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Hopf bifurcation at k-fold resonances in reversible systems

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Abstract

We study the bifurcation of families of periodic orbits near a symmetric equilibrium in a reversible system, when for a critical value of the parameters the linearization at the equilibrium has a pair of purely imaginary eigenvalues of geometric multiplicity one but algebraic multiplicity k — we call this a k-fold resonance. Combining the general reduction results of [2] with a particular normal form result for linear reversible operators we can reduce the problem to a scalar polynomial bifurcation equation. The problem has codimension $k-1$, and the resulting bifurcation set is a cuspoid of order k. When crossing the codimension one strata of the bifurcation set families of periodic orbits disappear or merge in a way which is similar to what happens at a Krein instability in Hamiltonian systems.

1. Introduction

One of the main properties of reversible systems is the following version of the classical Liapunov Center Theorem which holds in such systems (see e.g. [1]): under appropriate non-resonance conditions there corresponds to each pair of simple purely imaginary eigenvalues of the linearization at a symmetric equilibrium a one-parameter family of symmetric periodic orbits; this family originates at the equilibrium and generates a two-dimensional invariant manifold filled with periodic orbits surrounding the equilibrium (hence the name "center theorem"). This picture is stable: under a sufficiently small (reversible) perturbation of the system the simple purely imaginary eigenvalues remain simple and on the imaginary axis (this is a consequence of the reversibility), and each of them still generates a one-parameter family of symmetric periodic orbits. The only way one can destroy this local picture is by creating resonances. In particular, under a change of parameters a number of such simple purely imaginary eigenvalues can coalesce and move off the imaginary axis; in this paper we want to study what happens to the associated families of periodic orbits in such a scenario. To obtain the bifurcation picture we will combine the general reduction method explained in our earlier paper [2] with a result on linear normal forms which we prove in Section 2.

To be more precise, we consider an autonomous system of the form

$$\dot{x} = f(x, \lambda), \tag{1.1}$$

where $f : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^{2n}$ is a smooth vectorfield which is *reversible*, i.e.

(**R**) there exists a linear involution $R_0 \in \mathcal{L}(\mathbb{R}^{2n})$ (i.e. $R_0^2 = I$) for which $\dim \mathrm{Fix}(R_0) = \dim \{x \in \mathbb{R}^{2n} \mid R_0 x = x\} = n$ and such that

$$f(R_0 x, \lambda) = -R_0 f(x, \lambda). \tag{1.2}$$

Now suppose that for some parametervalue $\lambda_0 \in \mathbb{R}^m$ the system (1.1) has a symmetric equilibrium $x_0 \in \mathrm{Fix}(R_0)$; assume also that the linearization $A_0 := D_x f(x_0, \lambda_0) \in \mathcal{L}(\mathbb{R}^{2n})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$), and no other eigenvalues of the form $\pm il\omega_0$ ($l = 0, 2, 3, \ldots$). Then the Liapunov Center Theorem mentionned above gives us, for $\lambda = \lambda_0$, the existence of a one-parameter family of periodic solutions $x_\rho(t)$ ($0 \leq \rho < \rho_0$) such that $R_0 x_\rho(-t) = x_\rho(t)$ and $x_{\rho=0}(t) \equiv 0$; clearly the corresponding periodic orbits $\gamma_\rho := \{x_\rho(t) \mid t \in \mathbb{R}\}$ are invariant under R_0. This picture persists when we change the parameter λ near λ_0: A_0 is non-singular by the non-resonance condition, and therefore the system has for each λ near λ_0 a symmetric equilibrium x_λ near x_0; moreover, the linearization A_λ at this equilibrium anti-commutes with R_0, and therefore A_λ will also have a pair of simple purely imaginary eigenvalues (the relation $A_\lambda R_0 = -R_0 A_\lambda$ prevents *simple* eigenvalues to move off the imaginary axis); finally, from x_λ there originates a one-parameter family of symmetric periodic orbits.

Here we want to consider the case where the eigenvalues $\pm i\omega_0$ of A_0 are no longer simple, but non-semisimple; more precisely, we will assume that the eigenvalues $\pm i\omega_0$ have geometric multiplicity 1 but algebraic multiplicity $k \geq 2$; we say that we have a *k-fold resonance*. Unfolding the corresponding linear operator A_0 (see Section 3 for details) shows that such A_0 arises when we move k pairs of simple purely imaginary eigenvalues together (we need $k-1$ parameters for that) and let them coalesce. The full unfolding contains surfaces in parameter space along which we have p-fold resonances for any p between 2 and k; in particular (taking $p = 2$) we have for any two adjacent pairs a codimension one surface such that when crossing this surface the two pairs coalesce and split off the imaginary axis. The problem we want to study is about what happens to the k one-parameter families of periodic orbits (one associated to each of the simple pairs) under such change of parameters. For the case $k = 2$ the situation is well known (see e.g. [5]) and similar to what happens at a so-called *Krein collision* (also called a *Hamiltonian Hopf bifurcation* — see e.g. [6]) in Hamiltonian systems: when crossing the codimension one surface just mentionned the two families either connect to each other and detach from the equilibrium (the *hyperbolic* case), or the two families are locally connected and shrink as a whole down to the equilibrium and then disappear (the *elliptic* case). We will see that in the general case these elementary elliptic and hyperbolic bifurcations form the building stones of the full bifurcation picture; the bifurcation sets in parameter space are so-called cuspoids, well known from catastrophe theory.

Before we give a precise formulation of our hypotheses we remark that by appropriate translations and by a time rescale we can without loss of generality assume that $\lambda_0 = 0$, $x_\lambda = 0$ and $\omega_0 = 1$. Also, our assumptions will imply that A_0 must be non-singular: observe that since A_0 anti-commutes with R_0 this is only possible if $\dim\{x \mid R_0 x = x\} = \dim\{x \mid R_0 x = -x\}$; this motivates the dimension hypotheses made in (**R**). With the foregoing in mind we now formulate our main hypothesis:

(**H**) (i) $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^m$;
(ii) the operator $A_0 := D_x f(0,0) \in \mathcal{L}(R^{2n})$ has the eigenvalues $\pm i$, and no other eigenvalues of the form $\pm li$, with $l = 0, 2, 3, \ldots$;
(iii) the subspace $\ker(A_0^2 + I)$ is irreducible under A_0.

The hypothesis (**H**)(iii) means that the eigenvalues $\pm i$ of A_0 have geometric multiplicity equal to 1; their algebraic multiplicity is given by the smallest integer k such that $\ker(A_0^2 + I)^{k+1} =$

$\ker(A_0^2+I)^k$. It also follows that A_0 is reversible:

$$A_0R_0 = -R_0A_0. \tag{1.3}$$

Under the hypotheses (**R**) and (**H**) we want to study the following problem:

(**P**) Describe, for all sufficiently small $\lambda \in \mathbb{R}^m$, all small periodic solutions of (1.1) with period T near 2π.

Our approach to this problem will be based on the general reduction method described in [2]; we now give the main results of this paper as they apply to the particular situation considered here.

Let $A_0 = S_0+N_0$ be the (unique) semisimple-nilpotent decomposition of A_0, i.e. $S_0 \in \mathcal{L}(\mathbb{R}^{2n})$ is semisimple (that means complex diagonalizable), $N_0 \in \mathcal{L}(\mathbb{R}^{2n})$ is nilpotent and $S_0N_0 = N_0S_0$. It follows from (1.3) (see also Lemma 1) that also S_0 and N_0 are reversible:

$$S_0R_0 = -R_0S_0 \quad \text{and} \quad N_0R_0 = -R_0N_0. \tag{1.4}$$

Define the *reduced phase space* for the problem (**P**) as the subspace

$$U := \ker(\exp(2\pi S_0) - I) \subset \mathbb{R}^{2n}; \tag{1.5}$$

it follows from (**H**)(ii) that U coincides with the generalized eigenspace corresponding to the eigenvalue pair $\pm i$ of A_0, i.e. $U = \ker(A_0^2+I)^k = \ker(S_0^2+I)$. Also, U is invariant under each of the operators A_0, S_0, N_0 and R_0; we denote the restrictions of these operators to U by respectively A, S, N and $R \in \mathcal{L}(U)$. Clearly $A = S+N$, $U = \ker(S^2+I_U) = \ker(A^2+I_U)^k$, $SR = -RS$, $NR = -RN$, $R^2 = I_U$, $\dim U = 2k$ and $\dim Fix(R) = k$. Moreover, it follows from the definition (1.5) that S generates on U an S^1-action given by

$$\varphi \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \longmapsto \exp(\varphi S) \in \mathcal{L}(U). \tag{1.6}$$

The main result of [2] then states that for each (λ, T) near $(0, 2\pi)$ there exists a one-to-one relation between the small T-periodic solutions of (1.1) and the small T-periodic solutions of a *reduced system*

$$\dot{u} = g(u, \lambda), \tag{1.7}$$

where $g : U \times \mathbb{R}^m \to U$ is smooth (see Remark 2 further on) and has the following properties:

(a) $g(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}^m$ and $D_ug(0,0) = A = S+N$;

(b) $g(Ru,\lambda) = -Rg(u,\lambda)$, i.e. g is reversible;

(c) g is S^1-equivariant:

$$g(\exp(\varphi S)u, \lambda) = \exp(\varphi S)g(u,\lambda), \quad \forall \varphi \in S^1. \tag{1.8}$$

Moreover,

(d) for small λ all small periodic solutions of (1.7) have the form

$$\tilde{u}(t) = \exp((1+\sigma)S\,t)u \tag{1.9}$$

for some small $(\sigma, u) \in \mathbb{R} \times U$; these periodic solutions can therefore be obtained by solving the equation

$$(1+\sigma)Su = g(u,\lambda); \tag{1.10}$$

(e) if the original vectorfield is in normal form (with respect to S_0) up to order $q \geq 1$, i.e. if

$$f(x,\lambda) = f_{NF}(x,\lambda) + \mathrm{O}(\|x\|^{q+1}) \quad \text{as} \quad x \to 0,$$

with f_{NF} reversible and such that

$$f_{NF}(\exp(S_0 t)x,\lambda) = \exp(S_0 t) f_{NF}(x,\lambda), \quad \forall t \in \mathbb{R}, \tag{1.11}$$

then

$$g(u,\lambda) = f_{NF}(u,\lambda) + \mathrm{O}(\|u\|^{q+1}); \tag{1.12}$$

(f) the relation between the periodic solutions of (1.1) and those of (1.7) respects the reversibility properties of these solutions; that is, a periodic solution of (1.1) is symmetric if and only if the corresponding solution (1.9) of (1.7) is such that $Ru = u$.

Remark 1. The solutions of (1.10) come in S^1-orbits; two solutions on the same orbit correspond to two periodic solutions of (1.1) which are related to each other by a phase shift. So solution orbits of (1.10) correspond in a one-to-one way to periodic orbits of (1.1); such periodic orbit is symmetric if and only if the corresponding solution orbit of (1.10) intersects $\mathrm{Fix}(R)$ (see the property (f) above).

Remark 2. If the original vectorfield f is C^∞-smooth then the reduced vectorfield g is C^∞-smooth outside of the origin $u = 0$. As for the differentiability at $u = 0$ one can perform a sequence of near-identity transformations to bring the original vectorfield f in normal form up to any chosen order $q \geq 1$; the only drawback for choosing higher values of q is that the parameter range for which the normal form reduction is valid may shrink with q. The transformations which bring f in normal form are C^∞-smooth, and therefore the same holds for the normal form f_{NF} itself. Then (1.12) shows that the reduced vectorfield g is at least of class C^q.

For our further analysis we will assume that g is sufficiently smooth (at least $q \geq 3$), but we will only calculate explicitly the normal form up to order $q = 1$ (we will call this the *linear normal form*). Actually, we will show that one can not only impose the condition (1.11) on this linear normal form, but also a further condition which involves the nilpotent part N_0 of A_0. In the next section we prove a general result on such linear normal forms, and in Section 3 we apply this result to the particular situation which we study here. This will give us sufficient information on the reduced vectorfield g to complete the bifurcation analysis in Section 4. In a forthcoming paper [4] we will give a similar analysis for the case of a conservative system.

2. A general result on linear normal forms

Let Γ be a compact group acting linearly on a finite-dimensional vectorspace V, and let $\chi : \Gamma \to \mathbb{R}$ be a *real character* over Γ, i.e. χ is continuous, $\chi(\gamma) \neq 0$ for all $\gamma \in \Gamma$, and $\chi(\gamma_1^{-1}\gamma_2) = \chi(\gamma_1)^{-1}\chi(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$; the compactness of Γ then implies that for all

$\gamma \in \Gamma$ we have $|\chi(\gamma)| = 1$, i.e. $\chi(\gamma) = \pm 1$. We define the following spaces of linear operators over V:

$$\mathcal{L}_\Gamma(V) := \{A \in \mathcal{L}(V) \mid A\gamma = \gamma A,\ \forall \gamma \in \Gamma\} \tag{2.1}$$

and

$$\mathcal{L}_\chi(V) := \{A \in \mathcal{L}(V) \mid A\gamma = \chi(\gamma)\gamma A,\ \forall \gamma \in \Gamma\}\,; \tag{2.2}$$

of course $\mathcal{L}_\chi(V) = \mathcal{L}_\Gamma(V)$ if χ is trivial ($\chi(\gamma) = 1$ for all $\gamma \in \Gamma$), but in all other cases the two spaces $\mathcal{L}_\chi(V)$ and $\mathcal{L}_\Gamma(V)$ are linearly independent. In what follows we will assume that χ is nontrivial, i.e. there exists some $\gamma \in \Gamma$ such that $\chi(\gamma) = -1$; then $\Gamma_0 := \{\gamma \in \Gamma \mid \chi(\gamma) = 1\}$ forms a normal subgroup of Γ, and we can identify Γ with the semidirect product $\Gamma_0 \rtimes \mathbb{Z}_2$. We leave it to the reader to verify that the results which follow remain valid in the (simpler) case that χ is trivial. In Section 3 we will apply the results which follow to the case $V = U$, $\Gamma = \{I_U, R\}$ and $\chi(R) = -1$.

For each $\Psi \in \mathcal{L}(V)$ we define linear operators $\mathrm{Ad}\,(\Psi) \in \mathcal{L}(\mathcal{L}(V))$ and $\mathrm{ad}\,(\Psi) \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathrm{Ad}\,(\Psi)\cdot A := e^{-\Psi} A e^{\Psi} \quad \text{and} \quad \mathrm{ad}\,(\Psi)\cdot A := A\Psi - \Psi A, \quad \forall A \in \mathcal{L}(V); \tag{2.3}$$

using the fact that

$$\frac{d}{dt}\mathrm{Ad}\,(t\Psi)\cdot A = \mathrm{ad}\,(\Psi)\cdot(\mathrm{Ad}\,(t\Psi)\cdot A), \quad \forall t \in \mathbb{R},$$

one easily shows that

$$\mathrm{Ad}\,(\Psi) = e^{\mathrm{ad}\,(\Psi)}, \quad \forall \Psi \in \mathcal{L}(V). \tag{2.4}$$

If $\Psi \in \mathcal{L}_\Gamma(V)$ then $\mathrm{Ad}\,(\Psi)$ and $\mathrm{ad}\,(\Psi)$ map $\mathcal{L}_\chi(V)$ into itself, and we can (and will) consider $\mathrm{Ad}\,(\Psi)$ and $\mathrm{ad}\,(\Psi)$ as elements of $\mathcal{L}(\mathcal{L}_\chi(V))$. Similarly, if $\Psi \in \mathcal{L}_\chi(V)$ then $\mathrm{ad}\,(\Psi)$ maps $\mathcal{L}_\Gamma(V)$ into $\mathcal{L}_\chi(V)$ and $\mathcal{L}_\chi(V)$ into $\mathcal{L}_\Gamma(V)$. We start with a technical result.

Lemma 1. *Let $S_0 \in \mathcal{L}_\chi(V)$ be semisimple (when considered as an element of $\mathcal{L}(V)$). Then there exists a scalar product $\langle\cdot,\cdot\rangle$ on V such that when we denote the transpose of a linear operator $A \in \mathcal{L}(V)$ with respect to this scalar product by A^T then the following holds:*

- (i) *the action of Γ on V is orthogonal, i.e. $\gamma^T\gamma = I_V$ for all $\gamma \in \Gamma$;*
- (ii) $\ker(\mathrm{ad}\,(S_0^T)) = \ker(\mathrm{ad}\,(S_0))$.

As a consequence we have $A^T \in \mathcal{L}_\chi(V)$ for each $A \in \mathcal{L}_\chi(V)$.

Proof . Fix some $R_0 \in \Gamma$ such that $\chi(R_0) = -1$; it follows then from $S_0R_0 = -R_0S_0$ that if $\mu \in \mathbb{C}$ is an eigenvalue of S_0, then so is $-\mu$. Let (α_i, β_i) $(1 \le i \le \ell)$ be different elements of $\mathbb{R}_+ \times \mathbb{R}_+$ such that the set of eigenvalues of S_0 is given by $\{\pm\alpha_i \pm i\beta_i \mid 1 \le i \le \ell\}$. Since S_0 is semisimple we can then write

$$V = \sum_{i=1}^{\ell} V_i, \qquad V_i := \ker\Big(((S_0 - \alpha_i I_V)^2 + \beta_i^2 I_V)((S_0 + \alpha_i I_V)^2 + \beta_i^2 I_V)\Big)\,. \tag{2.5}$$

Each of the subspaces V_i $(1 \leq i \leq \ell)$ is invariant under S_0 and under the action of Γ; moreover, if $A \in \ker(\mathrm{ad}(S_0))$ then each V_i is also invariant under A. Let $\langle\cdot,\cdot\rangle$ be a scalar product on V such that the subspaces V_i $(1 \leq i \leq \ell)$ are mutually orthogonal, i.e. such that

$$\langle v_1, v_2\rangle := \sum_{i=1}^{\ell} \langle \pi_i v_1, \pi_i v_2\rangle_i, \quad \forall v_1, v_2 \in V, \tag{2.6}$$

where $\pi_i \in \mathcal{L}(V, V_i)$ is the projection from V onto V_i associated with the decomposition (2.5), and where $\langle\cdot,\cdot\rangle_i$ is some scalar product on V_i. Let $A \in \mathcal{L}(V)$ be such that $A(V_i) \subset V_i$ for $1 \leq i \leq \ell$; it then follows that $\pi_i A = A_i \pi_i$ for some $A_i \in \mathcal{L}(V_i)$, and that $\pi_i A^T = A_i^T \pi_i$, where $A_i^T \in \mathcal{L}(V_i)$ is the transpose of A_i with respect to $\langle\cdot,\cdot\rangle_i$. This shows that it is sufficient to prove the existence of a convenient scalar product $\langle\cdot,\cdot\rangle_i$ within each of the subspaces V_i, i.e. we are reduced to the case where

$$V = \ker\Big(((S_0 - \alpha I_V)^2 + \beta^2 I_V)((S_0 + \alpha I_V)^2 + \beta^2 I_V)\Big) \tag{2.7}$$

for some $\alpha, \beta \geq 0$. We consider now several cases.

Case (1): $\alpha > 0$ and $\beta > 0$. We can then rewrite (2.7) as

$$V = V_+ \times V_-, \quad V_\pm := \ker\Big((S_0 \mp \alpha I_V)^2 + \beta^2 I_V)\Big). \tag{2.8}$$

The subspaces V_+ and V_- are invariant under S_0 and under the action of Γ_0, while R_0 maps V_+ isomorphically onto V_- and V_- isomorphically onto V_+; we denote by $R_+ \in \mathcal{L}(V_+, V_-)$ (respectively $R_- \in \mathcal{L}(V_-, V_+)$) the restriction of R_0 to V_+ (respectively V_-). By definition of V_+ we have that

$$S_0 v_+ = \alpha v_+ + \beta J v_+, \quad \forall v_+ \in V_+, \tag{2.9}$$

where $J := \frac{1}{\beta}(S_0|_{V_+} - \alpha I_{V_+}) \in \mathcal{L}(V_+)$ is such that $J^2 = -I_{V_+}$. In combination with $S_0 R_0 = -R_0 S_0$ it follows that

$$S_0 v_- = -\alpha v_- - \beta (R_-)^{-1} J R_- v_-, \quad \forall v_- \in V_-. \tag{2.10}$$

The operator J generates on V_+ an S^1-action, given by

$$\varphi \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \longmapsto e^{J\varphi} \in \mathcal{L}(V_+).$$

Since this action commutes with the action of Γ_0 we have in fact the compact group $\Gamma_0 \times S^1$ acting on V_+.

Let $\langle\cdot,\cdot\rangle_+$ be any scalar product on V_+ for which this $\Gamma_0 \times S^1$-action is orthogonal, and define a scalar product $\langle\cdot,\cdot\rangle$ on $V = V_+ \times V_-$ by setting

$$\Big\langle (v_+, v_-), (v_+', v_-')\Big\rangle := \Big\langle v_+, v_+'\Big\rangle_+ + \Big\langle R_- v_-, R_- v_-'\Big\rangle_+, \quad \forall (v_+, v_-), (v_+', v_-') \in V_+ \times V_-. \tag{2.11}$$

We have then for each $\gamma \in \Gamma_0$ that

$$\begin{aligned}
\Big\langle \gamma(v_+, v_-), \gamma(v_+', v_-')\Big\rangle &= \Big\langle \gamma v_+, \gamma v_+'\Big\rangle_+ + \Big\langle R_- \gamma v_-, R_- \gamma v_-'\Big\rangle_+ \\
&= \Big\langle v_+, v_+'\Big\rangle_+ + \Big\langle R_- v_-, R_- v_-'\Big\rangle_+ \\
&= \Big\langle (v_+, v_-), (v_+', v_-')\Big\rangle, \qquad \forall (v_+, v_-), (v_+', v_-') \in V_+ \times V_-,
\end{aligned}$$

where we have used the fact that $R_-\gamma(R_-)^{-1}$ belongs to Γ_0 (or more precisely: the restriction of Γ_0 to V_+). Also

$$\begin{aligned}
\left\langle R_0(v_+,v_-),R_0(v_+',v_-')\right\rangle &= \left\langle (R_-v_-,R_+v_+),(R_-v_-',R_+v_+')\right\rangle \\
&= \left\langle R_-v_-,R_-v_-'\right\rangle_+ + \left\langle R_-R_+v_+,R_-R_+v_+'\right\rangle_+ \\
&= \left\langle R_-v_-,R_-v_-'\right\rangle_+ + \left\langle v_+,v_+'\right\rangle_+ \\
&= \left\langle (v_+,v_-),(v_+',v_-')\right\rangle, \qquad \forall (v_+,v_-),(v_+',v_-')\in V_+\times V_-,
\end{aligned}$$

using this time the fact that R_-R_+ acts on V_+ as an element of Γ_0, since $R_0^2\in\Gamma_0$. The foregoing relations show that the action of Γ on $V=V_+\times V_-$ is orthogonal with respect to the scalar product (2.11).

Since $J\in\mathcal{L}(V_+)$ generates an orthogonal S^1-action on V_+ it follows that J is anti-symmetric, i.e. $\left\langle Jv_+,v_+'\right\rangle_+ = -\left\langle v_+,Jv_+'\right\rangle_+$ for all $v_+,v_+'\in V_+$. Using (2.9) and (2.10) we find then for all $(v_+,v_-),(v_+',v_-')\in V_+\times V_-$ that

$$\begin{aligned}
&\left\langle S_0^T(v_+,v_-),(v_+',v_-')\right\rangle \\
&= \left\langle (v_+,v_-),S_0(v_+',v_-')\right\rangle \\
&= \left\langle (v_+,v_-),(\alpha v_+'+\beta Jv_+',-\alpha v_-'-\beta(R_-)^{-1}JR_-v_-')\right\rangle \\
&= \alpha\left\langle v_+,v_+'\right\rangle_+ + \beta\left\langle v_+,Jv_+'\right\rangle_+ - \alpha\left\langle R_-v_-,R_-v_-'\right\rangle_+ - \beta\left\langle R_-v_-,JR_-v_-'\right\rangle_+ \\
&= \alpha\left\langle v_+,v_+'\right\rangle_+ - \beta\left\langle Jv_+,v_+'\right\rangle_+ - \alpha\left\langle R_-v_-,R_-v_-'\right\rangle_+ + \beta\left\langle JR_-v_-,R_-v_-'\right\rangle_+ \\
&= \left\langle (\alpha v_+-\beta Jv_+,-\alpha v_-+\beta(R_-)^{-1}JR_-v_-),(v_+',v_-')\right\rangle.
\end{aligned}$$

Therefore

$$S_0^T(v_+,v_-)=(\alpha v_+-\beta Jv_+,-\alpha v_-+\beta(R_-)^{-1}JR_-v_-),\quad \forall (v_+,v_-)\in V_+\times V_-. \tag{2.12}$$

Suppose now that $A\in\ker(\mathrm{ad}(S_0))$; then A leaves the subspaces V_+ and V_- invariant, and we can write

$$A(v_+,v_-)=(A_+v_+,A_-v_-),\quad \forall (v_+,v_-)\in V_+\times V_-,$$

where $A_+\in\mathcal{L}(V_+)$ commutes with the restriction of S_0 to V_+, and $A_-\in\mathcal{L}(V_-)$ commutes with the restriction of S_0 to V_-. It follows then from (2.9) and (2.10) that A_+ commutes with J, while A_- commutes with $(R_-)^{-1}JR_-$; by (2.12) this implies that A commutes with S_0^T, i.e. $A\in\ker(\mathrm{ad}(S_0^T))$ and $\ker(\mathrm{ad}(S_0))\subset\ker(\mathrm{ad}(S_0^T))$. To prove the opposite inclusion we observe that (2.12) implies that

$$V_\pm := \ker\left((S_0^T\mp\alpha I_V)^2+\beta^2 I_V\right).$$

Hence each $A\in\ker(\mathrm{ad}(S_0^T))$ leaves the subspaces V_+ and V_- invariant, and the same argument we just used shows that such A also belongs to $\ker(\mathrm{ad}(S_0))$. This proves the property (ii) for the case when $\alpha>0$ and $\beta>0$.

Case (2): $\alpha > 0$ and $\beta = 0$. This time we have $V = V_+ \times V_-$ with

$$V_\pm := \ker\,(S_0 \mp \alpha I_V)\,,$$

and hence

$$S_0(v_+, v_-) = (\alpha v_+, -\alpha v_-), \quad \forall (v_+, v_-) \in V_+ \times V_-.$$

It follows that $S_0^T = S_0$ when the transpose is taken with respect to any scalar product on V such that the subspaces V_+ and V_- are orthogonal. Therefore the conditions (i) and (ii) of the lemma will be satisfied for this case when we define the scalar product $\langle \cdot, \cdot \rangle$ on V by (2.11), with $\langle \cdot, \cdot \rangle_+$ any scalar product on V_+ for which the action of Γ_0 on V_+ is orthogonal: the proof that the action of Γ on V is orthogonal is then the same as for case (1).

Case (3): $\alpha = 0$ and $\beta > 0$. In this case we have $S_0 = \beta J$ with $J \in \mathcal{L}(V)$ such that $J^2 = -I_V$; it follows that J generates an S^1-action on V, which combines with the action of Γ into an action of the compact group $\Gamma \ltimes S^1 \cong (\Gamma_0 \times S^1) \rtimes \mathbb{Z}_2$ on V (the S^1-action commutes with the Γ_0-action, and $\Gamma_0 \times S^1$ forms a normal subgroup of this semi-direct product). If $\langle \cdot, \cdot \rangle$ is any scalar product on V for which this $\Gamma \rtimes S^1$ is orthogonal then J is anti-symmetric, and hence $S_0^T = -S_0$ and $\ker(\mathrm{ad}\,(S_0^T)) = \ker(\mathrm{ad}\,(S_0))$.

Case (4): $\alpha = \beta = 0$. Since then $S_0 = 0$ this case is trivial: it is sufficient to take any scalar product on V for which the Γ-action is orthogonal.

We conclude the proof with the observation that (i) implies that together with A also A^T belongs to $\mathcal{L}_\chi(V)$: this follows by taking the transpose of the relation $A\gamma = \chi(\gamma)\gamma A$ ($\gamma \in \Gamma$) and using the fact that $\gamma^T = \gamma^{-1}$. ∎

Corollary 2. *Let $A_0 = S_0 + N_0$ be the semisimple-nilpotent decomposition of $A_0 \in \mathcal{L}_\chi(V)$. Then S_0 and N_0 belong to $\mathcal{L}_\chi(V)$. Moreover, if $\langle \cdot, \cdot \rangle$ is a scalar product on V associated to S_0 as in Lemma 1, then also A_0^T, S_0^T and N_0^T belong to $\mathcal{L}_\chi(V)$, and*

$$\ker(\mathrm{ad}\,(A_0^T)) = \ker(\mathrm{ad}\,(S_0)) \cap \ker(\mathrm{ad}\,(N_0^T)). \tag{2.13}$$

Proof . We have for each $\gamma \in \Gamma$ that $\gamma^{-1}A_0\gamma = \gamma^{-1}S_0\gamma + \gamma^{-1}N_0\gamma$ is the semisimple-nilpotent decomposition of $\gamma^{-1}A_0\gamma$; the uniqueness of the decomposition combined with the relation $A_0 = \chi(\gamma)\gamma^{-1}A_0\gamma$ then shows that $S_0 = \chi(\gamma)\gamma^{-1}S_0\gamma$ and $N_0 = \chi(\gamma)\gamma^{-1}N_0\gamma$, i.e. S_0 and N_0 belong to $\mathcal{L}_\chi(V)$.

Next let $\langle \cdot, \cdot \rangle$ be a scalar product on V such that the conditions of Lemma 1 are satisfied. It follows then immediately from Lemma 1 that A_0^T, S_0^T and N_0^T belong to $\mathcal{L}_\chi(V)$. Moreover, if $A = S + N$ is the semisimple-nilpotent decomposition of $A \in \mathcal{L}(V)$ then

$$\ker(\mathrm{ad}\,(A)) = \ker(\mathrm{ad}\,(S)) \cap \ker(\mathrm{ad}\,(N)); \tag{2.14}$$

indeed, the inclusion $\ker(\mathrm{ad}\,(S)) \cap \ker(\mathrm{ad}\,(N)) \subset \ker(\mathrm{ad}\,(A))$ is obvious, while the opposite inclusion follows from the fact that S and N can be written as polynomial expressions in A. Applying this result to $A = A_0^T$ (with Jordan decomposition $A_0^T = S_0^T + N_0^T$) shows that

$$\ker(\mathrm{ad}\,(A_0^T)) = \ker(\mathrm{ad}\,(S_0^T)) \cap \ker(\mathrm{ad}\,(N_0^T)),$$

which in turn implies (2.13), since $\ker(\mathrm{ad}\,(S_0^T)) = \ker(\mathrm{ad}\,(S_0))$ by the choice of the scalar product. ∎

Our main result is the following theorem on normal forms of linear operators $A \in \mathcal{L}_\chi(V)$.

Theorem 3. *Let $A_0 = S_0 + N_0$ be the semisimple-nilpotent decomposition of $A_0 \in \mathcal{L}_\chi(V)$, and let $\langle\cdot,\cdot\rangle$ be a scalar product on V associated with S_0 as in Lemma 1. Then there exists a neighborhood Ω of A_0 in $\mathcal{L}_\chi(V)$ and a C^∞-smooth mapping $\tilde{\Psi} : \Omega \to \mathcal{L}_\Gamma(V)$ with $\tilde{\Psi}(A_0) = 0$ and such that for each $A \in \Omega$ we have*

$$\mathrm{Ad}\,(\tilde{\Psi}(A)) \cdot A - A_0 \in \ker\,(\mathrm{ad}\,(S_0)) \cap \ker\,(\mathrm{ad}\,(N_0^T)). \tag{2.15}$$

Proof . Define $F : \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V) \to \mathcal{L}_\chi(V)$ by

$$F(\Psi, A) := \mathrm{Ad}\,(\Psi) \cdot A, \quad \forall(\Psi, A) \in \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V). \tag{2.16}$$

The mapping F is C^∞-smooth, with $F(0, A_0) = A_0$ and $D_\Psi F(0, A_0) \cdot \Psi = A_0\Psi - \Psi A_0 = \mathrm{ad}\,(\Psi) \cdot A_0 = -\mathrm{ad}\,(A_0) \cdot \Psi$, i.e.

$$D_\Psi F(0, A_0) = -\mathrm{ad}\,(A_0) \in \mathcal{L}(\mathcal{L}_\Gamma(V), \mathcal{L}_\chi(V)). \tag{2.17}$$

We will show further on that

$$\mathcal{L}_\chi(V) = \mathrm{Im}\,(\mathrm{ad}\,(A_0)) \oplus \ker\,(\mathrm{ad}\,(A_0^T)), \tag{2.18}$$

where $\mathrm{ad}\,(A_0)$ is considered as a linear operator from $\mathcal{L}_\Gamma(V)$ into $\mathcal{L}_\chi(V)$ (as in (2.17)), and $\mathrm{ad}\,(A_0^T)$ as a linear operator from $\mathcal{L}_\chi(V)$ into $\mathcal{L}_\Gamma(V)$. Let $\pi \in \mathcal{L}(\mathcal{L}_\chi(V))$ be the projection in $\mathcal{L}_\chi(V)$ onto $\mathrm{Im}\,(\mathrm{ad}\,(A_0))$ and parallel to $\ker\,(\mathrm{ad}\,(A_0^T))$ (see (2.18)), and define a C^∞-smooth mapping $G : \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V) \to \mathrm{Im}\,(\mathrm{ad}\,(A_0))$ by

$$G(\Psi, A) := \pi(F(\Psi, A) - A_0) = \pi(\mathrm{Ad}\,(\Psi) \cdot A - A_0), \quad \forall(\Psi, A) \in \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V). \tag{2.19}$$

Then $G(0, A_0) = 0$ and $D_\Psi G(0, A_0)$ is (by (2.17)) a surjective linear operator from $\mathcal{L}_\Gamma(V)$ onto the subspace $\mathrm{Im}\,(\mathrm{ad}\,(A_0))$ of $\mathcal{L}_\chi(V)$; therefore we can use the implicit function theorem to conclude that there exists a neighborhood Ω of A_0 in $\mathcal{L}_\chi(V)$ and a C^∞-smooth mapping $\tilde{\Psi} : \Omega \to \mathcal{L}_\Gamma(V)$ with $\tilde{\Psi}(A_0) = 0$ and such that

$$G(\tilde{\Psi}(A), A) = 0, \quad \forall A \in \Omega.$$

By the definition (2.19) of G this means that $\mathrm{Ad}\,(\tilde{\Psi}(A)) \cdot A - A_0$ belongs to $\ker\,(\mathrm{ad}\,(A_0^T))$ for all $A \in \Omega$, which in combination with Corollary 2 proves the theorem. Observe that the mapping $\tilde{\Psi}$ can be made uniquely defined by imposing the additional condition that this mapping should take its values in a given complement of $\ker\,(\mathrm{ad}\, A_0)$ in $\mathcal{L}_\Gamma(V)$ (where $\mathrm{ad}\,(A_0)$ is again considered as a linear operator from $\mathcal{L}_\Gamma(V)$ into $\mathcal{L}_\chi(V)$).

It remains to prove the decomposition (2.18) of $\mathcal{L}_\chi(V)$. To do so we define a scalar product $\ll\cdot,\cdot\gg$ on $\mathcal{L}(V)$ by

$$\ll A, B \gg := \mathrm{trace}\,(A^T B), \quad \forall A, B \in \mathcal{L}(V); \tag{2.20}$$

we will denote the induced scalar products on the subspaces $\mathcal{L}_\Gamma(V)$ and $\mathcal{L}_\chi(V)$ by respectively $\ll \cdot,\cdot \gg_\Gamma$ and $\ll \cdot,\cdot \gg_\chi$. Considering again $\mathrm{ad}\,(A_0)$ as a linear operator from $\mathcal{L}_\Gamma(V)$ into $\mathcal{L}_\chi(V)$ we have then by a classical algebra result that

$$\mathcal{L}_\chi(V) = \mathrm{Im}\,(\mathrm{ad}\,(A_0)) \oplus \ker\,((\mathrm{ad}\,(A_0))^*), \tag{2.21}$$

where $(\mathrm{ad}\,(A_0))^* \in \mathcal{L}(\mathcal{L}_\chi(V), \mathcal{L}_\Gamma(V))$ is uniquely defined by the relation

$$\ll \mathrm{ad}\,(A_0)\cdot A\,, B \gg_\chi = \ll A\,, (\mathrm{ad}\,(A_0))^* \cdot B \gg_\Gamma, \quad \forall (A,B) \in \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V). \tag{2.22}$$

A direct calculation shows that for all $(A,B) \in \mathcal{L}_\Gamma(V) \times \mathcal{L}_\chi(V)$ we have

$$\begin{aligned} \ll \mathrm{ad}\,(A_0)\cdot A\,, B \gg_\chi &= \ll AA_0 - A_0A\,, B \gg \;=\; \mathrm{trace}\,((AA_0 - A_0A)^T B) \\ &= \mathrm{trace}\,(A_0^T A^T B - A^T A_0^T B) \;=\; \mathrm{trace}\,(A^T(BA_0^T - A_0^T B)) \\ &= \ll A\,, \mathrm{ad}\,(A_0^T)\cdot B \gg_\Gamma; \end{aligned}$$

comparing with (2.22) we see that

$$(\mathrm{ad}\,(A_0))^* = \mathrm{ad}\,(A_0^T) \in \mathcal{L}(\mathcal{L}_\chi(V), \mathcal{L}_\Gamma(V)), \tag{2.23}$$

which in combination with (2.21) proves (2.18). ■

Remark 3. Under the conditions of Theorem 3 we have for each $A \in \Omega$ that

$$\exp(-\widetilde{\Psi}(A))\, A\, \exp(\widetilde{\Psi}(A)) \in \ker\,(\mathrm{ad}(S_0)) \,\cap\, \mathcal{L}_\chi(V). \tag{2.24}$$

This means that by a linear transformation which depends smoothly on A and which preserves the symmetry properties of A we can make each $A \in \Omega$ commute with S_0.

3. The linear normal form at a k-fold resonance

In this Section we apply the results of Section 2 to the linearizations of the reversible vectorfields introduced in Section 1; in particular we will obtain an explicit form for the linearization $D_u g(0,\lambda) \in \mathcal{L}(U)$ of the reduced vectorfield $g(u,\lambda)$.

Under the hypotheses of Section 1 let $A_0 := D_x f(0,0) \in \mathcal{L}(\mathbb{R}^{2n})$ have the semisimple-nilpotent decomposition $A_0 = S_0 + N_0$. To apply the results of Section 2 we take for Γ the group $\mathbb{Z}_2 = \{e,\kappa\}$ with $\kappa^2 = e$, and we define a character χ on $\mathbb{Z}_2$ by $\chi(\kappa) = -1$. We also define a group action of $\mathbb{Z}_2$ on both the phase space $\mathbb{R}^{2n}$ and the reduced phase space U by setting respectively $\kappa \cdot x := R_0 x$ for $x \in \mathbb{R}^{2n}$ and $\kappa \cdot u := Ru$ for $u \in U$; to indicate these action we will write $\mathbb{Z}_2(R)$ instead of just $\mathbb{Z}_2$. A linear operator $A \in \mathcal{L}(\mathbb{R}^{2n})$ belongs to $\mathcal{L}_\chi(\mathbb{R}^{2n})$ if and only if A is reversible, and similarly for operators in $\mathcal{L}(U)$. In particular, each of the linearizations $D_x f(0,\lambda)$ belongs to $\mathcal{L}_\chi(\mathbb{R}^{2n})$; it follows then from Remark 3 that by a linear Γ-equivariant transformation depending smoothly on the parameter λ we may assume that $D_x f(0,\lambda)$ commutes with S_0 for all sufficiently small $\lambda \in \mathbb{R}^m$. This means that

f is in normal form up to order 1, and hence (by property (e) of Section 1) the linearization $A_\lambda := D_u g(0,\lambda)$ of the reduced vectorfield $g(u,\lambda)$ is given by

$$A_\lambda = D_x f(0,\lambda)|_U \in \mathcal{L}_\chi(U). \tag{3.1}$$

An application of Theorem 3 to the family of linear operators $\{A_\lambda \mid \lambda \in \mathbb{R}^m\} \subset \mathcal{L}_\chi(U)$ shows that by a further parameter-dependent linear transformation of the reduced equation we may assume that

$$A_\lambda = S + N + B_\lambda, \tag{3.2}$$

with $B_{\lambda=0} = 0$ and

$$B_\lambda \in W := \mathcal{L}_\chi(U) \cap \ker(\mathrm{ad}(S)) \cap \ker(\mathrm{ad}(N^T)); \tag{3.3}$$

in (3.2) the transpose N^T of N has to be taken with respect to a scalar product on U for which the $S^1 \rtimes \mathbb{Z}_2(R) \cong O(2)$-action generated on U by S and R is orthogonal (see the proof of Lemma 1, case (3)). It is the aim of the remainder of this Section to obtain an explicit characterization of the subspace W of $\mathcal{L}(U)$ (see Corollary 8).

Let us start by repeating that $U = \ker(S^2 + I_U)$, while k is the smallest integer such that $U = \ker((A^2 + I_U)^k)$; also, $U_1 := \ker(A^2 + I_U)$ is $S^1 \rtimes \mathbb{Z}_2(R)$-invariant, and we know from (**H**)(iii) that U_1 is irreducible under A. But $A^2 + I_U = N(2S + N)$, with $(2S + N) \in \mathcal{L}(U)$ non-singular; it follows that $U_1 = \ker N$, and that U_1 is irreducible under S, and hence also under the $S^1 \rtimes \mathbb{Z}_2(R)$-action. Since $\mathrm{span}\{u, Su\}$ forms for each nonzero $u \in U_1$ an S-invariant subspace of U_1 we conclude that $\dim U_1 = 2$. Also, k is the smallest integer such that $N^k = 0$. The following lemma gives a more detailed structure of U.

Lemma 4. *There exist $S^1 \rtimes \mathbb{Z}_2(R)$-invariant and S-irreducible subspaces U_j of U such that*

(i) $U = U_1 \oplus \cdots \oplus U_k$;

(ii) $U_1 = \ker N$ *and N is an isomorphism of U_j onto U_{j-1} for $2 \le j \le k$.*

Proof . The subspaces $V_j := \ker((A^2 + I_U)^j) = \ker N^j$ $(0 \le j \le k)$ are $S^1 \rtimes \mathbb{Z}_2(R)$-invariant, since $A^2 + I_U$ is $S^1 \rtimes \mathbb{Z}_2(R)$-equivariant; they form a strictly increasing sequence $(V_j)_{0 \le j \le k}$ of subspaces of U, with $V_0 = \{0\}$, $V_1 = U_1$ and $V_k = U$. The operator N^{j-1} maps each $S^1 \rtimes \mathbb{Z}_2(R)$-invariant complement $\hat{U}_j$ of V_{j-1} in V_j injectively onto a $S^1 \rtimes \mathbb{Z}_2(R)$-invariant subspace of U_1; since U_1 is S-irreducible it follows that $\dim \hat{U}_j = \dim U_1$ and that $\hat{U}_j$ must also be S-irreducible. The result then follows by choosing an $S^1 \rtimes \mathbb{Z}_2(R)$-invariant complement U_k of V_{k-1} in V_k, and setting $U_j := N^{k-j}(U_k)$ for $1 \le j < k$. ■

Next we construct a particular scalar product on U.

Lemma 5. *There exists a scalar product on U for which the $S^1 \rtimes \mathbb{Z}_2(R)$-action is orthogonal and such that $U_k = \ker N^T$ and*

$$N^T(U_j) = U_{j+1}, \qquad N^T\Big|_{U_j} = \left(N\Big|_{U_{j+1}}\right)^{-1}, \qquad 1 \le j \le k-1. \tag{3.4}$$

Proof . Let $\langle \cdot, \cdot \rangle_1$ be a scalar product on U_1 for which the $S^1 \rtimes \mathbb{Z}_2(R)$-action on U_1 is orthogonal; then use Lemma 4 to define a scalar product $\langle \cdot, \cdot \rangle$ on U by

$$\left\langle \sum_{j=1}^{k} u_j, \sum_{j=1}^{k} u_j' \right\rangle := \sum_{j=1}^{k} \left\langle N^{j-1} u_j, N^{j-1} u_j' \right\rangle_1, \quad \forall u_j, u_j' \in U_j,\ 1 \le j \le k. \tag{3.5}$$

One verifies then immediately that the $S^1 \rtimes \mathbb{Z}_2(R)$-action on U is orthogonal with respect to this scalar product (just use the fact that N commutes with S and anti-commutes with R). Moreover, we have for each $u_j \in U_j$ $(1 \le j \le k-1)$ and for all $u_i' \in U_i$ $(1 \le i \le k)$ that

$$\left\langle N^T u_j, \sum_{i=1}^{k} u_i' \right\rangle = \left\langle u_j, \sum_{i=1}^{k} N u_i' \right\rangle = \left\langle N^{j-1} u_j, N^j u_{j+1}' \right\rangle_1 = \left\langle \hat{u}_{j+1}, \sum_{i=1}^{k} u_i' \right\rangle,$$

where $\hat{u}_{j+1} \in U_{j+1}$ is such that $N \hat{u}_{j+1} = u_j$. This proves the lemma. ■

Observe that N^T commutes with S and anti-commutes with R: this follows from the general theory of Section 2, but can also be seen directly from (3.4) The following lemma will be useful in the proof of Theorem 7; it is a special case of Lemma 2.3 in [5].

Lemma 6. *Let $1 \le j \le k$, $S_j := S|_{U_j}$ and $A \in \mathcal{L}(U_j)$. Then A commutes with S_j if and only if it has the form*

$$A = \alpha I_j + \beta S_j \qquad (I_j := I_{U_j}) \tag{3.6}$$

for some $\alpha, \beta \in \mathbb{R}$.

Proof . We will use the fact that U_j is S_j-irreducible (see Lemma 4). Clearly, if $A \in \mathcal{L}(U_j)$ has the form (3.6) then A commutes with S_j. Conversely, suppose that $A \in \mathcal{L}(U_j)$ commutes with S_j, and let $\alpha + i\beta$ be an eigenvalue of A. If $\beta = 0$ then $\ker(A - \alpha I_j)$ is a nontrivial and S_j-invariant subspace of U_j; it follows that $U_j = \ker(A - \alpha I_j)$ and $A = \alpha I_j$. If $\beta \ne 0$ then

$$\ker((A - \alpha I_j)^2 + \beta^2 I_j) = \ker((A - \alpha I_j)^2 - \beta S_j^2) = \ker((A - \alpha I_j - \beta S_j)(A - \alpha I_j + \beta S_j))$$

is nontrivial; hence either $\ker(A - \alpha I_j - \beta S_j)$ or $\ker(A - \alpha I_j + \beta S_j)$ must be nontrivial and S_j-invariant, and therefore equal to U_j. This proves (3.6) (replace β by $-\beta$ in the second case). ■

The following theorem forms the main result of this section.

Theorem 7. *A linear operator $B \in \mathcal{L}(U)$ belongs to $\ker(\mathrm{ad}(S)) \cap \ker(\mathrm{ad}(N^T))$ (i.e. commutes with both S and N^T) if and only if it has the form*

$$B = \sum_{j=1}^{k} (\alpha_j I_U + \beta_j S)(N^T)^{j-1} \tag{3.7}$$

for some $\alpha_j, \beta_j \in \mathbb{R}$.

Proof . It is trivial to see that the condition is sufficient. So suppose that $B \in \mathcal{L}(U)$ commutes with both S and N^T. We claim that there exist numbers $\alpha_j, \beta_j \in \mathbb{R}$ $(1 \leq j \leq k)$ such that if we set

$$B_i := B - \sum_{j=1}^{i} (\alpha_j I_U + \beta_j S)(N^T)^{j-1} \in \mathcal{L}(U) \tag{3.8}$$

then

$$\ker((N^T)^i) = U_{k-i+1} \oplus \cdots \oplus U_k \subset \ker(B_i), \quad (1 \leq i \leq k). \tag{3.9}$$

For $i = k$ this means that $B_k = 0$, such that (3.7) follows from (3.8).

We use induction to prove our claim. Since B commutes with N^T it maps $U_k = \ker N^T$ into itself. Since B also commutes with S it follows from Lemma 6 that the restriction of B to U_k must have the form $\alpha_1 I_k + \beta_1 S_k$ for some $\alpha_1, \beta_1 \in \mathbb{R}$. Then consider $B_1 := B - \alpha_1 I_U - \beta_1 S \in \mathcal{L}(U)$. Clearly B_1 commutes with both S and N^T, and by the way it is defined we also have that $B_1(U_k) = \{0\}$; this proves (3.9) for $i = 1$.

Next let $1 \leq i < k$, and suppose that we have found numbers $\alpha_j, \beta_j \in \mathbb{R}$ $(1 \leq j \leq i)$ such that (3.9) holds for this particular value of i. Then B_i commutes with S and N^T, and therefore

$$N^T(B_i(U_{k-i})) = B_i(N^T(U_{k-i})) \subset B_i(\ker((N^T)^i)) = \{0\}.$$

This proves that $B_i(U_{k-i})$ is contained in $\ker(N^T) = U_k$, which in combination with Lemma 4 implies that $B_i N^i$ maps U_k into itself. By Lemma 6 there exist numbers α_{i+1} and β_{i+1} such that

$$B_i N^i \Big|_{U_k} = \alpha_{i+1} I_k + \beta_{i+1} S_k. \tag{3.10}$$

Setting $B_{i+1} := B_i - (\alpha_{i+1} I_U + \beta_{i+1} S)(N^T)^i$ it is clear from the induction hypothesis that $B_{i+1}(\ker((N^T)^i)) = \{0\}$; moreover, we can write each $u_{k-i} \in U_{k-i}$ as $u_{k-i} = N^i u_k$ for some $u_k \in U_k$, such that using Lemma 5 it follows that

$$B_{i+1}(u_{k-i}) = B_{i+1}(N^i u_k) = B_i N^i u_k - (\alpha_{i+1} I_k + \beta_{i+1} S_k)(N^T)^i N^i u_k = 0.$$

Since $\ker((N^T)^{i+1}) = U_{k-i} \oplus \ker((N^T)^i)$ this proves (3.9) for $i + 1$. ■

Remark 4. The foregoing proof also shows that the representation (3.7) of linear operators $B \in \ker(\mathrm{ad}(S)) \cap \ker(\mathrm{ad}(N^T))$ is unique; indeed, if one starts with $B = 0$ then it is easily seen that at each step in the induction one finds that $\alpha_i = \beta_i = 0$. Using this uniqueness we obtain the following characterization of the subspace W.

Corollary 8. *The space W defined by (3.3) consists of those linear operators $B \in \mathcal{L}(U)$ which have the form*

$$B = \sum_{j=1}^{k} \delta_j S^j (N^T)^{j-1} \tag{3.11}$$

for some $\delta_j \in \mathbb{R}$ $(1 \leq j \leq k)$.

Proof . The space W consists of those operators $B \in \ker(\mathrm{ad}(S)) \cap \ker(\mathrm{ad}(N^T))$ which also belong to $\mathcal{L}_\chi(U)$, i.e. which are R-reversible. So we have to consider operators B of the form (3.7) and impose the additional condition $BR = -RB$, or equivalently $B = -RBR$. Using the fact that both S and N^T are R-reversible in combination with the uniqueness of the representation (3.7) then gives $\alpha_j = 0$ if j is odd, and $\beta_j = 0$ if j is even. The resulting simplified expression for B then takes the form (3.11) by using $S^2 = -I_U$ and setting $\delta_{2\ell+1} := (-1)^\ell \beta_{2\ell+1}$ and $\delta_{2\ell} := (-1)^\ell \alpha_{2\ell}$. ■

4. The bifurcation analysis

In this section we study the small periodic solutions of the reduced system (1.7), i.e. we analyse the solution set of the *bifurcation equation* (1.10) near the origin $(u, \lambda, \sigma) = (0, 0, 0)$. But first we perform some further simplifications.

Using the results of Sections 2 and 3 we know that under the hypotheses (**R**) and (**H**) and assuming $k \geq 2$ the linearization $A_\lambda = D_u g(0, \lambda)$ of the reduced vectorfield g has the form

$$A_\lambda = S + N + \sum_{j=1}^{k} \delta_j(\lambda) S^j (N^T)^{j-1} = (1 + \delta_1(\lambda))S + N + \sum_{j=2}^{k} \delta_j(\lambda) S^j (N^T)^{j-1}, \qquad (4.1)$$

for some C^q-functions $\delta_j : \mathbb{R}^m \to \mathbb{R}$ $(1 \leq j \leq k)$ satisfying $\delta_j(0) = 0$ (and where $q \geq 3$ can be chosen arbitrarily). Without loss of generality one can assume that $\delta_1(\lambda) \equiv 0$ in (4.1). Indeed, define $C_\lambda \in \mathcal{L}(U)$ by

$$C_\lambda u = C_\lambda(u_1, u_2, \ldots, u_k) := ((1 + \delta_1(\lambda))u_1, (1 + \delta_1(\lambda))^2 u_2, \ldots, (1 + \delta_1(\lambda))^k u_k),$$

and replace $u(t)$ in the equation (1.7) by $C_\lambda u((1+\delta_1(\lambda))t)$. This results in a similar equation which is still S^1-equivariant and R-reversible (because C_λ commutes with S and R), but with A_λ replaced by $\tilde{A}_\lambda := (1 + \delta_1(\lambda))^{-1} C_\lambda^{-1} A_\lambda C_\lambda$. Using Lemma 4 and Lemma 5 one verifies easily that

$$C_\lambda^{-1} N C_\lambda = (1 + \delta_1(\lambda))N \quad \text{and} \quad C_\lambda^{-1} N^T C_\lambda = (1 + \delta_1(\lambda))^{-1} N^T;$$

therefore the linearization $\tilde{A}_\lambda$ of the new reduced vectorfield takes the form

$$\tilde{A}_\lambda = S + N + \sum_{j=2}^{k} \frac{\delta_j(\lambda)}{(1 + \delta_1(\lambda))^j} S^j (N^T)^{j-1}.$$

Redefining the $\delta_j(\lambda)$ $(2 \leq j \leq k)$ appropriately returns us to the original expression (4.1) for A_λ, but with $\delta_1(\lambda) \equiv 0$, as claimed. The resulting form of A_λ then motivates the following *transversality condition* which we impose now:

(**T**) The mapping $\Delta : \mathbb{R}^m \longrightarrow \mathbb{R}^{k-1}$, $\lambda \longmapsto (\delta_2(\lambda), \ldots, \delta_k(\lambda))$ is transversal to the origin at $\lambda = 0$, i.e. $D\Delta(0) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{k-1})$ is surjective.

This implies that $m \geq k-1$, and without loss of generality we can assume that

$$\frac{\partial(\delta_2,\ldots,\delta_k)}{\partial(\lambda_1,\ldots,\lambda_{k-1})}(0) \neq 0.$$

By a change of parameters we can then put the functions δ_j in the explicit form $\delta_j(\lambda) = \lambda_{j-1}$ $(2 \leq j \leq k)$, which gives us our final form for A_λ:

$$A_\lambda = S + N + \sum_{j=1}^{k-1} \lambda_j S^{j+1}(N^T)^j. \tag{4.2}$$

The bifurcation equation (1.10) then takes the form

$$\sigma S u = N u + \sum_{j=1}^{k-1} \lambda_j S^{j+1}(N^T)^j u + \hat{g}(u,\lambda), \tag{4.3}$$

where the C^q-mapping $\hat{g} : U \times \mathbb{R}^m \to U$ has the following properties: $\hat{g}(0,\lambda) = 0$, $D_u\hat{g}(0,\lambda) = 0$, $\hat{g}(\exp(\varphi S)u,\lambda) = \exp(\varphi S)\hat{g}(u,\lambda)$ for all $\varphi \in S^1$, and $\hat{g}(Ru,\lambda) = -R\hat{g}(u,\lambda)$. In particular we have $\hat{g}(u,\lambda) = \mathrm{O}(\|u\|^3)$ as $u \to 0$, since $\hat{g}(-u,\lambda) = -\hat{g}(u,\lambda)$.

We have to determine all solutions $(u,\lambda,\sigma) \in U \times \mathbb{R}^m \times \mathbb{R}$ of (4.3) near $(0,0,0)$. Denote by $\pi_j \in \mathcal{L}(U)$ $(1 \leq j \leq k)$ the projections in U onto U_j associated with the splitting $U = U_1 \oplus \ldots \oplus U_k$; these projections commute with S and R. It follows from Lemma 4 and Lemma 5 that $\pi_j N = N\pi_{j+1}$ $(1 \leq j \leq k-1)$, $\pi_k N = 0$, $\pi_1 N^T = 0$ and $\pi_j N^T = N^T \pi_{j-1}$ $(2 \leq j \leq k)$; also $N^T N = I_U - \pi_1$. We can write $u \in U$ as $u = u_1 + \hat{u}$, with $u_1 := \pi_1 u \in U_1$ and $\hat{u} := (I_U - \pi_1)u \in \ker \pi_1 = U_2 \oplus \ldots \oplus U_k$. Finally, the equation (4.3) is equivalent to a system of two equations which one obtains by applying respectively N^T and $S^{-k}N^{k-1}$ to (4.3) (we use the fact that for $u \in U$ we have $u = 0$ if and only if $N^T u = 0$ and $S^{-k}N^{k-1}u = 0$; the reason for this somewhat complicated scheme should be clear from what follows). Working this out in detail gives the equations

$$\hat{u} = \sigma S N^T(u_1 + \hat{u}) - \sum_{j=1}^{k-1} \lambda_j S^{j+1}(N^T)^{j+1}(u_1+\hat{u}) - N^T\hat{g}(u_1+\hat{u},\lambda) \tag{4.4}$$

and

$$\sigma S^{-k+1}N^{k-1}\pi_k u - \sum_{j=1}^{k-1} \lambda_j S^{-k+j+1}N^{k-j-1}\pi_{k-j}u - S^{-k}N^{k-1}\hat{g}(u,\lambda) = 0 \tag{4.5}$$

which have to be solved simultaneously. The equation (4.4) forms an $S^1 \rtimes \mathbb{Z}_2(R)$-equivariant fixed point equation for $\hat{u} \in \ker \pi_1$, depending on the parameters (u_1,λ,σ). For small values of these parameters it can be solved by the implicit function theorem, giving $\hat{u} = \hat{u}^*(u_1,\lambda,\sigma)$. The solution mapping $\hat{u}^* : U_1 \times \mathbb{R}^m \times \mathbb{R} \to \ker \pi_1$ is of class C^q, with $\hat{u}^*(0,\lambda,\sigma) = 0$ and $D_1\hat{u}^*(0,0,0) = 0$; also, $\hat{u}^*$ is $S^1 \rtimes \mathbb{Z}_2(R)$-equivariant:

$$\hat{u}^*(\exp(\varphi S)u_1,\lambda,\sigma) = \exp(\varphi S)\,\hat{u}^*(u_1,\lambda,\sigma), \quad \forall \varphi \in S^1, \tag{4.6}$$

and

$$\hat{u}^*(Ru_1,\lambda,\sigma) = R\hat{u}^*(u_1,\lambda,\sigma). \tag{4.7}$$

Replacing u by $u_1 + \hat{u}^*(u_1, \lambda, \sigma)$ in (4.5) gives us the final bifurcation equation

$$H(u_1, \lambda, \sigma) = 0 \tag{4.8}$$

whose solution set lifts (via all the foregoing reduction steps) to the solution set for our original problem (**P**). The C^q-mapping $H : U_1 \times \mathbb{R}^m \times \mathbb{R} \to U_1$ is $S^1 \rtimes \mathbb{Z}_2(R)$-equivariant, with $H(0, \lambda, \sigma) = 0$ and $D_1H(0, 0, 0) = 0$. Before we calculate the linear part of $H(u_1, \lambda, \sigma)$ we first prove the following.

Theorem 9. *Under the hypotheses* (**R**) *and* (**H**) *all solutions* $\tilde{x}(t)$ *of the problem* (**P**) *are symmetric, i.e. they satisfy*

$$R_0 \{\tilde{x}(t) \mid t \in \mathbb{R}\} = \{\tilde{x}(t) \mid t \in \mathbb{R}\}. \tag{4.9}$$

Proof . By Remark 1 and the equivariance of $\hat{u}^*$ it is sufficient to show that each S^1-orbit of solutions of (4.8) intersects $\mathrm{Fix}(R)$; however, this is not only true for solution orbits, but for any S^1-orbit in U_1. Indeed, let $\bar{u}_1 \in U_1$ be an eigenvector of the restriction of R to U_1, corresponding to the eigenvalue $\epsilon = \pm 1$; then $S\bar{u}_1$ is also an eigenvector, with eigenvalue $-\epsilon$. So the restriction of R to U_1 has both $+1$ and -1 as an eigenvalue, and since U_1 is S-irreducible we conclude that U_1 has a basis $\{u_1^0, Su_1^0\}$ such that $Ru_1^0 = u_1^0$. Then we can write any $u_1 \in U_1$ in the form $u_1 = (\rho\cos\varphi)\, u_1^0 + (\rho\sin\varphi)\, Su_1^0 = \exp(\varphi S)(\rho u_1^0)$ for some $\rho \geq 0$ and some $\varphi \in S^1$; it follows that $\exp(-\varphi S)u_1 = \rho u_1^0$, i.e. the S^1-orbit through u_1 intersects $\mathrm{Fix}(R)$. ■

The foregoing proof also shows that it is sufficient to find the solutions of (4.8) in $\mathrm{Fix}(R)$, i.e. we can put $u_1 = \rho u_1^0$. The equivariance of H implies that $H(\rho u_1^0, \lambda, \sigma) = \beta(\rho, \lambda, \sigma)u_1^0$ for some C^q-function $\beta : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$; we have $\beta(0, \lambda, \sigma) = 0$, $D_1\beta(0, 0, 0) = 0$ and $\beta(-\rho, \lambda, \sigma) = -\beta(\rho, \lambda, \sigma)$. This means that in the end we have to solve the *scalar* equation

$$\beta(\rho, \lambda, \sigma) = 0. \tag{4.10}$$

Solutions of (4.10) come in pairs $(\pm\rho, \lambda, \sigma)$, with both elements of the pair generating the same periodic orbit of (1.1).

We now return to the mapping H, in order to determine its linear part $D_1H(0, \lambda, \sigma) \in \mathcal{L}(U_1)$. This can be done by forgetting the higher order term $\hat{g}(u, \lambda)$ in the reduction from (4.3) to (4.8). Then (4.4) reduces to

$$\hat{u} = \sigma S N^T(u_1 + \hat{u}) - \sum_{j=1}^{k-1} \lambda_j S^{j+1}(N^T)^{j+1}(u_1 + \hat{u}); \tag{4.11}$$

its solution $\hat{u}$ equals $D_1\hat{u}^*(0, \lambda, \sigma) \cdot u_1$. Applying consecutively each of the projections π_i $(i = 2, \ldots, k)$ to (4.11) shows that

$$\pi_i\hat{u} = h_{i-1}(\sigma, \lambda)S^{i-1}(N^T)^{i-1}u_1, \qquad (2 \leq i \leq k), \tag{4.12}$$

where we define the polynomials $h_i(\sigma,\lambda)$ $(0 \le i \le k)$ by the following iteration scheme:

$$h_0(\sigma,\lambda) := 1,\ h_1(\sigma,\lambda) := \sigma,\ h_i(\sigma,\lambda) := \sigma h_{i-1}(\sigma,\lambda) - \sum_{j=1}^{i-1} \lambda_j h_{i-1-j}(\sigma,\lambda)\ \ (2 \le i \le k). \quad (4.13)$$

Bringing (4.12) into the left hand side of (4.5) (in which we put again $\hat{g} = 0$) we find

$$D_1 H(0,\lambda,\sigma)\cdot u_1 = h_k(\sigma,\lambda)\, u_1, \quad \forall u_1 \in U_1, \qquad (4.14)$$

where $h_k(\sigma,\lambda)$ is the polynomial (of degree k in σ) which comes out of the iteration scheme (4.13). Observe that $h_k(\sigma,\lambda)$ only depends on the first $k-1$ components $(\lambda_1,\ldots,\lambda_{k-1})$ of $\lambda \in \mathbb{R}^m$.

Remark 5. The polynomial $h_k(\sigma,\lambda)$ appearing in (4.14) also comes out of the following problem. Consider, for small λ, the operator A_λ given by (4.2), and suppose we want to determine the purely imaginary eigenvalues of this operator; such eigenvalues will be close to $\pm i$, so let us denote them by $\pm(1+\sigma)i$. We have then to determine the kernel of

$$A_\lambda^2 + (1+\sigma)^2 I_U = A_\lambda^2 - (1+\sigma)^2 S^2 = (A_\lambda - (1+\sigma)S)(A_\lambda + (1+\sigma)S).$$

For λ and σ small $A_\lambda + (1+\sigma)S$ is close to $(2S+N)$, and hence invertible. It follows that A_λ has the eigenvalues $\pm(1+\sigma)i$ if and only if $\ker(A_\lambda - (1+\sigma)S)$ is nontrivial, that is, if and only if the equation

$$\sigma S u = N u + \sum_{j=1}^{k-1} \lambda_j S^{j+1} (N^T)^j u \qquad (4.15)$$

has a nonzero solution $u \in U$. Repeating the calculations which we just made shows that this is equivalent to

$$h_k(\sigma,\lambda) = 0. \qquad (4.16)$$

We conclude that $\pm(1+\sigma)i$ are eigenvalues of A_λ if and only if (4.16) holds. ■

It follows from the foregoing that the function β appearing in (4.10) has the form

$$\beta(\rho,\lambda,\sigma) = h_k(\sigma,\lambda)\rho + \tilde{\beta}(\rho,\lambda,\sigma)\rho^3 \qquad (4.17)$$

for some C^{q-3}-function $\tilde{\beta} : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ which is even in ρ. We now make our final hypothesis, which is a non-degeneracy condition:

(N-D) $b := \tilde{\beta}(0,0,0) \neq 0$.

Under this condition we can replace ρ by $|\tilde{\beta}(\rho,\lambda,\sigma)|^{-1/2}\rho$ in (4.17); setting also $\epsilon := -\mathrm{sgn}(b)$ it follows that for nontrivial solutions the equation (4.10) reduces to

$$h_k(\sigma,\lambda) - \epsilon\rho^2 = 0; \qquad (4.18)$$

this is a polynomial equation whose solutions can be studied more or less explicitly, at least for low values of k. In what follows we will discuss the global solution set of (4.18), but it is clear that only solutions (ρ,λ,σ) near $(0,0,0)$ correspond to solutions of our original problem. Also, if (ρ, λ, σ) is a solution then so is $(-\rho, \lambda, \sigma)$, but both correspond to the same periodic orbit of (1.1). Finally, we observe that the equation (4.18) only depends on the *essential parameters* $\lambda_1, \ldots, \lambda_{k-1}$. For simplicity of formulation we will therefore assume that $m = k-1$, i.e. that $\lambda = (\lambda_1, \ldots, \lambda_{k-1})$; it should be obvious how to reformulate the results in the case $m > k-1$.

For each $\lambda \in \mathbb{R}^{k-1}$ we define

$$\mathcal{S}_\lambda^{(k)} := \{\sigma \in \mathbb{R} \mid \epsilon h_k(\sigma, \lambda) \geq 0\}. \tag{4.19}$$

Then the solution set of (4.18) is given by

$$\Sigma_k = \{(\pm(\epsilon h_k(\sigma, \lambda))^{\frac{1}{2}}, \lambda, \sigma) \mid \lambda \in \mathbb{R}^{k-1}, \sigma \in \mathcal{S}_\lambda^{(k)}\}. \tag{4.20}$$

The set $\mathcal{S}_\lambda^{(k)}$ is completely determined by ϵ and by the zeros of $h_k(\cdot, \lambda)$, and consists of at most $[\frac{k}{2}] + 1$ closed intervals. These intervals merge or disappear together with the zeros of $h_k(\cdot, \lambda) = 0$, which allows us to define the *bifurcation set* for (4.18) as

$$\mathcal{B}_k := \{\lambda \in \mathbb{R}^{k-1} \mid h_k(\sigma, \lambda) = D_\sigma h_k(\sigma, \lambda) = 0 \ \text{ for some } \ \sigma \in \mathbb{R}\}. \tag{4.21}$$

The following result characterizes this bifurcation set.

Theorem 10. *The bifurcation set $\mathcal{B}_k$ is diffeomorphic to the standard cuspoid of order k, i.e. to the fold for $k = 2$, the cusp for $k = 3$, the swallowtail for $k = 4$, the butterfly for $k = 5$, etc.*

Proof . The standard versal unfolding of (the germ at zero of) the function $\sigma \mapsto \sigma^k$ $(k \geq 2)$ is given by $H_k(\sigma, \lambda) := \sigma^k + \sum_{j=1}^{k-1} \lambda_j \sigma^{k-1-j}$ $(\lambda \in \mathbb{R}^{k-1})$, and the cuspoid of order k is defined as the bifurcation set B_k for the zeros of $H_k(\cdot, \lambda)$, i.e.

$$B_k := \{\lambda \in \mathbb{R}^{k-1} \mid H_k(\sigma, \lambda) = D_\sigma H_k(\sigma, \lambda) = 0 \ \text{ for some } \ \sigma \in \mathbb{R}\}.$$

To prove the theorem we will show that we can write the function $h_k(\sigma, \lambda)$ in the form

$$h_k(\sigma, \lambda) = \sigma^k - \sum_{j=1}^{k-1} \phi_j(\lambda) \sigma^{k-1-j}, \tag{4.22}$$

where the functions $\phi_j : \mathbb{R}^{k-1} \to \mathbb{R}$ $(1 \leq j \leq k-1)$ are such that the mapping $\Phi : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ defined by

$$\Phi(\lambda) := (\phi_1(\lambda), \ldots, \phi_{k-1}(\lambda)), \quad \forall \lambda \in \mathbb{R}^{k-1}, \tag{4.23}$$

forms a diffeomorhism of $\mathbb{R}^{k-1}$ onto itself. The theorem then follows from the relation $h_k(\sigma, \lambda) = H_k(\sigma, -\Phi(\lambda))$.

To prove the statements about $h_k(\sigma, \lambda)$ we observe that a direct calculation using (4.13) shows that

$$\begin{aligned}
h_2(\sigma, \lambda) &= \sigma^2 - \lambda_1, \\
h_3(\sigma, \lambda) &= \sigma^3 - 2\lambda_1 \sigma - \lambda_2, \\
h_4(\sigma, \lambda) &= \sigma^4 - 3\lambda_1 \sigma^2 - 2\lambda_2 \sigma - (\lambda_3 - \lambda_1^2), \\
h_5(\sigma, \lambda) &= \sigma^5 - 4\lambda_1 \sigma^3 - 3\lambda_2 \sigma^2 - (2\lambda_3 - 3\lambda_1^2)\sigma - (\lambda_4 - 2\lambda_1 \lambda_2), \text{ etc.}\ldots
\end{aligned}$$

This motivates us to put

$$h_i(\sigma,\lambda)=\sigma^i-\sum_{j=1}^{i-1}(i-j)\lambda_j\sigma^{i-1-j}+\sum_{j=3}^{i-1}\psi_j^{(i)}(\lambda_1,\ldots,\lambda_{j-2})\sigma^{i-1-j} \tag{4.24}$$

in the iteration scheme (4.13); in (4.24) the last term is absent for $i\leq 3$, and the polynomials $\psi_j^{(i)}(\lambda_1,\ldots,\lambda_{j-2})$ $(3\leq j\leq i-1)$ are to be determined. After some straightforward rearrangements one finds the following iteration scheme for these polynomials:

$$\begin{aligned}\psi_j^{(i)}(\lambda_1,\ldots,\lambda_{j-2}) &= \psi_j^{(i-1)}(\lambda_1,\ldots,\lambda_{j-2})+(i-j)\sum_{m=1}^{j-2}\lambda_m\lambda_{j-1-m}\\ &\quad -\sum_{m=1}^{j-4}\lambda_m\psi_{j-1-m}^{(i-1-m)}(\lambda_1,\ldots,\lambda_{j-3-m}),\end{aligned} \tag{4.25}$$

valid for $3\leq j\leq i-1$; at the right hand side of (4.25) the first term is absent if $j=i-1$, and the last term if $j\leq 4$. It follows from these calculations that (4.22) holds, with

$$\phi_j(\lambda)=\begin{cases}(k-j)\lambda_j & \text{if } j=1,2,\ j\leq k-1,\\ (k-j)\lambda_j-\psi_j^{(k)}(\lambda_1,\ldots,\lambda_{j-2}) & \text{if } 3\leq j\leq k-1.\end{cases} \tag{4.26}$$

The triangular structure of these expressions immediately imply that the mapping Φ given by (4.23) is indeed a diffeomorphism of $\mathbb{R}^{k-1}$ onto itself. ■

It follows from the foregoing theorem and from the well known properties (see e.g. [7]) of the cuspoids that $\mathcal{B}_k$ forms a stratified set which is equal to the closure of its codimension one strata. More precisely, we call a parameter value $\lambda\in\mathcal{B}_k$ a *simple bifurcation point* if two conditions are satisfied: (1) there exists a *unique* $\sigma=\sigma_\lambda\in\mathbb{R}$ such that $h_k(\sigma,\lambda)=D_\sigma h_k(\sigma,\lambda)=0$; and (2) $D_\sigma^2 h_k(\sigma_\lambda,\lambda)\neq 0$. We call such simple bifurcation point *elliptic* if $\epsilon D_\sigma^2 h_k(\sigma_\lambda,\lambda)<0$, and *hyperbolic* if $\epsilon D_\sigma^2 h_k(\sigma_\lambda,\lambda)>0$. The set $\mathcal{B}_k^{(1)}$ of simple bifurcation points forms a codimension one submanifold of $\mathbb{R}^{k-1}$ (in general not connected), and $\mathcal{B}_k=\mathrm{cl}(\mathcal{B}_k^{(1)})$. For example, for $k=2$ we have $\mathcal{B}_2=\mathcal{B}_2^{(1)}=\{0\}$, and the only bifurcation value $\lambda=0$ is elliptic if $\epsilon=-1$, and hyperbolic if $\epsilon=1$. In general the complement of $\mathcal{B}_k$ consists of a finite number of open connected components, each of which has the origin in its closure; we call parameter values belonging to this complement *regular*. We now describe the solution set of (4.18) for such regular parameter values and the transitions which take place when one moves from one connected component of $\mathbb{R}^{k-1}\setminus\mathcal{B}_k$ to another by crossing $\mathcal{B}_k$ at a simple bifurcation point.

Fix a regular parameter value $\lambda\in\mathbb{R}^{k-1}\setminus\mathcal{B}_k$; then the connected components of $\mathcal{S}_\lambda^{(k)}$ have one of the following forms:

(i) A compact interval $[\alpha,\beta]$, with $h_k(\alpha,\lambda)=h_k(\beta,\lambda)=0$ and $\epsilon h_k(\sigma,\lambda)>0$ for $\sigma\in]\alpha,\beta[$; we call the corresponding branch of periodic solutions of (1.1) a *local loop*. At both sides the loop ends at the equilibrium $x=0$, forming there the two invariant manifolds given by the Liapunov Center Theorem and corresponding to the eigenvalue pairs $\pm(1+\alpha)i$ and $\pm(1+\beta)i$ of A_λ.

(ii) A semi-infinite closed interval of the form $]-\infty, \alpha]$ or $[\alpha, \infty[$, with $h_k(\alpha, \lambda) = 0$ and $\epsilon h_k(\sigma, \lambda) > 0$ for either $\sigma < \alpha$ or $\sigma > \alpha$, depending on the case; we call the corresponding branch of periodic solutions of (1.1) a *global branch* (although the term "global" has to be taken with a grain of salt — see the observation after (4.18)). Such global branch ends at one side at the equilibrium $x = 0$, forming there the invariant manifold given by the Liapunov Center Theorem and corresponding to the eigenvalue pair $\pm(1+\alpha)i$ of A_λ. If k is odd then there is exactly one global branch for each regular value of λ; if k is even and $\epsilon = -1$ then there are no global branches.

(iii) The whole real line $\mathbb{R}$, which means that $\epsilon h_k(\sigma, \lambda) > 0$ for all $\sigma \in \mathbb{R}$; this is only possible if k is even and $\epsilon = 1$. The corresponding branch of periodic solutions does not pass through the equilibrium; we therefore call this a *detached branch.* Observe that in this case the linear operator A_λ has no purely imaginary eigenvalues, and so the Liapunov Center Theorem cannot be applied to get invariant manifolds filled with periodic orbits.

(iv) If k is even and $\epsilon = -1$ then there is the possibility that $\mathcal{S}_\lambda^{(k)} = \emptyset$ (i.e. $\epsilon h_k(\sigma, \lambda) < 0$ for all σ). In this case there are no periodic orbits besides the equilibrium $x = 0$.

The number and type of the connected components of $\mathcal{S}_\lambda^{(k)}$ do not change as λ moves within the same connected component of $\mathbb{R}^{k-1} \backslash \mathcal{B}_k$. As λ crosses from one such connected component to another via a simple bifurcation point $\lambda_0 \in \mathcal{B}_k^{(1)}$ and in the appropriate direction then one of the following changes will occur:

(a) If λ_0 is elliptic then one of the compact components of $\mathcal{S}_\lambda^{(k)}$ shrinks to the point σ_{λ_0} for $\lambda = \lambda_0$ and then disappears; this corresponds to one of the local loops shrinking down to the equilibrium and then disappearing.

(b) If λ_0 is hyperbolic then we have the following possibilities:

  (b1) Two compact components of $\mathcal{S}_\lambda^{(k)}$ merge into one single compact component; this corresponds to two local loops merging into a single one.

  (b2) A compact component of $\mathcal{S}_\lambda^{(k)}$ merges with a semi-infinite component; this corresponds to a local loop merging with a global branch, resulting in a single global branch.

  (b3) $\mathcal{S}_\lambda^{(k)}$ consists of just two semi-infinite intervals which merge for $\lambda = \lambda_0$ to result in $\mathcal{S}_\lambda^{(k)} = \mathbb{R}$ (this is only possible if k is even and $\epsilon = 1$); this means that two global branches of periodic solutions become tangent to each other and then merge into a detached branch.

For example, if $k = 2$ then scenario (a) occurs if $\epsilon = -1$, and scenario (b3) if $\epsilon = 1$. In general, passing from one connected component of $\mathbb{R}^{k-1} \setminus \mathcal{B}_k$ to another via a higher codimension bifurcation point several of the foregoing transitions will happen at the same time.

We conclude this paper with a brief description of the bifurcation picture in the case $k = 3$. From $h_3(\sigma, \lambda) = \sigma^3 - 2\lambda_1\sigma - \lambda_2$ one easily calculates that the bifurcation set for $k = 3$ is given by the cusp $\mathcal{B}_3 = \{(\lambda_1, \lambda_2) \mid 32\lambda_1^3 = 27\lambda_2^2\}$. All bifurcation points except the origin $\lambda = 0$ are simple; a bifurcation point $(\lambda_1, \lambda_2) \in \mathcal{B}_3$ is elliptic if $\epsilon\lambda_2 > 0$, and hyperbolic if $\epsilon\lambda_2 < 0$. The complement $\mathbb{R}^2 \setminus \mathcal{B}_3$ has two connected components: for parameter values in

$\mathcal{C}_1 := \{(\lambda_1, \lambda_2) \mid 32\lambda_1^3 < 27\lambda_2^2\}$ there is just one global branch of periodic solutions, while for parameter values in the other component $\mathcal{C}_2 := \{(\lambda_1, \lambda_2) \mid 32\lambda_1^3 > 27\lambda_2^2\}$ we have one global branch and one local loop of periodic solutions. Crossing the bifurcation set from $\mathcal{C}_2$ to $\mathcal{C}_1$ (i.e. in the direction of decreasing λ_1) at an elliptic point the local loop shrinks and disappears; crossing in the same direction at a hyperbolic point the local loop merges with the global branch. Passing from $\mathcal{C}_2$ to $\mathcal{C}_1$ via the origin $\lambda = 0$ we see at the same time the shrinking of the local loop and its merging with the global branch.

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