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Computation and Continuation of Quasiperiodic Solutions

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Abstract. We consider periodically forced ODEs which exhibit quasiperiodic oscillations. These oscillations are investigated by an approximation and continuation of the associated invariant torus with respect to free system parameters. For the invariant torus we derive an uncomplicated invariance equation whereby we do not require the system to be partitioned or an a-priori-coordinate transformation to be applied. This equation is solved by semidiscretisation methods where Fourier-Galerkin methods especially in the case of periodically forced "weakly nonlinear" ODEs lead to low dimensional autonomous systems which can be treated by standard algorithms.
Also in the general case it turns out that this approach allows an efficient computation and continuation of quasiperiodic solutions. A number of problems has been analysed successfully and an example is given in this paper.

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Keywords. Dynamical Systems, Quasiperiodic oscillations,
Numerical methods

1 Introduction

We consider periodically forced ordinary differential equations (ODEs) of order $n \geq 2$

$$\frac{dx}{dt} = f(x,t), \qquad f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n \tag{1}$$

under the following assumptions:

1. $f \in \mathcal{C}^r(\mathbb{R}^n \times \mathbb{R})$ with $r \geq 1$ sufficiently large. f is $2\pi-$periodic with respect to t:
$$f(x, t + 2\pi) = f(x, t) \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$
2. System (1) has a locally unique quasiperiodic solution $y \in \mathcal{C}^1(\mathbb{R})$ with p ($2 \leq p \leq n$) rationally independent basic frequencies $\omega_1 = 1, \omega_2, \ldots, \omega_p$.

These p frequencies form a rationally independent (incommensurate) frequency basis $\varOmega = (\omega_1, \omega_2, \ldots, \omega_p)$ with $\omega_1 = 1$ because of the $2\pi-$periodicity of f. The following methods can be applied to autonomous systems as well, but then the basic frequency ω_1 is also unknown and an additional phase condition must be introduced.

During the last 15 years very different approaches have been developed for approximating quasiperiodic solutions and invariant tori. The method of invariance equations tries to compute the parametrisation of an invariant torus by solving the quasilinear partial differential equations. While [4], [5], [6], [1], [2], [16], [10] use special forms of difference methods, in [12], [9], [3] multidimensional Fourier methods are applied. The drawback of these approaches is that an a-priori transformation from Cartesian to radius-angle coordinates (u, θ) is required and in most applications such a global parametrisation is neither possible nor numerically feasible.

As an alternative approach, [8], [14], [15], [11], [9] consider a suitable Poincaré map P and try to compute an invariant manifold of P as a solution of a functional equation. The performance of these methods strongly depends on discretisation and interpolation techniques and also on the stability of solutions.

For special 2nd order systems and systems with small perturbations the following additional methods are widely used in enineering:

– the averaging method of Krylov and Bogoljubov,
– the method of amplitudes of Van der Pol,
– the harmonic balance method and generalizations of C. Hayashi and
– multiscale methods.

In principle, they "reduce the quasiperiodicity" of the solutions by transforming periodic solutions into equilibria and, if possible, quasiperiodic to periodic solutions. Unfortunalety they are restricted to special ODEs and do not extend to the general case of system (1).

The aim of our approach is a numerical approximation of quasiperiodic solutions $x(t) = u(\varOmega t)$ of the *original system* (1) without using a-priori transformations into radius-angle coordinates (u, θ). By means of a suitable torus system, we can analyse quasiperiodic solutions $x(t, \lambda)$ and their corresponding torus solutions $u(\theta, \lambda)$ depending on parameters $\lambda \in \mathbb{R}^m$ by *methods for periodic solutions*. So we are able to use existing continuation methods and methods for bifurcation analysis.

2 Transformation into the torus equation

For the quasiperiodic solution y of (1) the representation

$$y(t) \; = \; u(\mathit{\Omega} t) \; = \; u(t, \omega_2 t, \ldots, \omega_p t)$$

with the associated torus function $u = u(\theta) : \mathbb{T}^p \to \mathbb{R}^n$ is used. u is assumed to be continously differentiable and $2\pi-$periodic in every variable $\theta_i,\ i = 1, 2, \ldots, p$. Inserting this formulation for y into (1) yields

$$\omega_1 \frac{\partial u}{\partial \theta_1}(\mathit{\Omega} t) + \sum_{j=2}^{p} \omega_j \frac{\partial u}{\partial \theta_j}(\mathit{\Omega} t) \; = \; f(u(\mathit{\Omega} t), t) \; , \tag{2}$$

which by using the vector-valued function $g : \mathbb{R} \to \mathbb{R}^n$

$$g(t) = \frac{\partial u}{\partial \theta_1}(\mathit{\Omega} t) + \sum_{j=2}^{p} \omega_j \frac{\partial u}{\partial \theta_j}(\mathit{\Omega} t) - f(u(\mathit{\Omega} t), t) \; , \tag{3}$$

becomes equivalent to the equation

$$g(t) = 0 \qquad \forall t \in \mathbb{R} \; . \tag{4}$$

According to assumptions 1 and 2, $g \in C(\mathbb{R}, \mathbb{R}^n)$ is also quasiperiodic with basic frequencies $\omega_1 = 1,\ \omega_2, \ldots,\ \omega_p$. Its associated torus function $G : \mathbb{T}^p \to \mathbb{R}^n$ with $g(t) = G(\mathit{\Omega} t) = G(t, \omega_2 t, \ldots, \omega_p t)$ is defined by

$$G(\theta) = \frac{\partial u}{\partial \theta_1}(\theta) + \sum_{j=2}^{p} \omega_j \frac{\partial u}{\partial \theta_j}(\theta) - f(u(\theta), \theta_1) \; . \tag{5}$$

For $G \in C(\mathbb{T}^p, \mathbb{R}^n)$ the range of the quasiperiodic function $g(t) = G(\omega t)$ is dense in the range of the torus function $G(\theta)$, $\theta \in \mathbb{T}^p$ (see [12], p.10). With scalar product and norm in $\mathbb{C}^n$

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j} \; , \qquad |x|^2 = \langle x, x \rangle_2 = \sum_{j=1}^{n} |x_j|^2$$

the identity

$$\sup_{t \in \mathbb{R}} |g(t)| = \max_{\theta \in \mathbb{T}^p} |G(\theta)| \tag{6}$$

holds ([12], p.11). As a consequence it follows that

$$g(t) = 0 \quad \forall t \in \mathbb{R} \quad \Longleftrightarrow \quad G(\theta) = 0 \quad \forall \theta \in \mathbb{T}^p \tag{7}$$

which leads to in the invariance equation (the torus system) on $\mathbb{T}^p$

$$\frac{\partial u}{\partial \theta_1}(\theta) + \sum_{j=2}^{p} \omega_j \frac{\partial u}{\partial \theta_j}(\theta) \; = \; f(u(\theta), \theta_1) \; . \tag{8}$$

Any solution u of this system yields a quasiperiodic solution $x(t) = u(\Omega t)$ of $g(t) = 0$, $t \in \mathbb{R}$. Note that (8) is a special case of the general invariance equation of an invariant p-torus $u(\theta)$

$$\sum_{j=1}^{p} \Psi_j(u(\theta),\theta)\frac{\partial u}{\partial \theta_j}(\theta) \;=\; f(u(\theta),\theta). \tag{9}$$

In our approach, the basic frequencies ω_j for $j > 1$ are unknowns and may be determined by appropriate extensions to system (8).

For simplicity we consider the 2-dimensional case, but all the ideas can also be generalized to p-tori. In the case $p = 2$ semidiscretisation methods with respect to θ_1 may be applied, for example:

- Fourier-Galerkin,
- finite differences or
- collocation.

System (8) becomes thereby transformed into an autonomous system of ordinary differential equations for functions $u_k(\theta_2) : \mathbb{T}^1 \to \mathbb{R}^n, k \in \mathbb{Z}_k \subset \mathbb{Z}$. We get an initial value for the frequency ω_2 at a Neimark-Sacker bifurcation point (nonresonant case) of a periodic solution when a quasiperiodic solution is born. The first two approaches are discussed in more detail here.

3 Semidiscretisation by Fourier-Galerkin methods

We define the nonlinear operator $F : H^1 \to H^0$ as

$$F(u) \;\equiv\; \frac{\partial u}{\partial \theta_1} + \omega_2 \frac{\partial u}{\partial \theta_2} - f(u,\theta_1)\,, \tag{10}$$

where $H^s = H^s(\mathbb{T}^2)$, $s = 0, 1$ are the Sobolew spaces of torus functions $F : \mathbb{T}^2 \to \mathbb{C}^n$ with generalized derivatives $D^\alpha F \in L_2(\mathbb{T}^2)$ up to order s. Especially let $H^0 = L_2(\mathbb{T}^2)$. Scalar product and norm in H^s are defined by

$$(F,G)_s = \sum_{0\le|\alpha|\le s} \iint\limits_{\mathbb{T}^2} \langle D^\alpha F(\theta), D^\alpha G(\theta)\rangle \, d\theta_1 \, d\theta_2 \;,$$

$$||F||_s^2 = (F,F)_s \;=\; \sum_{0\le|\alpha|\le s} \iint\limits_{\mathbb{T}^2} |D^\alpha F(\theta)|^2 \, d\theta_1 \, d\theta_2$$

With (10) we obtain a zero problem which is equivalent to $G(\theta) = 0 \;\; \forall \theta \in \mathbb{T}^2$ in operator form

$$F(u) = 0\,, \quad u \in H^s(\mathbb{T}^2). \tag{11}$$

For simplicity we now replace $t = \theta_1$ because of $\omega_1 = 1$ and $\theta = \theta_2$ with frequency $\omega = \omega_2$. Let $\varphi_i(t)$, $i = -N, \ldots, -1, 0, 1, \ldots, N$, be an orthonormal system of a linear subspace of $L_2(\mathbb{T}^1)$ for which

$$\dot{\varphi}_k(t) = \sum_{j=-N}^{N} c_{kj}\varphi_j(t) \tag{12}$$

holds. In case of the trigonometric functions

$$\varphi_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}\ , \quad k = -N, -N+1, \dots, N$$

the constant matrix $C = (c_{kj})$ is especially

$$C = \mathrm{diag}(-iN, \dots, -2i, -i, 0, i, 2i, \dots, iN)\ .$$

We discretise $u(t,\theta)$ by projection onto the subspace $H_N = (span\{\varphi_k,\ |k| \leq N\})^n$ with projector

$$P_N u(t,\theta) = \sum_{k=-N}^{N} u_k(\theta)\cdot\varphi_k(t) \tag{13}$$

where the Fourier coefficients are

$$u_k(\theta) = \int\limits_{\mathbb{T}^1} u(t,\theta)\overline{\varphi_k(t)}\, dt. \tag{14}$$

Inserting $u^N \in H_N$ into (10) and applying P_N to (11) yields the Galerkin or spectral system

$$P_N F(u^N) \ = \ 0\,, \quad u^N \in H_N\ . \tag{15}$$

This Galerkin procedure can be defined component-wise by introducing the vector function

$$\varphi(t) = (\varphi_{-N}(t), ..., \varphi_0(t), \varphi_1(t), ..., \varphi_N(t))^T$$

and the matrix function

$$U(\theta) = (u_{-N}(\theta), ..., u_{-1}(\theta), u_0(\theta), u_1(\theta), ..., u_N(\theta))\ .$$

Then (13) in the representation

$$u^N(t,\theta) = U(\theta)\varphi(t)\ , \quad (t,\theta) \in \mathbb{T}^2 \tag{16}$$

can be inserted into (10) by using (12)

$$F(u^N(t,\theta)) = U(\theta)C\varphi(t) + \omega U'(\theta)\varphi(t) - f(U(\theta)\varphi(t), t)\ . \tag{17}$$

We now expand $f(U(\theta)\varphi(t), t)$ in (17) into a $\theta-$dependent Fourier series

$$f(U(\theta)\varphi(t), t) \ = \ \Gamma(U(\theta))\varphi(t) + R_N(\theta, t)\,,$$

where $R_N(\theta,t)$ is the remainder for $|k| > N$ and the coefficients are $\Gamma(U(\theta)) = (\gamma_{jk}) \in \mathbb{C}^{n\times(2N+1)}$. Applying the scalar product (14) to (17) in $L_2(\mathbb{T}^1)$ yields the component-wise representation

$$\omega \cdot u'_{il}(\theta) + (U(\theta)\cdot C)_{il} - \gamma_{il}(U(\theta)) = 0\,, \quad i = 1(1)n,\ |l| \leq N\,.$$

In vector notation, the *spectral system (Galerkin system)* is now

$$\omega \cdot U'(\theta) + U(\theta)\cdot C = \Gamma(U(\theta))\,. \tag{18}$$

If we consider the Fourier series for $N \to \infty$, then the periodic solutions of spectral system (18) will obviously yield quasiperiodic solutions of the original system (1). The following theorem can be proven (see [13]):

Theorem 1. *With assumptions 1 and 2 it holds for $N \to \infty$:*

(i) $U(\theta)$ *is a* $2\pi-$*periodic solution of system (18) if and only if* $u(t) = U(\omega t)\varphi(t)$ *is a quasiperiodic solution of the original system (1).*
(ii) $A = (a_{kl})$, $k = 1\ldots n$, $l \in \mathbb{Z}$, *is an equilibrium point of system (18) with*

$$A\cdot C = \Gamma(A) \tag{19}$$

iff $u(t) = A\varphi(t)$ *is* $2\pi-$*periodic solution of the original system (1).*

The Galerkin system (18) is an autonomous system with $n(2N+1)$ equations. For harmonically forced "weakly nonlinear" systems which frequently appear in electrical engineering we already achieve in practice good approximations for small $N = 1, 2, 3$. Applying the transformation to the independent variable

$$\theta = \omega\tau \quad \text{with} \quad U(\theta) = U(\omega\tau) = Y(\tau)\,,$$

to (18) we can eliminate the unknown parameter ω and obtain the spectral system

$$Y'(\tau) \;=\; \Gamma(Y(\tau)) - Y(\tau)\cdot C \tag{20}$$

for periodic solutions $Y(\tau)$ with unknown period T. This standard problem can now be solved by software tools for periodic oscillations and is an efficient way to compute and continue quasiperiodic solutions.

With such an approximation at hand the invariant closed curves γ_1 and γ_2 of the two Poincaré sections P_1 and P_2 of a quasiperiodic solution can easily be calculated. Using (13)

$$u^N(t,\theta) = \sum_{k=-N}^{N} u_k(\theta)\cdot\varphi_k(t) \tag{21}$$

we get immediately the approximations:

$$\gamma_1^N(\theta) = \frac{1}{\sqrt{2\pi}}\sum_{k=-N}^{N} u_k(\theta) \qquad \text{for } t = 2\pi m,\ m \in \mathbb{N} \text{ and} \tag{22}$$

$$\gamma_2^N(t) = \frac{1}{\sqrt{2\pi}}\sum_{k=-N}^{N} u_k(0)\cdot\varphi_k(t) \quad \text{for } \theta = Tm,\ m \in \mathbb{N}. \tag{23}$$

4 Semidiscretisation by finite difference methods

Let the nonlinear operator $F : B^1 \to B^0$ with

$$F(u) \;\equiv\; \frac{\partial u}{\partial \theta_1} + \omega_2 \frac{\partial u}{\partial \theta_2} - f(u, \theta_1)\,, \tag{24}$$

now act in Banach spaces $B^k = \mathcal{C}^k(\mathbb{T}^2),\ k = 0, 1$. The zero problem $G(\theta) = 0$ on $\mathbb{T}^2$ is then in operator form

$$F(u) = 0\,, \quad u \in \mathcal{C}^1(\mathbb{T}^2)\,. \tag{25}$$

If this problem is semidisretised by finite differences on θ_1 the resulting ODE system could also be treated by standard software. Here we use our own continuation methods and therefore we first linearise (24) and then discretise the linear systems.

Applying Newton's method to (25) yields the linearised problem

$$\begin{aligned} F'(u^k)\, v^k &= F(u^k) \\ u^{k+1} &= u^k - v^k \end{aligned} \tag{26}$$

where the Frechét derivative is

$$F'(u^k) = \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} - f_x(u^k, \theta_1)\,. \tag{27}$$

f_x is the Jacobian of f with respect to x. Then we have to solve linear systems of the form

$$\frac{\partial v}{\partial \theta_1} + \omega_2 \frac{\partial v}{\partial \theta_2} - A(\theta_1, \theta_2) \cdot v \;=\; r(\theta_1, \theta_2). \tag{28}$$

with quasiperiodic vector function r and matrix function A. Replacing $t = \theta_1$, $\theta = \theta_2$ and $\omega = \omega_2$, we define a mesh in t by

$$G_N = \left\{ t_i \,|\, t_i = i \cdot h;\ i = 0, 1, \ldots, N;\ h = \frac{2\pi}{N} \right\}$$

and discretise $v(t, \theta)$ on G_N by

$$v(t_i, \theta) = v_i(\theta), \quad v_0(\theta) \equiv v_N(\theta), \quad i = 0, 1, \ldots, N.$$

Approximation of the derivative with respect to t by finite differences

$$\frac{\partial v}{\partial t}(t_i, \theta) \approx \{D_N v\}_i = \frac{1}{h} \sum_{j=-k}^{l} c_j \cdot v_{i+j}(\theta)\,. \tag{29}$$

and insertion into (28) leads to

$$\omega v_i' + \frac{1}{h} \sum_{j=-k}^{l} c_j \cdot v_{i+j} - A_i(\theta) \cdot v_i \;=\; r_i(\theta) \tag{30}$$

where $A_i(\theta) = A(t_i, \theta)$ and $r_i(\theta) = r(t_i, \theta)$. Using $\omega = \frac{2\pi}{T}$ and isolating v_i' yields the cyclic periodically forced linear ODE system for the Newton corrections

$$v_i' \;=\; \frac{T}{2\pi}\left(A_i(\theta)\cdot v_i - \frac{1}{h}\sum_{j=-k}^{l} c_j \cdot v_{i+j} + r_i(\theta)\right) \tag{31}$$

with unknown period T. This linear system can now be solved by standard methods for periodic solutions.

Again, we can easily obtain approximations to the invariant closed curves γ_1 and γ_2 by:

$$\gamma_1(\theta) = u(0, \theta) \approx u_0(\theta) \tag{32}$$

$$\gamma_2(t_i) = u(t_i, 0) \approx u_i(0) \tag{33}$$

5 Application to an electrical circuit

As an example we study a dynamical system given by H. Kawakami and T. Yoshinaga in [17]. The Duffing-type system of order 3 is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 x_2 - \frac{1}{8}(x_1^2 + 3x_3^2)x_1 + B\cos t \\
\dot{x}_3 &= -\frac{1}{8}k_2(3x_1^2 + x_3^2)x_3 + B_0\,,
\end{aligned} \tag{34}$$

which describes a resonant electric circuit with two saturable inductors. The period of the Poincaré map is $T = 2\pi$ and we explore the system for the parameter values $B_0 = 0.03$, $B = 0.22$, $k_2 = 0.05$ and $k_1 \in [0.04, 0.15]$.

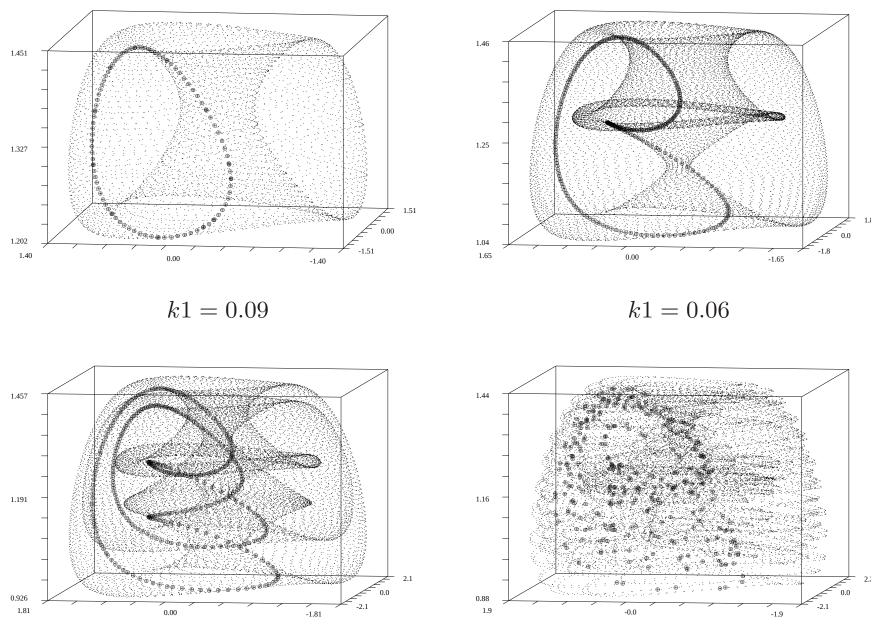
By numerical integration ("brute force" method), a bifurcation of the $2\pi-$periodic solution into an invariant torus can be observed at $k_1^* \approx 0.1214$. A stable quasiperiodic solution arises for smaller values $k_1 < k_1^*$, which is continued in figure 1. Obviously a cascade of period doublings with respect to one basic frequency (torus doublings) arises and finally a strange attractor can be seen. The $2\pi-$stroboscopic Poincaré map is displayed by bold dots.

1. Solution via spectral system. We choose a truncated Fourier series of order 1

$$\begin{aligned}
x_1(t) &= y_1(\omega t) + y_2(\omega t)\sin(t) + y_3(\omega t)\cos(t) \\
x_2(t) &= y_4(\omega t) + y_5(\omega t)\sin(t) + y_6(\omega t)\cos(t) \\
x_3(t) &= y_7(\omega t) + y_8(\omega t)\sin(t) + y_9(\omega t)\cos(t)
\end{aligned}$$

with the real functions

$$Y(\omega t) = \begin{pmatrix} y_1(\omega t) & y_2(\omega t) & y_3(\omega t) \\ y_4(\omega t) & y_5(\omega t) & y_6(\omega t) \\ y_7(\omega t) & y_8(\omega t) & y_9(\omega t) \end{pmatrix} \quad \text{and} \quad \varphi(t) = \begin{pmatrix} 1 \\ \sin(t) \\ \cos(t) \end{pmatrix}.$$



$k1 = 0.09$ $k1 = 0.06$

$k1 = 0.05$ $k1 = 0.043$

Fig. 1. Solution scenario of system (34)

As the right hand sides of (34) are polynomials in x_1, x_2, x_3, we can use a computer algebra system to generate the spectral system. With *Maple 5.1* we obtain the following 9-dimensional spectral system with symbolic parameters B_0, B, k_1 and k_2 (The dots denote derivatives to $\tau = \omega t$).

$$
\begin{aligned}
\dot{y}_1 &= y_4 \\
\dot{y}_2 &= y_5 + y_3 \\
\dot{y}_3 &= y_6 - y_2 \\
\dot{y}_4 &= -0.1875\, y_1 {y_8}^2 - k_1\, y_4 - 0.1875\, y_1 {y_9}^2 - 0.1875\, y_1 {y_2}^2 - 0.375\, y_1 {y_7}^2 \\
&\quad - 0.375\, y_2 y_7 y_8 - 0.375\, y_3 y_7 y_9 - 0.1875\, y_1 {y_3}^2 - 0.125\, {y_1}^3 \\
\dot{y}_5 &= -0.375\, y_2 {y_1}^2 - 0.375\, y_2 {y_7}^2 - 0.75\, y_1 y_7 y_8 - 0.28125\, y_2 {y_8}^2 + y_6 \\
&\quad - 0.09375\, y_2 {y_3}^2 - 0.09375\, y_2 {y_9}^2 - k_1\, y_5 - 0.09375\, {y_2}^3 \\
&\quad - 0.1875\, y_3 y_9 y_8 \\
\dot{y}_6 &= -0.375\, y_3 {y_7}^2 - 0.375\, y_3 {y_1}^2 - 0.28125\, y_3 {y_9}^2 - 0.09375\, y_3 {y_8}^2 \\
&\quad - 0.1875\, y_2 y_9 y_8 + B - y_5 - 0.09375\, {y_3}^3 - k_1\, y_6 - 0.75\, y_1 y_7 y_9 \\
&\quad - 0.09375\, y_3 {y_2}^2 \\
\dot{y}_7 &= -0.375\, k_2\, y_9 y_1 y_3 - 0.1875\, k_2\, y_7 {y_2}^2 - 0.1875\, k_2\, {y_9}^2 y_7 \\
&\quad - 0.1875\, k_2\, y_7 {y_3}^2 - 0.375\, k_2\, y_7 {y_1}^2 - 0.125\, k_2\, {y_7}^3 + B_0 \\
&\quad - 0.1875\, k_2\, {y_8}^2 y_7 - 0.375\, k_2\, y_8 y_1 y_2 \\
\dot{y}_8 &= -0.375\, k_2\, y_8 {y_1}^2 - 0.09375\, k_2\, {y_8}^3 - 0.75\, k_2\, y_7 y_1 y_2 + y_9 \\
&\quad - 0.1875\, k_2\, y_9 y_3 y_2 - 0.09375\, k_2\, y_8 {y_9}^2 - 0.09375\, k_2\, y_8 {y_3}^2 \\
&\quad - 0.375\, k_2\, y_8 {y_7}^2 - 0.28125\, k_2\, y_8 {y_2}^2 \\
\dot{y}_9 &= -0.75\, k_2\, y_7 y_1 y_3 - 0.09375\, k_2\, {y_9}^3 - 0.09375\, k_2\, {y_8}^2 y_9 \\
&\quad - 0.09375\, k_2\, y_9 {y_2}^2 - 0.375\, k_2\, y_9 {y_7}^2 - 0.1875\, k_2\, y_8 y_3 y_2 - y_8 \\
&\quad - 0.375\, k_2\, y_9 {y_1}^2 - 0.28125\, k_2\, y_9 {y_3}^2.
\end{aligned}
$$

This autonomous system can now be analysed by the continuation and bifurcation code AUTO 97 of E.J.Doedel et al. [7]. Some of the results are displayed in figures 2 and 3.

In figure 2 the spectral system is displayed for parameter $k_1 \in [0.025, 0.20]$. Hopf bifurcations arise at $k_1 \approx 0.1315$ and 0.1281 (labels 2 and 3). The branch arising at label 3 for $k_1 \approx 0.1281$ has been followed. A cascade of period doublings (labels 7, 13, 15) and further bifurcations occur. Figure 3 displays periodic orbits of the spectral system. Obviously a sequence of period doublings arises. For $k_1 = 0.043$ a phase portrait is given together with its Poincaré map.

An interpretation of the periodic orbits of the spectral system in connection with the quasiperiodic solutions of the original system can be found in table 1.

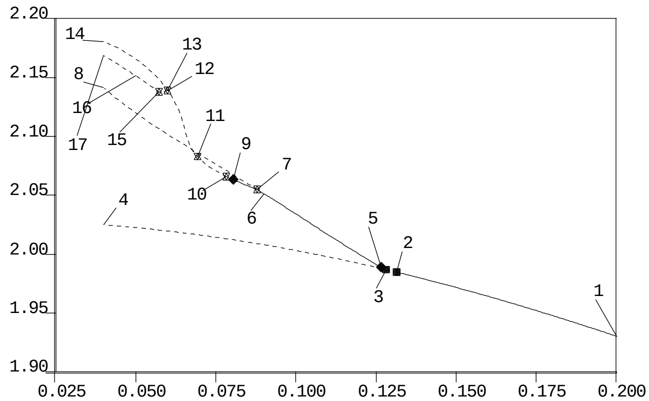


Fig. 2. Bifurcation diagram of the spectral system

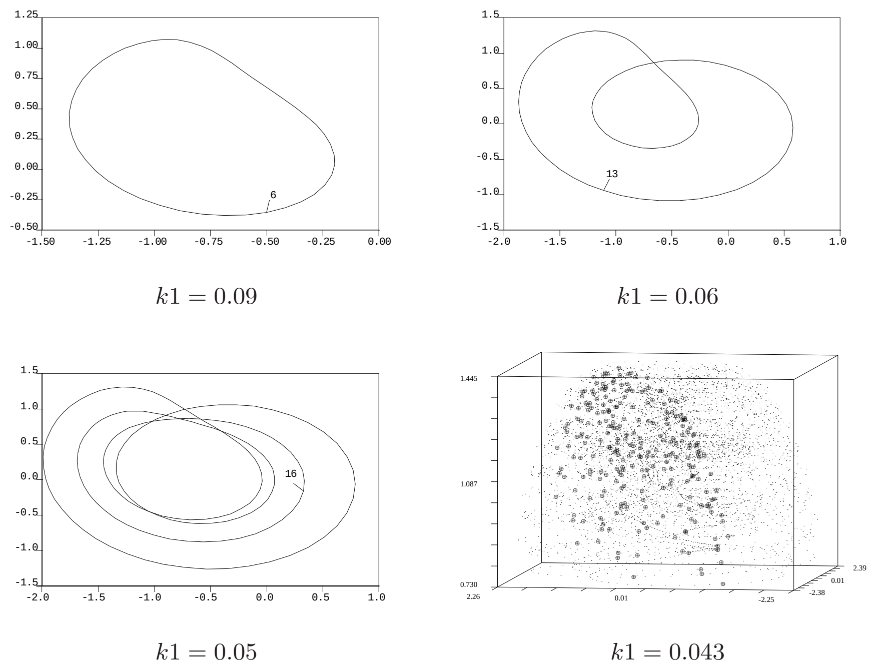


Fig. 3. Periodic orbits of the spectral system.

No. of label	spectral system	original system
1	stable equilibrium	stable periodic solution
3	Hopf bifurcation	torus bifurcation
6	stable periodic solution	stable invariant torus
7	period doubling	torus doubeling
12	unstable 2-per. solution	stable inv. double torus
13	period doubling	torus doubling
16	unstable 4-per. solution	stable inv. 4-fold torus
15	period doubling	torus doubling
—	strange attractor	strange attractor

Table 1. Interpretation of bifurcation diagram in figure 2.

2. Solution via finite differences. For semidiscretisation we used the central difference formula of 4th order:

$$\frac{\partial v_i}{\partial t} \approx \frac{1}{12h}\left(v_{i-2} - 8v_{i-1} + 8v_{i+1} - v_{i+2}\right). \tag{35}$$

In this example the resulting linear differential equations (31) were solved by the same finite difference method. The simple torus was continued on a 20 times 20 (t, θ)-grid and the double-torus on a 20 times 40 grid. Using standard continuation techniques we obtained the bifurcation diagram in figure 4 with the following special points:

bifurcation point	bifurcation type	emerging solution type
$k_1^* \in [0.12, 0.1225]$	torus bifurcation	asymptotically stable 2-torus
$k_1^* \in [0.08, 0.0825]$	torus-flip bifurcation	asymptotically stable doubled 2-torus

Figure 5 shows approximations of the invariant torus arising at $k_1^* \in [0.12, 0.1225]$ and figure 6 shows cross-sections of the doubled invariant torus for different parameter values.

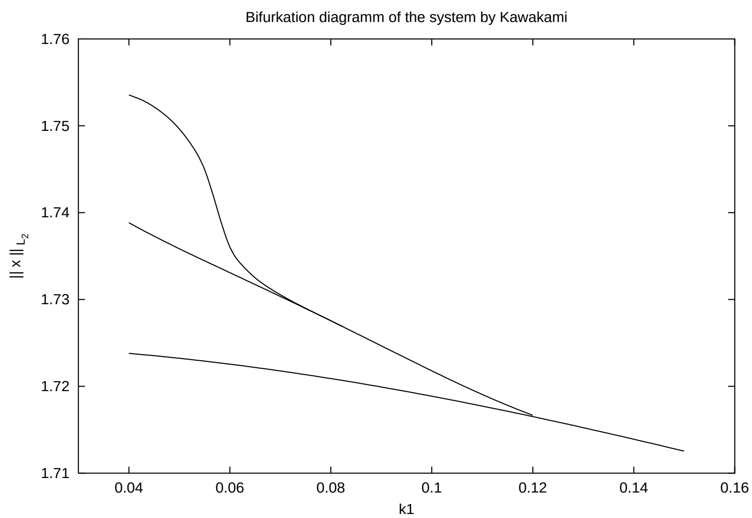
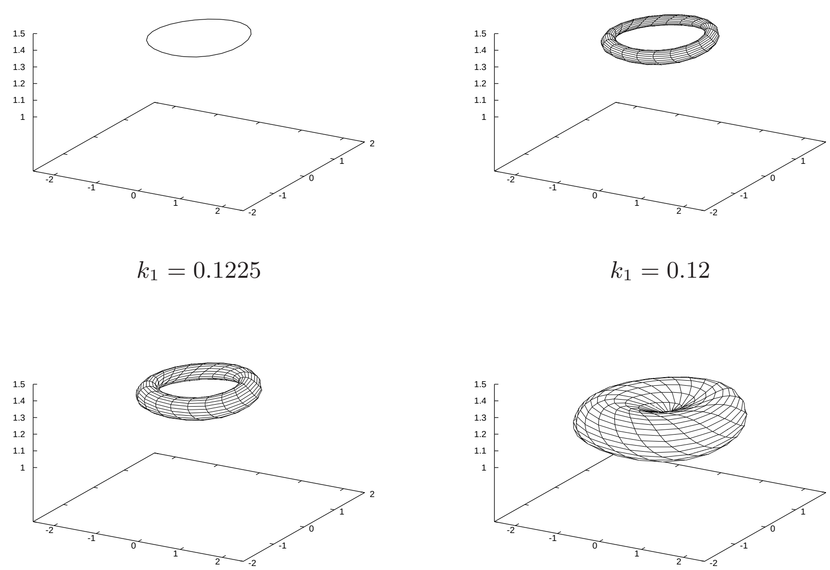


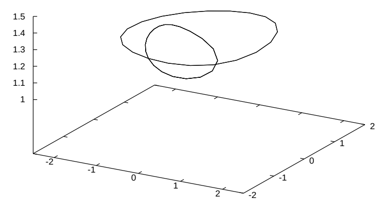
Fig. 4. Bifurcation diagram obtained by the finite difference method.



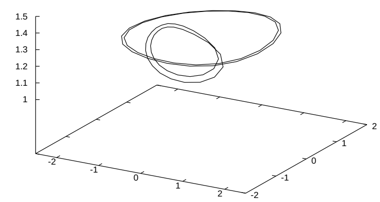
$k_1 = 0.1225$ $k_1 = 0.12$

$k_1 = 0.1175$ $k_1 = 0.04$

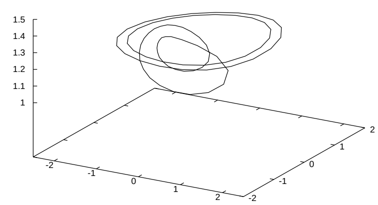
Fig. 5. Approximations of the emerging invariant 2-torus.



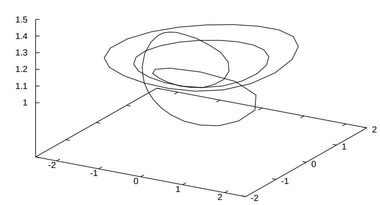
$k_1 = 0.0825$



$k_1 = 0.08$



$k_1 = 0.075$



$k_1 = 0.04$

Fig. 6. Approximations of the cross-sections γ_1 and γ_2 of the emerging invariant doubled 2-torus.

References

1. Bernet, K.: Ein Beitrag zur numerischen Approximation und Verfolgung von Toruslösungen parameterabhängiger dynamischer Systeme. TU Ilmenau, Shaker Verlag 1996.
2. Bernet, K.; Vogt, W.: Anwendung finiter Differenzenverfahren zur direkten Bestimmung invarianter Tori. ZAMM 74 (1994), No. 6, T577-T579.
3. Chen, Y.; Leung, A.: Bifurcation and Chaos in Engineering. Springer, London 1998.
4. Dieci,L.; Lorenz,J.; Russell,R.D.: Numerical Calculation of Invariant Tori. SIAM J. Sci. Stat. Comput. 12 (1991), 607–647.
5. Dieci,L.; Lorenz,J.: Block M-Matrices and Computation of Invariant Tori. SIAM J. Sci. Stat. Comput. 13 (1992), 885–903.
6. Dieci,L.; Bader,G.: On Approximating Invariant Tori : Block Iterations and Multigrid Methods for the Associated Systems. Preprint Nr. 658, University Heidelberg, 1992.
7. Doedel, E.J.; Champneys, A.R.; Fairgrieve, Th.F.; Kuznetsov Y.A.; Sandstede, B.; Wang, X.: AUTO 97: Continuation and Bifurcation Software for Ordinary Differential Equations (with HomCont), 1997
8. Kevrekidis, I.G., Aris, R., Schmidt, L.D., Pelikan, S.: Numerical Computations of Invariant Circles of Maps. Physica 16 D (1985), 243-251.
9. Mingyou, H.; Küpper, T.; Masbaum, N.: Computation of Invariant Tori by the Fourier Methods. SIAM J. Sci. Comput. 18 (1997), 918–942.
10. Moore, G.: Computation and Parametrisation of Invariant Curves and Tori. SIAM J. Numer. Anal., 33 (1996), 2333–2358.
11. Nedwal, F. Eine direkte Methode zur numerischen Approximation geschlossener Invarianzkurven von Poincaré–Abbildungen, Diplomarbeit TU Ilmenau, 1995.
12. Samoilenko, A.M.: Elements of the Mathematical Theory of Multi-Frequency Oscillations. Kluwer Academic Publishers, Dordrecht u.a. 1991.
13. Schilder, F.; Vogt,W.: Eine Spektralmethode zur Verfolgung und Analyse periodischer und quasiperiodischer Lösungen. Preprint No. M 27/00, TU Ilmenau 2000.
14. Van Veldhuizen,M.: A new algorithm for the numerical approximation of an invariant curve. SIAM J. Sci. Stat. Comput. 8 (1987), 951–962.
15. Van Veldhuizen,M.: Convergence Results for invariant curve algorithms. Math. Comp. 51 (1988), No. 184, 677–697.
16. Vogt, W.; Bernet, K.: A Shooting Method for Invariant Tori. Preprint No. M 3/95, TU Ilmenau 1995.
17. Yoshinaga, T. and Kawakami, H.: Bifurcations and Chaotic States in Forced Oscillatory Circuits containing Saturable Inductors. In: Carroll, T.; Pecora, L. Nonlinear Dynamics In Circuits. World Scientific Publishing, 1995.