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# Universal Adaptive Stabilization of Nonlinear Systems

ACHIM ILCHMANN ilchmann@maths.exeter.ac.uk
*Department of Mathematics, University of Exeter, North Park Road, Exeter Ex4 4QF, U.K.*

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**Abstract.** Mårtensson's result "The order of a stabilizing regulator is sufficient a priori information for adaptive stabilization" is proved to be valid also for a class of nonlinear, time–varying systems and the feedback strategy is simplified.

**Keywords:** Adaptive control, feedback, nonlinear systems, stabilization, universal stabilization

**Nomenclature**

| | |
|---|---|
| $\mathbb{C}_-$ | open left–half complex plane |
| $\|x\|$ | Euclidean norm |
| $\|K\|$ | $= \sup_{\|x\|\neq 0} \frac{\|K x\|}{\|x\|}, \quad K \in \mathbb{R}^{m\times p}$ |
| $\sigma(A)$ | the spectrum of the matrix $A \in \mathbb{C}^{n\times n}$ |
| $\mathcal{B}_\varepsilon(K_0)$ | $= \{K \in \mathbb{R}^{m\times p} \mid \|K - K_0\| < \varepsilon\}$ for $\varepsilon > 0$ |
| $L_p(I)$ | the vector space of measurable functions $f : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval, $n$ being defined by the context, such that $\|f(\cdot)\|_{L_p(I)} < \infty$, where |

$$\|f(\cdot)\|_{L_p(I)} = \begin{cases} \left[\int_I \|f(s)\|^p ds\right]^{1/p} & \text{for} \quad p \in [1,\infty) \\ \operatorname*{ess\,sup}_{s\in I} \|f(s)\| & \text{for} \quad p = \infty \end{cases}$$

## 1. Introduction

Since more than a decade, questions relating to minimal a priori information for adaptive stabilization and existence of universal controllers for various classes of (uncertain) dynamical systems are studied. In 1985, Mårtensson [6] published his famous result "The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization" and in 1993, Mårtensson and Polderman [7] published a "correction and simplification" of Mårtensson's [6] proof.

In the present note, it is proved that Mårtensson's result is also valid for a class of nonlinear, time–varying systems and, in addition the adaptation law $\dot{k}(t) = \|y(t)\|^2 + \|u(t)\|^2$ is simplified and generalized to $\dot{k}(t) = \|y(t)\|^q$, for arbitrary $q \geq 1$.

An anonymous referee brought to my attention that the paper by Pomet [9] is closely related to the present one. Pomet considers full state feedback and similar conditions on the right hand side of the given unknown system.

Our result is as follows. Consider the class of nonlinear systems

$$\left.\begin{aligned} \dot{x}(t) &= f(t,x(t)) + g(t,x(t))\,u(t) \\ y(t) &= h(t,x(t))\,, \end{aligned}\right\} \tag{1}$$

where

$$\begin{aligned} f &: [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ g &: [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m} \\ h &: [0,\infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^p \end{aligned}$$

are assumed to be *Carathéodory functions*, i.e. $\alpha : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ is called a Carathéodory function, if $\alpha(\cdot,x) : t \mapsto \alpha(t,x)$ is measurable on $\mathbb{R}$ for each $x \in \mathbb{R}^q$, $\alpha(t,\cdot) : x \mapsto \alpha(t,x)$ is continuous on $\mathbb{R}^q$ for all $t \in \mathbb{R}$, and for each compact set $S \subset \mathbb{R} \times \mathbb{R}^q$ there exists an integrable function $m_S(t)$ such that $\|\alpha(t,x)\| \leq m_S(t)$ for all $(t,x) \in S$.

Roughly speaking, systems of the form (1) are supposed to satisfy that the input $u(\cdot)$ enters linearly, the actuator function $g(\cdot,\cdot)$ is globally bounded, the sensor function $h(t,x)$ is uniformly linearly bounded in $x$ and, most importantly, there exists a time-invariant, linear output feedback $u(t) = K_0\,y(t)$ such that the system is globally uniformly exponentially stable. Note that all of these assumptions are only structural, no bounds need to be known.

To be more precise, the following conditions are assumed.

**Class of systems:**

- there exist some $\hat{g}, \hat{h} > 0$ such that

$$\left.\begin{aligned} \|g(t,x)\| &\leq \hat{g} && \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}^n \\ \|h(t,x)\| &\leq \hat{h}\|x\| && \text{for all } (t,x) \in [0,\infty) \times \mathbb{R}^n\,, \end{aligned}\right\} \tag{2}$$

- there exists a $K_0 \in \mathbb{R}^{m\times p}$ such that

$$\begin{aligned} \dot{\eta}(t) &= f_0(t,\eta(t)), \\ f_0(t,\eta(t)) &:= f(t,\eta(t)) + g(t,\eta(t))\,K_0\,h(t,\eta(t)) \end{aligned} \tag{3}$$

  possesses a continuously differentiable Lyapunov-function

$$V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$$

  with constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, $p \geq 1$ such that, for all $t \geq 0$ and all $x \in \mathbb{R}^n$,

$$\alpha_1\|x\|^p \leq V(t,x) \leq \alpha_2\|x\|^p \tag{4}$$

$$\frac{\partial}{\partial t} V(t,x) + \left\langle \frac{\partial}{\partial x} V(t,x),\ f_0(t,x) \right\rangle \le -\alpha_3 \|x\|^p \tag{5}$$

$$\left\| \frac{\partial}{\partial x} V(t,x) \right\| \le \alpha_4 \|x\|^{p-1} . \tag{6}$$

By Lyapunov's direct method, see e.g. [10], p. 173, (4) and (5) ensure that (3) is globally uniformly exponentially stable, i.e. for some (unknown) $M, \lambda > 0$ the solutions of (3) satisfy

$$\|\eta(t)\| \le M e^{-\lambda(t-t_0)} \|\eta(t_0)\| \quad \text{for all} \quad t \ge t_0,\ t_0 \ge 0, \quad \text{and for all} \quad \eta(t_0) \in \mathbb{R}^n . \tag{7}$$

Conversely, if (7) is satisfied, $\left\| \frac{\partial}{\partial x} f_0(t,x) \right\|$ is globally uniformly bounded, and $f_0(\cdot, 0) \equiv 0$, then by the Inverse Lyapunov theory, see e.g. [10], p. 244, (4)–(6) hold true.

For the class of systems (1)–(6) we present a universal adaptive stabilizer of the form

$$\boxed{\begin{aligned} u(t) &= (K \circ \beta \circ k)(t)\, y(t) \\ \dot{k}(t) &= \|y(t)\|^q \quad , \quad k(0) = k_0 \ge 0,\ q \ge 1 . \end{aligned}} \tag{8}$$

(8) is universal in the sense: Whenever the single controller (8) is applied to any system (1) which satisfies (2)–(6), then the closed–loop nonlinear system

$$\left.\begin{aligned} \dot{x}(t) &= f(t,x(t)) + g(t,x(t))\,(K \circ \beta \circ k)(t)\, h(t,x(t)) \ , \quad x(0) = x_0 \\ \dot{k}(t) &= \|h(t,x(t))\|^q \ , \quad k(0) = k_0 \end{aligned}\right\} \tag{9}$$

possesses the properties:

- finite escape time does not occur,
- all states are bounded,
- $k(t)$ converges to a finite value as $t \to \infty$,
- $\lim_{t\to\infty} x(t) = 0$ and $x(\cdot) \in L_i(0,\infty)$ for all $i \in [q, \infty]$.

Note that (8) is of striking simplicity. The crucial part plays the time–varying feedback matrix $(K \circ \beta \circ k)(t)$, where

$$K(\cdot) : (0,\infty) \to \mathbb{R}^{m\times p} \quad \text{is a dense curve in } \mathbb{R}^{m\times p}$$

and

$$\beta(\cdot) : [k_0, \infty) \to (0,\infty) , \qquad k \mapsto \log^{\ell}(k+2)$$

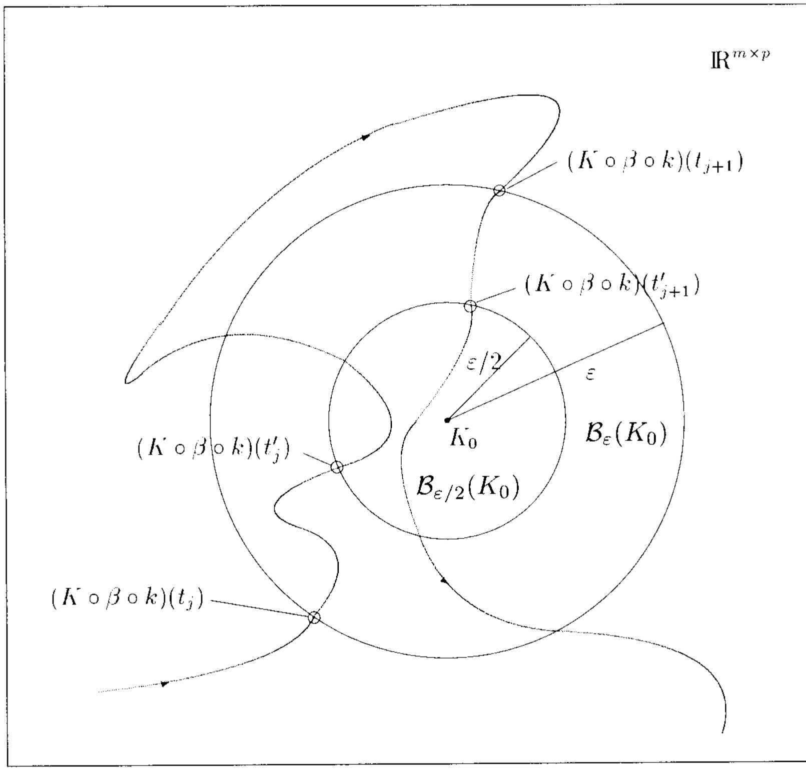


*Figure 1.*

is strictly, but extremely slowly, increasing. $l$ depends on $q$ and needs to be positive and smaller than $[1 + 3q/2]^{-1}$, see Example 2.7.

The intuition behind the control strategy (8) is as follows: Rewrite the first equation in (9) as

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))\, K_0\, h(t, x(t)) \\ &\quad + g(t, x(t))\, [(K \circ \beta \circ k)\,(t) - K_0]\, h(t, x(t))\,. \end{aligned} \tag{10}$$

Then global uniform exponential stability of $\dot{\eta}(t) = f_0(t, \eta(t))$ together with a perturbation result yields exponential decay of the solution of (10) on intervals over which the perturbation in (10) satisfies

$$\begin{aligned} \|g(t, x(t))\,[(K \circ \beta \circ k)\,(t) - K_0]\, h(t, x(t))\| &\le \hat{g}\ \|(K \circ \beta \circ k)\,(t) - K_0\|\, \hat{h}\ \|x(t)\| \\ &\le \varepsilon\, \hat{g}\hat{h} \|x(t)\| \end{aligned}$$

for $\varepsilon > 0$ sufficiently small.

As long as the solution component $x(t)$ of (9) resp. (10) (for sake of simplicity, we do not bother about finite escape time) does not tend to $0$, the gain $k(t) = k_0 + \int_0^t \|y(\tau)\|^q \, d\tau$ is increasing, therefore $\beta(k(t))$ is increasing as well and density of $K((0,\infty))$ in $\mathbb{R}^{m\times p}$ ensures that $(K \circ \beta \circ k)(t)$ will come close to $K_0$. Hence there exists an interval $[t_j, t_j')$ such that

$$(K \circ \beta \circ k)(t) \in \mathcal{B}_\varepsilon(K_0) \setminus \mathcal{B}_{\varepsilon/2}(K_0) \quad \text{for all } t \in [t_j, t_j') .$$

See Figure 1.

Now, by the perturbation result, $x(t)$ is exponentially descreasing as long as $t \in [t_j, t_j')$. However, $\int_{t_j}^\infty \|y(\tau)\|^q \, d\tau + k(t_j)$ may be too large so that $(K \circ \beta \circ k)(t)$ leaves $\mathcal{B}_\varepsilon(K_0)$ for some $t > t_j$. Then the perturbation result is no longer valid and $x(t)$ goes unstable again. But density of $K(0,\infty)$ ensures that the same procedure as described above occurs again. $(K \circ \beta \circ k)(t)$ will stay in $\mathcal{B}_\varepsilon(K_0) \setminus \mathcal{B}_{\varepsilon/2}(K_0)$ for another interval $[t_{j+1}, t_{j+1}')$. This time, $(K \circ \beta \circ k)(t)$ stays within $\mathcal{B}_\varepsilon(K_0) \setminus \mathcal{B}_{\varepsilon/2}(K_0)$ for a longer period, since $\beta(k)$ has become slower. Finally, for some interval $[t_\ell, t_\ell')$, $\beta(k)$ is sufficiently slow so that the system has enough time to settle down.

## 2. Universal adaptive stabilization

In the following theorem we present a universal adaptive stabilizer for the class of nonlinear, time–varying systems which satisfy (2)–(7). The same comment as in [6] is valid here: "The regulator [...] is absolutely useless for every practical purpose, and its value is only on the level of existence proofs, to show that the adaptive control with a certain amount of a priori information is possible."

If the class of systems to be stabilized is more restrictive, e.g. minimum phase, then the search in the parameter space $\mathbb{R}^{m\times p}$ can be chosen more efficiently. This is the topic of a large field of research over the last decade, see e.g. the references in [4].

### 2.1. Theorem

Let $q \geq 1$ and suppose

(A1) $\beta(\cdot) : [k_0, \infty) \to (0, \infty)$ is a differentiable, monotonically increasing and unbounded function,

(A2) $K(\cdot) : (0, \infty) \to \mathbb{R}^{m \times p}$ is a Lebesque integrable matrix–valued function,

$$\{K(\beta) \mid \beta \in (\gamma, \infty)\} \quad \text{is dense in} \quad \mathbb{R}^{m\times p} \quad \text{for every} \quad \gamma > 0\,,$$

and for every compact set $\mathcal{K} \subset \mathbb{R}^{m\times p}$ there exists some $\overline{\beta} = \overline{\beta}(\mathcal{K})$ such that

$$\|K(\beta_1) - K(\beta_2)\| \le \overline{\beta}\, |\beta_1 - \beta_2| \quad \text{for all} \quad \beta_1, \beta_2 \in \{\beta > 0 \mid K(\beta) \in \mathcal{K}\}\,,$$

(A3) $\lim_{k\to\infty} k \cdot \sup_{\kappa \ge k} \left|\frac{d}{d\kappa}\beta(\kappa)\right| \cdot \sup_{\kappa \in [0,k]} \|(K \circ \beta)(\kappa)\|^q = 0.$

Then, the adaptive control strategy (8) applied to any system (1) satisfying (2)–(6) yields a closed–loop system (9) such that there exists a solution $(x(\cdot), k(\cdot)) : [0, \omega) \to \mathbb{R}^{n+1}$ and every solution has on its maximal interval of existence $[0, \omega)$ the properties:

(i) $\omega = \infty$;

(ii) $\lim_{t\to\infty} k(t) = k_\infty \in \mathbb{R}$;

(iii) $\lim_{t\to\infty} x(t) = 0$, and $x(\cdot) \in L_i(0, \infty)$ for all $i \in [q, \infty]$.

### 2.2. Remark

The assumption on the system class that there exists an exponentially stabilizing, time–invariant, linear output feedback $u(t) = K_0\, y(t)$ can be weakened to the existence of a dynamic, exponentially stabilizing compensator of the form

$$\begin{aligned} \dot z(t) &= F\, z(t) + G\, y(t) \\ u(t) &= H\, z(t) + J\, y(t) \end{aligned}$$

for some $(F, G, H, J) \in \mathbb{R}^{\ell\times\ell} \times \mathbb{R}^{\ell\times p} \times \mathbb{R}^{m\times p} \times \mathbb{R}^{p\times p}$ where $\ell$ must be known. See [7] where is it shown how to rearrange the system equations so that Theorem 2.1 can be applied.

### 2.3. Remark

(i) The growth condition on $K(\cdot)$ stated in (A2) is exactly what Polderman and Mårtensson meant in their definition (I), see [7] p. 466. (Private communication with Jan Willem Polderman.)

(ii) The growth condition in (A3) on $\dot{\beta}(k)$ related to $(K \circ \beta \circ k)$ and $k$ is similar to (IV) on p. 466 in [7]. In case of $q = 2$, the additional term $\sup_{\kappa \in [0,k]} \|K \circ \beta(\kappa)\|^2$ is due to the simplification of $\dot{k} = \|y\|^2 + \|u\|^2$ to $\dot{k} = \|y\|^2$.

(iii) That the adaptation law $\dot{k} = \|y\|^2 + \|u\|^2$ can be simplified to $\dot{k} = \|y\|^2$ is a consequence of a general result stated by Morse [8] (the text below Ex.3). However it is not said how to do this and how the feedback law, here the assumption on the growth of $\beta(\cdot)$, has to be modified.

For the proof of Theorem 2.1 a perturbation result on global uniform exponential stable stystems is needed and proved in the following lemma. It can be viewed as a substitute of the Variation-of-Constants formula for linear systems and may be of independent interest.

### 2.4. Lemma

Suppose (3)–(6) are satisfied. Then every solution of the perturbed system

$$\dot{z}(t) = f_0(t, z(t)) + \psi(t), \qquad z(t_0) = z_0 \,, \tag{11}$$

where $\psi(\cdot) : [0, \infty) \to \mathrm{IR}^n$ denotes some locally integrable function, satisfies for some $M, \lambda > 0$ and all $t \geq t_0$, $t_0 \geq 0$,

$$\|z(t)\| \leq M\, e^{-\lambda(t-t_0)}\, \|z_0\| + M \int_{t_0}^{t} e^{-\lambda(t-s)}\, \|\psi(s)\|\, ds \,. \tag{12}$$

**Proof:** Without loss of generality one might assume that $p$ in (4)–(6) equals 1: Set $W(t,x) := V(t,x)^{1/p}$ and (4)–(6) hold true for $W(t,x)$, some $\alpha_1', \ldots, \alpha_4' > 0$ and $p = 1$.

The derivative of $V(t, z(t))$ along the solution of (11) satisfies

$$\begin{aligned} \frac{d}{dt}\, V(t, z(t)) &= \frac{\partial}{\partial t}\, V(t, z(t)) + \left\langle \frac{\partial}{\partial z}\, V(t, z(t)), \dot{z}(t) \right\rangle \\ &\leq -\alpha_3 \|z(t)\| + \alpha_4 \|\psi(t)\| \\ &\leq -\frac{\alpha_3}{\alpha_2}\, V(t, z(t)) + \alpha_4 \|\psi(t)\| \end{aligned}$$

and hence, by using a result on differential inequalities, see e.g. [2], p. 31,

$$V(t,z(t)) \le e^{-\frac{\alpha_3}{\alpha_2}(t-t_0)}\, V(t_0,z(t_0)) + \int_{t_0}^{t} e^{-\frac{\alpha_3}{\alpha_2}(t-s)}\, \alpha_4 \|\psi(s)\|\, ds\,.$$

Applying (4) once more yields (12) for

$$M = \frac{\alpha_2}{\alpha_1} + \frac{\alpha_4}{\alpha_1} \quad \text{and} \quad \lambda = \alpha_3/\alpha_2\,.$$

□

**Proof of Theorem 2.1:** We proceed in several steps. (a) is a general result on the existence of a solution. In (b), we derive three upper bounds on the state $x(\cdot)$ by using only the bounds (2) on $g(t,x)$ and $h(t,x)$, and the existence of an exponentially stabilizing feedback $u(t) = K_0\, y(t)$. In (c), we show that the inequalities in (b) already yield (i)–(iii) if $k(\cdot)$ is bounded. The difficult part of the proof is to show that unboundedness of $k(\cdot)$ yields a contradiction to the growth condition (A3). To this end, in (d) we derive an upper bound on $\|y(\cdot)\|_{L_q(t_0,t)}$ in case that the search of $(K\circ\beta\circ k)(\tau)$ is in a neighbourhood of the stabilizing matrix $K_0$. This is a disturbance result, not based on (A1)–(A3). Only in the final step (e) we use the growth assumption (A1)–(A3) on $K(\cdot)$ and $\beta(\cdot)$.
(a): Since the right hand side of (9) is a Carathéodory function, there exists a solution

$$(x(\cdot),k(\cdot)) \,:\, [0,\omega) \to \mathbb{R}^{n+1} \quad \text{for some maximal } \omega \in (0,\infty]\,.$$

This is a consequence of the classical theory of ordinary differential equations, see e.g. [1].
(b): Applying Lemma 2.4 to (10) yields, for some $M,\lambda > 0$ and arbitrary $0 \le t_0 \le t < \omega$,

$$\begin{aligned}\|x(t)\| &\le M\, e^{-\lambda(t-t_0)}\, \|x(t_0)\| \\ &\quad + \hat{g}\, \|(K\circ\beta\circ k)(\cdot) - K_0\|_{L_\infty(t_0,t)} \int_{t_0}^{t} M\, e^{-\lambda(t-s)} \|y(s)\| ds.\end{aligned} \tag{13}$$

An application of Hölder's inequality, for $\frac{1}{r} + \frac{1}{q} = 1$ and $q > 1$, to (13) yields

$$\begin{aligned}\|x(t)\| &\le M\, e^{-\lambda(t-t_0)}\, \|x(t_0)\| \\ &\quad + M\hat{g}\, \|(K\circ\beta\circ k)(\cdot) - K_0\|_{L_\infty(t_0,t)}\, (r\lambda)^{-1/r}\, \|y(\cdot)\|_{L_q(t_0,t)}\end{aligned} \tag{14}$$

and hence, by using the second equation in (8) and the fact that for positive constants $A, B$ we have $(A+B)^q \le 2^q(A^q + B^q)$, see [5], p. 311,

$$\begin{aligned}\|x(t)\|^q &\le 2^q\, M^q\, e^{-\lambda q(t-t_0)}\, \|x(t_0)\|^q \\ &\quad + 2^q\, M^q\, \hat{g}^q\, \|(K\circ\beta\circ k)(\cdot) - K_0\|^q_{L_\infty(t_0,t)}\, (r\lambda)^{-\frac{q}{r}}\, k(t).\end{aligned} \tag{15}$$

If $q = 1$, replace $(r\lambda)^{-1/r}$ in (14) by 1.

(c): We shall show that $k(\cdot) \in L_\infty(0,\omega)$ implies (i)–(iii).
Suppose $k(\cdot) \in L_\infty(0,\omega)$. If $\omega < \infty$, then $x(\cdot) \in L_\infty(0,\omega)$ by (14) and (8). However, $k(\cdot) \in L_\infty(0,\omega)$ and $x(\cdot) \in L_\infty(0,\omega)$ contradicts maximality of $\omega$. Therefore, $\omega = \infty$. This proves (i), and (ii) follows since $k(t)$ is monotonically increasing and bounded.
It remains to prove (iii). The left hand term in the sum of (14) tends, for arbitrary $t_0$, to 0 as $t$ goes to $\infty$; since $k(\cdot) \in L_\infty(0,\infty)$ is equivalent to $y(\cdot) \in L_q(0,\infty)$, the right hand term can be made arbitrarily small as $t_0 \to \infty$. Therefore, $\lim_{t\to\infty} x(t) = 0$. This proves the first statement in (iii). Since $y(\cdot) \in L_q(0,\infty)$, we know that the convolution $t \mapsto \int_0^t e^{-\lambda(t-s)} \|y(s)\| \, ds$ belongs to $L_q(0,\infty)$, as well, see e.g. [5], p. 374. Hence, by (13), $x(\cdot) \in L_q(0,\infty)$. Now the second statement is a trivial consequence of $x(\cdot) \in L_q(0,\infty) \cap L_\infty(0,\infty)$.

(d): In order to prove boundedness of $k(\cdot)$ on $[0,\omega)$, it will be proved that the inequality

$$\|y(\cdot)\|^q_{L_q(t_0,t)} \;\le\; \hat{h}^q\, M^q\, \frac{2}{(2q-1)\lambda}\, \|x(t_0)\|^q \tag{16}$$

holds true if

$$(K \circ \beta \circ k)(\tau) \in \mathcal{B}_\varepsilon(K_0) \quad \text{for all} \quad \tau \in [t_0,t) \quad \text{and} \quad \varepsilon := \frac{\lambda}{2q\hat{g}\, M\, \hat{h}}\,. \tag{17}$$

To prove this, note that (2) and (17) applied to (13) gives, for all $\tau \in [t_0,t)$,

$$\|x(\tau)\| \;\le\; M\, e^{-\lambda(\tau-t_0)}\, \|x(t_0)\| \;+\; \hat{g}\,\varepsilon\, M\, \hat{h} \int_{t_0}^{\tau} e^{-\lambda(\tau-s)}\, \|x(s)\|\, ds$$

or, equivalently,

$$\|e^{\lambda\tau}\, x(\tau)\| \;\le\; M \|e^{\lambda t_0}\, x(t_0)\| \;+\; \frac{\lambda}{2q} \int_{t_0}^{\tau} \|e^{\lambda s}\, x(s)\|\, ds\,,$$

and hence, by the Bellman–Gronwall–Inequality,

$$\|x(\tau)\| \;\le\; M\, e^{\left[-\lambda+\frac{\lambda}{2q}\right](\tau-t_0)}\, \|x(t_0)\| \;=\; M\, e^{\frac{-2q+1}{2q}\,\lambda\,(\tau-t_0)}\, \|x(t_0)\|\,,$$

whence

$$\int_{t_0}^{t} \|y(\tau)\|^q\, d\tau \;\le\; \hat{h}^q\, M^q \int_{t_0}^{t} e^{\frac{-2q+1}{2}\,\lambda\,(\tau-t_0)}\, \|x(t_0)\|^q\, d\tau$$

and (16) follows.

(e): Seeking a contradiction, suppose now $k(\cdot) \notin L_\infty(0,\omega)$. Let $\varepsilon$ be defined as in (17). Then, by assumption (A1) and (A2), $(K \circ \beta \circ k)(t)$ will travel through

$\mathcal{B}_\varepsilon(K_0) \setminus \mathcal{B}_{\varepsilon/2}(K_0)$ again and again as $t \to \omega$. More precisely, there exists a sequence of intervals

$$(t_j, t_j') \quad \text{with} \quad 0 < t_j < t_j' < t_{j+1} < t_{j+1}', \qquad j \in \mathbb{N}$$

with $\lim_{j\to\infty} t_j = \omega$ and

$$(K \circ \beta \circ k)(\tau) \in \mathcal{B}_\varepsilon(K_0) \setminus \mathcal{B}_{\varepsilon/2}(K_0) \quad \text{for all} \quad \tau \in (t_j, t_j'). \tag{18}$$

See Figure 1. Therefore, by (A2), there exists some $\overline{\beta} = \overline{\beta}(\mathcal{B}_\varepsilon(K_0)) > 0$, such that

$$\begin{aligned}
\frac{\varepsilon}{2} &\leq \left\| (K \circ \beta \circ k)(t_j') - (K \circ \beta \circ k)(t_j) \right\| \\
&\leq \overline{\beta} |(\beta \circ k)(t_j') - (\beta \circ k)(t_j)| \\
&= \overline{\beta} \left| \int_{t_j}^{t_j'} \frac{d}{d\tau} (\beta \circ k)(\tau)\, d\tau \right| \\
&\leq \overline{\beta} \sup_{k \in I_j} \left| \frac{d}{dk} \beta(k) \right| \int_{t_j}^{t_j'} \dot{k}(\tau)\, d\tau
\end{aligned} \tag{19}$$

where

$$I_j := (k(t_j), k(t_j')), \qquad j \in \mathbb{N}.$$

Since (17) is satisfied by (18), applying (16) to (19) yields

$$\frac{\varepsilon}{2} \leq \overline{\beta} \sup_{k \in I_j} \left| \frac{d}{dk} \beta(k) \right| \frac{2\hat{\beta}^q M^q}{(2q-1)\lambda} \|x(t_j)\|^q \tag{20}$$

whence, by (15),

$$\frac{\varepsilon}{2} \leq \frac{\overline{\beta}\, 2\hat{\beta}^q M^q}{(2q-1)\lambda} \sup_{k \in I_j} \left| \frac{d}{dk} \beta(k) \right| 2^q M^q$$

$$\cdot \left[ \|x(0)\|^q + \hat{g}^q (r\lambda)^{-\frac{q}{r}} \|(K \circ \beta \circ k)(\cdot) - K_0\|^q_{L_\infty(0,t_j)} k(t_j) \right]. \tag{21}$$

Since $k(\cdot) \notin L_\infty(0, \omega)$, by assumption (A3) the right hand side of (21) tends to $0$ as $j \to \infty$, which contradicts that it is greater or equal than $\frac{\varepsilon}{2}$. Therefore, $k(\cdot) \in L_\infty(0, \omega)$ is proved and the proof of the theorem is complete. □

## 2.5. Example (SISO)

If $m = p = 1$, i.e. in case of single–input single–output systems, functions $K(\cdot)$ and $\beta(\cdot)$ which satisfy the assumptions (A1)–(A3) in Theorem 2.1 are given by

$$\beta(k) \; := \; \sqrt{\log(k+2)} \quad \text{and} \quad K(\beta) \; := \; \beta^{\frac{1}{2q}} \sin \beta \, .$$

In order to present an example for the multivariable case, the following technical lemma is needed.

## 2.6. Lemma

Set

$$\varphi(\beta) \; := \; \sum_{i=1}^{N} \sin^2\left(\pi \sqrt{p_i}\, \beta\right) ,$$

where $p_1 = 1$, and $p_2, \ldots, p_N$ denote the $i$–th prime, resp. Then, for any $\alpha > 2$, we have

$$\lim_{\beta \to \infty} \varphi(\beta)\, \beta^{\alpha} \; = \; \infty \, .$$

**Proof:** It suffices to prove the claim for $N = 2$, however this does not simplify the proof. Suppose the contrary of the statement, i.e. the existence of some $c > 0$ and a sequence $\{\beta_n\}_{n \in \mathbb{N}}$ with $\beta_n < \beta_{n+1}$, $\lim_{n\to\infty} \beta_n = \infty$ such that

$$\varphi(\beta_n)\, \beta_n^{\alpha} \; \leq \; c \quad \text{for all} \quad n \in \mathbb{N} \, . \tag{22}$$

Choose a decomposition

$$\begin{aligned} \beta_n \; &= \; g_n + \varepsilon_n \quad \text{for} \quad g_n \in \mathbb{N} \, , \quad \varepsilon_n \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \, , \\ \sqrt{p_N}\, g_n \; &= \; \gamma_n + \delta_n \quad \text{for} \quad \gamma_n \in \mathbb{N} \, , \quad \delta_n \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \, . \end{aligned}$$

From the proof of Liouville's Theorem as given in Hardy and Wright [3], p. 161, we conclude that there exists a constant $A$, depending on $\sqrt{p_N}$, such that

$$\left| \frac{\delta_n}{g_n} \right| \; = \; \left| \sqrt{p_N} \; - \; \frac{\gamma_n}{g_n} \right| \; > \; \frac{A}{g_n^2}$$

and hence

$$\frac{A}{g_n} \; < \; |\delta_n| \, . \tag{23}$$

Since, by assumption (22),

$$\sin^2(\pi\,\varepsilon_n) \;=\; \sin^2(\pi\,\sqrt{p_1}\,\beta_n) \;\le\; \varphi(\beta_n) \;\le\; \frac{c}{\beta_n^{\alpha}}\,,$$

it follows that $\lim_{n\to\infty}\varepsilon_n = 0$. Hence, the inequality

$$\sin^2\,(\pi\,\sqrt{p_N}\,\beta_n) \;=\; \sin^2\,(\pi\,\delta_n + \pi\,\sqrt{p_N}\,\varepsilon_n) \;\le\; \frac{c}{\beta_n^{\alpha}}$$

yields as well $\lim_{n\to\infty}\delta_n = 0$. This gives, for some $n_0 \in \mathbb{N}$ sufficiently large,

$$\frac{\pi}{2}\,|\varepsilon_n| \;\le\; \sin(\pi|\varepsilon_n|) \;\le\; \frac{\sqrt{c}}{\beta_n^{\alpha/2}} \qquad \text{for all} \quad n \ge n_0 \tag{24}$$

and

$$\frac{\pi}{2}\,|\delta_n + \sqrt{p_N}\,\varepsilon_n| \;\le\; \sin\,(|\pi\,\delta_n + \pi\,\sqrt{p_N}\,\varepsilon_n|) \;\le\; \frac{\sqrt{c}}{\beta_n^{\alpha/2}} \qquad \text{for all} \quad n \ge n_0\,.$$

Therefore,

$$\left|\beta_n^{\alpha/2}\,\delta_n + \sqrt{p_N}\,\beta_n^{\alpha/2}\,\varepsilon_n\right| \;\le\; \frac{2\sqrt{c}}{\pi} \qquad \text{for all} \quad n \ge n_0\,. \tag{25}$$

However, consider the sum in (25): The right term is, by (24), bounded. An application of (23) yields

$$\beta_n^{\alpha/2}\,|\delta_n| \;>\; \beta_n^{\alpha/2}\,\frac{A}{g_n} \;=\; A\,\frac{(g_n+\varepsilon_n)^{\alpha/2}}{g_n}\,.$$

Since $\lim_{n\to\infty} g_n = \infty$ and $\alpha > 2$, the left term is unbounded, which contradicts (25). Hence the proof is complete. □

In the following example, it is shown that for multi-input, multi-output systems (1) belonging to the class (2)–(6) there actually exists a feedback satisfying (A1)–(A3), and hence the statement of Theorem 2.1 is non-vacuous.

### 2.7. Example (MIMO)

To present an example of functions $K(\cdot)$ and $\beta(\cdot)$ which satisfy (A1)–(A3) in case of $mp > 1$, identify $\mathbb{R}^{m\times p}$ with $\mathbb{R}^N$, $N := mp$, and set

$$\beta(k) \;:=\; \log^{\ell}(k+2), \qquad \text{where} \quad \ell \in \left(0, \tfrac{2}{2+3q}\right)\,,$$

and

$$K(\beta) \;:=\; (\chi\circ\xi\circ\mu)\,(\beta) \;=\; \left[\frac{1}{\sum_{i=1}^{N}\sin^2 a_i\,\beta} - 1\right]\,(\sin a_1\,\beta,\ldots,\sin a_N\,\beta)^T$$

and

$$\begin{array}{llll} \mu(\cdot): & (0,\infty) & \to & T := \mathbb{R}^N/2\pi\mathbb{Z} \\ & \beta & \mapsto & (a_1\beta \bmod 2\pi, \dots, a_N\beta \bmod 2\pi)^T, \\ \xi(\cdot): & T & \to & [-1,1]^N \\ & (\mu_1,\dots,\mu_N) & \mapsto & (\sin\mu_1,\dots,\sin\mu_N)^T, \\ \chi(\cdot): & [-1,1]^N\setminus\{0\} & \to & \mathbb{R}^N \\ & \xi & \mapsto & \left(\frac{1}{\|\xi\|} - \|\xi\|\right)\frac{\xi}{\|\xi\|}, \end{array}$$

and $a_i = \pi\sqrt{p_i}$ for $i = 1,\dots,N$ are chosen as in Lemma 2.6. Then $a_1,\dots,a_N$ are linearly independent over $\mathbb{Q}$.

The function $K(\cdot)$ has been introduced by [7] and proved to satisfy the density condition (A2) as follows: By Kronecker's theorem (see e.g. Theorem 444 in [3]) $\mu((0,\infty))$ is dense in $\mathbb{R}^N$. $\|(\xi\circ\mu)(\beta)\| \neq 0$ since $a_1,\dots,a_N$ are linearly independent. $\chi(\cdot)$ is surjective since $\left(\frac{1}{\|\xi\|} - \|\xi\|\right)$ takes all values in $\mathbb{R}$ as $\|\xi\| \in [-1,1]\setminus\{0\}$. Since $\xi(\cdot)$ resp. $\chi(\cdot)$ is surjective, it easily follows that density of $\mu((0,\infty))$ implies density of $\xi\circ\mu((0,\infty))$ resp. $\chi\circ\xi\circ\mu((0,\infty))$.

We shall show that the remaining condition in (A2) is satisfied as well:
Using the notation of Lemma 2.6, for every compact set $\mathcal{K}\subset\mathbb{R}^N$ there exists some $L_1 = L_1(\mathcal{K}) > 0$ such that

$$\|K(\beta)\| = \left|\frac{1}{\sqrt{\varphi(\beta)}} - \sqrt{\varphi(\beta)}\right| < L_1 \quad \text{for all } \beta > 0 \text{ such that } \quad K(\beta)\subset\mathcal{K}.$$

Therefore, there exists some $L_2 > 0$ such that

$$L_2 < \varphi(\beta) \quad \text{for all} \quad \beta > 0 \quad \text{with} \quad K(\beta)\in\mathcal{K}.$$

Now the inequality in (A2) is satisfied since

$$\begin{aligned} \left\|\frac{d}{d\beta}K(\beta)\right\| &\le \frac{2\sqrt{N}\sum_{i=1}^N a_i}{\varphi(\beta)^2} + \left|\frac{1}{\varphi(g\beta)} - 1\right|\sqrt{\sum_{i=1}^N a_i^2} \\ &\le \frac{2\sqrt{N}\sum_{i=1}^N a_i}{L_2^2} + \left(\frac{1}{L_2}+1\right)\sqrt{\sum_{i=1}^N a_i^2}. \end{aligned}$$

It remains to prove (A3). By Lemma 2.6, we have, for $\xi = (\xi \circ \mu \circ \beta)(k)$ and $\bar{k} > 0$ sufficiently large,

$$\|(K \circ \beta)(k)\| = \left\| \left( \tfrac{1}{\|\xi\|} - \|\xi\| \right) \tfrac{\xi}{\|\xi\|} \right\| \leq \tfrac{1}{\|\xi\|} + \|\xi\| \leq \frac{1}{\sqrt{\varphi(\beta(k))}} + \sqrt{N}$$
$$\leq \beta(k)^{3/2} + \sqrt{N}$$

for all $k \geq \bar{k}$.

Therefore, for $k \geq \bar{k}$,

$$\sup_{\kappa \in [0,k]} \|(K \circ \beta)(\kappa)\|^q \leq q^2 \beta^{\frac{3q}{2}} + q^2 N^{\frac{q}{2}},$$

and since

$$\frac{d}{dk} \beta(k) \leq \frac{1}{2} \log^{\ell-1}(k+2) \frac{1}{k+2}$$

and $\ell - 1 + \frac{3}{2} q\ell < 0$, we may conclude that

$$\lim_{k \to \infty} k \cdot \sup_{\kappa \geq k} \left\| \tfrac{d}{d\kappa} \beta(\kappa) \right\| \cdot \sup_{\kappa \in [0,k]} \|(K \circ \beta)(\kappa)\|^q$$
$$\leq \lim_{k \to \infty} k \log^{\ell-1}(k+2) \tfrac{1}{k+2} q^2 \left[ \beta(k)^{\frac{3q}{2}} + N^{\frac{q}{2}} \right] = 0.$$

This proves (A3) and hence $\beta(k)$ and $K(\beta)$ as defined above satisfy (A1)–(A3).

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