambda)v + h(t, v, \lambda)\,, \quad t \in I^-_T := [-T, 0]\,,
\end{aligned}
\tag{3.42}
$$

where $h(t, v, \lambda) = f(\gamma^{s(u)}(\lambda)(t) + v, \lambda) - f(\gamma^{s(u)}(\lambda)(t), \lambda) - D_1 f(\gamma^{s(u)}(\lambda)(t), \lambda)v$. The properties (P 3.3) can be rewritten in terms of $v^\pm$ as follows

(P 3.4)
(i) $\|v^\pm(t)\|_{\mathbb{R}^{2n+2}}$ are small for all $t \in I^\pm_T$;
(ii) $v^+(0)\,,\, v^-(0) \in Y^s \oplus Y^u \oplus Z\,;$
(iii) $v^+(T) \in W^{cs}_\lambda \cap B_{\frac{\delta}{2}}(0)$ and $v^-(-T) \in W^{cu}_\lambda \cap B_{\frac{\delta}{2}}(0)\,;$
(iv) $v^+(0) - v^-(0) \in Z.$

We want to remark that the conditions (P 3.3)(iii) and (P 3.4)(iii) are equivalent because in $B_\delta(0)$ W^{cs}_λ and W^{cu}_λ are flat.
In order to solve ((3.42), (P 3.4)) we use that the equations

$$
\dot v = D_1 f(\gamma^{s(u)}(\lambda)(t), \lambda)v \tag{3.43}
$$

have exponential trichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. This is clear by the statements of Section A.2 and (P 3.1)(iii). Let $P^\pm_u(t, \lambda)$, $P^\pm_s(t, \lambda)$ and $P^\pm_c(t, \lambda)$ be the corresponding projections and let us denote the transition matrix of (3.43) by $\Phi^{\pm}(t,s,\lambda)$. Then

$$\Phi^{\pm}(t,s,\lambda)P_i^{\pm}(s,\lambda) = P_i^{\pm}(t,\lambda)\Phi^{\pm}(t,s,\lambda)\,,\; i = u,s,c\,, \tag{3.44}$$

and moreover

$$T_{\gamma^s(\lambda)(t)}W_\lambda^s = \operatorname{im} P_s^+(t,\lambda)\;, \quad T_{\gamma^u(\lambda)(t)}W_\lambda^u = \operatorname{im} P_u^-(t,\lambda)\,,$$

$$T_{\gamma^s(\lambda)(t)}W_\lambda^{cs} = \operatorname{im} P_{cs}^+(t,\lambda)\;, \quad T_{\gamma^u(\lambda)(t)}W_\lambda^{cu} = \operatorname{im} P_{cu}^-(t,\lambda)\,.$$

To define these projections completely we set further:

$$\ker P_s^+(0,\lambda) = Y^u \oplus Y^c \oplus \hat{Z}\;, \quad \ker P_u^-(0,\lambda) = Y^s \oplus Y^c \oplus \hat{Z}\,,$$

$$\ker P_{cs}^+(0,\lambda) = Y^u \oplus \hat{Z}\;, \quad \ker P_{cu}^-(0,\lambda) = Y^s \oplus \hat{Z}\,.$$

Indeed, $T_{\gamma^s(\lambda)(t)}W^{cs}$ and $Y^u \oplus \hat{Z}$ are complementary, as well as $T_{\gamma^u(\lambda)(t)}W^{cu}$ and $Y^s \oplus \hat{Z}$ (see [Kno00] for a comparable consideration). These determinations are in accordance with Appendix A.2. Note further, that all projections depend smoothly on λ.

Obviously

$$\ker P_{cs}^+(0,\lambda) \subset \ker P_s^+(0,\lambda)\;.$$

In accordance with the explanations in A.2 we may define

$$P_c^+(0,\lambda) := P_{cs}^+(0,\lambda) - P_s^+(0,\lambda)\;,$$

$$P_c^-(0,\lambda) := P_{cu}^-(0,\lambda) - P_u^-(0,\lambda)\;.$$

In particular for $\lambda = 0$ we have:

$$\operatorname{im} P_s^+(0,0) = T_{\gamma(0)}W^s = \operatorname{span}\{f(\gamma(0),0)\} \oplus Y^s\;, \quad \ker P_s^+(0,0) = Y^u \oplus Y^c \oplus \hat{Z}\;,$$

$$\operatorname{im} P_{cs}^+(0,0) = T_{\gamma(0)}W^{cs} = \operatorname{span}\{f(\gamma(0),0)\} \oplus Y^s \oplus Y^c\;, \quad \ker P_{cs}^+(0,0) = Y^u \oplus \hat{Z}\;.$$

In the same way we find

$$\operatorname{im} P_u^-(0,0) = T_{\gamma(0)}W^u = \operatorname{span}\{f(\gamma(0),0)\} \oplus Y^u\;, \quad \ker P_u^+(0,0) = Y^s \oplus Y^c \oplus \hat{Z}\;,$$

$$\operatorname{im} P_{cu}^-(0,0) = T_{\gamma(0)}W^{cu} = \operatorname{span}\{f(\gamma(0),0)\} \oplus Y^u \oplus Y^c\;, \quad \ker P_{cu}^+(0,0) = Y^s \oplus \hat{Z}\;.$$

Therefore we have

$$\operatorname{im} P_c^+(0,0) = \operatorname{im} P_c^-(0,0) = Y^c\;. \tag{3.45}$$

Note that the projections $P_s^+(0,0)$ and $P_u^-(0,0)$ coincide with those defined in Section 3.1:

$$P_s^+(0,0) = P_s^+(0)\,, \quad P_u^-(0,0) = P_u^-(0)\,.$$

Let us consider (3.42). For the term h we find

$$h(t,0,\lambda) \equiv 0 \quad \text{and} \quad D_2 h(t,0,\lambda) \equiv 0\,. \tag{3.46}$$

Similar to our results in Section 3.1 we first discuss the "linearised equations" (3.42):

$$\begin{aligned} \dot v &= D_1 f(\gamma^s(\lambda)(t),\lambda)v + g(t)\,, \quad t \in I_T^+\,, \\ \dot v &= D_1 f(\gamma^u(\lambda)(t),\lambda)v + g(t)\,, \quad t \in I_T^-\,, \end{aligned} \tag{3.47}$$

where $g(\cdot) \in C^0(I_T^\pm, \mathbb{R}^{2n+2})$. We find that

$$L^+(t,g,\lambda) := \int_0^t \Phi^+(t,s,\lambda)P_{cs}^+(s,\lambda)g(s)ds - \int_t^T \Phi^+(t,s,\lambda)(id - P_{cs}^+(s,\lambda))g(s)ds\,,$$

$$L^-(t,g,\lambda) := -\int_t^0 \Phi^-(t,s,\lambda)P_{cu}^-(s,\lambda)g(s)ds + \int_{-T}^t \Phi^-(t,s,\lambda)(id - P_{cu}^-(s,\lambda))g(s)ds$$

are solutions of (3.47), which are of course bounded on any compact interval. Therefore any solution of (3.47) can be written in the form

$$v^\pm(t) = \Phi^\pm(t,0,\lambda)\eta^\pm + L^\pm(t,g,\lambda)\,, \quad t \in I_T^\pm\,. \tag{3.48}$$

Remark 3.2.1 Indeed the integral $\int_t^\infty \Phi^+(t,s,\lambda)(id - P_{cs}^+(s,\lambda))g(s)ds$ is convergent, but the limit $\lim_{t\to\infty} \int_0^t \Phi^+(t,s,\lambda)P_{cs}^+(s,\lambda)g(s)ds$ does not exist. An analogous assertion is true for the integrals appearing in the representation of L^-. □

Next we determine $\eta^\pm$ such that $v^\pm$ represented by (3.48) satisfy (P 3.4). First, (P 3.4)(ii) and the representation of $L^\pm$ imply that $\eta^\pm \in Y^s \oplus Y^u \oplus Z$. Further, because of (H 3.2) the demand (P 3.4)(iii) is equal to

$$\begin{aligned} v^+(T) &\in T_{\gamma^s(\lambda)(T)} W_\lambda^{cs} \cap B_{\frac{\delta}{2}}(0) = \operatorname{im} P_{cs}^+(T,\lambda) \cap B_{\frac{\delta}{2}}(0)\,, \\ v^-(-T) &\in T_{\gamma^u(\lambda)(-T)} W_\lambda^{cu} \cap B_{\frac{\delta}{2}}(0) = \operatorname{im} P_{cu}^-(-T,\lambda) \cap B_{\frac{\delta}{2}}(0)\,. \end{aligned} \tag{3.49}$$

Thus we ask for conditions such that

$$(id - P_{cs}^+(T,\lambda))v^+(T) = 0 \quad \text{and} \quad (id - P_{cu}^-(-T,\lambda))v^-(-T) = 0\,. \tag{3.50}$$

Exemplarily, we consider $(id - P_{cs}^+(T,\lambda))v^+(T)$ (having in mind (3.44) and (3.48))

$$\begin{aligned} &(id - P_{cs}^+(T,\lambda))v^+(T) \\ &= (id - P_{cs}^+(T,\lambda))\Phi^+(T,0,\lambda)\eta^+ + (id - P_{cs}^+(T,\lambda)) \int_0^T \Phi^+(T,s,\lambda)P_{cs}^+(s,\lambda)g(s)ds \\ &= \Phi^+(T,0,\lambda)(id - P_{cs}^+(0,\lambda))\eta^+ + (id - P_{cs}^+(T,\lambda))P_{cs}^+(T,\lambda) \int_0^T \Phi^+(T,s,\lambda)g(s)ds \end{aligned}$$

and conclude that (3.50) is satisfied if and only if

$$\eta^+ \in \ker(id - P^+_{cs}(0,\lambda)) = \operatorname{im} P^+_{cs}(0,\lambda) = T_{\gamma^s(\lambda)(0)} W^{cs}_\lambda \quad \text{and}$$
$$\eta^- \in \ker(id - P^-_{cu}(0,\lambda)) = \operatorname{im} P^-_{cu}(0,\lambda) = T_{\gamma^u(\lambda)(0)} W^{cu}_\lambda \,.$$

Summarising the above facts we have to demand

$$\eta^+ \in (Y^s \oplus Y^u \oplus Z) \cap T_{\gamma^s(\lambda)(0)} W^{cs}_\lambda \quad \text{and}$$
$$\eta^- \in (Y^s \oplus Y^u \oplus Z) \cap T_{\gamma^u(\lambda)(0)} W^{cu}_\lambda \,.$$

We can describe $\Sigma \cap W^{cs}_\lambda$ locally around $\gamma^s(\lambda)(0)$ as graph of a function $h^{cs}(\cdot,\lambda)$, more precisely $\Sigma \cap W^{cs}_\lambda = \gamma^s(\lambda)(0) + \operatorname{graph} h^{cs}(\cdot,\lambda)$. Here

$$h^{cs}(\cdot,\lambda) \,:\, Y^c \oplus Y^s \;\to\; Y^u \oplus \hat{Z}, \quad h^{cs}(0,\lambda) = 0, \quad D_1 h^{cs}(0,0) = 0\,. \tag{3.51}$$

Then the graph of the map $D_1 h^{cs}(0,\lambda)(\cdot)$ describes $(Y^c \oplus Y^s \oplus Y^u \oplus \hat{Z}) \cap T_{\gamma^s(\lambda)(0)} W^{cs}_\lambda$. Analogously, we find h^{cu} such that its derivative describes $(Y^s \oplus Y^c \oplus Y^u \oplus \hat{Z}) \cap T_{\gamma^u(\lambda)(0)} W^{cu}_\lambda$. Thus $\eta^\pm$ have the form

$$\begin{aligned} \eta^+ &= \eta^+(y_c^+, y_s, \lambda) = \underbrace{y_c^+}_{\in Y^c} + \underbrace{y_s}_{\in Y^s} + \underbrace{D_1 h^{cs}(0,\lambda)(y_c^+, y_s)}_{\in Y^u \oplus \hat{Z}}\,, \\ \eta^- &= \eta^-(y_c^-, y_u, \lambda) = \underbrace{y_c^-}_{\in Y^c} + \underbrace{y_u}_{\in Y^u} + \underbrace{D_1 h^{cu}(0,\lambda)(y_c^-, y_u)}_{\in Y^s \oplus \hat{Z}}\,. \end{aligned} \tag{3.52}$$

Therefore

$$v^+(t) = \Phi^+(t,0,\lambda)\eta^+(y_c^+, y_s, \lambda) + L^+(t,g,\lambda)\,, \quad t \in I_T^+\,,$$
$$v^-(t) = \Phi^-(t,0,\lambda)\eta^-(y_c^-, y_u, \lambda) + L^-(t,g,\lambda)\,, \quad t \in I_T^-$$

solve (3.47), (P 3.4)(i)-(iii).

Now we discuss the non-linear problem (3.42), (P 3.4). Replacing $g(\cdot)$ with $h(\cdot, v^\pm(\cdot), \lambda)$ we find all solutions of (3.42) by solving the fixed point equations

$$\begin{aligned} v^+(t) &= \Phi^+(t,0,\lambda)\eta^+(y_c^+, y_s, \lambda) + L^+(t, h(\cdot, v^+(\cdot), \lambda), \lambda)\,, \quad t \in I_T^+\,, \\ v^-(t) &= \Phi^-(t,0,\lambda)\eta^-(y_c^-, y_u, \lambda) + L^-(t, h(\cdot, v^-(\cdot), \lambda), \lambda)\,, \quad t \in I_T^-\,. \end{aligned} \tag{3.53}$$

We solve these equations by the Implicit Function Theorem. For this we consider the right-hand sides of these equations as maps

$$Y^c \times Y^s \times \mathbb{R}^2 \times C^0(I_T^+, \mathbb{R}^{2n+2}) \to C^0(I_T^+, \mathbb{R}^{2n+2})\,,$$
$$Y^c \times Y^u \times \mathbb{R}^2 \times C^0(I_T^-, \mathbb{R}^{2n+2}) \to C^0(I_T^-, \mathbb{R}^{2n+2})\,.$$

We know that $(y_c^+, y_s, \lambda, v^+) = (0,0,0,0)$ and $(y_c^-, y_u, \lambda, v^-) = (0,0,0,0)$ are solutions of these equations. Similar to Section 3.1 all assumptions of the Implicit Function Theorem are met. Invoking this theorem we can solve Equation (3.53) for

$$v^+ = \hat{v}^+(y_c^+, y_s, \lambda)(\cdot), \quad v^- = \hat{v}^-(y_c^-, y_u, \lambda)(\cdot). \tag{3.54}$$

The solving functions $\hat{v}^+$ and $\hat{v}^-$ depend smoothly on y_c^+, y_s and λ and y_c^-, y_u and λ, respectively. Hence, for sufficiently small (y_c^+, y_s, λ) and (y_c^-, y_u, λ) we have $\hat{v}^+(y_c^+, y_s, \lambda)(T) \in B_{\frac{\delta}{2}}(0)$ and $\hat{v}^-(y_c^-, y_u, \lambda)(-T) \in B_{\frac{\delta}{2}}(0)$.

The functions $\hat{v}^\pm$ solve (3.42), (P 3.4)(i)-(iii). So it remains to ensure condition (iv) in Problem (P 3.4). For this we consider

$$\begin{aligned} \hat{v}^+(y_c^+, y_s, \lambda)(0) &= \eta^+(y_c^+, y_s, \lambda) \\ &\quad -(id - P_{cs}^+(0,\lambda)) \int_0^T \Phi^+(0,s,\lambda) h(s, \hat{v}^+(y_c^+, y_s, \lambda)(s), \lambda) ds, \\ \hat{v}^-(y_c^-, y_s, \lambda)(0) &= \eta^-(y_c^-, y_s, \lambda) \\ &\quad +(id - P_{cu}^-(0,\lambda)) \int_{-T}^0 \Phi^-(0,s,\lambda) h(s, \hat{v}^-(y_c^-, y_s, \lambda)(s), \lambda) ds. \end{aligned} \tag{3.55}$$

In accordance with the direct sum decomposition (3.36) and (3.52) there are functions y_u^+, y_s^- and $z^\pm$ such that

$$\begin{aligned} \hat{v}^+(y_c^+, y_s, \lambda)(0) &= \underbrace{y_c^+}_{\in Y^c} + \underbrace{y_s}_{\in Y^s} + \underbrace{y_u^+(y_c^+, y_s, \lambda)}_{\in Y^u} + \underbrace{z^+(y_c^+, y_s, \lambda)}_{\in \hat{Z}}, \\ \hat{v}^-(y_c^-, y_u, \lambda)(0) &= \underbrace{y_c^-}_{\in Y^c} + \underbrace{y_u}_{\in Y^u} + \underbrace{y_s^-(y_c^-, y_u, \lambda)}_{\in Y^s} + \underbrace{z^-(y_c^-, y_u, \lambda)}_{\in \hat{Z}}. \end{aligned} \tag{3.56}$$

Hence $v^\pm$ satisfy (P 3.4)(iv) if

$$y_s = y_s^-(y_c^-, y_u, \lambda) \quad \text{and} \quad y_u = y_u^+(y_c^+, y_s, \lambda). \tag{3.57}$$

Lemma 3.2.2

(i) $y_s^-(0,0,0) = 0\,, \quad y_u^+(0,0,0) = 0.$

(ii) $D_2 y_s^-(0,0,0) = 0\,, \quad D_2 y_u^+(0,0,0) = 0.$

Proof We have gained $\hat{v}^\pm$ via the Implicit Function Theorem from (3.53). From that we know $\hat{v}^+(0,0,0)(0) = 0$ and $\hat{v}^-(0,0,0)(0) = 0$. Now (i) of the lemma follows from (3.56).

Plugging (3.52) and (3.54) in (3.53) yields

$$\hat{v}^+(y_c^+, y_s, \lambda)(0) = y_c^+ + y_s + D_1 h^{cs}(0,\lambda)(y_c^+, y_s) + L^+(0, h(\cdot, v^+(y_c^+, y_s, \lambda)(\cdot), \lambda), \lambda)\,.$$

Note that $D_1 h^{cs}(0,0) = 0$. Moreover, because of $D_2 h(t,0,\lambda) \equiv 0$ we have

$$\frac{\partial}{\partial y_s} L^+(0, h(\cdot, v^+(0,0,0)(\cdot), 0), 0) = 0 \ .$$

Therefore $D_2 \hat{v}^+(0,0,0)(0) = id$. On the other hand, differentiating the representation of $\hat{v}^+$ which is given in (3.56) yields $D_2 y_u^+(0,0,0) = 0$.
A similar consideration gives $D_2 y_s^-(0,0,0) = 0$. ∎

Due to the above lemma we can solve (3.57) for

$$y_s = y_s(y_c^+, y_c^-, \lambda) \quad \text{and} \quad y_u = y_u(y_c^+, y_c^-, \lambda) \tag{3.58}$$

by the Implicit Function Theorem. Combining this with (3.54) we get

$$\begin{aligned} v^+ = v^+(y_c^+, y_c^-, \lambda) &:= \hat{v}^+(y_c^+, y_s(y_c^+, y_c^-, \lambda), \lambda), \\ v^- = v^-(y_c^+, y_c^-, \lambda) &:= \hat{v}^-(y_c^-, y_u(y_c^+, y_c^-, \lambda), \lambda) \, . \end{aligned} \tag{3.59}$$

Summarising our results achieved up to now we can formulate the following lemma:

Lemma 3.2.3 *Assume (H 3.2). For $y_c^+, y_c^- \in Y^c$ and $\lambda \in \mathbb{R}^2$, sufficiently small, there is a unique one-homoclinic Lin solution $\{(\gamma^+(y_c^+, y_c^-, \lambda), \gamma^-(y_c^+, y_c^-, \lambda))\}$ to the centre manifold.*
Moreover, the mappings $\gamma^\pm(\cdot,\cdot,\cdot) : Y^c \times Y^c \times \mathbb{R}^2 \to C(\mathbb{R}^\pm, \mathbb{R}^n)$ are as smooth as the vector field.

Proof With $\gamma^+(y_c^+, y_c^-, \lambda)(\cdot) = \gamma^s(\lambda)(\cdot) + v^+(y_c^+, y_c^-, \lambda)(\cdot)$ and $\gamma^-(y_c^+, y_c^-, \lambda)(\cdot) = \gamma^u(\lambda)(\cdot) + v^-(y_c^+, y_c^-, \lambda)(\cdot)$ the assertions of the lemma follows from the above considerations. ∎

By solving the bifurcation equation

$$\xi^\infty(y_c^+, y_c^-, \lambda) := \gamma^+(y_c^+, y_c^-, \lambda)(0) - \gamma^-(y_c^+, y_c^-, \lambda)(0) = 0 \tag{3.60}$$

we compute one-homoclinic orbits to the centre manifold. So ξ^∞ is a smooth mapping

$$\xi^\infty : Y^c \times Y^c \times \mathbb{R}^2 \to Z.$$

Because of the representations $\gamma^+(0) = \gamma^s(\lambda)(0) + v^+(0)$ and $\gamma^-(0) = \gamma^u(\lambda)(0) + v^-(0)$ (see (3.41)) we have

$$\xi^\infty(y_c^+, y_c^-, \lambda) = \xi(\lambda) + v^+(y_c^+, y_c^-, \lambda)(0) - v^-(y_c^+, y_c^-, \lambda)(0).$$

Together with (3.56) this can be written as

$$\begin{aligned} \xi^\infty(y_c^+, y_c^-, \lambda) = \xi(\lambda) + (y_c^+ - y_c^-) + z^+(y_c^+, y_s(y_c^+, y_c^-, \lambda), \lambda) \\ - z^-(y_c^-, y_u(y_c^+, y_c^-, \lambda), \lambda). \end{aligned} \tag{3.61}$$

3.3 Discussion of the bifurcation equation

Now, we will compute all one-homoclinic orbits to the centre manifold as solutions of the bifurcation equation $\gamma^+(y_c^+, y_c^-, \lambda)(0) - \gamma^-(y_c^+, y_c^-, \lambda)(0) = 0$. In our considerations we have to distinguish elementary and non-elementary primary homoclinic orbits. In both cases the reversibility allows to reduce the three-dimensional equation to a one-dimensional one.

The next considerations are devoted to the precise analysis leading to solutions of the bifurcation equation.

3.3.1 The non-elementary case

Now we want to discuss the bifurcation equation $\xi^\infty = 0$ for the case of a non-elementary primary homoclinic orbit Γ. That is:

(H 3.3) W^{cs} intersects $\operatorname{Fix} R$ non-transversally at $\gamma(0)$.

This hypothesis is equivalent to (3.39). Together with the definition and properties of Y^c, which are presented in Section 3.2, we get

$$Y^c \subset \operatorname{Fix} R\,.$$

So, with Lemma 3.1.2 we obtain also

$$\hat{Z} \subset \operatorname{Fix}(-R)\,.$$

For the discussion of the bifurcation equation we consider (3.61). In the case under consideration $y_c^+ - y_c^-$ is the $\operatorname{Fix} R$-component of $\xi^\infty(y_c^+, y_c^-, \lambda)$, recall that $\xi(\lambda) \in \operatorname{Fix}(-R)$ (see (3.30)). Hence

$$\xi^\infty(y_c^+, y_c^-, \lambda) = 0 \quad \Longrightarrow \quad y_c^+ = y_c^-\,.$$

Altogether

$$\xi^\infty(y_c^+, y_c^-, \lambda) = 0 \quad \Longleftrightarrow \quad \hat{\xi}^\infty(y_c, \lambda_1, \lambda_2) := \xi^\infty(y_c, y_c, \lambda) = 0\,. \tag{3.62}$$

Moreover the representation (3.61) of ξ^∞ provides

$$\hat{\xi}^\infty(y_c, \lambda_1, \lambda_2) \in \operatorname{Fix}(-R) \cap Z. \tag{3.63}$$

Essentially, after introducing appropriate coordinates, $\hat{\xi}^\infty$ can be seen as a smooth mapping $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(y_c, \lambda_1, \lambda_2) \to \hat{\xi}^\infty(y_c, \lambda_1, \lambda_2)$ with

$$\hat{\xi}^\infty(0,0,0) = 0\,, \quad D_1\hat{\xi}^\infty(0,0,0) = 0\,. \tag{3.64}$$

Before discussing the structure of the solution set of the bifurcation equation we prove that all orbits corresponding to $\hat{\xi}^\infty = 0$ are symmetric ones.

Lemma 3.3.1 *Assume (H 3.2) and (H 3.3). Let y_c and $\lambda = (\lambda_1, \lambda_2)$ be such that $\hat{\xi}^\infty(y_c, \lambda_1, \lambda_2) = 0$. Further let $x(y_c, \lambda)(\cdot)$ be the corresponding solution of (1.1) with orbit $\mathcal{O}(y_c, \lambda)$. Then $\mathcal{O}(y_c, \lambda)$ is symmetric.*

Proof Clearly we have

$$x(y_c, \lambda)(0) = \gamma^+(y_c, y_c, \lambda)(0) = \gamma^-(y_c, y_c, \lambda)(0)\ . \tag{3.65}$$

In order to prove the symmetry of the solutions of the bifurcation equation we show that $\gamma^+(y_c, \lambda)(0) \in \operatorname{Fix} R$. That means, by (3.65) we have to show that

$$R\gamma^+(y_c, y_c, \lambda)(0) = \gamma^-(y_c, y_c, \lambda)(0)\ . \tag{3.66}$$

This is equivalent to

$$R\gamma^s(\lambda)(0) + Rv^+(y_c, y_s(y_c, \lambda), \lambda)(0) = \gamma^u(\lambda)(0) + v^-(y_c, y_u(y_c, \lambda), \lambda)(0)\ .$$

With (3.29) it remains to show $Rv^+(y_c, y_s(y_c, \lambda), \lambda)(0) = v^-(y_c, y_u(y_c, \lambda), \lambda)(0)$. Similar to Lemma 3.1.5 we find

$$Rv^+(y_c, y_s(y_c, \lambda), \lambda)(0) = v^-(Ry_c, Ry_s(y_c, \lambda), \lambda)(0). \tag{3.67}$$

Finally, the symmetry of the system (3.57) will be transferred to its solving functions (3.58):

$$Ry_s(y_c, \lambda) = y_u(Ry_c, \lambda).$$

Now $y_c \in \operatorname{Fix} R$ completes the proof. ■

Corollary 3.3.2 *Let $\mathcal{O}(y_c, \lambda)$ be an orbit according to $\hat{\xi}^\infty(y_c, \lambda_1, \lambda_2) = 0$. Then $\mathcal{O}(y_c, \lambda)$ is a homoclinic orbit to the equilibrium or to a periodic orbit in W^c_{loc}.*

Proof Due to the symmetry of the orbit it holds $\alpha(\mathcal{O}(y_c, \lambda)) = R\omega(\mathcal{O}(y_c, \lambda))$, where α and ω are the α- and ω-limit set, respectively. On the other hand it is clear that the ω-limit set either is the equilibrium or it coincides with one of the periodic orbits filling the (local) centre manifold. Finally the symmetry of these orbits gives the result. ■

For discussing the bifurcation equation $\hat{\xi}^\infty = 0$ we will presume several further assumptions. There the geometrical meaning of the parameters λ_1, λ_2 (which was explained in Section 1) will be reflected. First we assume

(H 3.4) $\quad D_2\hat{\xi}^\infty(0, 0, 0) \neq 0.$

This assumption underpins, in accordance with Hypothesis (H 3.1) and (3.31), that λ_1 is responsible for the drift of the stable and unstable manifold. By assuming (H 3.4) we can solve $\hat{\xi}^\infty = 0$ for $\lambda_1 = \lambda_1^*(y_c, \lambda_2)$ near $(y_c, \lambda_1, \lambda_2) = (0, 0, 0)$. Hence

$$\hat{\xi}^\infty(y_c, \lambda_1, \lambda_2) = 0 \Longleftrightarrow \lambda_1 = \lambda_1^*(y_c, \lambda_2). \tag{3.68}$$

Lemma 3.3.3 *$\lambda_1^*(0,0) = 0$, $D_1\lambda_1^*(0,0) = 0$, and $D_2^k\lambda_1^*(0,0) = 0$, for all $k \geq 1$.*

Proof The first two statements, $\lambda_1^*(0,0) = 0$ and $D_1\lambda_1^*(0,0) = 0$, follow directly from (3.68) and (3.64).
The parameter λ_1 describes, independently on λ_2, the splitting of the stable and unstable manifolds W^s and W^u, see (3.31). For $\lambda_1 = 0$ both manifolds intersect each other. So, for $\lambda_1 = 0$ and for any λ_2 we have at least one solution of the bifurcation equation $\hat{\xi}^\infty(y_c, 0, \lambda_2) = 0$, namely a solution corresponding to the intersection of the stable and unstable manifold of the equilibrium. Such a solution corresponds to

$$v^\pm = v^+(y_c, y_c, \lambda_1 = 0, \lambda_2)(0) = 0.$$

From (3.56) and (3.59) we see that $v^\pm(y_c, y_c, \lambda_1 = 0, \lambda_2)(0) = 0$ implies $y_c = 0$. Altogether this means $\hat{\xi}^\infty(0, 0, \lambda_2) \equiv 0$. Then (3.68) provides $\lambda_1^*(0, \lambda_2) \equiv 0$. This eventually implies that $D_2^k\lambda_1^*(0,0) = 0$, for all $k \geq 1$. ■

So it makes sense to assume

(H 3.5) $D_1^2\lambda_1^*(0,0)$ is non-singular.

Assumption (H 3.5) says that W^{cs} and W^{cu} have quadratic tangency. The second derivative $D_1^2\lambda_1^*(0,0)$ can be seen as a 2×2-matrix. In the further discussion we distinguish the cases that $D_1^2\lambda_1^*(0,0)$ is definite (both eigenvalues have the same sign) and $D_1^2\lambda_1^*(0,0)$ is indefinite (the eigenvalues have different signs).

Now, we consider the structure of the solution set of the bifurcation equation $\hat{\xi}^\infty = 0$. A more detailed interpretation of the dynamical consequences is given in Section 3.3.1.1 below. First we assume that $D_1^2\lambda_1^*(0,0)$ is definite.

Theorem 3.3.4 *Assume (H 3.2) – (H 3.5). Further let $D_1^2\lambda_1^*(0,0)$ be positive definite. Then there is a curve $\mathfrak{C}$ in the (λ_2, λ_1)-plane, where $\mathfrak{C}$ is graph of a function $\mathfrak{c} : \lambda_2 \mapsto \lambda_1 = \mathfrak{c}(\lambda_2)$ having a maximum at $(\lambda_1, \lambda_2) = (0,0)$ such that:*
For each sufficiently small $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 > \mathfrak{c}(\lambda_2)$ there is a closed curve κ_λ in Y^c such that for each $y_c \in \kappa_\lambda$ there is a symmetric one-homoclinic orbit $\mathcal{O}(y_c, \lambda)$ asymptotic to the centre manifold. The mapping $y_c \mapsto \mathcal{O}(y_c, \lambda)$ is injective. The curve κ_λ contracts as λ tends to $\mathfrak{C}$ - degenerates to a point for $\lambda \in \mathfrak{C}$ and disappears for λ below $\mathfrak{C}$, i.e., $\lambda_1 < \mathfrak{c}(\lambda_2)$.
All one-homoclinic orbits to the centre manifold near Γ are assigned to an element of κ_λ. So, for λ below $\mathfrak{C}$ there are no such homoclinic orbits.

Proof Near Γ all one-homoclinic orbits to the centre manifold are related to solutions of the bifurcation equation (3.60). Due to (3.68) the bifurcation equation is equivalent to

$$\lambda_1 = \lambda_1^*(y_c, \lambda_2).$$

So for given $\lambda = (\lambda_1, \lambda_2)$ we study the level set $\kappa_\lambda := \{y_c : \lambda_1 = \lambda_1^*(y_c, \lambda_2\}$ of $\lambda_1^*(\cdot, \lambda_2)$. Exactly the elements $y_c \in \kappa_\lambda$ correlate with homoclinic orbits $\mathcal{O}(y_c, \lambda)$. The symmetry of the orbits $\mathcal{O}(y_c, \lambda)$ has been proved by Lemma 3.3.1.

Because of (H 3.5) the equation $D_1\lambda_1^*(y_c, \lambda_2) = 0$ can be solved for $y_c = y_c^e(\lambda_2)$ near $(y_c, \lambda_2) = (0, 0)$. We define

$$\mathfrak{c} : \lambda_2 \mapsto \mathfrak{c}(\lambda_2) := \lambda_1^*(y_c^e(\lambda_2), \lambda_2).$$

From the derivation of $y_c^e(\cdot)$ we see that $y_c^e(0) = 0$. Together with Lemma 3.3.3 this provides $\mathfrak{c}(0) = 0$ and $D\mathfrak{c}(0) = 0$.
Because $D_1^2\lambda_1^*(0, 0)$ is positive definite, the function $\lambda_1^*(\cdot, \lambda_2)$ has a minimum in $y_c^e(\lambda_2)$. So the level set κ_λ of $\lambda_1^*(\cdot, \lambda_2)$ is a closed curve for λ above the curve $\mathfrak{C}$, or it consists of just one point for $\lambda \in \mathfrak{C}$, respectively. Finally, κ_λ is empty for λ below the curve $\mathfrak{C}$.
Recall that $\kappa_{\lambda=(0,\lambda_2)}$ is non-empty. Therefore $\mathfrak{c}(\lambda_2) \leq 0$ for all λ_2. ■

For the visualisation in a bifurcation diagram we refer to Figure 3.3.

Remark 3.3.5 We obtain qualitatively the same results if $D_1^2\lambda_1^*(0, 0)$ is negative definite. □

We now turn to the case that $D_1^2\lambda_1^*(0, 0)$ is indefinite.

Theorem 3.3.6 *Assume (H 3.2) – (H 3.5). Moreover we assume that* $D_1^2\lambda_1^*(0, 0)$ *is indefinite. Then for each sufficiently small* λ *there are two curves* κ_λ^1, κ_λ^2 *in* Y^c *such that exactly for* $y_c \in \kappa_\lambda^1 \cup \kappa_\lambda^2$ *there is a one-homoclinic orbit* $\mathcal{O}(y_c, \lambda)$ *to the centre manifold. The orbits* $\mathcal{O}(y_c, \lambda)$ *are symmetric.*
Furthermore there is a smooth curve $\mathfrak{C}$ *in the* (λ_2, λ_1)*-plane which is tangent to the* λ_2*-axis at* $(\lambda_1, \lambda_2) = (0, 0)$*: for* $\lambda \in \mathfrak{C}$ *the curves* κ_λ^1 *and* κ_λ^2 *intersect transversally; for* $\lambda \notin \mathfrak{C}$ *the curves* κ_λ^i, $i = 1, 2$, *are hyperbola-like and do not intersect.*

Proof The proof runs completely parallel to that one of Theorem 3.3.4. Only the indefiniteness of $D_1^2\lambda_1^*(0, 0)$ gives another structure of the level sets of $\lambda_1^*(\cdot, \lambda_2)$. ■

For the visualisation in a bifurcation diagram we refer to Figure 3.4.

3.3.1.1 Bifurcation scenario

We want to explain the consequences of the Theorems 3.3.4 and 3.3.6 for the dynamics more closely. We will do that only for the $\mathbb{R}^4$-case, i.e. $n = 1$. Although the results for $n > 1$ are the same, the arguments are easier in the case we want to look at.
We start by considering the centre-stable manifold W_λ^{cs} as stable manifold of the (local) centre manifold. So we can see W_λ^{cs} as the union of the stable fibres $M_{x,\lambda}$, with base points x in $W_{loc,\lambda}^c$:

$$W_\lambda^{cs} = \bigcup_{x \in W_{loc,\lambda}^c} M_{x,\lambda}.$$

These fibres are one-dimensional ($\mathbb{R}^4$-case) and depend smoothly on $x \in W_{loc,\lambda}^c$. Especially we have $M_{x=0,\lambda} = W_\lambda^s$. So, for sufficiently small x these fibres intersect

the cross section Σ transversally, that means $M_{x,\lambda} \cap \Sigma$ consists of exactly one point. The union of these points is nothing else but the trace of W_λ^{cs} within Σ. This trace can again be represented as graph of a function h_λ^{cs} defined on Y^c (see (3.51) and have in mind that $\dim Y^s = 0$). So we have a one-to-one relation between Y^c and $(\bigcup_{x \in W^c_{loc,\lambda}} M_{x,\lambda}) \cap \Sigma$. Moreover, let $\mathcal{P}$ be a periodic orbit in $W^c_{loc,\lambda}$ then $(\bigcup_{x \in \mathcal{P}} M_{x,\lambda}) \cap \Sigma$ forms a closed curve in $W_\lambda^{cs} \cap \Sigma$.
Let, as defined in the proof of Theorem 3.3.4, $\kappa_\lambda = \{y_c : \lambda_1 = \lambda_1^*(y_c, \lambda_2)\}$. Then $h_\lambda^{cs}(\kappa_\lambda)$ is a curve in $W_\lambda^{cs} \cap \Sigma$. (In the case under consideration we have even $h_\lambda^{cs}(\kappa_\lambda) \equiv 0$.) Projecting this curve along the stable fibres in $W^c_{loc,\lambda}$ gives a curve κ_λ^c. The projection is depicted in Figure 3.1. Indeed, such a projection is of class C^{k-1} if the vector field is in C^k, see [SSTC98].

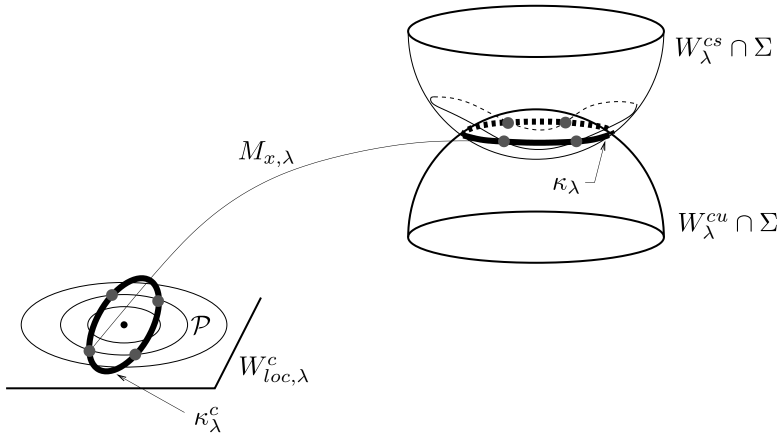

Figure 3.1: Projection along stable fibres

Now, exactly those periodic orbits in $W^c_{loc,\lambda}$ which are intersected by κ_λ^c are the limit sets of one-homoclinic orbits to the centre manifold. This is due to the invariance of the fibres, which asserts that for each $x \in W^c_{loc,\lambda}$

$$\varphi(t, M_{x,\lambda}, \lambda) \subset M_{\varphi(t,x,\lambda),\lambda} , \quad t > 0 ,$$

where $\varphi(t, \cdot, \lambda)$ denotes the flow, see Figure 3.2. Hence, if $x \in W^c_{loc,\lambda}$ belongs to a periodic orbit then each point in $M_{x,\lambda}$ will be "transported" to this orbit under the flow.
Under the assumptions of Theorem 3.3.4 κ_λ^c is a closed curve for those $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 > \mathfrak{c}(\lambda_2)$; κ_λ^c shrinks down as λ tends to $\mathfrak{C}$, merges to a point for $\lambda \in \mathfrak{C}$ and disappears as λ_1 becomes less than $\mathfrak{c}(\lambda_2)$. The position of κ_λ^c (bold lines) in W_λ^c is depicted in Figure 3.3. In order to retrace the presented bifurcation diagram, realise that $\kappa^c_{(\lambda_1>0,0)}$ is a closed curve surrounding $\gamma^s(\lambda)(0)$ within $W_\lambda^{cs} \cap \Sigma$. Consequently $\kappa^c_{(\lambda_1>0,0)}$ is a closed curve encircling the equilibrium $\mathring{x} = 0$. The rest of the bifurcation diagram presented in Figure 3.3 stems from the fact that κ_λ^c moves continuously by changing λ, and that κ_λ^c contains the equilibrium if and only if $\lambda_1 = 0$.

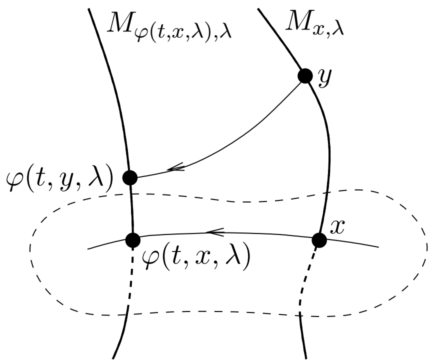

Figure 3.2: Invariance of the stable fibres

Note, that we have no further information concerning the exact shape of κ_λ^c. However we want to mention that different intersection points of κ_λ^c with one periodic orbit correspond to different homoclinic orbits to this periodic orbit. From the bifurcation diagram one may expect bifurcations of homoclinic orbits to a (distinguished) periodic orbit while moving κ_λ^c. But those bifurcations are beyond the scope of this thesis.

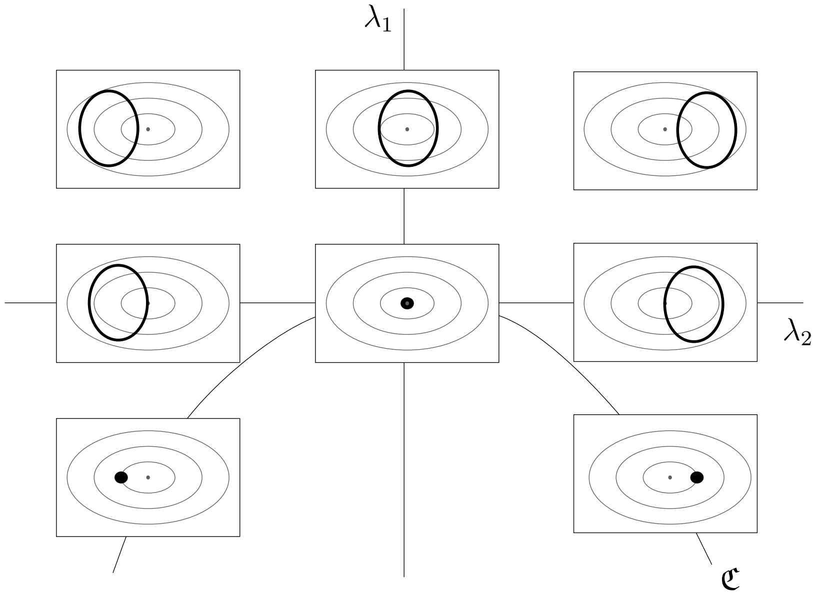

Figure 3.3: Bifurcation diagram corresponding to Theorem 3.3.4.

Analogously to the above considerations we can discuss the consequences for the dynamics under the assumptions of Theorem 3.3.6. Projection of κ_λ^i along the stable fibres gives curves $\kappa_\lambda^{i,c}$ in W_{loc}^c of the same structure as κ_λ^i. The relative position of these curves is depicted in Figure 3.4.

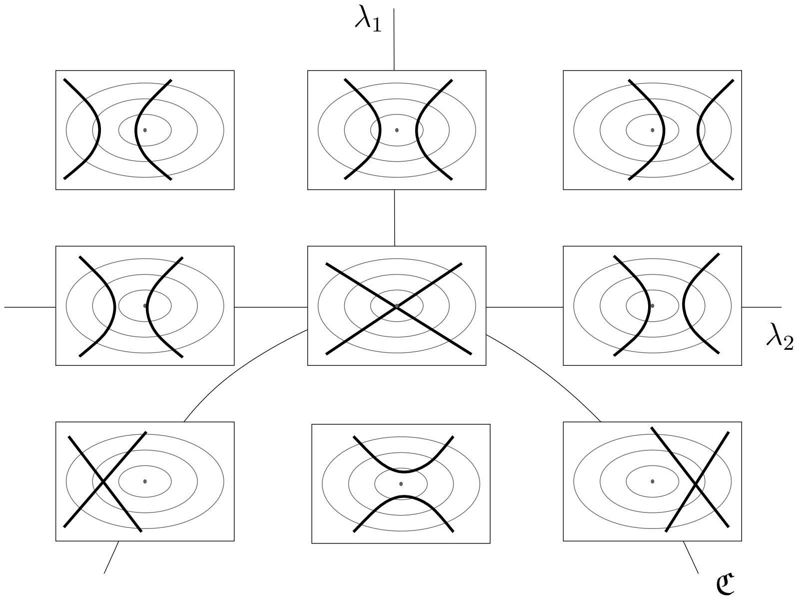

Figure 3.4: Bifurcation diagram corresponding to Theorem 3.3.6.

3.3.2 The elementary case

We assume that the primary homoclinic orbit Γ is elementary:

(H 3.6) W^{cs} intersects $\operatorname{Fix} R$ transversally at $\gamma(0)$.

This hypothesis is equivalent to (3.40). Together with the definition and properties of Y^c, which are presented in Section 3.2, we get

$$\dim(Y^c \cap \operatorname{Fix} R) = 1.$$

Hence Y^c is spanned by a one-dimensional subspace of $\operatorname{Fix} R$ and a one-dimensional subspace of $\operatorname{Fix}(-R)$. With Lemma 3.1.2 we obtain

$$\hat{Z} \subset \operatorname{Fix} R\ .$$

For discussing the bifurcation equation $\xi^\infty(y_c^+, y_c^-, \lambda) = 0$ we assort the components of ξ^∞ (see (3.61)) regarding their affiliation to $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$, respectively. For that reason we decompose

$$y_c^\pm = y_R^\pm + y_{-R}^\pm, \tag{3.69}$$

with $y_R^\pm \in Y^c \cap \operatorname{Fix} R$ and $y_{-R}^\pm \in Y^c \cap \operatorname{Fix}(-R)$. Plugging (3.69) into the representation (3.61) we can regard all functions as functions of $(y_R^+, y_R^-, y_{-R}^+, y_{-R}^-, \lambda)$. To denote these functions we add a " ˜ " to the original symbol of these functions.

In accordance with the direct sum decomposition (3.36) we see immediately that $\tilde{\xi}^\infty(y_R^+, y_R^-, y_{-R}^+, y_{-R}^-, \lambda) = 0$ is equivalent to

$$y_R^+ = y_R^- =: y_R \,,$$
$$\tilde{z}^+(y_R^+, y_R^-, y_{-R}^+, y_{-R}^-, \lambda) - \tilde{z}^-(y_R^+, y_R^-, y_{-R}^+, y_{-R}^-, \lambda) = 0 \,,$$
$$\xi(\lambda) + y_{-R}^+ - y_{-R}^- = 0.$$

Because of $\xi(\lambda) \equiv \lambda_1$ (see Remark 3.31) the last equation can be solved for $y_{-R}^- = y_{-R}^+ + \lambda_1$. A transformation T in $\mathrm{Fix}\,(-R) \cap Y^c$

$$y_{-R} := T(y_{-R}^+, \lambda_1) := y_{-R}^+ + \lambda_1/2$$

gives

$$y_{-R}^- = y_{-R} + \lambda_1/2.$$

This way we arrive again at a reduced bifurcation equation

$$\begin{aligned}\hat{\xi}^\infty(y_R, y_{-R}, \lambda) := \;& \tilde{z}^+(y_R, y_R, y_{-R} - \lambda_1/2, y_{-R} + \lambda_1/2, \lambda) \\ & - \tilde{z}^-(y_R, y_R, y_{-R} - \lambda_1/2, y_{-R} + \lambda_1/2, \lambda) = 0.\end{aligned} \tag{3.70}$$

Essentially $\hat{\xi}^\infty$ can be seen as a smooth mapping $\hat{\xi}^\infty : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ with

$$\hat{\xi}^\infty(0,0,0) = 0, \quad D_i\hat{\xi}^\infty(0,0,0) = 0,\; i = 1,2. \tag{3.71}$$

Moreover we have

Lemma 3.3.7 *Assume (H 3.1), (H 3.2) and (H 3.6). The function* $\hat{\xi}^\infty$ *is odd with respect to* y_{-R}*, i.e.,*

$$\hat{\xi}^\infty(y_R, y_{-R}, \lambda) = -\hat{\xi}^\infty(y_R, -y_{-R}, \lambda)\,.$$

Proof Similar to the proof of the symmetry properties (3.26) and (3.27) we get

$$Ry_u^+(y_c^+, y_s, \lambda) = y_s^-(Ry_c^+, Ry_s, \lambda), \quad Rz^+(y_c^+, y_s, \lambda) = z^-(Ry_c^+, Ry_s, \lambda)\,. \tag{3.72}$$

Exploiting this during the solving mechanism we find

$$Ry_u(y_c^+, y_c^-, \lambda) = y_s(Ry_c^-, Ry_c^+, \lambda)\,.$$

This way we obtain:

$$R(\tilde{z}^+(y_R, y_R, y_{-R} - \lambda_1/2, y_{-R} + \lambda_1/2, \lambda) - \tilde{z}^-(y_R, y_R, y_{-R} - \lambda_1/2, y_{-R} + \lambda_1/2, \lambda))$$
$$= \tilde{z}^-(y_R, y_R, -y_{-R} - \lambda_1/2, -y_{-R} + \lambda_1/2, \lambda) - \tilde{z}^+(y_R, y_R, -y_{-R} - \lambda_1/2, -y_{-R} + \lambda_1/2, \lambda).$$

On the other hand both $\tilde{z}^+$ and $\tilde{z}^-$ are elements of $\mathrm{Fix}\,R$. This together with the definition of $\hat{\xi}^\infty$ (see (3.70)) provides the lemma. ■

Lemma 3.3.7 says that

$$\hat{\xi}^{\infty}(y_R, 0, \lambda) \equiv 0. \tag{3.73}$$

Lemma 3.3.8 *Assume (H 3.1), (H 3.2) and (H 3.6). Exactly the orbits* $\mathcal{O}(y_R, y_{-R} = 0, \lambda)$ *corresponding to solutions* $(y_R, y_{-R} = 0, \lambda)$ *of* $\hat{\xi}^{\infty} = 0$ *are symmetric.*

Proof The principle set up is as in the proof of Lemma 3.3.1. So it remains to show $Rv^{+}(y_c^{+}, y_s(y_c^{+}, y_c^{-}, \lambda), \lambda)(0) = v^{-}(y_c^{-}, y_u(y_c^{+}, y_c^{-}, \lambda), \lambda)(0)$. Using relations similar to (3.67) and (3.72) we see that the above equation is equivalent to

$$\begin{aligned} &v^{-}(y_R - y_{-R}^{+}, y_u(y_R - y_{-R}^{-}, y_R - y_{-R}^{+}, \lambda), \lambda)(0) \\ &= v^{-}(y_R + y_{-R}^{-}, y_u(y_R + y_{-R}^{+}, y_R + y_{-R}^{-}, \lambda), \lambda)(0)\ . \end{aligned} \tag{3.74}$$

Of course this is true if $y_{-R}^{+} = -y_{-R}^{-}$. By the definition of y_{-R} this is equivalent to $y_{-R} = 0$. On the other hand: The representation (3.56) of v^{-} tells that the Y^c-component of $v^{-}(y_c^{-}, y_u(y_c^{+}, y_c^{-}, \lambda), \lambda)(0)$ is y_c^{-}. So (3.74) implies $y_{-R}^{+} = -y_{-R}^{-}$ and hence $y_{-R} = 0$. ■

Remark 3.3.9 In the original coordinates, before performing the λ-dependent transformation T, $y_{-R} = 0$ corresponds to $y_{-R}^{-} = \lambda_1/2$, $y_{-R}^{+} = -\lambda_1/2$. □

Because of (3.73) there is a smooth function

$$\psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (y_R, -y_{-R}, \lambda_1, \lambda_2) \mapsto \psi(y_R, -y_{-R}, \lambda_1, \lambda_2)$$

with

$$\hat{\xi}^{\infty}(y_R, y_{-R}, \lambda) = y_{-R}\psi(y_R, y_{-R}, \lambda_1, \lambda_2)\ . \tag{3.75}$$

The function ψ is even with respect to y_{-R}: $\psi(y_R, y_{-R}, \lambda_1, \lambda_2) = \psi(y_R, -y_{-R}, \lambda_1, \lambda_2)$. From (3.71) follows

$$\psi(0, 0, 0, 0) = 0. \tag{3.76}$$

We assume

(H 3.7) $\quad D_1\psi(0, 0, 0, 0) \neq 0.$

Hence the equation $\psi = 0$ can be solved for $y_R = y_R^{*}(y_{-R}, \lambda_1, \lambda_2)$.

Altogether, for fixed λ the set of zeros of $\hat{\xi}^{\infty}$ consists of two intersecting curves, $\kappa_{sym} := \{y_{-R} = 0\}$ and $\kappa_{asym} := \{y_R = y_R^{*}(y_{-R}, \lambda_1, \lambda_2)\}$. The curve κ_{sym} is related to symmetric one-homoclinic orbits to the centre manifold, cf. Lemma 3.3.8, while κ_{asym} corresponds to two families of non-symmetric one-homoclinic orbits to the centre manifold, which are R-images of each other. We summarise our results as follows:

Theorem 3.3.10 *Assume (H 3.1), (H 3.2), (H 3.6) and (H 3.7). Moreover let* $\xi(\lambda) \equiv \lambda_1$*. Then for sufficiently small* $\lambda \in \mathbb{R}^2$ *there exist exactly one one-parameter family of symmetric and exactly two one-parameter families of non-symmetric one-homoclinic orbits to the centre manifold. The two families of non-symmetric homoclinic orbits are* R*-images of each other.*

Finally we discuss one-homoclinic orbits to the centre manifold which involve the equilibrium. Those are either homoclinic orbits to the equilibrium or heteroclinic cycles between the equilibrium and a periodic orbit in the centre manifold. Such cycles consist of a heteroclinic orbit lying in the intersection of the stable manifold of the equilibrium and the unstable manifold of some periodic orbit, and its R-image. So it is sufficient to determine one-homoclinic orbits to the centre manifold lying in the stable manifold of the equilibrium. Those orbits correspond to solutions of

$$\xi^\infty(y_c^+, y_c^-, \lambda) = 0, \quad v^+(y_c^+, y_c^-, \lambda)(0) = 0; \tag{3.77}$$

the second equation, $v^+(y_c^+, y_c^-, \lambda)(0) = 0$, guarantees that the orbit is in W^s.
In our further considerations we restrict to $\mathbb{R}^4$ ($n = 1$). Because in that case $\dim Y^s = \dim Y^u = 0$ the formulas (3.56) to (3.59) provide that actually v^+ does not depend on y_c^-. So we can view $v^+(y_c^+, y_c^-, \lambda)(t)$ as a function $\hat{v}^+(y_c^+, \lambda)(t)$.

Lemma 3.3.11 $\hat{v}^+(y_c^+, \lambda)(0) = 0$ *if and only if* $y_c^+ = 0$.

Proof Let $\hat{v}^+(y_c^+, \lambda)(0) = 0$. Then (3.56) implies that $y_c^+ = 0$.
On the other hand for each λ the functions $\tilde{v}^\pm(\lambda)(t) \equiv 0$ solve (3.42),(P 3.4). So, in particular there is a y_c^+ such that $\tilde{v}^+(\lambda)(t) = \hat{v}(y_c^+, \lambda)(t)$. This implies that for all λ we have $0 = \tilde{v}^+(\lambda)(0) = \hat{v}(y_c^+, \lambda)(0)$. And again from (3.56) we conclude that $y_c^+ = 0$. Altogether $\hat{v}(0, \lambda)(0) \equiv 0$. ∎

Therefore in $\mathbb{R}^4$ the System (3.77) reduces to

$$\xi^\infty(0, y_c^-, \lambda) = 0.$$

More detailed this equation reads, see (3.61),

$$\lambda_1 - y_c^- - z^-(y_c^-, \lambda) = 0. \tag{3.78}$$

Exploiting the affiliation of the single terms on the right-hand side in the last equation to the subspaces of the direct sum decomposition (3.36), we find that (3.78) is equivalent to

$$\lambda_1 = y_c^- \quad \text{and} \quad z^-(y_c^-, \lambda) = 0. \tag{3.79}$$

Taking also into consideration that $y_c^+ = 0$ we find for the coordinates y_R and y_{-R} that $y_R = 0$, $y_{-R} = \lambda_1/2$ and thus we get

$$\hat{\xi}^\infty(0, \lambda_1/2, \lambda) = \lambda_1/2\psi(0, \lambda_1/2, \lambda_1, \lambda_2) = 0. \tag{3.80}$$

The solutions $\lambda = (0, \lambda_2)$ of this equation correspond to homoclinic orbits to the equilibrium. Therefore heteroclinic cycles between the equilibrium and a periodic orbit in W^c_{loc} exist for parameter values which satisfy

$$\psi(0, \lambda_1/2, \lambda_1, \lambda_2) = 0.$$

Lemma 3.3.12 *Assume (H3.1), (H3.2), (H3.6), (H3.7),* $D_4\psi(0,0,0,0) \neq 0$, $\xi(\lambda) \equiv \lambda_1$ *and* $n = 1$. *Then there is a mapping* $\lambda_2^* : \mathbb{R} \to \mathbb{R}$, $\lambda_1 \mapsto \lambda_2^*(\lambda_1)$ *with* $\lambda_2^*(0) = 0$ *such that for all parameter values* $(\lambda_1, \lambda_2) = (\lambda_1 \neq 0, \lambda_2^*(\lambda_1))$ *there is a heteroclinic cycle between the equilibrium and a periodic orbit in* W^c_{loc}.

Proof Under the assumptions of the lemma the equation $\psi(0, \lambda_1/2, \lambda_1, \lambda_2) = 0$ can be solved for $\lambda_2 = \lambda_2^*(\lambda_1)$. ∎

3.3.2.1 Bifurcation scenario

To describe the bifurcation scenario we proceed as in Section 3.3.1.1. For that reason we restrict again to the $\mathbb{R}^4$-case.

Let $\varkappa_{sym} := h^{cs}(\kappa_{sym}, \lambda)$ and $\varkappa_{asym} := h^{cs}(\kappa_{asym}, \lambda)$; the mapping h^{cs} has been defined in (3.51). Figure 3.5 depicts the curves κ and $\varkappa$.

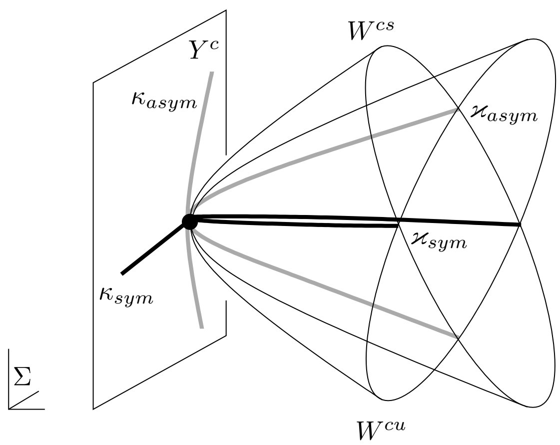

Figure 3.5: The intersection of the centre-stable and the centre-unstable manifold in Σ: $\varkappa_{sym}$ and $\varkappa_{asym}$

Projecting the curve $\varkappa_{sym} \subset W^{cs}_\lambda \cap W^{cu}_\lambda$ along stable fibres into the centre manifold yields a curve κ^c_{sym}. Each intersection of κ^c_{sym} with an orbit in the centre manifold is related to a symmetric homoclinic orbit to the centre manifold. From the dynamical point of view we distinguish whether κ^c_{sym} intersects the equilibrium or not, see Figure 3.6.

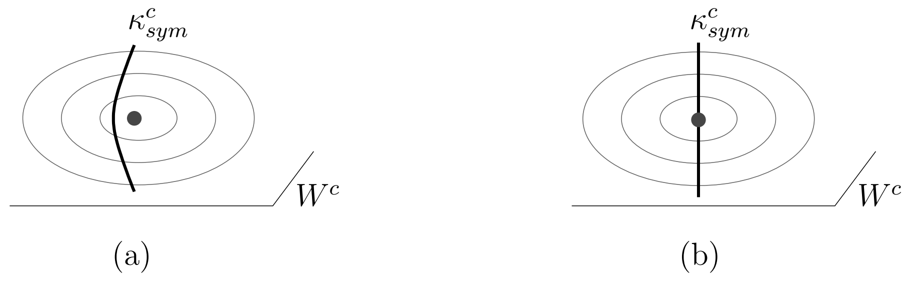

Figure 3.6: The intersection of κ^c_{sym} with an orbit in W^c corresponds to a symmetric homoclinic orbit to exclusively periodic orbits (a), or to all periodic orbits and to the equilibrium (b)

Next we consider $\varkappa_{asym}$. First we remark that this curve is R-symmetric (but not located in $\operatorname{Fix} R$). The projection of $\varkappa_{asym}$ along stable fibres into W^c gives a curve κ^c_{asym} intersecting all those orbits in W^c being approached (forward in time) by a non-symmetric homoclinic orbit to the centre manifold. Analogously we get the curve $R\kappa^c_{asym} \subset W^c$ by projecting $\varkappa_{asym}$ along unstable fibres (this is due to the R-symmetry of $\varkappa_{asym}$). So each $p \in \varkappa_{asym}$ belongs to an orbit connecting (in general different) orbits in the centre manifold. However, we cannot read from the curves κ^c_{asym} which orbits in the centre manifold are connected by a heteroclinic orbit. But together with its R-image any heteroclinic orbit forms a symmetric heteroclinic cycle. For possible positions of κ^c_{asym} and its R-image within the centre manifold we refer to Figure 3.7. In particular these curves can intersect in the equilibrium, see Figure 3.7(b). In this case there is a symmetric heteroclinic cycle connecting the equilibrium and an orbit in W^c.

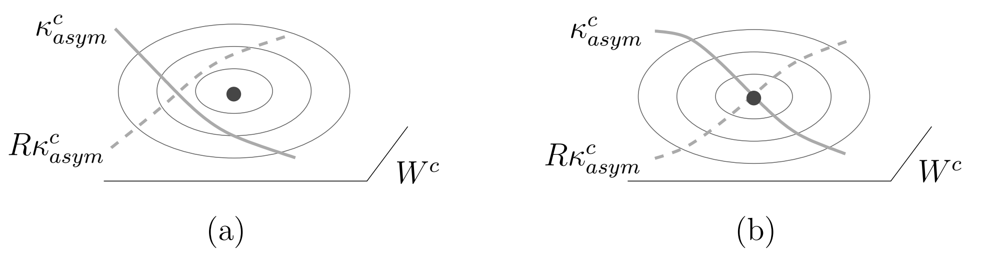

Figure 3.7: Relative positions of the projected curves κ^c_{asym}, $R\kappa^c_{asym}$ and the equilibrium

While changing λ we get different situations with respect to the relative position of the projected curves and the orbits of the centre manifold (see Figure 3.8). These situations arise as combinations of the cases depicted in Figure 3.6 and Figure 3.7. Particular scenarios are those, where the equilibrium lies on κ^c_{sym} or κ^c_{asym} and $R\kappa^c_{asym}$: These correspond to symmetric homoclinic orbits to the equilibrium $\mathring{x}$ for $\lambda_2 = 0$ ($\mathring{x} \in \kappa^c_{sym}$) or a heteroclinic cycle connecting the equilibrium with an orbit in W^c ($\mathring{x} \in \kappa^c_{asym}$).

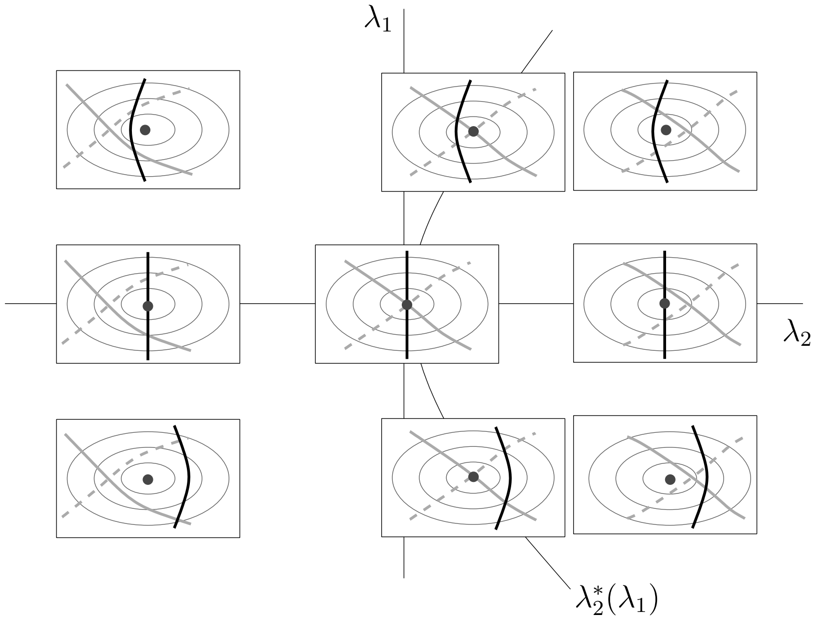

Figure 3.8: Bifurcation diagram corresponding to Theorem 3.3.10 and Lemma 3.3.12

3.4 Transformation flattening centre-stable and centre-unstable manifolds

We will perform global transformations $\mathcal{T}^\lambda$ mapping for all λ locally (around $\mathring{x} = 0$) the stable, unstable, centre-stable and centre-unstable manifold simultaneously in the corresponding subspaces of $\dot{x} = D_1 f(0,0)x$. To ensure that the transformed vector field $\mathcal{T}^\lambda_* f(x,\lambda) := D\mathcal{T}^\lambda({\mathcal{T}^\lambda}^{-1}(x)) f({\mathcal{T}^\lambda}^{-1}(x),\lambda)$ is again reversible we will construct a $\mathcal{T}^\lambda$ commuting with R. The whole procedure will be done in two steps: In the first step we create a local transformation $\mathcal{T}^\lambda_{loc}$ acting on an R-invariant ball B_δ around $\mathring{x} = 0$ with radius δ. In the second step we globalise $\mathcal{T}^\lambda_{loc}$ by means of an appropriate cut-off function.

In order to construct $\mathcal{T}^\lambda_{loc}$ we first flatten the centre manifold. This will be done by means of a transformation $\mathcal{T}^\lambda_1$. Let X^i, $i = s, u, c, cs, cu$, be the stable, unstable, centre, centre-stable and centre-unstable subspace of $\dot{x} = D_1 f(0,0)x$, respectively. The reversibility implies, see Appendix A.1,

$$RX^c = X^c, \quad RX^s = X^u, \quad RX^{cs} = X^{cu}. \tag{3.81}$$

Parallel to (3.81) we have

$$RW^c_{loc,\lambda} = W^c_{loc,\lambda}, \quad RW^s_{loc,\lambda} = W^u_{loc,\lambda}, \quad RW^{cs}_{loc,\lambda} = W^{cu}_{loc,\lambda}. \tag{3.82}$$

The local manifolds $W^c_{loc,\lambda}$, $W^s_{loc,\lambda}$ and $W^u_{loc,\lambda}$ can be understood as graphs of appropriate functions:

$$W^c_{loc,\lambda} = \{(x_c, h^c_\lambda(x_c)), x_c \in X^c, \|x_c\| \text{ small}\}, \quad h^c_\lambda : X^c \to X^s \oplus X^u,$$

$$W^s_{loc,\lambda} = \{(x_s, h^s_\lambda(x_s)), x_s \in X^s, \|x_s\| \text{ small}\}, \quad h^s_\lambda : X^s \to X^c \oplus X^u,$$

$$W^u_{loc,\lambda} = \{(x_u, h^u_\lambda(x_u)), x_u \in X^u, \|x_u\| \text{ small}\}, \quad h^u_\lambda : X^u \to X^c \oplus X^s.$$

Then (3.81) and (3.82) imply

$$Rh^c_\lambda = h^c_\lambda R, \quad Rh^s_\lambda = h^u_\lambda R. \tag{3.83}$$

We define a transformation

$$\mathcal{T}_1^\lambda : B_\delta \subset \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}, \quad x \mapsto x + h^c(x_c) + h^s(x_s) + h^u(x_u)\,.$$

This diffeomorphism maps X^c into $W^c_{loc,\lambda}$, X^s into $W^s_{loc,\lambda}$ and X^u into $W^u_{loc,\lambda}$. Due to (3.83) the transformation $\mathcal{T}_1^\lambda$ commutes with R and ${\mathcal{T}_1^\lambda}^{-1}(W^i) = X^i \cap B_\delta$, $i = c, s, u$.

Now we assume that we have already performed the transformation $\mathcal{T}_1^{-1}$. For the manifolds under consideration we keep the original notations. We obtain for all λ

$$h^c_\lambda(x_c) \equiv 0, \quad h^s_\lambda(x_s) \equiv 0, \quad h^u_\lambda(x_u) \equiv 0.$$

As above we may read the local manifolds W^{cs}_{loc} and W^{cu}_{loc} as graphs of mappings $h^{cs} : X^c \oplus X^s \to X^u$ and $h^{cu} : X^c \oplus X^u \to X^s$, respectively, satisfying

$$Rh^{cs}_\lambda = h^{cu}_\lambda R. \tag{3.84}$$

Lemma 3.4.1 *Let $W^c_{loc,\lambda} = X^c \cap B_\delta$. Then*

$$h^{cs}_\lambda|_{X^c} = 0, \quad h^{cs}_\lambda|_{X^s} = 0 \quad \text{and} \quad h^{cu}_\lambda|_{X^c} = 0, \quad h^{cu}_\lambda|_{X^u} = 0.$$

Proof Because of $W^c_{loc,\lambda} = W^{cs}_{loc,\lambda} \cap W^{cu}_{loc,\lambda}$, for each $x_c \in W^c_{loc} \subset X^c$ there are $x_s \in X^s$ and $x_u \in X^u$ such that $x_c = x_c + x_s + h^{cs}_\lambda(x_c + x_s) = x_c + x_u + h^{cu}_\lambda(x_c + x_u)$. Because of $X^s \oplus X^u \oplus X^c = \mathbb{R}^{2n+2}$ there is $x_s = x_u = 0$ and $h^{cs}_\lambda(x_c) = h^{cu}_\lambda(x_c) = 0$.

We have $(0, x^s, h^{cs}_\lambda(x^s)) \in W^{cs}_\lambda$. On the other hand $(0, x^s, 0) \in W^s \subset W^{cs}_\lambda$. Hence $(0, x^s, h^{cs}_\lambda(x^s)) = (0, x^s, 0)$ and therefore $h^{cs}_\lambda(x^s) = 0$. In the same way we find $h^{cu}_\lambda(x^u) = 0$. ∎

Next we define transformations $\mathcal{T}_2^\lambda$ which flatten the (local) centre-stable and centre-unstable manifolds simultaneously:

$$\mathcal{T}_2^\lambda : B_\delta \cap \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}, \quad x \mapsto x + h^{cs}_\lambda(x_c + x_s) + h^{cu}_\lambda(x_c + x_u).$$

Indeed, $\mathcal{T}_2^\lambda$ is a local diffeomorphism commuting with R (see (3.84)). Note that the transformations $\mathcal{T}_2^\lambda$ leave the manifolds $W^c_\lambda = X^c$, $W^s_\lambda = X^s$ and $W^u_\lambda = X^u$ untouched. This is an immediate consequence of Lemma 3.4.1. However, the main property of $\mathcal{T}_2^\lambda$ is established in the following corollary:

Corollary 3.4.2 *Let* $W^c_{loc,\lambda} = X^c \cap B_\delta$*. Then* $\mathcal{T}_2^\lambda(X^i \cap B_\delta) = W^i_{loc,\lambda}$*,* $i = cs, cu$*.*

Proof Lemma 3.4.1 provides $\mathcal{T}_2^\lambda(x_c + x_s) = x_c + x_s + h^{cs}_\lambda(x_c + x_s) \in W^{cs}_{loc,\lambda}$ and $\mathcal{T}_2^\lambda(x_c + x_u) = x_c + x_u + h^{cu}_\lambda(x_c + x_u) \in W^{cu}_{loc,\lambda}$. ■

Finally

$$\mathcal{T}^\lambda_{loc} := \mathcal{T}_2^{\lambda^{-1}} \circ \mathcal{T}_1^{\lambda^{-1}}$$

is the desired local transformation.

On the way of globalising $\mathcal{T}^\lambda_{loc}$ we first notice that $\mathcal{T}^\lambda_{loc}$ has the form $\mathcal{T}^\lambda_{loc} = id + \tilde{\mathcal{T}}^\lambda$, with $D\tilde{\mathcal{T}}^\lambda(0) = 0$. Let χ be a C^∞-cut-off function with $\chi(x) = 1$ for $\|x\| \leq 1$ and $\chi(x) = 0$ for $\|x\| \geq 2$. We may assume that χ is R-invariant, see Appendix A.1. With

$$\chi_\delta(x) := \chi\left(\frac{3x}{\delta}\right)$$

we can globalise $\mathcal{T}^\lambda_{loc}$ to $\mathcal{T}^\lambda$ as follows:

$$\mathcal{T}^\lambda(x) := \begin{cases} x + \chi_\delta(x)\tilde{\mathcal{T}}^\lambda(x) & , \quad x \in B_\delta\ , \\ x & , \quad \text{otherwise}\ . \end{cases}$$

The map $\mathcal{T}^\lambda$ is a global transformation commuting with R and accomplishing the desired flattening. The regularity of $D\mathcal{T}^\lambda$ follows from $\sup_{x\in B_\delta} \|D\chi_\delta(x)\tilde{\mathcal{T}}^\lambda(x)\| \to 0$, as $\delta \to 0$, similar to considerations in [Van89].

4 The Existence of Symmetric One-Periodic Orbits

In this chapter we study symmetric one-periodic orbits near the primary homoclinic orbit Γ. Throughout we assume our standing Hypotheses (H 1.1)-(H 1.6). (As usual within this thesis we do not mention this assumptions in our assertions.) Furthermore we restrict our considerations to $\mathbb{R}^4$. That means, we assume

(H 4.1) $\quad f : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4$.

This implies that the cross section Σ coincides with the "jump direction" Z. Thus, the existence of symmetric one-periodic Lin orbits is equivalent to the existence of symmetric partial orbits connecting Σ with itself. Note further, that in $\mathbb{R}^4$ our approach is also applicable for the detection of k-periodic Lin orbits ($k \in \mathbb{N}$). But within this thesis we do not deal with this issue.
One of the main problems in constructing the Lin orbits is the description of the flow near the centre manifold. For that purpose we make several assumptions regarding the dynamics locally around the equilibrium, see Hypotheses (H 4.2) and (H 4.3).

In Section 4.1 we prove the existence of symmetric one-periodic Lin solutions $\mathcal{X} = \{(x^+, x^{loc}, x^-)\}$ and provide estimates which are useful in the discussion of the bifurcation equation $x^+(0) - x^-(0)=0$. This discussion is done in Section 4.2. There we assume that the primary homoclinic orbit Γ is non-elementary, i.e., we assume (H 3.3) and consider the bifurcation equation as a perturbation of the one for one-homoclinic Lin orbits to the centre manifold.

4.1 Symmetric one-periodic Lin orbits

This section is devoted to the detection of symmetric one-periodic Lin orbits $\mathcal{L} = \{X\}$ of Equation (1.1). As described in Section 2.1 a periodic Lin orbit $\mathcal{L} = \{X\}$ consists of three parts: $X = (X^+, X^{loc}, X^-)$.
Due to the restriction to $\mathbb{R}^4$ the jump of each partial orbit, see Definition 2.1.1, is always in Z-direction. That means, every partial orbit is a one-periodic Lin orbit. For that reason the detection of symmetric one-periodic orbits reduces to the construction of X^{loc}: We obtain the orbit X^+ by integrating backward in time from the starting point $x^{loc}(0)$ of X^{loc} until reaching Σ the first time. Similarly, X^- can be achieved from the endpoint $x^{loc}(T)$ of X^{loc} by forward integration.
In order to construct X^{loc} we proceed as follows: Let Σ_{loc} be a local cross-section of the flow containing the origin of the phase space $\mathbb{R}^4$. That means, that any

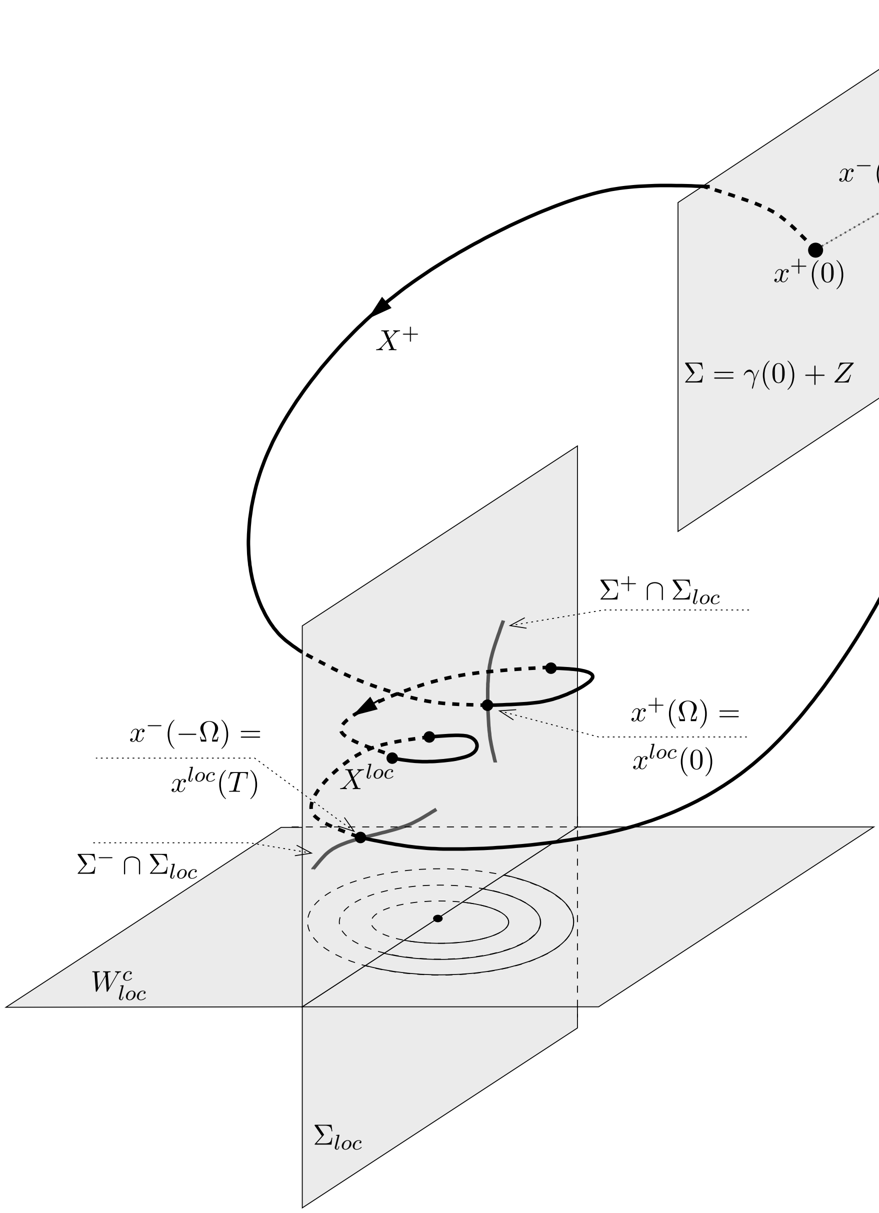

Figure 4.1: The construction of symmetric one-periodic Lin orbits in $\mathbb{R}^4$

orbit, except the equilibrium, intersects Σ_{loc} transversally. Let further $\Omega \in \mathbb{R}^+$ be sufficiently large. Then we define manifolds Σ^+ as the forward evolution of Σ at the time Ω and Σ^- as the backward evolution of Σ at the time $-\Omega$:

$$\Sigma^+ = \varphi(\Omega, \Sigma, \lambda) \quad \text{and} \quad \Sigma^- = \varphi(-\Omega, \Sigma, \lambda)\ .$$

We search orbits X^{loc} connecting $\Sigma^+ \cap \Sigma_{loc}$ and $\Sigma^- \cap \Sigma_{loc}$. We call $x^{loc}(\cdot)$ a *local solution*.

Although, in a strict sense, we cannot address Σ_{loc} as a Poincaré section, we will use

this notion, since we adapt the classical concept (see [Ama95], [Irw80]) and setting up a return map on Σ_{loc} defined by a smooth return time. Then we describe the local orbit X^{loc} by means of this return map. There we prescribe the number N of "windings along the centre manifold".
For each local solution $x^{loc}(\cdot)$, defined on $[0,T]$, we find *global solutions* $x^{\pm}(\cdot)$ by simply integrating the vector field as outlined above. Then, due to the definition of Σ^+ the solution $x^+(\cdot)$ is defined on $[0,\Omega]$ and satisfies $x^+(0) \in \Sigma$ and $x^+(\Omega) = x^{loc}(0)$. Similarly, $x^-(\cdot)$ is defined on $[-\Omega,0]$ with $x^-(-\Omega) = x^{loc}(T)$ and $x^-(0) \in \Sigma$. The entire procedure is illustrated in Figure 4.1.

Finally, using assertions from [Den88], we develop useful estimates for the symmetric one-periodic Lin orbits.

4.1.1 Existence

The following considerations result in a precise definition of the Poincaré section Σ_{loc} and a corresponding return map Π based on the definition of a smooth return time.
Let X^s_λ, X^u_λ and X^c_λ be the stable, unstable and centre eigenspace, respectively, of $D_1 f(0,\lambda)$. For $\lambda = 0$ we omit the index and write only X^s, X^u and X^c. Let further $X^c = X^c_R \oplus X^c_{-R}$, where X^c_R and X^c_{-R} are the intersections of X^c with $\mathrm{Fix}\, R$ and $\mathrm{Fix}\,(-R)$, respectively. Then, $\mathbb{R}^4$ can be decomposed

$$\mathbb{R}^4 = X^s \oplus X^u \oplus X^c_R \oplus X^c_{-R}\,. \tag{4.1}$$

For more detailed remarks concerning these subspaces see Section A.1. Henceforth, we make the following assumption.

(H 4.2) Let $\lambda \in \mathbb{R}^2$ be sufficiently small. Then

$$X^s_\lambda = X^s\,,\ X^u_\lambda = X^u\,,\ X^c_\lambda = X^c \quad \text{and} \quad W^s_{loc,\lambda} \subset X^s\,,\ W^u_{loc,\lambda} \subset X^u\,.$$

Furthermore, locally around zero, the subspace

$$X^h := X^s \oplus X^u \tag{4.2}$$

is invariant with respect to the flow of (1.1).

The first assumption of (H 4.2) is not a restriction because, for each small $\lambda \in \mathbb{R}^2$, there is a transformation (described in Section 3.4), which leads to such a situation. Further, a generalised Hartman-Grobman-Theorem (see [Van89]) ensures the existence of a C^0 manifold that contains the stable and the unstable manifold of the equilibrium and is invariant with respect to the flow of Equation (1.1) for $\lambda = 0$. In (H 4.2) we assume the C^r smoothness of this manifold.

Let $U(0) \subset \mathbb{R}^4$ be a neighbourhood of the equilibrium. Then, as depicted in Figure 4.2, we define Σ_{loc} by

$$\Sigma_{loc} := (X^h \oplus X^c_R) \cap U(0) \tag{4.3}$$

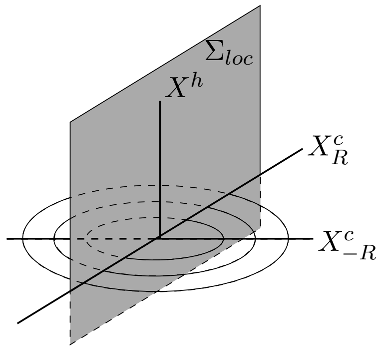

Figure 4.2: Position of the Poincaré section Σ_{loc}

Next we show the existence of a smooth return time t^+ with respect to Σ_{loc} and consequently the existence of a smooth return map.

Lemma 4.1.1 *Assume (H4.1) and (H4.2). Then, for all sufficiently small $x \in \Sigma_{loc}$ and $\lambda \in \Lambda_0$ there exists a C^r smooth function $t^+ : \Sigma_{loc} \times \Lambda_0 \to \mathbb{R}^+$ such that $\varphi(t^+, x, \lambda) \in \Sigma_{loc}$.*

Proof We view the flow φ as

$$\varphi(t, (\cdot, \cdot, \cdot, \cdot), \lambda) : X^s \times X^u \times X_R^c \times X_{-R}^c \to \mathbb{R}^4 .$$

Let $\tau > \pi$ and choose an open set $W \subset \Sigma_{loc}$ such that for all $(x^s, x^u, x_R^c) \in W$ we have: $\varphi(t, (x^s, x^u, x_R^c, 0), \lambda) \subset U(0)$ for all $t \in [-\tau, \tau]$ and $\lambda \in \Lambda_0$. We define a function

$$\begin{aligned} F : \ (-\tau, \tau) \times W \times \Lambda_0 \ &\to \ X_{-R}^c \\ (t, (x^s, x^u, x_R^c), \lambda) \ &\mapsto \ \mathcal{P} \circ \varphi(t, (x^s, x^u, x_R^c, 0), \lambda) \end{aligned}$$

where $\mathcal{P}$ projects $\mathbb{R}^4$ on X_{-R}^c along Σ_{loc}. Because the cross-section Σ_{loc} contains the invariant manifold $X^h \cap U(0)$ we obtain

$$F(t, (x^s, x^u, 0), \lambda) \equiv 0 .$$

Hence there is a function $H : (-\tau, \tau) \times W \times \Lambda_0 \to X_{-R}^c$, such that F can be written as

$$F(t, (x^s, x^u, x_R^c), \lambda) = x_R^c H(t, (x^s, x^u, x_R^c), \lambda) .$$

Now, we consider

$$H(t, (x^s, x^u, x_R^c), \lambda) = 0 . \tag{4.4}$$

Our aim is to solve (4.4) for $t = t^+((x^s, x^u, x_R^c), \lambda)$ near $(\pi, (0, 0, 0), 0) =: (\pi, 0, 0)$. Thus we have to show

$$H(\pi, 0, 0) = 0 , \tag{4.5}$$

$$D_t H(\pi, 0, 0) \neq 0 . \tag{4.6}$$

Obviously we have

$$D_{x_R^c} F(\pi, 0, 0) = H(\pi, 0, 0) ,$$

$$D_t D_{x_R^c} F(\pi, 0, 0) = D_t H(\pi, 0, 0) .$$

Thus, in order to prove (4.5) and (4.6) we show

$$\mathcal{P} \circ D_{x_R^c} \varphi(\pi, 0, 0) = 0 , \tag{4.7}$$

$$\mathcal{P} \circ D_t D_{x_R^c} \varphi(\pi, 0, 0) \neq 0 . \tag{4.8}$$

To check (4.7) and (4.8) we observe that $D_{x_R^c} \varphi(\cdot, 0, 0)$ solves the equation $\dot{x} = D_1 f(0, 0)x$ and we get

$$D_{x_R^c} \varphi(t, 0, 0) = e^{D_1 f(0,0)t} D_{x_R^c} \varphi(0, 0, 0) . \tag{4.9}$$

With

$$D_1 f(0, 0) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(see (A.10) with $A = 1$) Equation (4.9) reads

$$D_{x_R^c} \varphi(t, 0, 0) = \begin{pmatrix} e^{-\mu t} & 0 & 0 & 0 \\ 0 & e^{\mu t} & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} D_{x_R^c} \varphi(0, 0, 0) . \tag{4.10}$$

With $D_{x_R^c} \varphi(0, (0, 0, x_R^c, 0), 0) = (0, 0, 1, 0)^\top$ for all $x_R^c \in \mathbb{R}$ we have in particular

$$D_{x_R^c} \varphi(0, 0, 0) = (0, 0, 1, 0)^\top . \tag{4.11}$$

This equation and (4.10) for $t = \pi$ yield $D_{x_R^c} \varphi(\pi, 0, 0) = (0, 0, -1, 0)^\top$. This gives (4.7). Further we get $D_t D_{x_R^c} \varphi(\pi, 0, 0) = (0, 0, 0, 1)^\top$ which proves (4.8).

Thus, by the Implicit Function Theorem we can solve (4.4) for $t = t^+((x^s, x^u, x_R^c), \lambda)$ near $(\pi, 0, 0)$. Further, the Implicit Function Theorem implies that t^+ is C^r smooth. Hence t^+ is the desired return time. ■

With the smooth return time t^+ we define a return map

$$\begin{aligned} \tilde{\Pi}(\cdot, \lambda) : \quad \Sigma_{loc} &\rightarrow \Sigma_{loc} \\ x &\mapsto \varphi(t^+(x, \lambda), x, \lambda) =: \tilde{\Pi}(x, \lambda). \end{aligned}$$

Note that each periodic orbit in W^c_λ intersects Σ_{loc} twice. Since we want to work in the frame of "common" Poincaré maps we define

$$\Pi := \tilde{\Pi}^2 \tag{4.12}$$

as the Poincaré map. Of course, this map is C^r smooth, too.

Solving (4.4) near $(-\pi, 0, 0)$ we find parallel to the above considerations

Lemma 4.1.2 *Assume (H4.1) and (H4.2). Then, for all sufficiently small $x \in \Sigma_{loc}$ and $\lambda \in \Lambda_0$ there is a C^r smooth return time $t^- : \Sigma_{loc} \times \Lambda_0 \to \mathbb{R}^-$ such that $\varphi(t^-, x, \lambda) \in \Sigma_{loc}$ and*

$$-t^+((x^s, x^u, x^c_R), \lambda) = t^-((Rx^s, Rx^u, x^c_R), \lambda) \; . \tag{4.13}$$

Proof It remains to show (4.13): Invoking the R-reversibility of the flow φ, easy computations yield

$$H(-t, (Rx^s, Rx^u, x^c_R), \lambda) = RH(t, (x^s, x^u, x^c_R), \lambda) \; .$$

Thus, the uniqueness of $t^\pm(\cdot, \lambda)$ (see proof of Lemma 4.1.1) implies (4.13). ∎

It makes sense to define

$$\begin{aligned} \tilde{\Pi}^{-1}(\cdot, \lambda) : \quad \Sigma_{loc} &\to \Sigma_{loc} \\ x &\mapsto \varphi(t^-(x, \lambda), x, \lambda) =: \tilde{\Pi}^{-1}(x, \lambda) \end{aligned}$$

and

$$\Pi^{-1} := \tilde{\Pi}^{-2} \; .$$

As a direct consequence of (4.13) we get

$$R\Pi(x, \lambda) = \Pi^{-1}(Rx, \lambda) \quad \forall x \in \Sigma_{loc} \; . \tag{4.14}$$

Now, instead of looking for local solutions $x^{loc}(\cdot)$ of (1.1), we are searching solutions $z(\cdot)$ on Σ_{loc} of the discrete problem

$$z(n+1) = \Pi(z(n), \lambda) \; . \tag{4.15}$$

Due to (4.14) Equation (4.15) is R-reversible (see Section A.1).

Let henceforth Λ_0 be a neighbourhood of 0 in $\mathbb{R}^2$ and X^s_{loc}, X^u_{loc}, $X^c_{R,loc}$ and $X^c_{-R,loc}$ be neighbourhoods of 0 in X^s, X^u, X^c_R and X^c_{-R}, respectively, such that in particular $RX^h_{loc} = X^h_{loc}$ with $X^h_{loc} := X^s_{loc} + X^u_{loc}$. For $X^{s(u)}_{loc} + X^c_{loc}$ (with $X^c_{loc} = X^c_{R,loc} + X^c_{-R,loc}$) we write $X^{cs(cu)}_{loc}$. We assume these neighbourhoods and Σ_{loc} to be as small as necessary for the following considerations.

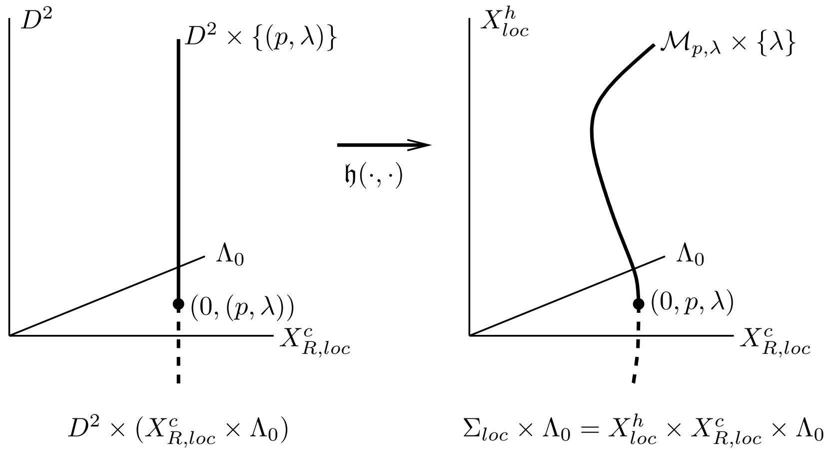

Figure 4.3: The foliation of $\Sigma_{loc} \times \Lambda_0$

As defined in [HPS77] a ***C^r foliation*** ($r \geq 0$) of an m-dimensional manifold M with leaves of dimension l is a disjoint decomposition of M into l-dimensional injectively immersed connected submanifolds (leaves) such that M is covered by C^r charts

$$\mathfrak{h} : D^l \times D^{m-l} \to M$$

and $\mathfrak{h}(D^l, y)$ is contained in the leaf through $\mathfrak{h}(0, y)$. Here D^l and D^{m-l} denote disks in $\mathbb{R}^l$ and $\mathbb{R}^{m-l}$, respectively.
In the following we assume a smooth foliation of $\Sigma_{loc} \times \Lambda_0$. There we use the notations D^2 and D^3 for disks in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

(H 4.3) There is a C^r foliation of $\Sigma_{loc} \times \Lambda_0$ with leaves $\tilde{\mathcal{M}}_{p,\lambda}$. That means, there is a C^r chart

$$\mathfrak{h} : D^2 \times D^3 \to \Sigma_{loc} \times \Lambda_0$$

such that $\mathfrak{h}(D^2, y)$ is contained in the leave through $\mathfrak{h}(0, y)$. We assume that $\mathfrak{h}$ can be chosen such that $D^3 = X^c_{R,loc} \times \Lambda_0$ and $\mathfrak{h}(0, \cdot) = id$. Further, the leaves $\tilde{\mathcal{M}}_{p,\lambda}$ have the form

$$\tilde{\mathcal{M}}_{p,\lambda} = \mathcal{M}_{p,\lambda} \times \{\lambda\} ,$$

where $\mathcal{M}_{p,\lambda} \subset \Sigma_{loc}$ is a $\Pi(\cdot, \lambda)$ invariant manifold and contains the fixed point p of $\Pi(\cdot, \lambda)$. Further,

$$R\mathcal{M}_{p,\lambda} = \mathcal{M}_{p,\lambda} .$$

For fixed p and λ Hypothesis (H 4.3) yields the C^r smoothness of $\tilde{\mathcal{M}}_{p,\lambda}$. Consequently, $\mathcal{M}_{p,\lambda}$ is a C^r manifold, too.

For each $\lambda \in \Lambda_0$ the set $X^c_{R,loc}$ consists of hyperbolic fixed points p of the Poincaré map $\Pi(\cdot, \lambda)$. The non-zero fixed points correspond to periodic orbits in the centre manifold of (1.1). Applying a generalised Hartman-Grobman-Theorem for maps (see [KP90]) to (4.15), for each $\lambda \in \Lambda_0$ there is a C^0 foliation of Σ_{loc}. The leaves of this foliation are two-dimensional and invariant with respect to $\Pi(\cdot, \lambda)$. (See [FL01] for the discussion of an analogous problem for vector fields.) By the above hypothesis even the C^r smoothness of such a foliation is supposed. Note, that as we will explain in Section 5, the smoothness of this foliation can be guaranteed for an important class of systems.

The following two lemmas state that the leaves $\mathcal{M}_{p,\lambda}$ and the stable and unstable manifolds therein can be described by smooth functions.

Lemma 4.1.3 *Assume (H4.1)-(H4.3). Then, for $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$, $\mathcal{M}_{p,\lambda}$ can be locally described as graph of a C^r smooth function $h_\mathcal{M}(\cdot, p, \lambda) : X^h_{loc} \to X^c_R$. That means,*

$$\mathcal{M}_{p,\lambda} = \{(x^h, h_\mathcal{M}(x^h, p, \lambda)),\ \ x^h \in X^h_{loc}\}\ .$$

Further, $h_\mathcal{M}(\cdot,\cdot,\cdot) : X^h_{loc} \times X^c_{R,loc} \times \Lambda_0 \to X^c_R$ is C^r smooth.

Proof For fixed $\lambda \in \Lambda_0$ we consider $\mathfrak{h}(\cdot, (\cdot, \lambda)) =: \mathfrak{h}_\lambda(\cdot,\cdot)$. Then Hypothesis (H 4.3) gives

$$\mathfrak{h}_\lambda(\cdot,\cdot) : D^2 \times X^c_{R,loc} \to \Sigma_{loc} \times \{\lambda\} \quad \text{with} \quad \mathfrak{h}_\lambda(D^2, p) \subset \mathcal{M}_{p,\lambda} \times \{\lambda\}\ .$$

Further, the family $\{\mathfrak{h}_\lambda(\cdot,\cdot),\ \lambda \in \Lambda_0\}$ is of class C^r. Let $\mathfrak{h}_\lambda = (\mathfrak{h}^h_\lambda, \mathfrak{h}^c_\lambda, \lambda)$ with $\mathfrak{h}^h_\lambda(\xi, p) \in X^h_{loc}$ and $\mathfrak{h}^c_\lambda(\xi, p) \in X^c_{R,loc}$ for all $(\xi, p) \in D^2 \times X^c_{R,loc}$. (Here we read Σ_{loc} as $\Sigma_{loc} = X^h_{loc} \times X^c_{R,loc}$.) Further let without loss of generality $\mathfrak{h}_0(0,0) = (0,0,0)$. The invariant manifold $\mathcal{M}_{0,0}$ contains the stable and unstable manifold of $x = 0$. Hence, the tangent space of $\mathcal{M}_{0,0}$ in $x = 0$ coincides with the subspace X^h containing X^h_{loc}. From that we conclude the non-singularity of $D_1\mathfrak{h}^h_0(0,0)$. The smoothness of the family $\{\mathfrak{h}_\lambda(\cdot,\cdot),\ \lambda \in \Lambda_0\}$ gives then that $D_1\mathfrak{h}^h_\lambda(0,0)$ is non singular, too.
Thus, due to the Inverse Mapping Theorem, for each p there is a C^r smooth map

$$\xi^*_p(\cdot) : X^h_{loc} \to D^2$$

such that $\xi^*_p(\mathfrak{h}^h_\lambda(\xi, p)) = \xi$. Further, $\xi^*(\cdot,\cdot)$ defined by $\xi^*(x^h, p) := \xi^*_p(x^h)$ is of class C^r. The map $\tilde{h}_\mathcal{M}(\xi, p, \lambda) := \mathfrak{h}^c_\lambda(\xi^*_p(\mathfrak{h}^h_\lambda(\xi, p)), p)$ describes $\mathcal{M}_{p,\lambda} \times \{\lambda\}$ and hence the desired map can be defined by

$$h_\mathcal{M}(x^h, p, \lambda) := \mathfrak{h}^c_\lambda(\xi^*_p(x^h), p)\ .$$

The C^r smoothness of this map is clear by construction. ■

Let $W^s_{\Pi(\cdot,\lambda)}(p)$ and $W^u_{\Pi(\cdot,\lambda)}(p)$ be the stable and unstable manifold of $p \in X^c_{R,loc}$ with respect to $\Pi(\cdot, \lambda)$. We can consider them as the intersection of the leaf $\mathcal{M}_{p,\lambda}$ and the centre-(un)stable manifold $W^{cs(cu)}_\lambda$:

$$W^{s(u)}_{\Pi(\cdot,\lambda)}(p) = \mathcal{M}_{p,\lambda} \cap W^{cs(cu)}_\lambda\ . \tag{4.16}$$

Lemma 4.1.4 *Assume (H4.1)-(H4.3). Then, for* $p \in X^c_{R,loc}$ *and* $\lambda \in \Lambda_0$, $W^s_{\Pi(\cdot,\lambda)}(p)$ *and* $W^u_{\Pi(\cdot,\lambda)}(p)$ *can be locally described as graph of* C^r *smooth functions* $h^s_\Pi(\cdot,p,\lambda) : X^s_{loc} \to X^u \times X^c_R$ *and* $h^u_\Pi(\cdot,p,\lambda) : X^u_{loc} \to X^s \times X^c_R$*, respectively. That means,*

$$W^{s(u)}_{\Pi(\cdot,\lambda)}(p) = \{(x^{s(u)}, h^{s(u)}_\Pi(x^{s(u)},p,\lambda)),\;\; x^{s(u)} \in X^{s(u)}_{loc}\}\,.$$

Further, $h^{s(u)}_\Pi(\cdot,\cdot,\cdot) : X^{s(u)}_{loc} \times X^c_{R,loc} \times \Lambda_0 \to X^{u(s)} \times X^c_R$ *is* C^r *smooth.*

Proof It is well-known that W^{cu}_λ is locally the graph of a C^r function $h^{cu}(\cdot,\lambda) : (X^c_{loc} \times X^u_{loc}) \to X^s$, where

$$D_{(x^c,x^u)}h^{cu}(0,\lambda) = 0 \quad \forall \lambda \in \Lambda_0 \tag{4.17}$$

and $h^{cu}(\cdot,\cdot) : (X^c_{loc} \times X^u_{loc}) \times \Lambda_0 \to X^s$ is of class C^r.
Let p and λ be fixed. Due to Lemma 4.1.3 the manifold $\mathcal{M}_{p,\lambda}$ is the graph of a C^r function $h_\mathcal{M}(\cdot,p,\lambda) : (X^s_{loc} \times X^u_{loc}) \to X^c_R$, where

$$D_{(x^s,x^u)}h_\mathcal{M}(0,0,0) = 0 \tag{4.18}$$

and $h_\mathcal{M}(\cdot,\cdot,\cdot) : (X^s_{loc} \times X^u_{loc}) \times X^c_{R,loc} \times \Lambda_0 \to X^c_R$ is of class C^r. Equation (4.18) is due to the fact that the tangent space of $\mathcal{M}_{0,0}$ in $x = 0$ coincides with $X^s \oplus X^u$.

Because of (4.16) $W^u_{\Pi(\cdot,\lambda)}(p)$ is characterised by

$$x^c_{-R} = 0\,,\;\; x^c_R = h_\mathcal{M}((x^s,x^u),p,\lambda)\,,\;\; x^s = h^{cu}(((x^c_R,x^c_{-R}),x^u),\lambda)\,.$$

For that reason we consider the system

$$\begin{aligned} x^c_R &= h_\mathcal{M}((x^s,x^u),p,\lambda) &=:\; &\tilde{h}_\mathcal{M}(x^s,x^u,p,\lambda)\,, \\ x^s &= h^{cu}(((x^c_R,0),x^u),\lambda) &=:\; &\tilde{h}^{cu}(x^u,x^c_R,\lambda)\,. \end{aligned} \tag{4.19}$$

Obviously, $(x^s,x^u,x^c_R,p,\lambda) = (0,0,0,0,0)$ is a solution of (4.19). Further, by the Implicit Function Theorem (4.17) and (4.18) imply that we can solve (4.19) for

$$(x^s,x^c_R) = (x^s,x^c_R)(x^u,p,\lambda)\,.$$

Therefore, we can define the desired function h^u_Π by

$$h^u_\Pi(x^u,p,\lambda) := (\tilde{h}^{cu}(x^u,x^c_R(x^u,p,\lambda),\lambda),\, \tilde{h}_\mathcal{M}(x^s(x^u,p,\lambda),x^u,p,\lambda)) \in X^s \times X^c_R\,.$$

The proof for $W^s_{\Pi(\cdot,\lambda)}(p)$ runs completely parallel to the above explanations. ■

Recall, that in order to find solutions of Equation (1.1) we search for solutions of Equation (4.15). For that purpose, for fixed $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$, we construct curves $\mathcal{C}^+$ and $\mathcal{C}^-$ which are within $\mathcal{M}_{p,\lambda}$ transversal to the stable manifold $W^s_{\Pi(\cdot,\lambda)}(p)$ and the unstable manifold $W^u_{\Pi(\cdot,\lambda)}(p)$, respectively. By application of a λ-lemma we will conclude the existence of solutions $z^{loc}(\cdot)$ of (4.15) connecting these two curves.

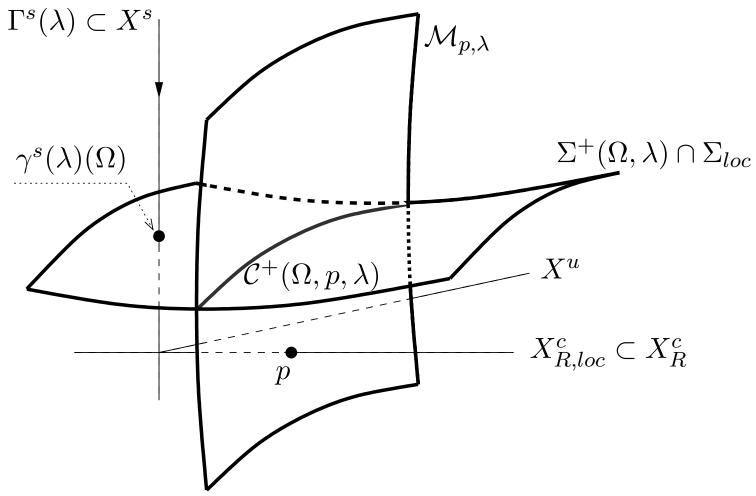

Figure 4.4: Construction of $\mathcal{C}^+(\Omega, p, \lambda)$

First, we explain how to get the curve $\mathcal{C}^+$ (see Figure 4.4 for illustration). Let $\Omega \in \mathbb{R}^+$ be such that $\gamma^s(\lambda)(\Omega) \in \Sigma_{loc}$, where $\gamma^s(\lambda)$ belongs to the homoclinic Lin solution $\{(\gamma^s(\lambda), \gamma^u(\lambda))\}$ found in Section 3.1. Consider the forward evolution of the cross section Σ along the orbit $\Gamma^s(\lambda) \subset W^s_\lambda$ at time Ω and define

$$\Sigma^+(\Omega, \lambda) := \varphi(\Omega, V_\Sigma, \lambda) \; , \tag{4.20}$$

where $\varphi(t, \cdot, \lambda)$ denotes the flow of (1.1) and V_Σ is a small neighbourhood of $\gamma^s(\lambda)(0)$ in Σ. We will show that the intersection of $\Sigma^+(\Omega, \lambda)$ and $\mathcal{M}_{p,\lambda}$ is a smooth curve $\mathcal{C}^+$ which is transversal to $W^s_{\Pi(\cdot,\lambda)}(p)$ within $\mathcal{M}_{p,\lambda}$. Further, this curve depends smoothly on Ω, p and λ. Preparing a corresponding lemma (see Lemma 4.1.6) we summarise some assertions about $\Sigma^+(\Omega, \lambda)$.

Lemma 4.1.5 *Assume (H4.1)-(H4.3). Let $\Omega_0 \in \mathbb{R}^+$ be sufficiently large and $U(\Omega_0)$ be a neighbourhood of Ω_0 in $\mathbb{R}^+$. Then, for $\Omega \in U(\Omega_0)$ and $\lambda \in \Lambda_0$, $\Sigma^+(\Omega, \lambda)$ can be locally described as graph of a C^r smooth function $h_\Sigma(\cdot, \Omega, \lambda) : X^{cu}_{loc} \to X^s$. That means,*

$$\Sigma^+(\Omega, \lambda) = \{(x^{cu}, h_\Sigma(x^{cu}, \Omega, \lambda)), \;\; x^{cu} \in X^{cu}_{loc}\} \, .$$

Further, $h_\Sigma(\cdot, \cdot, \cdot) : X^{cu}_{loc} \times U(\Omega_0) \times \Lambda_0 \to X^s$ is C^r smooth.

Proof For fixed Ω and λ let us first identify V_Σ with $\mathbb{R}^3$ (and in particular $\gamma^s(\lambda)(0)$ with 0). Further, let us write $\varphi(\Omega, \xi, \lambda)$ (with $\xi \in V_\Sigma$) as

$$\varphi(\Omega, \xi, \lambda) = (h^{cu}_\varphi(\Omega, \xi, \lambda), h^s_\varphi(\Omega, \xi, \lambda)) \in X^{cu} \times X^s \, .$$

Obviously $\varphi(\Omega_0, V_\Sigma, 0) \pitchfork_{\gamma(\Omega_0)} W^s_{loc}$ and thus (see (H 4.2)) $\varphi(\Omega_0, V_\Sigma, 0) \pitchfork_{\gamma(\Omega_0)} X^s$. This yields the non-singularity of $D_\xi h^{cu}_\varphi(\Omega_0, 0, 0)$.

Now, the proof can follow the same line as in the proof of Lemma 4.1.3. That means, by the Inverse Mapping Theorem we find a C^r function

$$\xi^*(\Omega,\lambda)(\cdot) : X^{cu} \to \mathbb{R}^3 \cong V_\Sigma$$

such that $\xi^*(\Omega,\lambda)(h^{cu}_\varphi(\Omega,\xi,\lambda)) = \xi$ and $\xi^*(\cdot,\cdot)(\cdot)$ is of class C^r.
With that we can define the function $h_\Sigma(\cdot,\cdot,\cdot)$ by

$$h_\Sigma(x^{cu},\Omega,\lambda) = h^s_\varphi(\Omega,\xi^*(\Omega,\lambda)(x^{cu}),\lambda)\ .$$

■

Now we can prove

Lemma 4.1.6 *Assume (H4.1)-(H4.3). Let $\Omega_0 \in \mathbb{R}^+$ be sufficiently large, $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$. Then, there exists a neighbourhood $U(\Omega_0) \subset \mathbb{R}^+$ of Ω_0 such that for all $\Omega \in U(\Omega_0)$*

$$\Sigma^+(\Omega,\lambda) \cap \mathcal{M}_{p,\lambda} =: \mathcal{C}^+(\Omega,p,\lambda) \tag{4.21}$$

is the image of a C^r smooth function $c^+(\Omega,p,\lambda)(\cdot) : X^u_{loc} \to X^s \times X^c_R$.
Moreover, the function $c^+(\cdot,\cdot,\cdot)(\cdot) : U(\Omega_0) \times X^c_{R,loc} \times \Lambda_0 \times X^u_{loc} \to X^s \times X^c_R$ is C^r smooth.

Proof Within this proof we follow the same ideas as in the proof of Lemma 4.1.4. Lemma 4.1.3 and Lemma 4.1.5 give functions

$$h_\Sigma(\cdot,\Omega,\lambda) : (X^u_{loc} \times X^c_{R,loc} \times X^c_{-R,loc}) \to X^s \text{ and}$$

$$h_\mathcal{M}(\cdot,p,\lambda) : (X^s_{loc} \times X^u_{loc}) \to X^c_R$$

whose graphs describe the manifolds $\Sigma^+(\Omega,\lambda)$ and $\mathcal{M}_{p,\lambda}$ for fixed Ω, p and λ. We further know that

$$h_\Sigma(\cdot,\cdot,\cdot) : (X^u_{loc} \times X^c_{R,loc} \times X^c_{-R,loc}) \times U(\Omega_0) \times \Lambda_0 \to X^s \text{ and}$$

$$h_\mathcal{M}(\cdot,\cdot,\cdot) : (X^s_{loc} \times X^u_{loc}) \times X^c_{R,loc} \times \Lambda_0 \to X^c_R$$

are of class C^r.
Because of its definition in (4.21) $\mathcal{C}^+(\Omega,p,\lambda)$ is characterised by

$$x^c_{-R} = 0,\ \ x^s = h_\Sigma((x^u,x^c_R,x^c_{-R}),\Omega,\lambda),\ \ x^c_R = h_\mathcal{M}((x^s,x^u),p,\lambda)\ .$$

For that reason we consider the system

$$\begin{aligned} x^s &= h_\Sigma((x^u,x^c_R,0),\Omega,\lambda) &&=: \tilde{h}_\Sigma(x^u,x^c_R,\Omega,\lambda)\ ,\\ x^c_R &= h_\mathcal{M}((x^s,x^u),p,\lambda) &&=: \tilde{h}_\mathcal{M}(x^s,x^u,p,\lambda)\ . \end{aligned} \tag{4.22}$$

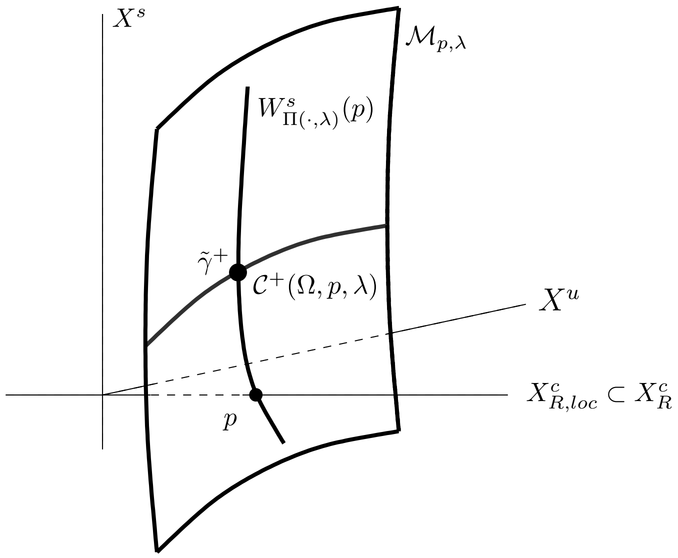

Figure 4.5: Transversal intersection of $\mathcal{C}^+(\Omega, p, \lambda)$ and $W^s_{\Pi(\cdot,\lambda)}(p)$ in $\tilde{\gamma}^+$ within $\mathcal{M}_{p,\lambda}$

By construction it is clear that there is a solution

$$(x^s, x^u, x^c_R, \Omega_0, p, \lambda) = (x^s_\gamma, x^u_\gamma, x^c_\gamma, \Omega_0, 0, 0)$$

of (4.22). This solution corresponds to the point $\gamma(\Omega_0) \in \Sigma^+(\Omega_0, 0) \cap \mathcal{M}_{0,0}$. Furthermore, $\gamma(\Omega_0) \in X^s_{loc}$ and $X^s_{loc} \subset \mathcal{M}_{0,0}$ give $X^s_{loc} \subset T_{\gamma(\Omega_0)}\mathcal{M}_{0,0}$. This yields

$$D_{x^s}\tilde{h}_{\mathcal{M}}(x^s_\gamma, x^u_\gamma, 0, 0) = 0 \; .$$

Therefore we can solve (4.22) by the Implicit Function Theorem for $(x^s, x^c_R) = (x^s, x^c_R)(x^u, \Omega, p, \lambda)$. So we can define the desired function $c^+(\cdot, \cdot, \cdot)$ by

$$c^+(\Omega, p, \lambda)(x^u) := \\ (\tilde{h}_\Sigma(x^u, x^c_R(x^u, \Omega, p, \lambda), \Omega, \lambda), \tilde{h}_{\mathcal{M}}(x^s(x^u, \Omega, p, \lambda), x^u, p, \lambda) \in X^s \times X^c_R \; .$$

■

As shown in Figure 4.5 and proved in the following lemma the curve $\mathcal{C}^+(\Omega, p, \lambda)$ and the stable manifold $W^s_{\Pi(\cdot,\lambda)}(p)$ intersect transversally within $\mathcal{M}_{p,\lambda}$.

Lemma 4.1.7 *Assume (H4.1)-(H4.3). Let $\Omega \in \mathbb{R}^+$ be sufficiently large, $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$. Then within $\mathcal{M}_{p,\lambda}$*

$$\mathcal{C}^+(\Omega, p, \lambda) \pitchfork_{\tilde{\gamma}^+} W^s_{\Pi(\cdot,\lambda)}(p) \; ,$$

with $\tilde{\gamma}^+ = \tilde{\gamma}^+(\Omega, p, \lambda)$ near $\gamma(\Omega)$.

Proof First, we observe that

$$\mathcal{C}^+(\Omega,0,0) \pitchfork_{\gamma(\Omega)} W^s_{\Pi(\cdot,0)}(0)\ . \tag{4.23}$$

This is clear due to $\Gamma = W^s = W^s_{\Pi(\cdot,0)}(0)$ (locally around zero), $\mathcal{C}^+(\Omega,0,0) \subset \Sigma^+(\Omega,0)$ and $\Sigma^+(\Omega,0) \pitchfork_{\gamma(\Omega)} \Gamma$.

Because of Lemma 4.1.4 and Lemma 4.1.6, $W^s_{\Pi(\cdot,\lambda)}(p)$ and $\mathcal{C}^+(\Omega,p,\lambda)$ can be understood as C^r perturbations of $W^s_{\Pi(\cdot,0)}(0)$ and $\mathcal{C}^+(\Omega,0,0)$, respectively. Transversal intersections persist under smooth perturbations (see [Hir76]). Thus, from (4.23) we conclude $\mathcal{C}^+(\Omega,p,\lambda) \pitchfork_{\tilde\gamma} W^s_{\Pi(\cdot,\lambda)}(p)$, with $\tilde\gamma$ near $\gamma(\Omega)$. ■

The same type of construction can be performed to gain a curve $\mathcal{C}^-$. We consider the manifold

$$\Sigma^-(\Omega,\lambda) := \varphi(-\Omega, RV_\Sigma, \lambda)\ . \tag{4.24}$$

The property $R\varphi(t,x,\lambda) = \varphi(-t,Rx,\lambda)$ (see (A.4)) and the definitions of $\Sigma^+(\Omega,\lambda)$ and $\Sigma^-(\Omega,\lambda)$ (see (4.20) and (4.24)) imply

$$R\Sigma^+(\Omega,\lambda) = \Sigma^-(\Omega,\lambda)\ . \tag{4.25}$$

We define

$$\mathcal{C}^-(\Omega,p,\lambda) := \Sigma^-(\Omega,\lambda) \cap \mathcal{M}_{p,\lambda}\ . \tag{4.26}$$

Then we find analogously to the above argumentation

Corollary 4.1.8 *Assume (H4.1)-(H4.3). Let $\Omega \in \mathbb{R}^+$ be sufficiently large, $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$.*
Then, there exists a neighbourhood $U(\Omega_0) \subset \mathbb{R}^+$ of Ω_0 such that for all $\Omega \in U(\Omega_0)$ $\mathcal{C}^-(\Omega,p,\lambda)$ is the image of a C^r smooth function $c^-(\Omega,p,\lambda)(\cdot) : U(0) \to \mathbb{R}^4$, where $U(0)$ is a neighbourhood of zero in $\mathbb{R}$.
Moreover, the function $c^-(\cdot,\cdot,\cdot)(\cdot) : U(\Omega_0) \times X^c_{R,loc} \times \Lambda_0 \times U(0) \to R^4$ is of class C^r.
Further, we have that within $\mathcal{M}_{p,\lambda}$

$$\mathcal{C}^-(\Omega,p,\lambda) \pitchfork_{\tilde\gamma^-} W^u_{\Pi(\cdot,\lambda)}(p)\ ,$$

with $\tilde\gamma^- = \tilde\gamma^-(\Omega,p,\lambda)$ near $\gamma(-\Omega)$.

Within $\mathcal{M}_{p,\lambda}$, we are in a situation which allows us to apply a λ-lemma for maps with hyperbolic fixed point as formulated for instance in [PdM82], [Wig90], [Den88] and [Rob95].
Here, we quote the λ-lemma from [Wig90]: Consider a C^r diffeomorphism $(r \geq 1)$ $\mathfrak{f} : \mathbb{R}^2 \to \mathbb{R}^2$ having a hyperbolic fixed point $x = 0$. Let $W^s(0)$ and $W^u(0)$ denote the stable and unstable manifold of this fixed point. Let $q \in W^s(0) \setminus \{0\}$ and κ be a curve intersecting $W^s(0)$ transversally at q. We denote the connected component of $\mathfrak{f}^N(\kappa) \cap U$ to which $\mathfrak{f}^N(q)$ belongs by κ^N. Here U denotes a neighbourhood of 0 in $\mathbb{R}^2$. Then we have

Lemma 4.1.9 (λ-lemma [Wig90]) *Let U be sufficiently small. Then for each $\varepsilon > 0$ there exists a positive integer N_0 such that for $N > N_0$ the connected component κ^N is C^1 ε-close to $W^u(0) \cap U$.*

Note, that the phrase C^1 ε-close implies that tangent vectors of κ^N are ε-close to tangent vectors of $W^u(0) \cap U$.

Using the λ-lemma we find orbits of (4.15) connecting $\mathcal{C}^+(\Omega, p, \lambda)$ and $\mathcal{C}^-(\Omega, p, \lambda)$ as stated in the following lemma.

Lemma 4.1.10 *Assume (H4.1)-(H4.3). Let $\Omega \in \mathbb{R}^+$ and $N \in \mathbb{N}$ be sufficiently large. Further, let $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$. Then, there is a unique orbit $Z^{loc}(\Omega, p, \lambda, N)$ connecting the curves $\mathcal{C}^+(\Omega, p, \lambda)$ and $\mathcal{C}^-(\Omega, p, \lambda)$ in N steps.*
The orbit $Z^{loc}(\Omega, p, \lambda, N)$ is symmetric.

Proof Let $U(p)$ be a sufficiently small neighbourhood of p in $\mathbb{R}^4$ and let $\varepsilon > 0$ be given. Let the pair (N^+, N^-) with sufficiently large N^+, $N^- \in \mathbb{N}$ be given. For fixed $\lambda \in \Lambda_0$, we denote the connected component of $\Pi^{N^+}(\mathcal{C}^+(\Omega, p, \lambda), \lambda)$ which contains $\Pi^{N^+}(\tilde{\gamma}^+, \lambda)$ by $\mathcal{C}^+_{N^+}(\Omega, p, \lambda)$. Similarly, we define $\mathcal{C}^-_{-N^-}(\Omega, p, \lambda)$ as the connected component of $\Pi^{-N^-}(\mathcal{C}^-(\Omega, p, \lambda), \lambda)$ which contains $\Pi^{-N^-}(\tilde{\gamma}^-, \lambda)$.
Then, the λ-lemma yields that within $U(p)$ the curve $\mathcal{C}^+_{N^+}(\Omega, p, \lambda)$ is C^1 ε-close to $W^u_{\Pi(\cdot,\lambda)}(p)$ and can therefore be considered as a C^1 perturbation of $W^u_{\Pi(\cdot,\lambda)}(p)$. Similarly, we get the curve $\mathcal{C}^-_{-N^-}(\Omega, p, \lambda)$ as a C^1 perturbation of $W^s_{\Pi(\cdot,\lambda)}(p)$. (For an illustration see Figure 4.6.)

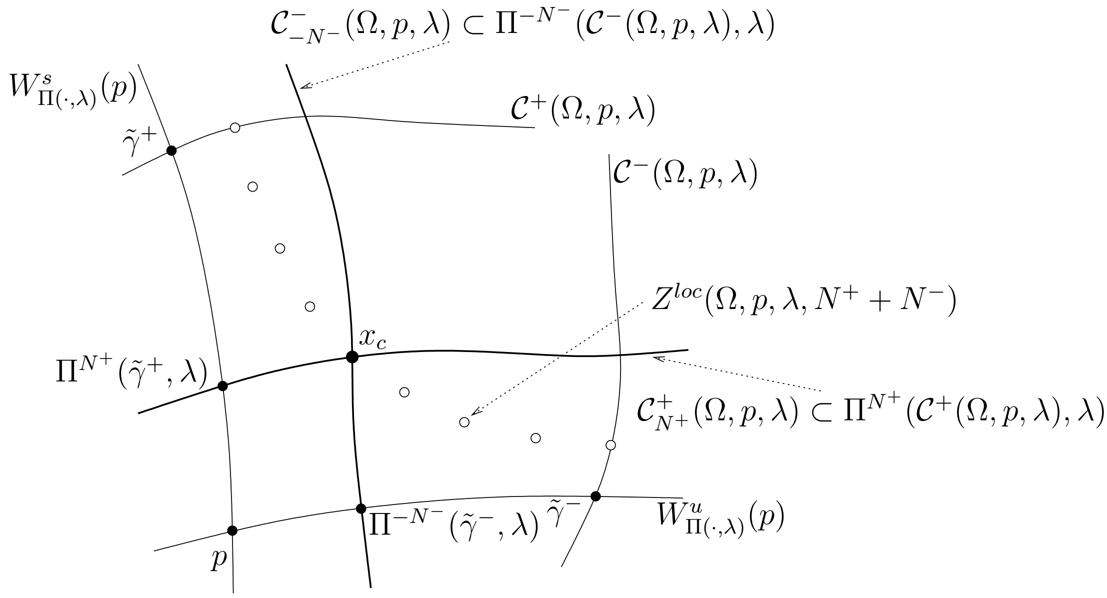

Figure 4.6: The existence of a unique orbit $Z^{loc}(\Omega, p, \lambda, N)$ connecting $\mathcal{C}^+(\Omega, p, \lambda)$ and $\mathcal{C}^-(\Omega, p, \lambda)$ for given N ($N = N^+ + N^-$).

Obviously, stable manifold $W^s_{\Pi(\cdot,\lambda)}(p)$ and unstable manifold $W^u_{\Pi(\cdot,\lambda)}(p)$ intersect transversally in p. This transversal intersection persists under C^1 perturbation (see [Hir76]). Thus, we can conclude that within $U(p)$

$$\mathcal{C}^+_{N^+}(\Omega,p,\lambda)\cap\mathcal{C}^-_{-N^-}(\Omega,p,\lambda)=\{x_{\mathcal{C}}\}\ ,$$

where $x_{\mathcal{C}}\in\mathcal{M}_{p,\lambda}$. (Recall that $\mathcal{M}_{p,\lambda}$ is $\Pi(\cdot,\lambda)$ invariant.)

Let $N^+ + N^- =: N$, then $x_{\mathcal{C}}$ corresponds to an orbit $Z^{loc}(\Omega,p,\lambda,N)$ of (4.15) connecting $\mathcal{C}^+(\Omega,p,\lambda)$ and $\mathcal{C}^-(\Omega,p,\lambda)$ in N steps. In fact for a given sufficiently large $N\in\mathbb{N}$ each pair (N^+,N^-) with $N^++N^-=N$ gives the same orbit Z^{loc}. This can be seen as follows. Let (N_1^+,N_1^-) and (N_2^+,N_2^-) be different pairs of natural numbers such that $N_1^++N_1^-=N_2^++N_2^-=N$. Then, without loss of generality, there is a number $d>0$ such that $N_1^++d=N_2^+$ and hence $N_2^-+d=N_1^-$.

Let $\{x^1_{\mathcal{C}}\}=\mathcal{C}^+_{N_1^+}(\Omega,p,\lambda)\cap\mathcal{C}^-_{-N_1^-}(\Omega,p,\lambda)$ and $\{x^2_{\mathcal{C}}\}=\mathcal{C}^+_{N_2^+}(\Omega,p,\lambda)\cap\mathcal{C}^-_{-N_2^-}(\Omega,p,\lambda)$. We consider

$$\Pi^d(x^1_{\mathcal{C}})\in\mathcal{C}^+_{N_1^++d}(\Omega,p,\lambda)\cap\mathcal{C}^-_{-N_1^-+d}(\Omega,p,\lambda)=\mathcal{C}^+_{N_2^+}(\Omega,p,\lambda)\cap\mathcal{C}^-_{-N_2^-}(\Omega,p,\lambda)=\{x^2_{\mathcal{C}}\}\ .$$

Hence, $\Pi^d(x^1_{\mathcal{C}})=x^2_{\mathcal{C}}$ and therefore $x^1_{\mathcal{C}}$ and $x^2_{\mathcal{C}}$ belong to one and the same orbit $Z^{loc}(\cdot)$.

It remains to prove the symmetry of the orbit Z^{loc}. We consider $\mathcal{C}^\pm(\Omega,p,\lambda)=\mathcal{M}_{p,\lambda}\cap\Sigma^\pm(\Omega,\lambda)$. Then with (4.25) and the R-invariance of $\mathcal{M}_{p,\lambda}$ assumed in (H 4.3) we get

$$R\mathcal{C}^+(\Omega,p,\lambda)=\mathcal{C}^-(\Omega,p,\lambda)\ . \tag{4.27}$$

Let $Z^{loc}(\Omega,p,\lambda,N)$ be the uniquely determined orbit of (4.15) which connects $\mathcal{C}^+(\Omega,p,\lambda)$ and $\mathcal{C}^-(\Omega,p,\lambda)$ in N steps. Then, due to the R-reversibility of Equation (4.15) $RZ^{loc}(\Omega,p,\lambda,N)$ is an orbit of (4.15), too. Furthermore, because of (4.27) $RZ^{loc}(\Omega,p,\lambda,N)$ connects $\mathcal{C}^+(\Omega,p,\lambda)$ and $\mathcal{C}^-(\Omega,p,\lambda)$ in N steps. Hence, the uniqueness of $Z^{loc}(\Omega,p,\lambda,N)$ gives

$$RZ^{loc}(\Omega,p,\lambda,N)=Z^{loc}(\Omega,p,\lambda,N)\ ,$$

which means that $Z^{loc}(\Omega,p,\lambda,N)$ is symmetric. ■

Let $z^{loc}(\Omega,p,\lambda,N)(\cdot)$ be the solution corresponding to $Z^{loc}(\Omega,p,\lambda,N)$ which is defined on $\{0,1,\dots,N\}$ and satisfies

$$z^{loc}(\Omega,p,\lambda,N)(0)\in\mathcal{C}^+(\Omega,p,\lambda)\qquad\text{and hence}$$

$$z^{loc}(\Omega,p,\lambda,N)(N)\in\mathcal{C}^-(\Omega,p,\lambda)\ .$$

Then we get the following lemma.

Lemma 4.1.11 *Assume (H 4.1)-(H 4.3). Let both $N\in\mathbb{N}$ and $\Omega_0\in\mathbb{R}^+$ be sufficiently large. Then, there exists a neighbourhood $U(\Omega_0)\subset\mathbb{R}^+$ of Ω_0 such that*

$$z^{loc}(\cdot,\cdot,\cdot,N)(0):U(\Omega_0)\times X^c_{R,loc}\times\Lambda_0\to\Sigma_{loc}$$

is of class C^r.

Proof Let $N \in \mathbb{N}$ and a decomposition $N = N^+ + N^-$, with $N^\pm \in \mathbb{N}$ and $N, N^\pm$ sufficiently large, be given.
In order to show the assertion of the lemma we proof that the intersection point of the connected components of $\mathcal{C}^+(\Omega, p, \lambda)$ and $\mathcal{C}^-(\Omega, p, \lambda)$ depends smoothly on Ω, p and λ. Then, the smoothness of the Poincaré map yields the smoothness assertion of the above lemma.

First, we show that $\mathcal{C}^+_{N^+}(\Omega, p, \lambda)$ can be written as graph of a C^r function

$$c^+_{N^+}(\cdot, \Omega, p, \lambda) : X^u_{loc} \to X^s \times X^c_R ,$$

where $c^+_{N^+}(\cdot, \cdot, \cdot, \cdot)$ is of class C^r. We can write $\mathcal{C}^+_{N^+}(\Omega, p, \lambda)$ as

$$\begin{aligned} \mathcal{C}^+_{N^+}(\Omega, p, \lambda) &= \{\Pi^{N^+}((x^u, c^+(\Omega, p, \lambda)(x^u)), \lambda),\ x^u \in X^u_{loc}\} \\ &=: \{\hat{\Pi}^{N^+}(x^u, \Omega, p, \lambda),\ x^u \in X^u_{loc}\} . \end{aligned}$$

Here $c^+(\Omega, p, \lambda)(\cdot)$ is the function introduced in Lemma 4.1.6.
Let $\hat{\Pi}^{N^+}(\cdot, \Omega, p, \lambda) = (\hat{\Pi}^{N^+,u}(\cdot, \Omega, p, \lambda), \hat{\Pi}^{N^+,cs}(\cdot, \Omega, p, \lambda))$ with $\hat{\Pi}^{N^+,u}(x^u, \Omega, p, \lambda) \in X^u$ and $\hat{\Pi}^{N^+,cs}(x^u, \Omega, p, \lambda) \in X^s \times X^c_R$ for all $x^u \in X^u_{loc}$. Because $\operatorname{im} \hat{\Pi}^{N^+}(\cdot, \Omega_0, 0, 0)$ is a C^1 perturbation of $W^u_{\Pi(\cdot,0)}(0) = W^u_{loc} = X^u_{loc}$ we have that $D_{x^u} \hat{\Pi}^{N^+,u}(0, \Omega_0, 0, 0)$ is non-singular. Thus, by the Inverse Mapping Theorem we find a C^r function

$$\xi^*(\Omega, p, \lambda)(\cdot) : X^u_{loc} \to X^u_{loc}$$

such that $\xi^*(\Omega, p, \lambda)(\hat{\Pi}^{N^+,u}(x^u, \Omega, p, \lambda)) = x^u$. Further, $\xi^*(\cdot, \cdot, \cdot)(\cdot)$ is of class C^r. So we get

$$\begin{aligned} \mathcal{C}^+_{N^+}(\Omega, p, \lambda) &= \{(x^u, \hat{\Pi}^{N^+,cs}(\xi^*(\Omega, p, \lambda)(\hat{\Pi}^{N^+,u}(x^u, \Omega, p, \lambda)), \Omega, p, \lambda)),\ x^u \in X^u_{loc}\} \\ &=: \{(x^u, c^+_{N^+}(x^u, \Omega, p, \lambda)),\ x^u \in X^u_{loc}\} . \end{aligned}$$

By similar arguments we find a C^r function

$$c^-_{-N^-}(\cdot, \Omega, p, \lambda) : X^s_{loc} \to X^u \times X^c_R$$

which describes $\mathcal{C}^-_{-N^-}(\Omega, p, \lambda)$ and where $c^-_{-N^-}(\cdot, \cdot, \cdot, \cdot)$ is of class C^r.

Now, we can describe the intersection point of the connected components of $\mathcal{C}^+(\Omega, p, \lambda)$ and $\mathcal{C}^-(\Omega, p, \lambda)$ in the following way. Let $c^+_{N^+} = (c^{+,s}_{N^+}, c^{+,c}_{N^+})$ such that $c^{+,s}_{N^+}(x^u, \Omega, p, \lambda) \in X^s$ and $c^{+,c}_{N^+}(x^u, \Omega, p, \lambda) \in X^c_R$ for all $x^u \in X^u_{loc}$. In an analogous way we decompose $c^-_{-N^-} = (c^{-,u}_{-N^-}, c^{-,c}_{-N^-})$ and $x^N_{\mathcal{C}} = (x^{N,s}_{\mathcal{C}}, x^{N,u}_{\mathcal{C}}, x^{N,c}_{\mathcal{C}})$, where

$$\{x^N_{\mathcal{C}}\} := \mathcal{C}^+_{N^+}(\Omega, p, \lambda) \cap \mathcal{C}^-_{-N^-}(\Omega, p, \lambda) . \tag{4.28}$$

So the following system describes $x_{\mathcal{C}}^N$

$$
\begin{aligned}
x^s &= c_{N^+}^{+,s}(x^u, \Omega, p, \lambda), \\
x^u &= c_{-N^-}^{-,u}(x^s, \Omega, p, \lambda), \\
c_{N^+}^{+,c}(x^u, \Omega, p, \lambda) &= c_{-N^-}^{-,c}(x^s, \Omega, p, \lambda)\,.
\end{aligned}
\tag{4.29}
$$

Let us consider first

$$
x^s = c_{N^+}^{+,s}(x^u, \Omega, p, \lambda), \quad x^u = c_{-N^-}^{-,u}(x^s, \Omega, p, \lambda)\,. \tag{4.30}
$$

Due to Lemma 4.1.10 there is a solution $(x^s, x^u, \Omega, p, \lambda) = (\bar{x}_{\mathcal{C}}^{N,s}, \bar{x}_{\mathcal{C}}^{N,u}, \Omega_0, 0, 0)$ of (4.30).
Further, it is known that $\mathcal{C}_{N^+}^{+}(\Omega_0, 0, 0) \pitchfork_{\bar{x}_{\mathcal{C}}^N} \mathcal{C}_{-N^-}^{-}(\Omega_0, 0, 0)$ within $\mathcal{M}_{0,0}$. Hence, the direct sum of the tangent spaces of $\mathcal{C}^+(\Omega_0, 0, 0)$ and $\mathcal{C}^-(\Omega_0, 0, 0)$ in $(\bar{x}_{\mathcal{C}}^{N,s}, \bar{x}_{\mathcal{C}}^{N,u}, \bar{x}_{\mathcal{C}}^{N,c})$ equals $X^s \oplus X^u$ and we can conclude that

$$
\begin{pmatrix} 1 & -D_{x^u} c_{N^+}^{+,s}(\bar{x}_{\mathcal{C}}^{N,u}, \Omega_0, 0, 0) \\ -D_{x^s} c_{-N^-}^{-,u}(\bar{x}_{\mathcal{C}}^{N,s}, \Omega_0, 0, 0) & 1 \end{pmatrix} \quad \text{is non-singular}\,.
$$

Thus, by the Implicit Function Theorem, (4.30) can be solved for $(x^s, x^u) = (x^s, x^u)^*(\Omega, p, \lambda)$, where $(x^s, x^u)^*(\cdot, \cdot, \cdot)$ is of class C^r.

The curves $\mathcal{C}_{N^+}^{+}(\Omega, p, \lambda)$ and $\mathcal{C}_{-N^-}^{-}(\Omega, p, \lambda)$ intersect in exactly one point (see Lemma 4.1.10). That is why $c_{N^+}^{+,c}(x^{u,*}(\Omega, p, \lambda), \Omega, p, \lambda) = c_{-N^-}^{-,c}(x^{s,*}(\Omega, p, \lambda), \Omega, p, \lambda)$ is automatically satisfied. That means, we found a solution of (4.29)

$$
(x^{s,*}(\Omega, p, \lambda), x^{u,*}(\Omega, p, \lambda), c_{N^+}^{+,c}(x^{u,*}(\Omega, p, \lambda), \Omega, p, \lambda)) =: x_{\mathcal{C}}^N(\Omega, p, \lambda)
$$

where $x_{\mathcal{C}}^N(\cdot, \cdot, \cdot)$ is of class C^r. ■

We can ensure the existence of local solutions $x^{loc}(\Omega, p, \lambda, N)(\cdot)$ of (1.1) suspended in $z^{loc}(\Omega, p, \lambda, N)(\cdot)$ under the conditions (H 1.1)-(H 1.6), (H 4.1)-(H 4.3). A simple integration process (as described in the beginning of this section) yields global solutions $x^+(\Omega, p, \lambda, N)(\cdot)$ and $x^-(\Omega, p, \lambda, N)(\cdot)$. Hypothesis (H 4.1) implies that the partial solution

$$
x(\Omega, p, \lambda, N)(\cdot) = (x^+(\Omega, p, \lambda, N)(\cdot), x^{loc}(\Omega, p, \lambda, N)(\cdot), x^-(\Omega, p, \lambda, N)(\cdot))
$$

is a one-periodic Lin solution. Further, we know that this Lin solution depends smoothly on Ω, p and λ.

We close this section with the following lemma, which summarise the result.

Lemma 4.1.12 *Assume (H4.1) – (H4.3). Let* $p \in X^c_{R,loc}$ *and* $\lambda \in \Lambda_0$*. Further, let both* $N \in \mathbb{N}$ *and* $\Omega_0 \in \mathbb{R}^+$ *be sufficiently large. Then, there exists a neighbourhood* $U(\Omega_0) \subset \mathbb{R}^+$ *of* Ω_0 *such that for all* $\Omega \in U(\Omega_0)$ *there is a unique one-periodic symmetric Lin orbit* $\mathcal{L}(\Omega, p, \lambda, N) = \{X(\Omega, p, \lambda, N)\}$ *of Equation (1.1), where* $X(\Omega, p, \lambda, N)$ *is composed of three orbits* $X^+(\Omega, p, \lambda, N)$*,* $X^{loc}(\Omega, p, \lambda, N)$ *and* $X^-(\Omega, p, \lambda, N)$*.*
Let $\mathcal{X}(\Omega, p, \lambda, N) = \{(x^+(\Omega, p, \lambda, N), x^{loc}(\Omega, p, \lambda, N), x^-(\Omega, p, \lambda, N))\}$ *be the corresponding one-periodic symmetric Lin solution. Then*

$$x^{loc}(\cdot, \cdot, \cdot, N)(0) : U(\Omega_0) \times X^c_{R,loc} \times \Lambda_0 \to \mathbb{R}^4$$

is of class C^r*.*

4.1.2 Estimates

In the following we provide estimates for the local solutions of Equation (4.15) which are useful for solving the bifurcation equation for the detection of symmetric one-periodic orbits.

We consider the restriction $\Pi(\cdot, p, \lambda) := \Pi(\cdot, \lambda)|_{\mathcal{M}_{p,\lambda}}$ of the Poincaré map to the invariant leaf $\mathcal{M}_{p,\lambda}$:

$$z(n+1) = \Pi(z(n), p, \lambda) \ . \tag{4.31}$$

By construction, each $p \in X^c_{R,loc}$ is a hyperbolic fixed point of $\Pi(\cdot, p, \lambda)$ with the stable and the unstable manifold $W^s_{\Pi(\cdot,\lambda)}(p)$ and $W^u_{\Pi(\cdot,\lambda)}(p)$, respectively. For $\lambda = 0$ and $p = 0$ these manifolds coincide with the stable manifold W^s and the unstable manifold W^u of $\mathring{x} = 0$ with respect to the flow of (1.1). Each leaf $\mathcal{M}_{p,\lambda}$ can be identified with $\mathbb{R}^2$. So, we henceforth interpret Equation (4.31) as a system in $\mathbb{R}^2$, i.e., $\Pi(\cdot, p, \lambda) : \mathbb{R}^2 \to \mathbb{R}^2$. Due to the reversibility of $\Pi(\cdot, p, \lambda)$ we have

$$\sigma(D_1\Pi(0, p, \lambda)) = \{\nu(p, \lambda)^{-1}, \nu(p, \lambda)\}, \quad 1 < \nu(p, \lambda), \quad \nu(\cdot) \in C^r. \tag{4.32}$$

The solution $z^{loc}(\Omega, p, \lambda, N)(\cdot)$ of (4.15) lies within the leaf $\mathcal{M}_{p,\lambda}$. So, it is a solution of (4.31). The goal of our further analysis is to prove estimates of z^{loc} as stated in Lemma 4.1.15 below. For that we consider z^{loc} as so-called Shilnikov solutions, see [Den88]. We note that this approach allows a generalisation of the following analysis to higher dimensions.
Using the notation $\theta := (p, \lambda) \in X^c_{R,loc} \times \Lambda_0 =: \Theta$, a C^r transformation $\mathcal{T}(p, \lambda)$ brings us into the setup of [Den88]. This transformation is defined on a neighbourhood of $(z_1, z_2) = 0$ in $\mathbb{R}^2$ and brings Equation (4.31) in the form

$$\begin{aligned} z_1(n+1) &= \nu(\theta)^{-1} z_1(n) + g_1(z_1(n), z_2(n), \theta), \\ z_2(n+1) &= \nu(\theta) z_2(n) + g_2(z_1(n), z_2(n), \theta), \end{aligned} \tag{4.33}$$

where the functions g_1 and g_2 are of class C^r and satisfy

$$
\begin{aligned}
&g_1(0, z_2, \theta) = 0 \quad \forall\,(0, z_2) \in U,\ \theta \in \Theta\,, \\
&g_2(z_1, 0, \theta) = 0 \quad \forall\,(z_1, 0) \in U,\ \theta \in \Theta\,, \\
&D_{(z_1,z_2)} g_1(0,0,\theta) = 0\,,\ \ D_{(z_1,z_2)} g_2(0,0,\theta) = 0 \quad \forall\,\theta \in \Theta\,.
\end{aligned}
\tag{4.34}
$$

Note, that for each $\theta \in \Theta$ the stable and the unstable manifold can be written as $\{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : z_2 = 0\}$ and $\{(z_1, z_2) \in \mathbb{R} \times \mathbb{R} : z_1 = 0\}$, respectively.

We now consider solutions of (4.33) that connect the hyperplanes $z_1 = \xi$ and $z_2 = \eta$ for given $\xi, \eta \in \mathbb{R}$ in a given time $N \in \mathbb{N}$. Such solutions are known as Shilnikov solutions. Applying the theory worked out in [Den88] (see Theorem 8.1 and the remarks in Section 9 therein) we find the following assertion.

Lemma 4.1.13 *There exists a constant $C > 0$ such that for every given $N \in \mathbb{N}$ and sufficiently small ξ, η and θ Equation (4.33) with* (4.32) *and* (4.34) *has a unique Shilnikov solution $(z_1, z_2)(\xi, \eta, \theta, N)(\cdot) : [0, n_0] \to U(0) \subset \mathbb{R}^2$. That means, $z_1(\xi, \eta, \theta, N)(0) = \xi$ and $z_2(\xi, \eta, \theta, N)(N) = \eta$, where n_0 depends on N, ξ, η and θ and $n_0 > N$. Here $U(0)$ is a small neighbourhood of zero in $\mathbb{R}^2$.*
Moreover, $(z_1, z_2)(\cdot,\cdot,\cdot,\cdot)(\cdot)$ is of class C^r and satisfies for $0 \le n \le N$ and $1 < \tilde\nu(\theta) < \nu(\theta)$, and $i = 1, 2$

(i) $\| z_1(\xi, \eta, \theta, N)(n) \| \le C\,\nu(\theta)^{-n}$, $\| z_2(\xi, \eta, \theta, N)(n) \| \le C\,\nu(\theta)^{(n-N)}$,

(ii) $\| D^i_{(\xi,\eta)} z_1(\xi, \eta, \theta, N)(n) \| \le C\,\nu(\theta)^{-n}$, $\| D^i_{(\xi,\eta)} z_2(\xi, \eta, \theta, N)(n) \| \le C\,\nu(\theta)^{(n-N)}$,

(iii) $\| D^i_\theta z_1(\xi, \eta, \theta, N)(n) \| \le C\,\tilde\nu(\theta)^{-n}$, $\| D^i_\theta z_2(\xi, \eta, \theta, N)(n) \| \le C\,\tilde\nu(\theta)^{(n-N)}$.

Let, as described in [Den88], $\mathcal{D}$ be the graph of a smooth function h of z_2 in a small neighbourhood of zero. We refer to $\mathcal{D}$ as a one-dimensional disc. The disc $\mathcal{D}$ is said to be C^r if $h(\cdot)$ is of class C^r. Let $\mathcal{D}_N := \phi^N(\mathcal{D}, \theta) \cap U(0)$ (where $\phi^n(\cdot, \theta)$ denotes the flow of (4.33)).
Using Lemma 4.1.13 one can prove a discrete version of Corollary 3.2 in [Den88]. The proof runs completely parallel to the one in [Den88]. Nevertheless, we present the proof of the Lemma because we will use assertions of it in our further analysis.

Corollary 4.1.14 (λ-Lemma) *Let $\theta \in \Theta$, and let $\mathcal{D}$ be a one-dimensional disc of class C^r. Then $\mathcal{D}_N$ is the graph of a C^r function h_N defined on $U(0)$, where $U(0)$ is a neighbourhood of 0 in $\mathbb{R}$. Moreover, there exist constants $n_0 > 0$, $\nu(\theta) > 1$ and $C > 0$ such that all derivatives of h_N up to order r are bounded by $C\nu(\theta)^{-N}$ for all $N \ge n_0$.*

Proof The proof of the above assertion is illustrated in Figure 4.7. We describe the disc $\mathcal{D}$ by

$$
\mathcal{D} = \{(z_1, z_2) :\ z_1 = z_1(h(z_2^0), z_2^0, \theta, 0)(0),\ z_2 = z_2(h(z_2^0), z_2^0, \theta, 0)(0),\ z_2^0 \text{ small}\}\,.
$$

Then $\mathcal{D}_N$ has the form

$$\mathcal{D}_N = \{(z_1, z_2) : \; z_1 = z_1(h(z_2^0), z_2^0, \theta, 0)(N), \; z_2 = z_2(h(z_2^0), z_2^0, \theta, 0)(N), \; z_2^0 \text{ small}\} \, .$$

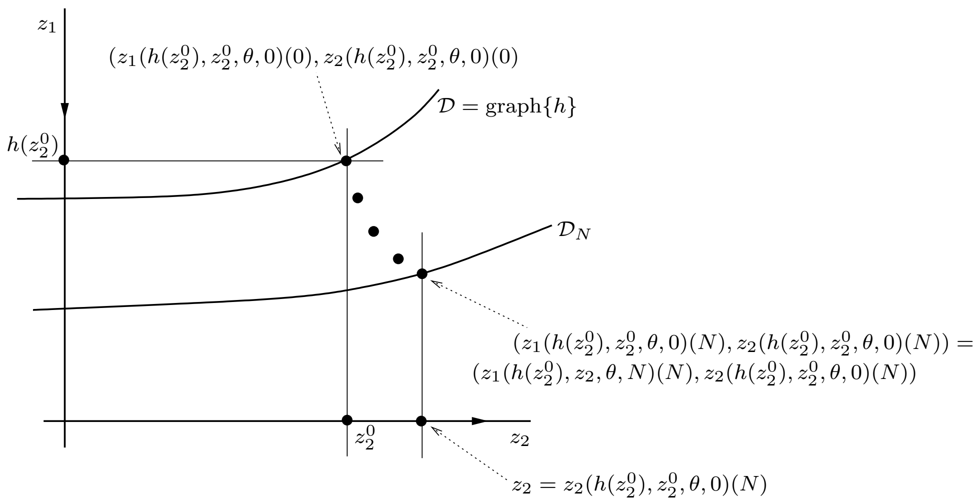

Figure 4.7: Illustration of the proof of the λ-Lemma

We rewrite this description by

$$\mathcal{D}_N = \{(z_1, z_2) : \; z_1 = z_1(h(z_2^0), z_2, \theta, N)(N), \; z_2 = z_2(h(z_2^0), z_2^0, \theta, 0)(N), \; z_2^0 \text{ small}\} \, .$$

Now, Lemma 4.1.13 implies

$$\| \, z_1(h(z_2^0), z_2, \theta, N)(N) \, \| \leq C \, \nu(\theta)^{-N} \tag{4.35}$$

provided that $\mathfrak{z}_2(\cdot) := z_2(h(\cdot), \cdot, \theta, 0)(N)$ is a diffeomorphism from a small neighbourhood U_{z_2} of $z_2 = 0$ onto itself. Further, $\mathfrak{z}_2^{-1}$ and all of its derivatives up to order r have to be uniformly bounded in N and $z_2 \in U_{z_2}$. That this condition is satisfied can be shown as discussed in [Den88].

With that, $\mathcal{D}_N$ can be expressed as the graph of a C^r function over the unstable manifold. For sufficiently large N this function is C^r exponentially small bounded by $C\nu(\theta)^{-N}$ (where C is independent on N). ■

In the same way as in Lemma 4.1.13 (ii) and (iii) we get for $i = 1, 2$

$$\| \, D^i_{(\xi,\eta)} z_1(h(z_2^0), z_2, \theta, N)(N) \, \| \leq C \, \nu(\theta)^{-N} \, , \tag{4.36}$$

$$\| \, D^i_\theta z_1(h(z_2^0), z_2, \theta, N)(N) \, \| \leq C \, \tilde{\nu}(\theta)^{-N} \, , \tag{4.37}$$

Recall, that $\mathcal{T}(p, \lambda)$ is the transformation leading to (4.33). Using the Mean Value

Theorem we find constants $L(p,\lambda)$ such that

$$\begin{aligned} & \| \mathcal{T}^{-1}(p,\lambda)(\mathcal{T}(p,\lambda)(z^{loc}(\Omega,p,\lambda,N))) \| \\ = \quad & \| \mathcal{T}^{-1}(p,\lambda)(\mathcal{T}(p,\lambda)(z^{loc}(\Omega,p,\lambda,N))) - \mathcal{T}^{-1}(p,\lambda)(\mathcal{T}(p,\lambda)(0)) \| \\ \leq \quad & L(p,\lambda) \| \mathcal{T}(p,\lambda)(z^{loc}(\Omega,p,\lambda,N)) \| \ . \end{aligned}$$

This shows that we get qualitatively the same estimates for $\| z^{loc}(\Omega,p,\lambda,N) \|$ as for $\| \mathcal{T}(p,\lambda)(z^{loc}(\Omega,p,\lambda,N)) \|$. In the same way we can consider derivatives of z^{loc} and its $\mathcal{T}(p,\lambda)$ image. So, in our further analysis we do not need to distinguish the solution $z^{loc}(\Omega,p,\lambda,N)$ and its $\mathcal{T}(p,\lambda)$ image.
Further, we do not distinguish between the curves $\mathcal{C}^{\pm}(\Omega,p,\lambda)$ and its $\mathcal{T}(p,\lambda)$ images.

Lemma 4.1.15 *Assume (H4.1) – (H4.3). Let both $\Omega \in \mathbb{R}^+$ and $N \in \mathbb{N}$ be sufficiently large, $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$. Let further*

$$z^{loc}(\Omega,p,\lambda,N)(\cdot) = (z_1^{loc}(\Omega,p,\lambda,N)(\cdot), z_2^{loc}(\Omega,p,\lambda,N)(\cdot))$$

be the unique solution of (4.31) which connects the curves $\mathcal{C}^+(\Omega,p,\lambda)$ and $\mathcal{C}^-(\Omega,p,\lambda)$ in N steps. Then we have for $i = 1,2$

(i) $\| z_1^{loc}(\Omega,p,\lambda,N)(N) \| \leq C\nu(\theta)^{-N}$,

(ii) $\| D^i_{\Omega} z_1^{loc}(\Omega,p,\lambda,N)(N) \| \leq C\nu(\theta)^{-N}$,

(iii) $\| D^i_{p} z_1^{loc}(\Omega,p,\lambda,N)(N) \| \leq C\tilde{\nu}(\theta)^{-N}$, *for* $1 < \tilde{\nu}(\theta) < \nu(\theta)$.

Proof
(i) Recall that, $\tilde{\gamma}^+(\Omega,p,\lambda) = \mathcal{C}^+(\Omega,p,\lambda) \cap W^s_{\Pi(\cdot,\lambda)}(p)$. As depicted in Figure 4.8, $\mathcal{C}^+(\Omega,p,\lambda)$ is a smooth curve intersecting the z_1-axes (that means the stable manifold) transversally in $\tilde{\gamma}^+(\Omega,p,\lambda)$. Thus, $\mathcal{C}^+(\Omega,p,\lambda)$ is the graph of a C^r function $h^+(\Omega,p,\lambda)(\cdot) : U_{z_2}(0) \subset \mathbb{R} \to U_{z_1}(0) \subset \mathbb{R}$ (where $U_{z_i}(0)$ are neighbourhoods of zero in $\mathbb{R}$ for $i = 1,2$). So, $\mathcal{C}^+(\Omega,p,\lambda)$ is a C^r disc $\mathcal{D}$. We define $\mathcal{D}_N := \Pi^N(\mathcal{D},\theta) = \Pi^N(\mathcal{C}^+(\Omega,p,\lambda),\theta)$. With that we have $z^{loc}(\Omega,p,\lambda,N)(N) \in \mathcal{D}_N$. Therefore, $z_1^{loc}(\Omega,p,\lambda,N)(N)$ can be estimated analogously to (4.35), and the assertion of the Lemma follows.

(ii) This estimate follows with (4.36). For that realise that in the present context the derivative of h^+ with respect to Ω remains bounded.

(iii) This estimate follows with (4.37) (in the present context we have $\theta = (p,\lambda)$), and (4.36) (see (ii), note that also the derivative of h^+ with respect to p remains bounded). ■

Now, we use the above lemma to provide the mentioned estimates, which we will use to discuss the bifurcation equation for symmetric periodic orbits.

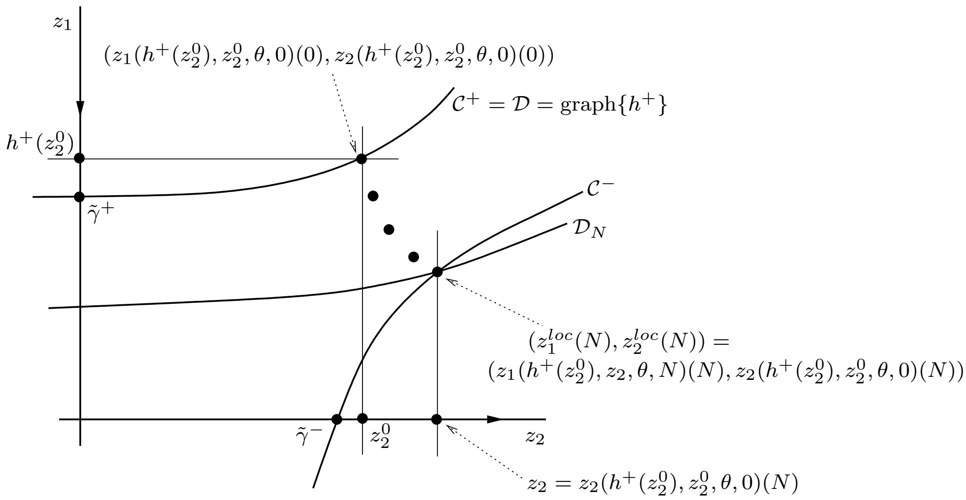

Figure 4.8: Using Deng's λ-Lemma to obtain estimates for the local solutions

Recall, that $\tilde{\gamma}^-(\Omega, p, \lambda) = \mathcal{C}^-(\Omega, p, \lambda) \cap W^u_{\Pi(\cdot,\lambda)}(p)$. For a moment we omit the arguments Ω, p and λ. The smooth curve $\mathcal{C}^-$ crossing the z_2-axes (that means the unstable manifold) transversally in $\tilde{\gamma}^-$ can be described as graph of a C^r function $h^- : U_{z_1}(0) \subset \mathbb{R} \to U_{z_2}(0) \subset \mathbb{R}$. Then we can write

$$z^{loc}(N) = (z_1^{loc}(N), z_2^{loc}(N)) = (z_1^{loc}(N), h^-(z_1^{loc}(N))) \quad \text{and} \quad \tilde{\gamma}^- = (0, h^-(0)) \, .$$

Further, Lemma 4.1.15(i), the C^r smoothness of $h^-(\cdot)$ and the Mean Value Theorem applied to h^- yield

$$\| h^-(0) - h^-(z_1^{loc}(N)) \| \leq C\nu(p,\lambda)^{-N} \, . \tag{4.38}$$

Finally, Lemma 4.1.15(i), (4.38) give

$$\begin{aligned} \| z^{loc}(\Omega, p, \lambda, N)(N) - \tilde{\gamma}^-(\Omega, p, \lambda) \| &= \| (z_1^{loc}(N), h^-(z_1^{loc}(N))) - (0, h^-(0)) \| \\ &\leq C\nu(p,\lambda)^{-N} = Ce^{-\mu(p,\lambda)N} \, , \end{aligned}$$

with $\mu(p, \lambda) := \ln \nu(p, \lambda) > 0$. Similarly, with Lemma 4.1.15(ii),(iii), and $\tilde{\mu}(p, \lambda) := \ln \tilde{\nu}(p, \lambda) > 0$ we get

$$\begin{aligned} &\| D^i_\Omega(z^{loc}(\Omega, p, \lambda, N)(N) - \tilde{\gamma}^-(\Omega, p, \lambda)) \| \leq Ce^{-\mu(p,\lambda)N} \, , \\ &\| D^i_p(z^{loc}(\Omega, p, \lambda, N)(N) - \tilde{\gamma}^-(\Omega, p, \lambda)) \| \leq Ce^{-\tilde{\mu}(p,\lambda)N} \, , \end{aligned} \tag{4.39}$$

$i = 1, 2$. By a similar discussion we get for $i = 0, 1, 2$

$$\begin{aligned} &\| D^i_\Omega(z^{loc}(\Omega, p, \lambda, N)(0) - \tilde{\gamma}^+(\Omega, p, \lambda)) \| \leq Ce^{-\mu(p,\lambda)N} \, , \\ &\| D^i_p(z^{loc}(\Omega, p, \lambda, N)(0) - \tilde{\gamma}^+(\Omega, p, \lambda)) \| \leq Ce^{-\tilde{\mu}(p,\lambda)N} \, . \end{aligned} \tag{4.40}$$

So, we finish this section with the following lemma.

Lemma 4.1.16 *Assume (H 4.1) – (H 4.3). Let both* $\Omega \in \mathbb{R}^+$ *and* $N \in \mathbb{N}$ *be sufficiently large,* $p \in X^c_{R,loc}$ *and* $\lambda \in \Lambda_0$. *Then, there exists N-independent positive constants* C *and* $\bar{\mu}(p,\lambda)$ *such that for* $i = 0, 1, 2$

$$\| D^i_\Omega(x^+(\Omega,p,\lambda,N)(0) - \varphi(-\Omega,\tilde{\gamma}^+(\Omega,p,\lambda),\lambda)) \| \leq Ce^{-\bar{\mu}(p,\lambda)N} ,$$

$$\| D^i_p(x^+(\Omega,p,\lambda,N)(0) - \varphi(-\Omega,\tilde{\gamma}^+(\Omega,p,\lambda),\lambda)) \| \leq Ce^{-\bar{\mu}(p,\lambda)N} ,$$

$$\| D^i_\Omega(x^-(\Omega,p,\lambda,N)(0) - \varphi(\Omega,\tilde{\gamma}^-(\Omega,p,\lambda),\lambda)) \| \leq Ce^{-\bar{\mu}(p,\lambda)N} ,$$

$$\| D^i_p(x^-(\Omega,p,\lambda,N)(0) - \varphi(\Omega,\tilde{\gamma}^-(\Omega,p,\lambda),\lambda)) \| \leq Ce^{-\bar{\mu}(p,\lambda)N} .$$

Proof Due to the smoothness of the flow φ of (1.1) and

$$\varphi(-\Omega, x^-(\Omega,p,\lambda,N)(0),\lambda) = z^{loc}(\Omega,p,\lambda,N)(N) ,$$

$$\varphi(\Omega, x^+(\Omega,p,\lambda,N)(0),\lambda) = z^{loc}(\Omega,p,\lambda,N)(0)$$

the estimates follow directly from (4.39) and (4.40). ■

4.2 Discussion of the bifurcation equation

Within this section we solve the bifurcation equation for symmetric one-periodic orbits near the non-elementary primary homoclinic orbit Γ in $\mathbb{R}^4$. So the main assumptions are (H 3.3) and (H 4.1).

Let both $\Omega \in \mathbb{R}^+$ and $N \in \mathbb{N}$ be sufficiently large, $p \in X^c_{R,loc}$ and $\lambda \in \Lambda_0$. We assume further (H 4.2) and (H 4.3). Then, as stated in Lemma 4.1.12, there exists a symmetric one-periodic Lin solution $\mathcal{X}(\Omega,p,\lambda,N) = \{(x^+(\Omega,p,\lambda,N), x^{loc}(\Omega,p,\lambda,N), x^-(\Omega,p,\lambda,N))\}$. With that the bifurcation equation for symmetric one-periodic orbits has the form

$$\xi_{per}(\Omega,p,\lambda,N) := x^+(\Omega,p,\lambda,N)(0) - x^-(\Omega,p,\lambda,N)(0) = 0 . \tag{4.41}$$

An illustration of the meaning of this bifurcation equation is given by Figure 2.6.

The symmetry of the Lin solution $\mathcal{X}(\Omega,p,\lambda,N)$ means $Rx^+(\Omega,p,\lambda,N)(0) = x^-(\Omega,p,\lambda,N)(0)$. This proves

Lemma 4.2.1 *Assume (H 4.1) – (H 4.3). Let both* $\Omega \in \mathbb{R}^+$ *and* $N \in \mathbb{N}$ *be sufficiently large, further* $p \in X^c_{R,loc}$ *and* $\lambda \in \Lambda_0$. *Then*

$$\xi_{per}(\Omega,p,\lambda,N) \in \mathrm{Fix}\,(-R) .$$

Hence, recalling that in $\mathbb{R}^4$ $\dim(\Sigma \cap \mathrm{Fix}\,(-R)) = 1$, Equation (4.41) is one-dimensional.

4.2.1 Preparations

Let $y \in Y^c$ be sufficiently small and $\lambda \in \Lambda_0$. Then, under Hypothesis (H 3.2), due to Lemma 3.2.3 there exists a uniquely determined one-homoclinic (symmetric) Lin solution asymptotic to the centre manifold

$$\{(\gamma^+(y,\lambda), \gamma^-(y,\lambda))\} := \{(\gamma^+(y,y,\lambda), \gamma^-(y,y,\lambda))\} . \tag{4.42}$$

Due to Equation (3.66) we have

$$R\gamma^+(y,\lambda)(0) = \gamma^-(y,\lambda)(0) . \tag{4.43}$$

It turns out that it is of advantage to consider Equation (4.41) as perturbation of the reduced bifurcation equation for one-homoclinic orbits to the centre manifold (see Section 3.3.1 and Figure 2.6)

$$\hat{\xi}^\infty(y,\lambda_1,\lambda_2) = \gamma^+(y,\lambda)(0) - \gamma^-(y,\lambda)(0) = 0 , \quad \lambda = (\lambda_1, \lambda_2) .$$

In order to do that, our first aim is to define an appropriate $y \in Y^c$ in dependence on $(\Omega, p, \lambda) \in \mathbb{R}^+ \times X^c_{R,loc} \times \Lambda_0$. More precisely, that means we will define a mapping $(\Omega, p, \lambda) \mapsto y^*(\Omega, p, \lambda)$ that implies that the norm of

$$\xi_r(\Omega,p,\lambda,N) := \xi_{per}(\Omega,p,\lambda,N) - \hat{\xi}^\infty(y^*(\Omega,p,\lambda),\lambda_1,\lambda_2)$$

becomes "exponentially small" for increasing N.

Let $\lambda \in \Lambda_0$, $p \in X^c_{R,loc}$ and a sufficiently large $\Omega \in \mathbb{R}^+$ be given. The manifolds $\Sigma^+(\Omega,\lambda)$ and $W^s_{\Pi(\cdot,\lambda)}(p)$ intersect transversally in a point $\tilde{\gamma}^+(\Omega,p,\lambda)$ (see (4.21) and Lemma 4.1.7):

$$\Sigma^+(\Omega,\lambda) \cap W^s_{\Pi(\cdot,\lambda)}(p) = \{\tilde{\gamma}^+(\Omega,p,\lambda)\} . \tag{4.44}$$

So, the point $\tilde{\gamma}^+(\Omega,p,\lambda)$ lies in the centre-stable manifold of $\mathring{x} = 0$ and due to the definition of $\Sigma^+(\Omega,\lambda)$ it is clear that

$$\varphi(-\Omega, \tilde{\gamma}^+(\Omega,p,\lambda), \lambda) \in W^{cs}_\lambda \cap \Sigma .$$

Hence, with the description of $W^{cs}_\lambda \cap \Sigma$ on page 36, there exists a unique $y \in Y^c$ such that

$$\varphi(-\Omega, \tilde{\gamma}^+(\Omega,p,\lambda), \lambda) = \gamma^s(\lambda)(0) + y + h^{cs}(y,\lambda) . \tag{4.45}$$

So we can define a mapping determining this y by

$$(\Omega,p,\lambda) \mapsto y^*(\Omega,p,\lambda) := \mathcal{P}_{Y^c}(\varphi(-\Omega, \tilde{\gamma}^+(\Omega,p,\lambda),\lambda) - \gamma^s(\lambda)(0)) , \tag{4.46}$$

where $\mathcal{P}_{Y^c}$ is the projection of $Z = Y^c \oplus \hat{Z}$ on Y^c along $\hat{Z}$. The smoothness of the mapping defined in (4.46) follows from the smoothness of $\tilde{\gamma}^+(\cdot,\cdot,\cdot)$, and the smoothness of the flow φ of (1.1). The next lemma shows that the defined mapping has the desired property. Here, see (4.42) for the definition of $\gamma^\pm(y,\lambda)$.

Lemma 4.2.2 *Assume (H 3.2), (H 3.3) and (H 4.1) – (H 4.3). Let both* $\Omega \in \mathbb{R}^+$ *and* $N \in \mathbb{N}$ *be sufficiently large,* $p \in X^c_{R,loc}$ *and* $\lambda \in \Lambda_0$*. Then, there exist* N*-independent positive constants* C *and* $\bar{\mu}(p,\lambda)$ *such that for* $i = 0, 1, 2$

$$\| D^i_\Omega(x^\pm(\Omega, p, \lambda, N)(0) - \gamma^\pm(y^*(\Omega, p, \lambda), \lambda)(0)) \| \leq C e^{-\bar{\mu}(p,\lambda)N} ,$$

$$\| D^i_p(x^\pm(\Omega, p, \lambda, N)(0) - \gamma^\pm(y^*(\Omega, p, \lambda), \lambda)(0)) \| \leq C e^{-\bar{\mu}(p,\lambda)N} .$$

Proof Let $\tilde{\gamma}^+(\Omega, p, \lambda)$ be defined as in (4.44) and let further

$$\{\tilde{\gamma}^-(\Omega, p, \lambda)\} = \Sigma^-(\Omega, \lambda) \cap W^u_{\Pi(\cdot,\lambda)}(p) .$$

From the definitions and properties of the manifolds $\Sigma^\pm(\Omega, \lambda)$ and $W^{s(u)}_{\Pi(\cdot,\lambda)}(p)$ within Section 4.1 we get

$$R\tilde{\gamma}^+(\Omega, p, \lambda) = \tilde{\gamma}^-(\Omega, p, \lambda) . \tag{4.47}$$

Recalling the description of $\gamma^+(y, \lambda)(0)$ by (3.41) and (3.56) we have

$$\begin{aligned} \gamma^+(y^*(\Omega, p, \lambda), \lambda)(0) &= \gamma^s(\lambda)(0) + y^*(\Omega, p, \lambda) + z^+(y^*(\Omega, p, \lambda), \lambda) \\ &\in W^{cs}_\lambda \cap \Sigma . \end{aligned}$$

On the other hand , (4.45) and (4.46) yield

$$\begin{aligned} \varphi(-\Omega, \tilde{\gamma}^+(\Omega, p, \lambda), \lambda) &= \gamma^s(\lambda)(0) + y^*(\Omega, p, \lambda) + h^{cs}(y^*(\Omega, p, \lambda), \lambda) \\ &\in W^{cs}_\lambda \cap \Sigma . \end{aligned}$$

Because both points $\gamma^+(y^*(\Omega, p, \lambda), \lambda)(0)$ and $\varphi(-\Omega, \tilde{\gamma}^+(\Omega, p, \lambda), \lambda)$ of $W^{cs}_\lambda \cap \Sigma$ have the same Y^c- component we conclude

$$\gamma^+(y^*(\Omega, p, \lambda), \lambda)(0) = \varphi(-\Omega, \tilde{\gamma}^+(\Omega, p, \lambda), \lambda) .$$

So, by the reversibility of the flow and with (4.43) and (4.47) we obtain

$$\gamma^-(y^*(\Omega, p, \lambda), \lambda)(0) = \varphi(\Omega, \tilde{\gamma}^-(\Omega, p, \lambda), \lambda)$$

and the lemma follows as a direct consequence of Lemma 4.1.16. ∎

Our considerations show that ξ_{per} can be written as

$$\xi_{per}(\Omega, p, \lambda, N) = \hat{\xi}^\infty(y^*(\Omega, p, \lambda), \lambda_1, \lambda_2) + \xi_r(\Omega, p, \lambda, N)$$

where $\| \xi_r(\Omega, p, \lambda, N) \| \leq C e^{-\bar{\mu}(p,\lambda)N}$.

In order to use results from Section 3.3.1 it would be favourable to express Ω and p as functions of y and λ in such a way that $y^*(\Omega(y, \lambda), p(y, \lambda)) = y$. Let

$$H^{cs}(y, \lambda) := \gamma^s(\lambda)(0) + y + h^{cs}(y, \lambda) , \tag{4.48}$$

where h^{cs} is defined as on page 36. By definition $H^{cs}(y,\lambda) \in W^{cs}_{\lambda} \cap \Sigma$. Then roughly speaking there exists $\hat{\Omega} \in \mathbb{R}^+$ such that $\Omega(y,\lambda) \in U(\hat{\Omega})$ is the "first-hit-time" of $H^{cs}(y,\lambda)$ under the flow. Here $U(\hat{\Omega})$ denotes a neighbourhood of $\hat{\Omega}$ in $\mathbb{R}^+$. Further, $p(y,\lambda)$ is determined by the leaf $\mathcal{M}_{p,\lambda}$ in which $H^{cs}(y,\lambda)$ attains Σ_{loc}. One of the main obstacles is that $\Omega(0,\lambda)$ is not uniquely determined. ($H^{cs}(0,\lambda)$ belongs to the stable manifold. So, for all sufficiently large t it follows that $\varphi(t, H^{cs}(0,\lambda),\lambda)$ belongs to Σ_{loc}.) For that reason we resort to introducing polar coordinates (ϱ,ϑ) in Y^c. This approach finally gives functions $\Omega^*(\varrho,\vartheta,\lambda)$ and $p^*(\varrho,\vartheta,\lambda)$. Note, that due to the above mentioned non-uniqueness of $\Omega(y,\lambda)$, $\Omega^*(0,\vartheta,\lambda) \not\equiv \Omega^*(0,0,\lambda)$.

For fixed sufficiently large Ω_0 we consider the map

$$\varphi(\Omega_0,\cdot,0) : W^{cs} \cap \Sigma \to \Sigma_{loc} \oplus X^c_{-R} \ ,$$

with $\dim(W^{cs} \cap \Sigma) = 2$ and $\dim \Sigma_{loc} = 3$. Then $D_2\varphi(\Omega_0,\gamma(0),0)$ is a map

$$D_2\varphi(\Omega_0,\gamma(0),0) : T_{\gamma(0)}(W^{cs} \cap \Sigma) \to \Sigma_{loc} \oplus X^c_{-R} \ .$$

Note further that in $\mathbb{R}^4$ $T_{\gamma(0)}(W^{cs} \cap \Sigma) = Y^c$. Because $D_2\varphi(\Omega_0,\gamma(0),0)(Y^c)$ is two-dimensional we have

$$\dim(D_2\varphi(\Omega_0,\gamma(0),0)(Y^c) \cap \Sigma_{loc}) \geq 1 \ .$$

Let $Y^c_{loc}(\Omega_0) \subset D_2\varphi(\Omega_0,\gamma(0),0)(Y^c) \cap \Sigma_{loc}$ be a one-dimensional linear manifold. We define

$$Y^c_{\Sigma}(\Omega_0) := (D_2\varphi(\Omega_0,\gamma(0),0))^{-1}(Y^c_{loc}(\Omega_0)) \subset Y^c \ . \tag{4.49}$$

Let $\operatorname{span}\{\hat{y}_1,\hat{y}_2\} = Y^c$ such that $\| \hat{y}_i \| = 1$, $i = 1,2$, and

$$\operatorname{span}\{\hat{y}_1\} = Y^c_{\Sigma}(\Omega_0) \ .$$

Then $y \in Y^c$ can be decomposed with respect to the basis $\{\hat{y}_1,\hat{y}_2\}$:

$$y = y^1\hat{y}_1 + y^2\hat{y}_2 \ .$$

Remark 4.2.3 The basis $\{\hat{y}_1,\hat{y}_2\}$ depends on the choice of Ω_0. Further, due to (4.49), there are $\omega_i \in \mathbb{R}^+$, $i \in \mathbb{N}$, and $\omega_i \approx 2\pi$, such that

$$D_2\varphi(\Omega_k,\gamma(0),0)(\hat{y}_1) \subset \Sigma_{loc} \ , \quad \Omega_k := \Omega_0 + \sum_{i=1}^{k} \omega_i \ . \tag{4.50}$$

□

We represent $(y^1,y^2) \in \mathbb{R}^2$ by means of polar coordinates $(\varrho,\vartheta) \in \mathbb{R}^2$:

$$y^1 = \varrho \cos\vartheta \quad \text{and} \quad y^2 = \varrho \sin\vartheta \ .$$

Note, that we explicitly allow ϱ to be negative. Further, by construction we have in particular

$$Y^c_\Sigma(\Omega_0) = \{(\varrho, \vartheta) : \vartheta = 0,\ \varrho \in \mathbb{R}\}\ . \tag{4.51}$$

Altogether y can be seen as a smooth function

$$\begin{aligned} y(\cdot,\cdot) :\ \ \mathbb{R} \times \mathbb{R} &\ \to\ Y^c \\ (\varrho, \vartheta) &\ \mapsto\ \varrho \cos\vartheta\, \hat{y}_1 + \varrho \sin\vartheta\, \hat{y}_2\ . \end{aligned} \tag{4.52}$$

With the smooth function $H^{cs}(\cdot,\cdot) : Y^c \times \Lambda_0 \to \mathbb{R}^4$ defined in (4.48) and the above definition of $y(\cdot,\cdot)$ we can prove the following lemma.

Lemma 4.2.4 *Assume (H3.2), (H3.3) and (H4.1) – (H4.3). Then, there are positive numbers $\hat{\varepsilon}_\varrho$ and $\hat{\varepsilon}_\vartheta$ such that there exists a C^r smooth function $\Omega^*(\cdot,\cdot,\cdot)$: $(-\hat{\varepsilon}_\varrho, \hat{\varepsilon}_\varrho) \times (-\hat{\varepsilon}_\vartheta, \hat{\varepsilon}_\vartheta) \times \Lambda_0 \to \mathbb{R}^+$ satisfying*

$$\varphi(\Omega^*(\varrho, \vartheta, \lambda), H^{cs}(y(\varrho, \vartheta), \lambda), \lambda) \in \Sigma_{loc}\ .$$

Proof Let Ω_0 be sufficiently large and let Ω_k be as introduced in (4.50). We define a smooth function

$$\begin{aligned} F :\ \ \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \Lambda_0 &\ \to\ X^c_{-R} \\ (\Omega, \varrho, \vartheta, \lambda) &\ \mapsto\ \mathcal{P} \circ \varphi(\Omega, H^{cs}(y(\varrho, \vartheta), \lambda), \lambda)\ , \end{aligned}$$

where $\mathcal{P}$ is the projection on X^c_{-R} along $X^h \oplus X^c_R$. In order to get the function Ω^* we consider

$$F(\Omega, \varrho, \vartheta, \lambda) = 0\ .$$

Because of $y(0, \vartheta) = 0$, $H^{cs}(0, \lambda) = \gamma^s(\lambda)(0) \in W^s_\lambda$ and $\varphi(\Omega, \gamma^s(\lambda)(0), \lambda) \in W^s_{\lambda,loc} = X^s_{\lambda,loc} = X^s_{loc} \subset \Sigma_{loc}$ we have

$$F(\Omega, 0, \vartheta, \lambda) = 0\ .$$

Thus, there exists a function $H : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \Lambda_0 \to X^c_{-R}$ such that

$$F(\Omega, \varrho, \vartheta, \lambda) = \varrho H(\Omega, \varrho, \vartheta, \lambda)\ .$$

Next we show that there is a $k \in \mathbb{N}$ such that the equation

$$H(\Omega, \varrho, \vartheta, \lambda) = 0$$

can be solved near $(\Omega_k, 0, 0, 0)$ by means of the Implicit Function Theorem. For that we show

$$H(\Omega_k, 0, 0, 0) = 0\ , \tag{4.53}$$

$$D_1 H(\Omega_k, 0, 0, 0) \neq 0 \tag{4.54}$$

for some suitable $k \in \mathbb{N}$.

For all $k \in \mathbb{N}$ we have

$$\begin{aligned} H(\Omega_k, 0, 0, 0) &= D_2 F(\Omega_k, 0, 0, 0) \\ &= \mathcal{P} \circ \left(\tfrac{\partial}{\partial \varrho} \varphi(\Omega_k, \gamma(0) + y(\varrho, 0) + h^{cs}(y(\varrho, 0), 0), 0) \right) \Big|_{\varrho=0} \end{aligned}$$

and

$$\begin{aligned} &\tfrac{\partial}{\partial \varrho} \varphi(\Omega_k, \gamma(0) + y(\varrho, 0) + h^{cs}(y(\varrho, 0), 0), 0) \Big|_{\varrho=0} \\ &= D_2\varphi(\Omega_k, \gamma(0)+y(0,0)+h^{cs}(y(0,0),0),0)(D_1 y(0,0)+D_1 h^{cs}(y(0,0),0) D_1 y(0,0)) \\ &= D_2 \varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) \ . \end{aligned}$$

The latter equality follows from

$$D_1 h^{cs}(y(0,0),0) = D_1 h^{cs}(0,0) = 0 \quad \text{and} \quad D_1 y(0,0) = \hat{y}_1 \ .$$

Hence

$$H(\Omega_k, 0, 0, 0) = \mathcal{P} \circ D_2 \varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) \tag{4.55}$$

and thus (4.53) follows by (4.50). So, it remains to prove (4.54). With (4.55) we find that

$$D_1 H(\Omega_k, 0, 0, 0) = \mathcal{P} \circ D_1 D_2 \varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) \ .$$

Exploiting that φ is the flow of the vector field f gives

$$D_1 D_2 \varphi(\Omega, \gamma(0), 0)(\hat{y}_1) = D_1 f(\gamma(\Omega), 0) D_2 \varphi(\Omega, \gamma(0), 0)(\hat{y}_1) \ . \tag{4.56}$$

Hence in order to prove (4.54) we have to show

$$\mathcal{P} \circ D_1 f(\gamma(\Omega_k), 0) D_2 \varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) \neq 0 \ . \tag{4.57}$$

Let $D_1 f(\gamma(\Omega), 0) = (a_{ij}(\Omega))_{i,j=1,\ldots,4}$. Further, because of (4.53) and (4.55) we can write

$$D_2 \varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) =: \left(x^1(\Omega_k), x^2(\Omega_k), x^3(\Omega_k), 0 \right)^\top \ . \tag{4.58}$$

Thus (4.57) reads

$$\sum_{j=1}^{3} a_{4j}(\Omega_k) x^j(\Omega_k) \neq 0 \ . \tag{4.59}$$

Equation (4.56) means that $D_2 \varphi(\Omega, \gamma(0), 0)(\hat{y}_1)$ satisfies

$$\dot{x} = D_1 f(\gamma(\Omega), 0) x \ . \tag{4.60}$$

The corresponding transition matrix is $\Phi(\cdot, \cdot)$. From the considerations in A.2 we know that this equation has an exponential trichotomy on $\mathbb{R}^+$ ($t_0 = 0$) with constants $-\alpha_s = \alpha_u =: \alpha$ and α_c. Note, that due to Remark A.2.5 any α and α_c with

$0 < \alpha_c < \alpha < \mu$ can be chosen. (Recall, that $-\mu$ and μ are the leading eigenvalues of the stable and the unstable spectrum of $D_1 f(0,0)$, respectively.) The projections $P_c^+(t)$, defined in Section 3.1, are associated to this exponential trichotomy. In accordance with the considerations on page 34 we are free to choose these projections such that $\operatorname{im} P_c^+(0) = Y^c$. So, due to $\hat{y}_1 \in Y^c$ we have

$$\hat{y}_1 \in \operatorname{im} P_c^+(0) . \tag{4.61}$$

Furthermore, taking notice of (3.14), there is a positive constant K such that

$$\| \Phi(t,s) P_c^+(s) \| \leq K e^{\alpha_c (t-s)} \quad \forall \alpha_c \in (0,\alpha) ,\ t \geq s \geq 0 , \tag{4.62}$$

$$\| \Phi(t,s) P_c^+(s) \| \leq K e^{-\alpha_c (t-s)} \quad \forall \alpha_c \in (0,\alpha) ,\ s \geq t \geq 0 . \tag{4.63}$$

Let Q_c denote the spectral projection corresponding to $\dot{x} = D_1 f(0,0) x$, with

$$\operatorname{im} Q_c = X^c . \tag{4.64}$$

Note, that Q_c is an associated projection to the exponential trichotomy of $\dot{x} = D_1 f(0,0)x$. Then, Lemma A.2.10 yields that there are positive constants C_1 and γ_c such that

$$\| P_c^+(t) - Q_c \| \leq C_1 e^{-\gamma_c t} . \tag{4.65}$$

Keeping γ_c fixed, due to Remark A.2.5 we can choose $\hat{\alpha}_c \in (0,\alpha)$ as a constant corresponding to the exponential trichotomy of (4.60) such that

$$\hat{\alpha}_c < \gamma_c . \tag{4.66}$$

Because Φ commutes with P_c^+ (see (3.13)) and (4.61)

$$\begin{aligned} D_2\varphi(\Omega_k, \gamma(0), 0)(\hat{y}_1) &= \Phi(\Omega_k, 0) D_2\varphi(0, \gamma(0), 0)(\hat{y}_1) = \Phi(\Omega_k, 0)\hat{y}_1 \\ &= \Phi(\Omega_k, 0) P_c^+(0) \hat{y}_1 = P_c^+(\Omega_k) \Phi(\Omega_k, 0) \hat{y}_1 . \end{aligned} \tag{4.67}$$

From that we get with $\| \hat{y}_1 \| = 1$ and (4.62)

$$\| \Phi(\Omega_k, 0)\hat{y}_1 \| \leq \| \Phi(\Omega_k, 0) P_c^+(0) \| \leq K e^{\hat{\alpha}_c \Omega_k} . \tag{4.68}$$

Further, (4.58) and (4.67) yield

$$\Phi(\Omega_k, 0)\hat{y}_1 = \big(x^1(\Omega_k), x^2(\Omega_k), x^3(\Omega_k), 0\big)^\top \tag{4.69}$$

and hence due to (4.64)

$$Q_c \Phi(\Omega_k, 0)\hat{y}_1 = \big(0, 0, x^3(\Omega_k), 0\big)^\top . \tag{4.70}$$

Recall that (4.67) in particular says

$$P_c^+(\Omega_k) \Phi(\Omega_k, 0)\hat{y}_1 = \Phi(\Omega_k, 0)\hat{y}_1 .$$

From (4.69) and (4.70) we obtain

$$P_c^+(\Omega_k)\Phi(\Omega_k,0)\hat{y}_1 - Q_c\Phi(\Omega_k,0)\hat{y}_1 = \left(x^1(\Omega_k), x^2(\Omega_k), 0, 0\right)^\top . \tag{4.71}$$

Hence with (4.65) and (4.68) we find that there is a constant C_2 such that

$$\| \left(x^1(\Omega_k), x^2(\Omega_k), 0, 0\right)^\top \| \leq C_2 e^{(\hat{\alpha}_c - \gamma_c)\Omega_k} . \tag{4.72}$$

This implies

$$|x^1(\Omega_k)|,\ |x^2(\Omega_k)| \leq C_2 e^{(\hat{\alpha}_c - \gamma_c)\Omega_k} . \tag{4.73}$$

In what follows we give a lower estimate for $|x^3(\Omega_k)|$. The linear mapping

$$L_k := P_c^+(\Omega_k)\Phi(\Omega_k,0) : \operatorname{im} P_c^+(0) \to \operatorname{im} P_c^+(\Omega_k)$$

is invertible, $L_k^{-1} = \Phi(0,\Omega_k)P_c^+(\Omega_k)$. Basic linear functional analysis tells

$$\frac{1}{\| L_k^{-1} \|} \| \hat{y}_1 \| \leq \| L_k \hat{y}_1 \| . \tag{4.74}$$

Because of (4.63) we have

$$\| \Phi(0,\Omega_k)P_c^+(\Omega_k) \| \leq K e^{\alpha_c \Omega_k} \quad \forall \alpha_c \in (0,\alpha)$$

and thus using $\| \hat{y}_1 \| = 1$ and (4.74) we get

$$\frac{1}{K} e^{-\alpha_c \Omega_k} \leq \| P_c^+(\Omega_k)\Phi(\Omega_k,0)\hat{y}_1 \| .$$

The latter estimate in combination with (4.70)–(4.73) implies

$$|x^3(\Omega_k)| \geq C_3 e^{-\alpha_c \Omega_k} \tag{4.75}$$

for some constant C_3.

For sufficiently large Ω the linearisation $D_1 f(\gamma(\Omega),0)$ is close to $D_1 f(0,0)$. Then, in particular, for each $\epsilon > 0$ there exists $\bar{\Omega} \in \mathbb{R}^+$ such that for all Ω with $\Omega > \bar{\Omega}$ we have

$$|a_{4j}(\Omega)| < \varepsilon \quad \text{for } j \in \{1,2,4\} \ \text{ and } \ |a_{43}(\Omega) - 1| < \varepsilon . \tag{4.76}$$

So, with (4.73), (4.75) and (4.76), we get

$$\sum_{j=1}^{3} a_{4j}(\Omega_k)x^j(\Omega_k) = \underbrace{a_{41}(\Omega_k)x^1(\Omega_k) + a_{42}(\Omega_k)x^2(\Omega_k)}_{\leq C_4 e^{(\hat{\alpha}_c - \gamma_c)\Omega_k}} + \underbrace{a_{43}(\Omega_k)x^3(\Omega_k)}_{\geq C_5 e^{-\alpha_c \Omega_k}}$$

for a suitable constants C_4, C_5. Hence, with (4.66) and $\alpha_c > 0$, (4.59) follows for sufficiently large $k \in \mathbb{N}$.

Finally we define $\hat{\Omega} := \Omega_k$. The foregoing considerations show that we can solve $H(\Omega,\varrho,\vartheta,\lambda) = 0$ near $(\hat{\Omega},0,0,0)$ for $\Omega = \Omega^*(\varrho,\vartheta,\lambda)$. ■

Furthermore we get

Lemma 4.2.5 *Assume (H3.2), (H3.3) and (H4.1) – (H4.3). Then, for* $\hat{\varepsilon}_\varrho$ *and* $\hat{\varepsilon}_\vartheta$ *given by Lemma 4.2.4, there is a* C^r *smooth function* $p^*(\cdot,\cdot,\cdot) : (-\hat{\varepsilon}_\varrho, \hat{\varepsilon}_\varrho) \times (-\hat{\varepsilon}_\vartheta, \hat{\varepsilon}_\vartheta) \times \Lambda_0 \to X^c_{R,loc}$ *such that*

$$\gamma^+(y(\varrho,\vartheta),\lambda)(\Omega^*(\varrho,\vartheta,\lambda)) \in \mathcal{M}_{p^*(\varrho,\vartheta,\lambda),\lambda} \,.$$

Proof For each small $(\varrho,\vartheta) \in \mathbb{R}^2$ there is a $y = y(\varrho,\vartheta)$ as defined in (4.52). From Lemma 4.2.4 we get

$$\begin{aligned}
&\gamma^+(y(\varrho,\vartheta),\lambda)(\Omega^*(\varrho,\vartheta,\lambda)) \\
&= \varphi(\Omega^*(\varrho,\vartheta,\lambda), \gamma^+(y(\varrho,\vartheta),\lambda)(0),\lambda) \\
&= \varphi(\Omega^*(\varrho,\vartheta,\lambda), \gamma^s(\lambda)(0) + y(\varrho,\vartheta) + h^{cs}(y(\varrho,\vartheta),\lambda),\lambda) \in \Sigma_{loc} \,.
\end{aligned}$$

The section Σ_{loc} is smoothly foliated into leaves $\mathcal{M}_{p,\lambda}$. So, there exists $p \in X^c_{R,loc}$ such that

$$\gamma^+(y(\varrho,\vartheta),\lambda)(\Omega^*(\varrho,\vartheta,\lambda)) \in \mathcal{M}_{p,\lambda} \,.$$

This defines the stated smooth correspondence $(\varrho,\vartheta,\lambda) \mapsto p^*(\varrho,\vartheta,\lambda)$. The smoothness is clear due to the smoothness of $\gamma^+(\cdot,\cdot)$, $y(\cdot,\cdot)$, $\Omega^*(\cdot,\cdot,\cdot)$ and of the foliation of Σ_{loc} into leaves $\mathcal{M}_{p,\lambda}$. ∎

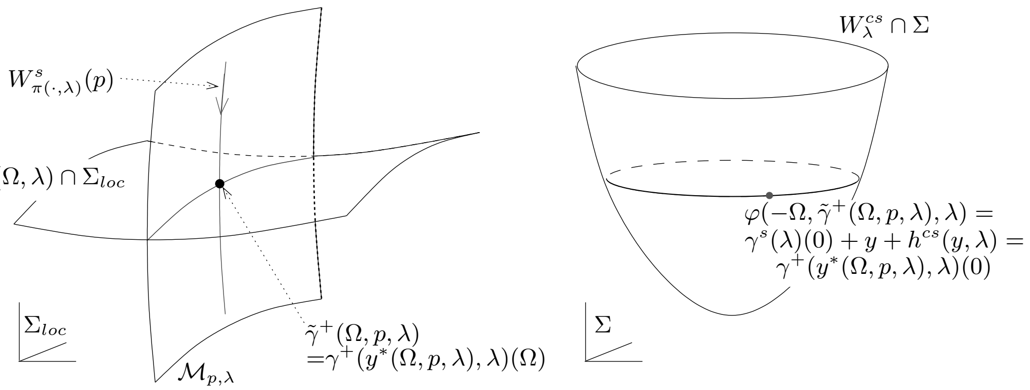

Figure 4.9: For $\Omega = \Omega^*(\varrho,\vartheta,\lambda)$ and $p^*(\varrho,\vartheta,\lambda)$ it holds $y = y^*(\Omega,p,\lambda)$

In Figure 4.9 we illustrate the meaning of the following lemma.

Lemma 4.2.6 *Assume (H3.2), (H3.3) and (H4.1) – (H4.3). Then, for all sufficiently small* $(\varrho,\vartheta) \in \mathbb{R}^2$ *and* $\lambda \in \Lambda_0$ *we have*

$$y(\varrho,\vartheta) = y^*(\Omega^*(\varrho,\vartheta,\lambda), p^*(\varrho,\vartheta,\lambda),\lambda) \,.$$

Proof From Lemma 4.2.4, Lemma 4.2.5 and $\gamma^s(\lambda)(0) + y(\varrho,\vartheta) + h^{cs}(y(\varrho,\vartheta),\lambda) \in W^{cs}_\lambda$ it is clear that

$$\begin{aligned}&\varphi(\Omega^*(\varrho,\vartheta,\lambda),\gamma^s(\lambda)(0) + y(\varrho,\vartheta) + h^{cs}(y(\varrho,\vartheta),\lambda),\lambda)\\ &\qquad \in W^s_{\Pi(\cdot,\lambda)}(p^*(\varrho,\vartheta,\lambda)) \cap \Sigma^+(\Omega^*(\varrho,\vartheta,\lambda),\lambda) = \{\tilde{\gamma}^+(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda)\}\ .\end{aligned}$$

On the other hand the definition of y^* yields

$$\begin{aligned}&\varphi(-\Omega^*(\varrho,\vartheta,\lambda),\tilde{\gamma}^+(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)\\ &= \gamma^s(\lambda)(0) + y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda) + h^{cs}(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)\,.\end{aligned}$$

Hence

$$y(\varrho,\vartheta) = y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda)\ .$$

■

4.2.2 Solutions

In what follows we consider the bifurcation equation for symmetric one-periodic orbits as a perturbation of the reduced bifurcation equation for one-homoclinic orbits $\hat{\xi}^\infty(y,\lambda_1,\lambda_2) = 0$, see Section 3.3.1. Moreover we assume that $\hat{\xi}^\infty$ has the form, see (3.68),

$$\hat{\xi}^\infty(y,\lambda_1,\lambda_2) = \lambda_1 - \lambda_1^*(y,\lambda_2) = 0\,.$$

This equation describes the complete bifurcation scenario for one-homoclinic orbits to the centre manifold near a non-elementary homoclinic orbit, see also (4.77). Lemma 3.3.3 and Hypothesis (H 3.5) allow to apply a generalised Morse lemma, [Nir01], to λ_1^*. Thus there is a transformation which brings λ_1^* in the form

$$\lambda_1^*(y,\lambda_2) = \tilde{\mathfrak{c}}(\lambda_2) + D_1^2\lambda_1^*(y(\lambda_2),\lambda_2)(y,y)\ ,$$

for some appropriate functions $\tilde{\mathfrak{c}}(\lambda_2)$ and $y(\lambda_2)$. Assuming the positive definiteness of $D_1^2\lambda_1^*(0,0)$ (see (H 3.5)) there is a further transformation which allows us to write

$$\lambda_1^*((y_1,y_2),\lambda_2) = \tilde{\mathfrak{c}}(\lambda_2) + y_1^2 + y_2^2\ , \quad y = (y_1,y_2)\ .$$

In the new coordinates the curve $\{(-\tilde{\mathfrak{c}}(\lambda_2),\lambda_2)\}$ plays the same role as $\mathfrak{C} = \{(\mathfrak{c}(\lambda_2),\lambda_2)\}$ in the discussion of one-homoclinic orbits, see Theorem 3.3.4. For that reason we assume that $\hat{\xi}^\infty$ has the form

$$\hat{\xi}^\infty((y_1,y_2),\lambda_1,\lambda_2) = \lambda_1 + \mathfrak{c}(\lambda_2) - y_1^2 - y_2^2\ . \tag{4.77}$$

Due to Lemma 4.2.6 and (4.77) we have

$$\begin{aligned}&\hat{\xi}^\infty((y_1^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),y_2^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda)),\lambda_1,\lambda_2)\\ &\ = \lambda_1 + \mathfrak{c}(\lambda_2) - (y_1^*)^2(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda) - (y_2^*)^2(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda)\\ &\qquad = \lambda_1 + \mathfrak{c}(\lambda_2) - (\varrho\cos\vartheta)^2 - (\varrho\sin\vartheta)^2 = \lambda_1 + \mathfrak{c}(\lambda_2) - \varrho^2\ .\end{aligned}$$

These considerations motivate the following assumption

(H 4.4) $\quad \hat{\xi}^{\infty}(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda_1,\lambda_2)=\lambda_1+\mathfrak{c}(\lambda_2)-\varrho^2.$

Recall that

$$\begin{aligned}\hat{\xi}^{\infty}&(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda_1,\lambda_2)\\&=\gamma^+(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)(0)\\&\qquad-\gamma^-(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)(0)\,.\end{aligned}$$

Then, under Hypothesis (H 4.4) we can write the bifurcation equation (4.41) in the following form:

$$\lambda_1+\mathfrak{c}(\lambda_2)-\varrho^2+\hat{\xi}_r(\varrho,\vartheta,\lambda,N)=0\,, \tag{4.78}$$

where we define

$$\begin{aligned}\hat{\xi}_r(\varrho,\vartheta,\lambda,N):=&\;x^+(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda,N)(0)\\&-\gamma^+(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)(0)-x^-(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda,N)(0)\\&+\gamma^-(y^*(\Omega^*(\varrho,\vartheta,\lambda),p^*(\varrho,\vartheta,\lambda),\lambda),\lambda)(0)\end{aligned} \tag{4.79}$$

We consider

$$\lambda_1+\mathfrak{c}(\lambda_2)=\varrho^2-\hat{\xi}_r(\varrho,\vartheta,\lambda,N)=:\mathcal{F}(\varrho,\vartheta,\lambda,N)\,. \tag{4.80}$$

Note, that due to the smoothness of all involved functions, the function $\mathcal{F}(\cdot,\cdot,\cdot,N)$ is of class C^r. Now, our aim is to show that there is a transformation in ϱ depending on ϑ, λ and N such that the real valued function $\mathcal{F}$ reads in the new coordinates $\tilde{\varrho}$

$$\mathcal{F}(\tilde{\varrho},\vartheta,\lambda,N):=\tilde{\varrho}^2+\tilde{\xi}_r(\vartheta,\lambda,N)\,.$$

Preparing the proof of the existence of the mentioned transformation we summarise useful properties of $\hat{\xi}_r(\varrho,\vartheta,\lambda,N)$.

Corollary 4.2.7 *Assume (H 3.2), (H 3.3) and (H 4.1) – (H 4.3). Let both $\varrho\in\mathbb{R}$ and $\vartheta\in\mathbb{R}$ be sufficiently small, $N\in\mathbb{N}$ be sufficiently large and $\lambda\in\Lambda_0$. Then there exist positive constants $\tilde{C}$ and $\mu^*(\varrho,\vartheta,\lambda)$, such that*

$$\|\ \hat{\xi}_r(\varrho,\vartheta,\lambda,N)\ \|\le\tilde{C}e^{-\mu^*(\varrho,\vartheta,\lambda)N}\,, \tag{4.81}$$

$$\|\ D_1\hat{\xi}_r(\varrho,\vartheta,\lambda,N)\ \|\le\tilde{C}e^{-\mu^*(\varrho,\vartheta,\lambda)N}\,, \tag{4.82}$$

$$\|\ D_1^2\hat{\xi}_r(\varrho,\vartheta,\lambda,N)\ \|\le\tilde{C}e^{-\mu^*(\varrho,\vartheta,\lambda)N}\,. \tag{4.83}$$

Proof Because $\Omega^*(\cdot,\cdot,\cdot)$ and $p(\cdot,\cdot,\cdot)$ are of class C^r, the above assertions follow directly from Lemma 4.2.2, (4.79) and $\mu^*(\varrho,\vartheta,\lambda):=\bar{\mu}(p^*(\varrho,\vartheta,\lambda),\lambda)$. ■

Using Corollary 4.2.7 we get the following Lemma. Here, we will allow solutions of (4.78) of the type $(\varrho,\vartheta,\lambda,N)=(\varrho,\vartheta,\lambda,\infty)$. Due to (4.81) these solutions correspond to symmetric one-homoclinic orbits to the centre manifold.

Lemma 4.2.8 *Let*

$$\mathcal{F}(\varrho,\vartheta,\lambda,N)=\varrho^2-\hat{\xi}_r(\varrho,\vartheta,\lambda,N)\ ,$$

where $\hat{\xi}_r(\varrho,\vartheta,\lambda,N)$ *is a* C^r *smooth real valued function satisfying* (4.82) *and* (4.83). *Then, for sufficiently small* $\vartheta\in\mathbb{R}$, $\lambda\in\Lambda_0$ *and sufficiently large* $N\in\mathbb{N}$ *there is a function*

$$\varrho^*(\cdot,\cdot,\cdot):(\vartheta,\lambda,N)\mapsto\varrho^*(\vartheta,\lambda,N)$$

such that

$$D_1\mathcal{F}(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)\equiv 0\ .$$

The function $\varrho^*(\cdot,\cdot,N)$ *is* C^r *smooth and there are positive constants* $\hat{C}$ *and* $\hat{\mu}(\vartheta,\lambda)$, *such that*

$$\parallel\varrho^*(\vartheta,\lambda,N)\parallel\leq\hat{C}e^{-\hat{\mu}(\vartheta,\lambda)N}\ . \tag{4.84}$$

Furthermore, there is a real valued C^r *smooth transformation* $\mathcal{A}(\cdot,\vartheta,\lambda,N)$ *defined on a small neighbourhood of zero in* $\mathbb{R}$, *such that*

$$\mathcal{F}(\varrho,\vartheta,\lambda,N)\ =\ \mathcal{F}(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)+(\mathcal{A}(\varrho-\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N))^2\ . \tag{4.85}$$

Proof Within the first part of the proof we show that we can solve

$$D_1\mathcal{F}(\varrho,\vartheta,\lambda,N)=0 \tag{4.86}$$

for $\varrho=\varrho^*(\vartheta,\lambda,N)$ near $(0,0,0,\infty)$. This can be done by means of a procedure which takes its pattern from the proof of the Implicit Function Theorem (see [Zei93]). For that we construct a fixed point equation which is equivalent to (4.86). We will do this similar to the discussion in [Kno04].

Equation (4.86) reads more detailed

$$2\varrho-D_1\hat{\xi}_r(\varrho,\vartheta,\lambda,N)=0\ .$$

We write this equation as a fixed point equation

$$\varrho=\frac{1}{2}D_1\hat{\xi}_r(\varrho,\vartheta,\lambda,N):=\Xi(\varrho,\vartheta,\lambda,N)\ . \tag{4.87}$$

In the following we want to show that $\Xi(\cdot,\vartheta,\lambda,N)$ is contractive on a ball around zero. The definition of Ξ and assertion (4.83) imply that there are positive numbers ε_ϱ, ε_ϑ, ε_λ and ε_N such that for all ϱ, ϑ, λ and N with $\parallel\varrho\parallel<\varepsilon_\varrho$, $\parallel\vartheta\parallel<\varepsilon_\vartheta$, $\parallel\lambda\parallel<\varepsilon_\lambda$ and $N>\frac{1}{\varepsilon_N}$ we have

$$\parallel D_1\Xi(\varrho,\vartheta,\lambda,N)\parallel<c<1\ . \tag{4.88}$$

Invoking the mean value theorem we find

$$\Xi(\varrho',\vartheta,\lambda,N)-\Xi(\varrho'',\vartheta,\lambda,N)=D_1\Xi(\tilde{\varrho},\vartheta,\lambda,N)(\varrho'-\varrho'')\ ,$$

with $\tilde{\varrho}\in(\varrho',\varrho'')$. With (4.88) this gives that the mapping $\Xi(\cdot,\vartheta,\lambda,N)$ is contractive on the ball $B(0,\varepsilon_\varrho)\subset\mathbb{R}$ around 0 with radius ε_ϱ.

It remains to show that for λ and ϑ, both sufficiently small, and sufficiently large N the mapping $\Xi(\cdot,\vartheta,\lambda,N)$ maps the closed ball $\bar{B}(0,\varepsilon_\varrho)$ into itself. (Here $\bar{B}(0,\varepsilon_\varrho)$ denotes the closure of $B(0,\varepsilon_\varrho)\subset\mathbb{R}$.) Consider for $\varrho\in\bar{B}(0,\varepsilon_\varrho)$ and $\tilde{\varrho}\in(0,\varrho)$

$$\Xi(\varrho,\vartheta,\lambda,N)=\Xi(0,\vartheta,\lambda,N)+D_1\Xi(\tilde{\varrho},\vartheta,\lambda,N)\varrho$$

which gives with (4.88)

$$\begin{aligned}\| \Xi(\varrho,\vartheta,\lambda,N)\| &\leq \| \Xi(0,\vartheta,\lambda,N)\| + \| D_1\Xi(\tilde{\varrho},\vartheta,\lambda,N)\| \| \varrho \| \\ &\leq \| \Xi(0,\vartheta,\lambda,N)\| + c\,\varepsilon_\varrho\,.\end{aligned} \tag{4.89}$$

Because of (4.82) there are $\tilde{\varepsilon}_\vartheta<\varepsilon_\vartheta$, $\tilde{\varepsilon}_\lambda<\varepsilon_\lambda$ and $\tilde{\varepsilon}_N<\varepsilon_N$ such that for all ϑ with $\|\vartheta\|<\tilde{\varepsilon}_\vartheta$, λ with $\|\lambda\|<\tilde{\varepsilon}_\lambda$ and all N with $N>\frac{1}{\tilde{\varepsilon}_N}$

$$\| \Xi(0,\vartheta,\lambda,N)\| \leq (1-c)\varepsilon_\varrho$$

and hence the right-hand side of (4.89) is less than or equal to ε_ϱ. Thus the above assertion is proved.

Now, we can conclude by the Banach fixed point theorem that there is a unique solution $\varrho=\varrho^*(\vartheta,\lambda,N)$ of Equation (4.87) and hence of Equation (4.86). Applying the Implicit Function Theorem to the fixed point equation (4.87) at a solution point $(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)$ of this equation gives that $\varrho=\varrho^*(\cdot,\cdot,N)$ is differentiable.

Next we prove (4.84). For the solution $\varrho^*(\vartheta,\lambda,N)$ of (4.87) the mean value theorem and (4.88) yield

$$\begin{aligned}\| \varrho^*(\vartheta,\lambda,N)\| &= \| \Xi(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)\| \\ &\leq \| \Xi(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)-\Xi(0,\vartheta,\lambda,N)\| + \| \Xi(0,\vartheta,\lambda,N)\| \\ &\leq c\,\| \varrho^*(\vartheta,\lambda,N)\| + \| \Xi(0,\vartheta,\lambda,N)\|\end{aligned}$$

for some $c<1$ and hence

$$\| \varrho^*(\vartheta,\lambda,N)\| \leq \frac{1}{1-c}\| \Xi(0,\vartheta,\lambda,N)\| \;.$$

Now assertion (4.84) follows from (4.82).

In the last part of the proof we show the existence of the transformation $\mathcal{A}$. Because $\varrho^*(\vartheta,\lambda,N)$ solves Equation (4.86) there is a real valued, smooth function g such that

$$\begin{aligned}\mathcal{F}(\varrho,\vartheta,\lambda,N) &= \mathcal{F}(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)\\ &\quad + g(\varrho-\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)(\varrho-\varrho^*(\vartheta,\lambda,N))^2\,.\end{aligned}$$

Assertion (4.83) and

$$D_1^2\mathcal{F}(\varrho,\vartheta,\lambda,N) = 2 - D_1^2\hat{\xi}_r(\varrho,\vartheta,\lambda,N)$$

show that $D_1^2\mathcal{F}(0,0,0,N) > 0$ for sufficiently large N and hence

$$g(0,0,0,N) > 0\ .$$

We define

$$\mathcal{A}(\varrho,\vartheta,\lambda,N) := (g(\varrho,\vartheta,\lambda,N))^{\frac{1}{2}}\,\varrho$$

and get

$$\mathcal{F}(\varrho,\vartheta,\lambda,N) = \mathcal{F}(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N) + (\mathcal{A}(\varrho - \varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N))^2\ .$$

■

In order to find the symmetric one-periodic orbits we finally discuss the bifurcation equation (4.80). We define

$$\begin{aligned}\tilde{\xi}_r(\vartheta,\lambda,N) &:= \mathcal{F}(\varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)\ ,\\ \tilde{\varrho} &:= \mathcal{A}(\varrho - \varrho^*(\vartheta,\lambda,N),\vartheta,\lambda,N)\ .\end{aligned} \tag{4.90}$$

Invoking the above definitions and (4.85) the bifurcation equation (4.80) has the form

$$\lambda_1 + \mathfrak{c}(\lambda_2) - \tilde{\xi}_r(\vartheta,\lambda,N) = \tilde{\varrho}^2\ . \tag{4.91}$$

Henceforth, let $\tilde{\varrho} \geq 0$, because only for these values there is a meaningful geometrical interpretation of the bifurcation equation. Note further, that due to (4.80), (4.81) and (4.84), for sufficiently small $\vartheta \in \mathbb{R}$ and $\lambda \in \Lambda_0$ we have

$$\|\ \tilde{\xi}_r(\vartheta,\lambda,N)\ \| \to 0 \quad \text{for}\ \ N \to \infty\ . \tag{4.92}$$

We want to remark that our considerations do not provide any information regarding the sign of $\tilde{\xi}_r(\vartheta,\lambda,N)$.

Let us recall the bifurcation scenario concerning the existence of one-homoclinic orbits to the centre manifold in the case of a non-elementary primary homoclinic orbit Γ, where $D_1^2\lambda_1^*(0,0)$ is positive definite. This scenario is precisely described within Section 3.3.1. The bifurcation equation has the form $\lambda_1 + \mathfrak{c}(\lambda_2) = y_1^2 + y_2^2$ (see (4.77)). So, the existence of one-homoclinic solutions depends on the sign of $\lambda_1 + \mathfrak{c}(\lambda_2)$. Obviously there are solutions only for $\lambda_1 + \mathfrak{c}(\lambda_2) \geq 0$. Because the graph of $(y_1, y_2) \mapsto y_1^2 + y_2^2$ forms a paraboloid we know the number of one-homoclinic solutions: If $\lambda_1 + \mathfrak{c}(\lambda_2) > 0$ the level set $\lambda_1 + \mathfrak{c}(\lambda_2) = y_1^2 + y_2^2$ is a closed curve corresponding to infinitely many one-homoclinic orbits to the centre manifold. For $\lambda_1 + \mathfrak{c}(\lambda_2) = 0$ there is only one such orbit.

Henceforth we further assume that for all sufficiently small $\vartheta \in \mathbb{R}$, $\lambda \in \Lambda_0$ and sufficiently large $N \in \mathbb{N}$

(H 4.5) $\tilde{\xi}_r(\vartheta, \lambda, N) > 0$.

Let $N \in \mathbb{N}$ be sufficiently large. If $(\lambda_1, \lambda_2) \in \Lambda_0$ be such that $\lambda_1 + \mathfrak{c}(\lambda_2) > 0$ then (4.92) yields

$$\lambda_1 + \mathfrak{c}(\lambda_2) - \tilde{\xi}_r(\vartheta, \lambda, N) > 0 .$$

This means, that there exists a positive number $N_\lambda \in \mathbb{N}$ such that for all $N > N_\lambda$ we find a $\tilde{\varrho} \in \mathbb{R}$ such that the bifurcation equation (4.91) is fulfilled.
Consider now $(\lambda_1, \lambda_2) \in \Lambda_0$ with $\lambda_1 + \mathfrak{c}(\lambda_2) \leq 0$. Due to (H 4.5) we have

$$\lambda_1 + \mathfrak{c}(\lambda_2) - \tilde{\xi}_r(\vartheta, \lambda, N) < 0 .$$

This implies, that for those parameter values there are no solutions of the bifurcation equation.

So, concerning the existence of symmetric one-periodic orbits we are able to make the following statement. Here $U_\vartheta(0)$ is a neighbourhood of zero in $\mathbb{R}$. Recall, that $\mathfrak{C}$ is the graph of $\mathfrak{c} : \lambda_2 \mapsto \lambda_1 = \mathfrak{c}(\lambda_2)$.

Theorem 4.2.9 *Assume (H 3.2), (H 3.3) and (H 4.1) – (H 4.5). Then, for all parameter values $\lambda \in \Lambda_0$ above $\mathfrak{C}$, i.e., $\lambda_1 > \mathfrak{c}(\lambda_2)$, there exists $N_\lambda \in \mathbb{N}$, such that there is a two-parameter family $\{\mathcal{O}_{\vartheta,N}(\lambda) : \vartheta \in U_\vartheta(0),\, N \in \mathbb{N}\,,\, N > N_\lambda\}$ of symmetric one-periodic orbits. The difference of the periods of $\mathcal{O}_{\vartheta,N}(\lambda)$ and $\mathcal{O}_{\vartheta,N+1}(\lambda)$ is approximately 2π. If λ tends to $\mathfrak{C}$ then the period of the symmetric one-periodic orbits converges to infinity.*
For parameter values on $\mathfrak{C}$ as well as such values below this curve, i.e., $\lambda_1 \leq \mathfrak{c}(\lambda_2)$, symmetric one-periodic orbits do not exist.

In Figure 4.10 the whole bifurcation scenario of a non-elementary homoclinic orbit concerning the occurrence of one-homoclinic orbits to the centre manifold and symmetric one-periodic orbits is depicted: As stated in Theorem 3.3.4, for each parameter value above $\mathfrak{C}$ there exists a family of infinitely many one-homoclinic orbits $\mathcal{O}(y, \lambda)$ to the centre manifold (bold dotted lines). As in Figure 2.7 orbits of the centre manifold (dotted lines) that have an intersection point with a bold dotted line are limit set of a one-homoclinic orbit.
For fixed sufficiently small ϑ consider the subfamily $\{\mathcal{O}_{\vartheta,N}(\lambda) : N > N_\lambda\}$ (bold solid lines). As $N \to \infty$ the intersection points of the corresponding periodic orbits and Σ converge to a point in $\Sigma \cap W^{cs}_\lambda \cap W^{cu}_\lambda$ which belongs to a one-homoclinic orbit. In this sense we may consider the addressed subfamily being attached to this distinguished one-homoclinic orbit.
For parameter values on $\mathfrak{C}$ there are only one-homoclinic orbits while below $\mathfrak{C}$ there are neither one-homoclinic orbits nor symmetric one-periodic orbits.

We close this section with some further remarks concerning the globalisation of the (in terms of ϑ) local picture of the bifurcation behaviour of (1.1). The restriction of the two-parameter family of periodic orbits to parameter values ϑ near zero originates from the definition of Ω^* in Lemma 4.2.4, p^* in Lemma 4.2.5 and ϱ^* in

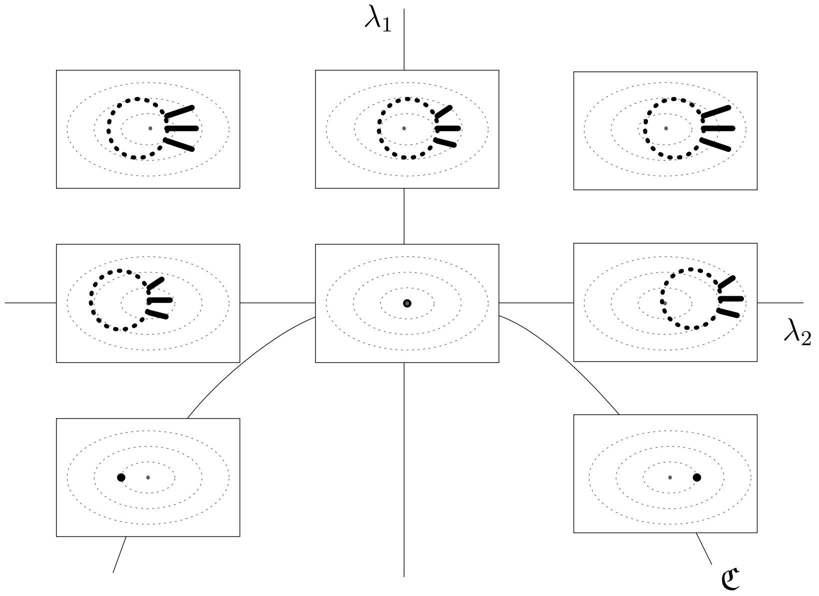

Figure 4.10: Bifurcation diagram concerning the existence of one-homoclinic orbits to the centre manifold and symmetric one-periodic orbits

Lemma 4.2.8. In the following discussion we give some ideas how to extend the functions Ω^*, p^* and ϱ^* to $\vartheta \in [0, 2\pi)$. First we do that for Ω^*.

Corollary 4.2.10 *Assume (H3.2), (H3.3) and (H4.1) – (H4.3). Let $\hat{\varepsilon}_\varrho$ and $\hat{\varepsilon}_\vartheta$ be as in Lemma 4.2.4. Let further $\tilde{\vartheta} \in \mathbb{R}$ such that $|\tilde{\vartheta}| < \hat{\varepsilon}_\vartheta$. Then there is a C^r smooth function $\tilde{\Omega}^* : (-\hat{\varepsilon}_\varrho, \hat{\varepsilon}_\varrho) \times (\tilde{\vartheta} - \hat{\varepsilon}_\vartheta, \tilde{\vartheta} + \hat{\varepsilon}_\vartheta) \times \Lambda_0 \to \mathbb{R}^+$ such that*

$$\varphi(\tilde{\Omega}^*(\varrho, \vartheta, \lambda), H^{cs}(y(\varrho, \vartheta), \lambda), \lambda) \in \Sigma_{loc} \quad \text{and} \quad \tilde{\Omega}^*(\varrho, \tilde{\vartheta}, \lambda) = \Omega^*(\varrho, \tilde{\vartheta}, \lambda) \; .$$

Proof First note, that for all (sufficiently small) $\tilde{\vartheta} \in (-\hat{\varepsilon}_\vartheta, \hat{\varepsilon}_\vartheta)$ there exists a number $\tilde{\Omega}_0$ near Ω_0 such that

$$Y^c_\Sigma(\tilde{\Omega}_0) = \operatorname{span}\{\hat{y}_1(\tilde{\Omega}_0)\} \; , \text{ where } \quad \hat{y}_1(\tilde{\Omega}_0) := \cos\tilde{\vartheta}\, \hat{y}_1 + \sin\tilde{\vartheta}\, \hat{y}_2 \; ,$$

where

$$D_2\varphi(\tilde{\Omega}_0, \gamma(0), 0)(\hat{y}_1(\tilde{\Omega}_0)) \subset \Sigma_{loc} \; .$$

Then, due to Remark 4.2.3 there are $\omega_i(\tilde{\Omega}_0)$ such that

$$D_2\varphi(\tilde{\Omega}_k, \gamma(0), 0)(\hat{y}_1(\tilde{\Omega}_0)) \subset \Sigma_{loc} \; , \quad \tilde{\Omega}_k := \tilde{\Omega}_0 + \sum_{i=1}^{k} \omega_i(\tilde{\Omega}_0) \; . \tag{4.93}$$

Now, proceeding as in the proof of Lemma 4.2.4 gives a uniquely determined function $\tilde{\Omega}^*(\cdot,\cdot,\cdot) : (-\hat{\varepsilon}_\varrho, \hat{\varepsilon}_\varrho) \times (\tilde{\vartheta} - \hat{\varepsilon}_\vartheta, \tilde{\vartheta} + \hat{\varepsilon}_\vartheta) \times \Lambda_0 \to \mathbb{R}^+$ such that $\varphi(\tilde{\Omega}^*(\varrho, \vartheta, \lambda), H^{cs}(y(\varrho, \vartheta), \lambda), \lambda) \in \Sigma_{loc}$.
The uniqueness of the functions Ω^* and $\tilde{\Omega}^*$ gives $\tilde{\Omega}^*(\varrho, \tilde{\vartheta}, \lambda) = \Omega^*(\varrho, \tilde{\vartheta}, \lambda)$. ■

In this way $\Omega^*(\varrho, \cdot, \lambda)$ can successively be extended to $[0, 2\pi]$. Consequently also $p^*(\varrho, \cdot, \lambda)$ can be extended to $[0, 2\pi]$, see Lemma 4.2.5. The extension of $\varrho^*(\cdot, \lambda, N)$ to $[0, 2\pi]$ can be achieved in principle by the same considerations as in Corollary 4.2.10. Hence, for each one-homoclinic orbit to the centre manifold $\mathcal{O}_{hom}(\varrho, \vartheta, \lambda)$, with $\vartheta \in [0, 2\pi]$, there is a corresponding one-parameter family $\{\mathcal{O}_{\vartheta,N}(\lambda) : N > N_\lambda\}$ of symmetric one-periodic orbits.

5 Discussion

Within this thesis we considered bifurcations from a homoclinic orbit Γ to a saddle-centre equilibrium $\mathring{x}$ of the reversible system (1.1). More precisely, we assumed that the linearisation of the vector field in $\mathring{x}$ has a pair of simple purely imaginary eigenvalues and real simple leading stable and unstable eigenvalues. Thus, due to the Liapunov Centre Theorem, the equilibrium has a two-dimensional centre manifold which is filled with symmetric periodic orbits, and hence it is uniquely determined.

Our issue is closely related to the work of Mielke, Holmes and O'Reilly, [MHO92], Koltsova and Lerman, [Ler91, KL95, KL96], and Champneys and Härterich, [CH00]. There the investigation of the dynamics near a homoclinic orbit to a saddle-centre equilibrium is based of return maps. Whereas in this thesis we used Lin's method to describe the bifurcation behaviour of (1.1). We refer to Chapter 2 for a detailed survey of the main ideas. Lin's method was developed for the discussion of the bifurcation scenario near heteroclinic cycles connecting hyperbolic equilibria. So, our aim was to find out how far Lin's method could be adapted to the present non-hyperbolic case. Our considerations make clear which geometrical assumptions are necessary for this approach.
Using Lin's method we could prove the existence of one-homoclinic orbits to the centre manifold in $\mathbb{R}^{2n+2}$, for $n \geq 1$. Notice, that the corresponding results are published in [KK03] and that our ideas have been applied successfully in [Wag02] and [WC02]. Furthermore, we detected symmetric one-periodic orbits in $\mathbb{R}^4$.

In the first part of this chapter we will address several problems which arose in the course of our analysis. Then we make some bibliographical notes. In particular we discuss the relation of our work to the one of Mielke, Holmes and O'Reilly, Koltsova and Lerman and Champneys and Härterich.

5.1 Open problems

The dynamics near a homoclinic orbit to a saddle-centre in reversible systems is not satisfyingly described yet. A corresponding discussion could be ordered by the codimension of the primary homoclinic orbit Γ; Γ is of codimension-one if W^{cs} and W^{cu} intersect transversally, it is of codimension-two if these manifolds intersect non-transversally. In the latter case one has to distinguish elementary and non-elementary homoclinic orbits. Recall, that a homoclinic orbit is called non-elementary if the intersection of W^{cs} and $\operatorname{Fix} R$ is non-transversal, otherwise we speak of an elementary homoclinic orbit.
In all cases the existence of k-homoclinic and k-periodic orbits as well as the oc-

currence of shift dynamics is largely unsolved. Symmetric one-periodic and two-homoclinic orbits have only been considered in $\mathbb{R}^4$, see Chapter 4 in this thesis and [CH00], respectively. A study of these objects in higher dimensions has still to be done.
In Section 5.2 we compare the dynamics investigated in this thesis with the one in related Hamiltonian scenarios. Similar considerations for more complicated dynamics are still pending.

As already mentioned we adapted Lin's method to our non-hyperbolic case. Some of the restrictions we made during our considerations have technical reasons. We explain this in the following.
The search for one-homoclinic orbits to the centre manifold, see Chapter 3, runs to a large extent parallel to the corresponding procedure in the classical version of Lin's method. Here we had "merely" to take into account that the linearisation of the system (1.1) has an exponential trichotomy.
In order to obtain symmetric one-periodic orbits we had to adapt Lin's method comprehensively, see Chapter 4. As the essential modification in comparison with the original method we described the dynamics near $\mathring{x}$ by a Poincaré map Π. This Poincaré map exists under assumption (H 4.2) which ensures the existence of a smooth locally invariant manifold that contains the stable and unstable manifold of (1.1). A statement from Delshams and Lázaro, [DL05], implies that in $\mathbb{R}^4$ this assumption is fulfilled if the vector field is analytic. Indeed, their results show that for each sufficiently small λ there is an analytic transformation which brings (1.1) in normal form

$$\dot{x} = \begin{pmatrix} \dot{x}^s \\ \dot{x}^u \\ \dot{x}^c_R \\ \dot{x}^c_{-R} \end{pmatrix} = \begin{pmatrix} x^s\, a_1(x^s x^u, (x^c_R)^2 + (x^c_{-R})^2, \lambda) \\ -x^u\, a_1(x^s x^u, (x^c_R)^2 + (x^c_{-R})^2, \lambda) \\ x^c_{-R}\, a_2(x^s x^u, (x^c_R)^2 + (x^c_{-R})^2, \lambda) \\ -x^c_R\, a_2(x^s x^u, (x^c_R)^2 + (x^c_{-R})^2, \lambda) \end{pmatrix} \tag{5.1}$$

with $x = (x^s, x^u, x^c_R, x^c_{-R}) \in X^s_{loc} \times X^u_{loc} \times X^c_{R,loc} \times X^c_{-R,loc}$ and scalar valued functions a_1 and a_2 satisfying $a_1(0,0,0) = -\mu$ and $a_2(0,0,0) = 1$. Hence, $X^s \oplus X^u$ is obviously flow invariant.
We want to mention that the considerations in [DL05] have been carried out for single vector fields only. In our context it is obvious that the assumptions for the normal form transformation are fulfilled for each λ, but the smooth dependence of the transformation on λ has not been proved. Nevertheless, we use this normal form to make clear that the assumption (H 4.2) can be satisfied. However, it would be desirable to have a λ-dependent version of the addressed normal form result.
Further, this normal form approach for proving the existence of the above mentioned invariant hyperbolic manifold is inapplicable for C^k-vector fields. To our knowledge it is an open problem whether the existence of such an invariant manifold can be concluded from the reversibility of a C^k-vector field.

The Poincaré map defined in the thesis can be seen as an R-reversible map $\Pi(\cdot,\lambda)$: $U(0) \subset \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, with $\dim \operatorname{Fix} R = n+1$ (and $U(0)$ is a neighbourhood of 0). The map $\Pi(\cdot,\lambda)$ possesses a family of symmetric fixed points $\{p(\zeta),\ \zeta \in \mathbb{R}\} \subset X^c_{R,loc}$. This family is related to the family of periodic orbits of the centre manifold. For each small $p \in X^c_{loc}$ the linearisation $D_1\Pi(p,\lambda)$ has a simple eigenvalue one.
For the detection of periodic Lin orbits we used also assumption (H 4.3), which ensures the existence of a smooth foliation of $\mathbb{R}^{2n+1}$ into Π- and R-invariant leaves $\mathcal{M}_{p,\lambda}$, $p \in X^c_{R,loc}$. The Poincaré map restricted to each such leaf is hyperbolic. Thus, for the consideration of $\Pi(\cdot,\lambda)|_{\mathcal{M}_{p,\lambda}}$ we could use techniques which are applicable to hyperbolic maps.
If the vector field is analytic we may assume that it is in normal form (5.1). Then the assumption (H 4.3) is satisfied because $(x^c_R)^2 + (x^c_{-R})^2$ is a first integral of (5.1): The manifolds

$$\{(x^s, x^u, x^c_R, x^c_{-R}) \in X^s_{loc} \times X^u_{loc} \times X^c_{R,loc} \times X^c_{-R,loc} : (x^c_R)^2 + (x^c_{-R})^2 = c\}\ ,$$

where c is a positive real number, are invariant with respect to the flow of (5.1). Therefore, for each $\lambda \in \Lambda_0$ (that means for each sufficiently small $\lambda \in \mathbb{R}^2$), the linear manifolds

$$M^{\pm}_c := \{(x^s, x^u, x^c_R, x^c_{-R}) \in X^s_{loc} \times X^u_{loc} \times X^c_{R,loc} \times X^c_{-R,loc} :\ x^c_R = \pm\sqrt{c}\,,\ x^c_{-R} = 0\}$$

are invariant with respect to $\Pi(\cdot,\lambda)$ and we can define

$$\mathcal{M}_{p,\lambda} := X^h_{loc} \times \{p\} \quad \text{with}\ \ p \in X^c_{R,loc}$$

as the desired manifold. Further, it is clear that

$$\bigcup_{p \in X^c_{R,loc},\ \lambda \in \Lambda_0} X^h_{loc} \times \{p\} \times \{\lambda\}$$

is a smooth foliation of a neighbourhood of the origin in $\mathbb{R}^{2n+1} \times \mathbb{R}^2$. Finally, the leaves $\mathcal{M}_{p,\lambda}$ are R-invariant, because $p \in X^c_{R,loc} \subset \operatorname{Fix} R$ and $RX^h_{loc} = X^h_{loc}$.

So, altogether the normal form (5.1) ensures both (H 4.2) and (H 4.3). Following up our remarks concerning the normal form we want to pose the question for an alternative approach for proving the existence of the mentioned smooth foliation. Besides a procedure for constructing the partial orbits is conceivable which does not rely on the Π-invariant hyperbolic leaves $\mathcal{M}_{p,\lambda}$. Recall, that we considered the hyperbolic map $\Pi(\cdot,\lambda)|_{\mathcal{M}_{p,\lambda}}$ to conclude to the existence of partial orbits by using a λ-lemma (which is only true near hyperbolic equilibria). So, the existence of the hyperbolic leaves would be dispensable if we had a reversible λ-lemma (for non-hyperbolic equilibria) at our disposal. Recently we could take first steps towards such a λ-lemma, [KKW06].
During the discussion of the existence of symmetric periodic orbits near Γ we restricted our considerations to vector fields in $\mathbb{R}^4$. In that case the cross-section Σ

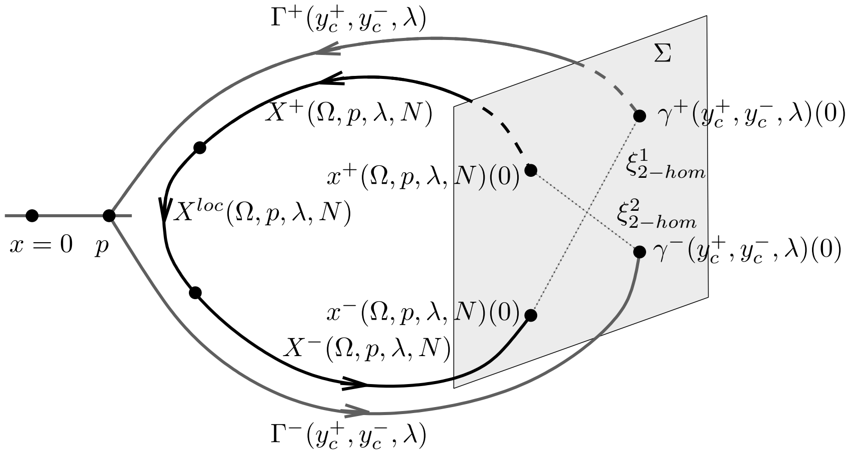

Figure 5.1: Illustration of the bifurcation equations of two-homoclinic orbits to the centre manifold

coincides with the direction Z of the jumps, more precisely $\Sigma = \gamma(0) + Z$. Therefore each partial orbit is automatically a Lin orbit. This allows to give bifurcation equations for k-homoclinic and k-periodic orbits for each $k \in \mathbb{N}$. For example, as depicted in Figure 5.1, two-homoclinic orbits to the centre manifold are determined by the bifurcation equations

$$\xi^1_{2-hom} = \gamma^+(y_c^+, y_c^-, \lambda)(0) - x^-(\Omega, p, \lambda, N)(0) = 0 \,,$$

$$\xi^2_{2-hom} = \gamma^-(y_c^+, y_c^-, \lambda)(0) - x^+(\Omega, p, \lambda, N)(0) = 0 \,,$$

where $\gamma^\pm$ are solutions corresponding to a one-homoclinic Lin orbit to the centre manifold (see Lemma 3.1.4) and $x^\pm$ are parts of a solution corresponding to a symmetric one-periodic Lin orbit (see Lemma 4.1.12). In case of a symmetric orbit ξ^1_{2-hom} and ξ^2_{2-hom} are R-images of each other and consequently the corresponding bifurcation equations reduce to a single equation.
In $\mathbb{R}^{2n+2}$, $n > 1$, no longer any consecutive partial orbits build a Lin orbit. In order to construct Lin orbits in this case we had to consider a hybrid system. This system consists of a discrete system describing the dynamics in a neighbourhood U of the equilibrium and a differential equation for the description of the solutions near Γ outside U. In this way one should be able to prove that for each (Ω, p, λ, N) there is a unique symmetric one-periodic Lin orbit $\mathcal{L}(\Omega, p, \lambda, N)$. Then our results concerning periodic orbits remain true in higher dimensions.
Similar hybrid systems are considered in [Rie06] for the case of a primary heteroclinic cycle connecting a hyperbolic periodic orbit and a hyperbolic equilibrium.

In this thesis we discussed symmetric one-periodic orbits near a non-elementary homoclinic orbit Γ in $\mathbb{R}^4$. We considered the corresponding bifurcation equation as perturbation of the bifurcation equation $\hat{\xi}^\infty(y, \lambda_1, \lambda_2) = 0$ for one-homoclinic orbits to the centre manifold. The equation could be solved for $\lambda_1^*(\cdot,\cdot) : (y,\lambda_2) \mapsto \lambda_1^*(y,\lambda_2)$, with $\lambda_1^*(0,0) = 0$ and $D_1\lambda_1^*(0,0) = 0$. We assumed that $D_1^2\lambda_1^*(0,0)$ is positive definite. This finally allowed to assume, that for an appropriate function $\mathfrak{c}(\cdot)$ and $y = (y_1, y_2)$, $\hat{\xi}^\infty$ is given by

$$\hat{\xi}^\infty = \lambda_1 + \mathfrak{c}(\lambda_2) - y_1^2 - y_2^2.$$

The main argument is a generalised (that means parameter dependent) Morse lemma, see [Nir01].

We did not discuss the case where $D_1^2\lambda_1^*(0,0)$ is indefinite. But here we could act as described above discussing perturbations of $\lambda_1 + \mathfrak{c}(\lambda_2) + y_1^2 - y_2^2$. In a similar manner, the case of an elementary homoclinic orbit Γ could be discussed. The corresponding bifurcation equation for one-homoclinic orbits has the form $\hat{\xi}^\infty(y_1,y_2,\lambda_1,\lambda_2) = y_2\psi(y_1,y_2,\lambda_1,\lambda_2)$, for an appropriate function ψ satisfying $\psi(0,0,0,0) = 0$. Further we assumed $D_1\psi(0,0,0,0) \neq 0$. Therefore, the leading term in the Taylor expansion of $\hat{\xi}^\infty$ is an indefinite quadratic form. So, again applying the parameter dependent Morse lemma, we could assume $\hat{\xi}^\infty$ to be in the form y_1y_2 and discuss perturbations of $y_1y_2 = 0$ to gain periodic orbits.

5.2 Bibliographical notes

One aim of this section is to compare reversible and Hamiltonian dynamics. We point out the differences in the underlying geometry. Thereby we discuss the possible relative positions of the traces of the centre-stable manifold W^{cs}, the centre-unstable manifold W^{cu} and the fixed space $\operatorname{Fix} R$ within a cross-section Σ.

We summarise briefly the results of Mielke, Holmes and O'Reilly, Koltsova and Lerman and Champneys and Härterich. Mielke, Holmes and O'Reilly, [MHO92], considered a two-parameter unfolding of a homoclinic orbit to a saddle-centre in a Hamiltonian system in $\mathbb{R}^4$, which is additionally reversible. Using a return map and a special normal form they detected k-homoclinic orbits to the equilibrium. Further, they concluded to the existence of horse shoes. Besides Koltsova and Lerman, [Ler91, KL95, KL96], made similar consideration in purely Hamiltonian systems. In addition they found families of homoclinic orbits to periodic orbits in the centre manifold. This could be done also in higher dimensions. Champneys and Härterich, [CH00], focussed on bifurcating two-homoclinic orbits to a saddle-centre equilibrium in purely reversible systems in $\mathbb{R}^4$. In $\mathbb{R}^4$ one-homoclinic and two-homoclinic orbits to the equilibrium cannot exist simultaneously because stable and unstable manifolds to the equilibrium are one-dimensional. Note, that in reversible systems the splitting of the stable and unstable manifolds (along a homoclinic orbit to a saddle-centre) will be controlled by only one parameter. This is due to the fact that the stable and unstable manifolds are R-images of each other. Consequently, the stable and unstable manifolds (within Σ) can only split off in a direction which is transversal to $\operatorname{Fix} R$. In other words Champneys and Härterich study a generic one-parameter unfolding of a homoclinic orbit to a saddle-centre.

In the following we discuss which geometrical scenarios admit an additional Hamiltonian structure of the vector field. In this case the stable and the unstable manifold of the equilibrium lie in the same level set. By counting dimensions we see that the splitting of these manifolds can be controlled by two parameters. (In other words, in Hamiltonian systems a homoclinic orbit to a saddle-centre is a codimension-two object, while in reversible systems it is of codimension one.) Further, due to the structure of the centre manifold (Liapunov family of periodic orbits) the restriction of the Hamiltonian H to this manifold has an extremum (say a minimum) in the equilibrium $\mathring{x}$. Let $H(\mathring{x}) = 0$. Then H takes only non-negative values on both W^{cs}_{loc} and W^{cu}_{loc}. This situation can be carried to a neighbourhood of $\gamma(0)$. So, the traces of both W^{cs} and W^{cu} in Σ are located on the same side of the $2n$-dimensional submanifold $\{H = 0\} \cap \Sigma$ of Σ. (We assume that H is non-singular along Γ.) Thus, only in the elementary case we are in a situation which is comparable with the one for Hamiltonian systems. Furthermore, these considerations show that in contrast to the reversible case, a transverse intersection of W^{cs} and W^{cu} in $\gamma(0)$ is impossible; the tangent spaces $T_{\gamma(0)}W^{cs}$ and $T_{\gamma(0)}W^{cu}$ are subsets of $T_{\gamma(0)}\{H = 0\}$.

As already mentioned Koltsova and Lerman, [KL96], focussed on one-homoclinic orbits to the centre manifold. They proved that on each level of the Hamiltonian $H = c$ with $c > 0$, the unperturbed system contains four homoclinic orbits to a periodic orbit in the centre manifold. Under perturbation these homoclinic orbits persist in level sets bounded away from that where the saddle-centre is located. Notice, that different orbits in the centre manifold are in different level sets of the Hamiltonian. In particular only for the critical parameter value there is a one-homoclinic orbit to the equilibrium. It turns out that their results agree to a large extent with our results obtained in the elementary case, see Theorem 3.3.10. There we found, that correspondingly periodic orbits in the centre manifold are limit sets of four homoclinic orbits to the centre manifold. We know that in each case two of them are symmetric, and therefore they are homoclinic to this periodic orbit. The both remaining homoclinic orbits are R-images of each other; they may connect different periodic orbits of the centre manifold. Further, in contrast to the Hamiltonian case in our case there is a line in the parameter plane for which one-homoclinic orbits to the equilibrium occur. This difference stems from the fact that in reversible systems a transversal intersection of W^{cs} and W^{cu} is possible, while the Hamiltonian structure prevents such a constellation.

We present a typical bifurcation scenario for one-homoclinic orbits to the centre manifold in the case of a transversal intersection of W^{cs}_{λ} and W^{cu}_{λ} as a part of the following scenario. Let Γ be elementary and the leading term of the function h^{cs}, which describes $W^{cs} \cap \Sigma$, be of third order. Indeed, W^{cs} and W^{cu} intersect non-transversally in $\gamma(0)$, see Figure 5.2. But, the announced transversal intersection appears in the unfolding of this situation. The corresponding analysis has not been worked out within this thesis. However, we obtain a bifurcation diagram for one-homoclinic orbits to the centre manifold as depicted in Figure 5.3. There (for the

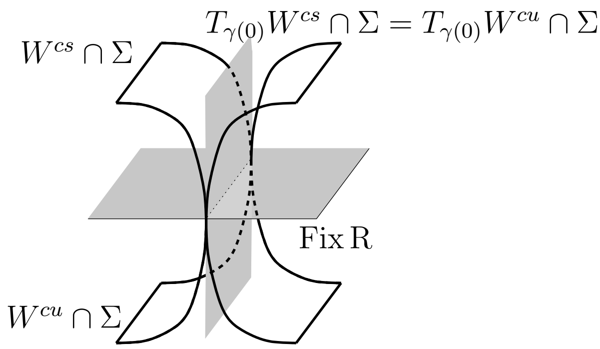

Figure 5.2: Intersection of W^{cs} and W^{cu} within Σ if the leading term of h^{cs} is of third order

$\mathbb{R}^4$-case) the relative positions of $W^{cs}_\lambda \cap \Sigma$ and $W^{cu}_\lambda \cap \Sigma$ are drawn. The intersection points of these manifolds (dotted lines) correspond to one-homoclinic orbits to the centre manifold. Near $\lambda = 0$, there exists exactly one one-parameter family of symmetric one-homoclinic orbits to the centre manifold. For all parameter values above $\mathcal{K}$ additionally two one-parameter families of non-symmetric one-homoclinic orbits to the centre manifold exist. In particular, the traces of W^s_λ and W^u_λ within Σ are indicated by the black rhombuses. Then, for fixed $\lambda_2 < 0$, the lower line in Figure 5.3 represents the typical bifurcation scenario in the case of a transversal intersection.

With this thesis we supplement the considerations of Champneys and Härterich in [CH00]. They proved the existence of families of two-homoclinic orbits to the equilibrium. Thereby, they made an assumption, condition (H5) in [CH00], which roughly speaking says that by varying the parameter the trace of the stable manifold intersects the trace of W^{cu} transversally. This transversality condition is fulfilled if W^{cs} and W^{cu} intersect transversally – see last "line" in Figure 5.3. Let there $\lambda_2 \neq 0$ be fixed, then tracking the rhombus representing the stable manifold while moving λ_1 through zero shows that condition (H5) of [CH00] is satisfied. This transversality condition is also fulfilled if Γ is non-elementary. In this case the verification of this non-zero speed condition can be read from Figure 1.2. Therefore in these cases we have (additionally to our results) families of two-homoclinic orbits to the equilibrium as stated in [CH00].

Condition (H5) in [CH00] is incompatible with a Hamiltonian structure, see [CH00] or explanations given above concerning Hamiltonian systems.

Homoclinic orbits to a saddle centre are often encountered in applications. So, first numerical investigations of homoclinic orbits to a saddle-centre are known from the study of the three-body problem, [DH65]. Further, the discussion of reversible Hamiltonian systems with a saddle-centre is relevant in studies of certain water wave equations, [AK89, Mie91].

The discussion of purely reversible systems with a saddle-centre is important in

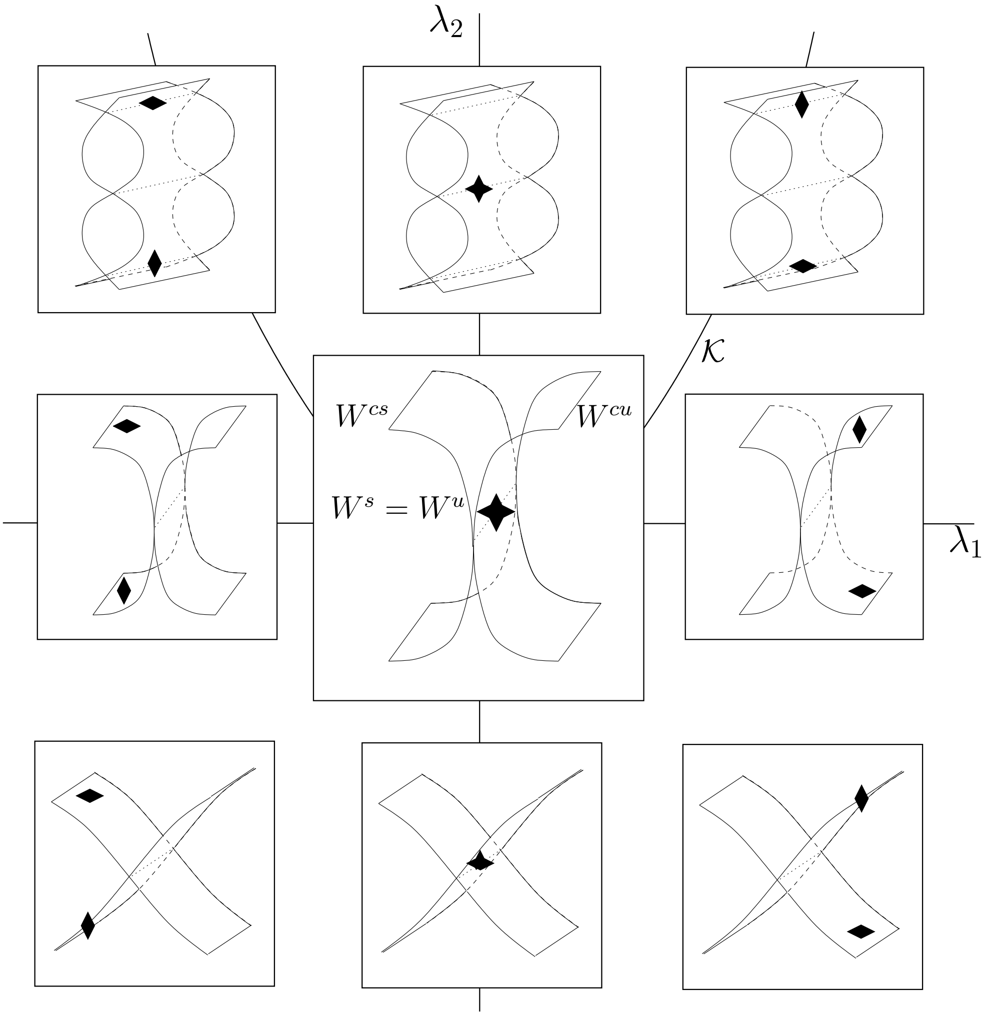

Figure 5.3: Bifurcation diagram for an elementary homoclinic orbit Γ, where the leading term of h^{cs} is of third order, concerning the occurrence of one-homoclinic orbits to the centre manifold

non-linear optics and water waves since homoclinic orbits to a saddle-centre describe *embedded solitons* in systems of partial differential equations representing physical models. Except for special cases their existence has been studied only numerically. However, in [WY06] an analytical approach based on an idea similar to that one of the Melnikov method is presented. Within the context of travelling wave equations homoclinic orbits to periodic orbits of the centre manifold are of interest because they correspond to *quasi-solitons* in the governing partial differential equation, [KCBS02].

Saddle-centre equilibria can occur during local bifurcations of equilibria, as discussed in [Wag03]. There bifurcations of a homoclinic orbit in a reversible system with an equilibrium with double, non-semisimple eigenvalue 0 and stable and unstable eigenvalues are considered. The theory is related to a model system which stems from a problem in non-linear optics.

Finally, we want to mention that there are papers of Bernard, Grotta Ragazzo and Salomao, [GrR97, BGRS03], where families of Hamiltonian systems on a 4-dimensional symplectic manifold possessing a homoclinic orbit asymptotic to a saddle-centre equilibrium are considered. Among other objects they proved the existence of families of periodic orbits and homoclinic orbits which are asymptotic to periodic orbits in the centre manifold

A Appendix

In this chapter we present a collection of important, general results used in this thesis. First we present a collection of results about reversible systems. There are various introductions to this subject, hence our presentation here is short and is only included to keep the thesis self-contained. We concentrate on results that have been of immediate use for the analysis. We often omit proofs and refer to the literature. Further we introduce the notion of exponential trichotomies. This is closely related to exponential dichotomies introduced by Coppel in [Cop78]. We give a slight generalisation of Coppel's definition and prove a corresponding roughness theorem. We apply this assertion to define a roughness theorem for exponential trichotomies. With that it becomes clear that variational equations of the type which we consider within this thesis have exponential trichotomies.

A.1 Reversible systems

To a large extent the following explanations follow Vanderbauwhede and Fiedler (see [VF92]). For further information about reversible systems we refer to [Dev76] and [LR98].

Let R be a linear involution on $\mathbb{R}^m$, that is a linear mapping satisfying $R^2 = id$. By R two subspaces of $\mathbb{R}^m$ are distinguished:

$$\operatorname{Fix} R := \{x \in \mathbb{R}^m : Rx = x\} \quad \text{and} \quad \operatorname{Fix}(-R) := \{x \in \mathbb{R}^m : Rx = -x\} . \tag{A.1}$$

We have the decomposition

$$\mathbb{R}^m = \operatorname{Fix} R \oplus \operatorname{Fix}(-R) .$$

Note, that $\frac{1}{2}(id + R)$ is a projection which realises this decomposition. Restricting $\frac{1}{2}(id + R)$ to a subspace $U \subset \mathbb{R}^m$ we get

Lemma A.1.1 *Let R be an involution on $\mathbb{R}^m$, and let $U \subset \mathbb{R}^m$ be an R-invariant subspace. Then, the subspace U has a direct sum decomposition into subspaces of* $\operatorname{Fix} R$ *and* $\operatorname{Fix}(-R)$

$$U = U^R \oplus U^{-R} , \quad U^R \subset \operatorname{Fix} R , \quad U^{-R} \subset \operatorname{Fix}(-R) .$$

Proof Define $\mathcal{P} := \frac{1}{2}(id + R)\big|_U$, then $\operatorname{im} \mathcal{P} = U^R$ and $\ker \mathcal{P} = U^{-R}$. ■

Lemma A.1.2 *Let R be an involution on $\mathbb{R}^m$, and let $W_1, \ldots, W_l$ be R-invariant subspaces, i.e., $RW_i = W_i$ for $i \in \{1, \ldots, l\}$, with $W_1 \oplus \ldots \oplus W_l = \mathbb{R}^m$. Then, there is an R-invariant scalar product $\langle \cdot, \cdot \rangle_R$ defined on $\mathbb{R}^m$, i.e., $\langle Rx, Ry \rangle_R = \langle x, y \rangle_R$ $\forall x, y \in U$, such that*

$$\langle w_i, w_j \rangle_R = 0 \quad \forall w_i \in W_i,\ w_j \in W_j,\ i \neq j\ ,\ i, j \in \{1, \ldots, l\}\ .$$

Proof Choose a scalar product $(\cdot, \cdot)$ on $\mathbb{R}^m$ such that

$$(w_i, w_j) = 0 \quad \forall w_i \in W_i\ ,\ w_j \in W_j\ . \tag{A.2}$$

Then $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle_R := (\cdot, \cdot) + (R\cdot, R\cdot)$$

is an R-invariant scalar product on U. Because of $RW_i = W_i$ and (A.2)

$$\langle w_i, w_j \rangle_R = 0 \quad \forall w_i \in W_i\ ,\ w_j \in W_j\ .$$

■

***R*-reversibility of differential equations** The equation

$$\dot{x} = f(x), \quad f \in C^k(\mathbb{R}^m), \ \ k \geq 2\ , \tag{A.3}$$

is said to be ***R-reversible*** if

$$Rf(x) = -f(Rx)\ , \quad \forall x \in \mathbb{R}^m\ .$$

Then, the flow $\varphi(t, \cdot)$ of (A.3) satisfies

$$R\varphi(t, x) = \varphi(-t, Rx)\ , \quad \forall t \in \mathbb{R}\ \ \forall x \in \mathrm{dom}(\varphi(t, \cdot))\ . \tag{A.4}$$

The reversibility of Equation (A.3) yields that if X is an orbit of (A.3), then so is RX. An orbit X is called ***symmetric*** if $RX = X$.

Furthermore, we find that an orbit X of an R-reversible system is symmetric if and only if

$$X \cap \mathrm{Fix}\, R \neq \emptyset\ .$$

More precisely, a symmetric orbit has either one or two intersection points with $\mathrm{Fix}\, R$. If an orbit has two intersection points with $\mathrm{Fix}\, R$ then this orbit is periodic. Hence a symmetric homoclinic orbit intersects $\mathrm{Fix}\, R$ in exactly one point.

Let $x_0 \in \mathbb{R}^m$ be a symmetric equilibrium of (A.3). Then we find that

$$RDf(x_0) = -Df(x_0)R\ . \tag{A.5}$$

Let $\sigma(Df(x_0))$ denote the spectrum of $Df(x_0)$, then with $\mu \in \sigma(Df(x_0))$ we also have $-\mu \in \sigma(Df(x_0))$. If furthermore $Df(x_0)$ is non-singular then (A.5) implies

$$\dim \mathrm{Fix}\, R = \dim \mathrm{Fix}\, (-R)\ .$$

Note here, that x_0 not necessarily has to be hyperbolic.

Reversible systems with a saddle-centre equilibrium In accordance with the main assumption of this thesis we consider the R-reversible equation (A.3) with $m = 2n + 2$. Further, we assume the existence of an equilibrium $\mathring{x} = 0$ such that

$$\begin{aligned} \sigma(Df(0)) &= \sigma^c \cup \sigma^u \cup \sigma^s \\ &= \{i, -i\} \cup \{\mu_1 =: \mu, \mu_2, \dots, \mu_n\} \cup \{-\mu, -\mu_2, \dots, -\mu_n\} \end{aligned} \tag{A.6}$$

with $\Re\mu$, $\Re\mu_i > 0$, $\Re\mu < \Re\mu_i$, $i = 2, \dots, n$.

Let X^s, X^u and X^c be the generalised eigenspaces of $Df(0)$ corresponding to σ^s, σ^u and σ^c, respectively. Note, that the stable and unstable eigenspaces, X^s and X^u, are R-images of each other. Moreover, the centre eigenspace is R-invariant:

$$RX^s = X^u \quad \text{and} \quad RX^c = X^c \,. \tag{A.7}$$

Due to Lemma A.1.1, every R-invariant subspace of $\mathbb{R}^{2n+2}$ can be decomposed into a subspace of $\operatorname{Fix} R$ and a subspace of $\operatorname{Fix}(-R)$. Thus we can write

$$X^s \oplus X^u = X^h_R \oplus X^h_{-R} \,, \tag{A.8}$$

$$X^c = X^c_R \oplus X^c_{-R} \,, \tag{A.9}$$

where X^h_R, $X^c_R \subset \operatorname{Fix} R$ and X^h_{-R},$X^c_{-R} \subset \operatorname{Fix}(-R)$. Let further $\{e^s_1, \dots, e^s_n\}$ be a basis of X^s. Then, $\{e^s_i + Re^s_i,\ i = 1, \dots, n\}$ is a basis of $\operatorname{Fix} R \cap (X^s \oplus X^u)$ and $\{e^s_i - Re^s_i,\ i = 1, \dots, n\}$ is a basis of $\operatorname{Fix}(-R) \cap (X^s \oplus X^u)$, and we get $\dim X^h_R = \dim X^h_{-R} = n$, and hence by counting dimensions $\dim X^c_R = \dim X^c_{-R} = 1$.

Due to the above considerations we can choose the following basis in $\mathbb{R}^{2n+2}$:

$$\{e^s_1, \dots, e^s_n, Re^s_1, \dots, Re^s_n, e^c_R, e^c_{-R}\} \,,$$

with $e^s_i \in X^s$, $i = 1, \dots, n$, $e^c_R \in X^c_R$ and $e^c_{-R} \in X^c_{-R}$. In these coordinates R can be identified with the matrix

$$R = \begin{pmatrix} 0 & id & 0 & 0 \\ id & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Furthermore $Df(0)$ has the form

$$Df(0) = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \tag{A.10}$$

where A is the matrix representation of $Df(0)|_{X^u}$ with respect to the above basis.

In the following we outline the definition of centre, centre-stable and centre-unstable manifold of $\mathring{x}$ and some useful properties. First, we restrict our considerations to a neighbourhood of $\mathring{x}$ in $\mathbb{R}^{2n+2}$ and define the local manifolds.
Following [Van89] we call a manifold W^{cs}_{loc} a local centre-stable manifold if the following conditions are satisfied:

(i) $\varphi(t, W^{cs}_{loc}) \subset W^{cs}_{loc}$, $t \in \mathbb{R}$;

(ii) $T_{\mathring{x}} W^{cs}_{loc} = X^c \oplus X^s$.

Similarly, we can define a local centre-unstable manifold W^{cu}_{loc}.

Let W^{cs}_{loc} be a local centre-stable manifold, then due to (A.4) and (A.7)

$$W^{cu}_{loc} := RW^{cs}_{loc} \tag{A.11}$$

is a local centre-unstable manifold. With that a local centre manifold can be defined by

$$W^{c}_{loc} = W^{cs}_{loc} \cap W^{cu}_{loc} \ . \tag{A.12}$$

The behaviour in a neighbourhood of the equilibrium $\mathring{x}$ is described by the *Liapunov Centre Theorem* (see [Dev76]). This theorem says that under the condition (A.6) there exists a smooth, two-dimensional, invariant manifold containing $\mathring{x}$. This manifold consists of a nested, one-parameter family of symmetric periodic orbits. Moreover, the periods of the closed orbits tend to 2π as the initial conditions tend to $\mathring{x}$.
The centre manifold contains all bounded orbits, so the Liapunov Centre Theorem gives the uniqueness of the local centre manifold. Thus the local centre-stable manifold and the centre-unstable manifold are uniquely determined, too.

The centre-stable and the centre-unstable manifold can be globalised by

$$W^{cs} = \bigcup_{t \in \mathbb{R}^-} \varphi(t, W^{cs}_{loc}) \quad \text{and} \quad W^{cu} = \bigcup_{t \in \mathbb{R}^+} \varphi(t, W^{cu}_{loc}) \ . \tag{A.13}$$

Equations (A.4), (A.11) – (A.13) imply

$$RW^{cs} = W^{cu} \quad \text{and} \quad RW^{c}_{loc} = W^{c}_{loc} \ ,$$

$$RT_q W^{cs} = T_{Rq} W^{cu} \ \ \forall q \in W^{cs} \quad \text{and} \quad RT_q W^{c}_{loc} = T_{Rq} W^{c}_{loc} \ \ \forall q \in W^{c}_{loc} \ .$$

***R*-reversibility of discrete systems** Let $\Pi : \mathbb{R}^m \to \mathbb{R}^m$ be a C^r diffeomorphism. Then the discrete system

$$x(n+1) = \Pi(x(n)) \ , \quad n \in \mathbb{Z} \ , \tag{A.14}$$

is called ***R-reversible*** with respect to a linear involution R on $\mathbb{R}^m$ if

$$R\Pi(x) = \Pi^{-1}(Rx) \; .$$

Iterating this equation we obtain

$$R\Pi^n(x) = \Pi^{-n}(Rx) \quad \forall n \in \mathbb{Z}$$

as an analogue of Equation (A.4). Hence, with an orbit $x(n)$ of (A.14), RX is an orbit of (A.14), too. Again, if an orbit X satisfies $RX = X$ then the orbit is called ***symmetric***.
Let $x_0 \in \mathbb{R}^m$ be a symmetric fixed point of (A.14). Then obviously we have

$$RD\Pi(x_0) = D\Pi^{-1}(x_0)R \; . \tag{A.15}$$

Let $\sigma(D\Pi(x_0))$ denote the spectrum of $D\Pi(x_0)$, then with $\nu \in \sigma(D\Pi(x_0))$ we also have $\nu^{-1} \in \sigma(D\Pi(x_0))$.

A.2 Exponential Trichotomies

In what follows we explain the concept of *exponential trichotomies*, which generalises the idea of stable, centre and unstable subspaces to non-autonomous equations (see [HL86] and [CL90]). Moreover, this notion is closely related to the one of exponential dichotomies introduced by Coppel in [Cop78]. (For further information regarding exponential dichotomies we refer to [Pal84].)
We start by giving a slight generalisation of Coppel's definition of an exponential dichotomy. Consider the linear differential equation

$$\dot{x} = A(t)x \; , \quad x \in \mathbb{R}^k \; , \tag{A.16}$$

with transition matrix $\Phi(\cdot,\cdot)$.

Definition A.2.1 *Equation (A.16), or the transition matrix Φ, respectively, is said to have an **exponential dichotomy on $[t_0, \infty)$ with constants α, β** if there exist projections $P(t)$ on $\mathbb{R}^k$, $t \in [t_0, \infty)$, and constants α and β with $\alpha < \beta$ and a positive constant C such that $\Phi(t,\tau)P(\tau) = P(t)\Phi(t,\tau)$ for all $t, \tau \in [t_0, \infty)$, and*

$$\begin{aligned} \|\Phi(t,\tau)P(\tau)\| &\leq Ce^{\alpha(t-\tau)}, \quad t \geq \tau \geq t_0 \, , \\ \|\Phi(\tau,t)(id - P(t))\| &\leq Ce^{\beta(\tau-t)}, \quad t \geq \tau \geq t_0 \, . \end{aligned} \tag{A.17}$$

The projections are called associated to the exponential dichotomy.

If $t_0 = 0$ we say that (A.16) has an exponential dichotomy on $\mathbb{R}^+$. In the same way we define exponential dichotomies on $(-\infty, t_0]$ or $\mathbb{R}^-$, respectively. In our further explanations we confine to exponential dichotomies on $[t_0, \infty)$. Note, that the only

difference to Coppel's notion is that we do not require $\operatorname{sgn}\alpha \neq \operatorname{sgn}\beta$ (more precisely, in the classical version one has $\alpha < 0 < \beta$). Further, we want to remark that the projections $P(t)$ are determined by $P(t_0)$.
For $\xi \in \operatorname{im} P(\tau)$ the norm of solution $\Phi(t,\tau)\xi$ can be estimated by

$$\|\Phi(t,\tau)\xi\| \leq Ce^{\alpha(t-\tau)}\|\xi\|. \tag{A.18}$$

Now let $\xi \in \operatorname{im}(id - P(\tau))$. Then $\psi := \Phi(t,\tau)\xi \in \operatorname{im}(id - P(t))$. With that we find

$$\|\xi\| = \|\Phi(\tau,t)(id - P(t))\psi\| \leq Ce^{\beta(\tau-t)}\|\psi\| = Ce^{\beta(\tau-t)}\|\Phi(t,\tau)\xi\|,$$

and hence

$$\frac{1}{C}\|\xi\|e^{\beta(t-\tau)} \leq \|\Phi(t,\tau)\xi\|. \tag{A.19}$$

In other words solutions of (A.16) starting at time τ in $\operatorname{im} P(\tau)$ have an exponential upper bound while solutions of (A.16) starting in $\operatorname{im}(id - P(\tau))$ have an exponential lower bound. (In the case $\alpha < 0 < \beta$ solutions starting in $\operatorname{im} P(\tau)$ approach zero exponentially fast while solutions starting in $\operatorname{im}(id - P(\tau))$ escape exponentially fast.) These explanations show that for two families of associated projections $P(t)$ and $Q(t)$ necessarily $\operatorname{im} P(t) = \operatorname{im} Q(t)$. We shall even show, see Lemma A.2.2 and Lemma A.2.3 below, that any family of projections $Q(t) := \Phi(t,t_0)Q(t_0)\Phi(t_0,t)$ with $\operatorname{im} Q(t_0) = \operatorname{im} P(t_0)$ is associated to the exponential dichotomy.
The first estimate in (A.17) provides for $t = \tau$

$$\|P(t)\| \leq C, \quad \forall t \geq t_0. \tag{A.20}$$

That means that the family of projections which are associated to an exponential dichotomy is bounded. From a more geometrical point of view this means that the "angle" between the spaces $\operatorname{im} P(t)$ and $\operatorname{im}(id - P(t))$ remains bounded away from zero.

The next two lemmas are formulated in [Cop78] for classical dichotomies.

Lemma A.2.2 *Let $A(t) \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$. Assume that $\dot{x} = A(t)x$ with transition matrix Φ has an exponential dichotomy on $[t_0, \infty)$ with constants α, β and associated projections $P(t)$. Let $Q(t_0)$ be a projection on $\mathbb{R}^k$ with $\operatorname{im} P(t_0) = \operatorname{im} Q(t_0)$, and define projections $Q(t) := \Phi(t,t_0)Q(t_0)\Phi(t_0,t)$. Then there is a constant $\tilde{C}$ such that $\|P(t) - Q(t)\| \leq \tilde{C}e^{(\alpha-\beta)(t-t_0)}$.*

Proof Again we denote the transition matrix of $\dot{x} = A(t)x$ by Φ. Because of $\operatorname{im} P(t_0) = \operatorname{im} Q(t_0)$ we have

$$P(t_0) - Q(t_0) = P(t_0)(P(t_0) - Q(t_0))(id - P(t_0)).$$

Therefore, using (A.17), we get

$$\begin{aligned}\|P(t)-Q(t)\| &= \|\Phi(t,t_0)(P(t_0)-Q(t_0))\Phi(t_0,t)\| \\ &= \|\Phi(t,t_0)P(t_0)(P(t_0)-Q(t_0))(id-P(t_0))\Phi(t_0,t)\| \\ &\le \|\Phi(t,t_0)P(t_0)\|\,\|P(t_0)-Q(t_0)\|\,\|(id-P(t_0))\Phi(t_0,t)\| \\ &= \|\Phi(t,t_0)P(t_0)\|\,\|P(t_0)-Q(t_0)\|\,\|\Phi(t_0,t)(id-P(t))\| \\ &\le C^2\|P(t_0)-Q(t_0)\|\,e^{(\alpha-\beta)(t-t_0)}.\end{aligned}$$

■

Lemma A.2.3 *Let $A(t) \in \mathbb{L}(\mathbb{R}^k,\mathbb{R}^k)$. Assume that $\dot{x} = A(t)x$ with transition matrix Φ has an exponential dichotomy on $[t_0,\infty)$ with constants α, β and associated projections $P(t)$. Let $Q(t_0)$ be a projection on $\mathbb{R}^k$ with $\operatorname{im} P(t_0) = \operatorname{im} Q(t_0)$. Then the projections $Q(t) := \Phi(t,t_0)Q(t_0)\Phi(t_0,t)$ are associated to the exponential dichotomy.*

Proof By the definition it is clear that the projections $Q(t)$ commute with the transition matrix Φ. So it remains to verify the estimates (A.17).
First, note that due to (A.20) and Lemma A.2.2 the norm $\|Q(t)\|$ remains bounded. Further, by construction we have $\operatorname{im} P(t) = \operatorname{im} Q(t)$. Therefore

$$Q(t)P(t) = P(t) \quad \text{and} \quad (id-Q(t))(id-P(t)) = id-Q(t).$$

Hence for all $t \ge \tau \ge t_0$

$$\|\Phi(t,\tau)Q(\tau)\| = \|\Phi(t,\tau)P(\tau)Q(\tau)\| \le \|\Phi(t,\tau)P(\tau)\|\,\|Q(\tau)\|.$$

Now, because of the estimate (A.17) for $\|\Phi(t,\tau)P(\tau)\|$ there is a constant $\hat{C}$ such that

$$\|\Phi(t,\tau)Q(\tau)\| \le \hat{C}e^{\alpha(t-\tau)}.$$

Similarly

$$\begin{aligned}\|\Phi(\tau,t)(id-Q(t))\| &= \|\Phi(\tau,t)(id-Q(t))(id-P(t))\| \\ &= \|(id-Q(\tau))\Phi(\tau,t)(id-P(t))\|.\end{aligned}$$

Because of the estimate (A.17) for $\|\Phi(\tau,t)(id-P(t))\|$ there is a $\tilde{C}$ such that

$$\|\Phi(\tau,t)(id-Q(t))\| \le \tilde{C}e^{\beta(\tau-t)}.$$

■

Consider an autonomous equation

$$\dot{x} = Ax \, , \quad x \in \mathbb{R}^k \, , \tag{A.21}$$

where $A \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$. Let the spectrum $\sigma(A)$ be composed of non-empty sets σ^1 and σ^2; $\sigma(A) = \sigma^1 \cup \sigma^2$. If there are constants α, β such that

$$\Re\mu^1 < \alpha < \beta < \Re\mu^2, \quad \forall \mu^1 \in \sigma^1, \forall \mu^2 \in \sigma^2, \tag{A.22}$$

then (A.21) has an exponential dichotomy on $\mathbb{R}^+$ with constants α, β. And vice versa, if (A.21) has an exponential dichotomy on $\mathbb{R}^+$ with constants α, β, then there exists a decomposition $\sigma(A) = \sigma^1 \cup \sigma^2$ of the spectrum of A such that (A.22) is true. If in particular $\alpha < 0 < \beta$, then σ^1 is the stable spectrum of A and σ^2 is the unstable spectrum of A. Moreover, the projections $P(t)$ can be chosen identically, $P(t) \equiv P$, where P is the spectral projection according to the given decomposition of the spectrum of A. (Below we shall give a more detailed comment regarding the freedom in choosing the projections $P(t)$.)

One of the most important features of exponential dichotomies is the so-called roughness property. Roughly speaking this means that exponential dichotomies are preserved under perturbations. Here we restrict ourselves to perturbation of autonomous systems.

Lemma A.2.4 *Let $A \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$, and for $t \in [t_0, \infty)$ let $B(t) \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$. We assume that $\dot{x} = Ax$ has an exponential dichotomy on $[t_0, \infty)$ with constants α and β, and that there are positive constants K_B and δ such that*

$$\| B(t) \| \leq K_B e^{-\delta t} \, . \tag{A.23}$$

Then the equation

$$\dot{x} = (A + B(t))x \, , \quad x \in \mathbb{R}^k \, , \tag{A.24}$$

has an exponential dichotomy on $[t_0, \infty)$ with constants α and β.

Proof We use ideas of Sandstede (see [San93]) and Knobloch (see [Kno99]) instead of going along the lines of Coppel's proof, where $\operatorname{sgn} \alpha \neq \operatorname{sgn} \beta$ was explicitly used, see [Cop78, Lemma 4.1]. We want to remark that both Sandstede and Knobloch considered equations with a hyperbolic equilibrium but their arguments also apply to the case of a non-hyperbolic equilibrium. Nevertheless, here we give the full proof, thereby working out the generalisation concerning the constants α and β.

Let $\Phi(\cdot, \cdot)$ and $\Phi_B(\cdot, \cdot)$ be the transition matrices of (A.21) and (A.24), respectively. There exists a decomposition $\sigma(A) = \sigma^1 \cup \sigma^2$ of the spectrum of A such that (A.22) is fulfilled. Further, let Q denote the corresponding spectral projection, where Q projects on the (generalised) eigenspace of σ^1.

First we construct projections $\hat{Q}_s(t)$ on $\mathbb{R}^k$ which satisfy the first inequality in (A.17). For that purpose we consider for $\eta \in \operatorname{im} Q$ the fixed point equation

$$\begin{aligned} x(t,\tau) &= \Phi(t,\tau)\eta + \textstyle\int_\tau^t \Phi(t,s)QB(s)x(s,\tau)ds \\ &\qquad - \textstyle\int_t^\infty \Phi(t,s)(id-Q)B(s)x(s,\tau)ds \\ &=: (T_s(x,\eta))(t,\tau) \end{aligned} \tag{A.25}$$

in the Banach space

$$\mathbb{S}_\alpha := \{x : [t_0,\infty)\times[t_0,\infty) \to \mathbb{R}^k : \sup_{t\geq\tau\geq t_0} e^{\alpha(\tau-t)}\|x(t,\tau)\|_{\mathbb{R}^k} < \infty\}$$

with norm $\|x\|_\alpha := \sup_{t\geq\tau\geq t_0} e^{\alpha(\tau-t)}\|x(t,\tau)\|_{\mathbb{R}^k}$. Indeed, the Banach fixed point theorem can be applied to prove that (A.25) has a unique fixed point $x_s(\eta)(\cdot,\cdot)$. For that we exploit that for some constant K

$$\begin{aligned} \|\Phi(t,s)Q\| &\leq Ke^{\alpha(t-s)}, \quad t \geq s, \\ \|\Phi(t,s)(id-Q)\| &\leq Ke^{\alpha(t-s)}, \quad s \geq t, \\ \|x(s,\tau)\| &\leq \|x\|_\alpha e^{\alpha(s-\tau)}, \quad s \geq \tau. \end{aligned} \tag{A.26}$$

Together with (A.23) these estimates show that T_s maps $\mathbb{S}_\alpha$ into itself, and that (at least for sufficiently large t_0) T_s is a contraction. At this point we want to mention that this conclusion is true independently of the sign of α.

By construction $x_s(\eta)(\cdot,\tau)$ solves (A.24). On the other hand, if

$$x(\cdot,\cdot;\xi), \quad x(t,\tau;\xi) := \Phi_B(t,\tau)\Phi_B(\tau,t_0)\xi, \tag{A.27}$$

belongs to $\mathbb{S}_\alpha$, then it solves (A.25). Let $x(\cdot,\cdot;\xi) \in \mathbb{S}_\alpha$. Then by the uniqueness of the solutions of (A.25) we have

$$x(t,\tau;\xi) = x_s(Q\xi)(t,\tau)\ .$$

With that projections $\hat{Q}_s(\tau)(\cdot)$ can be defined:

$$\begin{aligned} \hat{Q}_s(\tau) : \ \mathbb{R}^k &\to \mathbb{R}^k \\ \xi &\mapsto x_s(Q\xi)(\tau,\tau)\ . \end{aligned}$$

Note first that $\hat{Q}_s(\tau)(\cdot)$ is linear, because $x_s(\eta)(\cdot,\cdot)$ depends linearly on η. Moreover, $Q\xi = Qx_s(Q\xi)(\tau,\tau)$, see (A.25), shows that $\hat{Q}_s(\tau)(\cdot)$ is idempotent. Further, by construction we have

$$x_s(Q\xi)(t,\tau) = \Phi_B(t,\tau)x_s(Q\xi)(\tau,\tau) = \Phi_B(t,\tau)\hat{Q}_s(\tau)\xi. \tag{A.28}$$

Hence for $x(\cdot,\cdot;\xi)$ defined in (A.27) we find

$$x(\cdot,\cdot;\xi) \in \mathbb{S}_\alpha \quad \Longleftrightarrow \quad x(\tau,\tau;\xi) = \Phi_B(\tau,t_0)\xi = \hat{Q}_s(\tau)\xi \in \operatorname{im}\hat{Q}_s(\tau). \tag{A.29}$$

In particular (setting $\tau = t_0$) that means that

$$x(\cdot,\cdot;\xi) \in \mathbb{S}_\alpha \quad \Longleftrightarrow \quad \xi \in \operatorname{im}\hat{Q}_s(t_0)\ .$$

Then with (A.29) it follows

$$\Phi_B(t,\tau)\operatorname{im}\hat{Q}_s(\tau) = \operatorname{im}\hat{Q}_s(t). \tag{A.30}$$

Finally, (A.28) and the third estimate in (A.26) yield that for some positive $\mathcal{K}_s$

$$\| \Phi_B(t,\tau)\hat{Q}_s(\tau) \| \ \leq \ \mathcal{K}_s e^{\alpha(t-\tau)}\ , \qquad t \geq \tau \geq t_0\ . \tag{A.31}$$

With that we can construct projections $P(t)$ that are associated to the exponential dichotomy of (A.24). Consider the direct sum decompositions of $\mathbb{R}^k$

$$\mathbb{R}^k = \operatorname{im}\hat{Q}_s(t) \oplus \Phi_B(t,t_0)(\ker \hat{Q}_s(t_0))\ .$$

By $P(t)$ we denote projections for which

$$\operatorname{im} P(t) = \operatorname{im}\hat{Q}_s(t).$$

Of course $P(t)$ commute with Φ_B:

$$P(t)\Phi_B(t,\tau) = \Phi_B(t,\tau)P(\tau).$$

It remains to verify estimates (A.17). For that we consider the fixed point equation

$$\begin{aligned} x(\tau,t) \ &= \ \Phi(\tau,t)(id-Q)\eta + \textstyle\int_{t_0}^{\tau} \Phi(\tau,s)QB(s)x(s,t)ds \\ &\qquad\qquad - \textstyle\int_{\tau}^{t} \Phi(\tau,s)(id-Q)B(s)x(s,t)ds \\ &=: \ (T_u(x,(id-Q)\eta))(\tau,t) \end{aligned} \tag{A.32}$$

in the Banach space

$$\mathbb{S}^\beta := \{x : [t_0,\infty) \times [t_0,\infty) \to \mathbb{R}^k : \sup_{t\geq\tau\geq t_0} e^{\beta(t-\tau)}\|x(\tau,t)\|_{\mathbb{R}^k} < \infty\}$$

with norm $\|x\|^\beta := \sup_{t\geq\tau\geq t_0} e^{\beta(t-\tau)}\|x(\tau,t)\|_{\mathbb{R}^k}$. In our further analysis we use the estimates

$$\begin{aligned} \|\Phi(t,s)Q\| &\leq Ke^{\beta(t-s)}, \quad t \geq s, \\ \|\Phi(t,s)(id-Q)\| &\leq Ke^{\beta(t-s)}, \quad s \geq t, \\ \|x(s,t)\| &\leq \|x\|^\beta e^{\beta(s-t)}, \quad t \geq s. \end{aligned} \tag{A.33}$$

Again the Banach fixed point theorem yields that for each $\eta \in \mathrm{im}\,(id - Q)$ Equation (A.32) has a unique fixed point $x_u(\eta)(\cdot,\cdot)$, which also solves (A.24). Further, it is again clear that

$$\begin{aligned} \hat{Q}_u(\tau) : \ \mathbb{R}^k &\rightarrow \mathbb{R}^k \\ \xi &\mapsto x_u((id - Q)\xi)(\tau,\tau) \,. \end{aligned}$$

are projections. From the fixed point equation (A.32) we read that $\mathrm{im}\,\hat{Q}_u(t_0) = \mathrm{im}\,(id - Q)$; and similarly we read from (A.25) that $\ker \hat{Q}_s(t_0) = \mathrm{im}\,(id - Q)$. This gives finally

$$\mathrm{im}\,\hat{Q}_u(t) = \Phi_B(t,t_0)(\ker \hat{Q}_s(t_0)) = \ker P(t).$$

The function $x_u(\eta)(\cdot,\tau)$ is a solution of (A.24). Therefore

$$x_u((id - Q)\xi)(t,\tau) = \Phi_B(t,\tau)x_u((id - Q)\xi)(\tau,\tau) = \Phi_B(t,\tau)\hat{Q}_u(\tau)\xi. \qquad \text{(A.34)}$$

With this we conclude similar to (A.31) that for some positive $\mathcal{K}_u$

$$\| \, \Phi_B(\tau,t)\hat{Q}_u(t) \, \| \ \leq \ \mathcal{K}_u e^{\beta(\tau - t)} \,, \qquad t \geq \tau \geq t_0 \,. \qquad \text{(A.35)}$$

Because of $\mathrm{im}\,P(t) = \mathrm{im}\,\hat{Q}_s(t)$ and $\mathrm{im}\,(id - P(t)) = \mathrm{im}\,\hat{Q}_u(t)$ we have for $t \geq \tau \geq t_0$

$$\begin{aligned} \Phi_B(t,\tau)P(\tau) &= \Phi_B(t,\tau)\hat{Q}_s(\tau)P(\tau), \\ \Phi_B(\tau,t)(id - P(t)) &= \Phi_B(\tau,t)\hat{Q}_u(t)(id - P(t)). \end{aligned}$$

If $\{\|P(t)\|, t \geq t_0\}$ is bounded then, due to (A.31) and (A.35), we find a positive constant C such that

$$\begin{aligned} \|\Phi_B(t,\tau)P(\tau)\| &\leq Ce^{\alpha(t-\tau)}, \quad t \geq \tau \geq t_0 \,, \\ \|\Phi_B(\tau,t)(id - P(t))\| &\leq Ce^{\beta(\tau - t)}, \quad t \geq \tau \geq t_0 \,. \end{aligned}$$

Then, see Definition A.2.1, Equation (A.24) has an exponential dichotomy as stated in the lemma.

We conclude the proof with the verification that $\{\|P(t)\|, t \geq t_0\}$ is indeed bounded. For $\xi \in \mathbb{R}^k$ we define $\xi_s := P(t)\xi$ and $\xi_u := (id - P(t))\xi$. Since $\mathrm{im}\,P(t) = \mathrm{im}\,\hat{Q}_s(t)$ we have

$$P(t)\xi = \hat{Q}_s(t)P(t)\xi = \hat{Q}_s(t)\xi_s = x_s(Q\xi_s)(t,t).$$

Then due to (A.25) we find

$$\begin{aligned} P(t)\xi &= Q\xi_s - \textstyle\int_t^\infty \Phi(t,s)(id - Q)B(s)x_s(Q\xi_s)(s,t)ds \\ &= Q\xi - Q\xi_u - \textstyle\int_t^\infty \Phi(t,s)(id - Q)B(s)x_s(Q\xi_s)(s,t)ds \\ &= Q\xi - Q\hat{Q}_u(t)\xi_u - \textstyle\int_t^\infty \Phi(t,s)(id - Q)B(s)x_s(Q\xi_s)(s,t)ds \\ &= Q\xi - Qx_u((id - Q)\xi_u)(t,t) - \textstyle\int_t^\infty \Phi(t,s)(id - Q)B(s)x_s(Q\xi_s)(s,t)ds. \end{aligned}$$

Now, exploiting (A.32) we get

$$\begin{aligned} P(t)\xi = Q\xi \quad & - \textstyle\int_{t_0}^{t} \Phi(t,s)QB(s)x_u((id-Q)\xi_u)(s,t)ds \\ & - \textstyle\int_{t}^{\infty} \Phi(t,s)(id-Q)B(s)x_s(Q\xi_s)(s,t)ds. \end{aligned} \tag{A.36}$$

Because of (A.28) and (A.34) it follows

$$\begin{aligned} P(t)\xi = Q\xi \quad & - \textstyle\int_{t_0}^{t} \Phi(t,s)QB(s)\Phi_B(s,t)\hat{Q}_u(t)(id-P(t))\xi ds \\ & - \textstyle\int_{t}^{\infty} \Phi(t,s)(id-Q)B(s)\Phi_B(s,t)\hat{Q}_s(t)P(t)\xi ds. \end{aligned} \tag{A.37}$$

Next we estimate the integral terms in the last equation: By means of (A.23), (A.26) and (A.35) we find

$$\begin{aligned} & \| \textstyle\int_{t_0}^{t} \Phi(t,s)QB(s)\Phi_B(s,t)\hat{Q}_u(t)(id-P(t))\xi ds\| \\ & \quad \leq \textstyle\int_{t_0}^{t} KK_B\mathcal{K}_u e^{(\alpha-\beta)(t-s)}e^{-\delta s}(1+\|P(t)\|)\|\xi\|\, ds. \end{aligned}$$

Because of $\alpha - \beta < 0$ we get

$$\begin{aligned} & \| \textstyle\int_{t_0}^{t} \Phi(t,s)QB(s)\Phi_B(s,t)\hat{Q}_u(t)(id-P(t))\xi ds\| \\ & \quad \leq \left(\textstyle\int_{t_0}^{t} KK_B\mathcal{K}_u e^{-\delta s}\, ds + \textstyle\int_{t_0}^{t} KK_B\mathcal{K}_u e^{-\delta s}\, ds\|P(t)\|\right)\|\xi\|. \end{aligned} \tag{A.38}$$

In the same way, but this time exploiting (A.23), (A.31) and (A.33) we get

$$\begin{aligned} & \| \textstyle\int_{t}^{\infty} \Phi(t,s)(id-Q)B(s)\Phi_B(s,t)\hat{Q}_s(t)P(t)\xi ds\| \\ & \quad \leq \left(\textstyle\int_{t}^{\infty} KK_B\mathcal{K}_s e^{-\delta s}\, ds\|P(t)\|\right)\|\xi\|. \end{aligned} \tag{A.39}$$

We choose t_0 large enough, such that $KK_B \max\{\mathcal{K}_s, \mathcal{K}_u\} \int_{t_0}^{\infty} e^{-\delta s}\, ds \leq \frac{1}{4}$. Finally, combining (A.37) – (A.39) we get

$$\|P(t)\| \leq 2\left(\|Q\| + KK_B\mathcal{K}_u \int_{t_0}^{\infty} e^{-\delta s}\, ds\right) \leq 2\|Q\| + \frac{1}{2}.$$

■

Remark A.2.5 Let $\sigma(A) = \sigma^1 \cup \sigma^2$ be the decomposition of the spectrum of A as introduced at the beginning of the proof. Then $\dot{x} = Ax$ and therefore also $\dot{x} = (A + B(t))x$ has an exponential dichotomy with constants $\tilde{\alpha}$ and $\tilde{\beta}$ if

$$\Re\mu^1 < \tilde{\alpha} < \tilde{\beta} < \Re\mu^2, \quad \forall\mu^1 \in \sigma^1, \forall\mu^2 \in \sigma^2.$$

Moreover the corresponding mappings x_s and x_u do not depend on the choice of $\tilde{\alpha}$ and $\tilde{\beta}$. □

Next we will show that under the assumptions of Lemma A.2.4 for any projections $P(t)$ which are associated to the exponential dichotomy of the perturbed equation (A.24) the norm $\|P(t) - Q\|$ tends exponentially fast to zero as t tends to infinity. Because of Lemma A.2.2 it remains to prove that this is true for the projections constructed in Lemma A.2.4. It turns out that the exponential rate depends on the spectral gap between σ^1 and σ^2.

Lemma A.2.6 *Assume the hypotheses of Lemma A.2.4. Let Q be the spectral projection of A associated to the exponential dichotomy of $\dot{x} = Ax$, and let $P(t)$ be the projections in accordance with Lemma A.2.4. Then there are positive constants γ and $\mathcal{K}$ such that $\|P(t) - Q\| \le \mathcal{K}e^{-\gamma t}$. In particular, any $\gamma > 0$ with*

$$\alpha - \beta + \gamma < 0, \quad \Re\mu^1 < \alpha - \gamma,\ \forall \mu^1 \in \sigma^1, \quad \gamma - \delta < 0 \tag{A.40}$$

is suitable.

Proof In order to estimate $\|P(t) - Q\|$ we start from (A.36)

$$\begin{aligned} \|P(t)\xi - Q\xi\| \le\ & \| \textstyle\int_{t_0}^{t} \Phi(t,s)QB(s)x_u((id - Q)\xi_u)(s,t)\,ds\| \\ & + \| \textstyle\int_{t}^{\infty} \Phi(t,s)(id - Q)B(s)x_s(Q\xi_s)(s,t)\,ds\|. \end{aligned} \tag{A.41}$$

For that purpose we estimate the integral terms in (A.41) exacting. Again we adapt ideas from [San93].
From (A.32) we see that

$$\begin{aligned} Qx_u(\eta)(\tau,t) &= \textstyle\int_{t_0}^{\tau} \Phi(\tau,s)QB(s)x_u(\eta)(s,t)\,ds \\ &= \textstyle\int_{t_0}^{\tau} \Phi(\tau,s)QB(s)Qx_u(\eta)(s,t)\,ds \\ &\quad + \textstyle\int_{t_0}^{\tau} \Phi(\tau,s)QB(s)(id - Q)x_u(\eta)(s,t)\,ds. \end{aligned} \tag{A.42}$$

Therefore we have

$$\begin{aligned} \|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \le\ & e^{\beta(t-\tau)}e^{\gamma\tau}\Big(\textstyle\int_{t_0}^{\tau} \|\Phi(\tau,s)Q\|\,\|B(s)\|\,\|Qx_u(\eta)(s,t)\|\,ds \\ & + \textstyle\int_{t_0}^{\tau} \|\Phi(\tau,s)Q\|\,\|B(s)\|\,\|(id - Q)\|\,\|x_u(\eta)(s,t)\|\,ds\Big). \end{aligned}$$

With (A.23) and (A.26) it follows

$$\begin{aligned} \|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \le\ & e^{\beta(t-\tau)}e^{\gamma\tau}\Big(\textstyle\int_{t_0}^{\tau} KK_B e^{\alpha(\tau-s)}e^{-\delta s}\,\|Qx_u(\eta)(s,t)\|\,ds \\ & + \textstyle\int_{t_0}^{\tau} KK_B e^{\alpha(\tau-s)}e^{-\delta s}\,\|(id - Q)\|\,\|x_u(\eta)(s,t)\|\,ds\Big), \end{aligned}$$

and by inserting a "factor one"

$$
\begin{aligned}
&\|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \\
&= e^{\beta(t-\tau)}e^{\gamma\tau}\Big(\textstyle\int_{t_0}^{\tau} KK_B e^{\alpha(\tau-s)}e^{-\delta s}\,\|Qx_u(\eta)(s,t)\|e^{\beta(t-s)}e^{\gamma s}\,e^{-\beta(t-s)}e^{-\gamma s}\,ds \\
&+\textstyle\int_{t_0}^{\tau} KK_B e^{\alpha(\tau-s)}e^{-\delta s}\,\|(id-Q)\|\,\|x_u(\eta)(s,t)\|e^{\beta(t-s)}e^{\gamma s}\,e^{-\beta(t-s)}e^{-\gamma s}\,ds\Big)\,.
\end{aligned}
$$

With (A.33) this yields

$$
\begin{aligned}
&\|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \\
&\le \textstyle\int_{t_0}^{\tau} KK_B e^{(\alpha-\beta+\gamma)(\tau-s)}e^{-\delta s}\,ds\,\big(\sup_{t_0\le s\le\tau}\|Qx_u(\eta)(s,t)\|e^{\beta(t-s)}e^{\gamma s}\big) \qquad (A.43)\\
&+\textstyle\int_{t_0}^{\tau} KK_B e^{(\alpha-\beta+\gamma)(\tau-s)}e^{(\gamma-\delta)s}\,\|(id-Q)\|\,\|x_u(\eta)\|^{\beta}\,ds
\end{aligned}
$$

In what follows we will keep t fixed and consider the last inequality for $\tau \le t$. First we observe that

$$\sup_{t_0\le\tau\le t}\|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} < \infty. \qquad (A.44)$$

Further, we remark that, because of (A.40), for sufficiently large t_0 (independently on the choice of $t > t_0$)

$$\int_{t_0}^{\tau} KK_B e^{(\alpha-\beta+\gamma)(\tau-s)}e^{-\delta s}\,ds \le \frac{1}{2}. \qquad (A.45)$$

With (A.43) and (A.45) we estimate

$$
\begin{aligned}
&\sup\nolimits_{t_0\le\tau\le t}\|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \\
&\le \tfrac{1}{2}\big(\sup\nolimits_{t_0\le\tau\le t}\sup\nolimits_{t_0\le s\le\tau}\|Qx_u(\eta)(s,t)\|e^{\beta(t-s)}e^{\gamma s}\big) \\
&+\sup\nolimits_{t_0\le\tau\le t}\textstyle\int_{t_0}^{\tau} KK_B e^{(\alpha-\beta+\gamma)(\tau-s)}e^{(\gamma-\delta)s}\,\|(id-Q)\|\,\|x_u(\eta)\|^{\beta}\,ds.
\end{aligned}
$$

Because of (A.44) and $\gamma-\delta<0$, see (A.40), this yields

$$
\begin{aligned}
&\sup_{t_0\le\tau\le t}\|Qx_u(\eta)(\tau,t)\|e^{\beta(t-\tau)}e^{\gamma\tau} \\
&\qquad\le 2\int_{t_0}^{t} KK_B e^{(\alpha-\beta+\gamma)(t-s)}\,\|(id-Q)\|\,\|x_u(\eta)\|^{\beta}\,ds\,. \qquad (A.46)
\end{aligned}
$$

Note, that the integral on the right-hand side in the last inequality remains bounded as $t\to\infty$. This means, that there is a constant $\mathcal{K}^u$ such that for all $t \ge t_0$

$$\int_{t_0}^{t} KK_B e^{(\alpha-\beta+\gamma)(t-s)}\,\|(id-Q)\|\,\|x_u(\eta)\|^{\beta}\,ds \le \mathcal{K}^u.$$

So, in particular for $\tau = t$ the estimate (A.46) reads

$$\|Qx_u(\eta)(t,t)\| \le e^{-\gamma t} 2\mathcal{K}^u.$$

We want to emphasise that the last estimate is true for all $\eta \in \mathbb{R}^k$. So, with the notation of (A.42) we arrive at

$$\left\| \int_{t_0}^{t} \Phi(t,s) Q B(s) x_u(\eta)(s,t)\, ds \right\| \le e^{-\gamma t} 2\mathcal{K}^u, \quad \forall \eta \in \mathbb{R}^k. \tag{A.47}$$

Now, we turn towards the estimate of the second integral term in (A.41). In principle these calculations run along the same lines as above. However the counterpart of (A.44) calls for an additional argument. For this reason we present the detailed estimates here, too.
In the same way as above, but this time exploiting (A.25), we find:

$$\begin{aligned}
(id - Q)x_s(\eta)(t,\tau) &= -\int_t^\infty \Phi(t,s)(id-Q)B(s)x_s(\eta)(s,\tau)\, ds \\
&= -\Big(\int_t^\infty \Phi(t,s)(id-Q)B(s)(id-Q)x_s(\eta)(s,\tau)\, ds \\
&\qquad + \int_t^\infty \Phi(t,s)(id-Q)B(s)Qx_s(\eta)(s,\tau)\, ds\Big).
\end{aligned}$$

By means of (A.23) and (A.33) we conclude

$$\begin{aligned}
&\|(id-Q)x_s(\eta)(t,\tau)\|\, e^{\alpha(\tau-t)} e^{\gamma t} \\
&\le e^{\alpha(\tau-t)} e^{\gamma t} \Big(\int_t^\infty K K_B e^{\beta(t-s)} e^{-\delta s} \|(id-Q)x_s(\eta)(s,\tau)\|\, e^{\alpha(\tau-s)} e^{\gamma s}\, e^{-\alpha(\tau-s)} e^{-\gamma s}\, ds \\
&\qquad + \int_t^\infty K K_B e^{\beta(t-s)} e^{-\delta s} \|Q\|\, \|x_s(\eta)(s,\tau)\|\, e^{\alpha(\tau-s)} e^{\gamma s}\, e^{-\alpha(\tau-s)} e^{-\gamma s}\, ds\Big).
\end{aligned}$$

With (A.26) we get

$$\begin{aligned}
&\|(id-Q)x_s(\eta)(t,\tau)\|\, e^{\alpha(\tau-t)} e^{\gamma t} \\
&\le \int_t^\infty K K_B e^{(\beta-\alpha+\gamma)(t-s)} e^{-\delta s}\, ds\, \left(\sup_{t_0\le\tau\le t\le s} \|(id-Q)x_s(\eta)(s,\tau)\|\, e^{\alpha(\tau-s)} e^{\gamma s}\right) \\
&\qquad + \int_t^\infty K K_B e^{(\beta-\alpha+\gamma)(t-s)} e^{(\gamma-\delta)s} \|Q\|\, \|x_s(\eta)\|_\alpha\, ds.
\end{aligned}$$

Again we find that for sufficiently large t_0 (recall that $t \ge t_0$)

$$\int_t^\infty K K_B e^{(\beta-\alpha+\gamma)(t-s)} e^{-\delta s}\, ds \le \frac{1}{2}.$$

Therefore

$$\begin{aligned}
&\sup_{t_0\le\tau\le t} \|(id-Q)x_s(\eta)(t,\tau)\|\, e^{\alpha(\tau-t)} e^{\gamma t} \\
&\le \tfrac{1}{2} \sup_{t_0\le\tau\le t}\ \sup_{t_0\le\tau\le t\le s} \|(id-Q)x_s(\eta)(s,\tau)\|\, e^{\alpha(\tau-s)} e^{\gamma s} \\
&\qquad + \sup_{t_0\le\tau\le t} \int_t^\infty K K_B e^{(\beta-\alpha+\gamma)(t-s)} e^{(\gamma-\delta)s} \|Q\|\, \|x_s(\eta)\|_\alpha\, ds.
\end{aligned}$$

Now we keep τ fixed. Then there is a constant $\mathcal{K}^s$ such that

$$\sup_{t_0 \le \tau \le t} \int_t^\infty K K_B e^{(\beta-\alpha+\gamma)(t-s)} e^{(\gamma-\delta)s} \|Q\| \, \|x_s(\eta)\|_\alpha \, ds \le \mathcal{K}^s.$$

Further we make clear that $\sup_{t_0 \le \tau \le t} \|(id - Q)x_s(\eta)(t,\tau)\| \, e^{\alpha(\tau-t)} e^{\gamma t} < \infty$: For this we rewrite $e^{\alpha(\tau-t)} e^{\gamma t} = e^{(\alpha-\gamma)(\tau-t)} e^{\gamma\tau}$. Because of (A.40) and Remark A.2.5 (set here $\tilde{\alpha} = \alpha - \gamma$) this supremum is indeed finite. Now, we can continue in the same way as for the estimate of Qx_u. First we find

$$\sup_{t_0 \le \tau \le t} \|(id - Q)x_s(\eta)(t,\tau)\| \, e^{\alpha(\tau-t)} e^{\gamma t} \le 2\,\mathcal{K}^s,$$

and for $t = \tau$ we get finally, again independently on the choice of η

$$\left\| \int_t^\infty \Phi(t,s)(id - Q)B(s)x_s(\eta)(s,t)\, ds \right\| \le e^{-\gamma t} 2\,\mathcal{K}^s \,. \tag{A.48}$$

So, (A.41), (A.47) and (A.48) result in the lemma. ■

Corollary A.2.7 *Assume the hypotheses of Lemma A.2.4. Let Q and $\tilde{P}(t)$ be projections associated to the exponential dichotomy with constants α, β of the equation $\dot{x} = Ax$ and $\dot{x} = (A + B(t))x$, respectively. Then there are positive constants C and γ such that $\|\tilde{P}(t) - Q\| \le Ce^{-\gamma(t-t_0)}$.*

Proof The proof results from the combination of Lemma A.2.6 and Lemma A.2.2. ■

Next we define exponential trichotomies as introduced in [HL86].

Definition A.2.8 *Let the map $A : [t_0, \infty) \to \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ be continuous. Equation (A.16), $\dot{x} = A(t)x$, or the transition matrix Φ, is said to have an **exponential trichotomy** on $[t_0, \infty)$ if there exist projections $P_s(t)$, $P_c(t)$ and $P_u(t)$, with $P_s(t) + P_c(t) + P_u(t) = id$, $t \in [t_0, \infty)$, and constants $\alpha_s < -\alpha_c < 0 < \alpha_c < \alpha_u$ and $K > 0$ such that*

$$\Phi(t,\tau)P_i(\tau) = P_i(t)\Phi(t,\tau), \quad i = s,\, c,\, u\;,$$

and

$$\|\Phi(t,\tau)P_s(\tau)\| \le Ke^{\alpha_s(t-\tau)}, \qquad \|\Phi(t,\tau)P_c(\tau)\| \le Ke^{\alpha_c(t-\tau)}, \quad t \ge \tau \ge t_0\,,$$

$$\|\Phi(t,\tau)P_c(\tau)\| \le Ke^{-\alpha_c(t-\tau)}, \quad \|\Phi(t,\tau)P_u(\tau)\| \le Ke^{\alpha_u(t-\tau)}, \quad \tau \ge t \ge t_0\,.$$

If $t_0 = 0$ we say that Equation (A.16) has an exponential trichotomy on $\mathbb{R}^+$. Of course an analogous definition can be made for an exponential trichotomy on $(-\infty, t_0]$ or $\mathbb{R}^-$, respectively. However, here we will restrict our explanations to the $[t_0, \infty)/\mathbb{R}^+$-case.

The existence of an exponential trichotomy for Φ means that a solution $x(\cdot)$ of

(A.16) starting in $\operatorname{im} P_s(\tau)$ decays with an exponential rate of at least α_s (as $t \to \infty$). If, on the other hand, $x(t) \in \operatorname{im} P_u(t)$ then $x(\tau)$ increases with an exponential rate of at least α_u (as $\tau \to \infty$). This can easily be seen by considering $x(t) = \Phi(t,\tau)P_u(\tau)\Phi(\tau,t)x(t)$. Finally, if $x(\tau) \in \operatorname{im} P_c(\tau)$ then $x(t)$ does not decay faster than $e^{-\alpha_c t}$ and simultaneously it does not increase faster than $e^{\alpha_c t}$, see also the equivalent of (A.18) and (A.19).
Thus, in this sense, the images of $P_s(t)$, $P_c(t)$ and $P_u(t)$ can be seen as stable, centre and unstable subspaces at time t corresponding to the non-autonomous equation (A.16). Note, that only the images of $P_s(t)$ and $P_s(t) + P_c(t) =: P_{cs}(t)$ are uniquely determined; in other words only the stable and centre-stable subspaces are fixed.

Now, let $A \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$ with spectrum $\sigma(A) = \sigma^s \cup \sigma^c \cup \sigma^u$, where

$$\sigma^s = \{\lambda \in \sigma(A) : \Re(\lambda) < 0\} ,$$
$$\sigma^c = \{\lambda \in \sigma(A) : \Re(\lambda) = 0\} ,$$
$$\sigma^u = \{\lambda \in \sigma(A) : \Re(\lambda) > 0\} .$$

If $\sigma^s, \sigma^c, \sigma^u \neq \emptyset$, then the equation $\dot{x} = Ax$ has an exponential trichotomy on both $\mathbb{R}^+$ and $\mathbb{R}^-$. The next lemma is a roughness theorem for exponential trichotomies.

Lemma A.2.9 *Let $A \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$, and for all $t \in [t_0, \infty)$ let $B(t) \in \mathbb{L}(\mathbb{R}^k, \mathbb{R}^k)$. We assume that $\dot{x} = Ax$ has an exponential trichotomy on $[t_0, \infty)$ with constants α_s, α_c and α_u, and that there are positive constants K_B and δ such that $\| B(t) \| \leq K_B e^{-\delta t}$. The equation $\dot{x} = (A + B(t))x$ has an exponential trichotomy on $[t_0, \infty)$ with the same constants α_s, α_c and α_u.*

Proof The idea is to define the projections $P_s(t)$ and $P_{cs}(t)$ by means of generalised exponential dichotomies of the equation under consideration.

Equation (A.24) has, in accordance with Lemma A.2.4 two exponential dichotomies: one with constants $\alpha = \alpha_s$, $\beta = -\alpha_c$ and associated projections $P_s(t)$ and another one with constants $\alpha = \alpha_c$, $\beta = \alpha_u$ and associated projections $P_{cs}(t)$. Because of $\alpha_s < \alpha_c$ we have $\operatorname{im} P_s(t) \subset \operatorname{im} P_{cs}(t)$.
Further, Lemma A.2.3 allows us to choose the kernels of these projections such that $\ker P_{cs}(0) \subset \ker P_s(0)$. This property is carried to $t > 0$ by the transition matrix. This yields that $P_s(t)$ and $P_{cs}(t)$ commute and therefore $P_c(t) := P_{cs}(t) - P_s(t)$ is a projection.
Hence, Equation $\dot{x} = (A + B(t))x$ has an exponential trichotomy with projections $P_s(t)$, $P_c(t)$ and $P_u(t) := id - P_{cs}(t)$ and constants α_s, α_c and α_u. ■

Applying Corollary A.2.7 to estimate $\|P_s(t) - Q_s\|$ and $\|(P_s(t) + P_c(t)) - (Q_s + Q_c)\|$ we obtain

Lemma A.2.10 *Assume the hypotheses of Lemma A.2.9. Let Q_s, Q_c and $P_s(t)$ and $P_c(t)$ be projections that are associated to the exponential trichotomy of $\dot{x} = Ax$*

and $\dot{x} = (A + B(t))x$*, respectively. Then there are positive constants* C_s*,* γ_s *and* C_c*,* γ_c *such that* $\|P_s(t) - Q_s\| \leq C_s e^{-\gamma_s(t-t_0)}$ *and* $\|P_c(t) - Q_c\| \leq C_c e^{-\gamma_c(t-t_0)}$*.*

We consider an equation $\dot{x} = f(x, \lambda)$ having a saddle-centre equilibrium $x = 0$ and corresponding (un)stable manifold $W_\lambda^{s(u)}$ and centre-(un)stable manifold $W_\lambda^{cs(cu)}$. Let $\psi(\cdot)$ be a solution of $\dot{x} = f(x, \lambda)$ with $\psi(\tau) \in W_\lambda^s$. Then the following property holds:

Lemma A.2.11 *The variational equation along* ψ*,*

$$\dot{x} = D_1 f(\psi(t), \lambda)x,$$

has an exponential trichotomy on $\mathbb{R}^+$ *with projections* $P_s(t,\lambda)$*,* $P_u(t,\lambda)$ *and* $P_c(t,\lambda)$*. Moreover* $\operatorname{im} P_s(t,\lambda) = T_{\psi(t)} W_\lambda^s$ *and* $\operatorname{im}(P_s(t,\lambda) + P_c(t,\lambda)) = T_{\psi(t)} W_\lambda^{cs}$*.*

Proof The first observation for proving the lemma is that the differential equation (A.2.11) can be written as

$$\dot{x} = D_1 f(0, \lambda)x + (D_1 f(\psi(t), \lambda) - D_1 f(0, \lambda))x. \tag{A.49}$$

Moreover, taking into account $f \in C^2$ we use the mean value theorem to show that the second term on the right-hand side of (A.49) tends exponentially fast to zero (for $t \to \infty$). We can apply Lemma A.2.9 to this equation and find that (A.49) has an exponential trichotomy with projections $P_s(t,\lambda)$, $P_c(t,\lambda)$ and $P_u(t,\lambda)$ and constants α_s, α_c and α_u.
Finally, by construction we have $(P_s(t,\lambda) + P_c(t,\lambda)) = P_{cs}(t,\lambda)$. Once having this we find that the images of $P_s(t,\lambda)$ and $P_{cs}(t,\lambda)$ are just the tangent spaces $T_{\psi(t)} W_\lambda^s$ and $T_{\psi(t)} W_\lambda^{cs}$, respectively. The proof runs parallel to the one in [Scha95] for the case of a hyperbolic equilibrium. ■

In a similar way we find that variational equations along solutions in the unstable manifold have exponential trichotomies on $\mathbb{R}^-$.

List of Notations

$\lambda = (\lambda_1, \lambda_2)$	parameter of Equation (1.1)
$f(\cdot, \lambda)$	vector field of Equation (1.1)
$\varphi(t, \cdot, \lambda)$	flow of Equation (1.1)
R	linear involution; a linear map satisfying $R^2 = id$
$\mathrm{Fix}\, R$, $\mathrm{Fix}\,(-R)$	Equation (A.1)
$\mathring{x}$	saddle-centre, Hypothesis (H 1.2)
$\gamma(\cdot)$	homoclinic solution approaching the equilibrium $\mathring{x}$, Hypothesis (H 1.3)
Γ	homoclinic orbit to $\mathring{x}$: $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$, Hypothesis (H 1.3)
	Both of the following notations are used in this thesis depending on what is more convenient.
$D_i^j f$	partial derivative of f with respect to the i^{th} variable of order j
$D_{x_i}^j f$	partial derivative of f with respect to the variable x_i of order j
$\sigma(A)$	spectrum of A
μ	leading unstable eigenvalue of $D_1 f(\mathring{x}, 0)$, Hypothesis (H 1.2)
$-\mu$	leading stable eigenvalue of $D_1 f(\mathring{x}, 0)$, Hypothesis (H 1.2)
$\Re(\mu)$	real part of the complex number μ
σ^{ss}, σ^{uu}	strong stable and strong unstable spectrum of $D_1 f(\mathring{x}, 0)$
$\mathcal{U}$	neighbourhood of $\Gamma \cup \{\mathring{x}\}$

	The following notations of the generalised eigenspaces and the invariant manifolds may possess subscripts λ and *loc*. Those indicates the dependence on λ or emphasise the local character; the absence of λ is equivalent to $\lambda = 0$.
X^i,	$i = s, c, u$, generalised eigenspace of the stable, centre and unstable eigenvalue, respectively, of $D_1 f(\mathring{x}, 0)$
X^h	hyperbolic subspace, Equation (4.2)
X^c_R, X^c_{-R}	Equation (4.1)
W^i	$i = s, c, u, cs, cu$, stable, centre, unstable, centre-stable and centre-unstable manifold, respectively

$T_q W^i$	$i = s, c, u, cs, cu$, tangent spaces of the stable, centre, unstable, centre-stable and centre-unstable manifold, respectively

Y^s, Y^u, Z, Y^c, $\hat{Z}$	subspaces related to a direct sum decomposition of $\mathbb{R}^{2n+2}$, Equation (3.1) and Equation (3.38)
Σ	transversal intersection of Γ in $\gamma(0)$, Equation (3.2)

$\mathcal{X}^0_{hom} = \{(\gamma^s, \gamma^u)\}$	one-homoclinic Lin solution tending to the equilibrium, Definition 2.1.6
$\mathcal{L}^0_{hom} = \{(\Gamma^s, \Gamma^u)\}$	one-homoclinic Lin orbit to the equilibrium, Definition 2.1.6
$\mathcal{X}_{hom} = \{(\gamma^+, \gamma^-)\}$	one-homoclinic Lin solution tending to the centre manifold, Definition 2.1.6
$\mathcal{L}_{hom} = \{(\Gamma^+, \Gamma^-)\}$	one-homoclinic Lin orbit to the centre manifold, Definition 2.1.6
$\mathcal{X} = \{(x^+, x^{loc}, x^-)\}$	one-periodic Lin solution, Definition 2.1.2 and page 13
$\mathcal{L} = \{(X^+, X^{loc}, X^-)\}$	one-periodic Lin orbit, Definition 2.1.2 and page 13
$\gamma^{s(u)}(\lambda)(\cdot)$	parts of $\mathcal{X}^0_{hom}$, Lemma 3.1.4
$\Gamma^{s(u)}(\lambda)$	parts of $\mathcal{L}^0_{hom}$, associated to Lemma 3.1.4
$\gamma^{\pm}(y^+_c, y^-_c, \lambda)(\cdot)$	parts of $\mathcal{X}_{hom}$, Lemma 3.2.3
$\Gamma^{\pm}(y^+_c, y^-_c, \lambda)$	parts of $\mathcal{L}_{hom}$, associated to Lemma 3.2.3

$x^{\pm}(\Omega, p, \lambda, N)(\cdot)$, $x^{loc}(\Omega, p, \lambda, N)(\cdot)$	parts of $\mathcal{X}$, Lemma 4.1.12
$X^{\pm}(\Omega, p, \lambda, N)$, $X^{loc}(\Omega, p, \lambda, N)$	parts of $\mathcal{L}$, Lemma 4.1.12

$\xi(\lambda)$	jump of the one-homoclinic Lin orbits to the equilibrium, (3.24)
$\xi^{\infty}(y_c^+, y_c^-, \lambda)$	jump of the one-homoclinic Lin orbits to the centre manifold, (3.60)
$\hat{\xi}^{\infty}(y_c, \lambda_1, \lambda_2)$	jump of the one-homoclinic Lin orbits to the centre manifold for $y_c := y_c^+ = y_c^-$, (3.62)
$\xi_{per}(\Omega, p, \lambda, N)$	jump of the symmetric one-periodic Lin orbits, (4.41)
$\hat{\xi}_r(\varrho, \vartheta, \lambda, N)$	Equation (4.79)

$\Phi(t, s)$	transition matrix of the variational equation along $\gamma(\cdot)$ (3.11)
$P_i^{\pm}(t)$	$i = s, c, u$, associated projections of the exponential trichotomies of Equation (3.11)
$\Phi^{\pm}(t, s, \lambda)$	transition matrix of the variational equation along $\gamma^{s(u)}(\cdot)$ (3.43)
$P_i^{\pm}(t, \lambda)$	$i = s, c, u, cs, cu$, associated projections of the exponential trichotomies of Equation (3.43)

α, α_c	constants satisfying $\mu > \alpha > \alpha_c > 0$
$V_{\bar{\alpha}}^{\pm}$	spaces of exponentially bounded functions defined for $\bar{\alpha} \in (\alpha_c, \alpha)$ in Equation (3.16)
Λ_0	small neighbourhood of 0 in $\mathbb{R}^2$

$h^{cs}(\cdot, \cdot)$, $h^{cu}(\cdot, \cdot)$	Equation (3.51) and page 36
Σ_{loc}	Poincaré section, Equation (4.3)
$\Pi(\cdot, \lambda)$	Poincaré map, Equation (4.12)
p	fixed points of the Poincaré map $\Pi(\cdot, \lambda)$, page 61
$\mathcal{M}_{p,\lambda}$	leaf of the C^r foliation of Σ_{loc} given in Hypothesis (H 4.3)

$W^{s(u)}_{\Pi(\cdot,\lambda)}(p)$	Equation (4.16)
$\mathcal{C}^+(\Omega, p, \lambda)$	Equation (4.21)
$\mathcal{C}^-(\Omega, p, \lambda)$	Equation (4.26)
$\Sigma^+(\Omega, \lambda)$	Equation (4.20)
$\Sigma^-(\Omega, \lambda)$	Equation (4.24)
$\tilde{\gamma}^+(\Omega, p, \lambda)$	Lemma 4.1.7
$\tilde{\gamma}^-(\Omega, p, \lambda)$	Corollary 4.1.8
θ, Θ	page 72
$(z_1, z_2)(\xi, \eta, \theta, N)(\cdot)$	Shilnikov solution connecting the hyperplanes $z_1 = \xi$ and $z_2 = \eta$ for given $\xi, \eta \in \mathbb{R}$ in a given time $N \in \mathbb{N}$, Lemma 4.1.13
$\gamma^\pm(y, \lambda)(\cdot)$	Equation (4.42)
$y^*(\cdot, \cdot, \cdot)$	Equation (4.46)
(ϱ, ϑ)	polar coordinates, page 80
$y(\cdot, \cdot)$	Equation (4.52)
$\Omega^*(\cdot, \cdot, \cdot)$	Lemma 4.2.4
$p^*(\cdot, \cdot, \cdot)$	Lemma 4.2.5
$\varrho^*(\cdot, \cdot, \cdot)$	Lemma 4.2.8
$\mathcal{A}(\cdot, \vartheta, \lambda, N)$	Lemma 4.2.8
$\tilde{\xi}_r(\vartheta, \lambda, N)$	Equation (4.90)
$\tilde{\varrho}$	Equation (4.90)
$M_1 \pitchfork_q M_2$	transversal intersection of M_1 and M_2 in q
$\alpha(\hat{\Gamma})$, $\omega(\hat{\Gamma})$	α and ω set of the orbit $\hat{\Gamma}$
id	identity map
$\langle \cdot, \cdot \rangle$	scalar product
$M^\perp$	orthogonal complement of M with respect to $\langle \cdot, \cdot \rangle$
$a^\top$	transposed of a

$\mathbb{L}(\mathbb{R}^n, \mathbb{R}^m)$	space of linear mappings $\mathbb{R}^n \to \mathbb{R}^m$

Bibliography

[AM67] Abraham, R. and Marsden, J. E., *Foundations of mechanics. A mathematical exposition of classical mechanics with an introduction to the qualitative theory of dynamical systems and applications to the three-body problem*, Benjamin, New York and Amsterdam 1967.

[AK89] Amick, C. J. an Kirchgässner, K., A theory of solitary water-waves in the presence of surface tension, *Arch. Ration. Mech. Anal.* 105 (1989) 1–49.

[Ama95] Amann, H., *Gewöhnliche Differentialgleichungen,* de Gruyter Berlin 1995.

[AL39] Andronov, A. and Leontovich, E., Some case of dependence of the limit cycles upon parameters, *Uchen. Zap. Gor'kov. Univ.* 6 (1939) 3–24.

[BGRS03] Bernard, P., Grotta Ragazzo, C. and Salomao, P. A. S., Homoclinic orbits near saddle-center fixed points of Hamiltonian systems with two degrees of freedom. (English) de Melo, Welington (ed.) et al., Geometric methods in dynamics. (I.) *Astérisque* 286 (2003) 151–165.

[Cha98] Champneys, A. R., Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics, *Physica D* 112 (1998) 158–186.

[Cha99] Champneys, A. R., Homoclinic orbits in reversible systems II: Multibumps and saddle-centres, *CWI Quarterly* 12(3-4) (1999) 185–212.

[CH00] Champneys, A. R. and Härterich, J., Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems, *Dyn. Stab. Syst.* 15(3) (2000) 231–252.

[CMYK01a] Champneys, A. R., Malomed, B. A., Yang, J. and Kaup, D. J., Embedded solitons: solitary waves in resonance with the linear spectrum, Physica D 152–153 (2001) 340–354.

[CMYK01b] Yang, J., Malomed, B. A., Kaup, D. J. and Champneys, A. R., Embedded solitons: a new type of solitary waves, *Mathematics and computers in Simulation* 56 (2001) 585–600.

[CL90] Chow, S.-N. and Lin, X.-B., Bifurcation of a homoclinic orbit with a saddle-node equilibrium, *J. Diff. and Int. Eq.* 3(3) (1990) 435–466.

[ChH82] Chow, S.-N. and Hale, J. K., *Methods of Bifurcation Theory*, Springer Verlag 1982.

[Cop78] Coppel, W. A., *Dichotomies in Stability Theory,* LNM No. 629, Springer Verlag 1978.

[Del95] Delatte, D., On normal forms in Hamiltonian dynamics, a new approach to some convergence questions, *Ergod. Th. and Dynam. Sys.* 15 (1995) 49–66.

[DH65] Deprit, A. and Henrard, J., Symmetric Doubly Asymptotic Orbits in the restricted Three-Body Problem, *Astronomical Journal* 70 (1965) 271.

[DL05] Delshams, A. and Lázaro, J. T., Pseudo-normal form near saddle-center or saddle-focus equilibria, *J. Diff. Eq* 208 (2005) 312–343.

[Den88] Deng, B., The Shilnikov Problem, Expansions, Strong λ-Lemma, C^1-Linearisation, and Homoclinic Bifurcation, *J. Diff. Eq* 79 (1988) 189–231.

[Den89] Deng, B., Exponential Expansions with Silnikovs's saddle-focus, *J. Diff. Eq.* 82 (1989) 156–173.

[Dev76] Devaney, R., Reversible diffeomorphisms and flows, *Trans. Amer. Math. Soc.* 218 (1976) 89–113.

[FL01] Fiedler, B. and Liebscher, S., Takens-Bogdanov bifurcations without parameters, In: *Global Analysis of Dynamical Systems* Ed.: H. Broer, B. Krauskopf, G. Vegter, IOP Publishing 2001.

[GrR97] Grotta Ragazzo, C., Irregular dynamics and homoclinic orbits to Hamiltonian saddle centers. *Commun. Pure Appl. Math.* 50(2) (1997) 105–147.

[HL86] Hale, J. and Lin, X.-B., Heteroclinic Orbits for retarded Functional Differential Equations, *J. Diff. Eq.* 65 (1986) 175–202.

[Hir76] Hirsch, M. W., *Differential Topology*, Springer Verlag New York Inc. 1976.

[HK06] Homburg, A.J. and Knobloch, J., Multiple homoclinic orbits in conservative and reversible systems, *Trans. Amer. Math. Soc.* 358 (2006) 1715-1740.

[HPS77] Hirsch, M. W., Pugh, C. C. and Shub, M., *Invariant Manifolds*, Lecture Notes in Mathematics, Eds.: A. Dold, B. Eckmann, Springer Verlag Berlin Heidelberg 1977.

[IL99] Ilyashenko, Yu. and Li, W., *Nonlinear Bifurcations*, Mathematical Surveys and Monographs, Vol. 66, American Mathematical Society 1999.

[Irw80] Irwin, M. C., *Smooth Dynamical Systems*, in: Pure and Applied Mathematics Vol. 94, Ed.: H. Bass, A. Borel, J. Moser and S.-T. Yau, Academic Press 1980.

[KCBS02] Kolossovski, K., Champneys, A. R., Buryak, A. V. and Sammut, R. A., Multi-pulse embedded solitons as bound states of quasi-solitons, *Physica D* 171 (2002) 153–177.

[KP90] Kirchgraber, U. and Palmer, K. J., *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, Longman Group UK Limited 1990.

[KK03] Klaus, J. and Knobloch, J., Bifurcation of homoclinic orbits to a saddle-centre in reversible systems, *Int. J. Bifurcation and Chaos* 13(9) (2003) 2606–2622.

[KKW06] Klaus, J., Knobloch, J. and Wagenknecht, T., *A reversible λ-Lemma*, in preparation

[KL95] Koltsova, O. Yu. and Lerman, L. M., Periodic and homoclinic orbits in a two-parameter unfolding of a Hamiltonian system with a homoclinic orbit to a saddle-centre, *Int. J. Bifurcation and Chaos* 5(2) (1995) 397–408.

[KL96] Koltsova, O. Yu. and Lerman, L. M., Families of transverse Poincaré homoclinic orbits in $2n$-dimensional Hamiltonian systems close to the system with a loop to a saddle-centre, *Int. J. Bifurcation and Chaos* 6(6) (1996) 991–1006.

[Kno97] Knobloch, J., Bifurcation of degenerate homoclinics in reversible and conservative systems, *J. Dyn. Diff. Eq.* Vol.9 No.3 (1997) 427–444.

[Kno99] Knobloch, J., *Jump estimates for Lin's method,* Preprint TU Ilmenau No M31/99.

[Kno00] Knobloch, J., Lin's method for discrete dynamical systems, *J. of Difference Eq. and Appl.* Vol. 6 (2000) 577–623.

[Kno04] Knobloch, J., *Lin's method for discrete and continuous dynamical systems and applications*, TU Ilmenau (2004).

[Kuz98] Kuznetsov, Y., *Elements of Applied Bifurcation Theory*, Springer Verlag 1998.

[LR98] Lamb, J. S. W. and Roberts, J. A. G., Time-reversal symmetry in dynamical systems: A survey, *Physica D* 112 (1998) 1–39.

[Ler91] Lerman, L. M., Hamiltonian systems with loops of a separatrix of a saddle-centre, *Selecta Mathematica Sovieta* 10(3) (1991) 297–306.

[Lin90] Lin, X.-B., Using Melnikovs method to solve Shilnikovs problems, *Proc. Roy. Soc. Edinburgh* 116A (1990) 295–325.

[Lin96] Lin, X.-B., Homoclinic bifurcations with weakly expanding center manifolds, in: *Dynamics Reported* Vol. 5 (New Series), Ed.: C. K. R. T. Jones, U. Kirchgraber, H. O. Walther, Springer Verlag, Berlin 1996.

[MHO92] Mielke, A., Holmes, P. and O'Reilly, O., Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle-centre, *J. Dyn. Differ. Equations* 4(1) (1992) 95–126.

[Mie91] Mielke, A., *Hamiltonian and Lagrangian Flows on Center Manifolds with Applications to Elliptic Variational Problems*, Lect. Notes Math. Vol. 1489, Springer Verlag New York (1991).

[Mos56] Moser, J. K., The analytic invariants of an area-preserving mapping near a hyperbolic fixed point, *Comm. Pure and Appl. Math.* Vol IX (1956) 673–692.

[Nir01] Nirenberg, L., *Topics in nonlinear functional analysis*, Courant Lecture Notes in Mathematics, Eds.: J. Shatah, P. D. Monsour, R. Goldsmith, J. Shearer (2001).

[PdM82] Palis, J. and de Melo, W., *Geometric Theory of Dynamical Systems: An Introduction*, Springer Verlag 1982.

[Pal84] Palmer, K. J., Exponential Dichotomies and Transversal Homoclinic Points, *J. Diff. Eq.* 55 (1984) 225-256.

[Po1890] Poincaré, H., Sur le problème des trois corps et les équations de la dynamique (Mémoire couronné du prise de S. M. le roi Oscar de Suède), *Acta Math.* 13 (1890) 1–270.

[Rem96] Remoissenet, M., *Waves called solitons: concepts and experiments*, Springer 1996.

[Rie06] Rieß, Th., *Using Lin's method for a heteroclinic cycle connecting a hyperbolic equilibrium and a hyperbolic periodic orbit*, in preparation

[Rob95] Robinson, C., *Dynamical systems: stability, symbolic dynamics, and chaos*, A volume in Studies in advanced mathematics, CRC Press, Inc. (1995).

[San93] Sandstede, B., *Verzweigungstheorie homokliner Verdopplungen,* IAAS Report No. 7 (1993).

[Scha95] Schalk, U., *Homokline Punkte in periodisch gestörten Systemen gewöhnlicher Differentialgleichungen,* PhD-Thesis TU Ilmenau 1995.

[Sch87] Schecter, S., The saddle-node separatrix-loop bifurcation, *SIAM J. Math. Anal.*, 18(4) (1987) 1142–1156.

[Sch93] Schecter, S., Numerical computation of saddle-node homoclinic bifurcation points, *SIAM J. Numer. Anal.*, 30(4) (1993) 1155–1178.

[Shi63] Shilnikov, L. P., Some cases of periodic motions from singular trajectories, *Mat. Sbornik* 61 (1963) 443–466.

[Shi65] Shilnikov, L. P., A case of the existence of a countable number of periodic motions, *Soviet Math. Dokl.* 6 (1965) 163 –166.

[Shi67] Shilnikov, L. P., On a poincaré-Birkhoff problem, *Math. USSR-Sb.* 3 (1967) 353–371.

[SSTC98] Shilnikov, L. P., Shilnikov, A. L., Turaev, D. V. and Chua, L. O., *Methods of qualitative theory in nonlinear dynamics, Part I*, World Scientific 1998.

[Shu87] Shub, M., *Global stability of dynamical systems*, Springer Verlag New York Inc. 1987.

[Van89] Vanderbauwhede, A., Centre manifolds, normal forms and elementary bifurcations, In: *Dynamics Reported* Vol.2, Ed.: U. Kirchgraber and H.-O. Walther, B.G. Teubner/ John Wiley & Sons 1989.

[Van92] Vanderbauwhede, A. Bifurcation of degenerate homoclinics, *Results in Mathematics* Vol.21 (1992) 211–223.

[VF92] Vanderbauwhede, A. and B. Fiedler, Homoclinic period blow-up in reversible and conservative systems, *ZAMP* 43 (1992) 292–318.

[Wag02] Wagenknecht, T., About a homoclinic pitchfork bifurcation in reversible systems with additional $\mathbb{Z}_2$-symmetry, *Nonlinearity* 15 (2002) 2097–2119.

[Wag03] Wagenknecht, T., *Homoclinic Bifurcations in Reversible Systems*, PhD-Thesis Tu Ilmenau (2003).

[WC02] Wagenknecht, T. and Champneys, A. R., When gap solitons become embedded solitons: a generic unfolding, *Physica D* 177 (2003) 50–70.

[WY06] Wagenknecht, T. and Yagasaki, K., Detection of symmetric homoclinic orbits to saddle-centres in reversible systems, *Physica D* 214 (2) (2006) 169–181.

[Wig90] Wiggings, S., *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer Verlag, New York 1990.

[Zei93] Zeidler, E., *Nonlinear Functional Analysis and its Applications, Vol. I Fixed point theorems*, Springer Verlag 1993.

Zusammenfassung in deutscher Sprache

Einleitung

Seit längerer Zeit besteht starkes Interesse an homoklinen Orbits und ihrem Bifurkationsverhalten in gewöhnlichen Differentialgleichungen. Homokline Orbits sind von mathematischer Bedeutung, da sie häufig organisierende Zentren für die Dynamik in ihrer Umgebung darstellen. Oft werden Homoklinen durch komplizierte Objekte wie k-Homoklinen und k-periodische Orbits begleitet, oder es tritt Shiftdynamik in ihrer Nähe auf. Bei k-Homoklinen und k-periodischen Orbits handelt es sich um Orbits, die einen Transversalschnitt zur Homokline k-mal schneiden. Desweiteren treten Homoklinen in Modellgleichungen auf. So finden sie zum Beispiel Anwendung als solitäre Wellenlösung partieller Differentialgleichungen. Die partielle Differentialgleichung wird durch einen Wellenansatz zu einer gewöhnlichen Differentialgleichung reduziert, Homklinen dieser Gleichung korrespondieren dann zu Solitonen der Ausgangsgleichung.
In Anwendungen spielen Hamiltonsche oder reversible dynamische Systeme eine große Rolle. So sind zum Beispiel die Bewegungsgleichungen mechanischer Systeme ohne Reibung Hamiltonsch und oftmals zusätzlich reversibel. Jedoch existieren auch rein reversible Systeme, wie Beispiele aus der nichtlinearen Optik zeigen. Hamiltonsche und reversible Systeme besitzen vor allem hinsichtlich des Auftretens von Homoklinen oder periodischen Orbits viele Gemeinsamkeiten, [Cha98, LR98]. In letzter Zeit gibt es jedoch auch Untersuchungen, die Unterschiede in der komplizierteren Dynamik, wie zum Beispiel Shiftdynamik, aufzeigen, [HK06]. Insgesamt ist es von großem Interesse, Hamiltonsche und reversible Systeme auf Unterschiede und Gemeinsamkeiten zu untersuchen.

Frühere Studien beschäftigten sich mit Homoklinen an hyperbolische Gleichgewichtslagen (GGL). In den letzten Jahren jedoch finden sich immer mehr Arbeiten zu nicht-hyperbolischen GGL. Im allgemeinen erwartet man in diesem Fall auch Bifurkationen der GGL, wie zum Beispiel bei der Sattel-Knoten-Bifurkation, [Sch87, Sch93, HL86, Lin96]. Unter bestimmten Bedingungen sind nicht-hyperbolische GGL robust, so bleiben zum Beispiel GGL vom Sattel-Zentrums-Typ (das zugehörige Spektrum besitzt ein Paar rein imaginärer Eigenwerte, während die Realteile der restlichen Eigenwerte ungleich Null sind) in Hamiltonschen oder reversiblen Systemen unter Störung erhalten. In beiden Arten von Systemen ist die Zentrumsmannigfaltigkeit der GGl mit einer Familie periodischer Orbits ausgefüllt, [AM67, Dev76].

In dieser Arbeit beschäftigen wir uns mit dem Bifurkationsverhalten von Homoklinen an ein Sattel-Zentrum in reversiblen Systemen. Hinsichtlich dieser Problematik sind die Arbeiten von Mielke, Holmes und O'Reilly, [MHO92], sowie

Koltsova und Lerman, [Ler91, KL95, KL96] von großem Interesse. Mielke, Holmes und O'Reilly untersuchten reversible Hamiltonsche Systeme im $\mathbb{R}^4$, die eine Kodimension-zwei Homokline an ein Sattel-Zentrum besitzen. Dabei konnten sie die Existenz von k-homoklinen Orbits zur GGL und Shiftdynamik nachweisen. Koltsova und Lerman stellten ähnliche Betrachtungen in rein Hamiltonschen Systemen an. über die Ergebnisse von Mielke u.a. hinaus zeigten sie das Auftreten von Homoklinen an Orbits der Zentrumsmannigfaltigkeit. Alle Autoren nutzen in starkem Maße die Hamiltonstruktur der Systeme aus. Dies wirft die Frage nach einer Betrachtung rein reversibler Systeme auf, [Cha98]. Dazu lieferten Champneys und Härterich, [CH00], erste Antworten. Sie untersuchten das Auftreten 2-homokliner Orbits in einer 1-parametrigen Familie rein reversibler Vektorfelder im $\mathbb{R}^4$.
In allen genannten Arbeiten, [MHO92, Ler91, KL95, KL96, CH00], basierte die Analysis auf der Konstruktion von Rückkehrabbildungen.

Problemstellung

In dieser Arbeit greifen wir die Frage nach der Dynamik in rein reversiblen Systemen auf. Wir betrachten einen Kodimension-zwei homoklinen Orbit Γ an eine Sattel-Zentrums-GGL in einem reversiblen System im $\mathbb{R}^{2n+2}$. Wir richten unser Augenmerk auf 1-homokline und 1-periodische Orbits. Im Gegensatz zu den Arbeiten [MHO92, Ler91, KL95, KL96, CH00] nutzen wir für unsere Betrachtungen Lins Methode, [Lin90]. Diese Methode wurde von Lin ursprünglich für die Analyse verbindender Orbits zu hyperbolischen GGL entwickelt. Auch Weiterentwicklungen anderer Autoren, [VF92, San93, Kno97], basierten auf der Annahme hyperbolischer GGL. Daher ist ein weiteres Ziel unserer Arbeit, Lins Methode an die Problematik nicht-hyperbolischer GGL anzupassen.

Wir betrachten ein System

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^{2n+2},\ \lambda \in \mathbb{R}^2\,, \tag{A.50}$$

mit f in C^r und zugehörigem Fluss $\varphi(t, \cdot, \lambda)$. Wir setzen voraus, dass das System reversibel ist, d.h. es existiert eine lineare Involution R ($R^2 = id$) und

(H 1.1) $Rf(x, \lambda) = -f(Rx, \lambda)$.

(Im Anhang der Arbeit haben wir wichtige Begriffe und Resultate für reversible Systeme zusammengetragen.)
Für $\lambda = 0$ besitze das System (A.50) ein Sattel-Zentrum $\mathring{x}$:

(H 1.2) $f(\mathring{x}, 0) = 0$ mit $\sigma(D_1 f(\mathring{x}, 0)) = \{\pm i\} \cup \{\pm\mu\} \cup \sigma^{ss} \cup \sigma^{uu}\,,$

wobei $\mu \in \mathbb{R}^+$ einfach ist; σ bezeichnet das Spektrum von $D_1 f(\mathring{x}, 0)$, σ^{ss} und σ^{uu} das streng stabile bzw. instabile Spektrum.

Diese Voraussetzungen bewirken, dass die zentrumsstabile Mannigfaltigkeit W^{cs} und die zentrumsinstabile Mannigfaltigkeit W^{cu} eindeutig bestimmt sind. Weiter setzen wir die Existenz einer symmetrischen Homokline voraus:

(H 1.3) Für $\lambda = 0$ existiert ein symmetrischer homokliner Orbit $\Gamma := \{\gamma(t) : t \in$

$\mathbb{R}\}$ zum Sattel-Zentrum $\mathring{x}$ mit $R\gamma(0) = \gamma(0)$.

Um Degeneriertheiten zwischen der stabilen Mannigfaltigkeit W^s und W^{cu} auszuschließen, fordern wir

(H 1.4) $\dim(T_{\gamma(0)}W^s \cap T_{\gamma(0)}W^{cu}) = 1$.

Durch folgende Bedingung sichern wir, dass eine typische Familie betrachtet wird.

(H 1.5) $\{W^s_\lambda, \lambda \in U(0)\}$ und der Fixraum $\operatorname{Fix} R$ von R schneiden sich transversal.

Hier bezeichnet $U(0) \subset \mathbb{R}^2$ eine Umgebung der Null im Parameterraum. Wir beschränken uns auf den Fall

(H 1.6) W^{cs} und W^{cu} schneiden sich nicht-transversal in $\gamma(0)$.

Anpassung von Lins Methode

Seien Σ ein transversaler Schnitt zu Γ in $\gamma(0)$ und Z, mit $\gamma(0) + Z \subset \Sigma$, ein Unterraum transversal zu $T_{\gamma(0)}W^s + T_{\gamma(0)}W^u$:

$$\mathbb{R}^{2n+2} = Z \oplus (T_{\gamma(0)}W^s + T_{\gamma(0)}W^u) .$$

Wegen **(H 1.4)** gilt $\dim Z = 3$.

Die im Folgenden definierten Begriffe sind der Frage nach 1-Homoklinen bzw. 1-periodischen Orbits angepasst. In der Arbeit wurde die Definition dieser Begriffe allgemeiner gefasst, so dass mit diesem Ansatz auch k-Homoklinen und k-periodische Orbits erfasst werden.

Wir bezeichnen einen Orbit $X := \{x(t) : t \in [0, \tau]\}$ als *partiellen Orbit, der* Σ *verbindet*, wenn gilt:

(i) $x(0), x(\tau) \in \Sigma$;

(ii) $x(t) \notin \Sigma \quad \forall t \in (0, \tau)$.

Die zugehörige Lösung $x(\cdot)$ nennen wir *partielle Lösung.* Ist zusätzlich die Sprungbedingung

$$\xi := x(\tau) - x(0) \in Z$$

erfüllt, so sprechen wir von einer *1-periodischen Lin-Lösung* $\mathcal{X} := \{x\}$ (mit zugehörigem *1-periodischem Lin-Orbit* $\mathcal{L} := \{X\}$). Lösungen von

$$\xi = x(\tau) - x(0) = 0 ,$$

korrespondieren zu 1-periodischen Orbits.

Wir suchen (symmetrische) partielle Orbits X in der Form $X = (X^+, X^{loc}, X^-)$. Die Orbits X^+ und X^- folgen dem Orbit Γ zwischen Σ und einer kleinen Umgebung um $\mathring{x}$. Der Orbit X^{loc} folgt dem Fluss in der Zentrumsmannigfaltigkeit. Die zugehörigen

Lösungen $x^+(\cdot)$, $x^{loc}(\cdot)$ und $x^-(\cdot)$ sind auf $[0,\Omega]$, $[0,\Omega^{loc}]$ bzw. $[-\Omega,0]$ definiert und erfüllen die folgenden Bedingungen:

$$x^+(0),\, x^-(0) \in \Sigma\ , \quad x^+(\Omega) = x^{loc}(0)\ , \ \ x^-(-\Omega) = x^{loc}(\Omega^{loc})\ .$$

Für die Konstruktion des lokalen Orbits X^{loc} beschreiben wir den lokalen Fluss in der Nähe von $\mathring{x}$ durch eine Rückkehrabbildung $\Pi(\cdot,\lambda)$ bezüglich eines Transversalschnittes Σ_{loc}. Wir setzen voraus, dass Σ_{loc} glatt in $\Pi(\cdot,\lambda)$-invariante Blätter $\mathcal{M}_{p,\lambda}$ geblättert ist. Eingeschränkt auf $\mathcal{M}_{p,\lambda}$ besitzt die Rückkehrabbildung $\Pi(\cdot,\lambda)$ einen hyperbolischen Fixpunkt p. Für hinreichend großes $\Omega \in \mathbb{R}^+$ schneiden sich die Spuren $\mathcal{C}^+(\Omega,p,\lambda)$ und $\mathcal{C}^-(\Omega,p,\lambda)$ von $\varphi(\Omega,\Sigma,\lambda)$ bzw. $\varphi(-\Omega,\Sigma,\lambda)$ in $\mathcal{M}_{p,\lambda}$ mit der (lokalen) stabilen bzw. instabilen Mannigfaltigkeit von p transversal. Ein Argument, welches auf einem λ-Lemma (für hyperbolische Fixpunkte) beruht, liefert für jedes hinreichend große $N \in \mathbb{N}$ ein Orbitsegment (bez. der Rückkehrabbildung), welches $\mathcal{C}^+$ und $\mathcal{C}^-$ in N Schritten verbindet. Dieses Orbitsegment korrespondiert zum gesuchten lokalen Orbit $X^{loc}(\Omega,p,\lambda,N)$. Dabei ist N die Anzahl der “Windungen um $\mathring{x}$”. Da die Kurven $\mathcal{C}^+$ und $\mathcal{C}^-$ R-Bilder voneinander sind, ist X^{loc} ein symmetrischer Orbit.
In der Arbeit beschränken wir uns auf Vektorfelder im $\mathbb{R}^4$. In diesem Falle gilt $\Sigma = \gamma(0) + Z$ und somit ist jeder partielle Orbit auch 1-periodischer Lin-Orbit. Daher erhalten wir $X^{\pm}(\Omega,p,\lambda,N)$ durch einen einfachen Integrationsprozess. Die angegebene Konstruktion liefert somit einen symmetrischen periodischen Lin-Orbit $\mathcal{L}(\Omega,p,\lambda,N)$.

In ähnlicher Art und Weise werden partielle Lösungen und Orbits definiert, die Σ und die lokale Zentrumsmannigfaltigkeit W^c_{loc} von $\mathring{x}$ verbinden. Dabei werden Γ^+ und Γ^- *partielle Orbits, die Σ mit der Zentrumsmannigfaltigkeit verbinden*, genannt, wenn die zugehörigen Lösungen $\gamma^+(\cdot)$ und $\gamma^-(\cdot)$, die folgenden Bedingungen erfüllen:

(i) $\gamma^-(0), \gamma^+(0) \in \Sigma$;

(ii) $\alpha(\Gamma^-), \omega(\Gamma^+) \subset W^c_{loc}$;

(iii) $\gamma^+(t) \notin \Sigma \quad \forall t \in (0,\infty)$, $\gamma^-(t) \notin \Sigma \quad \forall t \in (-\infty,0)$.

Mit $\alpha(\Gamma^-)$ und $\omega(\Gamma^+)$ bezeichnen wir die α- bzw. ω-Grenzmenge von Γ^- bzw. Γ^+. Gilt die Sprungbedingung

$$\xi^\infty := \gamma^+(0) - \gamma^-(0) \in Z\ ,$$

so bezeichnen wir $\mathcal{L}_{hom} := \{(\Gamma^+,\Gamma^-)\}$ als *1-homoklinen Lin-Orbits zur Zentrumsmannigfaltigkeit* und $\mathcal{X}_{hom} := \{(\gamma^+,\gamma^-)\}$ als *1-homokline Lin-Lösung zur Zentrumsmannigfaltigkeit*. Lösungen von

$$\xi^\infty = \gamma^+(0) - \gamma^-(0) = 0$$

korrespondieren zu 1-Homoklinen zur Zentrumsmannigfaltigkeit.

Bei der Suche nach 1-homoklinen Orbits zur Zentrumsmannigfaltigkeit weisen wir in einem ersten Schritt nach, dass für jedes λ eine eindeutig bestimmte Lin-Lösung $\mathcal{X}^0_{hom} = \{(\gamma^s(\lambda), \gamma^u(\lambda))\}$ zu $\mathring{x}$ existiert. Weiter können wir zeigen, dass für alle $y_c^+, y_c^- \in Y^c$ und $\lambda \in \mathbb{R}^2$ eine 1-homokline Lin-Lösung $\mathcal{X}_{hom} = \{(\gamma^+(y_c^+, y_c^-, \lambda), \gamma^-(y_c^+, y_c^-, \lambda))\}$ existiert, wobei Y^c ein 2-dimensionaler Unterraum von Z ist. Dabei betrachten wir $\mathcal{X}_{hom}$ als Störung von $\mathcal{X}^0_{hom}$. Das bedeutet, wir betrachten Variationsgleichungen entlang γ^s bzw. γ^u. Im Anhang der Arbeit zeigen wir, dass diese Gleichungen exponentielle Trichotomien besitzen. Diesen Fakt nutzen wir wesentlich in unserer Analysis aus.

Resultate

Existenz 1-homokliner Orbits zur Zentrumsmannigfaltigkeit: Lösungen der Bifurkationsgleichung

$$\xi(\lambda) := \gamma^s(\lambda)(0) - \gamma^u(\lambda)(0) = 0$$

führen zu 1-homoklinen Orbits zur GGL. Wir zeigen, dass in der Parameterebene eine Kurve durch $\lambda = 0$ existiert, die zu symmetrischen 1-homoklinen Orbits zu $\mathring{x}$ gehört.
Lösungen der Bifurkationsgleichung

$$\xi^\infty(y_c^+, y_c^-, \lambda) := \gamma^+(y_c^+, y_c^-, \lambda)(0) - \gamma^-(y_c^+, y_c^-, \lambda)(0) = 0$$

korrespondieren zu 1-Homoklinen zur Zentrumsmannigfaltigkeit. Bei der Diskussion dieser Verzweigungsgleichung unterscheiden wir bezüglich der relativen Lage von W^{cs} und $\operatorname{Fix} R$. So nennen wir einen symmetrischen homoklinen Orbit Γ *nicht-elementar*, wenn sich W^{cs} und $\operatorname{Fix} R$ nicht-transversal schneiden. Anderenfalls nennen wir den homoklinen Orbit *elementar*.

Sei Γ nicht-elementar. Dann sind alle 1-Homoklinen zur Zentrumsmannigfaltigkeit symmetrisch. Innerhalb unserer Betrachtungen unterscheiden wir wiederum zwei Fälle. Im $\mathbb{R}^4$ gibt es eine einfache geometrische Interpretation. Wir beschreiben die zentrumsstabile Mannigfaltigkeit als Graph einer Funktion h^{cs}. Generisch ist die Hessematrix D^2h^{cs} dieser Funktion nicht-degeneriert, daher ist sie entweder positiv (bzw. negativ) definit oder indefinit.
Ist D^2h^{cs} definit, existiert eine parabelförmige Kurve $\mathcal{C}$ in der Parameterebene, so dass für Parameterwerte auf einer Seite von $\mathcal{C}$ eine 1-parametrige Familie 1-homokliner Orbits zur Zentrumsmannigfaltigkeit existiert. Diese Familie degeneriert für $\lambda = 0$ zu einem Punkt und verschwindet, wenn die Kurve überschritten wird. Ist hingegen D^2h^{cs} indefinit, so existiert für jeden Parameter eine 1-parametrige-Familie von 1-homoklinen Orbits zur Zentrumsmannigfaltigkeit.

Betrachten wir eine elementare Homokline Γ, so finden wir für jeden Parameterwert eine 1-parametrige Familie symmetrischer 1-homokliner Orbits und zwei 1-parametrige Familien nicht-symmetrischer 1-homokliner Orbits zur Zentrumsmannigfaltigkeit. Die zwei Familien nicht-symmetrischer Orbits sind R-Bilder voneinander.

Existenz symmetrischer 1-periodischer Orbits: Das Auftreten symmetrischer 1-periodischer Orbits untersuchen wir im Falle einer nicht-elementaren Homokline Γ. Dabei beschränken wir uns auf Vektorfelder im $\mathbb{R}^4$. Untersuchungen der Bifurkationsgleichung

$$\xi_{per}(\Omega, p, \lambda, N) := x^+(\Omega, p, \lambda, N)(0) - x^-(\Omega, p, \lambda, N)(0) = 0$$

ergeben, dass nur für Parameterwerte auf einer Seite der oben angegebenen Kurve $\mathcal{C}$ symmetrische 1-periodische Orbits existieren. Diese Orbits treten als eine 2-parametrige Familie $\{\mathcal{O}_{\vartheta,N}(\lambda) : (\vartheta, N) \in \mathbb{R} \times \mathbb{N}\}$ auf.
Diese Aussage kann so interpretiert werden, dass jede 1-Homokline zur Zentrumsmannigfaltigkeit von einer 1-parametrigen Familie von symmetrischen 1-periodischen Orbits begleitet wird. Der Familienparameter ist der diskrete Parameter N.

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