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On a class of biomorphic motion systems

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On a Class of Biomorphic Motion Systems

0 Introduction

One task in engineering biomechanics is to derive technological advantage from nature. This means in particular to observe live objects and to apply the knowledge obtained about structure and dynamics in design of technical objects. Often such a technical object is to behave under certain aspects similar to the living prototype, and the design could possibly even create improvements by utilizing materials, power sources, and devices (e.g. wheels) that are not found in live world. There are no bounds on the designer's creative imagination.

Observing terrestrial locomotion of living beings one recognizes first a surface contact with the ground that is responsible for the conversion of (mostly periodic) internal and internally driven motions into change of external position (*undulatory locomotion* [7]). Where there is contact of bodies there is some kind of friction, its peculiarities strongly depend on the physical properties of the surfaces in contact. In particular, the friction may be anisotropic, i.e., depending on the direction of the (tendency of) relative displacement. In a limiting case friction could even prevent any relative motion in a certain direction. Finally, a surface that is covered with scales (like a snake's belly) could make the friction also orientation dependent (in sliding forward the frictional forces are minimal while in opposite direction the scales dig in and cause large friction), thus ultimately restricting the sign of a velocity coordinate, $v_x \geq 0$, say.
In [6] the author considers, in a computer graphics context, mass-spring systems with scales aiming at modeling virtual worms and snakes and their animation.
In our paper we describe and elaborate models of this type but having an eye upon eventual technical realization as certain worm- or snake-like motion systems of what size ever. As to the recent interest in such motion systems and their utility we refer, e.g., to [3].
Again, these systems transform internal reciprocating (possibly periodic) motion into unidirectional external motion, classifying themselves as *mechanical rectifiers* [2] - most (all?) live locomotion systems belong to that class.
The transformation of internal motions to external displacement via scales in the contact areas will be modeled by constancy of sign of output velocities. Therefore our considerations concern a particular species of *hybrid systems of complementary slackness type* investigated in [9], [10]. We will refer to these papers under slight modifications (w.r.t. heteronomic systems).
From an analytical mechanics point of view we shall be dealing with systems of finite degree (DOF) of freedom under differential equality and inequality constraints [4], [5]. For systems in rectilinear motion the constraints relate to holonomic velocities whereas for systems that are equipped with scales of steerable direction and move in the plane, anholonomic velocities are concerned, we refer to [8].
We do not claim this paper to be a mature and comprehensive report on worm-like motion systems. It is part of broader research activities on motion systems and it is to serve as a (in parts tutoring) working paper that, hopefully, will stimulate further investigations concerning structure, actuators, control, and realization, possibly applying microtechnology.

1 Simplest model in $\mathbb{R}^1$, elementary treatment

We consider two point masses in a common straight line. We suppose them to be equipped with scales contacting the ground which prevent any motion in, say, negative $x-$direction.

To start with, let the two point masses be interconnected by a massless linear spring of fixed stiffness c and controllable tension-free length l. So we have an oscillatory motion system that may perform a net locomotion in positive $x-$direction superimposed by harmonic oscillations.

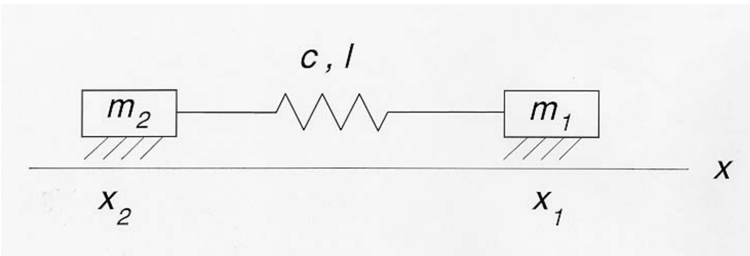

Fig. 1.1 Scaly 2mass - 1spring system.

The scales ensure that $\dot{x}_i(t) \geq 0,\ i = 1, 2$, for all t and they keep either point mass at rest as long as the resulting physical force acting upon that point is nonpositive. While this is happening, the point mass behaves like being constrained to a fixed position. Clearly, there are 4 types of motion of the system depending on which points stay at rest. The four respective dynamics will be called the *modes* of the system. The system may switch from one mode to another caused by (endogenous or exogenous) *events*. Positions x_i and velocities $\dot{x}_i$ as functions of time t are continuous under switches.
Let $m_1 = m_2 = m,\ \omega^2 := c/m$, and denote the initial data when entering a mode by $(t_o, \mathring{x}_1, \mathring{x}_2, \mathring{v}_1, \mathring{v}_2)$, of course, with speeds $\mathring{v}_i \geq 0$. Then the modes can be described as follows. Only for simplicity we suppose here constant spring length l between consecutive switches.

Mode 0: Neither point mass at permanent rest.

$$\begin{aligned} \ddot{x}_1 &= -\omega^2(x_1 - x_2 - l), \\ \ddot{x}_2 &= \omega^2(x_1 - x_2 - l), \end{aligned} \quad t \in (t_o, t_1), \tag{1.1}$$

$$x_1(t_o) = \mathring{x}_1,\ x_2(t_o) = \mathring{x}_2,\ \dot{x}_1(t_o) = \mathring{v}_1,\ \dot{x}_2(t_o) = \mathring{v}_2\ .$$

The DEs are equivalent to

$$\begin{aligned} &(x_1 + x_2)^{\cdot\cdot} = 0, \\ &(x_1 - x_2)^{\cdot\cdot} = -2\omega^2[(x_1 - x_2) - l], \end{aligned} \tag{1.2}$$

and this means that the center of mass is in uniform locomotion with speed

$$\bar{v} = \frac{1}{2}(\mathring{v}_1 + \mathring{v}_2)$$

while the point masses oscillate in counterphase with frequency $\sqrt{2}\omega$. The motion in mode 0 continues as long as $\dot{x}_i(t) > 0,\ i = 1, 2$. It terminates and a switch occurs if one of the velocities is going to be negative. Denote the right hand sides of (1.1) by f_1, f_2, respectively. Then mode 0 terminates at the smallest time $t_1 > t_o$

$$\begin{aligned} &\text{where (a) } \dot{x}_1 = 0 \wedge [f_1 < 0 \vee (f_1 = 0 \wedge \text{ decreasing for } t > t_1)] \\ &\text{or} \qquad \text{(b) } \dot{x}_2 = 0 \wedge [f_2 < 0 \vee (f_2 = 0 \wedge \text{ decreasing for } t > t_1)], \end{aligned} \tag{1.3}$$

motion in mode 1 or mode 2, respectively, follows. (It is easy to see that a switch to mode 12: both point masses resting, is not possible.)

Mode 1: Point mass 1 at permanent rest.

$$\begin{aligned} & x_1(t) = \mathring{x}_1,\ \dot{x}_1(t) = 0, \\ & \ddot{x}_2 = \omega^2(\mathring{x}_1 - x_2 - l), \end{aligned} \qquad t \in (t_o, t_1), \tag{1.4}$$

$$x_2(t_o) = \mathring{x}_2,\ \dot{x}_2(t_o) = \mathring{v}_2\ .$$

The point mass 2 moves sinusoidally. Mode 1 terminates at the smallest time $t_1 > t_o$ where $\dot{x}_2(t)$ is going to be negative or $f_1 = -f_2 = -\omega^2(\mathring{x}_1 - x_2 - l)$ is going to be positive, then driving point 1 out of rest into positive direction. A glance at the solution of (1.4), $\dot{x}_2(t) = \omega(\mathring{x}_1 - \mathring{x}_2 - l)\sin\omega(t - t_o) + \mathring{v}_1 \cos\omega(t - t_o)$, shows that the latter event comes first, so mode 1 terminates at the smallest $t_1 > t_o$ such that

$$\begin{aligned} -\omega(\mathring{x}_1 - \mathring{x}_2 - l)\cos\omega(t_1 - t_o) + \mathring{v}_2 \sin\omega(t_1 - t_o) &= 0, \\ \omega(\mathring{x}_1 - \mathring{x}_2 - l)\sin\omega(t_1 - t_o) + \mathring{v}_2 \cos\omega(t_1 - t_o) &> 0. \end{aligned} \tag{1.5}$$

Since $\dot{x}_1$ will be positive for $t > t_1$ while $\dot{x}_2(t_1)$ still is, motion in mode 0 will follow.

Mode 2: Point mass 2 at permanent rest.
Same as in mode 1 with indexes 1 and 2 interchanged.

Mode 12: Both point masses at permanent rest.
Obviously, this mode requires special initial data $\mathring{v}_1 = \mathring{v}_2 = 0$ and $\mathring{x}_1 - \mathring{x}_2 - l = 0$ (relaxed spring). A switch out of this mode can only be caused by exogenous events.

In the preceding considerations all events terminating a mode were endogenous: certain state functions cross certain tresholds. Exogenous events in the above system could be sudden changes of the spring characteristics c and l (used as controls).

Example 1. In the system considered above we let

$$l(t) = l_o(1 + \epsilon(t)), \quad \epsilon(t) = \begin{cases} \epsilon_o, & t \in (0, \tau) \\ 0 & \text{else} \end{cases}$$

with given constant $\epsilon_o \in (-1, 0)$ and $0 < \tau \leq +\infty$. So we have two exogenous events: at time $t = 0$ some muscle-like actuator instantaneously shortens the natural spring length and brings it back to the original value l_o at time τ (to be chosen later on): $l(\cdot)$ is a kind of bang-bang control.
For $t \leq 0$ we let $x_1(t) = l_o,\ x_2(t) = 0$: the system is in mode 12 until $t = 0$.
For $t = +0$ the spring acts at the point masses by

$$f_1(+0) = -\omega^2(\overset{\circ}{x}_1 - \overset{\circ}{x}_2 - l(+0)) = \omega^2 l_o \epsilon_o < 0,\ f_2(+0) = -f_1(+0) > 0.$$

Thus in mode 0 we would get an unfeasible $\dot{x}_1(t) < 0$ hence, for (small) $t > 0$ mode 1 is the only option. So for $t \in (0, t_1)$ the motion is governed by (1.4) and $\overset{\circ}{x}_1 = l_o,\ \overset{\circ}{x}_2 = 0,\ \overset{\circ}{v}_2 = 0$ which yields

$$x_2(t) = -l_o\epsilon_o(1 - \cos\omega t), \quad \dot{x}_2(t) = -l_o\epsilon_o\omega\sin\omega t.$$

The first zero of $f_1 = -\omega^2(\overset{\circ}{x}_1 - x_2 - l) = l_o\epsilon_o\omega^2\cos\omega t$ is at $t_1 = \pi/2\omega$ (where $\dot{x}_2(t)$ is maximal).
Assuming the switching time for the spring $\tau > t_1$, the following mode 0 motion is governed by (1.1) with initial values at t_1 : $\overset{\circ}{x}_1 = l_o,\ \overset{\circ}{x}_2 = -l_o\epsilon_o,\ \overset{\circ}{v}_1 = 0,\ \overset{\circ}{v}_2 = \dot{x}_2(t_1) = -l_o\epsilon_o\omega$. This yields the velocities

$$\dot{x}_1(t) = -\tfrac{1}{2}l_o\epsilon_o\omega(1 - \cos\sqrt{2}\omega(t - t_1)),$$
$$\dot{x}_2(t) = -\tfrac{1}{2}l_o\epsilon_o\omega(1 + \cos\sqrt{2}\omega(t - t_1)).$$

Since both velocities are non-negative for all t this mode will not be terminated by any endogenous event. There could be a switch to another mode at time τ.

1st choice: $\tau > t_1$ such that $\dot{x}_2(\tau) = 0$ (i.e. $\dot{x}_1(\tau)$ maximal):
$\sqrt{2}(\omega\tau - \frac{\pi}{2}) = (2\nu - 1)\pi$ with some $\nu \in \mathbb{N}$.
Now $l(t) = l_o$ for $t > \tau$, and $f_2(\tau + 0) = \omega^2(x_1(\tau) - x_2(\tau) - l_o) = \omega^2 l_o\epsilon_o < 0$. So $\dot{x}_2(t)$ would become negative, while m_1 gets an accelerating push: a mode 2 follows,

$$\dot{x}_1(t) = -l_o\epsilon_o\omega\sqrt{2}\sin(\omega(t - \tau) + \frac{\pi}{4}).$$

This mode terminates at $\tau_1 > \tau$ when $f_2 = -f_1 = -\ddot{x}_1$ is going to be positive, i.e.

$$\omega(\tau_1 - \tau) = \frac{\pi}{4}.$$

Again a mode 0 follows,

$$\dot{x}_1(t) = -\sqrt{2}\tfrac{1}{2}l_o\epsilon_o\omega(1 + \cos\sqrt{2}\omega(t - \tau_1)),$$
$$\dot{x}_2(t) = -\sqrt{2}\tfrac{1}{2}l_o\epsilon_o\omega(1 - \cos\sqrt{2}\omega(t - \tau_1)).$$

Mind that, compared with the first mode 0 phase of motion, the mean speed has increased by a factor $\sqrt{2}$.

Problem: If the bang-bang control is repeated that way, what is the sequence $(1, \sqrt{2}, \ldots)$ of factors to the mean speed in the mode 0 phases?

2nd choice: $\tau > t_1$ such that $\dot{x}_1(\tau) = \dot{x}_2(\tau)$:
$\sqrt{2}(\omega\tau - \frac{\pi}{2}) = \frac{1}{2}(2\kappa - 1)\pi$ with some $\kappa \in \mathbb{N}$.

Since both velocities at $t = \tau$ are positive, the system remains in mode 0 but now with

$$\dot{x}_1(t) = -\tfrac{1}{2}l_o\epsilon_o\omega[1 + (\sqrt{2} - (-1)^\kappa)\sin\sqrt{2}\omega(t - \tau)],$$
$$\dot{x}_2(t) = -\tfrac{1}{2}l_o\epsilon_o\omega[1 - (\sqrt{2} - (-1)^\kappa)\sin\sqrt{2}\omega(t - \tau)].$$

The mean speed has not changed. If κ is an even number then neither of the velocities will have a zero and this mode never terminates endogenously. But if κ is odd (large amplitude of antisynchronous oscillation) then at the first zero of $\dot{x}_2$ a switch to mode 2 occurs (compare 1st choice) followed (at maximum of $\dot{x}_1$) by another switch to mode 0 again.

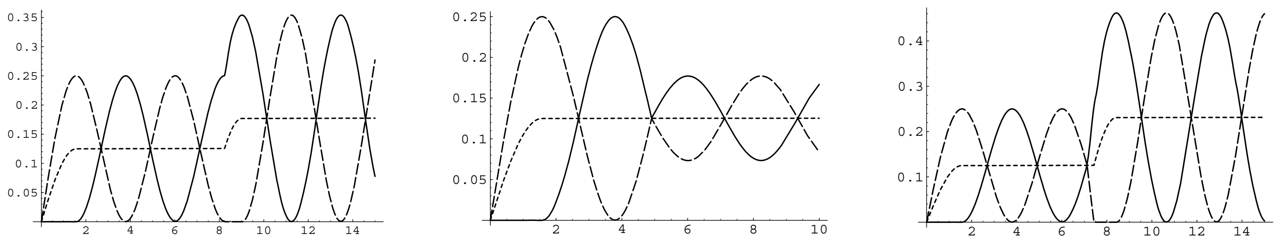

Fig. 1.2. Velocities vs.t. Left: 1st choice of τ. Middle: 2nd choice of τ, κ even. Right: 2nd choice of τ, κ odd.

Remark. A sudden change at time $t = \tau$ of the natural spring length entails an energy input

$$\Delta U = \frac{c}{2}\,[x_1(\tau) - x_2(\tau) - l(\tau + 0)]^2 - \frac{c}{2}\,[x_1(\tau) - x_2(\tau) - l(\tau - 0)]^2.$$

Writing for short $x_1 = x_1(\tau), l^+ = l(\tau + 0)$, etc.,

$$\Delta U = c(l^+ - l^-)\left[\frac{1}{2}(l^+ + l^-) - (x_1 - x_2)\right].$$

Under contraction, $l^+ < l^-$, the energy input is positive iff $\frac{1}{2}(l^+ + l^-) < x_1 - x_2 =$ actual spring length at $t = \tau$. In order to maximize ΔU at given values l^+, l^-, the switching time τ has to be chosen so that $x_1 - x_2$ is maximal at $t = \tau$, i.e., $\dot{x}_1(\tau) - \dot{x}_2(\tau) = 0 \wedge \ddot{x}_1(\tau - 0) - \ddot{x}_2(\tau - 0) < 0$. Analogously, under expansion, $l^+ > l^-$, a maximal $\Delta U > 0$ is gained iff

$$\dot{x}_1(\tau) - \dot{x}_2(\tau) = 0 \wedge \ddot{x}_1(\tau - 0) - \ddot{x}_2(\tau - 0) > 0.$$

This becomes evident by Fig. 1.2.(right).
The above rules could be used for the design of a ΔU maximizing feedback control.

Problem: Is there any piecewise constant or continuous control $t \mapsto l(t)$ that makes the system stop?

2 Finite DOF scaly straight worm

We consider now n point masses in a common straight line which interact via interconnecting elastic or viscous or muscle-like devices. Let some ($k \leq n$) of the point masses contact the ground through scales (of common orientation) preventing velocities from being negative and causing friction when in motion.
A typical worm-like system of this kind with DOF $n = 4$ is sketched in Figure 2.1. The slender ellipses indicate massless muscle-like devices (of any physical characteristics whatever). The springs above could model the worm's skin.

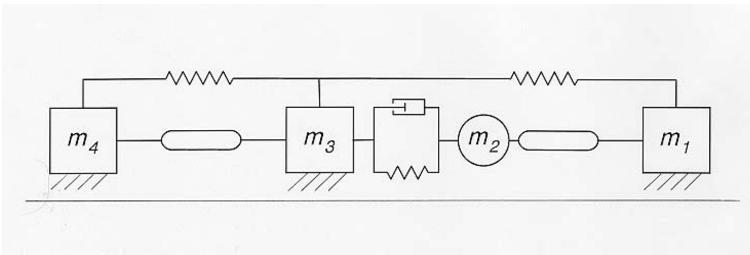

Fig. 2.1. Typical scaly straight worm

In order to formulate a comprehensive dynamics let x_ν, $\nu = 1, \ldots, n$, denote the (absolute) coordinates of the point masses, let $x = (x_1, \ldots, x_n)$. The total physical force acting at the point mass ν is denoted $f_\nu(x, \dot{x}, t)$. Let K be that subset of $\{1, \ldots, n\}$ with cardinality k, $1 \leq k \leq n$, which enumerates the scale-equipped point masses. Then the system is subject to the *differential inequality constraint*

$$\dot{x}_\kappa \geq 0, \ \kappa \in K.$$

The corresponding Lagrange multipliers, λ_κ, represent the reaction forces which, due to the scales, are either zero or act in positive $x-$direction.
The motions of the system are then governed by the following *differential-algebraic equations* supplemented by a set of *complementary-slackness conditions:*

$$\begin{aligned} \dot{x}_\nu &= v_\nu, \\ m_\nu \dot{v}_\nu &= f_\nu(x, v, t) + \sum_{\kappa \in K} \delta_\nu^\kappa \lambda_\kappa, \quad \nu = 1, \ldots, n, \end{aligned} \tag{2.1a}$$

$$v_\kappa \geq 0, \ \lambda_\kappa \geq 0, \ v_\kappa \lambda_\kappa = 0, \quad \kappa \in K \subset \{1, \ldots, n\}. \tag{2.1b}$$

(2.1b) means that $\lambda_\kappa(t)$ must be zero when the corresponding inequality constraint is strict ("has slack"), $v_\kappa(t) > 0$, while $\lambda_\kappa(t)$ may have arbitrary non-negative values when $v_\kappa(t) = 0$. It should be noticed that (2.1) is a special type of those more general *hybrid systems* that are investigated in [9], [10]. We will partly follow the lines of these papers.
Concerning the laws of force, $f_\nu(\cdot, \cdot, \cdot)$, we assume such analytical properties that ensure existence and uniqueness of solutions of certain ODEs which derive from (2.1) in the following (i.e. at least, $f_\nu(x, v, \cdot)$ piecewise continuous, $f_\nu(\cdot, \cdot, t)$ locally Lipschitz). Continuity w.r.t. t would be too strong in view of bang-bang control.

Now let I be any subset of K and suppose $v_i = 0$ for $i \in I$ and $\lambda_{i'} = 0$ for $i' \in K \backslash I$. So to every subset $I \subset K$ there corresponds a DAE

$$\left.\begin{array}{ll} \dot{x}_{\nu'} = v_{\nu'}, & \\ m_{\nu'}\dot{v}_{\nu'} = f_{\nu'}(x, v', t), & \nu' \in \{1, \ldots, n\}\backslash I, \\ \dot{x}_i = 0, & \\ 0 = f_i(x, v', t) + \lambda_i, & i \in I, \end{array}\right\} \Sigma_I \tag{2.2}$$

where v' stands for $(v_1, \ldots, v_n)$ with $v_i = 0,\ i \in I$. The dynamics of Σ_I is called *mode I* of the full system (2.1a). There are 2^k modes, each of which has to satisfy the remaining inequalities (2.1b) as feasibility conditions.

A point $(\mathring{x}, \mathring{v}, \mathring{\lambda}) \in \mathbb{R}^{2n+k}$ is called a *consistent point of* Σ_I *at* t_o if there exists a smooth (here that means at least: absolutely continuous) solution of Σ_I that passes that point at time t_o. The set of these points is, owing to our assumptions concerning f_ν,

$$\mathcal{V}_I(t_o) = \{(x, v, \lambda) \in \mathbb{R}^{2n+k} | v_i = 0,\ \lambda_i = -f_i(x, v, t_o)\ (i \in I),\ \lambda_{i'} = 0\ (i' \in K\backslash I)\}. \tag{2.3}$$

Since furthermore there is only one solution through every point of $\mathcal{V}_I(t_o)$, the system Σ_I is called *deterministic* (*autonomous* in [9], [10]). The subset of those points for which also the remaining inequality conditions (2.1b) are satisfied is called *feasible set of mode I at* t_o,

$$\mathcal{W}_I(t_o) = \{(x, v, \lambda) \in \mathcal{V}_I(t_o) | v_{i'} \geq 0,\ (i' \in K\backslash I),\ \lambda_i = -f_i(x, v, t_o) \geq 0,\ (i \in I)\}. \tag{2.4}$$

$\mathcal{W}_I(t_o)$ could be empty (e.g., if $f_i(x, v, t_o) > 0$ for some $i \in I$). Suppose $\mathcal{W}_I(t_o)$ to be nonvoid. Choose $(\mathring{x}, \mathring{v}, \mathring{\lambda}) \in \mathcal{W}_I(t_o)$, then there is a unique solution of Σ_I starting from $(\mathring{x}, \mathring{v}, \mathring{\lambda})$ at time t_o. Denote the point reached at time $t > t_o$ by $(x(t), v(t), \lambda(t))$. If there is an $\epsilon > 0$ such that $(x(t), v(t), \lambda(t)) \in \mathcal{W}_i(t)$ for all t with $t - t_o \in (0, \epsilon)$ then *smooth continuation is possible from* $(\mathring{x}, \mathring{v}, \mathring{\lambda})$ *in mode I*. This is *not* the case iff

$$\Gamma(I; t_o) := \Gamma_1((\mathring{x}, \mathring{v}, \mathring{\lambda}); I) \cup \Gamma_2((\mathring{x}, \mathring{v}, \mathring{\lambda}); I) \neq \emptyset \tag{2.5a}$$

where

$$\Gamma_1((\mathring{x}, \mathring{v}, \mathring{\lambda}); I) = \{i' \in K\backslash I \mid \exists \epsilon > 0\ \forall t \in (t_o, t_o + \epsilon)\ \ v_{i'}(t) < 0\}, \tag{2.5b}$$

$$\Gamma_2((\mathring{x}, \mathring{v}, \mathring{\lambda}); I) = \{i \in I \mid \exists \epsilon > 0\ \forall t \in (t_o, t_o + \epsilon)\ \ \lambda_i(t) = -f_i(x(t), v(t), t) < 0\}. \tag{2.5c}$$

While the system moves in mode I, $\Gamma(I; t)$ is empty. $\Gamma(I; t_o) \neq \emptyset$ marks an event that forces the system to switch to another mode J. Since in (2.5) the next future to t_o is crucial, $\Gamma(I; t_o)$ is like an "infinitesimally anticipant event explorer".

It is reasonable to choose

$$J := (I \backslash \Gamma_2((\mathring{x}, \mathring{v}, \mathring{\lambda}); I)) \cup \Gamma_1(\mathring{x}, \mathring{v}, \mathring{\lambda}); I) \tag{2.6}$$

and the initial values of x, v, and λ the new mode is to start with by continuity i.e., $\mathring{x}$ and $\mathring{v}$, and $\mathring{\lambda}_j := -f_j(\mathring{x}, \mathring{v}, t_o^+)$. (Call to mind that in mode I $v_i(t) = 0,\ \lambda_{i'}(t) = 0$, while $v_{i'} \geq 0$ and $\lambda_i \geq 0$ are governed by (2.2). Constructing J means to remove from I all those i for

which to maintain $v_i(t) = 0$ negative $\lambda_i(t)$ is required, and to add all those i' for which $v_{i'}(t)$ is going to be negative.)
Whether the mode J motion actually starts or not is decided by $\Gamma(J;t_o)$. If $\Gamma(J;t_o) \neq \emptyset$ then instantaneously another switch to some mode J' (by (2.6) with J instead of I) occurs at the same time t_o. It seems noteworthy that the events causing $\Gamma(I;t_o) \neq \emptyset$ can be described by

$$\Gamma_1 : v_{i'}(t_o) = 0 \wedge [f_{i'}(\mathring{x}, \mathring{v}, t_o^+) < 0 \vee (f_{i'}(\mathring{x}, \mathring{v}, t_o^+) = 0 \wedge f_{i'}(x,v,t)|_I \text{ decreasing for } t > t_o)],$$

$$\Gamma_2 : f_i(\mathring{x}, \mathring{v}, t_o^+) > 0 \vee [f_i(\mathring{x}, \mathring{v}, t_o^+) = 0 \wedge f_i(x,v,t)|_I \text{ increasing for } t > t_o].$$

Mind that it is the limit values for $t \to t_o + 0$ of $f_\kappa(x(t), v(t), t)|_I$ that are crucial since $f_\kappa(\mathring{x}, \mathring{v}, \cdot)$ could be discontinuous at t_o due to a jump of an open-loop control (possibly causing an exogenous event).
So far, we have seen that to describe the motion of a scaly straight worm is the unpleasant task to solve (in practice numerically by means of some integration routine) a certain chain of systems of type (2.2) and, simultaneously, to check the emptiness of the sets given by (2.5) in order to recognize the next system of the chain. In section 3 we will give a procedure that joins these two parts of the task. But before we will review the considered motion system under two different aspects.

System-theoretic view [9], [10]
Formally, the worm system (2.1) can be seen as a control system in $\mathbb{R}^{2n}$ with state (x,v), k scalar controls λ_κ and output $y_\kappa = v_\kappa$, $\kappa \in K$. The attached complementary slackness conditions make the system exhibit in fact as the family of systems $\{\Sigma_I \mid I \subset K\}$. The parameter I represents a *discrete state* of the worm, taking values in the power set of K. To each discrete state I a *continuous state* $(x_{\nu'}, v_{\nu'})$, $\nu' \in \{1,\ldots,n\}\backslash I$, is attached. In this sense, the worm is modeled as a *hybrid system*. The discrete states could be seen as the nodes of a (in principle complete) *transition graph*. A transition along the edge $I \to J$ is governed, dependent on the actual continuous state and control at mode I, by the explorer (2.5).

Differential-geometric view
The state space of a system of n point masses in a straight line is $T\mathbb{R}^n$, the tangent bundle of the configuration space $\mathbb{R}^n$. It can be visualized as $\mathbb{R}^n$ with the tangent space $T_x\mathbb{R}^n = \mathcal{R}^n$ attached to each point $x \in \mathbb{R}^n$. If the system is in configuration $x = (x_1,\ldots,x_n)$, then the actual velocity $v = (v_1,\ldots,v_n)$ has to be seen as an element of $T_x\mathbb{R}^n$. An inequality $v_\kappa \geq 0$ (κ fixed) singles out from $T_x\mathbb{R}^n$ a half-space with boundary, H_κ. The constraint $v_\kappa \geq 0$, $\kappa \in K$, thus restricts the feasible velocities to lie in $T_x\mathbb{R}^n \cap (\bigcap_{\kappa\in K} H_\kappa) =: T_x^c\mathbb{R}^n$ (if $k = n$, this is just the closed positive cone of $T_x\mathbb{R}^n$). The state space of the worm system with differential constraint $v_\kappa \geq 0$, $\kappa \in K$, has now become a manifold with boundary. Mode I with $I \neq \emptyset$ means motion within a certain part of the boundary ($x \in \mathbb{R}^n$, $v \in T_x\mathbb{R}^n : v_i = 0$) whereas mode $\emptyset$ means motion in the interior ($x \in \mathbb{R}^n$, $v \in T_x\mathbb{R}^n$, $v_\kappa > 0$). The complementary slackness condition in mode $I \neq \emptyset$, $\lambda_i = -f_i(x,v,t) \geq 0$, $\lambda_{i'} = 0$, then says that the system moves in mode I as long as the physical force $f(x,v,t)$ (that can be seen as an element of $T_x\mathbb{R}^n$, too) does not point inwards $T_x^c\mathbb{R}^n$; the set of points (x,v) where this holds at time t is nothing else but the projection of $\mathcal{W}_I(t)$ - see (2.4) - onto $T_x^c\mathbb{R}^n$.

3 Explicit switching law

Fortunately, the constraint $v_\kappa \geq 0$ of the worm is so simple, that the process of mode exploring and switching can be combined with the differential equations.

Proposition *If we set*

$$\lambda_\kappa = -\frac{1}{2}(1 - \ sign\ v_\kappa)(1 - \ sign\ f_\kappa) f_\kappa \tag{3.1}$$

then the complementary slackness conditions (2.1b) are satisfied and (2.1) reduces to an unconstrained system of ODEs.

Proof. (3.1) yields (at any t)

$$v_\kappa > 0 \Rightarrow \lambda_\kappa = 0,$$
$$v_\kappa = 0 \Rightarrow \lambda_\kappa = -\tfrac{1}{2}(f_\kappa - |f_\kappa|) \geq 0.$$

If at some instant $t = t_o$ we have $v_\kappa(t_o) = 0$ then due to (2.1a)

$$m_\kappa \dot{v}_\kappa(t_o) = \frac{1}{2}(f_\kappa + |f_\kappa|)|_{t=t_o} \geq 0$$

thus $v_\kappa(t)$ is not going to be negative.

With (3.1) the full equations of motion now are

$$\left.\begin{array}{rcll} \dot{x}_\nu &=& v_\nu, & \nu \in \{1, \dots, n\}, \\ m_{\nu'} \dot{v}_{\nu'} &=& f_{\nu'}(x, v, t), & \nu' \in \{1, \dots, n\} \backslash K, \\ m_\kappa \dot{v}_\kappa &=& [1 - \frac{1}{2}(1 - \text{sign } v_\kappa)(1 - \text{sign } f_\kappa)] f_\kappa(x, v, t), & \kappa \in K. \end{array}\right\} \tag{3.2}$$

$\square$

In the system theoretic view of section 2, (3.1) can be seen as a feedback that now does the job of switching between modes. Indeed, the feedback may also depend on the physical controls which are immanent to f_κ. In analytical mechanics view, (3.1) represents the evaluated actual reaction forces; in our context and due to the fact $\lambda_\kappa(\cdot) > 0 \Rightarrow v_\kappa \equiv 0 \Rightarrow \dot{v}_\kappa \equiv 0$ they do not depend on accelerations as they do in general.
Clearly, the initial data $(\mathring{x}, \mathring{v})$ have to obey $\mathring{v}_\kappa \geq 0$, $\kappa \in K$.

4 Simulations in $\mathbb{R}^1$

In the following we consider a loose collection of straight worms of various DOF, internal forces, and drives. We denote by $c_{\nu\mu}$ the stiffness of a linear spring, by $k_{\nu\mu}$ the coefficient of a linear damper connecting point masses ν and μ, k_ν is the external damping coefficient at point mass ν.

It should be emphasized that the values of all system parameters are arbitrarily chosen to the only purpose of visualizing diverse effects.

The first three figures refer to a system of two equal masses connected by spring and damper, and driven by time-varying spring-length. The left part shows the relative change of spring-length, in the middle the velocities (1 solid, 2 dashed, mean dotted) and on the right the positions (x_1 on top, center of mass dotted) are given - all vs. t.
Figure 4.1. clearly shows diverse switchings between different modes as it was detailed in Section 1. The following two figures point to damping effects.

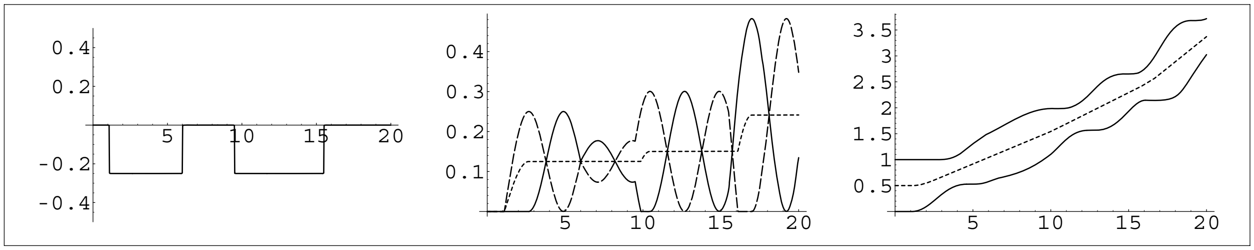

Fig. 4.1. $m_1 = m_2 = 1, c_{12} = 1, k_{12} = k_1 = k_2 = 0$.

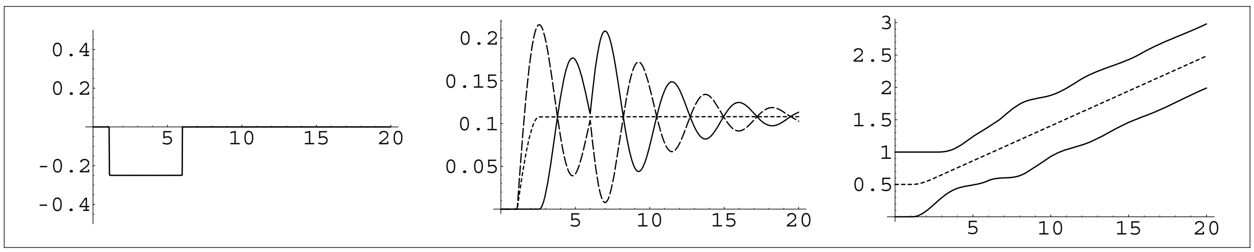

Fig. 4.2. $m_1 = m_2 = 1, c_{12} = 1, k_{12} = 0.2, k_1 = k_2 = 0$.

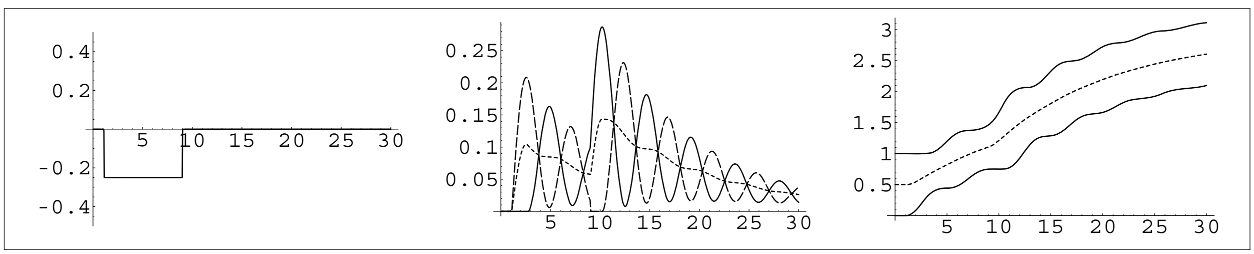

Fig. 4.3. $m_1 = m_2 = 1, c_{12} = 1, k_{12} = 0.075, k_1 = 0, k_2 = 0.175$.

Now we consider a 3mass-2spring worm ($m_1 = m_2 = m_3 = 1, c_{12} = c_{23} = 2$, no damping). Drive is by alternate sinusoidal contraction of the spring-lengths.

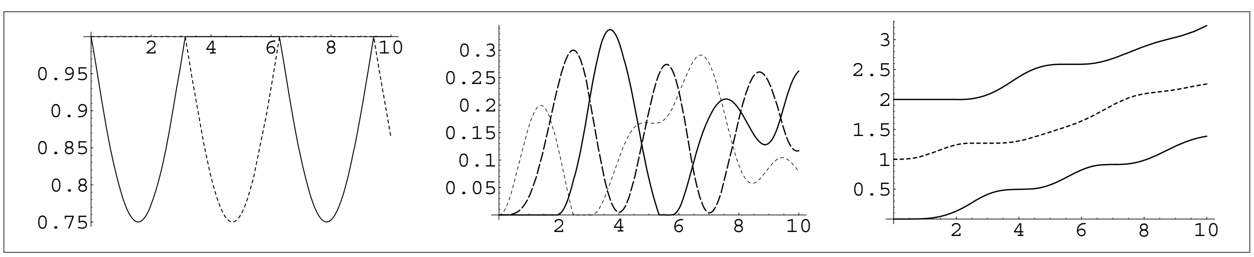

Fig. 4.4. left: natural (controlled) spring lengths (12 solid, 23 dashed); middle: velocities (1 solid, 2 dotted, 3 dashed); right: positions ($x1$ on top).

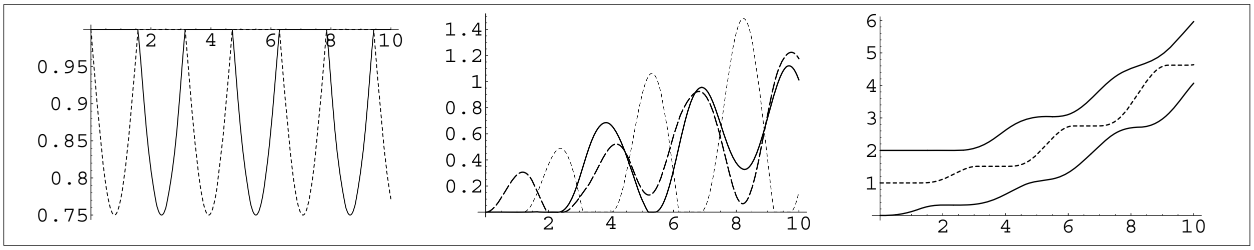

Fig. 4.5. Same as preceding, but higher frequency of excitation.

In the following example ($m_1 = m_2 = 1, m_3 = 3, c_{12} = c_{23} = 1$, no damping) the point mass 2 is without scales. It is elongated (spring 23 contracted by half) at the beginning.

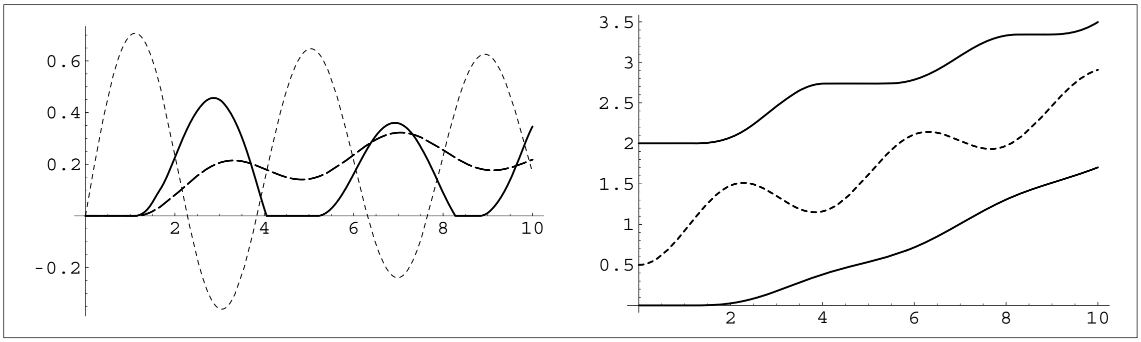

Fig. 4.6. left: velocities (1 solid, 2 dotted, 3 dashed); right: position (x_1 on top).

Finally, the positions vs. t (x_1 on top) of a $DOF = 4$ system are shown. Drive is by sinusoidal variation of the natural spring-length l_{23} (graph dashed).

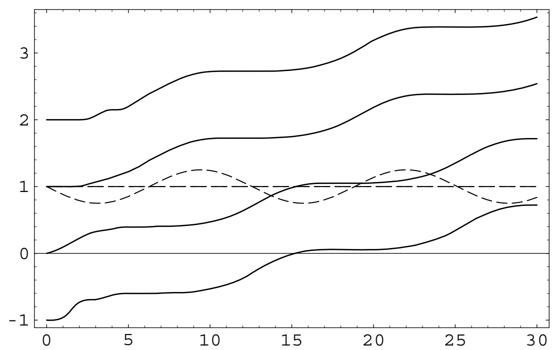

Fig. 4.7. $(m_1, m_2, m_3, m_4) = (0.5, 1, 1, 0.5)$
$(c_{12}, c_{23}, c_{34}) = (4, 9, 4)$
$(k_{12}, k_{23}, k_{34}) = (0, 0.5, 0)$
$(k_1, k_2, k_3, k_4) = (0.1, 0.2, 0.2, 0.1)$

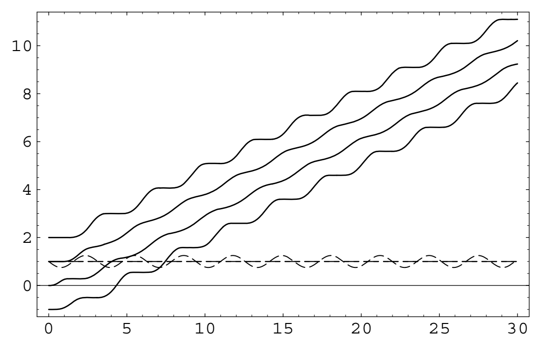

Fig. 4.8. $(m_1, m_2, m_3, m_4) = (1, 1, 1, 1)$
$(c_{12}, c_{23}, c_{34}) = (4, 9, 4)$
$(k_{12}, k_{23}, k_{34}) = (0, 0.5, 0)$
$(k_1, k_2, k_3, k_4) = (0.2, 0.3, 0.3, 0.2)$

Problems: (a) To look for parameter constellations which are optimal under certain aspects (e.g., fast worm, not too big reaction forces λ_ν, utilization of resonance effects) - also in view to possible realization.
(b) To investigate further principles for internal drive (e.g. by centrifugal forces of internal unbalanced rotators).
(c) To enlarge the number of point masses and investigate drive by a traveling wave of spring

contractions (cf. figures 4.4 and 4.5 above).

5 Finite DOF scaly worm in $\mathbb{R}^2$

The plane motion systems we are going to consider now consist of point masses each endowed with a massless runner (preventing side-slip) of controllable direction. Let the point masses be labeled 0 to n and suppose consecutive ones to be coupled by physical devices as shown in Figure 2.

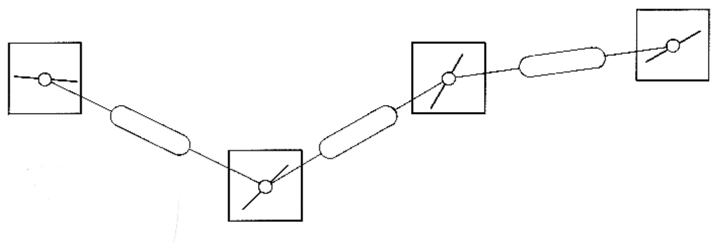

Fig. 5.1. Scaly worm in the plane

So far the system is typically worm-shaped with $DOF = 2(n+1)$, nonholonomically constrained by the $n+1$ no-side-slip conditions, and controllable by, e.g., variable spring lengths (internal drive) and variable runner directions (steer). Let, further, the runners be covered with scales so that each runner gets its unique "forward" orientation in which sliding is allowed while any backward velocity is constrained to be zero.

<u>Remark:</u> In realization, a runner can always be represented by a massless wheel, and a ratchet then mimics the effect of scales.

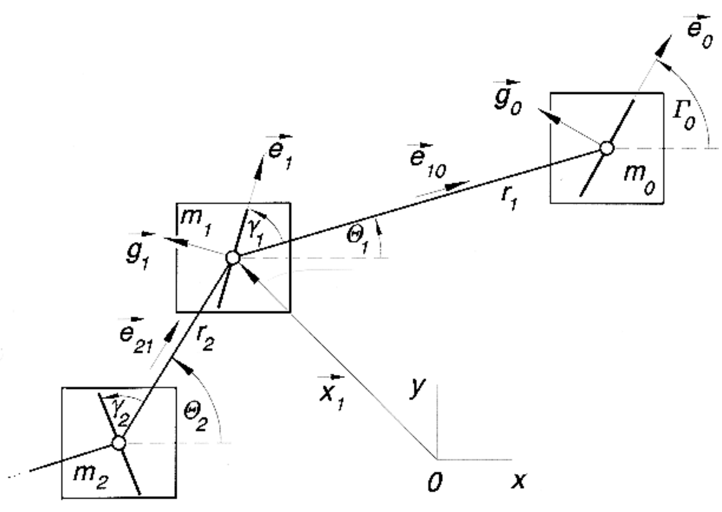

Fig. 5.2. Position of system w.r.t. inertial frame $\{0; x, y\}$.

Following Figure 5.2 let $\mathbf{x}_\nu$, $\nu = 0, 1, \ldots, n$, be the position vectors w.r.t. the Cartesian

inertial frame $\{0; x, y\}$, r_ν the distances of neighboring point masses, Θ_ν the respective heading angles. The unit vectors $\mathbf{e}_\nu = \cos\Gamma_\nu \mathbf{e}_x + \sin\Gamma_\nu \mathbf{e}_y$ describe the feasible sliding directions of the runners, we put $\Gamma_\nu = \Theta_\nu + \gamma_\nu$ and $\Gamma_o = \Theta_1 + \gamma_o$. Finally let $\mathbf{g}_\nu \perp \mathbf{e}_\nu$.
The no-side-slip conditions

$$0 = (\dot{\mathbf{x}}_\nu | \mathbf{g}_\nu) = -\dot{x}_\nu \sin\Gamma_\nu + \dot{y}_\nu \cos\Gamma_\nu \tag{5.1}$$

imply for the velocities and accelerations

$$\dot{\mathbf{x}}_\nu = v_\nu \mathbf{e}_\nu, \tag{5.2a}$$

$$\ddot{\mathbf{x}}_\nu = \dot{v}_\nu \mathbf{e}_\nu + v_\nu \dot{\Gamma}_\nu \mathbf{g}_\nu. \tag{5.2b}$$

State coordinates are, so far, $(x_\nu, y_\nu, v_\nu;\ \nu = 0, \ldots, n)$ but in regard to the internal forces it is appropriate to transform to $(x_o, y_o, r_\nu, \Theta_\nu, v_o, v_\nu;\ \nu = 1, \ldots, n) : \mathbf{x}_\nu = \mathbf{x}_{\nu-1} - r_\nu \mathbf{e}_{\nu\nu-1}$, $\mathbf{e}_{\nu\nu-1} = \cos\Theta_\nu \mathbf{e}_x + \sin\Theta_\nu \mathbf{e}_y$. Differentiation w.r.t. t then yields $\dot{r}_\nu, \dot{\Theta}_\nu$ and, from (5.2b) we obtain the following equations which govern the *kinematics* of the worm system:

$$\begin{aligned}
\dot{x}_o &= v_o \cos(\gamma_o + \Theta_1), \\
\dot{y}_o &= v_o \sin(\gamma_o + \Theta_1), \\
\dot{r}_\nu &= v_{\nu-1} \cos(\gamma_{\nu-1} + \Theta_{\nu-1} - \Theta_\nu) - v_\nu \cos\gamma_\nu, && \nu = 1, \ldots, n, \\
\dot{\Theta}_\nu &= [v_{\nu-1} \sin(\gamma_{\nu-1} + \Theta_{\nu-1} - \Theta_\nu) - v_\nu \sin\gamma_\nu]/r_\nu, && \nu = 1, \ldots, n.
\end{aligned} \tag{5.3}$$

For the scaly worm, (5.3) has to be supplemented by the inequalities

$$v_\nu \geq 0, \quad \nu = 0, 1, \ldots, n. \tag{5.4}$$

<u>Remark.</u> After adding $n+1$ integrators $\dot{\gamma}_\nu = \omega_\nu$ the kinematical equations are of the formal structure of a driftless control system with state $(x_o, y_o, r, \Theta, \gamma) \in \mathbb{R}^{3(n+1)}$ and control $(v, \omega) \in \mathbb{R}^{2(n+1)}$. Under this viewpoint however the system had to be seen as equipped with driven wheels (driving speed v) instead of the runners whereas in our context (regarding the actuators lying between consecutive point masses) the only reasonable kinematical drive were $t \mapsto r_\nu(t)$, $\nu = 1, \ldots, n$.

<u>Problem:</u> To find, at given $t \mapsto (r_\nu, \gamma_o, \gamma_\nu)(t)$, the speeds v_o, v_ν.

The physical forces at the point masses are composed of internal forces (coupling springs and dampers, actuators) and external ones (gravity, friction). In obvious notation we assume

$$\begin{aligned}
\mathbf{F}_\nu = F_{\nu\nu-1}(r_\nu, \dot{r}_\nu, t)\mathbf{e}_{\nu\nu-1} \quad & -F_{\nu+1\nu}(r_{\nu+1}, \dot{r}_{\nu+1}, t)\mathbf{e}_{\nu+1\nu} \\
& + E_\nu(x_\nu, y_\nu, v_\nu)\mathbf{e}_\nu + G_\nu(x_\nu, y_\nu)\mathbf{g}_\nu,
\end{aligned} \tag{5.5}$$

(where there is, of course, only one internal force for $\nu = 0$ and $\nu = n$). The $\dot{r}$ have to be eliminated by means of (5.3).
The reaction force is

$$\mathbf{R}_\nu = \lambda_\nu \mathbf{e}_\nu + \mu_\nu \mathbf{g}_\nu, \tag{5.6}$$

where μ_ν corresponds to the no-side-slip condition (5.1) whereas λ_ν belongs to the scale constraint $v_\nu \geq 0$.
The system's dynamics then is governed by

$$m_\nu \ddot{\mathbf{x}}_\nu = \mathbf{F}_\nu + \mathbf{R}_\nu, \quad \nu = 0, 1, \ldots, n, \tag{5.7}$$

together with the complementary slackness conditions

$$v_\nu \geq 0 \ \wedge \ \lambda_\nu \geq 0 \ \wedge \ v_\nu \lambda_\nu = 0, \ \nu = 0, 1, \ldots, n. \tag{5.8}$$

Due to (5.2b) the dynamical equations split into

$$m_\nu \dot{v}_\nu = f_\nu(x_o, y_o, r, \Theta, v, \gamma, t) + \lambda_\nu, \tag{5.9}$$

$$m_\nu v_\nu \dot{\Gamma}_\nu = g_\nu(x_o, y_o, r, \Theta, v, \gamma, t) + \mu_\nu, \tag{5.10}$$

where $f_\nu := (\mathbf{F}_\nu | \mathbf{e}_\nu)$, $g_\nu := (\mathbf{F}_\nu | \mathbf{g}_\nu)$.
The full system of equations of motion is now (5.3), (5.9), (5.8). It is similar to that of the straight worm (2.1) but with some more involved kinematical part and additional steering controls. Putting $\Theta_\nu = \gamma_\nu = 0$, (2.1) is recovered (with relative coordinates r_ν instead of x_ν). The occurence of λ_ν and the conditions (5.8) will cause the typical switching between different modes. Everything said in Section 2 about this behavior is, mutatis mutandis, hereditary to the systems in the plane. In particular we are again free to automatize the switching by adopting the "feedback" (3.1)

$$\lambda_\nu = -\frac{1}{2}(1 - \text{sign } v_\nu)(1 - \text{sign } f_\nu) f_\nu. \tag{5.11}$$

For any motion, (5.10) yields the forces μ_ν necessary to prevent side-slip of the runners. In this paper we will not be interested in a closer investigation of the μ's.

Remark. Concluding this Section we shall give a sketch how to incorporate into the above framework a point mass with no runner nor scales nor contact to the ground. Let this point mass be with number ν. Referring to Figure 5.2 there holds $x_\nu = x_{\nu-1} - r_\nu \cos\Theta_\nu$, $x_{\nu+1} = x_\nu - r_{\nu+1}\cos\Theta_{\nu+1}$, and similar for $y-$coordinates. Suppose point mass ν to be collinear with its neighbors (realized by a massless guiding tube, e.g.), then there is a constraint $\Theta_{\nu+1} - \Theta_\nu = 0$. Let $\mathbf{e}_\nu := \cos\Theta_\nu \mathbf{e}_x + \sin\Theta_\nu \mathbf{e}_y$, $\mathbf{g}_\nu := -\sin\Theta_\nu \mathbf{e}_x + \cos\Theta_\nu \mathbf{e}_y$, and $\dot{\mathbf{x}}_\nu = v_\nu \mathbf{e}_\nu + w_\nu \mathbf{g}_\nu$. Then it follows

$$\begin{aligned} \dot{r}_\nu &= v_{\nu-1}\cos(\gamma_{\nu-1} + \Theta_{\nu-1} - \Theta_\nu) - v_\nu, && (5.12a)\\ \dot{r}_{\nu+1} &= v_\nu - v_{\nu+1}\cos\gamma_{\nu+1}, && (5.12b)\\ r_\nu \dot{\Theta}_\nu &= v_{\nu-1}\sin(\gamma_{\nu-1} + \Theta_{\nu-1} - \Theta_\nu) - w_\nu, && (5.12c)\\ r_{\nu+1}\dot{\Theta}_\nu &= w_\nu - v_{\nu+1}\sin\gamma_{\nu+1}. && (5.12d) \end{aligned}$$

The dynamics of the νth point mass is governed by

$$\begin{aligned} m_\nu \ddot{\mathbf{x}}_\nu &\equiv m_\nu[(\dot{v}_\nu - w_\nu \dot{\Theta}_\nu)\mathbf{e}_\nu + (\dot{w}_\nu + v_\nu \dot{\Theta}_\nu)\mathbf{g}_\nu] \\ &= (F_{\nu\nu-1}(r_\nu, \dot{r}_\nu, t) - F_{\nu+1\nu}(r_{\nu+1}, \dot{r}_{\nu+1}, t))\mathbf{e}_\nu + \mu_\nu \mathbf{g}_\nu, \end{aligned} \tag{5.13}$$

(see (5.7), (5.5) with external forces dropped) where μ_ν now is the reaction to the collinearity constraint. The contribution to the overall equations of motion finally is

$$\begin{aligned}
\dot{r}_\nu &= v_{\nu-1}\cos(\gamma_{\nu-1}+\Theta_{\nu-1}-\Theta_\nu) - v_\nu, && (5.14a)\\
\dot{r}_{\nu+1} &= v_\nu - v_{\nu+1}\cos\gamma_{\nu+1}, && (5.14b)\\
\dot{\Theta}_\nu &= [v_{\nu-1}\sin(\gamma_{\nu-1}+\Theta_{\nu-1}-\Theta_\nu) - v_{\nu+1}\sin\gamma_{\nu+1}]/(r_\nu + r_{\nu+1}), && (5.14c)\\
m_\nu \dot{v}_\nu &= m_\nu[r_{\nu+1}\dot{\Theta}_\nu + v_{\nu+1}\sin\gamma_{\nu+1}]\dot{\Theta}_\nu && (5.14d)\\
&\quad + f_\nu(r_\nu, r_{\nu+1}, \Theta_{\nu-1}, \Theta_\nu, v_{\nu-1}, v_\nu, v_{\nu+1}, \gamma_{\nu-1}, \gamma_{\nu+1}, t),
\end{aligned}$$

where (5.14c) has to be plugged into the centrifugal term of the last equation.

6 Some simulations in $\mathbb{R}^2$

We restrict the simulations in the plane to systems of two equal point masses interconnected by spring and damper. Drive is by bang-bang control of the natural spring length.
The first system is with strong internal damping and fixed steering angles. The front point mass moves almost along a circle.

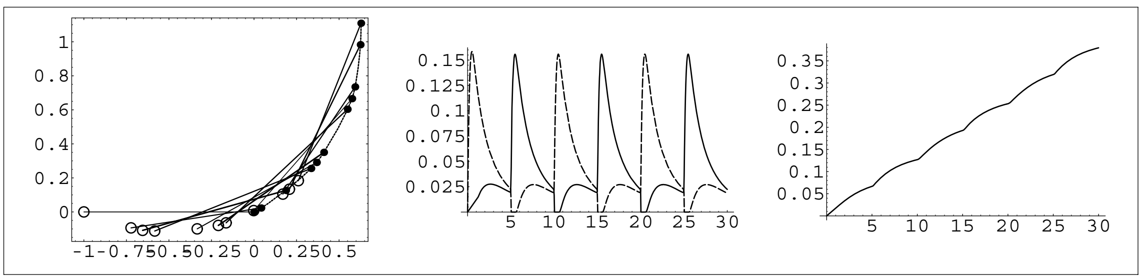

Fig. 6.1. Left: system in the plane; middle: velocities (0 solid, 1 dashed); right: Θ_1 vs.t.
$(m_o, m_1) = (1,1)$; $c_{10} = 4$; $(k_{10}; k_0, k_1) = (5; 0.3, 0.3)$;
$\gamma_0 = -\gamma_1 = \frac{\pi}{8}$. Spring extended in $(0,5) \cup (10,15) \cup (20,25)$.

In the next example the system nearly undergoes a parallel displacement.

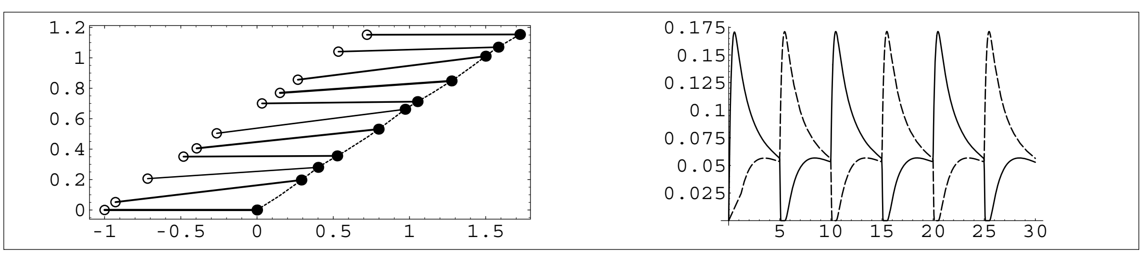

Fig. 6.2. Left: system in the plane; right: velocities (0 solid, 1 dashed).
$(m_0, m_1) = (1,1)$; $c_{10} = 4$; $(k_{10}; k_0, k_1) = (5; 0.1, 0.1)$.
$\gamma_0 = \gamma_1 = \frac{\pi}{6}$. Spring extended in $(0,5) \cup (10,15) \cup (15,20)$.

Finally we consider a weakly damped system with time-variant steering.

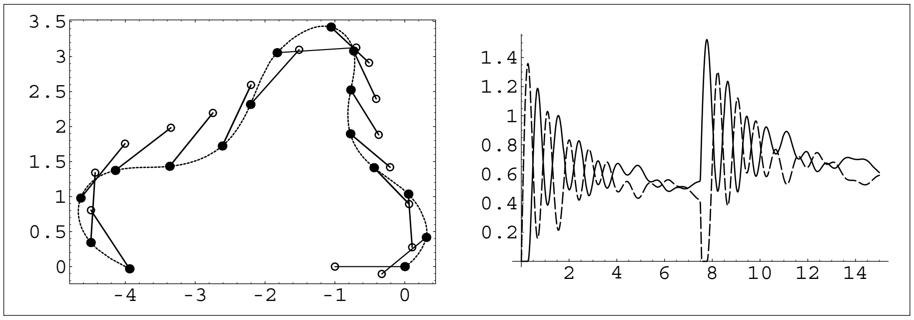

Fig. 6.3. Left: system in the plane; right: velocities (0 solid, 1 dashed).
$(m_0, m_1) = (1, 1)$; $c_{10} = 36$; $(k_{10}; k_0, k_1) = (0.75; 0.05, 0.05)$.
$\gamma_0 = \frac{\pi}{4}\sin t$, $\gamma_1 = -\frac{\pi}{8}$. Spring contracted in $(0, 7.5)$.

7 Conclusion

This paper presents some first theoretical investigations of worm-like motion systems that have the earthworm as live prototype. These systems are modeled as chains of pointmasses in a plane (or straight line). The pointmasses are equipped with massless steerable runners (described via knife-edge conditions), the latter ones covered with scales that prevent *backward* displacement of the points of ground contact (described by differential inequality constraint). The systems are driven by internal muscle-like actuators lying between consecutive point masses. The mathematical setting thus is Analytical Mechanics of (nonholonomic) systems with unilateral constraints or, more general, Theory of Hybrid Systems of complementary-slackness type. Under control-theoretic view (taking the to-be-constrained velocities as outputs and the corresponding Lagrange multipliers as controls) the systems exhibit a relative degree $\varrho = 1$ (differentiate the output once to see the control). This pleasant feature enables one to automatize the switching between different modes of the motion by discontinuous feedback thus coming to pure (closed-loop system) differential equations which then served as the basis for simulations using mathematica 3.0.
The presented, until now somewhat sporadic results show the principal behavior of such systems and its dependence on certain parameters.
Inevitable tasks for both theory and simulation within further investigations to come are

- a close and systematic examination of the influence of system parameters,
- an optimal (w.r.t. some goal to be formulated) tuning of spring-damper subsystems,
- to find (optimal) *gaits* (i.e., excitation patterns in space and time for the driving and steering actuators),
- to do all this with high DOF in order to come close to worm-shaped systems,
- to enlarge the class of utilizable actuator principles,
- to replace the pointmasses by rigid bodies (allowing now also rotations) interconnected by more than one (controlled) spring.

A strong cooperation with potential users of such systems and with hardware engineers responsible for technical realization will be necessary.

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