Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 06/12

Universal Confidence Sets for Solutions of Optimization Problems

Vogel, Silvia

Dezember 2006

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

Universal Confidence Sets for Solutions of Optimization Problems

Silvia Vogel

Abstract

We consider random approximations to deterministic optimization problems. The objective function and the constraint set can be approximated simultaneously. Relying on concentration-of-measure results we derive universal confidence sets for the constraint set, the optimal value and the solution set. Special attention is paid to solution sets which are not single-valued. Many statistical estimators being solutions to random optimization problems, the approach can also be employed to derive confidence sets for constrained estimation problems.

Keywords: random optimization problems, universal confidence sets, convergence rate, constrained estimation

MSC2000: 90C15, 90C31, 62F25, 62F30

1 Introduction

Random approximations of deterministic or random optimization problems come into play if unknown quantities are replaced with estimates or for numerical reasons. Qualitative stability results, which make assertions on the (semi-)convergence of the constraint sets, the optimal values, and the solution sets, are available for convergence almost surely, in probability and in distribution, c.f. [11], [16], [7], [6], [18], [2]. Furthermore, there are quantitative results which estimate the distance between the optimal values and/or solutions sets by suitable probability metrics, see [13] for an overview.

Confidence bounds for optimal values and solution sets provide valuable additional information. In the traditional way confidence bounds are derived from the distribution of the statistic under consideration. However, the exact distribution is available only in rare cases. Therefore, often knowledge about the limit distribution is used as a surrogate. Hence qualitative stability results for convergence in distribution can be employed to derive asymptotic confidence sets, see [11] and [2].

In [12] Pflug suggests an approach which can be used to derive even non-asymptotic confidence sets without knowledge about the exact distribution. The starting point are assertions on convergence in probability supplemented with a suitable convergence rate and tail behavior. In [12] uniform convergence with known convergence rate and tail behavior of the objective functions together with a growth condition are assumed and results on inner approximations of the solution sets are proved. In this way non-asymptotic confidence sets can be obtained if the solution set is single-valued.

We will pursue the way proposed in [12] farther, take into account also the approximation of the constraint set, and show how one can proceed if the solution set is not single-valued. The results are formulated in a general way, allowing e.g. for 'ε_n-relaxed' constraint sets and ε_n-optimal solutions. We will also assume that suitable assertions on the (one-sided) uniform convergence in probability of the objective functions and/or the constraint functions with a convergence rate and tail function are available, albeit these conditions are rather restrictive. It is, however, also possible to derive similar, yet more technical assertions if one assumes some kind of continuous convergence in probability with convergence rate and tail behavior. This so-called pointwise approach opens the possibility to employ directly suitable concentration inequalities. This method will be investigated elsewhere.

The results will be illustrated by three examples. Firstly, we discuss a simple example in order to show how one can deal with the uniform convergence assumptions. Secondly, we consider the approximation for a chance constraint. The third example was chosen to emphasize the applicability of our results in statistics. Many statistical estimators being solutions to random optimization problems, our assertions can be employed to derive confidence bounds for estimators, even if one does not have full knowledge about the distribution of the underlying statistic. We will provide universal confidence bounds for quantiles, allowing for distribution functions which are not continuous.

The paper is organized as follows. In Chapter 2 we introduce the mathematical model and show how universal confidence sets can be derived. In Chapter 3 and Chapter 4 we prove the needed convergence assertions for the constraint sets, the optimal values, and the solutions sets. Chapter 5 contains the examples.

2 Universal confidence sets

Let (E, d) be a complete separable metric space and $[\Omega, \Sigma, P]$ a complete probability space. We assume that a deterministic optimization problem
$(P_0)\ \min\limits_{x\in\Gamma_0} f_0(x)$
is approximated by a sequence of random problems
$(P_n)\ \min\limits_{x\in\Gamma_n(\omega)} f_n(x,\omega), \qquad n \in N.$
Additionally to (P_n), for a given $\varepsilon > 0$, we consider so-called ε-relaxations
$(P_{n,\varepsilon})\ \min\limits_{x\in\Gamma_{n,\varepsilon}(\omega)} f_{n,\varepsilon}(x,\omega), \qquad n \in N.$
The relaxed problems offer the possibility to deal with approximate constraint sets, objective functions and/or solution sets. Consequently, the approach can be applied, e.g., to constraint sets and functions which are obtained by Monte Carlo methods (c.f. [14], [10]) or to methods which use plug-in estimators. Moreover, the relaxed problems occur in a natural manner if we aim at deriving outer approximations of constraint sets and solution sets.

The following results will be formulated for $(P_{n,\varepsilon})$. The problem (P_n) is then regarded as a special case of $(P_{n,\varepsilon})$ with objective functions and constraint sets that do not depend on ε.

Γ_0 is a nonempty closed subset of E, and $f_0|E \to \bar{R}^1$ is a lower semicontinuous function. $\Gamma_{n,\varepsilon}|\Omega \to 2^E$ are closed-valued measurable multifunctions, and $f_{n,\varepsilon}|E \times \Omega \to \bar{R}^1$ are lower semicontinuous random functions which are supposed to be $(\mathcal{B}(E) \otimes \Sigma, \bar{\mathcal{B}}^1)$-measurable. $\mathcal{B}(E)$ denotes the Borel-σ-field of E and $\bar{\mathcal{B}}^1$ the σ-field of Borel sets of $\bar{R}^1$. The measurability conditions imposed here do not have the weakest form. We use them for sake of simplicity. They are satisfied in many applications and guarantee that all functions of ω needed in the following have the necessary measurability properties. Furthermore, it should be mentioned that the lower semicontinuity assumption of the objective functions $f_{n,\varepsilon}$ can be dropped. Imposing this condition, however, we can omit some technical details in the proofs. Eventually, we assume that all objective functions are (almost surely) proper functions, i.e. functions with values in $(-\infty, +\infty]$ which are not identically ∞.

Our main concern will be with the solution sets Ψ_0 of (P_0) and $\Psi_{n,\varepsilon}$ of $(P_{n,\varepsilon})$. We aim at proving assertions of the form

$$\forall \varepsilon > 0 : \sup_{n \geq n_0(\varepsilon)} P\{\omega : \ \Psi_{n,\varepsilon}(\omega) \setminus U_{\beta_{n,\varepsilon}} \Psi_0 \neq \emptyset\} \leq \mathcal{K}(\varepsilon) \tag{1}$$

and

$$\forall \varepsilon > 0 : \sup_{n \geq n_0(\varepsilon)} P\{\omega : \Psi_0 \setminus U_{\beta_{n,\varepsilon}} \Psi_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}(\varepsilon). \tag{2}$$

Here $(\beta_{n,\varepsilon})_{n\in N}$ is a sequence of nonnegative numbers which tends to zero for each $\varepsilon > 0$ and $\mathcal{K}|R^+ \to R^+$ is a function with $\lim\limits_{\varepsilon\to\infty} \mathcal{K}(\varepsilon) = 0$. $U_\alpha X$ denotes an open neighborhood of the set X with radius α:
$U_\alpha X := \{x \in E : d(x, X) < \alpha\}$.

Because of the similarity with inner approximations in probability of sequences of random sets (c.f. [7], [19]) we will call a sequence $(\Psi_{n,\varepsilon})_{n\in N}$ fulfilling the first relation an *inner approximation in probability to* Ψ_0 *with convergence rate* $\beta_{n,\varepsilon}$ *and tail behavior function* $\mathcal{K}$ (in short, an *inner* $(\beta_{n,\varepsilon}, \mathcal{K})$*-approximation*). Correspondingly a sequence $(\Psi_{n,\varepsilon})_{n\in N}$ fulfilling the second relation will be called an *outer approximation in probability to* Ψ_0 *with convergence rate* $\beta_{n,\varepsilon}$ *and tail behavior function* $\mathcal{K}$ (in short, an *outer* $(\beta_{n,\varepsilon}, \mathcal{K})$*-approximation*).

In order to derive *universal confidence sets to the level* ε_0, i.e. a sequence of random sets $(C_n)_{n\in N}$ with $\sup\limits_{n\geq n_0} P\{\omega : \Psi_0 \setminus C_n(\omega) \neq \emptyset\} \leq \varepsilon_0$, we can proceed as follows:

Suppose that an outer $(\beta_{n,\varepsilon}, \mathcal{K})$-approximation $(\Psi_{n,\varepsilon})_{n\in N}$ to Ψ_0 is available and choose, to $\varepsilon_0 > 0$, an $\varepsilon > 0$ such that $\mathcal{K}(\varepsilon) \leq \varepsilon_0$. The set

$$C_n := U_{\beta_{n,\varepsilon}} \Psi_{n,\varepsilon} \tag{3}$$

has the desired property. Of course, one is interested in small confidence sets, hence $(\beta_{n,\varepsilon})_{n\in N}$ should go to zero as fast as possible and $\mathcal{K}$ should converge to zero as fast as possible if ε tends to infinity.

If one considers approximating problems (P_n) without relaxation, then under reasonable conditions one obtains inner approximations only. This is satisfying if all approximating problems (P_n) have solutions which are uniformly bounded and the solution set to the problem (P_0) is single-valued, because in this case inner approximations are also outer approximations and we can proceed as above.

What can be done if the solution set to (P_0) is not single-valued ? Taking into account that we need knowledge about convergence rates anyway, we can exploit this knowledge to determine suitable relaxing sequences $(\kappa_{n,\varepsilon})_{n\in N}$, which tend to zero for each $\varepsilon > 0$ and consider $\kappa_{n,\varepsilon}$-optimal solutions $\Psi_{n,\varepsilon}^{\kappa_n}$.

The problem, that, without relaxing, only inner approximations can be obtained is also apparent for the constraint sets. There are problems, where $\kappa_{n,\varepsilon}$-relaxations are needed to obtain outer approximations of the constraint sets. An example will be considered in Chapter 5.

In order to obtain reasonable confidence sets one would like to have $\lim_{n\to\infty} \beta_{n,\varepsilon}^{(i)} = 0$ and $\lim_{\varepsilon\to\infty} \mathcal{K}_i \to 0$ for all sequences $(\beta_{n,\varepsilon}^{(i)})_{n\in N}$ and functions $\mathcal{K}_i$ which occur in the following. As mentioned, in order to obtain 'small' confidence sets, the limits should go to zero as fast as possible. These properties are, however, not needed to prove the results in Chapter 3 and Chapter 4. We only assume throughout the paper that the sequences $(\beta_{n,\varepsilon}^{(i)})_{n\in N}$ are non-increasing sequences of positive numbers and the functions $\mathcal{K}_i|R^+ \to R^+$ are non-increasing.

As a by-product, from the results presented in the following, one can also derive confidence sets for the optimal values, proceeding as described above for the solution sets. Hence the approach can help to assess the quality of a solution to an approximate optimization problem and may be of interest also for model selection.

3 Approximation of the constraint set

In this section we consider constraint sets, which are given by inequality constraints, and their approximations. Results of that kind are of course needed if approximation of the constraint set is inherent in the problem under consideration. Moreover, the statements will be employed to derive assertions on the behavior of the solution sets, regarding the difference between the true objective function and the optimal value as constraint function. As the objective values for problems which $\kappa_{n,\varepsilon}$-relaxed constraint sets may depend on ε, the resulting constraint function may depend on ε, too. Furthermore, when applying Theorem 1 below to the solution sets, the multifunctions Q_n will be interpreted as constraint sets, hence we have to allow that they depend on ε, too. Consequently, we have to make sure, that even the results for the constraint sets hold for relaxed constraint functions and multifunctions.

Let Q_0 a closed non-empty subset of E and $J = \{1, \ldots, j_M\}$ a finite index set. We consider functions $g_0^j|E \to R^1,\ j \in J$, which are lower semicontinuous in all points $x \in E$, and define $\Gamma_0 := \{x :\ g_0^j(x) \leq 0,\ j \in J\} \cap Q_0$. Γ_0 is assumed to be nonempty.

For each $\varepsilon > 0$, the set Q_0 is approximated by a sequence $(Q_{n,\varepsilon})_{n\in N}$ of closed-valued measurable multifunctions, and the functions $g_0^j,\ j \in J$, are approximated by sequences $(g_{n,\varepsilon}^j)_{n\in N}$ of functions $g_{n,\varepsilon}^j|E \to R^1,\ j \in J$, which are $(\mathcal{B}(E) \otimes \Sigma, \mathcal{B}^1)$-measurable. Furthermore, we assume that the functions $g_{n,\varepsilon}(\cdot,\omega)$ are lower semicontinuous for all $\omega \in \Omega$. Also these measurability and semicontinuity properties could be weakened.

Eventually $\Gamma_{n,\varepsilon}$ is defined by
$\Gamma_{n,\varepsilon}(\omega) := \{x \in E :\ g_{n,\varepsilon}^j(x,\omega) \leq 0,\ j \in J\} \cap Q_{n,\varepsilon}(\omega)$. Under our assumptions $\Gamma_{n,\varepsilon}$ is a closed-valued measurable multifunction.

If we have $Q_0 = Q_{n,\varepsilon}(\omega) \equiv E$, we will use the denotation $\hat{\Gamma}_0$ and $\hat{\Gamma}_{n,\varepsilon}$, respectively: $\hat{\Gamma}_0 = \{x \in E :\ g_0^j(x) \leq 0,\ j \in J\}$ and $\hat{\Gamma}_{n,\varepsilon}(\omega) := \{x \in E : g_{n,\varepsilon}^j(x,\omega) \leq 0,\ j \in J\}$.

In the following we use functions ν, μ and λ. We assume that they map R^+ into R^+, fulfill $\nu(0) = \mu(0) = \lambda(0) = 0$, are increasing and non-constant. By the superscript $^{-1}$ we denote their right inverses: $\nu^{-1}(y) := \inf\{x \in R : \nu(x) > y\}$.

Theorem 1 (Inner Approximation of the Constraint Set) *Assume that the following conditions are satisfied:*

(CI1) There exist a function $\mathcal{K}_1$ and to all $\varepsilon > 0$ a sequence $(\beta_{n,\varepsilon}^{(1)})_{n\in N}$ such that
$\sup_{n\in N} P\{\omega :\ Q_{n,\varepsilon}(\omega) \setminus U_{\beta_{n,\varepsilon}^{(1)}} Q_0 \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon).$

(CI2) There exist a function $\mathcal{K}_2$ and to all $\varepsilon > 0$ a sequence $(\beta_{n,\varepsilon}^{(2)})_{n\in N}$ such that
$\sup_{j\in J} \sup_{n\in N} P\{\omega :\ \inf_{x\in UQ_0\setminus\Gamma_0} (g_{n,\varepsilon}^j(x,\omega) - g_0^j(x)) \leq -\beta_{n,\varepsilon}^{(2)}\} \leq \mathcal{K}_2(\varepsilon)$
for a suitable neighborhood UQ_0.

(CI3) There exists a function ν such that for all $\varepsilon > 0$
$U_\varepsilon\Gamma_0 \supset U_{\nu(\varepsilon)}Q_0 \cap U_{\nu(\varepsilon)}\hat{\Gamma}_0.$

(CI4) There exists a function μ such that for all $\varepsilon > 0$
$\forall\, x \in U_\varepsilon Q_0 \setminus U_\varepsilon\hat{\Gamma}_0\ \ \exists j \in J :\ g_0^j(x) \geq \mu(\varepsilon).$

Then for all $\varepsilon > 0$, $\beta_{n,\varepsilon}^{(3)} = \max\{\nu^{-1}(\beta_{n,\varepsilon}^{(1)}), \nu^{-1}(\mu^{-1}(\beta_{n,\varepsilon}^{(2)}))\}$, and $n_0(\varepsilon) = \min\{k :\ U_{\beta_{k,\varepsilon}^{(3)}} Q_0 \subset UQ_0\}$ *the relation*

$$\sup_{n\geq n_0(\varepsilon)} P\{\omega:\ \Gamma_{n,\varepsilon}(\omega)\setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0\neq\emptyset\}\leq \mathcal{K}_1(\varepsilon)+j_M\mathcal{K}_2(\varepsilon)$$

holds.

Proof. Assume that for given $\varepsilon>0$, $n\geq n_0(\varepsilon)$, and $\omega\in\Omega$ the relation $\sup_{n\geq n_0(\varepsilon)} P\{\omega:\ \Gamma_{n,\varepsilon}(\omega)\setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0\neq\emptyset\}$ holds. Then there is $x_{n,\varepsilon}(\omega)\in\Gamma_{n,\varepsilon}(\omega)$ which does not belong to $U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0$. Because of (CI3) we have $x_{n,\varepsilon}(\omega)\notin U_{\nu(\beta^{(3)}_{n,\varepsilon})}Q_0$ or $x_{n,\varepsilon}(\omega)\in U_{\nu(\beta^{(3)}_{n,\varepsilon})}Q_0$ and $x_{n,\varepsilon}(\omega)\notin U_{\nu(\beta^{(3)}_{n,\varepsilon})}\hat{\Gamma}_0$. In the first case we obtain $Q_{n,\varepsilon}\setminus U_{\nu(\beta^{(3)}_{n,\varepsilon})}Q_0\neq\emptyset$. Hence, because of $\nu(\beta^{(3)}_{n,\varepsilon})\geq\beta^{(1)}_{n,\varepsilon}$, $Q_{n,\varepsilon}(\omega)\setminus U_{\beta^{(1)}_{n,\varepsilon}}Q_0\neq\emptyset$, and we can employ (CI1). In the second case we obtain by (CI4) for at least one $j\in J$, $g^j_0(x_{n,\varepsilon}(\omega))\geq\mu(\nu(\beta^{(3)}_{n,\varepsilon}))\geq\beta^{(2)}_{n,\varepsilon}$, hence, because of $U_{\nu(\beta^{(3)}_{n,\varepsilon})}Q_0\subset UQ_0$, $\inf_{x\in UQ_0\setminus\Gamma_0}(g^j_{n,\varepsilon}(x,\omega)-g^j_0(x))\leq-\beta^{(2)}_{n,\varepsilon}$. It remains to employ (CI2). □

The conditions (CI2) and (CI4) can be replaced using strict inequalities in the following form without changing the conclusion of Theorem 1. This holds correspondingly for further assertions.

(CI2-s) There exist a function $\mathcal{K}_2$ and to all $\varepsilon>0$ a sequence $(\beta^{(2)}_{n,\varepsilon})_{n\in N}$ such that
$\sup_{j\in J}\sup_{n\in N} P\{\omega:\ \inf_{x\in UQ_0\setminus\Gamma_0}(g^j_{n,\varepsilon}(x,\omega)-g^j_0(x))<-\beta^{(2)}_{n,\varepsilon}\}\leq\mathcal{K}_2(\varepsilon)$
for a suitable neighborhood UQ_0.

(CI4-s) There exists a function μ such that for all $\varepsilon>0$
$\forall\, x\in U_\varepsilon Q_0\setminus U_\varepsilon\hat{\Gamma}_0\ \ \exists j\in J:\ g^j_0(x)>\mu(\varepsilon)$.

If the set Q_0 remains fixed in all approximations, i.e. $Q_{n,\varepsilon}\equiv Q_0$, the assumptions in Theorem 1 can be weakened.

Corollary 1.1 *Assume that the following assumptions are satisfied:*

(CI2-W) There exist a function $\mathcal{K}_2$ and to all $\varepsilon>0$ a sequence $(\beta^{(2)}_{n,\varepsilon})_{n\in N}$ such that
$\sup_{j\in J}\sup_{n\in N} P\{\omega:\ \inf_{x\in Q_0\setminus\Gamma_0}(g^j_{n,\varepsilon}(x,\omega)-g^j_0(x))\leq-\beta^{(2)}_{n,\varepsilon}\}\leq\mathcal{K}_2(\varepsilon)$.

(CI4-W) *There exists a function* μ *such that for all* $\varepsilon > 0$

$$\forall\, x \in Q_0 \setminus U_\varepsilon \hat{\Gamma}_0 \;\; \exists j \in J: \;\; g_0^j(x) \geq \mu(\varepsilon).$$

Then for all $\varepsilon > 0$ *and* $\beta^{(3)}_{n,\varepsilon} = \mu^{-1}(\beta^{(2)}_{n,\varepsilon})$ *the relation*

$$\sup_{n\in N} P\{\omega: \; \Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0 \neq \emptyset\} \leq j_M \mathcal{K}_2(\varepsilon)$$

holds.

Proof. We can choose $\nu(\varepsilon) = \varepsilon$. (CI1) is satisfied with $\mathcal{K}_1 \equiv 0$ and an arbitrary $\beta^{(1)}_{n,\varepsilon}$. As we are interested in small values of $\beta^{(3)}_{n,\varepsilon}$, we use the form given in the assertion. □

If we want to employ Theorem 1 for solution sets we have to deal with one constraint function only. The following corollary is a specialization of Theorem 1 to $j_M = 1$ and $g_0^1 =: g_0$, $g^1_{n,\varepsilon} =: g_{n,\varepsilon}$. Additionally, we replace (CI4) with a growth condition.

Corollary 1.2 *Assume that (CI1), (CI2), (CI3), and the following condition (Gr-*g_0*) are satisfied.*

(Gr-g_0*)* *There exist an increasing function* $\psi_1 | R^+ \to R^+$ *and constants* $c_1 > 0$, $\delta_1 > 0$, *and* $\theta_1 > 0$ *such that*

$$\forall x \in UQ_0: \;\; g_0(x) \geq \psi_1(d(x, \hat{\Gamma}_0)) \;\; \textit{and}$$
$$\forall\, 0 < \theta < \theta_1: \psi_1(\theta) \geq c_1 \cdot \theta^{\delta_1}.$$

Then for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{\nu^{-1}(\beta^{(1)}_{n,\varepsilon}), \nu^{-1}((\frac{\beta^{(2)}_{n,\varepsilon}}{c_1})^{\frac{1}{\delta_1}})\}$, *and* $n_0(\varepsilon) = \min\{k: \; U_{\beta^{(3)}_{k,\varepsilon}} Q_0 \subset U_{\theta_1} Q_0 \cap UQ_0\}$ *the relation*

$$\sup_{n \geq n_0(\varepsilon)} P\{\omega: \; \Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0 \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon)$$

holds.

If the functions $g_{n,\varepsilon}$ do not depend on ε, $Q_{n,\varepsilon} \equiv Q_0$ and $\beta^{(2)}_{n,\varepsilon} = \frac{\varepsilon}{\gamma_n}$ holds, instead of (CI2) the following condition can be used:

(CI2') There exist a function $\mathcal{K}_2$ and a sequence $(\gamma_n)_{n\in N} \to \infty$ such that

$$\sup_{n\in N} P\{\omega: \; \gamma_n(\inf_{x \in Q_0 \setminus \Gamma_0} (g_n(x,\omega) - g_0(x)) \leq -\varepsilon\} \leq \mathcal{K}_2(\varepsilon).$$

In this case Corollary 1.2 can be simplified in the following way:

Corollary 1.3 *Assume that (CI2') and (Gr-g_0) with Q_0 instead of UQ_0 are satisfied. Then for all $\varepsilon > 0$ the relation*
$$\sup_{n\in N} P\{\omega: \ \Gamma_n(\omega) \setminus U_{\frac{\varepsilon}{(\gamma_n)^{\frac{1}{\delta_1}}}}\Gamma_0 \neq \emptyset\} \leq \mathcal{K}_2(c_1\varepsilon^{\delta_1})$$
holds.

Proof. With Corollary 1.1 and Corollary 1.2 we obtain
$\sup_{n\in N} P\{\omega: \ \Gamma_n(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0 \neq \emptyset\} \leq \mathcal{K}_2(\varepsilon)$ with $\beta^{(3)}_{n,\varepsilon} = (\frac{\varepsilon}{c_1\gamma_n})^{\frac{1}{\delta_1}}$. Replacing ε with $c_1\eta^{\delta_1}$ yields the conclusion. □

As mentioned in the introduction, in general, the sequence $(\Gamma_n)_{n\in N}$ approximates a subset of Γ_0 only. In order to obtain outer approximations, additional assumptions have to be imposed. Qualitative stability theory usually assumes that the condition $\Gamma_0 \subset \mathrm{cl}\{x \in Q_0: \ g_0^j(x) < 0, \ \forall j \in J\}$ is fulfilled where cl denotes the closure. In order to obtain also a convergence rate and a tail function, we impose a 'quantified version' of this assumption, see (CO3) below. Unfortunately, a condition of that kind is useless if one intends to employ the result for the solution set, because (CO3) can not be satisfied by $\tilde{g}_n := f_n - \Phi_n$ where Φ_n denotes the optimal value to the problem (P_n).

Hence we will provide a second approach which considers inequality constraints of the form $g^j_{n,\varepsilon}(x,\omega) < \kappa_{n,\varepsilon}, \ j \in J$, where $(\kappa_{n,\varepsilon})_{n\in N}$ is a suitable sequence of positive reals with $\lim_{n\to\infty} \kappa_{n,\varepsilon} = 0 \ \forall\varepsilon > 0$.

For the formulation of (CO3) we need the ε-interior of Γ_0. Let, for a given $\varepsilon > 0$, $CI(\varepsilon) := \Gamma_0 \setminus U_\varepsilon(E \setminus \Gamma_0)$. $\bar{U}_\varepsilon$ denotes the closure of U_ε.

Theorem 2 (Outer Approximation of the Constraint Set, Inner Point) *Assume that the following conditions are satisfied:*

(CO1) There exist a function $\mathcal{K}_1$ and to all $\varepsilon > 0$ a sequence $(\beta^{(1)}_{n,\varepsilon})_{n\in N}$ such that
$$\sup_{n\in N} P\{\omega: \ Q_0 \setminus U_{\beta^{(1)}_{n,\varepsilon}}Q_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon).$$

(CO2) There exist a function $\mathcal{K}_2$ and to all $\varepsilon > 0$ a sequence $(\beta^{(2)}_{n,\varepsilon})_{n\in N}$ such that
$$\sup_{j\in J}\ \sup_{n\in N} P\{\omega: \ \sup_{x\in\Gamma_0}(g^j_{n,\varepsilon}(x,\omega) - g^j_0(x)) \geq \beta^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_2(\varepsilon).$$

(CO3) There exist $\tilde{\varepsilon} > 0$ and a function μ such that for all $0 < \varepsilon \le \tilde{\varepsilon}$
$\Gamma_0 \subset \bar{U}_\varepsilon CI(\varepsilon)$ *and*
$\forall x \in CI(\varepsilon)\, \forall j \in J: \; g_0^j(x) \le -\mu(\varepsilon)$.

Then for all $0 < \varepsilon \le \tilde{\varepsilon}$, $\beta^{(3)}_{n,\varepsilon} = \max\{\beta^{(1)}_{n,\varepsilon}, \mu^{-1}(2\beta^{(2)}_{n,\varepsilon})\}$, and $n_0(\varepsilon) = \min\{k : \beta^{(3)}_{k,\varepsilon} \le 2\tilde{\varepsilon}\}$ the relation
$\sup\limits_{n \ge n_0(\varepsilon)} P\{\omega : \; \Gamma_0 \setminus (U_{\beta^{(3)}_{n,\varepsilon}} \hat{\Gamma}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)) \ne \emptyset\} \le \mathcal{K}_1(\varepsilon) + j_M \mathcal{K}_2(\varepsilon)$
holds.

Proof. Assume that for given $\varepsilon > 0$, $n \in N$, and $\omega \in \Omega$ the relation $\Gamma_0 \setminus (U_{\beta^{(3)}_{n,\varepsilon}} \hat{\Gamma}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)) \ne \emptyset$ holds. Then there is $x_{n,\varepsilon}(\omega) \in \Gamma_0$ which does not belong to $U_{\beta^{(3)}_{n,\varepsilon}} \hat{\Gamma}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)$. If $x_{n,\varepsilon}(\omega) \notin U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)$ we can employ (CO1). If $x_{n,\varepsilon}(\omega) \notin U_{\beta^{(3)}_{n,\varepsilon}} \hat{\Gamma}_{n,\varepsilon}(\omega)$ we can choose $\tilde{x}_{n,\varepsilon}(\omega) \in CI(\frac{\beta^{(3)}_{n,\varepsilon}}{2})$ with $\tilde{x}_{n,\varepsilon}(\omega) \notin \hat{\Gamma}_{n,\varepsilon}(\omega)$, i.e. $g^{j_0}_{n,\varepsilon}(\tilde{x}_{n,\varepsilon}(\omega), \omega) > 0$ for at least one $j_0 \in J$. Because of $g_0^{j_0}(\tilde{x}_{n,\varepsilon}(\omega)) \le -\mu(\frac{\beta^{(3)}_{n,\varepsilon}}{2}) \le -\beta^{(2)}_{n,\varepsilon}$ we can employ (CO2). □

Now we consider $\kappa_{n,\varepsilon}$-relaxed inequality constraints. Let $\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) := \{x \in E : \; g^j_{n,\varepsilon}(x, \omega) < \kappa_{n,\varepsilon}, \; j \in J\}$ and $\Gamma^{\kappa_n}_{n,\varepsilon}(\omega) = \hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap Q_{n,\varepsilon}$.

Theorem 3 (Outer Approximation of the Constraint Set, Relaxation)
Assume that (CO1) and (CO2) are satisfied.

Then for all $\varepsilon > 0$, $\kappa_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon}$ and $\beta^{(3)}_{n,\varepsilon} = \max\{\beta^{(1)}_{n,\varepsilon}, \beta^{(2)}_{n,\varepsilon}\}$ the relation
$\sup\limits_{n \in N} P\{\omega : \; \Gamma_0 \setminus (\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)) \ne \emptyset\} \le \mathcal{K}_1(\varepsilon) + j_M \mathcal{K}_2(\varepsilon)$
holds.

Proof: Assume that for given $\varepsilon > 0$, $n \in N$, and $\omega \in \Omega$ the relation $\Gamma_0 \setminus (\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)) \ne \emptyset$ holds. Then there is $x_{n,\varepsilon}(\omega) \in \Gamma_0$ which does not belong to $\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)$. Hence $g_0^j(x_{n,\varepsilon}(\omega)) \le 0 \; \forall j \in J$ and $x_{n,\varepsilon}(\omega) \in Q_0(\omega)$, but either $x_{n,\varepsilon}(\omega) \notin U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega)$ or $g^j_{n,\varepsilon}(x_{n,\varepsilon}(\omega), \omega) > \beta^{(2)}_{n,\varepsilon} = \kappa_{n,\varepsilon}$ for at least one $j \in J$. In the first case we obtain $Q_0 \setminus U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}(\omega) \ne \emptyset$. The second case yields $\sup\limits_{x \in \Gamma_0} (g^j_{n,\varepsilon}(x, \omega) - g_0^j(x)) > \beta^{(2)}_{n,\varepsilon}$ for at least one $j \in J$.
□

A corresponding result holds if $U_{\beta^{(1)}_{n,\varepsilon}} Q_{n,\varepsilon}$ is replaced with $Q_{n,\varepsilon}$ in the condition (CO1) and in the assertion.

The question arises under what conditions $(\Gamma^{\kappa_n}_{n,\varepsilon})_{n\in N}$ is also an inner approximation. Results of that kind will help to assess the quality of an outer approximation. An inspection of the proof to Theorem 1 shows that with $\kappa_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon}$ the following statement can be obtained.

Theorem 4 (Inner Approximation of the Constraint Set, Relaxation) *Assume that (CI1), (CI2), (CI3), and (CI4) are satisfied. Then for all* $\varepsilon > 0$, $\kappa_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon}$, $\beta^{(3)}_{n,\varepsilon} = \max\{\nu^{-1}(\beta^{(1)}_{n,\varepsilon}), \nu^{-1}(\mu^{-1}(2\beta^{(2)}_{n,\varepsilon}))\}$, *and* $n_0(\varepsilon) = \min\{k : U_{\beta^{(3)}_{k,\varepsilon}} Q_0 \subset UQ_0\}$ *the relation*
$\forall \varepsilon > 0 \sup_{n\geq n_0(\varepsilon)} P\{\omega : \Gamma^{\kappa_n}_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0 \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon) + j_M \mathcal{K}_2(\varepsilon)$
holds.

Finally, we will summarize what we obtain for one constraint function under a growth condition.

Theorem 5 (Approximation of the Constraint Set, Relaxation) *Assume that* $j_M = 1$ *and (CI1), (CO1) (CI2), (CO2), (CI3), and (Gr-*g_0*) are satisfied.*
Then for all $\varepsilon > 0$, $\kappa_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon}$, $\beta^{(3)}_{n,\varepsilon} = \max\{\beta^{(1)}_{n,\varepsilon}, \beta^{(2)}_{n,\varepsilon}, \nu^{-1}(\beta^{(1)}_{n,\varepsilon}), \nu^{-1}((\frac{2\beta^{(2)}_{n,\varepsilon}}{c_1})^{\frac{1}{\delta_1}})\}$, *and* $n_0(\varepsilon) = \min\{k : U_{\beta^{(3)}_{k,\varepsilon}} Q_0 \subset UQ_0\}$ *the relation*
$\sup_{n\geq n_0(\varepsilon)} P\{\omega : (\Gamma^{\kappa_n}_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0) \cup (\Gamma_0 \setminus (\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap Q_{n,\varepsilon}(\omega)) \neq \emptyset\} \leq 2\mathcal{K}_1(\varepsilon) +$
$2\mathcal{K}_2(\varepsilon)$
holds.

Proof. We have
$P\{\omega : (\Gamma^{\kappa_n}_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0) \cup (\Gamma_0 \setminus (\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap Q_{n,\varepsilon}(\omega)) \neq \emptyset\}$
$\leq P\{\omega : \Gamma^{\kappa_n}_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0 \neq \emptyset\} + P\{\omega : \Gamma_0 \setminus (\hat{\Gamma}^{\kappa_n}_{n,\varepsilon}(\omega) \cap Q_{n,\varepsilon}(\omega)) \neq \emptyset\}$.
The assumptions of Theorem 3 and Theorem 4 are satisfied with $\mu(\varepsilon) = c_1\varepsilon^{\delta_1}$ and $\tilde{\beta}^{(1)}_{n,\varepsilon} = \tilde{\beta}^{(2)}_{n,\varepsilon} = \beta^{(3)}_{n,\varepsilon}$. □

Unfortunately, the result is not 'symmetric'. In order to derive a result of the form

$$\sup_{n\geq n_0(\varepsilon)} P\{\omega:\ (\Gamma^{\kappa_n}_{n,\varepsilon}(\omega)\setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0)\cup(\Gamma_0\setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma^{\kappa_n}_{n,\varepsilon}(\omega))\neq\emptyset\}\leq\mathcal{K}(\varepsilon)$$

we would need functions $\nu_{n,\varepsilon}$ and conditions similar to (CI3) for each n.

Now assume that there is only one constraint function g_n, which does not depend on ε, and $Q_{n,\varepsilon}\equiv Q_0$ holds. If $\beta^{(2)}_{n,\varepsilon}=\frac{\varepsilon}{\gamma_n}$, instead of (CO2) the following condition can be used:

(CO2') There exist a function $\mathcal{K}_2$ and a sequence $(\gamma_n)_{n\in N}\to\infty$ such that $\sup_{n\in N} P\{\omega:\ \gamma_n \sup_{x\in\Gamma_0}(g_n(x,\omega)-g_0(x))\geq\varepsilon\}\leq\mathcal{K}_2(\varepsilon)$.

Then the following result can be derived.

Corollary 5.1 *Assume that (CI2'), (CO2'), and (Gr-g_0) with Q_0 instead of UQ_0 are satisfied. Then for all $\varepsilon>0$, $\kappa_{n,\varepsilon}=\frac{\varepsilon}{\gamma_n}$, $\beta^{(3)}_{n,\varepsilon}=\max\{\frac{\varepsilon}{\gamma_n},(\frac{2\varepsilon}{c_1\gamma_n})^{\frac{1}{\delta_1}}\}$, and $n_0(\varepsilon)=\min\{k:\ \beta^{(3)}_{k,\varepsilon}\leq\theta_1\}$ the relation*

$$\sup_{n\geq n_0(\varepsilon)} P\{\omega:\ (\Gamma^{\kappa_n}_{n,\varepsilon}(\omega)\setminus U_{\beta^{(3)}_{n,\varepsilon})}\Gamma_0)\cup(\Gamma_0\setminus\Gamma^{\kappa_n}_{n,\varepsilon}(\omega))\neq\emptyset\}\leq 2\mathcal{K}_2(\varepsilon)$$

holds.

Proof. The proof follows by Theorem 5 proceeding in a similar way as in the derivation of Corollary 1.3. □

If $\max\{\frac{\varepsilon}{\gamma_n},(\frac{2\varepsilon}{c_1\gamma_n})^{\frac{1}{\delta_1}}\}=(\frac{2\varepsilon}{c_1\gamma_n})^{\frac{1}{\delta_1}}$ the result can be rewritten as in Corollary 1.3.

4 Approximation of the optimal values and the solution sets

We turn to the optimal values and the solutions sets of the problems (P_0) and $(P_{n,\varepsilon})$. Let $\Phi_{n,\varepsilon}(\omega):=\inf_{x\in\Gamma_{n,\varepsilon}(\omega)} f_{n,\varepsilon}(x,\omega)$ and $\Psi_{n,\varepsilon}(\omega)$ denote the corresponding solution set. In the following, the constraint sets and their approximations are not supposed the have a special form. Especially, $\Gamma_{n,\varepsilon}$ can have the form which was used in Chapter 3, but it can also denote a set originating from a relaxation like $\Gamma^{\kappa_n}_{n,\varepsilon}$.

We do not impose compactness conditions on Γ_0 and $\Gamma_{n,\varepsilon}$. Instead we assume, for sake of simplicity, that the original and the approximating problems have a solution.

Theorem 6 (Lower Approximation of the Optimal Value) *Assume that the following conditions are satisfied:*

(VL1) There exist a function $\mathcal{K}_1$ and to all $\varepsilon > 0$ a sequence $(\beta^{(1)}_{n,\varepsilon})_{n\in N}$ such that
$\sup_{n\in N} P\{\omega : \Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(1)}_{n,\varepsilon}}\Gamma_0 \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon)$.

(VL2) There exist a function $\mathcal{K}_2$ and to all $\varepsilon > 0$ a sequence $(\beta^{(2)}_{n,\varepsilon})_{n\in N}$ such that
$\sup_{n\in N} P\{\omega : \inf_{x\in U\Gamma_0}(f_{n,\varepsilon}(x,\omega) - f_0(x)) \leq -\beta^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_2(\varepsilon)$ *for a suitable neighborhood $U\Gamma_0$.*

(VL3) There exists a function λ such that for all $\varepsilon > 0$
$\forall\, x \in U_{\lambda(\varepsilon)}\Gamma_0 \cap U\Gamma_0 : \ f_0(x) \geq \Phi_0 - \varepsilon$.

Then for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{2\lambda^{-1}(\beta^{(1)}_{n,\varepsilon}), 2\beta^{(2)}_{n,\varepsilon}\}$, and
$n_0(\varepsilon) = \min\{k : U_{\lambda(\frac{\beta^{(3)}_{k,\varepsilon}}{2})}\Gamma_0 \subset U\Gamma_0\}$ *the relation*
$$\sup_{n\geq n_0(\varepsilon)} P\{\omega : \Phi_{n,\varepsilon}(\omega) - \Phi_0 \leq -\beta^{(3)}_{n,\varepsilon}\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon)$$
holds.

Proof. Assume that for given $\varepsilon > 0$, $n \geq n_0(\varepsilon)$, and $\omega \in \Omega$ the relation $\Phi_{n,\varepsilon}(\omega) \leq \Phi_0 - \beta^{(3)}_{n,\varepsilon}$ holds. Then there exists $x_{n,\varepsilon}(\omega) \in \Gamma_{n,\varepsilon}(\omega)$ such that $f_{n,\varepsilon}(x_{n,\varepsilon}(\omega),\omega) = \Phi_{n,\varepsilon}(\omega) \leq \Phi_0 - \beta^{(3)}_{n,\varepsilon}$.

Firstly, let $x_{n,\varepsilon}(\omega) \in U_{\lambda(\frac{\beta^{(3)}_{n,\varepsilon}}{2})}\Gamma_0$. Then
$$\inf_{x\in U\Gamma_0}(f_{n,\varepsilon}(x,\omega) - f_0(x)) \leq f_{n,\varepsilon}(x_{n,\varepsilon}(\omega),\omega) - f_0(x_{n,\varepsilon}(\omega)) \leq \Phi_{n,\varepsilon}(\omega) - \Phi_0 + \frac{\beta^{(3)}_{n,\varepsilon}}{2}$$
$$\leq -\frac{\beta^{(3)}_{n,\varepsilon}}{2} \leq -\beta^{(2)}_{n,\varepsilon}.$$

Eventually, if $x_{n,\varepsilon}(\omega) \notin U_{\lambda(\frac{\beta^{(3)}_{n,\varepsilon}}{2})}\Gamma_0$, we have $\Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(1)}_{n,\varepsilon}}\Gamma_0 \neq \emptyset$. □

The proof shows that also the following assertion holds.

Corollary 6.1 *Assume that* $\Gamma_{n,\varepsilon} \equiv \Gamma_0$ *and the following condition is satisfied:*

(VL2') There exist a function $\mathcal{K}_2$ *and to all* $\varepsilon > 0$ *a sequence* $(\beta_{n,\varepsilon}^{(2)})_{n\in N}$ *such that*
$\sup_{n\in N} P\{\omega : \inf_{x\in\Gamma_0} (f_{n,\varepsilon}(x,\omega) - f_0(x)) \leq -\beta_{n,\varepsilon}^{(2)}\} \leq \mathcal{K}_2(\varepsilon).$

Then for all $\varepsilon > 0$ *the relation*
$\sup_{n\in N} P\{\omega : \Phi_{n,\varepsilon}(\omega) - \Phi_0 \leq -\beta_{n,\varepsilon}^{(2)}\} \leq \mathcal{K}_2(\varepsilon)$
holds.

In the following we consider upper approximations and distinguish two cases according to whether
$\forall\varepsilon > 0 : \sup_{n\geq n_0(\varepsilon)} P\{\omega : \Gamma_0 \setminus U_{\beta_{n,\varepsilon}^{(1)}}\Gamma_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon)$ or
$\forall\varepsilon > 0 : \sup_{n\geq n_0(\varepsilon)} P\{\omega : \Gamma_0 \setminus \Gamma_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon)$
is imposed. As expected, in the second case we obtain a better tail behavior.

Theorem 7 (Upper Approximation of the Optimal Value I) *Assume that the following conditions are satisfied:*

(VU1) There exist a function $\mathcal{K}_1$ *and to all* $\varepsilon > 0$ *a sequence* $(\beta_{n,\varepsilon}^{(1)})_{n\in N}$ *such that*
$\sup_{n\in N} P\{\omega : \Gamma_0 \setminus U_{\beta_{n,\varepsilon}^{(1)}}\Gamma_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon).$

(VU2) There exist a function $\mathcal{K}_2$ *and to all* $\varepsilon > 0$ *a sequence* $(\beta_{n,\varepsilon}^{(2)})_{n\in N}$ *such that*
$\sup_{n\in N} P\{\omega : \sup_{x\in U\Psi_0} (f_{n,\varepsilon}(x,\omega) - f_0(x)) \geq \beta_{n,\varepsilon}^{(2)}\} \leq \mathcal{K}_2(\varepsilon)$ *for a suitable neighborhood* $U\Psi_0$.

(VU3) There exists a function λ *such that for all* $\varepsilon > 0$
$\forall\, x \in U_{\lambda(\varepsilon)}\Psi_0 \cap U\Psi_0 : \; f_0(x) \leq \Phi_0 + \varepsilon.$

Then for all $\varepsilon > 0$, $\beta_{n,\varepsilon}^{(3)} = \max\{2\lambda^{-1}(\beta_{n,\varepsilon}^{(1)}), 2\beta_{n,\varepsilon}^{(2)}\}$, *and*
$n_0(\varepsilon) = \min\{k : U_{\lambda(\frac{\beta_{k,\varepsilon}^{(3)}}{2})}\Psi_0 \subset U\Psi_0\}$ *the relation*
$\sup_{n\geq n_0(\varepsilon)} P\{\omega : \Phi_{n,\varepsilon}(\omega) - \Phi_0 \geq \beta_{n,\varepsilon}^{(3)}\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon)$
holds.

Proof. Assume that for given $\varepsilon > 0$, $n \geq n_0(\varepsilon)$, and $\omega \in \Omega$ the relation $\Phi_{n,\varepsilon}(\omega) \geq \Phi_0 + \beta^{(3)}_{n,\varepsilon}$ holds. Then there exists $x_{n,\varepsilon}(\omega) \in \Gamma_{n,\varepsilon}(\omega)$ such that $f_{n,\varepsilon}(x_{n,\varepsilon}(\omega), \omega) = \Phi_{n,\varepsilon}(\omega) \geq \Phi_0 + \beta^{(3)}_{n,\varepsilon}$. To $x_{n,\varepsilon}(\omega)$ we select $\tilde{x}_{n,\varepsilon}(\omega) \in \Gamma_{n,\varepsilon}(\omega)$ such that $d(\tilde{x}_{n,\varepsilon}(\omega), \Psi_0) = \min\limits_{x \in \Gamma_{n,\varepsilon}(\omega)} d(x, \Psi_0)$.
Firstly, assume that $\tilde{x}_{n,\varepsilon}(\omega) \in U_{\lambda(\frac{\beta^{(3)}_{n,\varepsilon}}{2})} \Psi_0$. Then
$f_{n,\varepsilon}(\tilde{x}_{n,\varepsilon}(\omega), \omega) \geq \Phi_{n,\varepsilon}(\omega) \geq \Phi_0 + \beta^{(3)}_{n,\varepsilon} \geq f_0(\tilde{x}_{n,\varepsilon}(\omega)) + \frac{\beta^{(3)}_{n,\varepsilon}}{2}$
and consequently, $\sup\limits_{x \in U\Psi_0} (f_{n,\varepsilon}(x, \omega) - f_0(x)) \geq \frac{\beta^{(3)}_{n,\varepsilon}}{2} \geq \beta^{(2)}_{n,\varepsilon}$. If $\tilde{x}_{n,\varepsilon}(\omega) \notin U_{\lambda(\frac{\beta^{(3)}_{n,\varepsilon}}{2})} \Psi_0$, we have $\Gamma_0 \setminus U_{\beta^{(1)}_{n,\varepsilon}} \Gamma_{n,\varepsilon}(\omega) \neq \emptyset$. □

If we impose the special upper semicontinuity condition (UCon) for f_0, we obtain the following corollary.

Corollary 7.1 (Upper Approximation of the Optimal Value I) *Assume that (VU1), (VU2) and the following condition (UCon) are satisfied:*

(UCon) There exist an increasing function $\psi_2 | R^+ \to R^+$ and constants $c_2 > 0$, $\delta_2 > 0$ and $\theta_2 > 0$ such that
$\forall x \in U\Psi_0 : \ f_0(x) \leq \Phi_0 + \psi_2(d(x, \Psi_0))$ *and*
$\forall \ 0 < \theta < \theta_2 : \psi_2(\theta) \leq c_2 \theta^{\delta_2}$.

Then, for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{2c_2(\beta^{(1)}_{n,\varepsilon})^{\delta_2}, 2\beta^{(2)}_{n,\varepsilon}\}$, and
$n_0(\varepsilon) = \min\{k : \ U_{\lambda_{k,\varepsilon}} \Psi_0 \subset U\Psi_0 \cap U_{\delta_2}\Psi_0\}$ *with* $\lambda_{k,\varepsilon} = (\frac{\beta^{(3)}_{k,\varepsilon}}{2c_2})^{\frac{1}{\delta_2}}$
the relation
$\sup\limits_{n \geq n_0(\varepsilon)} P\{\omega : \ \Phi_{n,\varepsilon}(\omega) - \Phi_0 \geq \beta^{(3)}_{n,\varepsilon}\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon)$
holds.

Proof. We employ Theorem 7. Because of (UCon) we can choose $\lambda(\theta) = (\frac{\theta}{c_2})^{\frac{1}{\delta_2}}$ and consequently $\lambda^{-1}(\varepsilon) = c_2 \varepsilon^{\delta_2}$. □

A similar Corollary can be proved for the lower approximation of the optimal values. We give only the result for the upper approximation because we will use it in the following. Furthermore, the assertions can be supplemented by results similar to Corollary 1.3 and Corollary 5.1.

Theorem 8 (Upper Approximation of the Optimal Value II) *Assume that the following conditions are satisfied:*

(VU1-R) *There exist a function* $\mathcal{K}_1$ *and to all* $\varepsilon > 0$ *a sequence* $(\beta^{(1)}_{n,\varepsilon})_{n\in N}$ *such that*
$\sup_{n\in N} P\{\omega : \ \Gamma_0 \setminus \Gamma_{n,\varepsilon}(\omega) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon).$

(VU2-R) *There exist a function* $\mathcal{K}_2$ *and to all* $\varepsilon > 0$ *a sequence* $(\beta^{(2)}_{n,\varepsilon})_{n\in N}$ *such that*
$\sup_{n\in N} P\{\omega : \ \sup_{x\in\Psi_0} (f_{n,\varepsilon}(x,\omega) - f_0(x)) \geq \beta^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_2(\varepsilon).$

Then for all $\varepsilon > 0$ *the relation*
$\sup_{n\in N} P\{\omega : \ \Phi_{n,\varepsilon}(\omega) - \Phi_0 \geq \beta^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon)$
holds.

Proof. Assume that for given $\varepsilon > 0$, $n \in N$, and $\omega \in \Omega$ the relation $\Phi_{n,\varepsilon}(\omega) \geq \Phi_0 + \beta^{(2)}_{n,\varepsilon}$ holds. Then there exists $x_{n,\varepsilon}(\omega) \in \Gamma_{n,\varepsilon}(\omega)$ such that $f_{n,\varepsilon}(x_{n,\varepsilon}(\omega),\omega) = \Phi_{n,\varepsilon}(\omega) > \Phi_0 + \beta^{(2)}_{n,\varepsilon}$. To $x_{n,\varepsilon}(\omega)$ we select $\tilde{x}_{n,\varepsilon}(\omega) \in \Gamma_{n,\varepsilon}(\omega)$ such that $d(\tilde{x}_{n,\varepsilon}(\omega), \Psi_0) = \min_{x\in\Gamma_{n,\varepsilon}(\omega)} d(x, \Psi_0)$.

If $\tilde{x}_{n,\varepsilon}(\omega) \in \Psi_0$ we have
$f_{n,\varepsilon}(\tilde{x}_{n,\varepsilon}(\omega),\omega) \geq \Phi_{n,\varepsilon}(\omega) \geq \Phi_0 + \beta^{(2)}_{n,\varepsilon} = f_0(\tilde{x}_{n,\varepsilon}(\omega)) + \beta^{(2)}_{n,\varepsilon}$
and consequently, $\sup_{x\in\Psi_0} (f_{n,\varepsilon}(x,\omega) - f_0(x)) \geq \beta^{(2)}_{n,\varepsilon}$.

Otherwise we have $\Gamma_0 \setminus \Gamma_{n,\varepsilon}(\omega) \neq \emptyset$ and can employ the first assumption.
□

Now we turn to the solution sets. We use the abbreviation $\hat{\Psi} = \{x \in E : f_0(x) \leq \Phi_0\}$.

Theorem 9 (Inner Approximation of the Solution Set) *Assume that (VL1), (VL2) and the following assumptions are satisfied:*

(SI3) *There exist a function* $\mathcal{K}_3$ *and to all* $\varepsilon > 0$ *a sequence* $(\hat{\beta}^{(2)}_{n,\varepsilon})_{n\in N}$ *and* $n_0(\varepsilon)$ *such that*
$\sup_{n\geq n_0(\varepsilon)} P\{\omega : \ \Phi_{n,\varepsilon}(\omega) - \Phi_0 \geq \hat{\beta}^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_3(\varepsilon).$

(SI4) *There exists a function* ν *such that for all* $\varepsilon > 0$
$U_\varepsilon \Psi_0 \supset U_{\nu(\varepsilon)}\Gamma_0 \cap U_{\nu(\varepsilon)}\hat{\Psi}_0$.

(SI5) *There exists a function* μ *such that for all* $\varepsilon > 0$
$\forall\, x \in U_\varepsilon\Gamma_0 \setminus U_\varepsilon\hat{\Psi}_0 : \; f_0(x) \geq \Phi_0 + \mu(\varepsilon)$.

Then for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{\nu^{-1}(\beta^{(1)}_{n,\varepsilon}), \nu^{-1}(\mu^{-1}(\beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}))\}$, *and*
$n_1(\varepsilon) = \min\{k \geq n_0(\varepsilon) : \; U_{\beta^{(3)}_{k,\varepsilon}}\Gamma_0 \subset U\Gamma_0\}$ *the relation*
$\sup_{n \geq n_1(\varepsilon)} P\{\omega : \; \Psi_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}}\Psi_0 \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon) + \mathcal{K}_3(\varepsilon)$
holds.

Proof. Let $\tilde{g}_{n,\varepsilon}(x,\omega) := f_{n,\varepsilon}(x,\omega) - \Phi_{n,\varepsilon}(\omega)$, $\tilde{g}_0(x) := f_0(x) - \Phi_0$. Then $\Psi_{n,\varepsilon}(\omega) = \Gamma_{n,\varepsilon}(\omega) \cap \{x \in E : \; \tilde{g}_{n,\varepsilon}(x,\omega) \leq 0\}$ and
$\Psi_0 = \Gamma_0 \cap \{x \in E : \; \tilde{g}_0(x) \leq 0\}$. Furthermore,

$$\sup_{n \geq n_0(\varepsilon)} P\{\omega : \inf_{x \in U\Gamma_0 \setminus \Psi_0} (\tilde{g}_{n,\varepsilon}(x,\omega) - \tilde{g}_0(x)) \leq -\beta^{(2)}_{n,\varepsilon} - \hat{\beta}^{(2)}_{n,\varepsilon}\}$$
$$\leq \sup_{n \geq n_0(\varepsilon)} P\{\omega : \inf_{x \in U\Gamma_0 \setminus \Psi_0} (f_{n,\varepsilon}(x,\omega) - f_0(x)) \leq -\beta^{(2)}_{n,\varepsilon}\}$$
$$+ \sup_{n \geq n_0(\varepsilon)} P\{\omega : \; -\Phi_{n,\varepsilon}(\omega) + \Phi_0 \leq -\hat{\beta}^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_2(\varepsilon) + \mathcal{K}_3(\varepsilon) =: \tilde{\mathcal{K}}_2(\varepsilon).$$

It remains to apply Theorem 1 with $\tilde{\beta}^{(2)}_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}$ and $\tilde{\mathcal{K}}_2$. □

We emphasize that we can choose $\nu(\varepsilon) = \varepsilon$ if $\Gamma_{n,\varepsilon} \equiv \Gamma_0$.

Furthermore, if $U_{\tilde{\varepsilon}}\Psi_0 \subset \Gamma_0$ for a suitable $\tilde{\varepsilon} > 0$, we can also deal with $\nu(\varepsilon) = \varepsilon$ for all $\varepsilon \leq \tilde{\varepsilon}$.

Corollary 9.1 (Inner Approximation of the Solution Set)
Assume that (VL1), (VL2), (VU1), (VU2), (SI4), (UCon), and the following condition are satisfied:

(Gr-f_0) *There exist an increasing function* $\psi_1 | R^+ \to R^+$ *and constants* $c_1 > 0$, $\delta_1 > 0$, *and* $\theta_1 > 0$ *such that*
$\forall x \in U\Gamma_0 : \; f_0(x) - \Phi_0 \geq \psi_1(d(x, \hat{\Psi}_0))$ *and*
$\forall\, 0 < \theta < \theta_1 : \psi_1(\theta) \geq c_1 \cdot \theta^{\delta_1}$.

Then for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{\nu^{-1}(\beta^{(1)}_{n,\varepsilon}), \nu^{-1}((\frac{\hat{\beta}^{(2)}_{n,\varepsilon} + \beta^{(2)}_{n,\varepsilon}}{c_1})^{\frac{1}{\delta_1}})\}$,
$\hat{\beta}^{(2)}_{n,\varepsilon} = \max\{2c_2(\beta^{(1)}_{n,\varepsilon})^{\delta_2}, 2\beta^{(2)}_{n,\varepsilon}\}$, *and*
$n_1(\varepsilon) = \min\{k \geq n_0(\varepsilon) : \; U_{\beta^{(3)}_{k,\varepsilon}}\Gamma_0 \subset U\Gamma_0, \; U_{\beta^{(3)}_{k,\varepsilon}}\Psi_0 \subset U\Psi_0, \; \beta^{(3)}_{k,\varepsilon} \geq \max\{\theta_1, \theta_2\}\}$

the relation
$$\sup_{n\geq n_1(\varepsilon)} P\{\omega :\ \Psi_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}}\Psi_0 \neq \emptyset\} \leq 2\mathcal{K}_1(\varepsilon) + 2\mathcal{K}_2(\varepsilon)$$
holds.

Proof: We apply Theorem 9 together with Corollary 7.1. (SI3) is satisfied with $\hat{\beta}^{(2)}_{n,\varepsilon}$ and $\mathcal{K}_3 = \mathcal{K}_1 + \mathcal{K}_2$. (VL2) is satisfied with $\hat{\beta}^{(2)}_{n,\varepsilon}$ and $\mathcal{K}_2$. Theorem 9 with $\mu^{-1}(\varepsilon) = (\frac{\varepsilon}{c_1})^{\frac{1}{\delta_1}}$ yields the conclusion. □

If for all $\varepsilon > 0$ the condition

(CK-R) There exist a function $\mathcal{K}_1$ and to all $\varepsilon > 0$ a sequence $(\beta^{(1)}_{n,\varepsilon})_{n\in N}$ such that
$$\sup_{n\in N} P\{\omega :\ (\Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(1)}_{n,\varepsilon}}\Gamma_0) \cup (\Gamma_0 \setminus \Gamma_{n,\varepsilon}(\omega)) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon)$$

is satisfied, then also (VL1) and (VU1) are fulfilled with $\mathcal{K}_1$ and $\beta^{(1)}_{n,\varepsilon}$. Consequently, if $\Gamma_{n,\varepsilon} = \Gamma^{\kappa_n}_{n,\varepsilon}$ we can employ Theorem 5 or Corollary 5.1 in order to determine a suitable $\kappa_{n,\varepsilon}$ and formulate sufficient conditions for (VL1) and (VU1).

Now we consider outer approximations of the solution set via $\kappa_{n,\varepsilon}$-optimal solutions of the approximating problems.

Let $\hat{\Psi}^{\kappa_n}_{n,\varepsilon}(\omega) := \{x \in E :\ f_{n,\varepsilon}(x,\omega) < \Phi_{n,\varepsilon}(\omega) + \kappa_{n,\varepsilon}\}$. $\Gamma_{n,\varepsilon}$ can, e.g., be specified as $U_{\beta^{(1)}_{n,\varepsilon}}\Gamma_n$ or as $\Gamma^{\kappa_n}_{n,\varepsilon}$.

Theorem 10 (Outer Approximation of the Solution Set, Relaxation)
Assume that for all $\varepsilon > 0$ there exist sequences $(\beta^{(i)}_{n,\varepsilon})_{n\in N}$, $i = 1,2$, and functions $\mathcal{K}_i$, $i = 1,2$, such that (VU1-R), (VU2-R) and the following assumption are satisfied:

(SO3) There exist a function $\mathcal{K}_3$ and to all $\varepsilon > 0$ a sequence $(\hat{\beta}^{(2)}_{n,\varepsilon})_{n\in N}$ and $n_0(\varepsilon)$ such that
$$\sup_{n\geq n_0(\varepsilon)} P\{\omega :\ \Phi_{n,\varepsilon}(\omega) - \Phi_0 \leq -\hat{\beta}^{(2)}_{n,\varepsilon}\} \leq \mathcal{K}_3(\varepsilon).$$

Then for all $\varepsilon > 0$, $\kappa_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}$ and $\beta^{(3)}_{n,\varepsilon} = \max\{\beta^{(1)}_{n,\varepsilon}, \beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}\}$ the relation

$$\sup_{n\in N(\varepsilon)} P\{\omega:\ \Psi_0 \setminus (\hat{\Psi}^{\kappa_n}_{n,\varepsilon}(\omega) \cap U_{\beta^{(1)}_{n,\varepsilon}}\Gamma_{n,\varepsilon}(\omega)) \neq \emptyset\} \leq \mathcal{K}_1(\varepsilon) + \mathcal{K}_2(\varepsilon) + \mathcal{K}_3(\varepsilon)$$

holds.

Proof. We apply Theorem 3 and label the denotations in Theorem 3 with a tilde. With $\tilde{g}_{n,\varepsilon}(x,\omega) := f_{n,\varepsilon}(x,\omega) - \Phi_{n,\varepsilon}(\omega)$ and $\tilde{g}_0(x) := f_0(x) - \Phi_0$ we have $\Psi_0 = \{x \in \Gamma_0 :\ \tilde{g}_0(x) \leq 0\}$ and $\hat{\Psi}^{\kappa_n}_{n,\varepsilon}(\omega) := \{x \in E :\ \tilde{g}_{n,\varepsilon}(x,\omega) < \kappa_{n,\varepsilon}\}$. Furthermore, let $\tilde{Q}_{n,\varepsilon} = \Gamma_{n,\varepsilon}$, $\tilde{\hat{\Gamma}}^{\kappa_n}_{n,\varepsilon} = \hat{\Psi}^{\kappa_n}_{n,\varepsilon}$, $\tilde{Q}_0 = \Gamma_0$, and $\tilde{\hat{\Gamma}}_0 = \hat{\Psi}_0$.

Because of $P\{\omega:\ \inf_{x\in\Psi_0}((f_{n,\varepsilon}(x,\omega) - \Phi_{n,\varepsilon}(\omega)) - (f_0(x) - \Phi_0)) \geq \beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}\}$ $\leq P\{\omega:\ \inf_{x\in\Psi_0}(f_{n,\varepsilon}(x,\omega) - f_0(x)) \geq \beta^{(2)}_{n,\varepsilon}\} + P\{\omega:\ -\Phi_{n,\varepsilon}(\omega) + \Phi_0) \geq \hat{\beta}^{(2)}_{n,\varepsilon}\} \leq$ $\mathcal{K}_2(\varepsilon) + \mathcal{K}_3(\varepsilon) =: \tilde{\mathcal{K}}_2(\varepsilon)$ condition (CO2) is satisfied with $\tilde{\beta}^{(2)}_{n,\varepsilon} = \beta^{(2)}_{n,\varepsilon} + \hat{\beta}^{(2)}_{n,\varepsilon}$ and $\tilde{\mathcal{K}}_2$. □

5 Examples

We will illustrate the approach by three examples. When applying our results to problems in decision theory or estimation theory, the most critical assumption is probably (SI4). Fortunately, there are several important applications where some quantities do not vary with n and ν is not needed at all or, as in our third example, (SI4) is easy to verify.

In the general case, however, if the constraint set and the objective function are approximated simultaneously and the solution lies on the boundary of the constraint set, ν can not be ignored. Only in rare cases one should have enough knowledge to determine it exactly. One way out are adaptive methods for successive approximation of ν. However, even if one does not succeed in determining ν with a satisfactory accuracy, our results still yield assertions on the convergence rate, albeit without a reliable constant. Results of that kind can be used to derive asymptotic confidence sets if a limiting distribution is not available.

Firstly, we will discuss a simple example which is intended to give an idea of how one can deal with the uniform convergence assumption for the objective functions. At the first glance this assumption seems to be rather restrictive. There is, however, a growing number of results from probability theory yielding assertions of that kind, c.f. [15], [1], [12]. A more refined investigation, which relies on stability assertions adjusted to the direct utilization of concentration-of measure inequalities for sequences of random variables, will be given elsewhere. Moreover, we would like to refer the reader to [12] where several sufficient conditions are gathered.

We assume $E = R^p$ and consider a fixed compact constraint set K and a linear objective function $q(z)^T x$ with $x = (x_1, \dots, x_p)^T$, $q|R^m \to R^p$. z is the realization of a random vector Z with given distribution P_Z on the sigma-field of Borel sets of R^m. The range of $q(Z)$ is supposed to be bounded. For sake of simplicity we deal with the metric $d(x, y) = \max\limits_{i=1,\dots,p} |x_i - y_i|$. The problem

(P_0) $\quad \min\limits_{x \in K} \mathrm{E}q(Z)^T x$

is approximated, replacing the expectation with respect to P_Z by the expectation with respect to the empirical distribution based on a sequence $Z^{(j)}, i = 1, 2, \dots,$ of i.i. P_Z-distributed random vectors:

(P_n) $\quad \min\limits_{x \in K} \frac{1}{n} \sum\limits_{j=1}^{n} q(Z^{(j)})^T x.$

We assume $\mathrm{E}q(Z) \neq \mathbf{0}$, because otherwise the problem becomes trivial, and abbreviate $m := \max\limits_{i=1,\dots,p} \sup\limits_{\omega} |q_i(Z(\omega))|$.

We consider the sets $K_k = \{x \in K : \ k-1 \leq d(0, x) \leq k\}, \ k = 1, 2, \dots$ Let $I_K := \{k : \ K_k \cap K \neq \emptyset\}$. Hence we obtain by Hoeffding's inequality ([3], [1]):

$$P\{\omega : \sup_{x \in K} |\tfrac{1}{n} \sum_{j=1}^{n} q(Z^{(j)}(\omega))^T x - \mathrm{E}q(Z)^T x| \geq \tfrac{\varepsilon}{\sqrt{n}}\}$$

$$\leq \sum_{k \in I_K} P\{\omega : \sup_{x \in K_k} |\tfrac{1}{n} \sum_{j=1}^{n} q(Z^{(j)}(\omega))^T x - \mathrm{E}q(Z)^T x| \geq \tfrac{\varepsilon}{\sqrt{n}}\}$$

$$\leq \sum_{k \in I_K} P\{\omega : \max_{i=1,\dots,p} |\tfrac{1}{n} \sum_{j=1}^{n} q_i(Z^{(j)}(\omega)) - \mathrm{E}q_i(Z)| \geq \tfrac{\varepsilon}{k\sqrt{n}}\}$$

$$\leq 2 \sum_{k \in I_K} e^{-\frac{\varepsilon^2}{2k^2 m^2}} =: \mathcal{K}_2(\varepsilon) \ .$$

Of course this inequality can be further improved. For example, due to the linearity, it is enough to consider a cover of the boundary of K only. Employing other concentration inequalities, also the boundedness condition for $q(Z)$ can be overcome. Moreover, if the functions have a more involved form, one can often proceed in a similar way.

We aim at employing Theorem 9 and (in case of a non-unique solution) Theorem 10. (VL1), (VU1), and (VU1-R) are satisfied with $\mathcal{K}_1 \equiv 0$. (VL2), (VU2), and (VU2-R) are fulfilled with $\mathcal{K}_2$. (SI4) is satisfied with $\nu(\varepsilon) = \varepsilon$.

It remains to investigate the growth condition (Gr-f_0), where we can choose $U\Gamma_0 = \Gamma_0$, and the semicontinuity condition (UCon).
We have $f_0(x) - \Phi_0 \leq \max\limits_{i=1,\ldots,p} \mathrm{E}|q_i(z)|\; d(x, \Psi_0)$, hence (UCon) is satisfied with $c_2 = \max\limits_{i=1,\ldots,p} \mathrm{E}|q_i(z)|$ and $\delta_2 = 1$.
Let $I := \{i \in \{1, \ldots, p\} :\; \mathrm{E}q_i(Z) \neq 0\}$. According to our assumption I is not empty. To x with $d(x, \hat{\Psi}_0) > \varepsilon$ there are $\hat{x} \in \hat{\Psi}_0$ and $i \in I$ such that $|x_i - \hat{x}_i| > \varepsilon$. Consequently (Gr-$f_0$) is satisfied with $c_1 = \min\limits_{i \in I} \mathrm{E}|q_i(Z)|$ and $\delta_1 = 1$.

Secondly, we consider the approximation of a constraint set which is determined by a probabilistic constraint. Again, replacing the true probability measure with the empirical measure, we obtain a sequence of approximating constraint functions.

In detail, we assume that the constraint function has the special form $g_0(x) = \alpha - P_Z((-\infty, \gamma(x)]) = \alpha - F_Z(\gamma(x))$.

Here Z and P_Z are defined as above; F_Z denotes the corresponding distribution function. Additionally we restrict the considerations to $p = 1$. $\alpha \in (0, 1)$ denotes a probability level and $\gamma|E \to R^1$ a given concave function. The inequality constraint $g_0(x) \leq 0$ then reads as $P(Z \leq \gamma(x)) \geq \alpha$. We assume that $\Gamma_0 = \hat{\Gamma}_0 = \{x \in E :\; g_0(x) \leq 0\} \neq \emptyset$. The approximating constraint set has the form $\Gamma_n(\omega) = \{x \in E :\; \alpha - F_n(\gamma(x), \omega) \leq 0\}$ with the empirical distribution function F_n .

In order to fulfil (CI2') and (CO2') we can directly apply the Dvoretzky-Kiefer-Wolfowitz inequality with Massart's bound ([9], [1]) and we obtain
$P\{\omega :\; \sqrt{n} \sup\limits_{x \in R^1} |(\alpha - F_n(\gamma(x), \omega)) - (\alpha - F_Z(\gamma(x))| > \varepsilon\} \leq 2e^{-2\varepsilon^2}$.

(CI3) is not needed. In order to fulfill (Gr-g_0), we will impose growth conditions for F_Z and γ.

Assume that, for the given probability level α, the α-quantile q_α of F_Z is unique and consider a compact set $\tilde{K}$ such that $q_\alpha \in \mathrm{int}\tilde{K}$. Furthermore, let $X_{\tilde{K}} := \{x \in E :\; \gamma(x) \in \tilde{K}\}$ and suppose that the following conditions are satisfied:

(IG) There exist positive constants $c_{1,\gamma},\; c_{1,F},\; \delta_{1,\gamma},\; \delta_{1,F}$ such that
$\forall y \in \tilde{K}$ with $y < q_\alpha :\; \alpha - F_Z(y) > c_{1,F}(d(y, q_\alpha))^{\delta_{1,F}}$ and
$\forall x \in X_{\tilde{K}} :\; \gamma(x) < q_\alpha - c_{1,\gamma} d(x, \Gamma_0)^{\delta_{1,\gamma}}$.

(Gr-g_0) with $\tilde{K}$ instead of UQ_0 and a strict inequality is then satisfied with

$\tilde{c}_1 = c_F(c_{1,\gamma})^{\delta_F}$ and $\tilde{\delta}_1 = \delta_{1,\gamma} \cdot \delta_F$. Of course only one inequality in (IG) has to be strict.

If Γ_0 is single-valued, it remains to apply Theorem 1. Otherwise we employ Theorem 2 and assume that the following condition is satisfied:

(OG) There exist positive constants $c_{2,\gamma},\ c_{2,F},\ \delta_{2,\gamma},\ \delta_{2,F}$ such that
$\forall y \in \tilde{K}$ with $y > q_\alpha :\ F_Z(y) - \alpha > c_{2,F}(d(y, q_\alpha))^{\delta_{2,F}}$.
Furthermore, there exists an $\tilde{\varepsilon} > 0$ such that $CI(\tilde{\varepsilon}) \neq \emptyset$ and
$\forall x \in \Gamma_0 \setminus CI(\tilde{\varepsilon}) :\ \gamma(x) > q_\alpha + c_{2,\gamma} d(x, (E \setminus \Gamma_0)^{\delta_{2,\gamma}}$.

Hence, with respect to (CO3) we obtain for all $0 < \varepsilon \leq \tilde{\varepsilon}$
$\Gamma_0 \subset \bar{U}_\varepsilon CI(\varepsilon)$ and $\forall x \in \Gamma_0 \setminus CI(\tilde{\varepsilon}) :\ g_0(x) < -\tilde{c}_2(d(x, E \setminus \Gamma_0)^{\tilde{\delta}_2}$
with $\tilde{c}_2 = c_F(c_{2,\gamma})^{\delta_F}$ and $\tilde{\delta}_2 = \delta_{2,\gamma} \cdot \delta_F$. Thus in Theorem 2 we can choose $\mu(\varepsilon) = \tilde{c}_2 \varepsilon^{\tilde{\delta}_2}$.

Consequently, for all $\varepsilon \leq \tilde{\varepsilon}$, $\beta^{(3)}_{n,\varepsilon} = \max\{(\frac{\varepsilon}{\tilde{c}_1})^{\frac{1}{\tilde{\delta}_1}} n^{-\frac{1}{2\tilde{\delta}_1}}, (2\frac{\varepsilon}{\tilde{c}_2})^{\frac{1}{\tilde{\delta}_2}} n^{-\frac{1}{2\tilde{\delta}_2}}\}$, and $n_0(\varepsilon) = \min\{k : \beta^{(3)}_{k,\varepsilon} \leq 2\tilde{\varepsilon}, \gamma(\Gamma_0 \setminus CI(\frac{\beta^{(3)}_{k,\varepsilon}}{2})) \subset K\}$ the relation
$\sup_{n \geq n_0(\varepsilon)} P\{\omega :\ (\Gamma_n(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_0) \cup (\Gamma_0 \setminus (U_{\beta^{(3)}_{n,\varepsilon}} \Gamma_n(\omega)) \neq \emptyset\} \leq 4e^{-2\varepsilon^2}$ holds.

Eventually, we consider quantile estimation because here relaxation of the constraint set comes into play in a natural way. Papers dealing with quantile estimation usually assume that the distribution function is strictly increasing in a neighborhood of the quantile (c.f. [4], [5]). There are, however, applications where one can not a priori assume, that the lower and the upper quantile coincide.

We consider - as in the foregoing example - a real-valued random variable Z with distribution P_Z and distribution function F_Z. We will, for a fixed $\alpha \in (0, 1)$, investigate the lower α-quantile $q^l_\alpha := \inf\{x \in R^1 : F_Z(x) \geq \alpha\}$.

We consider the constraint set $\Gamma_0 := \{x \in R : F_Z(x) \geq \alpha\}$ and the optimization problem
$(P_0) \quad \min_{x \in \Gamma_0} x.$

As F_Z is upper semicontinuous by definition, the set Γ_0 is closed and the minimum q^l_α will be attained.

(P_0) could be approximated replacing F_Z by the empirical distribution function F_n. Unfortunately, the set $\{x \in R^1 : F_n(x) \geq \alpha\}$, in general, does not approximate the whole set Γ_0. In [19] we showed that with a suitable relaxation $\kappa_{n,\varepsilon}$ the solutions to the approximate problems convergence in probability to the desired quantile. Here we can proceed in a similar way, consider the modified constraint set $\Gamma_{n,\varepsilon}$ with

$\Gamma_{n,\varepsilon}(\omega) := \{x \in R : F_n(x,\omega) > \alpha - \frac{\varepsilon}{\sqrt{n}}\}$

and investigate the approximating optimization problems

$(P_{n,\varepsilon}) \quad \min_{x \in \Gamma_{n,\varepsilon}} x.$

$(P_{n,\varepsilon})$ has a unique solution, too.

In order to obtain a convergence rate, we need some knowledge about F_Z, e.g. a growth condition.

Theorem 11 (Quantile Estimation) *Assume that there exist constants* $c > 0$*,* $\delta > 0$*, and* $\theta > 0$ *such that*

$\forall\, \tilde{x}$ *with* $0 < d(\tilde{x}, \Gamma_0) \leq \theta :\ F_0(\tilde{x}) < \alpha - cd(\tilde{x}, \Gamma_0)^{\delta}$.

Then for all $\varepsilon > 0$, $\beta^{(3)}_{n,\varepsilon} = \max\{\varepsilon n^{-\frac{1}{2}}, (\frac{2\varepsilon}{c})^{\frac{1}{\delta}} n^{-\frac{1}{2\delta}}\}$*, and* $n_0(\varepsilon) = \min\{k : \beta^{(3)}_{k,\varepsilon} \leq \theta\}$ *the relations*

$\sup_{n \geq n_0(\varepsilon)} P\{\omega : (\Gamma_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}}\Gamma_0) \cup (\Gamma_0 \setminus \Gamma_{n,\varepsilon}(\omega)) \neq \emptyset\} \leq 2e^{-2\varepsilon^2}$ *and*

$\sup_{n \geq n_0(\varepsilon)} P\{\omega : \Psi_{n,\varepsilon}(\omega) \setminus U_{\beta^{(3)}_{n,\varepsilon}}\Psi_0 \neq \emptyset\} \leq 2e^{-2\varepsilon^2}$

hold.

Proof. We employ Corollary 5.1 and, since the solution is unique, Theorem 9. The objective function is not approximated, hence (VL2) and (VU2) are satisfied with $\mathcal{K}_2 \equiv 0$. The conditions (UCon) and (Gr-f_0) are fulfilled with $c_i = \delta_i = 1,\ i = 1, 2$. Due to the Dvoretzky-Kiefer-Wolfowitz inequality with Massart's bound ([9], [1]), 'strict' variants of (CI2') and (CO2') are satisfied with $\gamma_n = n^{\frac{1}{2}}$ and $\mathcal{K}_2(\varepsilon) = e^{-2\varepsilon^2}$. The first assertion then follows by Corollary 5.1. Furthermore, taking into account that (SI4) is satisfied with $\nu(\varepsilon) = \varepsilon$, the second assertion is implied by Theorem 9. □

References

[1] L. Devroye and G. Lugosi. Combinatorical Methods in Density Estimation. Springer, 2001.

[2] O. Gersch. Convergence in Distribution of Random Closed Sets and Applications in Stability Theory of Stochastic Optimisation. Dissertation thesis, Technical University Ilmenau 2006.

[3] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statistical Association.*, 58:13–30, 1963.

[4] A.I. Kibzun and Y.S. Kan. Stochastic Programming Problems. Wiley 1996.

[5] K. Knight. What are the limiting distibutions of quantile estimators? Technical Report, University of Toronto, 1999.

[6] P. Lachout, E. Liebscher and S. Vogel. Strong convergence of estimators as ε_n-estimators of optimization problems. *Ann. Inst. Statist. Math.*, 57:291–313, 2005.

[7] P. Lachout and S. Vogel. On continuous convergence and epi-convergence of random functions. Part I: Theory and relations. *Kybernetika*, 39:75–98, 2003.

[8] P. Lachout and S. Vogel. On continuous convergence and epi-convergence of random functions. Part II: Sufficient conditions and applications. *Kybernetika*, 39:99–118, 2003.

[9] P. Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *Ann. of Probability*, 18:1269–1293, 1990.

[10] A. Nemirovski and A. Shapiro. Scenario approximations of chance constraints. *SIAM J. Optimization*, to appear.

[11] G.C. Pflug. Asymptotic dominance and confidence for solutions of stochastic programs. *Czechoslovak J. Oper. Res.*, 1:21–30, 1992.

[12] G.C. Pflug. Stochastic Optimization and Statistical Inference. In: Stochastic Programming (A. Ruszczynski, A. Shapiro, eds.), Handbooks in Operations Research and Mangement Science Vol. 10, Elsevier 2003, 427–482.

[13] W. Römisch. Stability of Stochastic Programming. In: Stochastic Programming (A. Ruszczynski, A. Shapiro, eds.), Handbooks in Operations Research and Mangement Science Vol. 10, Elsevier 2003, 483 - 554.

[14] A. Shapiro. Monte Carlo Sampling Methods. In: Stochastic Programming (A. Ruszczynski, A. Shapiro, eds.), Handbooks in Operations Research and Mangement Science Vol. 10, Elsevier 2003, 353–425.

[15] A.W. van der Vaart and J.A. Wellner. Weak Convergence and Empirical Processes. Springer, 1996.

[16] S. Vogel. A stochastic approach to stability in stochastic programming. *J. Comput. and Appl. Math., Series Appl. Analysis and Stochastics*, 56: 65–96, 1994.

[17] S. Vogel. On stability in stochastic programming - Sufficient conditions for continuous convergence and epi-convergence. Preprint TU Ilmenau, 1995.

[18] S. Vogel. On semicontinuous approximations of random closed sets with application to random optimization problems. *Ann. Oper. Res.*, 142: 169–282, 2006.

[19] S. Vogel. Qualitative stability of stochastic programs with applications in asymptotic statistics. *Statistics and Decisions*, 23: 219–248, 2005.