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Time-varying polynomial matrix systems

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Differential polynomial matrices with coefficients meromorphic on an open interval $I \subseteq \mathbb{R}$ are studied first. The subclass of full matrices P (having the properties P is non-singular and every local solution of $Pf=0$ extends analytically to I) forms a sublattice with respect to right division of square matrices which is anti-isomorphic to the lattice of the corresponding solution spaces. This and further properties are then exploited in the study of time-varying systems in differential operator representation. Results on equivalence, state space models, controllability/observability and transfer functions are derived.

Nomenclature

$\mathscr{M}(\mathscr{M}_I)$	Set of meromorphic functions defined on $\mathbb{R}$ (on I).
$\mathscr{A}(\mathscr{A}_I)$	Set of analytic functions defined on $\mathbb{R}$ (on I).
$\mathscr{C}^\infty$	Set of infinitely differentiable functions.
$\mathscr{F}_I$, σ	Definition 2.1.
$\ker_{\mathscr{F},I} P$	Definition 3.5.
'full'	Definition 3.7.
$\mathscr{U}, \mathscr{X}, \mathscr{Y}$	See eqn. (1.3).
P_c	Theorem 3.1.
$P \sim Q$	Section 3.
rk P	Definition 3.2.
gcrd (p, q), lclm (p, q)	Lemma 2.6.
gcrd (P, Q), lclm (P, Q)	Lemma 3.11.
$\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2$	Definition 5.1.
$\Sigma_{\text{st}}{}^1 \overset{\text{s}}{\sim} \Sigma_{\text{st}}{}^2$	Example 5.2.
ord P	Definition 3.2.

1. Introduction

Our motivation for an algebraic treatment of time-varying systems in differential operator representation of the type

$$\left.\begin{aligned} Pz &= Qu \\ y &= Vz + Wu \end{aligned}\right\} \tag{1.1}$$

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(the specifications valid in this paper are given in (1.3)) is whether a polynomial framework exists which is suitable for the study of such systems. In the time-invariant case such a framework arose from the work of Rosenbrock (1970) and Wolovich (1974) and is well established.

In the few existing articles on the subject the entries of the differential polynomial matrices are usually considered as members of some skew polynomial ring $\mathscr{M}[D]$ and the coefficients of the polynomials belong to some differentially closed ring of functions $\mathscr{M}$ or generalizations of such a ring.

The choice of $\mathscr{M}$ represents a main decision with regard to the chances for a successful treatment of systems described by (1.1) and to the applicability of the results.

Ylinen (1975) collects the basic algebraic results necessary for a thorough analysis of the equation

$$Az = Bu \tag{1.2}$$

in the case where $\mathscr{M}$ is a ring of endomorphisms and A, B are matrices over $\mathscr{M}[D]$. His main source is Cohn (1971). He also discusses basic system theoretic problems (mainly transfer matrices, minimal realization, interconnections and observability). The concrete results still suffer from restrictive assumptions which in situations of interest turn out to be unrealistic in the time-varying case (e.g. Ylinen (1975), proposition 26, p. 61 ; proposition 37, p. 63)). Ylinen only considers equations of type (1.2), a special case of (1.1), but of course one can also consider (1.1) as a special case of (1.2) if we let

$$A = \begin{bmatrix} P & 0 \\ -V & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} Q \\ W \end{bmatrix}$$

Therefore some results also have a meaning for (1.1). Thus a right coprimeness criterion on P and V for observability of (1.1) is hidden in the discussion of Ylinen (1975, pp. 78–81).

Kamen (1976) assumes, for his main result, that $\mathscr{M}$ is noetherian. Under this hypothesis he constructs a state space representation for (1.2) with monic A. The Noether condition is required essentially to ensure that monic polynomials form an appropriate denominator set (see Cohn (1971)) thus making possible the inversion of A in (1.2) and the reduction to the scalar case. The Noether condition seems to be rather restrictive for the polynomial coefficients (see examples given by Kamen (1976)). The ring of all analytic functions is not noetherian. We mention also that, in general, monic polynomials do not form a denominator set.

In another and more recent report by Ylinen (1980), he concentrates mainly on the situation where $\mathscr{M}$ is a subring of $\mathscr{C}^\infty$ the space of infinitely differentiable complex-valued functions on an open real interval, and thus obtains more precise results for systems of type (1.2) and their interconnections. In addition to the topics in his report from 1975, controllability is now also treated and a coprimeness criterion similar to the one known from the time-invariant case is approached and partially established. The main restrictions required for the substantial results in Ylinen (1980) are : $\mathscr{M}$ must not contain zero-divisors of $\mathscr{C}^\infty$ and the composite matrix $[A \,\vdots\, -B]$ and all its right factors of the same format must be row equivalent to a matrix in upper triangular form with coefficients also in $\mathscr{M}$ and monic diagonal elements. It can be shown (see Example 3.9 (*c*)) that for a non-singular $n\times n$ matrix A with analytic coefficients, A only has solutions which can be analytically continued to all of $\mathbb{R}$. Such differential polynomial matrices will be called full (see Definition 3.7).

In fact the possibility for a time-varying differential polynomial matrix not to have ' sufficiently ' many solutions represents a main difference to the time-invariant case where the dimension of the solution space of a non-singular differential polynomial matrix is prescribed only by the degree of the determinant (see, for example, Hinrichsen and Prätzel-Wolters (1980)).

For this reason, in §§ 2 and 3 we study differential polynomials and non-singular polynomial matrices with full kernels. We will call such operators themselves full. In this paper we choose $\mathscr{M}$ to be the field of real-valued meromorphic functions defined on a fixed open interval of real numbers. The main result is : The principal left ideals of $n\times n$ differential polynomial matrices with a full generator form a sublattice of the lattice of all principal left ideals and this sublattice is anti-isomorphic to the lattice of finite dimensional real subvector spaces of $\mathscr{M}^n$. The correspondence is given as follows :

$$P\in\mathscr{M}[D]^{n\times n}\text{ full}\leftrightarrow\text{kernel of }P\text{ as an operator on }(\mathscr{C}^\infty)^n$$

We will also see that in the scalar case ($n=1$) the set of full polynomials of equal degree just represents a similarity class in the algebraic sense introduced by Ore (1933) (see also Cohn (1971)).

Thus it becomes clear that the notion ' full ' singles out a very natural subclass of differential operators. Full operators are important system theoretically because they lead to system trajectories which do not interrupt within the time interval under consideration.

In § 4 we begin to analyse the trajectory spaces described by (1.1) with full operator P and the additional condition im $Q\subseteq$ im P. The latter condition of course guarantees the existence of forced motions for any admitted u.

The specifications for (1.1) in detail will be

$$Pz=Qu,\quad Y=Vz+Wu$$

where

$$P\in\mathscr{M}[D]^{r\times r},\quad Q\in\mathscr{M}[D]^{r\times m},\quad V\in\mathscr{M}[D]^{p\times r},\quad W\in\mathscr{M}[D]^{p\times m}$$

$$u\in\mathscr{U}^m:=\{u\in(\mathscr{C}^\infty)^m\,|\,\text{supp }u\text{ bounded to the left}\}$$

$$z\in\mathscr{Z}^r:=(\mathscr{C}^\infty)^r$$

$$y\in\mathscr{Y}^p:=(\mathscr{C}^\infty)^p$$

The matrix

$$\mathbb{P}=\begin{bmatrix}P & -Q\\ V & W\end{bmatrix}\in\mathscr{M}[D]^{(r+p)\times(r+m)}\tag{1.3}$$

is called the system matrix corresponding to the above equations.

Remarks

(*a*) We do not admit piecewise continuous functions or distributions as inputs since the presence of jumps would destroy the following theory (cf. § 2). As a consequence concatenation of inputs, a common axiom in system theory (see Kalman *et al.* (1969)), is not present for arbitrary inputs in our context.

(*b*) The restriction $y \in (\mathscr{C}^{\infty})^p$ for the outputs is not essential. If poles occur they must be caused by the coefficients of V and W. Thus left multiplication by an invertible $p \times p$ diagonal matrix with components in $\mathscr{M}$ leads to a new set of equations producing no poles in the output. Mathematically there is no reason not to admit poles in the output y or the internal motions z, and nearly all results which follow are also valid if we admit quotients f/a where f is in $\mathscr{C}^{\infty}$ and a is non-zero and analytic.

In § 5 the concept of system equivalence is introduced. The equivalence classes are studied in detail and a state space representation is given. Section 6 discusses controllability and observability which are characterized by coprimeness conditions. Finally, in § 7 transfer functions are introduced. As in the time-invariant case the transfer function determines a system of type (1.3) up to equivalence if controllability and observability are imposed.

2. The full polynomials of $\mathscr{M}[D]$

Let $\mathscr{A}(\mathscr{M})$ be the $\mathbb{R}$-algebra of real-valued analytic (meromorphic) functions defined in $\mathbb{R}$. The ordinary differentiation operator D defined on $\mathscr{A}$ extends uniquely to all of $\mathscr{M}$ and therefore will be considered as a special endomorphism of $\mathscr{M}$, i.e. an element of End $(\mathscr{M})$, the algebra of $\mathbb{R}$-linear maps from $\mathscr{M}$ to $\mathscr{M}$. As usual $\mathscr{M}$ itself can be considered as a subset of $\mathscr{M}$ and one notes that D is an indeterminate over $\mathscr{M}$.

We denote by $\mathscr{M}[D]$ the set of (left-)polynomials

$$\sum_{i=0}^{n} f_i D^i$$

in D with coefficients f_i out of $\mathscr{M}$. $\mathscr{M}[D]$ is an $\mathbb{R}$-vector space. Considering also the multiplication in End $(\mathscr{M})$, we arrive at the skew polynomial ring $\mathscr{M}[D]$ (cf. Cohn (1971, pp. 34 and 64)) with the basic multiplication rule

$$Df = fD + \dot{f} \quad \text{for every } f \in \mathscr{M} \ (\dot{f} \text{ means } D(f))$$

It is easily proved that $\mathscr{M}[D]$ is a left- and right-euclidean domain with no non-trivial two-sided ideals (i.e. $\mathscr{M}[D]$ is simple) (see Cozzens and Faith (1975, p. 43)). We call the elements of $\mathscr{M}[D]$ differential polynomials. The differential polynomials with all coefficients in $\mathscr{A}$ form a subring which we denote by $\mathscr{A}[D]$.

If p is a differential polynomial, we are interested in the solutions $f \in \mathscr{M}$ of the equation $pf = 0$, or equivalently in the kernel of the endomorphism p. (Note that pf means $p(f)$, p viewed as an endomorphism of $\mathscr{M}$. Clearly the product pf also has a meaning within $\mathscr{M}[D]$ but the actual interpretation will always be clear from the context.) For technical reasons we introduce a slightly more general notation.

2.1. *Definition*

(*a*) Let I be an open real-interval. Then by $\mathscr{A}_I(\mathscr{M}_I)$ we denote the algebra of real-valued analytic (meromorphic) functions defined on I. (Note that $\mathscr{A}_I$ does not consist of the restriction of $\mathscr{A}$ to I only.)

(*b*) By $\mathscr{F}_I$ (or simply $\mathscr{F}$ if $I=\mathbb{R}$) we will always mean a member of a family $(\mathscr{F}_I)_{I\in O}$, where O is the set of real open intervals, which has to fulfil the following two properties: $\mathscr{A}_I \subseteq \mathscr{F}_I \subseteq \mathscr{M}_I$ for every $I\in O$ and $J\subseteq I \Rightarrow \mathscr{F}_{I|_J} \subseteq \mathscr{F}_J$. Here $\mathscr{F}_{I|_J}$ denotes the space of functions taken from $\mathscr{F}_I$ and restricted to J.

(*c*) Finally for $p\in\mathscr{M}_I[D]$ let $\ker_{\mathscr{F},I} p := \{f\in\mathscr{F}_I \mid pf=0\}$ where we omit I if $I=\mathbb{R}$.

If the leading coefficient of $p\in\mathscr{A}_I[D]$ is a unit in $\mathscr{A}_I$, the theory of ordinary differential equations (Herold 1975) guarantees that $\ker_{\mathscr{F},I} p$ is an n-dimensional subspace of $\mathscr{A}_I$, $n=\deg p$ being the degree of p. In general one can only be sure of the following obvious property.

2.2. *Property*

For $p\in\mathscr{M}[D]$, $p\neq 0$ and $I\in O$ we always have

$$\dim \ker_{\mathscr{F},I} p \leqslant \deg p$$

The example $p := tD$ demonstrates that the admission of distributions as solutions of the equation $pf=0$ would lead to a violation of Property 2.2 (Gel'Fand and Shilov (1969, p. 42)).

The examples $p_1=tD^2+D$, $p_2=tD+1$ and $p_3=t^2D+1$ illustrate some situations where equality does not hold in Property 2.2.

In detail, let $I\in O$ and $0\notin I$. Then $\ker_{\mathscr{M},I} p_1=\langle 1, \ln|t|\rangle_{\mathbb{R}}$ and $\ker_{\mathscr{M}} p_1=\langle 1\rangle_{\mathbb{R}}$; $\ker_{\mathscr{A},I} p_2=\ker_{\mathscr{M},I} p_2=\langle 1/t\rangle_{\mathbb{R}}$ but $\{0\}=\ker_{\mathscr{A}} p_2 \subsetneq \ker_{\mathscr{M}} p_2=\langle 1/t\rangle_{\mathbb{R}}$; $\ker_{\mathscr{M}} p_3=\{0\}$ but $\ker_{\mathscr{M},I} p_3=\langle \exp(1/t)\rangle_{\mathbb{R}}$.

If the differential polynomials p are to describe internal motions of a dynamical system one cannot expect to get reasonable results admitting any kind of such polynomials. The full polynomials are of main interest and are defined as follows.

2.3. *Definition*

$p\in\mathscr{M}[D]$ is called *full* (with reference to (wrt) $\mathscr{F}_I$) if $p\neq 0$ and $\dim\ker_{\mathscr{F},I} p = \deg p$.

Monic polynomials out of $\mathscr{A}[D]$ are examples of full polynomials wrt $\mathscr{A}$. Also for $f\in\mathscr{M}$, the polynomial $fD-\dot{f}$ is full wrt $\mathscr{M}$ since $\ker_{\mathscr{M}}(fD-\dot{f})=\langle f\rangle_{\mathbb{R}}$. A complete description of full differential polynomials will be given below.

Since we consider only meromorphic solutions the identity theorem for $\mathscr{A}$ extended to $\mathscr{M}$ causes the injectivity of the restriction mappings $\pi_{I,J}$ (for $J\subseteq I$; $J, I\in O$) defined as follows:

$$\pi_{I,J}:\ \ker_{\mathscr{F},I} p \to \ker_{\mathscr{F},J} p$$
$$f \to f|_J$$

In this context the property 'full' just means that surjectivity is also valid.

2.4. *Property*

Let $I\in O$. $p\in\mathscr{M}[D]$. Then p is full wrt $\mathscr{F}_I$ iff $\pi_{I,J}$ is an isomorphism for all $J\subseteq I$, $J\in O$.

In the following sections the next lemma will also become important.

2.5. *Lemma*

Let $p \in \mathcal{M}[D]$ be full wrt $\mathcal{M}$. Then, for $f \in \mathscr{C}^\infty$, $pf = 0$ implies $f \in \mathscr{A}$.

In the case where p is not full $p = t^2D + 1$ is a counter-example of the above lemma since the function f, with $f(t) = \exp(1/t)$ for $t < 0$ and $f(t) = 0$ for $t \geq 0$, is a solution which belongs to $\mathscr{C}^\infty \backslash \mathscr{A}$.

We now turn to the more algebraic properties of differential polynomials and in particular of the full ones. The next basic lemma is due to Ore (1933).

2.6. *Lemma*

For all $p, q \in \mathcal{M}[D]$ a monic greatest common right divisor $g = \text{gcrd}(p, q)$ and a monic least common left multiple $l = \text{lclm}(p, q)$ exist and are unique.

There exist $a, b \in \mathcal{M}[D]$ such that $g = ap + bq$ and the following degree formula is valid.

$$\deg l + \deg g = \deg p + \deg q$$

Analogous results hold if ' left ' and ' right ' are interchanged and if finitely many polynomials are considered.

The algebraic structure of a differential polynomial p depends very much on its kernel. This is expressed by the following lemma already known to Schlesinger (1895, p. 81).

2.7. *Lemma*

Let $p \in \mathcal{M}[D]$ and $0 \neq f \in \ker_{\mathcal{M}} p$. Then p has $fD - \dot{f}$ as a right factor, i.e. $(fD - \dot{f})|_r p$.

Proof

The right euclidean algorithm leads to the equation $p = q(fD - \dot{f}) + r$ where $r \in \mathcal{M}$. Since $pf = 0$ we conclude $r = 0$. □

We illustrate this by an example. Let $p = D^2 + 1$ and $0 \neq f \in \ker_{\mathscr{A}} p = \langle \sin, \cos \rangle_{\mathbb{R}}$. Then $p = f^{-1}D(fD - \dot{f})$.

2.8. *Proposition*

(*a*) Let $p \in \mathcal{M}[D]$ be full wrt $\mathscr{F}$ and $\ker_{\mathscr{F}} p = \langle f_1, \ldots, f_n \rangle_{\mathbb{R}}$. Then for some $u \in \mathcal{M}$ and $l := \underset{1 \leq i \leq n}{\text{lclm}} \{(f_i D - \dot{f}_i)\}$ we have $p = ul$.

(*b*) For any finite-dimensional subspace V of $\mathcal{M}$, there exists a full polynomial $p \in \mathcal{M}[D]$ such that $\ker_{\mathcal{M}} p = V$.

Proof

(*a*) Without restriction assume $f_1, \ldots, f_n$ to be $\mathbb{R}$-linearly independent. Since p is full and by the degree formula in Lemma 2.6 we conclude that $\deg l \leq n = \deg p$. By Property 2.2 we have a $n \leq \deg l$ and therefore $\deg l = \deg p$. Because of Lemma 2.7, l is a right divisor of p. This concludes the proof.

(*b*) Straightforward. □

Further properties of full polynomials follow.

2.9. *Proposition*

(*a*) For $p, q \in \mathscr{M}[D]$ where p is full wrt $\mathscr{F}$ we have $\ker_{\mathscr{F}} p \subseteq \ker_{\mathscr{F}} q$ iff $q = rp$ for some $r \in \mathscr{M}[D]$.

(*b*) If $p = rg$ is full wrt $\mathscr{F}$, then g is full wrt $\mathscr{F}$ and r is full wrt $\mathscr{M}$.

(*c*) lclm and gcrd of full polynomials are again full.

Proof

(*a*) Proposition 2.8 gives a representation for p which by Lemma 2.7 must right divide q if the inclusion of the kernels is valid. The converse is immediate.

(*b*) Assume $p = rg \neq 0$. Let J be such that all occurring polynomials are full wrt $\mathscr{A}_J$ and $\mathscr{A}_J \subseteq \operatorname{im} g$. We have $\ker_{\mathscr{A},J} p = \ker_{\mathscr{A},J} g \oplus V$, where V is an arbitrary complementing vector space. Now $g(V) = \ker_{\mathscr{A},J} r$ and g is injective on V. If p is full then $\ker_{\mathscr{A},J} p = \ker_{\mathscr{F}} p|_J$. Therefore all solutions out of $\ker_{\mathscr{A},J} g$ extend to solutions in $\ker_{\mathscr{F}} g$ and all solutions in $g(V)$ extend to solutions in $\ker_{\mathscr{M}} r$, i.e. r and g are full wrt $\mathscr{M}$ and $\mathscr{F}$ respectively.

(*c*) The statement for the lclm is a consequence of the associativity of this operation (cf. Ore (1933, p. 487)) and Proposition 2.8. The statement for the gcrd is a special case of (*b*). □

The example $p = D$ and $q = D + 1$ where

$$\operatorname{lclm}(p, q) = D^2 + \frac{t^2 - 1}{t} D$$

shows that the multiplicative semigroup of monic polynomials in $\mathscr{A}[D]$ is not closed under the formation gcrd and lclm. On the other hand the set of full polynomials which is closed under the operations lclm and gcrd does not form a multiplicative semigroup. This can be seen by the following example.

Let $p = tD + 1$ and $q = D$. Then $\ker_{\mathscr{M}} q = \langle 1 \rangle_{\mathbb{R}}$ and $\ker_{\mathscr{M}} p = \langle 1/t \rangle_{\mathbb{R}}$ but $\ker_{\mathscr{M},I} pq = \langle 1, \ln |t| \rangle_{\mathbb{R}}$ for every open interval I excluding zero.

For differential polynomials Ore (1933, p. 488) introduced the notion of similarity which will become important in § 3 in the context of a canonical form for matrices over $\mathscr{M}[D]$. For this reason we will discuss this notion and study its meaning for full differential polynomials.

2.10. *Definition*

$p, q \in \mathscr{M}[D]$ are called *similar* if they can be put in a *coprime relation*, i.e. if $pa = bq$ for some $a, b \in \mathscr{M}[D]$ where in addition p, b are left coprime and a, q are right coprime.

2.11. *Lemma*

Let $p, q \in \mathscr{M}[D]$.

(*a*) p, q are similar iff $\mathscr{M}[D]/p\mathscr{M}[D]$ is isomorphic to $\mathscr{M}[D]/q\mathscr{M}[D]$ as an $\mathscr{M}[D]$ right module.

(*b*) Similarity is an equivalence relation.

(*c*) If p, q are similar then $\deg p = \deg q$.

(*d*) In a coprime relation $pa = bq$, it can always be assumed without restriction that $\deg a < \deg q$ and $\deg b < \deg p$.

(*e*) The differential polynomials p, q of degree one are similar iff they are *associated*, i.e. if $pu = vq$ for some $u, v \in \mathscr{M}$.

Proof

A proof of (a) is given by Cohn (1971, p. 126). Of course (b) is a trivial consequence of (a). Ore (1933, p. 488) gives a direct and more elementary proof of (b).

(c) is also a consequence of (a) but is obtained in a more natural way by Ore (1933, p. 488).

To prove (d) we apply the euclidean right-algorithm as follows: $a = rq + a'$ where $\deg a' < \deg q$. This changes the given coprime relation to $pa' = b'q$ where $b' = b - pr$. The new relation is again coprime. The left algorithm now gives $b' = pa + b''$ and leads analogously to a new coprime relation for p and q with the properties required in (d). Finally (d) implies (e). □

For a complete description of some of the similarity classes in Proposition 2.13 we will need the following lemma also of importance in later sections.

2.12. *Lemma*

Let $f_1, \dots, f_n \in \mathscr{M}$ be linearly independent and $h_1, \dots, h_n \in \mathscr{M}$. Then there exists $a \in \mathscr{M}[D]$ with $\deg a = n - 1$ and $af_i = h_i$ for $i = 1, \dots, n$.

Proof

If $a := a_{n-1}D^{n-1} + \dots + a_0$ we have to show that

$$\begin{bmatrix} af_1 \\ \vdots \\ af_n \end{bmatrix} = \underbrace{\begin{bmatrix} f_1^{(n-1)} & \cdots & \dot{f}_1 & f_1 \\ \vdots & & \vdots & \vdots \\ f_n^{(n-1)} & \cdots & \dot{f}_n & f_n \end{bmatrix}}_{=:\,W} \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

has a solution. This is true if $\det W \neq 0$ in $\mathscr{M}$. Considering

$$l = \operatorname*{lclm}_{1 \leqslant i \leqslant n} \{(f_i D - \dot{f}_i)\}$$

we see that, at least locally, $\det W$ is a classical wronskian for the differential equation $lf = 0$. Thus it is non-zero. □

2.13. *Proposition*

Let $q \in \mathscr{M}[D]$ be full wrt $\mathscr{F}$ and $\deg q = n$. Then the similarity class of q consists of all full polynomials of degree n.

Proof

Let $pa = bq$ be a coprime relation and q be full. Recalling Lemma 2.7 we observe that a acts as a monomorphism on $\ker_{\mathscr{F}} q$ since a and q are right coprime. Therefore $\dim \ker_{\mathscr{F}} p \geqslant \dim \ker_{\mathscr{F}} q = \deg q$. By Lemma 2.11 ($c$), p and q have the same degree and therefore by Property 2.2 we conclude that p must be full, too.

It remains for us to demonstrate that any two full polynomials p and q of necessarily equal degree n can be put in a coprime relation. For this, let $f_1, \dots, f_n$ and $h_1, \dots, h_n$ be a basis of $\ker_{\mathscr{F}} q$ and $\ker_{\mathscr{F}} p$ respectively. Applying Lemma 2.12 we can construct $a \in \mathscr{M}[D]$ such that $\deg a = n - 1$ and $af_i = h_i$ for $i = 1, \dots, n$. Now $\ker_{\mathscr{F}} q \subseteq \ker_{\mathscr{F}} pa$. Because of Proposition 2.9 (a) we must have the relation $pa = bq$ for some $b \in \mathscr{M}[D]$. By construction, a and q are right coprime. If $p = lp'$ and $b = lb'$ the validity of $p'a = b'q$ together with the monomorphy of a restricted to $\ker_{\mathscr{F}} q$ would imply $\deg p' = n$ and therefore $l \in \mathscr{M}$. □

We conclude this section with two examples.

2.14. *Examples*

The polynomials $p = D^2$ and $q = D^2 + 1$ are similar but not associated. The latter is easily proved by calculation. It remains to show that the relation $pa = bq$ with

$$a = (t \sin(t) + 2\cos(t))\, D + 2\sin(t) - t\cos(t)$$

and

$$b = (t \sin(t) + 2\cos(t))\, D + t\cos(t)$$

is coprime. Assume a and q have a common right divisor which has (without restriction since q is full) the form $fD - \dot{f}$, $0 \neq f \in \ker_{\mathscr{F}} q$. Then there exist $x, y \in \mathbb{R}$ such that

$$f = x \sin(t) + y \cos(t) \in \ker_{\mathscr{F}} p$$

From $af = 0$ we deduce that $x = y = 0$. To show that p and b are left coprime we use the same method as above and notice that D as a left operator has the form $f \mapsto -\dot{f}$.

If two full polynomials $p, q \in \mathscr{M}[D]$ are in a coprime relation $pa = bq$ then a, b are not necessarily full. To see this, let $q = D^2$ and $a = t^2 D + 1$. Clearly a is not full wrt $\mathscr{M}$. For these polynomials the relation $pa = bq = \mathrm{lclm}\,(a, q)$ is coprime and p is full wrt $\mathscr{M}$.

Several further results on full polynomials not mentioned here are special cases of results in § 3.

3. The lattice of full matrices over $\mathscr{M}[D]$

As in the case of polynomials we shall see that also for matrices over $\mathscr{M}[D]$ the property that a ' reasonable ' kernel exists will single out the subclass of ' full matrices ' (see Definition 3.7) which is studied in detail.

The main result will be that full matrices form a lattice which, by considering kernels, becomes (anti-)isomorphic to a lattice of finite-dimensional function spaces.

The set $\mathscr{M}[D]^{m \times n}$ of matrices over $\mathscr{M}[D]$ is naturally a left and right $\mathscr{M}[D]$ module and in addition has a ring structure if $m = n$.

At first we discuss a generalization of the Smith form and the normed upper triangular form, known from the commutative theory, which both represent an important tool in the subsequent analysis.

Two matrices $P, Q \in \mathscr{M}[D]^{m \times n}$ are called *equivalent*, written $P \sim Q$, if $P = UQV$ for some invertible $U \in \mathscr{M}[D]^{m \times m}$ and $V \in \mathscr{M}[D]^{n \times n}$.

3.1. *Theorem*

(*a*) Every matrix $P \in \mathscr{M}[D]^{m \times n}$ is equivalent to a matrix

$$P_c = \mathrm{diag}\,\underbrace{(1, \ldots, 1, p}_{=:\, l}, 0, \ldots, 0), \quad p \in \mathscr{M}[D]$$

p is uniquely determined by P up to similarity.

(*b*) If $l > 1$, p can be chosen arbitrarily within its similarity class. We call P_c a *canonical form* of P.

The proof for a more general situation is given by Cohn (1971, p. 288). The result simplifies considerably for the ring $\mathscr{M}[D]$ since it is a simple one.

In general the elements of a canonical form are not unique up to left and right multiplication by units, as claimed by Ylinen (1975, p. 46). To see this we construct two equivalent canonical matrices whose elements are not pairwise associated. We choose $p, q \in \mathscr{M}[D]$, as in Example 2.14, which satisfy the coprime relation $pa = bq$ and are not associated. By Cohn (1971, p. 89) there exist $r, s, v, w \in \mathscr{M}[D]$ such that

$$U = \begin{bmatrix} p & b \\ r & s \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} v & -a \\ w & q \end{bmatrix}$$

Since $-ra + sq = 1$ for

$$P = \begin{bmatrix} 1 & -a \\ 0 & q \end{bmatrix}$$

we have

$$UP = \begin{bmatrix} p & 0 \\ r & 1 \end{bmatrix}$$

Since U is invertible we conclude

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \sim UP \sim P \sim \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$$

3.2. *Definition*

Given the situation of Theorem 3.1. The degree of p is called the order of P, ord P. The number of non-zero elements in P_c is called the rank of P, rk P. P is called *non-singular* if rk $P = \min(m, n)$.

The rank just defined has the familiar property that it equals the left column rank and right row rank of P. See Cohn (1971, p. 194) for details.

3.3. *Lemma*

Every matrix $P \in \mathscr{M}[D]^{m \times n}$ can be transformed by multiplication from the left by an invertible $U \in \mathscr{M}[D]^{m \times m}$ into a *normed upper triangular matrix*, i.e. the first non-zero element of every row is monic and the entries above it are of lower degree. An analogous result, which considers the transformation from the right, can be obtained as, in addition, can various further triangular forms if row and column permutations are considered.

The proof is completely analogous to the commutative case (Newman 1972, p. 15). We can use the canonical forms to derive the following rule.

3.4. *Cancellation rule*

Let $P \in \mathscr{M}[D]^{n \times m}$ be non-singular, $n \leqslant m$ $(n \geqslant m)$ and $A, B \in \mathscr{M}[D]^{r \times n}$ $(\in \mathscr{M}[D]^{m \times r})$. Then $AP = BP$ $(PA = PB)$ implies $A = B$.

From this rule it is easily seen that if a square matrix over $\mathscr{M}[D]$ has a right (left) inverse it is a left (right) inverse, too.

We now begin the study of the kernels for matrices out of $\mathscr{M}[D]^{r\times r}$ and the extension of the notion ' full ' to the matrix case.

3.5. *Definition*

Let $(\mathscr{F}_I)_{I\in O}$ be as in Definition 2.1 and $P\in\mathscr{M}_I[D]^{r\times r}$. Then $\ker_{\mathscr{F},I} P := \{f\in\mathscr{F}_I{}^r \mid Pf=0\}$. If $I=\mathbb{R}$ we omit I.

3.6. *Elementary properties of the kernels*

For $P, Q, R\in\mathscr{M}[D]^{r\times r}$, $P\sim P_c=\operatorname{diag}(1, \ldots, 1, p)$, $p\neq 0$ and $I\in O$ we have

(*a*) $\dim\ker_{\mathscr{F},I} P \leqslant \dim\ker_{\mathscr{F},J} P$ for $J\subseteq I$, $J\in O$.
(*b*) $\deg p \geqslant \dim\ker_{\mathscr{M},I} p = \dim\ker_{\mathscr{M},I} P_c = \dim\ker_{\mathscr{M},I} P$.
(*c*) $\operatorname{rk} Q = r$ iff $\dim\ker_{\mathscr{F},I} Q < \infty$.
(*d*) There exists $J\in O$ such that

$$P:\ \mathscr{A}_J{}^r \to \mathscr{A}_J{}^r$$
$$f \mapsto Pf$$

is surjective.

(*e*) For $R=QP$ we have

$$\dim\ker_{\mathscr{F},I} R \leqslant \dim\ker_{\mathscr{M},I} Q + \dim\ker_{\mathscr{F},I} P$$

If $\ker_{\mathscr{M},I} Q \subseteq \operatorname{im}(P|_I)$ then equality holds.

Proof

(*a*)–(*c*) is immediate.

(*d*) Without restriction we consider P_c and p to be monic. In an interval J where the coefficients of p are analytic the operator p restricted to $\mathscr{A}_J$ is surjective.

(*e*) Let V be a finite dimensional subvector space of $\mathscr{F}_I{}^r$ with $\ker_{\mathscr{F},I} R = \ker_{\mathscr{F},I} P \oplus V$. Now P acts injectively on V and $P(V)\subseteq\ker_{\mathscr{M},I} Q$. □

3.7. *Definition*

A non-singular $P\in\mathscr{M}_I[D]^{r\times r}$ is called *full* wrt $\mathscr{F}_I$ if the map

$$\pi_{I,J}:\ \ker_{\mathscr{F},I} P \to \ker_{\mathscr{F},J} P$$
$$f \mapsto f|_J$$

is surjective for every open interval $J\subseteq I$.

Note that $\pi_{I,J}$ is always injective.

We first derive basic properties of full differential polynomial matrices and then give some examples.

3.8. *Proposition*

For $P, Q\in\mathscr{M}[D]^{r\times r}$ and p as defined by Theorem 3.1 (*a*) we have

(*a*) If $P\sim Q$ and P is full wrt $\mathscr{M}$ then Q is full wrt $\mathscr{M}$, too.
(*b*) P is full wrt $\mathscr{F}$ iff $\dim\ker_{\mathscr{F}} P = \deg p = \operatorname{ord} P$.

(c) Let P be full wrt $\mathscr{F}$, $\dim \ker_{\mathscr{F}} P = n$ and $r > 1$. Then $P \sim \operatorname{diag}(1, \ldots, 1, D^n)$.

(d) If $P \in \mathbb{R}[D]^{r\times r}$ is non-singular, and $r > 1$ then in $\mathscr{M}[D]^{r\times r}$ we have $P \sim \operatorname{diag}(1, \ldots, 1, D^n)$ where $n = \deg \det P$.

Proof

We use the notation of Theorem 3.1.

(a) follows by Definition 3.7.

(b) '$\Rightarrow$': There exists an open interval $J \subseteq \mathbb{R}$ such that $\dim \ker_{\mathscr{F},J} P_c = \deg p$. By Property 3.6 (b) we have $\dim \ker_{\mathscr{F},J} P_c \leqslant \dim \ker_{\mathscr{U},J} P \leqslant \deg p$. Since P is full the assertion follows. '$\Leftarrow$': Once more by Property 3.6 (b) we have for every open interval $J \subseteq \mathbb{R}$,

$$\deg p \geqslant \dim \ker_{\mathscr{F},J} P \geqslant \dim \ker_{\mathscr{F}} P = \deg p$$

(c) By Proposition 2.13 and Theorem 3.1 for $r > 1$, p as a full polynomial can be chosen arbitrarily of degree n.

(d) Since any non-singular matrix P out of $\mathbb{R}[D]^{r\times r}$ is of course full (see Example 3.9 (a) with ord $P = \deg \det P = n$ any matrix out of $\mathbb{R}[D]^{r\times r}$ whose determinant has degree n is in the equivalence class of P. □

3.9. *Examples of full polynomial matrices*

(a) Let $P \in \mathbb{R}[D]^{r\times r}$ be non-singular. Then P is full wrt $\mathscr{A}$ and ord $P = \deg \det P$.

(b) Let $P \in (\mathscr{A}^{r\times r})[D]$ be monic. Then P is full wrt $\mathscr{A}$.

(c) Let $P \in \mathscr{A}[D]^{r\times r}$ be non-singular and in normed upper triangular form as described in Lemma 3.3. Then P is full wrt $\mathscr{A}$.

Of course there are full matrices which are not of the type (a), (b) or (c).

Proof

(a) We can transform P into Smith form $P_s = \operatorname{diag}(p_1, \ldots, p_r)$ by unimodular matrices over $\mathbb{R}[D]$. Since

$$\dim \ker_{\mathscr{A}} P = \dim \ker_{\mathscr{A}} P_s = \sum_{i=1}^{r} \deg p_i$$

the assertion follows.

(b) In the same way as a scalar differential equation is transformed to a first-order vector differential equation we can transform the equation $Pf = 0$ to an equation $(DI_{rn} - B)g = 0$, $B \in \mathscr{A}^{rn\times rn}$ such that the solution spaces are isomorphic. Now the assertion follows.

(c) A matrix $U \in \mathscr{A}[D]^{r\times r}$ with inverse in $\mathscr{A}[D]^{r\times r}$ can be constructed such that the entries of $P' = PU$ satisfy the following conditions for $1 \leqslant i, j \leqslant r$: $p'_{ij} = 0$ for $i > j$; $p'_{ii} = p_{ii}$; $\deg p'_{ij} < \min(\deg p'_{ii}, \deg p'_{jj})$, $i \neq j$. Let $s_0 = \max_{1\leqslant i\leqslant r} \deg p'_{ii}$, $s_i = \deg p'_{ii}$ for $1 \leqslant i \leqslant r$ and $Q = \operatorname{diag}(D^{s_0 - s_1}, \ldots, D^{s_0 - s_r})$. Then QP' is a monic element of $(\mathscr{A}^{r\times r})[D]$ and thus full wrt $\mathscr{A}$ by (b). The proof becomes complete if we apply the following proposition. □

3.10. *Proposition*

Let P, R, $G \in \mathscr{M}[D]^{r\times r}$ and $P = RG$.

(a) If P is full wrt $\mathscr{F}$ then G is full wrt $\mathscr{F}$ and R is full wrt $\mathscr{M}$.

(b) If P is non-singular we have

$$\text{ord } P = \text{ord } R + \text{ord } G$$

Proof

Use canonical form and the same arguments as in the proof of Proposition 2.9 (b). □

3.11. *Lemma* (and definition)

For $P \in \mathscr{M}[D]^{r_1\times r}$ and $Q \in \mathscr{M}[D]^{r_2\times r}$ we have the following.

(a) Let $r_1 = r$. In $\mathscr{M}[D]^{r\times r}$ there exists a greatest common right divisor (gcrd) G of P and Q which can be written as $G = AP + BQ$ for some $A \in \mathscr{M}[D]^{r\times r_1}$ and $B \in \mathscr{M}[D]^{r\times r_2}$. If P or Q is non-singular, a gcrd of P and Q is unique modulo left multiplication by an invertible matrix and non-singular too.

(b) P and Q are called *right coprime* if there exist $S \in \mathscr{M}[D]^{r\times r_2}$ and $T \in \mathscr{M}[D]^{r\times r_1}$ such that $I_r = TP + SQ$. Clearly, P and Q are right coprime iff every square crd of P and Q is invertible.

(c) Let $J \in O$. Then P and Q are right coprime iff $P|_J$ and $Q|_J$ are right coprime in $\mathscr{M}_J[D]^{r\times r}$.

(d) Let $r_1 = r_2 = r$ and P, Q be non-singular. Then there exists a non-singular least common left multiple (lclm) in $\mathscr{M}[D]^{r\times r}$ of P and Q which is unique modulo multiplication from the left by an invertible matrix.

Analogous results hold if ‘ left ’ and ‘ right ’ are interchanged.

Proof

(a) To prove the existence of a gcrd we can slightly modify the results of MacDuffee (1956, p. 35) who proved the statement for matrices over a commutative principal ideal domain. Within the same framework the uniqueness is proved by using Cancellation Rule 3.4.

(b) Use (a).

(c) Let $P|_J$ and $Q|_J$ be right coprime in $\mathscr{M}_J[D]^{r\times r}$ and G be a gcrd of P and Q. By (a) there exist A, B such that $G = AP + BQ$. A canonical form of G is diag $(1, \ldots, 1, p)$. Since G is invertible on J, $p \in \mathscr{M}\backslash\{0\}$ and by the identity theorem G is invertible.

(d) By Lemma 3.3 there exists an invertible

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \quad \text{and} \quad U^{-1} = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$$

such that

$$U \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}$$

A slightly modified version of MacDuffee (1956, p. 36) gives: $U_3P = -U_4Q =: L$ is an lclm of P and Q. Now for every $x \in \mathscr{M}[D]^{1\times r}$ the equations $xL = xU_3P = 0 = -xU_4Q$ imply $xU_3 = 0 = -xU_4$. Since $U_3V_2 + U_4V_4 = I_r$ necessarily $x = 0$ and thus L is non-singular. □

By Proposition 3.10 we already know that especially a gcrd of two polynomial matrices, one of them being full, is full too. In order to prove that an lclm of two full matrices is also full we need the following two propositions, which are of importance on their own.

3.12. *Proposition*

For $P \in \mathscr{M}[D]^{r\times r}$ full wrt $\mathscr{F}$ and $Q \in \mathscr{M}[D]^{r_1\times r}$ the following holds. $\ker_{\mathscr{F}} P \subseteq \ker_{\mathscr{F}} Q$ iff there exists $R \in \mathscr{M}[D]^{r_1\times r}$ such that $Q = RP$.

Proof

By Proposition 2.9 the result is obvious for $r = 1$, since $\mathscr{M}[D]$ is euclidian. Therefore let $r > 1$. We have to show '$\Rightarrow$' while the converse is trivial.

Choose two invertible matrices U and V such that $P = UP_cV$ where $P_c = \text{diag}(1, \ldots, 1, D^n)$. Since P is full we have for every $f \in \ker_{\mathscr{F}} P_c$ that $V^{-1}f \in \ker_{\mathscr{F}} P$ and with the assumption it follows that $\ker_{\mathscr{F}} P_c \subseteq \ker_{\mathscr{F}} QV^{-1}$. Since $\ker_{\mathscr{F}} D^n \subseteq \ker_{\mathscr{F}} (QV^{-1})_{ir}$ for $i = 1, \ldots, r_1$, by Proposition 2.9 $t_1, \ldots, t_{r_1} \in \mathscr{M}[D]$ such that $(QV^{-1})_{ir} = t_iD^n$. Let $(QV^{-1})_j$, $j = 1, \ldots, r$, denote the columns of QV^{-1}. Then we have

$$\underbrace{[(QV^{-1})_1, \ldots, (QV^{-1})_{r-1}, (t_1, \ldots, t_{r_1})^{\mathrm{T}}]}_{=:\,\tilde{T}} \text{diag}(1, \ldots, 1, D^n) = QV^{-1}$$

Therefore $(\tilde{T}U^{-1})UP_cV = Q$. □

3.13. *Proposition*

Let V be a finite n-dimensional subvector space of $\mathscr{F}^r$. Then $P \in \mathscr{M}[D]^{r\times r}$ full wrt $\mathscr{F}$ can be constructed such that $\ker_{\mathscr{F}} P = V$.

Proof

Let $V = \langle f_1, \ldots, f_n\rangle \subseteq \mathscr{F}^r$ with $f_1, \ldots, f_n$ linearly independent and let

$$A := \begin{bmatrix} f_{11} & \cdots & f_{n1} \\ \vdots & & \vdots \\ f_{1r} & \cdots & f_{nr} \end{bmatrix}$$

Without restriction assume that the first row of A is not zero otherwise multiply A from the left by an invertible matrix. Choose an $\mathbb{R}$ basis of the first row entries and by multiplication from the right by an invertible matrix $U_1 \in \mathbb{R}^{n\times n}$ we get

$$AU_1 = \begin{bmatrix} g_{11} & \cdots & g_{i_1,1} & 0 & \cdots & 0 \\ g_{12} & & \cdots & & & g_{n2} \\ \vdots & & & & & \vdots \\ g_{1r} & & \cdots & & & g_{nr} \end{bmatrix}$$

with $g_{11}, \ldots, g_{i_1,1}$ linearly independent. Of course the columns of AU_1 are still a basis of V. By Lemma 2.12 there exists $p_2 \in \mathcal{M}[D]$ such that $p_2 g_{k,1} = g_{k2}$ for $1 \leqslant k \leqslant i_1$.

Therefore with

$$P_2 = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ -p_2 & & \ddots & \\ 0 & & & \ddots \\ & & & 1 \end{bmatrix} \in \mathcal{M}[D]^{r\times r}$$

we have

$$P_2(AU_1) = \begin{bmatrix} g_{11} & \cdots & g_{i,1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & g_{i_1+1,2} & \cdots & g_{n2} \\ & & & * & & \end{bmatrix}$$

Defining $p_3, \ldots, p_r \in \mathcal{M}[D]$ and $P_3, \ldots, P_r \in \mathcal{M}[D]^{r\times r}$ in a similar way we get

$$(P_r \ldots P_2)(AU_1) = \begin{bmatrix} g_{11} & \cdots & g_{i_1,1} & 0 & \cdots & 0 \\ & & & g_{i_1+1,2} & \cdots & g_{n2} \\ & 0 & & \vdots & & \vdots \\ & & & g_{i_1+1,r} & \cdots & g_{nr} \end{bmatrix}$$

Applying this procedure successively on the remaining submatrices we finally find an invertible matrix $\tilde{P} \in \mathcal{M}[D]^{r\times r}$ and an invertible matrix $\tilde{U} \in \mathbb{R}^{n\times n}$ such that

$$\tilde{P}(A\tilde{U}) = \begin{bmatrix} g_{11} & \cdots & g_{i_1,1} & 0 & \cdots & & & 0 \\ & & & g_{1,2}{}^{(2)} & \cdots & g_{i_1,2}{}^{(2)} & 0 & \\ & & 0 & & \ddots & & \ddots & \\ & & & & & g_{1,k}{}^{(k)} & \cdots & g_{i_k,k}{}^{(k)} \\ 0 & & & \cdots & & & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & \cdots & & & & 0 \end{bmatrix}$$

for $1 \leqslant k \leqslant r$, and the elements $g_{c,j}{}^{(j)}$ in every row are linearly independent. Now define for $1 \leqslant j \leqslant k$

$$q_j = \underset{1 \leqslant c \leqslant i_j}{\operatorname{lclm}} \left(g_{c,j}{}^{(j)} D - \dot{g}_{c,j}{}^{(j)}\right)$$

For $Q := \operatorname{diag}(q_1, \ldots, q_k, 1, \ldots, 1) \in \mathcal{M}[D]^{r\times r}$ we have $Q\tilde{P}(A\tilde{U}) = 0$. Since Q is full wrt $\mathscr{F}$ the product $P = Q\tilde{P}$ is also full wrt $\mathscr{F}$ and $\ker_{\mathscr{F}} P = V$. □

3.14. *Proposition*

Every lclm of two full matrices $P, Q \in \mathcal{M}[D]^{r\times r}$ is also full.

Proof

Let $L = \operatorname{lclm}(P, Q)$ and $\ker_{\mathscr{F}} P + \ker_{\mathscr{F}} Q =: \langle f_1, \ldots, f_n \rangle_{\mathbb{R}} =: V$. By Proposition 3.13 there exists $L' \in \mathcal{M}[D]^{r\times r}$ full wrt $\mathscr{F}$ such that $\ker_{\mathscr{F}} L' = V$.

Together with Proposition 3.12 this implies that L' is a clm of P and Q. Therefore there exists $E\in\mathscr{M}[D]^{r\times r}$ such that $EL=L'$. By Proposition 3.10 it follows that L is full wrt $\mathscr{F}$. □

Let us define $(P)=\{RP \mid R\in\mathscr{M}[D]^{r\times r}\}$ for $P\in\mathscr{M}[D]^{r\times r}$. Then by Proposition 3.14 the set

$$\mathscr{L}_{\mathrm{f}} := \{(P) \mid P\in\mathscr{M}[D]^{r\times r} \text{ full wrt } \mathscr{F}\}$$

is a lattice with the operations lclm and gcrd as supremum and infimum ordered by inclusion.

If $\mathscr{L}_{\mathrm{v}}$ denotes the lattice of finite-dimensional subvector spaces of $\mathscr{F}^r$ with inclusion as the ordering and '$\cap$' and '$+$' as infimum and supremum respectively, the foregoing facts can be summarized by a theorem about $\mathscr{L}_{\mathrm{f}}$.

3.15. *Theorem*

$\mathscr{L}_{\mathrm{f}}$ is anti-isomorphic to $\mathscr{L}_{\mathrm{v}}$.

Proof

The map $h: \mathscr{L}_{\mathrm{f}} \to \mathscr{L}_{\mathrm{v}}$ with $(P)\mapsto \ker_{\mathscr{F}} P$ is well defined because of Proposition 3.12. Again by Propositions 3.12 and 3.13 we know that h is injective and surjective. It remains to show that for full P, $Q\in\mathscr{M}[D]^{r\times r}$ we have

(i) $$h((P)+(Q))=h((P))\cap h((Q))$$

and

(ii) $$h((P)\cap(Q))=h((P))+h((Q))$$

which imply that h is an anti-homomorphism.

(i) Let G be a gcrd of P and Q which by Lemma 3.11 (*a*) fulfils $(P)+(Q)=(G)$. Therefore we have to prove $\ker_{\mathscr{F}} G=\ker_{\mathscr{F}} P\cap\ker_{\mathscr{F}} Q$ which is evident.

(ii) Let L be an lclm of P and Q. Since $(L)=(P)\cap(Q)$ we have to prove $\ker_{\mathscr{F}} L=\ker_{\mathscr{F}} P+\ker_{\mathscr{F}} Q$. '$\supseteq$' is trivial while using the proof of Proposition 3.14 we know that $EL=L'$. Proposition 3.12 implies $\ker_{\mathscr{F}} L=\ker_{\mathscr{F}} L'$ which concludes the proof. □

3.16. *Remark*

Some of the results in Hinrichsen and Prätzel-Wolters (1980) in the context of Theorem 3.15 summarize as follows.

The non-singular $r\times r$-polynomial matrices over $\mathbb{R}[D]$ form a sublattice of $\mathscr{L}_{\mathrm{f}}$ which is anti-isomorphic to the sublattice of D-invariant spaces of $\mathscr{L}_{\mathrm{v}}$.

3.17. *Order formula*

Conditions (i) and (ii) in the proof of Theorem 3.15 enable us to generalize the degree formula (cf. Lemma 2.6) for non-singular square matrices $P, Q\in\mathscr{M}[D]^{r\times r}$. Let L denote an lclm of P and Q, and G a gcrd of P and Q, then

$$\operatorname{ord} L+\operatorname{ord} G=\operatorname{ord} P+\operatorname{ord} Q$$

Proof

Applying the foregoing results it is possible to choose an interval $J \in O$ such that the following equations are valid :

$$\begin{aligned}&\dim \ker_{\mathscr{M},J} P + \dim \ker_{\mathscr{M},J} Q\\&\quad = \dim (\ker_{\mathscr{M},J} P + \ker_{\mathscr{M},J} Q) + \dim (\ker_{\mathscr{M},J} P \cap \ker_{\mathscr{M},J} Q)\\&\quad = \dim \ker_{\mathscr{M},J} L + \dim \ker_{\mathscr{M},J} G\end{aligned}$$ □

4. Solution spaces and their homomorphisms

In this section, as a first step towards the study of system equivalence, we study the equation

$$Pz = Qu \tag{4.1}$$

with $P \in \mathscr{M}[D]^{r \times r}$, $Q \in \mathscr{M}[D]^{r \times m}$, $z \in \mathscr{Z}^r$, $u \in \mathscr{U}^m$.

The material is organized along the same lines as in Hinrichsen and Prätzel-Wolters (1980). As far as possible similar notations are used to make visible the far-reaching analogy between time-invariant and meromorphic equations of the form (4.1).

The solution space of the eqn. (4.1) is denoted by

$$M(P, Q) := \{(z, u)^{\mathrm{T}} \in \mathscr{Z}^r \times \mathscr{U}^m \mid Pz = Qu\}$$

$M(P, Q)$ contains the *subspace of forced motions starting from zero*

$$M_+(P, Q) := \{(z, u)^{\mathrm{T}} \in M(P, Q) \mid \text{supp } z \text{ is bounded to the left}\}$$

and the *subspace of free motions*

$$\ker P \times \{0\} = \{(z, 0)^{\mathrm{T}} \in M(P, Q)\}$$

In the following ' full ' will always mean ' full wrt $\mathscr{A}$ ' ; of course many of the following statements will also hold if full wrt some $\mathscr{F}$ is required.

4.1. *Proposition*

Given $P \in \mathscr{M}[D]^{r \times r}$ is full and $Q \in \mathscr{M}[D]^{r \times m}$ there exists a direct decomposition of $M(P, Q)$:

$$M(P, Q) = (\ker P \times \{0\}) \oplus M_+(P, Q)$$

Proof

$\ker P \times \{0\} \cap M_+(P, Q) = \{0\}$ follows directly from the definition of subspaces and the identity theorem for analytic functions. It remains to show ' $\subseteq$ '. Let $(z, u)^{\mathrm{T}} \in M(P, Q)$. Then there exists $I = (-\infty, t_0)$ such that $u|_I = 0$. Thus $(z, u)^{\mathrm{T}}|_I = (z|_I, 0)^{\mathrm{T}}$. $\ker P \cong \ker_I P$ implies the existence of a unique $z' \in \ker P$ with $z'|_I = z|_I$. Therefore we have

$$(z, u)^{\mathrm{T}} = (z', 0)^{\mathrm{T}} + (z - z', u)^{\mathrm{T}}, \ (z - z', u) \in M_+(P, Q)$$ □

If $(z_u, u) \in M_+(P, Q)$, z_u is called the *forced motion starting from zero under control* u. Of course z_u, with support bounded to the left, is always uniquely determined by u. If, moreover, im $Q \subseteq$ im P then z_u exists for every $u \in \mathscr{U}^m$.

If we omit the assumption ' P full ' in Proposition 4.1 the decomposition may not be possible. Choosing $p = t^2 D + 1$ and $q(D) = 1$, we see that p with

$\ker_{\mathscr{A},J} p = \langle \exp(1/t) \rangle_{\mathbb{R}}$ (if $0 \notin J$) is not full. Let $k(t)$ be zero for $t \geqslant 0$ and $\exp(1/t)$ for $t < 0$, then $k \in \mathscr{C}^\infty$. For $a > 0$ let $h \in \mathscr{C}^\infty$ such that

$$h(t) = \begin{cases} 1 & \text{for } t \leqslant -a \\ 0 & \text{for } t \geqslant 0 \end{cases}$$

Finally let $u = p(hk)$. Then $u \neq 0$ and $u \in \mathscr{U}$. But (hk, u) cannot be decomposed.

We now list basic properties of solution spaces.

4.2. *Proposition*

Let $P_i \in \mathscr{M}[D]^{r_i \times r_i}$, $Q_i \in \mathscr{M}[D]^{r_i \times m}$, $i = 1, 2$.

(*a*) Given $T \in \mathscr{M}[D]^{r_2 \times r_1}$ with $P_2 = TP_1$ and $Q_2 = TQ_1$, then $M(P_1, Q_1) \subseteq M(P_2, Q_2)$. If, moreover, T is left invertible then the equality holds.

(*b*) If P_1 is full, $M(P_1, Q_1) \subseteq M(P_2, Q_2)$ and $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$ there exists $T \in \mathscr{M}[D]^{r_2 \times r_1}$ with $P_2 = TP_1$ and $Q_2 = TQ_1$.

(*c*) Let $r_1 = r_2$, P_1 and P_2 full and $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$. Then $M(P_1, Q_1) = M(P_2, Q_2)$ iff there exists an invertible $T \in \mathscr{M}[D]^{r_1 \times r_1}$ with $P_2 = TP_1$ and $Q_2 = TQ_1$.

Proof

(*a*) is immediate.

(*b*) Since $\ker P_1 \subseteq \ker P_2$ and P_1 is full by Proposition 3.13 there exists $T \in \mathscr{M}[D]^{r_2 \times r_1}$ with $P_2 = TP_1$. $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$ implies $(TQ_1 - Q_2)u = 0$ for all $u \in \mathscr{U}^m$. Now Lemma A1 in the Appendix gives $TQ_1 = Q_2$.

(*c*) $\operatorname{im} Q_2 \subseteq \operatorname{im} P_2$ follows from $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$ and $M(P_1, Q_1) \subseteq M(P_2, Q_2)$. By (*b*) there exist matrices $T, T' \in \mathscr{M}[D]^{r_1 \times r_1}$ with $P_2 = TP_1$, $Q_2 = TQ_1$ and $P_1 = T'P_2$. Hence: $P_1 = T'TP_1$. The Cancellation Rule 3.4 implies that T is invertible. □

4.3. *Definition*

Given $P_i \in \mathscr{M}[D]^{r_i \times r_i}$ full, $Q_i \in \mathscr{M}[D]^{r_i \times m}$, $i = 1, 2$ and $T_1 \in \mathscr{M}[D]^{r_2 \times r_1}$, $Y \in \mathscr{M}[D]^{r_2 \times m}$ the map

$$f: M(P_1, Q_1) \to M(P_2, Q_2)$$

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} T_1 & Y \\ 0 & I_m \end{bmatrix} (z, u)^{\mathrm{T}}$$

is called a *solution homomorphism*.

If f is, in addition, bijective it is called a *solution isomorphism*.

An example of a solution isomorphism is already known from the theory of linear differential equations.

Given $p_1 = a_0 + a_1 D + \ldots + a_{n-1} D^{n-1} + D^n \in \mathscr{A}[D]$, $n \geqslant 1$, $q_1 \in \mathscr{A}$ and

$$P_2 = \left[DI_n - \begin{bmatrix} 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & \ddots & \ddots \\ & & & 0 & 1 \\ -a_0 & & & -a_{n-2} & -a_{n-1} \end{bmatrix} \right]$$

$$Q_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ q_1 \end{bmatrix} \in \mathscr{A}^{n \times 1}$$

then

$$f: \; M(p_1, q_1) \to M(P_2, Q_2)$$

defined by

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} \begin{bmatrix} 1 \\ D \\ \vdots \\ D^{n-1} \end{bmatrix} & \begin{matrix} 0 \\ \vdots \\ \vdots \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix} (z, u)^{\mathrm{T}}$$

is a solution isomorphism.

4.4. *Remark*

Every solution homomorphism f induces the following $\mathbb{R}$-linear maps.

$$f_1: \; M(P_1, Q_1) \to \mathscr{X}^{r_2}$$

$$(z, u)^{\mathrm{T}} \mapsto T_1 z + Yu$$

$$f_0: \quad \ker P_1 \to \ker P_2$$

$$z \mapsto T_1 z$$

Every solution homomorphism preserves the direct decomposition of the solution space $M(P_1, Q_1)$, i.e.

(*a*) $$f(\ker P_1 \times \{0\}) \subseteq \ker P_2 \times \{0\}$$

(*b*) $$f(M_+(P_1, Q_1)) \subseteq M_+(P_2, Q_2)$$

The existence of a solution homomorphism is described algebraically by the following proposition.

4.5. *Proposition*

Let P_i, Q_i, T_1, $\tilde{T}_1$, Y, $\tilde{Y}$, $i=1, 2$ be as in Definition 4.3 with im $Q_1 \subseteq$ im P_1 and let $f, \tilde{f}$ be described by

$$\begin{bmatrix} T_1 & Y \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} \tilde{T}_1 & \tilde{Y} \\ 0 & I \end{bmatrix}$$

respectively.

(*a*) f maps $M(P_1, Q_1)$ into $M(P_2, Q_2)$ iff there exists $T \in \mathcal{M}[D]^{r_2 \times r_1}$ with

$$T(P_1, Q_1) = (P_2, Q_2) \begin{bmatrix} T_1 & Y \\ 0 & I_m \end{bmatrix}$$

in this case :

(*b*) im $Q_1 \subseteq$ im P_1 results in im $Q_2 \subseteq$ im P_2.

(*c*) $f = \tilde{f}$ iff there exists a uniquely determined $L \in \mathcal{M}[D]^{r_2 \times r_1}$ such that $\tilde{T}_1 = T_1 - LP_1$ and $\tilde{Y} = Y + LQ_1$.

Proof

(*a*) follows from Proposition 4.2 (*b*) since

$$f(M(P_1, Q_1)) \subseteq M(P_2, Q_2) \Leftrightarrow M(P_1, Q_1) \subseteq M(P_2 T_1, -P_2 Y + Q_2)$$

(*b*) Given T as in (*a*), it can be shown that im $(P_2 Y + Q_2) \subseteq$ im P_2. Take $v \in$ im Q_2, i.e. $v = Q_2 u$, $u \in \mathcal{U}^m$. Now for $x := P_2 Y u + Q_2 u$ there exists $z' \in \mathcal{X}^{r_2}$ with $x = P_2 z'$. We see that $P_2(Yu - z') = Q_2 u = v$.

(*c*) ' $\Leftarrow$ ' is obvious from the definition of f and $\tilde{f}$. For ' $\Rightarrow$ ' the proof is analogous to Hinrichsen and Prätzel-Wolters (1980, p. 792). □

In the following, various properties of a solution homomorphism are expressed algebraically in terms of the corresponding matrices. Since, as in Proposition 4.5, the results are either elementary or their proofs are completely analogous to the time-invariant case, we skip all the proofs and refer to Hinrichsen and Prätzel-Wolters (1980, p. 793).

4.6. *Proposition*

Suppose that $f: M(P_1, Q_1) \to M(P_2, Q_2)$ is given as in Definition 4.3. Then

(*a*) f is injective $\Leftrightarrow$ f_0 is injective $\Leftrightarrow$ P_1 and T_1 are right coprime.

(*b*) If im $Q_1 \subseteq$ im P_1 and $T \in \mathcal{M}[D]^{r_2 \times r_1}$ as in Proposition 4.5 (*a*) then f is surjective $\Leftrightarrow$ f_0 is surjective $\Leftrightarrow$ P_2 and T are left coprime.

(*c*) If f is bijective f^{-1} is again of the form

$$\begin{bmatrix} T' & Y' \\ 0 & I_m \end{bmatrix}$$

with $T' \in \mathcal{M}[D]^{r_1 \times r_2}$, $Y' \in \mathcal{M}[D]^{r_1 \times m}$. □

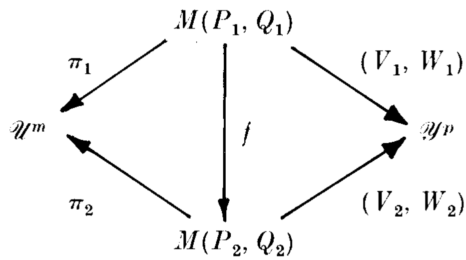
5. Systems and their homomorphisms

Now, for the systems described by a system matrix $\mathbb{P}$ as given in (1.3), the notion of equivalence is introduced and studied in detail. In particular we look for typical representatives in the equivalence classes, especially for a state space representation. As in § 4 we maintain as far as possible the analogy to Hinrichsen and Prätzel-Wolters (1980), though results and proofs will now differ much more. All through § 5 the matrices P and Q within the system matrix $\mathbb{P}$ will have the properties P full wrt $\mathscr{A}$ and im $Q \subseteq$ im P.

Given two system matrices

$$\mathbb{P}_i = \begin{bmatrix} P_i & -Q_i \\ V_i & W_i \end{bmatrix} \in \mathscr{M}[D]^{(r_i+p)\times(r_i+m)}, \quad i=1,2 \tag{5.1}$$

A system homomorphism f between the solution spaces $M(P_1, Q_1)$ and $M(P_2, Q_2)$ should not change the input u and should guarantee that the outputs of the systems described by $\mathbb{P}_i$ with respect to $(z, u)^{\mathrm{T}}$ and $f((z, u)^{\mathrm{T}})$ are identical, i.e. it should make the following diagram commute :



with

$$\left.\begin{aligned} \pi_i :\ M(P_i, Q_i) &\to \mathscr{U}^m \\ (z, u)^{\mathrm{T}} &\mapsto u \end{aligned}\right\} \quad i=1,2$$

This is expressed by the following definition.

5.1. *Definition*

Given two system matrices $\mathbb{P}_1$, $\mathbb{P}_2$ of the form (5.1), a solution homomorphism

$$f:\ M(P_1, Q_1) \to M(P_2, Q_2)$$

defined by

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} T_1 & Y \\ 0 & I \end{bmatrix} (z, u)^{\mathrm{T}}$$

with $T_1 \in \mathscr{M}[D]^{r_2\times r_1}$, $Y \in \mathscr{M}[D]^{r_2\times m}$ (see Definition 4.3) is called a *system homomorphism* if, in addition

$$V_1 z + W_1 u = (V_2, W_2) f((z, u)^{\mathrm{T}}) \quad \text{for all } (z, u)^{\mathrm{T}} \in M(P_1, Q_1) \tag{5.2}$$

f is called a *system isomorphism* if it is invertible as a system homomorphism.

$\mathbb{P}_1$ and $\mathbb{P}_2$ are called *system equivalent*, denoted by $\mathbb{P}_1 \overset{\mathrm{se}}{\sim} \mathbb{P}_2$, if there exists a system ismorphism $f:\ M(P_1, Q_1) \to M(P_2, Q_2)$.

5.2. *Example*

A special case of equivalent system matrices is represented by system matrices given by similar analytic state space systems Σ_{st}. We recall that two state space systems $\Sigma_{st}{}^1$, $\Sigma_{st}{}^2$ described by

$$\mathbb{P}_i = \begin{bmatrix} DI_{r_i} - A_i & -B_i \\ C_i & E_i(D) \end{bmatrix} \in \mathscr{A}[D]^{(r_i+p)\times(r_i+m)} \tag{5.3}$$

(where $i=1, 2$ and A_i, B_i, C_i are matrices over $\mathscr{A}$) are called *similar*, denoted by $\Sigma_{st}{}^1 \overset{s}{\sim} \Sigma_{st}{}^2$, if $r_1 = r_2$ and there exists an invertible matrix $T \in \mathscr{A}^{r_1 \times r_1}$ such that $A_2T - TA_1 = \dot{T}$, $B_2 = TB_1$, $C_2 = C_1T^{-1}$, $E_1(D) = E_2(D)$. Defining

$$f: \ M(DI_r - A_1, B_1) \to M(DI_r - A_2, B_2)$$

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} (z, u)^{\mathrm{T}}$$

we see that f is a system isomorphism. Therefore $\mathbb{P}_1 \overset{se}{\sim} \mathbb{P}_2$.

In contrast to the time-invariant case we see that every system $\Sigma_{st}{}^1$ with

$$\mathbb{P}_1 = \begin{bmatrix} DI_r - A_1 & -B_1 \\ C_1 & E_1(D) \end{bmatrix}$$

$\mathbb{P}_1 \in \mathscr{A}[D]^{(r+p)\times(r+m)}$, is similar to a system $\Sigma_{st}{}^2$ with system matrix

$$\mathbb{P}_2 = \begin{bmatrix} DI_r & -B_2 \\ C_2 & E_2(D) \end{bmatrix} \in \mathscr{M}[D]^{(r+p)\times(r+m)}$$

(Choose for the transformation the transition matrix $\Phi_1(t_0, \cdot)$ of $\Sigma_{st}{}^1$.)

Thus in the time-varying case there is much more freedom in the choice of the internal behaviour without changing the overall properties. In this context also, the following observation is of interest : Applying the well-known rank criteria for complete controllability and complete observability (Kalman *et al.* 1969, pp. 36 and 55) for time-invariant systems, it can be shown that two time-invariant completely controllable or completely observable state space systems which are not equivalent in the sense of the classical theory (i.e. transformation is a constant invertible matrix) cannot become equivalent if embedded in the set of time-varying systems.

A very useful criterion for the study of system equivalence is now derived. It is completely analogous to the time-invariant result of Hinrichsen and Prätzel-Wolters (1980, p. 794).

5.3. *Proposition*

Let $\mathbb{P}_i$ be two system matrices of the form (5.1). Then $\mathbb{P}_1 \overset{se}{\sim} \mathbb{P}_2 \Leftrightarrow$ there exist T, $T_1 \in \mathscr{M}[D]^{r_2 \times r_1}$, $X \in \mathscr{M}[D]^{p \times r_1}$ and $Y \in \mathscr{M}[D]^{r_2 \times m}$ such that

(*a*)
$$\begin{bmatrix} T & 0 \\ X & I_p \end{bmatrix} \mathbb{P}_1 = \mathbb{P}_2 \begin{bmatrix} T_1 & Y \\ 0 & I_m \end{bmatrix}$$

(*b*) T, P_2 are left coprime and P_1, T_1 are right coprime.

im $Q_1 \subseteq$ im P_1 is only needed for the implication ' $\Rightarrow$ '.

Proof

' $\Rightarrow$ ': Using the notations of Definition 5.1 and combining the results of Proposition 4.5 and 4.6, we have only to show the existence of $X \in \mathscr{M}[D]^{p \times r_1}$ such that $XP_1 + V_1 = V_2T_1$ and $XQ_1 + W_1 = V_2Y + W_2$.

Condition (5.2) gives

$$M(P_1, Q_1) \subseteq M(V_1 - V_2T_1, W_2 - W_1 + V_2Y)$$

Thus Proposition 4.2 gives the desired matrix X. ' $\Leftarrow$ ' follows from Proposition 4.5 (*a*) and 4.6. □

5.4. *Remark*

If in condition (*a*) in Proposition 5.3, $T_1 = I_{r_1}$ and $Y = 0$ then—if also (*b*) is valid—T is necessarily invertible. Therefore in the case where $M(P_1, Q_1) = M(P_2, Q_2)$ the identity map is a system isomorphism iff

$$\begin{bmatrix} T & 0 \\ X & I_p \end{bmatrix} \mathbb{P}_1 = \mathbb{P}_2$$

and $T \in \mathscr{M}[D]^{r_1 \times r_1}$ is invertible. If $M(P_1, Q_1) = M(P_2, Q_2)$ and if the identity map is a system isomorphism, then the systems described by $\mathbb{P}_1$, $\mathbb{P}_2$ are called *undistinguishable*.

In the Example 5.2 we saw that system matrices given by similar analytic state space systems Σ_{st} are always system equivalent. The following proposition shows that the converse is also true.

5.5. *Proposition*

Let $\Sigma_{st}{}^i$, $i = 1, 2$, be two state space systems described by system matrices $\mathbb{P}_i$ as in (5.3). Then

$$\mathbb{P}_1 \overset{se}{\sim} \mathbb{P}_2 \quad \text{iff} \quad \Sigma_{st}{}^1 \overset{s}{\sim} \Sigma_{st}{}^2$$

Proof

' $\Rightarrow$ ': By assumption ker $(DI_r - A_1) \simeq$ ker $(DI_r - A_2)$. Therefore $r_1 = r_2$. By the Example 5.2 and ' $\Leftarrow$ ' it suffices to consider $(r := r_1)$

$$\mathbb{P}_1 = \begin{bmatrix} DI_r & -F_1 \\ G_1 & E_1(D) \end{bmatrix} \overset{se}{\sim} \mathbb{P}_2 = \begin{bmatrix} DI_r & -F_2 \\ G_2 & E_2(D) \end{bmatrix}$$

Let

$$f: M(DI_r, F_1) \to M(DI_r, F_2)$$

described by

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} T_1 & Y \\ 0 & I_m \end{bmatrix} (z, u)^{\mathrm{T}}$$

with $T_1 \in \mathscr{M}[D]^{r\times r}$ and $Y \in \mathscr{M}[D]^{r\times m}$, be the system isomorphism. Choosing $Q \in \mathscr{M}[D]^{r\times r}$, $H \in \mathscr{M}^{r\times r}$ such that $T_1 = QDI_r + H$, we see that

$$f((z, u)^T) = (T_1 z + Yu, u)^T = ((QF_1 + Y)u + Hz, u)^T$$

Since $f:\ \ker DI_r \to \ker DI_r$ with $z \mapsto Hz$ is an isomorphism we have H is invertible over $\mathbb{R}$. Therefore

$$DI_r((QF_1 + Y)u + Hz) = F_2 u \quad \text{for all } (z, u)^T \in M(DI_r, F_1)$$

Defining $Y_1 := QF_1 + Y$ we see $DI_r z = H^{-1}(F_2 - DI_r Y_1)u$ for all $(z, u)^T \in M(DI_r, F_1)$. Thus $F_1 = H^{-1}(F_2 - DI_r Y_1)$ is given by Lemma A 1 in the Appendix. Hence $F_2 - DI_r Y_1 \in \mathscr{M}^{r\times m}$, which implies $Y_1 = 0$, i.e. $F_2 = HF_1$.

Condition (5.2) gives the implication

$$(G_2, E_2)f((z, u)^T) = (G_1, E_1)(z, u)^T \quad \text{for all } (z, u)^T \in M(DI_r, F_1)$$

Thus

$$G_1 z + E_1 u = G_2 Hz + E_2 u \quad \text{for all } (z, u)^T \in M(DI_r, F_1)$$

Since $\operatorname{im} F_1 \subseteq \operatorname{im} DI_r$ we can choose $u = 0$ and see $G_2 H = G_1$. Application of Lemma A 1 in the Appendix to $E_2 u = E_1 u$ gives $E_2 = E_1$. □

We can now begin to study the equivalence class of a system matrix

5.6. *Proposition*

Given a system matrix $\mathbb{P}_1$ of the form (5.1) with $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$ and matrices $P_j \in \mathscr{M}[D]^{r_j \times r_j}$, $j = 2, 3, 4, 5$ satisfying

(*a*)	$P_2 \sim P_1$	$(r_1 = r_2)$
(*b*)	$P_3 = \operatorname{diag}(I_l, P_1)$, $l \in \mathbb{N}$ arbitrary	$(r_3 = l + r)$
(*c*)	$P_4 = DI_n$ with $n = \operatorname{ord} P_1$	$(r_4 = n)$
(*d*)	$P_5 = D^n$ with $n = \operatorname{ord} P_1$	$(r_5 = 1)$

Then for $2 \leqslant j \leqslant 5$ there exist matrices Q_j, V_j, W_j such that

$$\mathbb{P}_j = \begin{bmatrix} P_j & -Q_j \\ V_j & W_j \end{bmatrix} \in \mathscr{M}[D]^{(r_j+p)\times(r_j+m)}$$

satisfies $\mathbb{P}_1 \overset{sc}{\sim} \mathbb{P}_j$.

In case (*b*) $\mathbb{P}_3$ is called a *trivial expansion* of $\mathbb{P}_1$.

Proof

We use the criteria and notation of Proposition 5.3.

(*a*) Let $P_2 = UP_1U'$, where U and U' are invertible $r_1 \times r_1$ matrices over $\mathscr{M}[D]$. Choose $T = U$, $X = 0$, $T_1 = U'^{-1}$, $Y = 0$, $V_2 = V_1 U'$, $Q_2 = UQ_1$, $W_1 = W_2$.

(*b*) The map

$$f:\ M(P_1, Q_1) \to M(P_3, Q_3)$$

with

$$(z, u)^T \mapsto \begin{bmatrix} 0_l \\ z \\ u \end{bmatrix}$$

defines an isomorphism with the required properties. P_3 is obviously full.

(*c*) By (*b*) we can assume $r_1 = n$. Since $DI_n \sim \operatorname{diag}(1, \ldots, 1, D^n)$ the assertion now is a special case of (*a*).

(*d*) is a consequence of (*a*) and (*b*). □

Of course there are more constructive procedures for the transition between the different equivalent systems of Proposition 5.6 which we do not develop here.

5.7. *Proposition* (state space representation for system matrices)

For every system matrix $\mathbb{P}_1$ of the form (5.1) there exists a system matrix

$$\mathbb{P}_2 = \begin{bmatrix} DI_n & -B \\ C & E(D) \end{bmatrix} \in \mathscr{A}[D]^{(n+p)\times(n+m)} \quad (n = \operatorname{ord} P_1) \tag{5.4}$$

with B, C defined over $\mathscr{A}$ such that $\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2$. $\mathbb{P}_2$ is uniquely determined up to a constant similarity transformation.

Proof

By Proposition 5.6 (*c*) we can assume that

$$\mathbb{P} = \begin{bmatrix} DI_n & -Q \\ V & W \end{bmatrix} \in \mathscr{M}[D]^{(n+p)\times(n+m)}$$

There exist $B \in \mathscr{M}^{n\times n}$, $C \in \mathscr{M}^{p\times n}$, $Y \in \mathscr{M}[D]^{n\times n}$, $X \in \mathscr{M}[D]^{p\times n}$ such that $-Q = DI_n Y - B$ and $V = XDI_n + C$. Setting $E(D) := W + XQ - CY$ we obtain

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \mathbb{P}_1 = \begin{bmatrix} DI_n & -B \\ C & E(D) \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$$

Therefore $\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2$.

It remains to show that B, C are matrices over $\mathscr{A}$ and E is a matrix over $\mathscr{A}[D]$. Let $I \subseteq \mathbb{R}$ be an open interval. For $u_i \in \mathscr{U}^m$ with $u_{i|I} = e_i$ for each $i = 1, \ldots, n$ (compare Lemma A 1 in the Appendix) there exists, by the condition $\operatorname{im} Q_1 \subseteq \operatorname{im} P_1$, $z \in (\mathscr{C}^\infty)^n$ such that $DI_n z = Be_i$ in I. Therefore B cannot have any poles on I. Since I was chosen arbitrarily we have $B \in \mathscr{A}^{n\times m}$. Setting $u = 0$ and $z = e_i \in \mathbb{R}^n = \ker DI_n$ and $y = Cz + E(D)u$ shows (because of the assumption that the outputs of the system belonging to $\mathbb{P}_2$ should have no poles) $C \in \mathscr{A}^{p\times n}$.

Now taking $u_{ik} \in \mathscr{U}^m$ such that $u_{ik} = t^k e_i$ on I, $e_i \in \mathbb{R}^m$, $i = 1, \ldots, m$, $k \in \mathbb{N}$, we see from the reasoning as above that $E(D) \in \mathscr{A}[D]^{p\times m}$. The uniqueness follows by the proof of Proposition 5.5 ' ⇒ '. □

Rosenbrock (1970) introduced the concept of strict system equivalence. In our context the definition appears as follows.

5.8. *Definition*

Two system matrices $\mathbb{P}_1$, $\mathbb{P}_2$ of the form (5.1) and satisfying $r_1 = r_2$, are called *strictly system equivalent* (sse) if there exist invertible matrices $(r := r_1)$

$M, N \in \mathscr{M}[D]^{r\times r}$, $R \in \mathscr{M}[D]^{p\times r}$ and $S \in \mathscr{M}[D]^{r\times m}$ such that

$$\begin{bmatrix} M & 0 \\ R & I_p \end{bmatrix} \mathbb{P}_1 \begin{bmatrix} N & S \\ 0 & I_m \end{bmatrix} = \mathbb{P}_2$$

As in the commutative case (see Hinrichsen and Prätzel-Wolters (1980)) the relation between ' se ' and ' sse ' can be clarified by the method of trivial expansion (see Proposition 5.6 (*b*)).

5.9. *Proposition*

Let $\mathbb{P}_1$, $\mathbb{P}_2$ be as in (5.1), then $\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2 \Leftrightarrow$ there exist trivial expansions $\overline{\mathbb{P}}_1$, $\overline{\mathbb{P}}_2$ satisfying $\overline{\mathbb{P}}_1 \overset{\text{sse}}{\sim} \overline{\mathbb{P}}_2$.

We omit the proof since it does not require new ideas in addition to those applied in the proofs of the foregoing propositions.

To conclude this section we derive a result on system equivalence which will be of use in § 6.

5.10. *Lemma*

Given two system matrices $\mathbb{P}_i$ as in (5.1), with $\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2$, then P_1 and Q_1 are left coprime iff P_2 and Q_2 are left coprime.

Proof

It suffices to show ' $\Rightarrow$ '. By Proposition 5.3 there exist $T, T_1 \in \mathscr{M}[D]^{r_2\times r_1}$, $X \in \mathscr{M}[D]^{p\times r_1}$, $Y \in \mathscr{M}[D]^{r_1\times m}$ such that $TP_1 = P_2T_1$ and $Q_2 = P_2Y + TQ_1$. Since T and P_2 are left coprime there exist $A \in \mathscr{M}[D]^{r_1\times r_2}$ and $B \in \mathscr{M}[D]^{r_2\times r_2}$ such that $I_{r_2} = TA + P_2B$. By assumption there exist $E \in \mathscr{M}[D]^{r_1\times r_1}$ and $F \in \mathscr{M}[D]^{m\times r_1}$ such that $I_{r_1} = P_1E + Q_1F$. Therefore

$$\begin{aligned} T &= TP_1E + TQ_1F \\ &= P_2T_1E + (Q_2 - P_2Y)F \\ &= P_2(T_1E - YF) + Q_2F \end{aligned}$$

Furthermore

$$\begin{aligned} TA &= P_2(T_1EA - YFA) + Q_2FA \\ I_{r_2} - P_2B &= P_2(T_1EA - YFA) + Q_2FA \\ I_{r_2} &= P_2(T_1EA + B - YFA) + Q_2FA \end{aligned}$$

□

6. Controllability and observability

The result given in this section is mainly a characterization of controllability by a coprimeness condition. A similar result also holds for observability. As in the time-invariant case the latter is much more straightforward since no inputs are involved.

In this section $P \in \mathscr{M}[D]^{r\times r}$ is always assumed to be full wrt $\mathscr{A}$.

6.1. *Definition*

Let

$$\mathbb{P} = \begin{bmatrix} P & -Q \\ V & W \end{bmatrix} \in \mathcal{M}_I[D]^{r+p,r+m}, \quad I \in O, \text{ be a system matrix}$$

(*a*) $\mathbb{P}$ is called (completely) *controllable* on $[t_0, t_1] \subseteq I$ if for any $z^0 \in \ker_{\mathcal{A},I} P$ a control $u \in \mathcal{U}^m$ exists with supp $u \subseteq [t_0, t_1]$ such that

$$(z^0 + z_u)(t) = \begin{cases} z^0(t) & \text{for } t \leqslant t_0 \\ 0 & \text{for } t \geqslant t_1 \end{cases}$$

(*b*) $\mathbb{P}$ is called (completely) *observable* (on I) if V acts as a monomorphism on $\ker_{\mathcal{A},I} P$.

For time-varying state space systems in the literature, various notions of controllability and also reachability are usually introduced which could be easily translated to the present arrangement. But since most of them coincide in the analytic case we concentrate on completely controllable system matrices and omit ' completely ' in the following. A similar remark is valid for observability. Without losing generality we can restrict our analysis to $I = \mathbb{R}$.

6.2. *Remark*

Controllability and observability are maintained under system equivalence. Given two system matrices $\mathbb{P}_i$ $(i = 1, 2)$ as in (5.1) with $\mathbb{P}_1 \overset{\text{se}}{\sim} \mathbb{P}_2$ then $\mathbb{P}_1$ is controllable (observable) on $[t_0, t_1] \subseteq \mathbb{R}$, $t_0 < t_1$ iff $\mathbb{P}_2$ is controllable (observable) on $[t_0, t_1] \subseteq \mathbb{R}$, $t_0 < t_1$.

Proof

Let the system equivalence be described by the system-isomorphism

$$f: \; M(P_1, Q_1) \to M(P_2, Q_2)$$

$$(z, u)^{\mathrm{T}} \mapsto \begin{bmatrix} T_1 & Y \\ 0 & I_m \end{bmatrix} (z, u)^{\mathrm{T}}$$

For given $z \in \ker_{\mathcal{A}} P_2$ we have to show that there exists $u \in \mathcal{U}^m$ with supp $u \subseteq [t_0, t_1]$ such that

$$(z + z_u^{\,2})(\tau) = \begin{cases} z(\tau) & \text{for } \tau \leqslant t_0 \\ 0 & \text{for } \tau \geqslant t_1 \end{cases}$$

By assumption there exists a unique $z^0 \in \ker_{\mathcal{A}} P_1$ with $T_1 z^0 = z$. Since $\mathbb{P}_1$ is controllable on $[t_0, t_1]$ there exists $u \in \mathcal{U}^m$ with supp $u \subseteq [t_0, t_1]$ such that

$$(z^0 + z_u)(\tau) = \begin{cases} z^0(\tau) & \text{for } \tau \leqslant t_0 \\ 0 & \text{for } \tau \geqslant t_1 \end{cases} \qquad (*)$$

By Remark 4.4 we have for every $(z_u, u)^{\mathrm{T}} \in M_+(P_1, Q_1)$ that $f((z_u, u)^{\mathrm{T}}) = (T_1 z_u + Yu, u)^{\mathrm{T}} \in M_+(P_2, Q_2)$. Since $z_u^{\,2} = T_1 z_u + Yu$ the assertion follows by (*).

The proof that observability is invariant under system equivalence is straightforward. □

It is easily proved that our definition of observability coincides with the usual ones if considering system matrices in state space form.

The same is not straightforward for controllability. The reason is that common definitions for state space systems involve a much larger control space namely piecewise-continuous vector functions. For simplicity let

$$\mathbb{P}=\begin{bmatrix} DI_r & -B \\ C & E(D)\end{bmatrix}\in\mathscr{M}[D]^{(r+p)\times(r+m)}$$

be in state space form. Define

$$\mathscr{U}_I=\{u\in\mathscr{U}\,|\,\operatorname{supp} u\subseteq I\}\quad I:=[t_0,t_1]$$

For such a system matrix our definition of controllability on I just means that for any $x^0\in\ker DI_r=\mathbb{R}^r$ there exists a control $u\in\mathscr{U}_I{}^m$ such that

$$-x^0=\int_{t_0}^{t_1}B(t)u(t)\,dt\;=:H(u) \tag{6.1}$$

Or equivalently : $\mathbb{P}$ is controllable iff $H:\ \mathscr{U}_I{}^m\to\mathbb{R}^r$ is surjective.

It can be proved (see Appendix, Lemma A 2) that the image of H does not depend on whether we admit, in addition, piecewise continuous controls or not. When this is done it becomes clear that for every system matrix $\mathbb{P}$ being controllable according to Definition 6.1, its corresponding state space system is controllable in the classical sense. Therefore we can apply the corresponding results from the state space theory, in particular the following criterion.

6.3. *Controllability criterion* (Silverman and Meadows 1967)

Let

$$\mathbb{P}=\begin{bmatrix} DI_r & -B \\ C & E(D)\end{bmatrix}\in\mathscr{A}[D]^{(r+p)\times(r+m)} \tag{6.2}$$

be in state space form. Then $\mathbb{P}$ is completely controllable on $[t_0,t_1]\subseteq\mathbb{R}$, $t_0<t_1$ iff

$$\operatorname{rk}\,(B(t),\dot{B}(t),\ldots,B^{(r-1)}(t))=r\quad\text{for every } t\in(t_0,t_1)\backslash N \tag{6.3}$$

where N is a discrete set and $B^{(k)}$ denotes the kth derivative of B.

6.4. *Theorem*

For a system matrix

$$\mathbb{P}=\begin{bmatrix} P & -Q \\ V & W\end{bmatrix}\in\mathscr{M}[D]^{(r'+p)\times(r'+m)}$$

with $\operatorname{im} Q\subseteq\operatorname{im} P$ the following statements are equivalent.

(*a*) $\mathbb{P}$ is controllable on $[t_0,t_1]\subseteq\mathbb{R}$, $t_0<t_1$.
(*b*) $\mathbb{P}$ is controllable on every closed subinterval of $\mathbb{R}$.
(*c*) P and Q are left coprime.

Proof

By Proposition 5.3 we know that left coprimeness of P and Q is maintained under system equivalence. Therefore by Proposition 5.9 and Remark 6.2 it suffices to show the assertion for a system matrix $\mathbb{P}$ of the form (6.2) where $r = \operatorname{ord} P$.

Let $\mathscr{K}_i := [B, \dot{B}, \dots, B^{(-1)}]$ for $i > 0$ and note that the rank condition (6.3) is equivalent to $\operatorname{rk}_{\mathscr{M}} \mathscr{K}_r = r$ where $rk_{\mathscr{M}}$ denotes the rank over $\mathscr{M}$.

$(a) \Leftrightarrow (b)$: Since B and its derivatives are analytic the assertion is obvious.

$(b) \Rightarrow (c)$: Since $\operatorname{rk}_{\mathscr{M}} \mathscr{K}_r = r$ there exist $Y_0, \dots, Y_{r-1} \in \mathscr{M}^{m\times r}$ such that for $Y^{\mathrm{T}} = [Y_0, \dots, Y_{r-1}]^{\mathrm{T}}$ we have $\mathscr{K}_r Y = I_r$. Since

$$BD^n = \sum_{\lambda=0}^{n-1} \binom{n}{\lambda} (-1)^\lambda D^{n-\lambda} B^{(\lambda)} + (-1)^n B^{(n)}$$

$$= D \underbrace{\left[\sum_{\lambda=0}^{n-1} \binom{n}{\lambda} (-1)^\lambda D^{n-\lambda-1} B^{(\lambda)} \right]}_{=:\, M_n(D)} + (-1)^n B^{(n)}$$

we calculate

$$B[Y_0 - DY_1 + \dots + (-1)^{r-1} D^{r-1} Y_{r-1}]$$

$$= BY_0 - [DM_1(D) - B] Y_1 + \dots$$

$$+ (-1)^{r-1} [DI_r M_{r-1}(D) + (-1)^{r-1} B^{(r-1)}] Y_{r-1}$$

$$= BY_0 + BY_1 + \dots + B^{(r-1)} Y_{r-1} + \sum_{\lambda=1}^{r-1} (-1)^\lambda D M_\lambda(D) Y_\lambda$$

$$= [B, \dots, B^{(r-1)}] Y + D \sum_{\lambda=1}^{r-1} (-1)^\lambda M_\lambda(D) Y_\lambda$$

$$= I_m + D \sum_{\lambda=1}^{r-1} (-1)^\lambda M_\lambda(D) Y_\lambda$$

By Lemma 3.11 we conclude that B and DI_r are left coprime.

$(c) \Rightarrow (b)$: By Lemma 3.11 and the assumption there exist $X \in \mathscr{M}[D]^{r\times r}$ and

$$Y = \sum_{i=0}^{m} D^i Y_i \in \mathscr{M}[D]^{m\times r}$$

such that $DI_r X + BY = I_r$. We now calculate as follows :

$$I_r = DX + BY$$

$$= DX + \sum_{i=0}^{n} \sum_{\lambda=0}^{i} (-1)^i \binom{i}{\lambda} D^{i-\lambda} B^{(\lambda)} Y_i$$

$$= DX + \sum_{i=0}^{n} \left(\sum_{\lambda=0}^{i-1} (-1)^i \binom{i}{\lambda} D^{i-\lambda} B^{(\lambda)} + (-1)^i B^{(i)} \right) Y_i$$

$$= DX + D \sum_{i=0}^{n} \sum_{\lambda=0}^{i-1} (-1)^i \binom{i}{\lambda} D^{i-\lambda-1} B^{(\lambda)} Y_i + \sum_{i=0}^{n} B^{(i)} (-1)^i Y_i$$

Comparing the coefficients we have

$$\mathscr{K}_n[Y_0, -Y_1, \dots, (-1)^n Y_n]^T = I_r$$

which implies $\mathrm{rk}_{\mathscr{M}} \mathscr{K}_n = r$. We have to prove that $\mathrm{rk}_{\mathscr{M}} \mathscr{K}_r = r$. Since $\mathrm{rk}_{\mathscr{M}} \mathscr{K}_i$ considered as a function of i can only be strictly monotonic within the set $\{0, \dots, r\}$ $\mathrm{rk}_{\mathscr{M}} \mathscr{K}_n = r$ implies $\mathrm{rk}_{\mathscr{M}} \mathscr{K}_r = r$. This proves (*b*). □

We conclude this section with a result for observability which is analogous to Theorem 6.4.

6.5. *Proposition*

For a system matrix

$$\mathbb{P} = \begin{bmatrix} P & -Q \\ V & W \end{bmatrix} \in \mathscr{M}[D]^{(r+p)\times(r+m)}$$

the following statements are equivalent :

(*a*) $\mathbb{P}$ is observable.

(*b*) P and V are right coprime.

Proof

Let G denote a gcrd of P and V.

(*a*) ⇒ (*b*) : Since $\ker_{\mathscr{A}} G = \ker_{\mathscr{A}} P \cap \ker_{\mathscr{A}} V$ we have $\ker_{\mathscr{A}} G = \{0\}$. Assume P and V are not right coprime. Then there exists a crd of P and V called G' which is not invertible. Therefore $0 \neq z \in \mathscr{A}^r$ exists such that $G'z = 0$ which implies $\ker_{\mathscr{A}} G \neq \{0\}$.

(*b*) ⇒ (*a*) : Since G is invertible we have $\ker_{\mathscr{A}} G = \{0\}$. □

Remark

Based on the unique anti-isomorphism ϕ of $\mathscr{M}[D]$ which maps fD onto $-Df$ one can define the generalized transpose $*P = (*p_{ij})$ for matrices $P = (p_{ij}) \in \mathscr{M}[D]^{m\times n}$ as follows

$$*p_{ij} := \phi(p_{ji})$$

Now the system matrix dual or adjoint to $\mathbb{P}$ is just $*\mathbb{P}$ and the duality of the criteria in Theorem 6.4 (*c*) and in Proposition 6.5 (*b*) is obvious. Note that if $p \in \mathscr{M}[D]$ is full wrt $\mathscr{A}_1$ then $*p$ is in most case only full wrt $\mathscr{M}$.

7. System homomorphisms and transfer functions

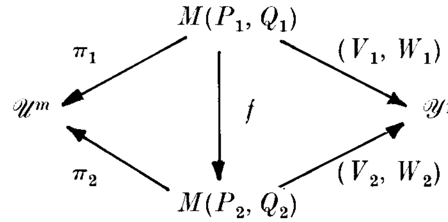
The following more intrinsic definition of a system homomorphism turns out to be equivalent to the one given in Definition 5.1.

7.1. *Definition*

Let $\mathbb{P}_i$ be defined as in (5.1) for $i = 1, 2$, im $Q_1 \subseteq$ im P_1 and P_i full wrt $\mathscr{A}$. The map $f:\ M(P_1, Q_1) \to M(P_2, Q_2)$ is called a system homomorphism if the following conditions are satisfied :

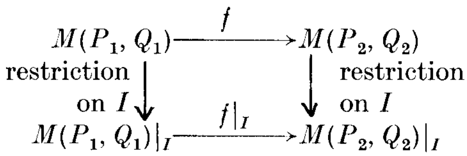
(*a*) f is $\mathbb{R}$ linear.

(*b*) f does not affect the inputs and outputs. More precisely : The diagram



commutes. $(\pi_i(z, u)^T := u$; $(V_i, W_i)(z, u)^T := V_iz + W_iu$ for $i = 1, 2.)$

(*c*) For every open interval I there exists an $\mathbb{R}$-linear map $f|_I$ such that



commutes.

(*d*) f is called time invariant if it commutes with all time shifts.

Condition (*c*) can be replaced by the condition

(*c'*) For every $(z, u)^T \in M(P_1, Q_1)$ we have

$$(z, u)^T|_I = 0 \Rightarrow f((z, u)^T)_I = 0$$

Clearly every system homomorphism according to Definition 5.1 satisfies (*a*), (*b*) and (*c*). Conversely a system homomorphism according to Definition 7.1 can be represented by a differential polynomial matrix as in Definition 5.1 which has constant coefficients if (*d*) is fulfilled. The proof of the latter statement is lengthy and is omitted here. Its main tool is a result of Peetre (1960) (see also Wells and De Prima (1973)).

The equivalence of the two definitions of system homomorphy enables us to clarify the relation between transfer function and system equivalence.

7.2. *Definition*

Let $\mathbb{P}$ be of the form (5.1) and im $Q \subseteq$ im P. Since for any $u \in \mathscr{U}^m$ the forced motion z_u starting from zero is unique. The map $\psi : \mathscr{U}^m \to \mathscr{X}^r$ with $u \mapsto z_u$ is well-defined. Therefore the *transfer function* T of $\mathbb{P}$ is introduced by

$$T : \mathscr{U}^m \to \mathscr{Y}^p$$

$$u \mapsto (V\psi + W)u$$

Algebraically there is no difficulty in also defining transfer matrices associated with a transfer function T over the quotient skew field of $\mathscr{M}[D]$. Of course, not all matrices $VP^{-1}Q + W$ can be interpreted as an operator on all of $\mathscr{U}^m$. For this paper there is no advantage in considering transfer matrices.

7.3. *Proposition*

For $i = 1, 2$, let $\mathbb{P}_i$ be defined as in (5.1), im $Q_i \subseteq$ im P_i, P_i full wrt $\mathscr{A}$, and T_i be the transfer functions of $\mathbb{P}_i$.

(*a*) If a system homomorphism exists between $\mathbb{P}_1$ and $\mathbb{P}_2$ then $T_1 = T_2$.

(*b*) If $T_1 = T_2$ and in addition both $\mathbb{P}_1$ and $\mathbb{P}_2$ are controllable on $J = [t_0, t_1]$ and observable then $\mathbb{P}_1 \overset{sc}{\sim} \mathbb{P}_2$.

Proof

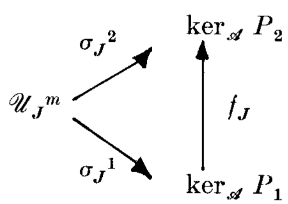
(*a*) is an immediate consequence of our definition.

To prove (*b*) we proceed in several steps :

(1) For every $u \in \mathscr{U}_J{}^m$ there is a unique $z \in \ker_{\mathscr{A}} P$ such that $z(t) = z_u(t)$ for $t \geqslant t_1$. Thus $\sigma_J : \mathscr{U}_J{}^m \to \ker_{\mathscr{A}} P$ with $\sigma_J(u) = z$ is a well defined $\mathbb{R}$-linear map (cf. Hinrichsen and Prätzel-Wolters (1980)).

(2) The controllability of $\mathbb{P}$ can now be characterized as follows : $\mathbb{P}$ is controllable on J iff σ_J is surjective.

(3) Now let $\sigma_J{}^i$ be the corresponding maps for $\mathbb{P}_i$ $(i = 1, 2)$ as defined in (1). We then have that if $\mathbb{P}_1$ is controllable, $\mathbb{P}_2$ is observable and $T_1 = T_2$ then $\sigma_J{}^1$ can be factorized by $\sigma_J{}^2$, i.e. there is a unique $\mathbb{R}$-linear map f_J such that the following diagram commutes :



To prove this we demonstrate that $\sigma_J{}^1$ is surjective and $\ker \sigma_J{}^1 \subseteq \ker \sigma_J{}^2$. The first property is fulfilled by (2). For the latter, assume $\sigma_J{}^1(u) = 0$ for $u \in \mathscr{U}_J{}^m$. Since $(z_u{}^1, u)|_{(t_1, \infty)} = 0$ and $V_1 z_u{}^1 + W_1 u = T_1 u = T_2 u = V_2 z_u{}^2 + W_2 u$ we conclude $V_2 z_u{}^2|_{(t_1, \infty)} = 0$. Since $\mathbb{P}_2$ is observable we conclude $z_u{}^2|_{(t_1, \infty)} = 0$ which implies $\sigma_J{}^2(u) = 0$.

(4) Now let $\mathbb{P}_i$ $(i = 1, 2)$ be given as in part (*b*) of Proposition 7.3. Applying (3) twice we see that the $\mathbb{R}$-linear map

$$f_J : \ker P_1 \to \ker P_2$$
$$z^1 \mapsto \sigma_J{}^2(u)$$

for some $u \in (\sigma_J{}^1)^{-1}(z^1)$ is in fact an isomorphism.

(5) Now we define the linear map

$$f : M(P_1, Q_1) \to M(P_2, Q_2)$$

by

$$f|_{\ker_{\mathscr{A}} P_1 \times \{0\}}(z^1, u)^{\mathrm{T}} = (f_J(z^1), 0)^{\mathrm{T}}$$
$$f|_{M_+(P_1, Q_1)}(z_u{}^1, u)^{\mathrm{T}} = (z_u{}^2, u)^{\mathrm{T}}$$

f is a system isomorphism if we prove the conditions (*b*) and (*c'*) of Definition 7.1.

(6) First we prove Definition 7.1 (*b*). We decompose $(z, u)^{\mathrm{T}} \in M(P_1, Q_1)$ by Proposition 4.1 as follows

$$(z, u) = (z^1, 0) + (z_u{}^1, u)$$

Therefore

$$f((z, u)^{\mathrm{T}}) = (f_J(z^1), 0)^{\mathrm{T}} + (z_u{}^2, u)^{\mathrm{T}}$$

It remains to show that

$$V_1 z^1 + V_1 z_u{}^1 + W_1 u = V_2 f_J(z^1) + V_2 z_u{}^2 + W_2 u$$

Since $T_1 = T_2$ it remains to show that $V_1 z^1 = V_2 f_J(z^1)$. Choose $u' \in \mathscr{U}_J{}^m$ such that $\sigma_J{}^1(u') = z^1$. Therefore $f_J(z^1) = \sigma_J{}^2(u')$ and by definition $(f_J(z^1), u)^{\mathrm{T}} = (z_{u'}{}^2, 0)^{\mathrm{T}}$ for $t \geqslant t_1$. Since $T_1 = T_2$ we have $V_1 z^1 = V_1 z_{u'}{}^1 = V_2 z_{u'}{}^2 = V_2 f_J(z^1)$ for $t \geqslant t_1$. By the identity property of analytic functions the assertion follows.

(7) It now remains to prove Definition 7.1 (c'). Let $(z, u)^{\mathrm{T}} \in M(P_1, Q_1)$ and $(z, u)|_I = 0$. We decompose (z, u) by Proposition 4.1 as follows:

$$(z, u) = (z^1, 0) + (z_u{}^1, u)$$

By (b) we already know that

$$V_1 z + W_1 u = V_1(z^1 + z_u{}^1) + W_1 u = V_2(f_J(z^1) + z_u{}^2) + W_2 u$$

For $t \in I$ therefore $V_2(f_J(z^1) + z_u{}^2) = 0$ which by observability of $\mathbb{P}_2$ gives $f((z, u)^{\mathrm{T}}) = 0$ in I. □

Appendix

Lemma A 1

Given

$$A = \sum_{i=0}^{n} A^i D^i \in \mathscr{M}^{1\times m}[D]$$

then $Au = 0$ for all $u \in \mathscr{U}^m$ results in $A_0 = A_1 = \ldots = A_n = 0$.

Proof

For $t_0 < t_1$, $t_0, t_1 \in \mathbb{R}$ there is $u \in \mathscr{C}^\infty$ satisfying

$$u(t) = \begin{Bmatrix} 0 & t \leqslant t_0 \\ > 0 & t_0 < t < t_1 \\ 1 & t_1 \leqslant t \end{Bmatrix}$$

Inserting successively

$$u_j(t) = t^k e_j u(t) \in \mathscr{U}^m$$

for $k = 0, 1, \ldots, n-1$, $j = 1, 2, \ldots, m$ and e_j the jth canonical basis vector of $\mathbb{R}^m$, gives the assertion. □

Lemma A 2

Let $B \in \mathscr{A}^{r\times m}$ and H be the map (6.2). Define

$$\mathscr{C}_{p,I} := \{\text{piecewise continuous functions } f:\ I \to \mathbb{R}\}$$

Then for the map $G:\ \mathscr{C}_{p,I}{}^m \to \mathbb{R}^r$ with

$$u \mapsto \int_{t_0}^{t_1} B(t)u(t)\, dt$$

we have $\operatorname{im} G = \operatorname{im} H$.

Proof

Trivially im $H \subseteq$ im G. Since the set of $\mathscr{C}^\infty$ functions is dense in $\mathscr{C}_{p,I}$ with reference to the L_1 norm and for every closed interval $K \subset (t_0, t_1)$, there exists $h \in \mathscr{C}^\infty$ such that supp $h \subsetneq I$, $h|_K = 1$ and $0 \leqslant h(t) \leqslant 1$ for $t \in I \backslash K$ we can prove that $\mathscr{U}_I{}^m$ (the set of $\mathscr{C}^\infty$-m-vector functions with supp $\subseteq I$) is L_1 dense in $\mathscr{C}_{p,I}{}^m$. Now let $x_1, \ldots, x_s$ be an $\mathbb{R}$ basis of im $G \subseteq \mathbb{R}^r$. Then by continuity of G for each $\epsilon > 0$ there are $u_1, \ldots, u_s \in \mathscr{U}_I{}^m$ such that $\|H(u_i) - x_i\| < \epsilon$ for $1 \leqslant i \leqslant s$. Choosing ϵ small enough the vectors $H(u_i)$ become linearly independent. The latter means im $G =$ im H. □

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