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Asymptotic tracking with prescribed transient behaviour for linear systems*

Achim Ilchmann† and **Eugene P. Ryan**‡

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Abstract

The problem of asymptotic tracking of reference signals is considered in the context of m-input, m-output linear systems (A, B, C) with the following structural properties: (i) CB is sign definite (but possibly of unknown sign), (ii) the zero dynamics are exponentially stable. The class $\mathcal{Y}_{\text{ref}}(\alpha)$ of reference signals is the set of all possible solutions of a fixed, stable, linear, homogeneous differential equation (with associated characteristic polynomial α). The first control objective is asymptotic tracking, by the system output $y = Cx$, of any reference signal $r \in \mathcal{Y}_{\text{ref}}(\alpha)$. The second objective is guaranteed error $e = y - r$ transient performance: e should evolve within a prescribed performance funnel $\mathcal{F}_\varphi$ (determined by a function φ). Both objectives are achieved simultaneously by an internal model in series with a proportional time-varying error feedback $t \mapsto u(t) = -k(t)e(t)$. The time-varying proportional factor $k(t)$ is generated via a nonlinear function of the product $\|e(t)\|\varphi(t)$. The feedback structure essentially exploits an intrinsic high-gain property of the system by ensuring that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact.

Keywords: Tracking, output feedback, transient behaviour, internal model, minimum phase

1 Introduction

In the precursor [4] to the present paper, the concept of a performance funnel was introduced in a context of tracking control for nonlinear systems. The basic problem addressed therein was that of approximate tracking (with prescribed transient behaviour), by the system output y, of any absolutely continuous and bounded function r with essentially bounded derivative: the terminology "approximate tracking" means that, for any prescribed $\lambda > 0$, a control structure can be determined which ensures that the tracking error $e = y - r$ is ultimately bounded by λ (that is, $\|e(t)\| \leq \lambda$ for all t sufficiently large); the terminology "with prescribed transient behaviour" means that,

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for some suitable prescribed function φ, the error function is required to satisfy $\|e(t)\| \le 1/\varphi(t)$ for all $t > 0$. The choice of φ determines the transient behaviour; moreover, by imposing the property $\liminf_{t\to\infty} \varphi(t) \ge \lambda > 0$, the approximate tracking objective is assured. For example, with $\varphi : t \mapsto \min\{t/T, 1\}/\lambda$, the approximate tracking objective is achieved in prescribed time $T > 0$. Figure 1 encapsulates the approach: the function φ determines the performance funnel $\mathcal{F}_\varphi$, which may be identified with the graph of the set-valued map $t \mapsto \{v \mid \varphi(t)\|v\| < 1\}$. Simply stated, the control objective is to maintain the evolution of the tracking error in the funnel $\mathcal{F}_\varphi$. For

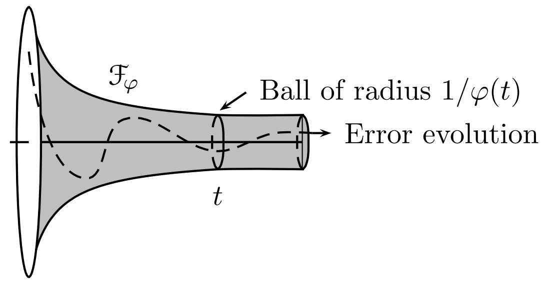


Figure 1: Performance funnel $\mathcal{F}_\varphi$

the general class of reference signals considered in [4], the condition $\liminf_{t\to\infty} \varphi(t) > 0$ cannot be relaxed, and so *exact* asymptotic tracking cannot be achieved. The purpose of the present note is to demonstrate that the condition may be relaxed if one confines attention to minimum-phase linear systems with sign-definite high frequency gain and restricts the class of reference signals to coincide with the set of solutions of a fixed, stable, linear, homogeneous differential equation. Under these restrictions, exact asymptotic tracking is achieved by adopting an internal model (capable of replicating the reference signals) in conjunction with a performance funnel with radius asymptotic to zero and an output feedback structure akin to that in [4, Section 6.3]. In an adaptive control context, the use of internal models in problems of asymptotic tracking for linear systems is well established (see, for example, [6, 7, 2, 3]). We emphasize that the approach adopted in the present paper is non-adaptive.

2 Class of systems

We consider the class of m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$) linear systems of the form

$$\left.\begin{array}{ll} \dot{x}(t) = A\,x(t) + B\,u(t), & x(0) = x^0 \in \mathbb{R}^n \\ y(t) = C\,x(t)\,, & \end{array}\right\} \tag{2.1}$$

where the triple $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ has the following properties:

P1: *strict relative degree one with sign-definite high-frequency gain*, that is,

$$\langle x, CBx\rangle = 0 \quad \Longleftrightarrow \quad x = 0,$$

P2: *minimum-phase*, that is,

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \qquad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re} s \geq 0.$$

Under assumption P1, the minimum-phase property P2 is equivalent to the assumption that the system (2.1) has exponentially stable zero dynamics (this equivalence can also be deduced from Lemma 3.3 below).

2.1 Control objectives, class of reference signals, and performance funnel

Let $\mathcal{M}$ denote the set of square real matrices having no eigenvalue with positive real part and such that every eigenvalue on the imaginary axis is semi-simple. The *reference signals* to be tracked are all functions $r : \mathbb{R}_+ \to \mathbb{R}^m$ the components r_i of which are solutions of the scalar differential equation $\alpha(\frac{d}{dt})r_i(\cdot) = 0$, where $\alpha \in \mathbb{R}[s]$ is the characteristic polynomial of some $M \in \mathcal{M}$ (and so every such function r is bounded). We denote this reference signal class by

$$\mathcal{Y}_{\text{ref}}(\alpha) \ := \ \left\{ r \in C^\infty(\mathbb{R}_+, \mathbb{R}^m) \ \middle| \ \alpha(\tfrac{d}{dt})r(\cdot) = 0, \ \alpha(s) = \det[sI - M], \quad M \in \mathcal{M} \right\} .$$

The first control objective is asymptotic (output) tracking of any reference signal $r \in \mathcal{Y}_{\text{ref}}(\alpha)$. By this we mean a (dynamic) output feedback strategy which incorporates an internal model (capable of replicating the reference signal) and which ensures that $\lim_{t\to\infty} \big(y(t) - r(t)\big) = 0$ whilst maintaining boundedness all the other signals. The second control objective is prescribed transient behaviour of the error signal $e = y - r$. We capture both objectives in the concept of a performance funnel

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_+ \times \mathbb{R}^m \middle| \ \varphi(t) \, \|e\| < 1 \right\} \tag{2.2}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) with the following properties

$$\left.\begin{array}{ll} \text{(a)} & \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is absolutely continuous and non-decreasing;} \\ \text{(b)} & \varphi(t) = 0 \iff t = 0; \\ \multicolumn{2}{l}{\text{there exists } c > 1 \text{ such that:}} \\ \text{(c)} & \varphi(t) \leq c\, \varphi(t/2) \quad \text{for all } t \in \mathbb{R}_+; \\ \text{(d)} & \dot\varphi(t) \leq c\,[1 + \varphi(t)] \quad \text{for almost all } t \in \mathbb{R}_+. \end{array}\right\} \tag{2.3}$$

For example, $t \mapsto \varphi(t) = t^a$, $a \geq 1$, satisfies (2.3) with $c = 2^a$. We record the following observation for later use.

Proposition 2.1 *Let φ be such that (2.3) holds. For every $p \geq \ln c / \ln 2$,*

$$0 \ < \ \varphi(t) \ \leq \ \varphi(1)\,[1 + c t^p] \quad \textit{for all } \ t > 0. \tag{2.4}$$

Proof: Since φ is non-decreasing with property (b), we have $0 < \varphi(t) \leq \varphi(1)$ for all $t \in (0,1]$. Now, let $t \in (1,\infty)$ be arbitrary and choose $n \in \mathbb{N}$ such that $2^{n-1} \leq t \leq 2^n$ or, equivalently, $1/2 \leq t/2^n \leq 1$. Then, by (b), (c) and the non-decreasing property,

$$0 < \varphi(t) \leq c\,\varphi(t/2) \leq \ldots \leq c^n\,\varphi(t/2^n) \leq c^n\,\varphi(1) = c\,\varphi(1)\,2^{(n-1)\ln c/\ln 2} \leq c\,\varphi(1)\,t^p\,.$$

The claim (2.4) follows. □

Proposition 2.1 implies, in particular, that exponentially contracting funnels are excluded.

3 The control

Let $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ be such that P1 and P2 hold, and define

$$s(CB) := \begin{cases} +1, & \text{if } CB \text{ is positive definite, i.e. } \langle x, CBx\rangle > 0 \ \ \forall\, x \neq 0 \\ -1, & \text{if } CB \text{ is negative definite, i.e. } \langle x, CBx\rangle < 0 \ \ \forall\, x \neq 0. \end{cases} \tag{3.5}$$

We will have occasion to consider the two possible cases: $s(CB)$ known or unknown *a priori* (the latter case is largely of academic interest).

3.1 Internal model

A body of work by Francis and Wonham in the 1970s (see, for example, [1, 10]) led to the so-called *Internal Model Principle*, succinctly summarized in the context of linear systems in [11, p. 210] as "every good regulator must incorporate a model of the outside world". Recent extensions of this "principle" to a nonlinear setting are contained in [9].

Let $\alpha \in \mathbb{R}[s]$ be the characteristic polynomial of some $M \in \mathcal{M}$ (and so every $r \in \mathcal{Y}_{\rm ref}(\alpha)$ is bounded). Let $\beta \in \mathbb{R}[s]$ be a monic Hurwitz polynomial (i.e. all zeros of β lie in the open left half complex plane) and such that α and β are coprime of degree $p := \deg\beta = \deg\alpha$. Then

$$\lim_{s\to\infty} \beta(s)/\alpha(s) = 1. \tag{3.6}$$

The *internal model* is now defined to be the m-input, m-output linear system with transfer function

$$G_m(s) := \frac{\beta(s)}{\alpha(s)}\,I_m\,. \tag{3.7}$$

Let $(\hat{A},\hat{b},\hat{c},1) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{p\times 1} \times \mathbb{R}^{1\times p} \times \mathbb{R}$ be a minimal state space realization of $\beta(s)/\alpha(s)$. Then a minimal state space realization of the internal model is given by

$$\left.\begin{aligned} \dot\xi(t) &= \widetilde{A}\,\xi(t) + \widetilde{B}\,w(t), \quad \xi(0) = \xi^0 \\ u(t) &= \widetilde{C}\,\xi(t) + I_m\,w(t) \end{aligned}\right\} \tag{3.8}$$

with

$$\widetilde{A} = \mathrm{diag}\{\hat{A},\ldots,\hat{A}\} \in \mathbb{R}^{mp\times mp}, \quad \widetilde{B} = \mathrm{diag}\{\hat{b},\ldots,\hat{b}\} \in \mathbb{R}^{mp\times m},$$
$$\widetilde{C} = \mathrm{diag}\{\hat{c},\ldots,\hat{c}\} \in \mathbb{R}^{m\times mp}, \qquad \xi^0 \in \mathbb{R}^{mp}.$$

We will refer to $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ as the internal model (although, strictly speaking, the use of "the" here is incorrect as any quadruple in the similarity orbit of $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ also qualifies for the title "internal model").

3.2 Feedback

Let φ be such that (2.3) holds and let $\mathcal{F}_\varphi$ be the associated performance funnel given by (2.2). Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any C^∞ function such that, for some strictly-increasing, unbounded sequence (k_j) in $(1, \infty)$,

$$\nu(k_j)\, s(CB) \to \infty \quad \text{as } j \to \infty\,. \tag{3.9}$$

If $s(CB)$ is known *a priori*, then $\nu : k \mapsto k\, s(CB)$ suffices. If $s(CB)$ is unknown *a priori*, then any C^∞ function ν with the following properties suffices:

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty\,. \tag{3.10}$$

A simple example of a function satisfying (3.10) is $\nu\colon k \mapsto k\,\cos k$. In the latter case of unknown $s(CB)$, the rôle of the function ν is similar to the concept of a "Nussbaum" function in adaptive control. Note, however, that the requisite properties (3.10) are less restrictive than (a) the "Nussbaum property"

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = \infty, \qquad \liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = -\infty,$$

as required in [12], for example, or (b) the stronger "scaling invariant" Nussbaum property, as required in [5], for example.

The control strategy is given by

$$w(t) = -\nu(k(t))\,[y(t) - r(t)], \qquad k(t) = \frac{1}{1 - \big(\varphi(t)\|y(t) - r(t)\|\big)^2}\,, \tag{3.11}$$

in series with the internal model (3.8).

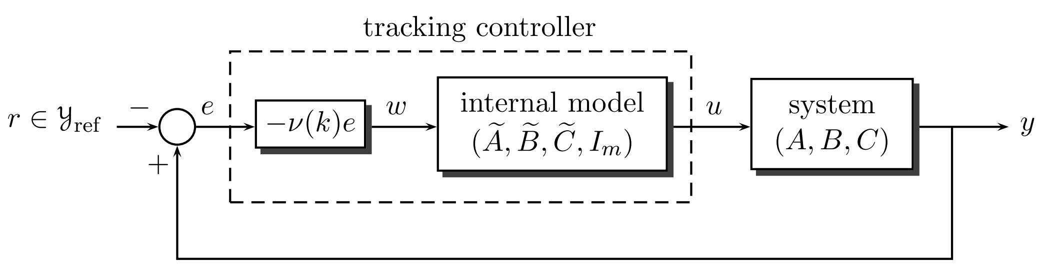


Figure 2: Asymptotic tracking controller with internal model

3.3 Closed-loop system

The conjunction of (2.1), (3.8), and (3.11) yields the closed-loop initial-value problem

$$\left.\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\,\bar{x}(t) - \nu(k(t))\,\bar{B}\,[y(t) - r(t)], \quad \bar{x}^0 = \begin{bmatrix} x^0 \\ \xi^0 \end{bmatrix} \\ y(t) &= \bar{C}\,\bar{x}(t), \\ k(t) &= \frac{1}{1 - \big(\varphi(t)\|\bar{C}\,\bar{x}(t) - r(t)\|\big)^2}\,, \end{aligned}\right\} \tag{3.12}$$

where

$$\bar{A} = \begin{bmatrix} A & B\widetilde{C} \\ 0 & \widetilde{A} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ \widetilde{B} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C\,, 0 \end{bmatrix}, \quad \bar{x}(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}. \tag{3.13}$$

Noting the potential singularity in the function k, some care must be exercised in defining the concept of a solution of (3.12): a function $\bar{x}\colon [0,\omega) \to \mathbb{R}^{n+mp}$, with $0 < \omega \le \infty$, is deemed a *solution* of (3.12) if, and only if, it is absolutely continuous, with $\bar{x}(0) = \bar{x}^0$, it satisfies the differential equations in (3.12) for almost all $t \in [0,\omega)$, and $\varphi(t)\,\|\bar{C}\bar{x}(t) - r(t)\| < 1$ for all $t \in [0,\omega)$. A solution is *maximal* if, and only if, it has no proper right extension that is also a solution.

3.4 Main result

Theorem 3.1 Let $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ have strict relative degree one, sign-definite high-frequency gain, and be minimum-phase. Let φ satisfy (2.3), let $\mathcal{F}_\varphi$ be the performance funnel (2.2) associated with φ, and let $r \in \mathcal{Y}_{\mathrm{ref}}(\alpha)$. Then the feedback (3.11) applied in series with the internal model (3.8) yields the initial-value problem (3.12) which, for every $x^0 \in \mathbb{R}^n$ and $\xi^0 \in \mathbb{R}^{mp}$, has a solution and every solution can be extended to a maximal solution. Every maximal solution $\bar{x}\colon [0,\omega) \to \mathbb{R}^{n+mp}$ has the properties:

(i) $\omega = \infty$;

(ii) the functions $\bar{x}$, k and u are bounded;

(iii) there exists $\varepsilon \in (0,1)$ such that, for all $t \ge 0$, $\varphi(t)\,\|y(t) - r(t)\| \le 1 - \varepsilon$;

(iv) if φ is unbounded, then $(y(t) - r(t), u(t)) \to (0,0)$ as $t \to \infty$.

Remark 3.2 In the specific case of positive-definite CB and zero reference signal $r \equiv 0$, it is shown in [4] that the assertions of Theorem 3.1 hold for the feedback $u = -ke$ without recourse to an internal model.

The proof of Theorem 3.1 invokes three lemmas; we briefly digress to present these.

3.5 Three technical lemmas

The first lemma is well known and is a re-statement of [3, Lemma 2.1.3].

Lemma 3.3 Assume that $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ has strict relative degree one. Let $V \in \mathbb{R}^{n\times(n-m)}$ be such that $\operatorname{im} V = \ker C$ (of dimension $n-m$) and write

$$N := (V^T V)^{-1} V^T \big[I_n - B(CB)^{-1} C \big].$$

Then

$$L = \begin{bmatrix} C \\ N \end{bmatrix}$$

is invertible, with inverse $L^{-1} = \big[B(CB)^{-1} \ , \ V \big]$ and

$$LAL^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad LB = \begin{bmatrix} CB \\ 0 \end{bmatrix}, \quad CL^{-1} = \begin{bmatrix} I_m & 0 \end{bmatrix}$$

where $A_1 \in \mathbb{R}^{m\times m}$ (with A_2, A_3, A_4 of conforming formats). Furthermore, (A,B,C) is minimum phase if, and only if, A_4 is Hurwitz.

Lemma 3.4 Let $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ be minimum phase with strict relative degree one and sign-definite high-frequency gain. If $(\widetilde{A}, \widetilde{B}, \widetilde{C}, I_m)$ is a minimal realization of the internal model as specified in Subsection 3.1, then $(\bar{A}, \bar{B}, \bar{C})$, as defined in (3.13), is minimum phase with strict relative degree one and sign-definite high-frequency gain.

Proof: Clearly, $\bar{C}\bar{B} = CB$ and so the system $(\bar{A}, \bar{B}, \bar{C})$ has strict relative degree one and sign-definite high-frequency gain.
It remains to show that

$$\det \begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} \neq 0 \qquad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re} s \geq 0.$$

Since $(\hat{A}, \hat{b})$ is a controllable pair, the Hautus condition implies that $[sI - \hat{A}, \hat{b}]$ has full rank p for all $s \in \mathbb{C}$, whence

$$\operatorname{rank} \begin{bmatrix} sI - \widetilde{A} & \widetilde{B} \end{bmatrix} = mp \quad \text{for all } s \in \mathbb{C}.$$

By the minimum-phase property of (A,B,C), we have

$$\operatorname{rank} \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + m \quad \text{for all } s \in \mathbb{C} \text{ with } \operatorname{Re}(s) \geq 0\,,$$

and so

$$\operatorname{rank} \begin{bmatrix} sI - \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} sI - A & -B\widetilde{C} & B \\ 0 & sI - \widetilde{A} & \widetilde{B} \\ C & 0 & 0 \end{bmatrix} = n + mp + m$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$, and the claim follows. □.

A proof of the following lemma can be found in [8], see also [3, Lem. 5.1.2].

Lemma 3.5 Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ be minimum phase with strict relative degree one and sign-definite high-frequency gain. If $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is a minimal realization of the internal model as specified in Subsection 3.1, then, for any $r \in \mathcal{Y}_{\mathrm{ref}}(\alpha)$, there exists $\rho^0 \in \mathbb{R}^{n+mp}$ such that

$$\left.\begin{aligned} \dot\rho(t) &= \bar A\,\rho(t), \quad \rho(0)=\rho^0 \\ r(t) &= \bar C\,\rho(t)\,, \end{aligned}\right\} \tag{3.14}$$

where $\bar A$ and $\bar C$ are given by (3.13).

3.6 Proof of Theorem 3.1

By Lemma 3.5, there exists $\rho^0 \in \mathbb{R}^{n+mp}$ such that $r(\cdot) = \bar C\rho(\cdot)$, where $\rho\colon t \mapsto \big(\exp \bar A t\big)\rho^0$. Writing

$$x_e(t) = \bar x(t) - \rho(t), \qquad e(t) = y(t) - r(r),$$

the closed-loop initial-value problem (3.12) may be expressed in the equivalent form

$$\left.\begin{aligned} \dot x_e(t) &= \bar A\,x_e(t) - \nu(k(t))\bar B\bar C\,e(t), \quad x_e(0) = x_e^0 := \bar x^0 - \rho^0, \\ e(t) &= \bar C\,x_e(t)\,, \\ k(t) &= \Big[1 - \big(\varphi(t)\|e(t)\|\big)^2\Big]^{-1}\,. \end{aligned}\right\} \tag{3.15}$$

Introducing the open set

$$\mathcal{D} := \Big\{(x_e, \eta) \in \mathbb{R}^{n+mp} \times \mathbb{R} \;\Big|\; \varphi(|\eta|)\,\|\bar C\,x_e\| < 1\Big\}\,,$$

and defining the function

$$d : \mathcal{D} \to \mathbb{R}_+, \quad (x_e, \eta) \mapsto d(x_e, \eta) := \frac{1}{1 - \big(\varphi(|\eta|)\|\bar C x_e\|\big)^2}\,, \tag{3.16}$$

then the non-autonomous closed-loop initial-value problem (3.12) (equivalently, (3.15)) may be recast on $\mathcal{D}$ as the following autonomous initial-value problem

$$\left.\begin{aligned} &\dot x_e(t) = \bar A x_e(t) - \nu\big(d(x_e(t), \eta(t))\big)\,\bar B\bar C\,x_e(t) \\ &\dot\eta(t) = 1 \\ &(x_e(0), \eta(0)) = (x_e^0, 0) \in \mathcal{D}\,. \end{aligned}\right\} \tag{3.17}$$

The standard theory of ordinary differential equations now applies to conclude the existence of a solution $t \mapsto (x_e(t), \eta(t)) \in \mathcal{D}$ to (3.17) and, moreover, every solution can be extended to a maximal solution $(x_e, \eta)\colon [0, \omega) \to \mathcal{D}$. We will make use of the following fact in due course: if there exists a compact set $\mathcal{C} \subset \mathcal{D}$ such that $(x_e(t), \eta(t)) \in \mathcal{C}$ for all $t \in [0, \omega)$, then $\omega = \infty$. To see this, assume that such a set $\mathcal{C}$ exists. Then $(x_e(\cdot), \eta(\cdot))$ and $\nu\big(d(x_e(\cdot), \eta(\cdot))\big)$ are bounded functions which, together with (3.17), implies that $(x_e(\cdot), \eta(\cdot))$ is uniformly continuous. Seeking a contradiction, suppose that $\omega < \infty$. By uniform continuity, it follows that the limit $(x_e^*, \omega) = \lim_{t\nearrow\omega}(x_e(t), \eta(t))$ exists and, by compactness, lies in $\mathcal{C} \subset \mathcal{D}$. By the existence theory, the initial-value problem (3.17), with initial data (x_e^*, ω) replacing $(x_e^0, 0)$ has a solution: concatenation of this solution with (x_e, η) yields a proper right extension of the latter, contradicting its maximality.

Clearly, if $(x_e, \eta)\colon [0,\omega) \to \mathcal{D}$ is a solution of (3.17), then $\bar{x} = x_e + \rho\colon [0,\omega) \to \mathbb{R}^{n+mp}$ is a solution of (3.12); conversely, if $\bar{x}\colon [0,\omega) \to \mathbb{R}^{n+mp}$ is a solution of (3.12), then $(x_e, \eta)\colon [0,\omega) \to \mathbb{R}^{n+mp} \times \mathbb{R}$, with $x_e = \bar{x} - \rho$ and component η given by $\eta(t) = t$, is a solution of (3.17). We may now conclude that, for each $\bar{x}^0 \in \mathbb{R}^{n+mp}$, (3.12) has a solution and every solution can be maximally extended.

Let $x^0 \in \mathbb{R}^n$ and $\xi^0 \in \mathbb{R}^{mp}$ be arbitrary and let $\bar{x}$ be a maximal solution of (3.12) with interval of existence $[0,\omega)$. Then, for $x_e^0 = \bar{x}^0 - \rho^0$, the function $t \mapsto (x_e(t), \eta(t)) = (\bar{x}(t) - \rho(t)\,,\, t)$ is a maximal solution of (3.17) with interval of existence $[0,\omega)$. By (3.16) and the first of equations (3.17), we now have

$$\dot{x}_e(t) = \bar{A}x_e(t) - \nu(k(t))\bar{B}\bar{C}e(t), \quad k(t) = \left[1 - \big(\varphi(t)\|e(t)\|\big)^2\right]^{-1} \quad \forall\, t \in [0,\omega). \tag{3.18}$$

By Lemma 3.4, $(\bar{A}, \bar{B}, \bar{C})$ is minimum phase with strict relative degree one, and so, by Lemma 3.3, there exists N such that

$$L := \begin{bmatrix} \bar{C} \\ N \end{bmatrix}$$

is invertible and the transformation

$$\begin{bmatrix} \bar{C} \\ N \end{bmatrix} x_e(t) = \begin{bmatrix} e(t) \\ z(t) \end{bmatrix}$$

converts (3.18) into the equivalent form

$$\left.\begin{aligned} \dot{e}(t) &= A_1\, e(t) + A_2\, z(t) - \nu(k(t))\, CB\, e(t) \\ \dot{z}(t) &= A_3\, e(t) + A_4\, z(t) \\ k(t) &= \left[1 - \big(\varphi(t)\|e(t)\|\big)^2\right]^{-1} \end{aligned}\right\} \quad \forall\, t \in [0,\omega), \tag{3.19}$$

wherein $A_4 \in \mathbb{R}^{(n+m(p-1))\times(n+m(p-1))}$ is Hurwitz and we have invoked the equality $\bar{C}\bar{B} = CB$. Since $(x_e(t), t) \in \mathcal{D}$ for all $t \in [0,\omega)$, we have

$$\varphi(t)\|e(t)\| < 1 \quad \forall\, t \in [0,\omega) \tag{3.20}$$

and so e is bounded, which, together with the Hurwitz property of A_4 and the second of equations (3.19), implies that z is bounded. It immediately follows that x_e is bounded, whence boundedness of $\bar{x} = x_e + \rho$.

Writing $e^0 = \bar{C}x_e^0$, $z^0 = Nx_e^0$ and defining

$$q_0(t) := A_2\, \exp(A_4 t)z^0, \quad q_1(t) := A_1 e(t) + A_2 \int_0^t \exp(A_4(t-s))A_3 e(s)ds, \quad \forall\, t \in [0,\omega), \tag{3.21}$$

then the first two equations in (3.19) are equivalent to

$$\dot{e}(t) = q_0(t) + q_1(t) - \nu(k(t))\, CB\, e(t) \quad \forall\, t \in [0,\omega)\,. \tag{3.22}$$

Since A_4 is Hurwitz, there exist $c_1, \mu > 0$ such that

$$\|q_0(t)\| = \|A_2 \exp(A_4 t) z^0\| \leq c_1 \, e^{-\mu t} \quad \forall \; t \in [0, \omega) \tag{3.23}$$

and

$$\begin{aligned} \|q_1(t)\| &\leq \|A_1\| \|e(t)\| + c_1 \left(\int_0^{t/2} + \int_{t/2}^t \right) e^{-\mu(t-s)} \|e(s)\| \, ds \\ &\leq \|A_1\| \|e(t)\| + \frac{c_1}{\mu} \left[e^{-\mu \, t/2} \max_{s \in [0\,,\,t/2]} \|e(s)\| + \max_{s \in [t/2\,,\,t]} \|e(s)\| \right] \quad \forall \; t \in [0, \omega). \end{aligned} \tag{3.24}$$

By boundedness of e, together with (3.20) and invoking property (2.3d) of φ, we may infer the existence of $c_2 > 0$ such that

$$\begin{aligned} &\varphi(t) \, \dot\varphi(t) \, \|e(t)\|^2 \leq c[1 + \varphi(t)] \, \varphi(t) \, \|e(t)\|^2 \\ &\qquad \leq c \, [1 + 2\varphi^2(t)] \, \|e(t)\|^2 \leq c \, [\|e(t)\|^2 + 2] \leq c_2 \quad \text{for almost all } t \in [0, \omega). \end{aligned} \tag{3.25}$$

Since CB is sign definite, there exists $c_3 > 0$ such that

$$\frac{1}{2} \, c_3 \, \|e\|^2 \;\leq\; |\langle e, CB \, e \rangle| \qquad \forall \, e \in \mathbb{R}^m. \tag{3.26}$$

Now we are in a position to prove boundedness of k.

Define $\tilde\nu : \mathbb{R} \to \mathbb{R}$ as follows

$$\tilde\nu(k) := \nu(k) \, s(CB).$$

By property (3.9) of ν, there exists a strictly-increasing unbounded sequence (k_j) in $(1, \infty)$ such that $\tilde\nu(k_j) \to \infty$ as $j \to \infty$. Passing to a subsequence if necessary, we may assume that the sequence $\big(\tilde\nu(k_j)\big)$ is in $(0, \infty)$ and is strictly increasing. Seeking a contradiction, suppose that k is unbounded. For each $j \in \mathbb{N}$, define

$$\begin{aligned} \tau_j &:= \inf\{t \in [0, \omega) | \; k(t) = k_{j+1}\} \\ \sigma_j &:= \sup\{t \in [0, \tau_j] | \; \tilde\nu(k(t)) = \tilde\nu(k_j)\} \\ \tilde\sigma_j &:= \sup\{t \in [0, \tau_j] | \; k(t) = k_j\} \leq \sigma_j \, . \end{aligned}$$

Observe that

$$k(\tau_j) > k(\sigma_j) \quad \forall \; j \in \mathbb{N} \, . \tag{3.27}$$

Furthermore, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, we have $k(t) \geq k_j$ and $\tilde\nu(k(t)) \geq \tilde\nu(k_j)$. Therefore,

$$1 > (\varphi(t) \|e(t)\|)^2 \geq 1 - \frac{1}{k_j} \geq 1 - \frac{1}{k_1} =: c_4 > 0 \quad \forall \; t \in [\sigma_j, \tau_j] \;\; \forall \; j \in \mathbb{N} \, , \tag{3.28}$$

and since φ is non-decreasing, we arrive at

$$\max_{s \in [t/2\,,t]} \|e(s)\| \; < \; \frac{1}{\varphi(t/2)} \; \leq \; \frac{\varphi(t)}{\sqrt{c_4} \, \varphi(t/2)} \, \|e(t)\| \quad \forall \; t \in [\sigma_j, \tau_j] \;\; \forall \; j \in \mathbb{N} \, . \tag{3.29}$$

By (3.24) and (3.29), together with boundedness of e and property (2.3c) of φ, we may infer the existence of $c_5 > 0$ such that

$$\|q_1(t)\| \;\le\; c_5 \left[e^{-\mu t/2} + \|e(t)\|\right] \quad \forall\, t \in [\sigma_j, \tau_j] \;\forall\, j \in \mathbb{N}\,. \tag{3.30}$$

Invoking (3.25), (3.23), (3.26), (3.30), (3.28), recalling that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, \omega)$, and noting that, by Proposition 2.1, the functions $t \mapsto \varphi(t)e^{-\mu t}$ and $t \mapsto \varphi(t)e^{-\mu t/2}$ are bounded, we may conclude the existence of $c_6 > 0$ such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\, k(t) =\; & k^2(t)\left[2\,\varphi(t)\dot\varphi(t)\,\|e(t)\|^2 + 2\,\varphi^2(t)\,\langle e(t), q_0(t) + q_1(t) - \nu(k(t))\, CB\, e(t)\rangle\right] \\
\le\; & k^2(t)\left[2\,c_2 + 2\,\varphi(t)\left[\|q_0(t)\| + \|q_1(t)\|\right] - 2\,\varphi^2(t)\,\tilde\nu(k(t))\,|\langle e(t),\, CB\, e(t)\rangle|\,\right] \\
\le\; & k^2(t)\left[2\,c_2 + 2\,c_1\varphi(t)\,e^{-\mu t} + 2\,c_5\,\varphi(t)\,\left[e^{-\mu t/2} + \|e(t)\|\right] - c_3\,\varphi^2(t)\,\tilde\nu(k(t))\,\|e(t)\|^2\right] \\
\le\; & k^2(t)\left[c_6 - c_3\, c_4\,\tilde\nu(k(t))\right] \qquad\qquad \text{for almost all } t \in [\sigma_j, \tau_j] \text{ and all } j \in \mathbb{N}\,.
\end{aligned}$$

Let $j^* \in \mathbb{N}$ be sufficiently large to that $c_6 - c_3\, c_4 \tilde\nu(k_{j^*}) < 0$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, k(t) < 0 \qquad \text{for almost all } t \in [\sigma_{j^*}, \tau_{j^*}]\,,$$

which contradicts (3.27). This proves boundedness of k.

Next we show boundedness of u. Since k is bounded, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1-\varepsilon$ for all $t \in [0, \omega)$. By boundedness of e, z, and k, it follows that u is bounded.

We proceed to prove that $\omega = \infty$. Suppose that ω is finite. Let $c_7 > 0$ be such that $\|x_e(t)\| \le c_7$ for all $t \in [0, \omega)$, and set

$$\mathcal{C} := \left\{(x_e,\, \eta) \in \mathcal{D} \,\middle|\, \varphi(|\eta|)\, \|\bar C\, x_e\| \le 1 - \varepsilon, \quad \|x_e\| \le c_7, \quad \eta \in [0, \omega]\right\}.$$

Then $\mathcal{C}$ is a compact subset of $\mathcal{D}$ and contains the trajectory of the maximal solution $t \mapsto (x_e(t), t)$ of (3.17). Therefore, the supposition that ω is finite is false. This completes the proof of Assertions (i)-(iii).

It remains only to establish Assertion (iv). Assume that φ is unbounded. Then $\|e(t)\| < 1/\varphi(t) \to 0$ as $t \to \infty$. By boundedness of k, we have $u(t) = -\nu(k(t))e(t) \to 0$ as $t \to \infty$. □

4 Example

Let (A, b, c) be a single-input, single-output minimum-phase system with positive high-frequency gain $cb > 0$. Assume that the class of reference signals $r : \mathbb{R}_+ \to \mathbb{R}$ comprises all linear combinations of constant functions and sinusoidal functions of period 2π. Choosing as internal model the linear system with transfer function

$$\frac{\beta(s)}{\alpha(s)} = \frac{(s+1)^3}{s(s^2+1)}\,,$$

and selecting the funnel function $t \mapsto \varphi(t) := t^2$, then the feedback

$$u(t) = -k(t)e(t), \quad k(t) = \frac{1}{1-(t^2\, e(t))^2}, \quad e(t) = y(t) - r(t),$$

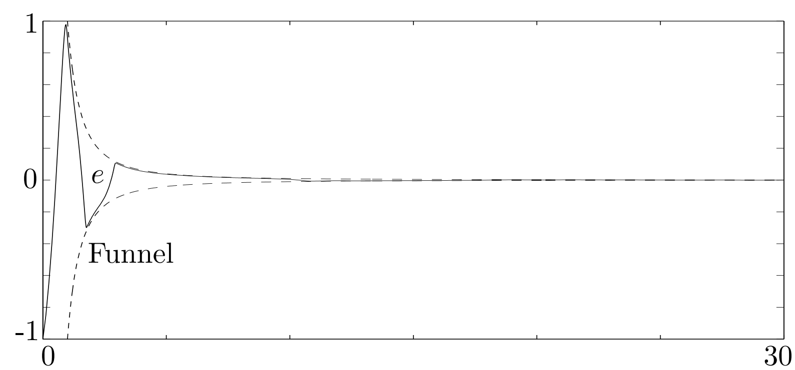
in series with the internal model, ensures asymptotic tracking of every admissible reference signal r and achieves a tracking error decay rate of the order t^{-2}. In the specific case

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad c = [1 \;\; 0 \;\; 0]\,,$$

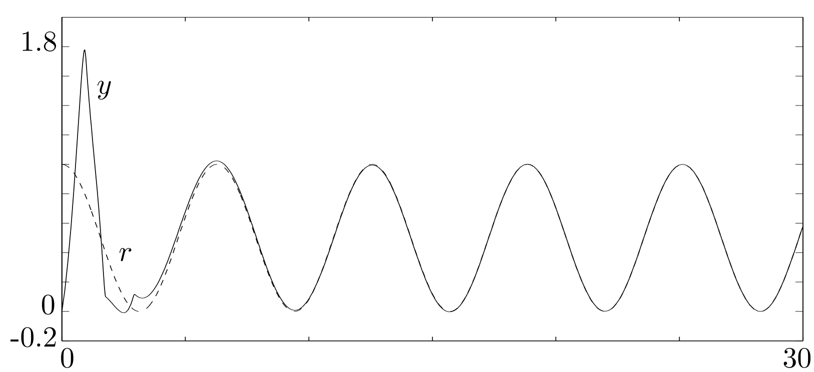
with zero initial conditions and reference signal

$$r : t \mapsto \frac{1}{2}\big[1 + \cos t\big]\,,$$

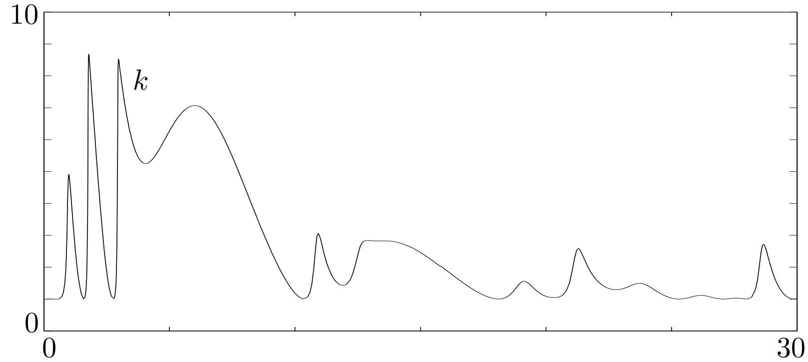
the behaviour of the feedback system is depicted in Figure 3(a-d).



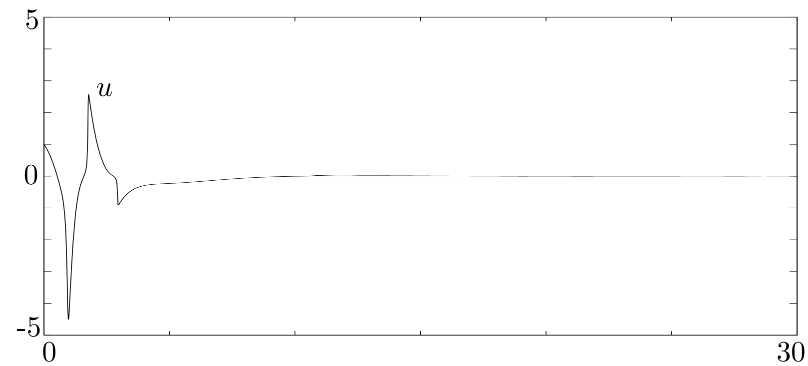
(a) The funnel and tracking error e



(b) The reference r and output y



(c) The gain function k



(d) The control u

Figure 3: Example

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