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Numerical Simulation of Subharmonically Reacting Nonlinear Electrical Circuits

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Abstract

The mathematical synthesis of nonlinear dynamical systems with subharmonic response is an interesting approach in the study of bifurcations of electrical circuits. We use an idea of E.S. PHLIPPOW [6] and construct a mathematical model for *periodically forced heteronomous systems* as a combination of an oscillating system with expected solution and a so-called *identity* which is coupeling the oscillator with the external force. For *parametrically forced systems*, the model equations can be obtained in an analogous way.
By numerical simulation we demonstrate the occurence of quasi-periodic solutions in these systems. In generic cases a stable invariant torus emerges as the closure of a quasi-periodic orbit. This leads to the idea of analyzing a quasi-periodic solution by direct computation of the invariant torus. Two different numerical approaches are presented which can approximate both stable and weakly unstable torus solutions.

Keywords: Numerical simulation, quasi-periodic orbits, invariant tori

AMS subject classification: 65C20, 65L10

1 The Principle of Mathematical Synthesis

The idea to find a mathematical model of subharmonically reacting systems is described by E.S. Philippow in [6]. The task is to develop an ideal frequency dividing network with input $\sigma(\tau) = \sin(n\tau)$ or $\sigma(\tau) = \cos(n\tau)$ and output $\zeta(\tau) = \sin(\tau)$ or $\zeta(\tau) = \cos(\tau)$. The input $\sigma(\tau)$ and the output $\zeta(\tau)$ are normalized currents or voltages, τ is a normalized time. The required behaviour can be described by a non-autonomous 2-nd order differential equation

$$\frac{d^2x}{d\tau^2} + F\left(x, \frac{dx}{d\tau}\right) = \sigma(\tau) \tag{1}$$

with the condition

$$\lim_{\tau\to\infty} \{x\,(x(0), \dot{x}(0), \tau) - \zeta(\tau)\} = 0 \tag{2}$$

for the solution $x(x_0, \dot{x}_0, \tau)$ of (1). For further considerations it is convenient to split the function $F(x, \dot{x})$ into two terms, a damping term

$$T_d = f\left(x, \frac{dx}{d\tau}\right)\frac{dx}{d\tau} \tag{3}$$

and a restoring term

$$T_r = g\left(x, \frac{dx}{d\tau}\right). \tag{4}$$

For the pure sinusoidal solution of (1), the function $f(x, \dot{x})$ in the $x - \dot{x}$ plane has to describe an exact circle

$$f(x, \dot{x}) = \varepsilon\left(x^2 + \dot{x}^2 - 1\right). \tag{5}$$

ε is a parameter determining the transient process from the inital point to the limit cycle. The restoring term saves the capability of the system to produce an oscillation. So the term should have the following properties:

$$g(x, \dot{x}) = -g(-x, \dot{x})\ , \quad x g(x, \dot{x})|_{|x|>\delta} = 0\ , \quad \delta < |\hat{x}|\ . \tag{6}$$

Then the model equation is

$$\ddot{x} - \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} + g(x, \dot{x}) = \sigma(\tau)\ . \tag{7}$$

If $g(x, \dot{x})$ fulfils the condition

$$g\left(x(\tau), \dot{x}(\tau)\right) - \sigma(\tau) = x(\tau) \tag{8}$$

the model equation becomes

$$\ddot{x} - \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} + x = 0 \tag{9}$$

with the exact sinusoidal solution $x = \cos\tau$ and $\hat{x} = 1$. Equation (8) yields a condition for $g(x, \dot{x})$

$$\begin{aligned} g(x, \dot{x}) &= \sigma(\tau) + x = \cos(n\tau) + x \\ \cos(n\tau) &= T_n(\cos\tau) = T_n(x) \\ g(x, \dot{x}) &= T_n(x) + x \end{aligned} \tag{10}$$

where $T_n(x)$ is the Chebyshev polynomial of order n. (10) describes an identity. The Chebyshev polynomials of even order $n = 2m$, $m = 1, 2, \ldots$, do not fulfil the requirements for the restortion term (6). They have to be transformed (usually by differentiation) into a suitable term (see Table 1). For $n = 2m$, $T_n(x, \dot{x})$ has the structure of a damping term.

The left hand side of the identity describes a circuit for the coupeling of the oscillator with the external force. The coefficients b and $\hat{B}$ allow to investigate the influence of the coupeling term and the amplitude of forcing term:

$$b\, T_n(x, \dot{x}) = \begin{cases} \hat{B}\cos(n\tau) \\ \hat{B}\sin(n\tau)\ . \end{cases} \tag{11}$$

n	$\sigma(\tau)$	$T_n(x,\dot{x})$	$\zeta(\tau)$
2	$\sin(2\tau)$	$2\,x\,\dot{x}$	$\sin\tau$
3	$\cos(3\tau)$	$2\,x^3 - 3\,x$	$\cos\tau$
4	$\sin(4\tau)$	$(8\,\dot{x}^2\,x - 4\,x)\,\dot{x}$	$\sin\tau$
5	$\cos(5\tau)$	$16\,x^5 - 20\,x^3 + 5\,x$	$\cos\tau$

Table 1: Structure of possible terms T_n

We use the term **identity** to call this part of the model equation and mean only a so-called identity.

The basic philosophy of modelling a subharmonically reacting system is

(1) to build an *autonomous model* equation with the desired subharmonic solution
(2) to find an *identity* which provides the harmonic signal, if we put the same solution into this identity
(3) to add the autonomous equation and the identity to get the *model of the subharmonic system.*

A parametric system is based on an equation of Matthieu type. So we take a Rayleigh equation to describe the oscillating system

$$\ddot{x} - \varepsilon(k_1 - \dot{x}^2)\dot{x} + x = 0\;. \tag{12}$$

To guarantee the stability of the investigated systems with an even subharmonic reaction we use as identity the modified Chebyshev polynomials of the form

$$-\frac{b}{n^2}\frac{dT_n}{dx}\dot{x} = \frac{\hat{B}}{n}\sin(n\tau) \tag{13}$$

respectively

$$\frac{b}{n}\frac{dT_n}{dx}\dot{x} = -\hat{B}\sin(n\tau)\;. \tag{14}$$

The identities are fulfilled if $b = \hat{B}$. To find a parametric system we have to multiply (13) or (14) by x. So we get two new forms of the identities

$$-\frac{b}{n^2}\frac{dT_n}{dx}\dot{x}x = \frac{bx}{n}\sin(n\tau) \tag{15}$$

$$\frac{b}{n}\frac{dT_n}{dx}\dot{x}x = -bx\sin(n\tau)\;. \tag{16}$$

In the following we consider two special cases of model equations. To obtain a mathematical model of a **3-fold subharmonically reacting system** we use a combination of the oscillating system

$$\ddot{x} - \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} + x = 0 \tag{17}$$

with the solution $x(\tau) = \sin\tau$ or $x(\tau) = \cos\tau$ and the so-called identity

$$b(4x^3 - 3x) = \hat{B}\cos(3\tau)\;. \tag{18}$$

The identity is fulfilled if $b = \hat{B}$ and $x(\tau) = \cos\tau$. By using (17) and (18) the model equation may be written as

$$\ddot{x} - \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} + x + b(4x^3 - 3x) = \hat{B}\cos(3\tau) \; . \tag{19}$$

If we consider $b = \hat{B} = 1$ we get the desired subharmonic solutions of (19)

$$x_1 = \cos\tau \;, \quad x_2 = \cos(\tau + 2/3\pi) \;, \quad x_3 = \cos(\tau + 4/3\pi) \; . \tag{20}$$

The solutions (20) can be considered within $5 < \varepsilon < 30$.

For the equation of a **2-fold subharmonically reacting system** we have to chose $T_2 = 2x^2 - 1$. If we use the identity (15) with $k_1 = 1$ we find the model equation

$$\ddot{x} + \varepsilon\dot{x}^3 - \varepsilon\dot{x} - bx^2\dot{x} + \left[1 - \frac{b}{2}\sin(2\tau)\right] x = 0 \; . \tag{21}$$

The desired solution of this equation is $x = \cos\tau$. For the network synthesis we may furthermore transform the term x^2 into $x^2 = 1 - \dot{x}^2$ and find

$$\ddot{x} + (\varepsilon + b)\dot{x}^3 - (\varepsilon + b)\dot{x} + \left[1 - \frac{b}{2}\sin(2\tau)\right] x = 0 \tag{22}$$

or more generally

$$\ddot{x} + \alpha\dot{x}^3 - \beta\dot{x} + \left[1 - \frac{b}{2}\sin(2\tau)\right] x = 0 \tag{23}$$

with $\alpha = \beta = (\varepsilon + b)$.

The choice of the parameter values is also based on the condition that the identity and the oscillating system must have an approximately identical harmonic solution. The solution of a Rayleigh type equation is sinusoidal only for small values of ε . The smaller the ε-values the smaller are the distortions of solutions too. The ε-value may be chosen within the boundaries $0 < \varepsilon < 0.5$.

2 Computation of Torus Solutions

We consider (19) and (23) as special dynamical systems of the general form

$$\frac{dx}{dt} = f(x) \quad , \quad f \; : \; \mathbb{R}^n \rightarrow \mathbb{R}^n \tag{24}$$

where $f \in C^r$ is a sufficiently smooth vector field. While reliable numerical tools are available for stationary and periodic solutions, there exist only first approaches dealing with multi-frequency oscillations as a further kind of equilibriums in dynamical systems. Two ways are generic in the occurence of quasi-periodic orbits:

- A stable biperiodic solution emerges by a Hopf (Neimark-Sacker) bifurcation in the neighbourhood of a periodic solution.
- A periodic solution on a torus undergoes a saddle-node bifurcation and so a quasi-periodic solution emerges.

In both cases a stable invariant torus emerges as the closure of the quasi-periodic orbit. This leads to the idea of analyzing quasi-periodic solutions by direct computation of their invariant tori. We present two numerical approaches to approximate invariant 2-tori:

- Transformation of problem (24) into torus coordinates and solution of the partial differential equations describing the invariant torus manifold (see [1], [9])
- Computation of the Poincaré map of 2-tori in $\mathbb{R}^n$ and approximation of the closed invariant curve as a polygon by integration of (24) and suitable addition of further points (see [3]).

Both approaches can be applied to general quasi-periodic orbits and to p-tori, but the numerical effort is considerable in case of $p > 2$. In our subharmonically reacting circuits we can restrict the considerations to invariant 2-tori and to periodic and biperiodic orbits laying on them.

2.1 Integration of the Torus Equations

We pass over from Cartesian coordinates $x = (x_1, \cdots, x_n)$ to radial coordinates $u = (u_1, \cdots, u_q)$ and torus coordinates $\theta = (\theta_1, \cdots, \theta_p)$ with $p + q = n$ by the *transformation*

$$x = B(\theta)u + b(\theta) \tag{25}$$

where $b : \mathbb{T}^p \to \mathbb{R}^n$ and $B : \mathbb{T}^p \to \mathbb{R}^{n \times q}$. Now we are looking for a parametrization of the torus $\mathcal{M}$ in (θ, u)-coordinates

$$\mathcal{M} = \{(\theta, u) \mid u = u(\theta) \ , \ \theta \in \mathbb{T}^p\} \tag{26}$$

Using transformation (25) in (24) yields

$$[B'(\theta)u + b'(\theta) \ , \ B(\theta)] \begin{bmatrix} \dot{\theta} \\ \dot{u} \end{bmatrix} = f(B(\theta)u + b(\theta)) \ , \tag{27}$$

and we obtain the partitioned system (see [7])

$$\begin{aligned} \dot{\theta} &= \Omega(\theta, u) \\ \dot{u} &= R(\theta, u) \ . \end{aligned} \tag{28}$$

We will assume that (28) has a locally unique invariant torus $\mathcal{M}$, which can be parametrized in the form (26) with the C^r-function $u(\theta)$ $(r \geq 1)$. The torus manifold $\mathcal{M} \subset \mathbb{T}^p \times \mathbb{R}^q$ is invariant under the given vector field $(\Omega(\theta, u), R(\theta, u))^T$ of (28) if and only if $u(\theta)$ satisfies the *quasilinear PDE system* with the same principle part

$$\sum_{j=1}^{p} \Omega_j(\theta, u) \frac{\partial u}{\partial \theta_j} = R(\theta, u) \tag{29}$$

for $\theta \in \mathbb{T}^p$. System (29) together with the torus conditions

$$u(\theta_1, \cdots, \theta_{j-1}, \theta_j + 2\pi, \theta_{j+1}, \cdots, \theta_p) = u(\theta_1, \cdots, \theta_p) \ , \quad j = 1 ... p \ , \tag{30}$$

for all $\theta_j \in \mathbb{R}$, describes the invariant torus $\mathcal{M}$. The aim of our approach is to reduce the two-dimensional problem to a one-dimensional periodic problem, because there exist reliable numerical methods for the treatment of periodic orbits. For it we have to select the variable θ_2 , for which $\Omega_2 \neq 0$ for $(\theta, u) \in \mathbb{T}^2 \times \mathbb{R}^q$. If we divide (29) by Ω_2 we obtain

$$\frac{\partial u}{\partial t} + \omega(\theta, t, u)\frac{\partial u}{\partial \theta} = r(\theta, t, u) \ , \quad (\theta, t) \in \mathbb{T}^2 \tag{31}$$

with $u \in C^1(\mathbb{T}^2)$, $\omega : \mathbb{T}^2 \times \mathbb{R}^q \to \mathbb{R}$, $r : \mathbb{T}^2 \times \mathbb{R}^q \to \mathbb{R}^q$ and torus conditions

$$u(\theta, t) = u(\theta + 2\pi, t) \ , \quad u(\theta, t) = u(\theta, t + 2\pi) \ . \tag{32}$$

For fixed $t \in I = [0, T]$, $T = 2\pi$, let $v(t) := u(\theta, t)$ be an element of the Banach space

$$\mathcal{B} = \{ w \in \mathbb{C}^1 (\mathbb{T}^1, \mathbb{R}^q) \mid w(\theta) = w(\theta + 2\pi) \ , \ \theta \in [0, 2\pi) \} \ .$$

With the differential operator $\mathcal{F}$ defined on $\mathcal{B} \times I$ by

$$\mathcal{F}(v, t) \ \equiv \ -\omega(\theta, t, u)\frac{\partial u}{\partial \theta} + r(\theta, t, u) \tag{33}$$

problem (31),(32) is equivalent to the two point boundary value problem

$$\frac{dv}{dt} = \mathcal{F}(v, t) \quad , \quad v(0) = v(T) \quad , \quad T = 2\pi \tag{34}$$

for $v : I \to \mathcal{B}$. The periodicity of the functions ω and r is transmitted to $\mathcal{F}$ with

$$\mathcal{F}(v, t + T) = \mathcal{F}(v, t) \quad , \quad (v, t) \in \mathcal{B} \times \mathbb{R} \quad , \tag{35}$$

hence we can consider (34) as a periodically forced system. Any periodic solution v^* corresponds to exact one torus solution u^* of (31) and conversely. Now it is an obvious approach to reduce this problem with the help of a shooting method to a convergent sequence of initial value problems. Therefore we consider the corresponding initial value problem

$$\frac{dv}{dt} = \mathcal{F}(v, t) \quad , \quad v(0) = g \in \mathcal{B} \tag{36}$$

for $t \in (0, T)$. Let $g \in \mathcal{B}$ be a fixed initial element. Then (36) yields the corresponding solution $v(t; g)$ on $I = [0, T]$. So the map $\varphi : \mathcal{B} \to \mathcal{B}$ can be defined by

$$\varphi(g) \equiv v(T, g) - g \ . \tag{37}$$

Obviously $v^*(0) = g^*$ is the initial condition of (36) if g^* solves the equation

$$\varphi(g) = 0 \ . \tag{38}$$

A first version of the shooting method is based on the fixed point (Picard) iteration for

$$g = v(T, g) \; . \tag{39}$$

To apply this method the torus $\mathcal{M}$ has to be orbitally stable. Otherwise we have to use Newton-like methods to solve (38). The numerical computations however must be carried out in finite dimensional spaces. System (31) can be discretized on a suitable grid $\mathbb{T}_h^2$ over the standard torus $\mathbb{T}$. On this discretized torus

$$\mathbb{T}_h^2 = \{(\theta_j, t_n) \mid \theta_j = jh, t_n = n\tau \; , \; j = 0..J \; , \; n = 0..N\} \tag{40}$$

a large number of explicit and linearly implicit difference methods of first order can be described in the general representation

$$\frac{1}{\tau}\left[\sum_{\mu=-1}^{1} S_\mu^*(\theta_j, t_n, u_j^n) u_{j+\mu}^{n+1} - \sum_{\mu=-1}^{1} S_\mu(\theta_j, t_n, u_j^n) u_{j+\mu}^n\right] = r(\theta_j, t_n, u_j^n) \tag{41}$$

with grid functions $u_j^n \sim u(\theta_j, t_n)$ and stepsizes h and τ (τ/h = const). S_μ^* and S_μ $(\mu = -1, 0, 1)$ are diagonal matrices where S_μ^* , $S_\mu \in \mathbb{R}^{q\times q}$.

Under suitable assumptions it can be verified that these one-step methods are consistent and convergent in h and τ (see [1]). Special methods implemented in the C code **TORUS** are the

- Explicit upwind-type method of order 1
- Linearly-implicit upwind-type mehod of order 1
- Upwind methods with global extrapolation of order 2
- Upwind methods with defect correction of order 2.

2.2 Approximation of Closed Invariant Curves

We now consider periodically forced systems

$$\frac{dx}{dt} = f(t, x) \; , \quad f \in C^r(\mathbb{R} \times \mathbb{R}^n) \; , \; r \geq 2 \tag{42}$$

with $f(t + T, x) = f(t, x) \; \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$, $T \geq 0$. The (stroboscopic) Poincaré map of (42) with time step T is defined by

$$\Phi \; : \; D \in \mathbb{R}^n \to \mathbb{R}^n \; , \quad \Phi(x_0) := x(T; x_0) \; . \tag{43}$$

Let us assume that Φ has a simple closed *invariant curve* $\gamma \subset \mathbb{R}^n$. Then there exists a map $u : I \to \mathbb{R}^n$, $I = [0, 2\pi)$ such that $\gamma = \{x | x = u(\tau) \; , \; \tau \in I \; , \; u(0) = u(2\pi)\}$ is *invariant under* Φ :

$$\Phi \, \gamma \; = \; \gamma \; . \tag{44}$$

The numerical solution of this equation is based on an idea of Van Veldhuizen [8], who approximates γ by polygons $\mathcal{P} \subset \mathbb{R}^n$ in the following steps:

- Let $x_1, x_2, \ldots, x_N$ be the N vertices of the polygon $\mathcal{P}(\{x_i\}_{i=1}^N)$ with edges $[x_1, x_2], [x_2, x_3], \ldots, [x_N, x_1]$.
- Application of the Poincaré map Φ yields the images Φx_i and hence a new polygon $\mathcal{P}(\{\Phi x_i\}_{i=1}^N)$.
- For asymptotically stable γ it can be assumed that $\mathcal{P}(\{\Phi x_i\}_{i=1}^N)$ will be a "better" approximation of γ than $\mathcal{P}(\{x_i\}_{i=1}^N)$ and finally

  $$\mathcal{P}(\{\Phi^n x_i\}_{i=1}^N) \longrightarrow \gamma \quad , \qquad n, N \to \infty$$

  if the initial polygon $\mathcal{P}(\{\Phi^0 x_i\}_{i=1}^N) = \mathcal{P}(\{x_i\}_{i=1}^N)$ is sufficiently close to γ .

The following problem however can arise: If the flow on the torus is periodic then the $\Phi^n x_i$, $n = 1, 2, \ldots$, converge to one point or to few points on the curve γ . A "homogeneous" distribution of the vertices $\Phi^n x_i$, $n = 1, 2, \ldots$ can be obtained by projection of the old vertices onto the new polygon $\mathcal{P}(\{\Phi x_i\}_{i=1}^N)$ and insertion of new vertices after each iteration step. The numerical map then consists of Φ and a projection Π by piecewise linear interpolation : $Kx_i = \Pi\Phi x_i$.
One step of the fixed point iteration implemented in the C code **ICURVE** now reads as follows:

1. Computation of the images of all vertices $\mathcal{P}(\{x_i\}_{i=1}^N)$ by numerical integration of the differential equations
2. Projection of the old vertices $\{x_i\}_{i=1}^N$ onto the polygon $\mathcal{P}(\{\Phi x_i\}_{i=1}^N)$
3. Insertion of further vertices in order to obtain a sufficiently dense approximation of the curve

An approximating polygon $\mathcal{P}(\{x_i^*\}_{i=1}^N)$ is defined as a solution of the equation

$$\mathcal{P}(\{x_i\}_{i=1}^N) = \mathcal{P}(\{Kx_i\}_{i=1}^N) \ . \tag{45}$$

Under suitable conditions of the problem (see [3]) it can be shown that this equation has a unique solution and the sequence of the polygons

$$\mathcal{P}(\{x_i\}_{i=1}^N), \mathcal{P}(\{Kx_i\}_{i=1}^N), \mathcal{P}(\{K^2 x_i\}_{i=1}^N), \ldots$$

converges to $\mathcal{P}(\{x_i^*\}_{i=1}^N)$.

3 Torus Solutions of the Model Equations

At first we consider the model equation (23)

$$\ddot{x} + \alpha\dot{x}^3 - \beta\dot{x} + (1 + B\sin 2\tau)x = 0$$

with

$$B = 0.1 \qquad \alpha = \varepsilon - B = \varepsilon - 0.1 \qquad \beta = \frac{\varepsilon}{2} - B = \frac{\varepsilon}{2} - 0.1 \ . \tag{46}$$

Automatic transformation by the code **TORUS** into polar coordinates yields

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \beta cs - sin^2\theta_1 - \alpha u^2 \sin^3\theta_1 \cos\theta_1 - (1 + B\sin 2\theta_2)\cos^2\theta_1 = \psi(\theta_1, \theta_2, u, \varepsilon) \\
\frac{d\theta_2}{dt} &= 1 \qquad (47) \\
\frac{du}{dt} &= u(cs + \beta\sin^2\theta_1) - \alpha u^3 \sin^4\theta_1 - u(1 + B\sin 2\theta_2)cs = r(\theta_1, \theta_2, u, \varepsilon)
\end{aligned}$$

with $cs = \cos\theta_1 \sin\theta_1$.
The torus equation of $\mathcal{M}$ is now

$$\frac{\partial u}{\partial \theta_2} + \psi(\theta_1, \theta_2, u, \varepsilon)\frac{\partial u}{\partial \theta_1} = r(\theta_1, \theta_2, u, \varepsilon) \ .$$

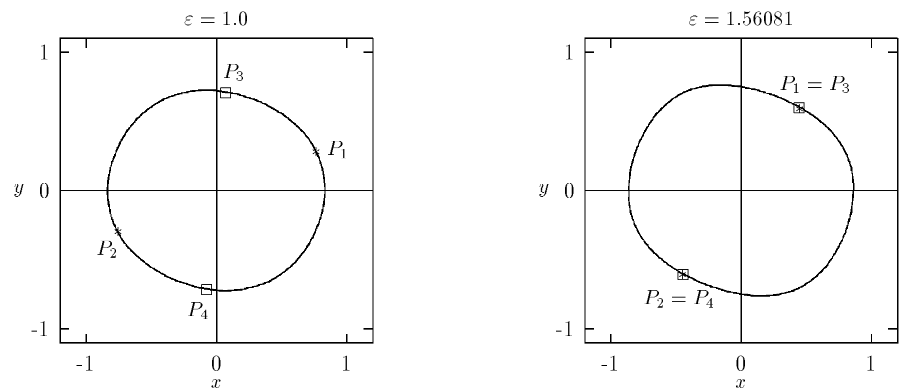

Figure 1: Torus sections : periodic solutions $\varepsilon = 1.0$, $\varepsilon = 1.56081$
$*$: stable periodic solution $\square$: periodic solution of saddle type

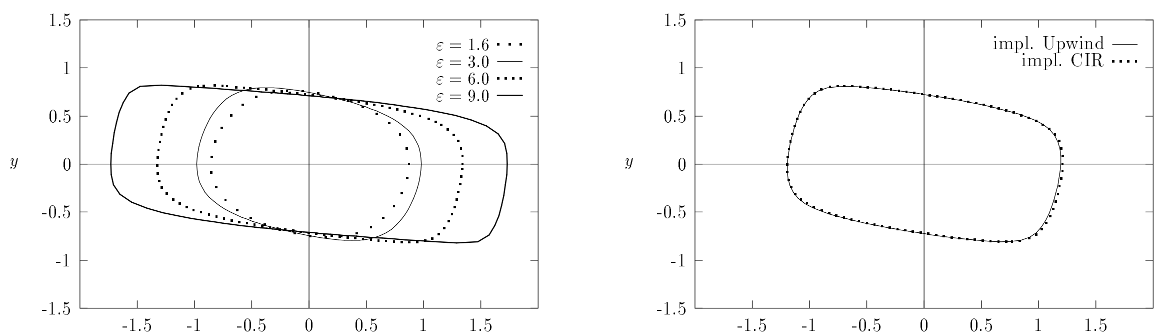

Figure 2: Torus sections : (a) quasi-periodic solutions; (b) $\varepsilon = 5.0$

The invariant tori were computed by the linear-implicit (smooth) upwind-type method including global defect corrections. In figure 3 the whole torus is displayed in two projections for $\varepsilon = 5.0$.

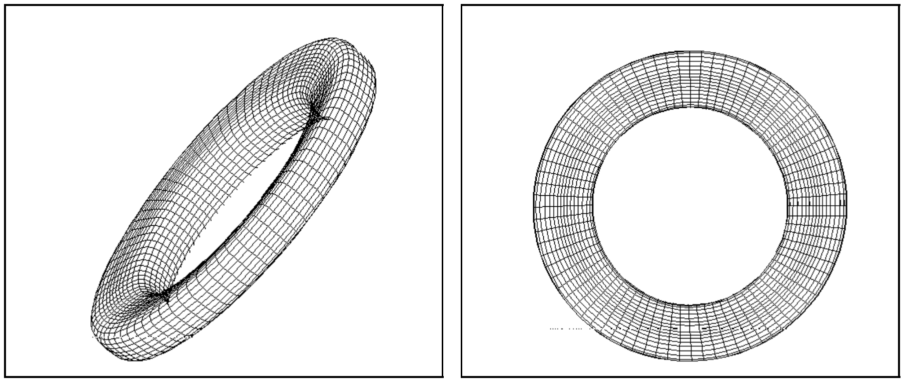

Figure 3: Grid model for $\varepsilon = 5.0$

The code **ICURVE** begins with an initial polygon of 40 points $x_1, x_2, ..., x_{40}$ on the unit circle. We obtained the following invariant curves:

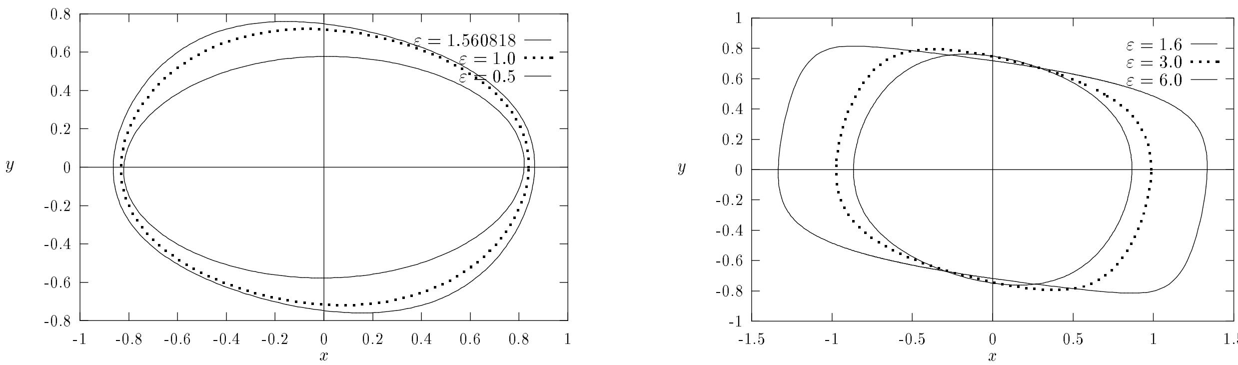

Figure 4: (a) Invariant curves : periodic solutions; (b)Invariant curves : quasi-periodic solutions

Now we apply the two methods to the parametric model (19)

$$\ddot{x} - \varepsilon(1 - x^2 - \dot{x}^2)\dot{x} + x + b(4x^3 - 3x) = B\cos 3\tau \ .$$

Automatic transformation by **TORUS** into polar coordinates yields the partitioned system with $\tau = \theta_2$

$$\begin{aligned}
\frac{d\theta_1}{dt} &= \varepsilon(1-u^2)\sin\theta_1\cos\theta_1 - 1 - b\cos^2\theta_1(4u^2\cos^2\theta_1 - 3) + \frac{1}{u}B\cos\theta_1\cos 3\theta_2 \\
&= \omega(\theta_1, \theta_2, u, \varepsilon, B) \\
\frac{d\theta_2}{dt} &= 1 \qquad (48) \\
\frac{du}{dt} &= \varepsilon(1-u^2)u\sin^2\theta_1 - b\sin\theta_1(4u^3\cos^3\theta_1 - 3u\cos\theta_1) + B\sin\theta_1\cos 3\theta_2 \\
&= r(\theta_1, \theta_2, u, \varepsilon, B) \ .
\end{aligned}$$

The invariant torus is described by the PDE

$$\frac{\partial u}{\partial \theta_2} + \omega(\theta_1, \theta_2, u, \varepsilon, B)\frac{\partial u}{\partial \theta_1} = r(\theta_1, \theta_2, u, \varepsilon, B) \tag{49}$$

and the torus conditions

$$\begin{aligned} u(\theta_1, \theta_2) &= u(\theta_1 + 2\pi, \theta_2) \\ u(\theta_1, \theta_2) &= u(\theta_1, \theta_2 + 2\pi) \ . \end{aligned} \tag{50}$$

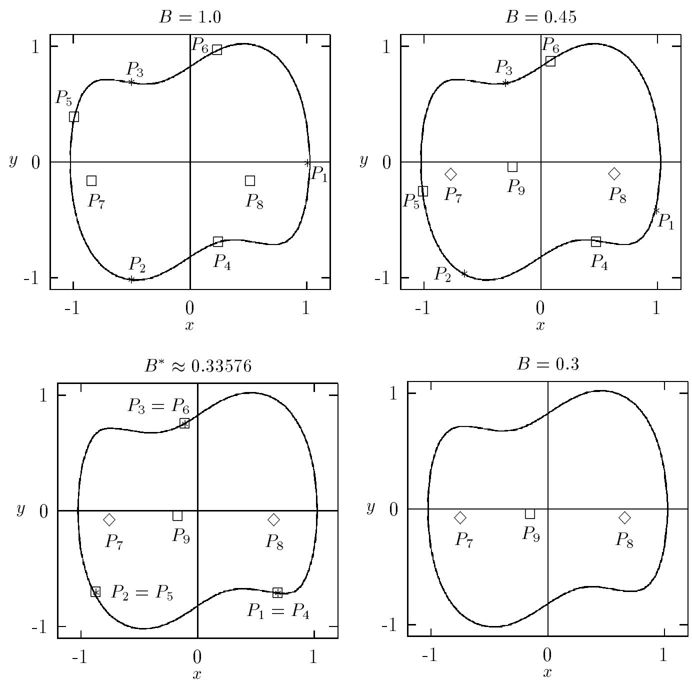

Figure 5: Torus sections : $\varepsilon = 2.0$, $b = 1.0$
$*$: stable periodic solution $\square$: periodic solution of saddle type
$\diamond$: unstable periodic solution

Figure 5 shows that both the torus with periodic solutions and the quasi-periodic torus can be computed by the method without numerical difficulties.

Further investigation with a new version **TORUS2** showed that in case of greater values of parameter $B = 1.0$ the torus will be "overlapping". The parameter studies in Figures 6 and 7 display this development.

Finally, we compute the invariant curves with parameter values $\varepsilon = 2.0$, $b = 1.0$, $B = 0.3$ by using **ICURVE**. The approximating polygons for $b = 1.0$, $B = 0.3$ and different values of ε are displayed in figure 8. With a bound of 0.05 of the maximal length of the edges and a tolerance of 10^{-6}, the method used 3 iterations. As initial polygon we used 100 points on the unit circle. The numbers of the vertices of the resulting polygons are between 180 and 250.

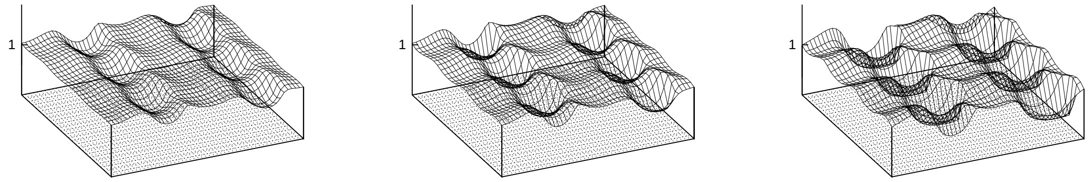

Figure 6: 3-fold subharmonically reacting system for $\varepsilon = 2$, $b = 1$ and different values of B.

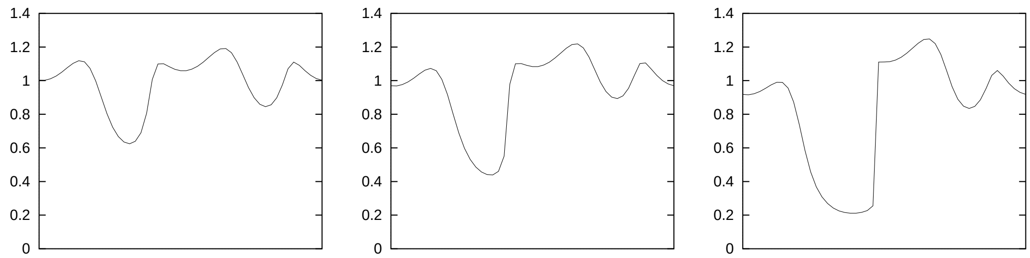

Figure 7: Torus sections of 3-fold subharmonically reacting system for $\varepsilon = 2$, $b = 1$ and different values of B

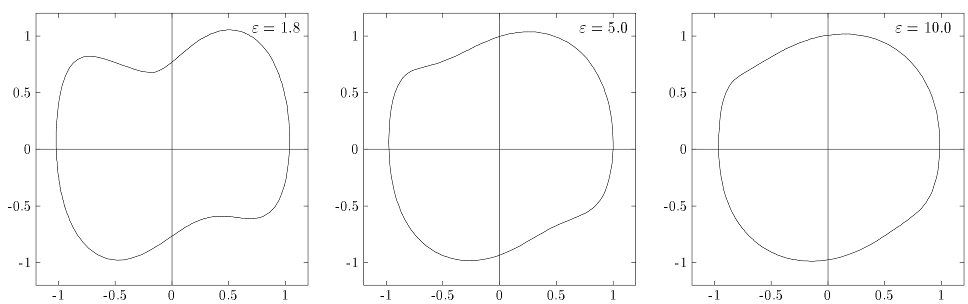

Figure 8: 3-fold subharmonically reacting system

The applications of the two completely different numerical methods to our model equations (19) and (23) show that in both models quasi-periodic responses can arise. So these methods can serve as reliable tools for numerical simulations of electrical circuits including variations of their system parameters.

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