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A stability radius for time-varying linear systems

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Notation

$\mathbb{R}_+ = \{z \in \mathbb{R} | z \geq 0\}$
$\mathbb{C}_- = \{z \in \mathbb{C} | Re z < 0\}$
$\sigma(A)$ spectrum of $A \in \mathbb{C}^{n \times n}$
$GL_n(\mathbb{C})$ the set of all invertible matrices $T \in \mathbb{C}^{n \times n}$
$\|x\|$ Euclidean norm of $x \in \mathbb{C}^n$
$\|D\|$ induced operator norm for $D \in \mathbb{C}^{m \times p}$
$\|D(\cdot)\|_{L_\infty} = \sup_{t_0 < t < t_1} \{\|D(t)\|\}$ for $D(\cdot) \in PC\left((t_0, t_1); \mathbb{C}^{m \times p}\right)$

$L_2(t_0, t_1; \mathbb{C}^m)$	space of functions $u : (t_0, t_1) \to \mathbb{C}^m$ s. t. $t \mapsto \|u(t)\|^2$ is integrable over (t_0, t_1)
$PC((t_0, t_1); \mathbb{C}^{n \times m})$	set of piecewise continuous matrix functions $D(\cdot) : (t_0, t_1) \to \mathbb{C}^{n \times m}$
$PC_b((t_0, t_1); \mathbb{C}^{n \times m})$	set of all bounded matrix functions in $PC((t_0, t_1); \mathbb{C}^{n \times m})$
$PC^1((t_0, t_1); GL_n(\mathbb{C}))$	set of all piecewise continuously differentiable functions $D(\cdot) : (t_0, t_1) \to GL_n(\mathbb{C})$
$C^1(t_0, t_1; \mathbb{C}^{n \times m})$	set of all continuously differentiable $D(\cdot) : (t_0, t_1) \to \mathbb{C}^{n \times m}$

1 Introduction

In recent years problems of robust stability have received a good deal of attention. Most of the work on time-invariant linear systems – including the successful H^∞-approach (see [4], [12]) – is based on transform techniques. However, in [7], [8] a state space approach via the concept of ***stability radius*** is proposed. In the present paper this approach is extended to a time-varying setting.

Consider a nominal system of the form

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0 \tag{1.1}$$

where $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$. Assume that (1.1) is ***exponentially stable***, i.e. there exist M , $w > 0$ so that

$$\|\phi(t,s)\| \leq Me^{-w(t-s)} \quad \text{for all } t \geq s \geq 0 \tag{1.2}$$

where $\phi(t,s)$ denotes the transition matrix of (1.1). Many authors (see [1], [2], [3], [5], [10]) have determined bounds $\delta > 0$ so that exponential stability of the disturbed system

$$\dot{x}(t) = [A(t) + D(t)]x(t) \quad ,t \geq 0 \tag{1.3}$$

is preserved whenever

$$\|D(\cdot)\|_{L_\infty} < \delta \qquad \text{for} \quad D(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n}). \tag{1.4}$$

These bounds are conservative. Our problem is to determine a ***sharp*** upper bound. We call this bound the (complex)[1] ***stability radius*** and define it by

$$r_{\mathbb{C}}(A) = \inf\{\|D(\cdot)\|_{L_\infty} | D \in PC_b(\mathbb{R}_+, \mathbb{C}^{n\times n}) \text{ and (1.3) is not exponentially stable }\} \tag{1.5}$$

We also consider the case where A is subjected to ***structured*** pertubations, so that the perturbed system is

$$\dot{x}(t) \;=\; [A(t) + B(t)D(t)C(t)]\,x(t), \quad t \geq 0 \tag{1.6}$$

where $D(\cdot) \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p})$ is an ***unknown*** bounded time-varying disturbance matrix and $B(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times m})$, $C(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{p\times n})$ are ***given*** "scaling matrices" defining the "structure" of the perturbation. Then the ***structured stability***

[1]The real stability radius is defined analogously. In spite of its prime importance it is not studied here, since even in the time-invariant setup only rudimentary results are available.

radius is

$$r_{\mathbb{C}}(A;B,C) = \inf\{\|D(\cdot)\|_{L_\infty} | D \in PC_b(\mathbb{R}_+, \mathbb{C}^{m\times p}) \text{ and (1.6) is not exponentially stable }\} \quad (1.7)$$

In the unstructured case $r_{\mathbb{C}}(A)$ is simply the distance of (1.1) from the set of not exponentially stable systems with respect to the L_∞-norm.

Remark 1.1 The following properties are easily obtained:

(a) $r_{\mathbb{C}}(A) = 0 \iff$ (1.1) is not exponentially stable

(b) $r_{\mathbb{C}}(\alpha A) = \alpha r_{\mathbb{C}}(A)$ for all $\alpha \geq 0$

(c) $A(\cdot) \mapsto r_{\mathbb{C}}(A)$ is continuous on $PC_b(\mathbb{R}_+, \mathbb{C}^{n\times n})$

2 Bohl exponent and Bohl transformation

For the stability behaviour of (1.1) the number

$$k_B(A) := \mathit{inf}\,\{-w \in \mathbb{R} | \exists M_w > 0 : t \geq s \geq 0 \Longrightarrow \|\phi(t,s)\| \leq M_w e^{-w(t-s)}\} \quad (2.1)$$

introduced by Bohl [1] is useful. We call $k_B(A)$ the *Bohl exponent* of (1.1). It is possible that $k_B(A) = \pm\infty$. The following properties are easily seen.

Proposition 2.1 Let $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$. Then

(a) $k_B(A) < 0 \iff$ (1.1) is exponentially stable

(b) If $A(\cdot) \equiv A \in \mathbb{C}^{n\times n}$ then

$$k_B(A) = \max_{i\in \underline{n}} Re\lambda_i(A), \quad \text{where } \lambda_i(A) \text{ are the eigenvalues of } A.$$

(c) In the scalar case, i.e. $n = 1$, we have

$$r_{\mathbb{C}}(A) = -k_B(A)$$

(d) For the matrix case only an inequality is valid:

$$r_{\mathbb{C}}(A) \leq -k_B(A)$$

Remark 2.2 We want to emphasize that $k_B(A)$ may be a bad indicator for the robustness margin of (1.1). Consider

$$A_k = -\begin{bmatrix} k & k^3 \\ 0 & k \end{bmatrix}, \quad D_k = k^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \text{for } k \in \mathbb{N}$$

Then $\lim_{k\to\infty} k_B(A_k) = -\infty$. However, $\sigma(A_k+D_k) = \{\frac{1}{k}, \frac{1}{k}-2k\}$ although $\lim_{k\to\infty} \|D_k\| = 0$. Thus $\lim_{k\to\infty} r_{\mathbb{C}}(A_k) = 0$.

The following properties of the Bohl exponent can be found in [3].

Proposition 2.3 Let $A(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}^{n\times n})$. Then

(a) $k_B(A)$ is finite if $A(\cdot)$ is *bounded.*

(b) $k_B(A)$ is finite iff $\sup_{0\le|t-s|\le 1} \|\phi(t,s)\| < \infty$.

(c) If $k_B(A) < \infty$ then

$$k_B(A) = \limsup_{s,t-s\to\infty} \frac{\log\|\phi(t,s)\|}{t-s}.$$

We now analyse the effect of time-varying linear coordinate transformations

$$z(t) = T(t)^{-1}x(t), \quad T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C})) \tag{2.2}$$

on the system (1.1) which yields

$$\dot z(t) = \hat A(t) z(t), \quad \text{where } \hat A = T^{-1}AT - T^{-1}\dot T \tag{2.3}$$

These transformations will not, in general, preserve exponential stability. Therefore we introduce the set of *Bohl transformations* $\mathcal{B}_n$, i.e. the set of all $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ such that

$$\inf\{\varepsilon \in \mathbb{R} | \exists M_\varepsilon > 0 : \forall t,s \ge 0 \Rightarrow \|T(t)^{-1}\| \cdot \|T(s)\| \le M_\varepsilon e^{\varepsilon|t-s|}\} = 0 \tag{2.4}$$

Remark 2.4 It is obvious that

(a) the set $\mathcal{B}_n$ forms a group with respect to (pointwise) multiplication

(b) $\mathcal{B}_n$ contains the group of *Lyapunov transformations*, i.e. all $T(\cdot) \in PC^1(\mathbb{R}_+, GL_n(\mathbb{C}))$ so that $T(\cdot), T(\cdot)^{-1}, \dot T(\cdot)$ are bounded,

(c) $k_B(A) = k_B(T^{-1}AT - T^{-1}\dot T)$ for all $T \in \mathcal{B}_n$

The following proposition shows that even in the time-invariant case a similarity transformation may drastically change the stability radius.

Proposition 2.5 [7] If $A \in \mathbb{C}^{n\times n}$ with $\sigma(A) \subset \mathbb{C}_-$ then $\{r_{\mathbb{C}}(T^{-1}AT);\ T \in GL_n(\mathbb{C})\}$ is equal to the interval $(0, -\max_{i\in\underline{n}} Re\lambda_i(A)]$ with possible exception of the right extremum.

In the scalar case we can prove

Proposition 2.5 If

$$\dot{x}(t) = a(t)x(t), \quad a(\cdot) \in PC(\mathbb{R}_+, \mathbb{C}) \tag{2.5}$$

has a *strict* Bohl exponent, i.e.

$$k_B(a) = \lim_{s,t-s\to\infty} \frac{\log \|\phi(t,s)\|}{t-s}$$

then there exists $\Theta \in \mathcal{B}_1$ so that $z(t) = \Theta(t)^{-1}x(t)$ converts (2.5) into

$$\dot{z}(t) = k_B(a)z(t)$$

3 The perturbation operator

In the time-invariant setup, where $(A, B, C) \in \mathbb{C}^{n\times n}\times\mathbb{C}^{n\times m}\times\mathbb{C}^{p\times n}$, the structured stability radius can be characterized via the convolution operator

$$\begin{aligned} L_0 : L_2(0,\infty;\mathbb{C}^m) &\to L_2(0,\infty;\mathbb{C}^p) \\ u(\cdot) &\mapsto (t \mapsto \int_0^t Ce^{A(t-s)}Bu(s)ds) \end{aligned} \tag{3.1}$$

as follows

Proposition 3.1 [8] If $\sigma(A) \subset \mathbb{C}_-$ and $G(s) := C(sI_n - A)^{-1}B$ then

$$r_{\mathbb{C}}(A,B,C) = \begin{cases} \|L_0\|^{-1} = \left[\max_{w\in\mathbb{R}} \|G(iw)\|\right]^{-1} & \text{if } G \neq 0 \\ \infty & \text{if } G = 0 \end{cases}$$

In order to explore the possibility of obtaining similar results for time-varying systems, we consider the parametrized family of *perturbation operators* $(L_{t_0}^{\Sigma})_{t_0\in\mathbb{R}_+}$ defined by

$$\begin{array}{rcl} L_{t_o}^{\Sigma} : L_2(t_0,\infty;\mathbb{C}^m) & \to & L_2(t_0,\infty;\mathbb{C}^p), \quad t_0 \geq 0 \\ u(\cdot) & \mapsto & (t \mapsto \int_{t_0}^{t} C(t)\phi(t,s)B(s)u(s)ds) \end{array} \tag{3.2}$$

associated with

$$\Sigma = (A,B,C) \in PC(\mathbb{R}_+,\mathbb{C}^{n\times n}) \times PC_b(\mathbb{R}_+,\mathbb{C}^{n\times m}) \times \\ \times PC_b(\mathbb{R}_+,\mathbb{C}^{p\times n}),\ k_B(A) < 0 \tag{3.3}$$

Basic properties of $L_{t_0}^{\Sigma}$ are summarized in the following

Proposition 3.2 [6]

(a) $L_{t_0}^{\Sigma}$ is a bounded operator
(b) $t_0 \mapsto \|L_{t_0}^{\Sigma}\|$ is monotonically decreasing on $\mathbb{R}_+$
(c) $\|L_{t_0}^{\Sigma}\| = \|L_{t_1}^{\Sigma}\|$ for all $t_0, t_1 \in \mathbb{R}_+$ if A, B, C are periodic with a common period
(d) $\|L_{t_0}^{\Sigma}\|^{-1} \leq r_{\mathbb{C}}(A;B,C)$
(e) For the unstructured case, i.e. $B(\cdot) = C(\cdot) = I_n$, if $M, w > 0$ satisfy (1.2) then

$$\tfrac{w}{M} \leq \|L_{t_0}^{\Sigma}\|^{-1} \leq \lim_{t_0\to\infty} \|L_{t_0}^{\Sigma}\|^{-1} \leq r_{\mathbb{C}}(A)$$

As opposed to the time-invariant case $\|L_{t_0}^{\Sigma}\|^{-1}$ or $\lim_{t_0\to\infty} \|L_{t_0}^{\Sigma}\|^{-1}$ do not necessarily coincide with $r_{\mathbb{C}}(A;B,C)$. Even in the simple case when $a(\cdot) \in PC(\mathbb{R}_+,\mathbb{R})$ is periodic and $b = c = 1$, we have worked out an example in [6] for which

$$\|L_{t_0}^{\Sigma}\|^{-1} = \|L_{t_1}^{\Sigma}\|^{-1} < r_{\mathbb{C}}(a) \quad \text{for all} \quad t_0, t_1 \in \mathbb{R}_+$$

However note that scalar Bohl transformations $\Theta \in \mathcal{B}_1$ do not change the stability radius but will change the norm of the perturbation operator. Let

$$\Sigma_{\Theta} := (A - \tfrac{\dot{\Theta}}{\Theta} I_n;\ B, C)$$

By using Proposition 2.5 and 3.1 one can show

Proposition 3.3 Suppose $a(\cdot) \in PC(\mathbb{R}_+,\mathbb{C})$ has a strict Bohl exponent $k_B(a) < 0$ and $b, c \in \mathbb{C}$. Then there exists a $\Theta \in \mathcal{B}_1$ such that

$$r_{\mathbb{C}}(a;b,c) = \|L_{t_0}^{\Sigma_{\Theta}}\|^{-1} \quad \text{for all} \quad t_0 \geq 0.$$

For the matrix case we have the following

Conjecture 3.4 Suppose $\Sigma = (A, B, C)$ satisfies (3.3), then

$$r_{\mathbb{C}}(A; B, C) = \sup_{\Theta \in \mathcal{B}_1} \left\{ \lim_{t_0 \to \infty} \|L_{t_0}^{\Sigma_\Theta}\|^{-1} \right\}$$

4 The associated parametrized differential Riccati equation

In the time-invariant setup another useful characterization of $r_{\mathbb{C}}(A; B, C)$ is possible via the parametrized *algebraic* Riccati equation, $\mathbf{ARE}_\rho$

$$A^* P + PA - \rho C^* C - PBB^* P = 0, \quad \rho \in \mathbb{R}$$

Proposition 4.1 [8] Suppose $(A, B, C) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times m}$ and $\sigma(A) \subset \mathbb{C}_-$:

(a) If $-\infty < \rho < r_{\mathbb{C}}^2(A; B, C)$ then there exists a unique stabilizing Hermitian solution P_ρ of $\mathbf{ARE}_\rho$, that is a solution $P_\rho = P_\rho^*$ which satisfies
$\sigma(A - BB^* P_\rho) \subset \mathbb{C}_-$.
If $\rho = r_{\mathbb{C}}^2$ then there exists a unique Hermitian solution $P_{r_{\mathbb{C}}^2}$ of $\mathbf{ARE}_{r_{\mathbb{C}}^2}$ having the property $\sigma(A - BB^* P_{r_{\mathbb{C}}^2}) \subset \overline{\mathbb{C}_-}$.

(b) If there exists a Hermitian solution P_ρ of $\mathbf{ARE}_\rho$ then necessarily $\rho \le r_{\mathbb{C}}^2(A; B, C)$.

Guided by this result we study, in the time-varying setting, the parametrized *differential* Riccati equation, $\mathbf{DRE}_\rho$

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - \rho C^*(t)C(t) - P(t)B(t)B^*(t)P(t) = 0, \quad t \ge t_0$$

associated with the system

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad & x(t_0) = x_0 \in \mathbb{C}^n \\ y(t) &= C(t)x(t), & t \ge t_0 \ge 0 \end{aligned}\right\} \tag{4.1}$$

Throughout this section we assume $\Sigma = (A, B, C)$ satisfies (3.3).

Kalman [9] and *Reghis and Megan* [11] among others, have studied differential Riccati equations, however their results cannot be applied to $\mathbf{DRE}_\rho$ if $\rho > 0$.

Just as in the time-invariant case we consider the parametrized optimal control problem $\mathbf{OCP}_\rho$:

Minimize over $u \in L_2(t_0, t_1; \mathbb{C}^m)$

$$J_\rho(x_0, (t_0, t_1), u(\cdot)) = \int_{t_0}^{t_1} [\|u(s)\|^2 - \rho \|y(s)\|^2] ds$$

where $y(\cdot)$ is defined via (4.1) and $\rho \in \mathbb{R}$. If $\rho < 0$ this is the usual linear quadratic regulator problem **LQR**, whereas in our situation $\rho > 0$, so that the state penalty is negative. To consider a cost functional with negative state penalty is quite natural in this context since we are concerned with a minimum norm *destabilization* problem while the classical **LQR** problem is concerned with stabilization.

The analysis of the $\mathbf{DRE}_\rho$ and its relation to the $\mathbf{OCP}_\rho$ is quite involved, details may be found in [6]. Here we only state the main results.

Proposition 4.2 (finite time) If $\rho < \|L_{t_0}^\Sigma\|^{-2}, \ 0 \le t_0 < t_1 < \infty$, then

(a) there exists a unique Hermitian solution $P^{t_1}(\cdot)$ of $\mathbf{DRE}_\rho$ on $[t_0, t_1]$ with $P^{t_1}(t_1) = 0$

(b) $P^{t_1}(t) \le 0$ (resp. ≥ 0) for all $t \in [t_1, t_0]$ if $\rho \ge 0$ (resp. $\rho \le 0$).

(c) the minimal cost of $\mathbf{OCP}_\rho$ is

$$\inf_{u \in L_2(t_0, t_1; \mathbb{C}^m)} J_\rho(x_0, (t_0, t_1), u(\cdot)) = < x_0, P^{t_1}(t_0) x_0 >$$

(d) the optimal control is given by

$$u(t) = -B^*(t) P^{t_1}(t) x(t)$$

where $x(\cdot)$ solves $\dot{x}(t) = [A - BB^* P^{t_1}](t) x(t), \ x(t_0) = x_0$.

The next proposition is obtained by studying what happens if $t_1 \to \infty$.

Proposition 4.3 (infinite time) If $\rho < \|L_{t_0}^\Sigma\|^{-2}, \ t_0 \ge 0$, then

(a) $P^+(t) = \lim_{t_1 \to \infty} P^{t_1}(t)$ exists for all $t \ge t_0$ and yields a bounded Hermitian solution of $\mathbf{DRE}_\rho$;

(b) $P^+(\cdot)$ is the only solution so that $k_B(A - BB^* P^+) < 0$;

(c) for any other bounded Hermitian solution $Q(\cdot) \in C^1(t_0', \infty; \mathbb{C}^{n \times n})$, $t_0' \ge t_0$ of $\mathbf{DRE}_\rho$ we have $Q(t) \le P^+(t)$ for all $t \ge t_0'$;

(d) the minimal cost is

$$\inf_{u \in L^2(t_0, \infty; \mathbb{C}^m)} J_\rho(x_0, (t_0, \infty), u(\cdot)) = < x_0, P^+(t_0) x_0 >;$$

(e) the optimal control is

$$u(t) = -B^*(t) P^+(t) x(t), \quad t \ge t_0$$

where $x(\cdot)$ solves

$$\dot{x}(t) = [A - BB^* P^+](t) x(t), \quad x(t_0) = x_0, \ t \ge t_0$$

As a partial converse of Proposition 4.3 we have

Proposition 4.4 If $Q(\cdot) \in C^1(t_0, \infty; \mathbf{C}^{n\times n})$ is a bounded Hermitian solution of $\mathbf{DRE}_\rho$ on (t_0, ∞) then necessarily $\rho \leq \|L_{t_0}^{\Sigma}\|^{-2}$.

While the previous two propositions yield a complete characterization of the norm $\|L_{t_0}^{\Sigma}\|$ in terms of the associated parametrized differential Riccati equation, they do not provide a full generalization of Proposition 4.1 to the time-varying case. To find a complete characterization of the stability radius $r_{\mathbb{C}}(A; B, C)$ for time-varying systems is an open problem.

5 Robust Lyapunov functions and nonlinear perturbations

The following proposition shows how solutions of the parametrized differential Riccati equation $\mathbf{DRE}_\rho$ can be used to construct *robust* Lyapunov functions for the system (1.1).

Proposition 5.1 Suppose $0 < \rho < \|L_{t_0}^{\Sigma}\|^{-2}$. If $P_\rho(\cdot)$ solves $\mathbf{DRE}_\rho$ then

$$V(t, x) \ := \ - < x, P_\rho(t)x >, \quad t \geq t_0, \ x \in \mathbb{C}^n$$

is a common Lyapunov function for all perturbed systems

$$\dot{x}(t) = [A + BDC](t)x(t), \quad t \geq t_0, \ x(t_0) = x_0$$

with $\|D(\cdot)\|_{L_\infty}^2 < \rho$.

If (A, B, C) are constant matrices and $\sigma(A) \subset \mathbb{C}_-$ a similar result to Proposition 5.1 holds true for $\|D(\cdot)\|_{L_\infty} < r_{\mathbb{C}}(A; B, C)$, see [8].

Using the above Lyapunov function it is possible to extend our robustness analysis to nonlinear perturbations of the form $\Delta(t) = B(t)N(C(t)x, t)$ so that the perturbed system is

$$\dot{x}(t) = A(t)x(t) + B(t)N(C(t)x(t), t), \quad x(t_0) = x_0, \ t \geq t_0 \tag{5.1}$$

where (A, B, C) satisfy (3.3) and $N : \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}^m$ is continuously differentiable. We assume $N(0, t) = 0$ so that 0 is an equilibrium state of (5.1). The following result shows that no nonlinear perturbation with global gain smaller than $\|L_{t_0}^{\Sigma}\|^{-1}$ can destabilize the system.

Proposition 5.2 Suppose $\gamma < \|L_{t_0}^{\Sigma}\|^{-1}$ and

$$\|N(y, t)\| \leq \gamma\|y\| \quad \text{for all} \quad t \geq t_0, \ y \in \mathbb{C}^p$$

Then the origin is globally exponentially stable for the system (5.1).

It is not clear whether an analogous statement holds (over a suitable time interval (t_0, ∞)) if the gain of the nonlinear perturbation is strictly less than $r_{\mathbb{C}}(A; B, C)$.

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