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Chapter 4

On Stability Radii of Slowly Time-Varying Systems

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ABSTRACT We consider robustness of exponential stability of time-varying linear systems with respect to structured dynamical nonlinear perturbations. Sufficient conditions in terms of L^2-stability are derived. It is shown that the infimum of the complex stability radii of a family of time-invariant linear systems provides a good estimate for the stability radius of a linear time-varying system if time variations are sufficiently slow.

Nomenclature

$\mathbb{K}$	=	$\mathbb{R}$ or $\mathbb{C}$
A^*	=	conjugate complex and transpose of $A \in \mathbb{C}^{p\times m}$
$\|M\|$	=	induced Euclidean norm for $M \in \mathbb{K}^{n\times m}$
$L^p(I;\mathbb{K}^{n\times m})$	=	the set of p-integrable functions $f: I \to \mathbb{K}^{n\times m}$, $I \subset \mathbb{R}$ an interval, $p \geq 1$
$L^2_{\text{loc}}(\mathbb{R}_{\geq 0};\mathbb{K}^p)$	=	the set of locally quadratic integrable functions $f: \mathbb{R}_{\geq 0} \to \mathbb{K}^p$
$\|f\|_{L^p(I;\mathbb{K}^{n\times m})}$	=	$\left(\int_I \|f(\tau)\|^p d\tau\right)^{1/p}$
$L^\infty(I;\mathbb{K}^{n\times m})$	=	the set of functions $f: I \to \mathbb{K}^{n\times m}$ that are essentially bounded on the interval $I \subset \mathbb{R}$
$\|f\|_{L^\infty(I;\mathbb{K}^{n\times m})}$	=	ess $\sup_{t\in I} \|f(t)\|$
$\mathcal{C}_{\text{pw,bdd}}(I;\mathbb{K}^{n\times m})$	=	set of piecewise continuous and bounded maps $M(\cdot): I \to \mathbb{K}^{n\times m}$, $I \subset \mathbb{R}$ an interval
$\pi_t(f)(\tau)$	=	$\begin{cases} f(\tau), & \tau \in [0,t] \\ 0, & \tau > t \end{cases}$

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4.1 Introduction

In this chapter we investigate the robustness of uniformly exponentially stable time-varying systems

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0, \tag{4.1}$$

where $A(\cdot) \in \mathcal{C}_{\text{pw,bdd}}(\mathbb{R}_{\geq 0}; \mathbb{K}^{n \times n})$ is assumed to be piecewise continuous and bounded.

Definition 4.1. *The system* (4.1) *is called* uniformly exponentially stable *if and only if there exist* $L, \lambda > 0$ *such that its transition matrix* $\Phi(\cdot,\cdot)$ *satisfies*

$$\|\Phi(t, t_0)\| \leq Le^{-\lambda(t-t_0)} \qquad \text{for all } \ t \geq t_0 \text{ and all } \ t_0 \geq 0. \tag{4.2}$$

We investigate the robustness of the stability of systems described by (4.1) with respect to additive nonlinear perturbations:

$$\dot{x}(t) = A(t)x(t) + B(t)\,\mathcal{D}(C(\cdot)x(\cdot))(t). \tag{4.3}$$

The structure of the perturbation is represented by piecewise continuous and bounded matrix valued functions

$$B(\cdot) \in \mathcal{C}_{\text{pw,bdd}}(\mathbb{R}_{\geq 0}; \mathbb{K}^{n \times m}), \quad C(\cdot) \in \mathcal{C}_{\text{pw,bdd}}(\mathbb{R}_{\geq 0}; \mathbb{K}^{p \times n}), \tag{4.4}$$

and by the nonlinear causal dynamical perturbation operator

$$\mathcal{D}(\cdot) \,:\, L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^p) \to L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^m).$$

Observe that we allow for infinite-dimensional perturbation systems. Precise definitions of the perturbation classes and global existence of the solution of the possibly nonlinear system (4.3) are given in Section 4.2.

In Section 4.3 we investigate several stability concepts — such as (global uniform) exponential, L^2, L^2-output, and asymptotic stability — for nonlinear systems of the form (4.3). These concepts are nested and the main results in this section are sufficient conditions for (global uniform) exponential stability of the zero solution.

In Section 4.4 we recall the concept of structured stability radius. Loosely speaking, the (complex) stability radius $r_{\mathbb{C}}$ is the sharp bound for the norm of the perturbation operator $\mathcal{D}$ so that the global L^2-stability of the perturbed system (4.3) is preserved as long as $\|\mathcal{D}\| < r_{\mathbb{C}}$ and might be lost if $\|\mathcal{D}\| = r_{\mathbb{C}}$. See Definition 4.5. The main result is that the structured stability radius of the time-varying system is close to the infimum of the structured stability radii of all "frozen" systems; that is, for fixed $\tau \geq 0$,

$$\dot{x}(t = A(\tau)x(t) + B(\tau)\,\mathcal{D}(C(\tau)x(\cdot))(t), \qquad t \geq 0, \tag{4.5}$$

if the time-variations of $A(\cdot), B(\cdot), C(\cdot)$ are sufficiently small.

Section 4.5 contains a useful lemma on the convolution of L^p-functions and another lemma collecting smoothness properties of the solution of the algebraic Riccati equation.

4.2 Perturbation Classes

In this section we introduce three different perturbation classes; we then define what we understand as a solution of the perturbed system. For the perturbation classes considered, it turns out that the solutions are well defined into the future and unique.

Definition 4.2. *We consider, for time-varying scaling matrices as in (4.4), the following three classes of the perturbed system (4.1),*

$$\dot{x}(t) = A(t)x(t) + B(t)\,\Delta(t)\,C(t)x(t), \tag{4.6}$$

$$\dot{x}(t) = A(t)x(t) + B(t)\,D\left(t, C(t)x(t)\right), \tag{4.7}$$

$$\dot{x}(t) = A(t)x(t) + B(t)\,\mathcal{D}\big(C(\cdot)x(\cdot)\big)(t), \tag{4.8}$$

where

(i) $\Delta(\cdot) \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{K}^{m\times p})$ *is a time-varying linear perturbation and*

$$\|\Delta(\cdot)\|_{L^\infty(0,\infty)} := \text{ess sup}_{t\geq 0}\|\Delta(t)\|\,;$$

(ii) $D(\cdot,\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{K}^p \to \mathbb{K}^m$ *is a time-varying nonlinear perturbation with:* $D(t,0) = 0$ *for all* $t \geq 0$, $t \mapsto D(t,y)$ *is* measurable *for almost all* $y \in \mathbb{K}^p$, $y \mapsto D(t,y)$ *is* Lipschitz continuous uniformly in t on compact intervals; *that is, for every* $T > 0$ *there exists some* $L_T > 0$ *such that*

$$\|D(t,y) - D(t,\bar{y})\| \leq L_T\|y - \bar{y}\| \quad \text{for all } y, \bar{y} \in \mathbb{K}^p \text{ and all } t \in [0,T],$$

and $D(\cdot,\cdot)$ *is of* finite gain; *that is,*

$$\|D(\cdot,\cdot)\|_{\text{nt}} := \inf\left\{\gamma > 0 \,\middle|\, \forall t \geq 0, \forall y \in \mathbb{K}^p : \|D(t,y)\| \leq \gamma\|y\|\right\} < \infty;$$

(iii) $\mathcal{D}(\cdot) : L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^p) \to L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^m)$ *is a dynamical perturbation satisfying* causality; *that is,*

$$\pi_t \mathcal{D} \pi_t = \pi_t \mathcal{D} \qquad \text{for all} \quad t \geq 0;$$

weakly L^2-Lipschitz continuity, *that is, for every* $(t_0, x_0, \varphi(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathbb{K}^n \times L^2(0, t_0; \mathbb{K}^p)$ *there exist* $r > 0, t_1 > t_0$ *and* $L \geq 0$ *such that for all* $y_1(\cdot), y_2(\cdot) \in L^2(0, t_1; \mathbb{K}^p)$ *satisfying* $y_1(\tau) = y_2(\tau) = \varphi(\tau)$ *for almost all* $\tau \in [0, t_0]$ *and* $\|y_i(\cdot) - \varphi(\cdot)\|_{L^\infty(t_0,t_1)} \leq r$ *for* $i = 1, 2$, *we have*

$$\|\mathcal{D}(y_1)(\cdot) - \mathcal{D}(y_2)(\cdot)\|_{L^2(t_0,t_1;\mathbb{K}^m)} \leq L\|y_1(\cdot) - y_2(\cdot)\|_{L^2(t_0,t_1;\mathbb{K}^p)},$$

and $\mathcal{D}$ *is of* finite gain; *that is,*

$$\|\mathcal{D}(\cdot)\|_{\text{dyn}} := \inf\left\{\gamma > 0 \,\middle|\, \begin{array}{l} \forall y(\cdot) \in L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^p) : \\ \|\mathcal{D}(y)(\cdot)\|_{L^2(\mathbb{R}_{\geq 0};\mathbb{K}^m)} \leq \gamma\|y(\cdot)\|_{L^2(\mathbb{R}_{\geq 0};\mathbb{K}^p)} \end{array}\right\} < \infty.$$

These three classes of linear time-varying, nonlinear time-varying, and dynamical perturbations are denoted $\mathcal{P}_{\mathrm{lt}}(\mathbb{K})$, $\mathcal{P}_{\mathrm{nt}}(\mathbb{K})$, *and* $\mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$, *respectively.*

The above perturbation classes were introduced by Hinrichsen and Pritchard [9] for time-invariant systems; it was shown by them that $D(\cdot,\cdot) \in \mathcal{P}_{\mathrm{nt}}(\mathbb{K})$ can be identified with $\mathcal{D}(\cdot) \in \mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$ by setting

$$\mathcal{D}(y(\cdot))(t) := D(t, y(t)) \qquad \text{for } t \geq 0 \text{ and } y(\cdot) \in L^2(\mathbb{R}_{\geq 0}; \mathbb{K}^p)$$

and the following chain of norm-preserving embeddings holds

$$\mathbb{K}^{m \times p} \subset \mathcal{P}_{\mathrm{lt}}(\mathbb{K}) \subset \mathcal{P}_{\mathrm{nt}}(\mathbb{K}) \subset \mathcal{P}_{\mathrm{dyn}}(\mathbb{K}). \tag{4.9}$$

Since $\mathcal{D}(y(\cdot))(t)$ depends not only on $y(t)$ but on the whole "past" $y(\cdot)|_{[0,t]}$ of $y(\cdot)$, Hinrichsen and Pritchard [9] introduced the following initial value problem, which we extend to the time-varying case in a straightforward manner.

Definition 4.3. *Suppose* (4.1) *is perturbed by* $\mathcal{D}(\cdot) \in \mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$ *with scaling matrices as in* (4.4) *so that we consider*

$$\dot{x}(t) = A(t)x(t) + B(t)\,\mathcal{D}\big(C(\cdot)x(\cdot)\big)(t). \tag{4.10}$$

Then

$$x(\cdot) = x(\cdot; t_0, x_0, \varphi(\cdot)) \;:\; I \to \mathbb{K}^n, \qquad I = [t_0, t_1),\; t_1 > t_0,$$

is said to be a solution of (4.10) with initial data $(t_0, x_0, \varphi(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathbb{K}^n \times L^2(0, t_0; \mathbb{K}^p)$ *if and only if* $x(\cdot)$ *is absolutely continuous on* I, $x(t_0) = x_0$, *and, for almost all* $t \in I$,

$$\dot{x}(t) = A(t)x(t) + B(t)\,\mathcal{D}\big([Cx]^{\varphi}(\cdot)\big)(t), \tag{4.11}$$

where

$$[Cx]^{\varphi}(\tau) := \begin{cases} \varphi(\tau), & \tau \in [0, t_0) \\ C(\tau)x(\tau), & \tau \in [t_0, t_1) \\ 0, & \tau \in [t_1, \infty). \end{cases}$$

Note that the perturbed system (4.8), and therefore also the less general ones (4.6) and (4.7), preserve the zero solution.

It turns out that the smoothness assumption on the dynamical perturbation operator $\mathcal{D}$ as defined in Definition 4.2*(iii)* is sufficient to guarantee existence, uniqueness, and that finite escape time does not exist for the initial value problem. One reason for this is, roughly speaking, that the perturbations are assumed to be linearly bounded; see in particular the definition of $\|D(\cdot,\cdot)\|_{\mathrm{nt}}$.

Theorem 4.1. *For any $\mathcal{D}(\cdot) \in \mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$ and any initial data $(t_0, x_0, \varphi(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathbb{K}^n \times L^2(0, t_0; \mathbb{K}^p)$, the initial value problem (4.10) possesses a solution $x(\cdot; t_0, x_0, \varphi(\cdot))$; and this solution is unique and exists on the whole of $[t_0, \infty)$.*

The proof of Theorem 4.1 is given for constant matrices $(A(\cdot), B(\cdot), C(\cdot)) \equiv (A, B, C) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m} \times \mathbb{K}^{p\times n}$ by Hinrichsen and Pritchard [9]. The extension to the time-varying case is straightforward and omitted here.

4.3 Stability of Nonlinearly Perturbed Linear Systems

We are interested in seeing to what extent the stability properties of the unperturbed system (4.1) are inherited by the perturbed systems (4.6) to (4.8). To this end we introduce the following different concepts of stability and investigate how they are related.

Definition 4.4. *The origin of the initial value problem (4.10) is called*

(i) globally uniformly exponentially stable *if and only if*

$$\|x(t; t_0, x_0, \varphi(\cdot))\| \leq Me^{-\omega(t-t_0)} \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right],$$

(ii) globally uniformly L^2-stable *if and only if*

$$\begin{aligned} \|x(t; t_0, x_0, \varphi(\cdot))\| &\leq M \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right] \quad \forall\, t \geq t_0, \\ \|x(\cdot; t_0, x_0, \varphi(\cdot))\|_{L^2(t_0,\infty)} &\leq M \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right], \end{aligned}$$

(iii) globally uniformly L^2-output stable *if and only if*

$$\begin{aligned} \|C(t)x(t; t_0, x_0, \varphi(\cdot))\| &\leq M \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right] \quad \forall\, t \geq t_0, \\ \|C(\cdot)x(\cdot; t_0, x_0, \varphi(\cdot))\|_{L^2(t_0,\infty)} &\leq M \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right], \end{aligned}$$

(iv) globally uniformly asymptotically stable *if and only if*

$$\begin{aligned} \|x(t; t_0, x_0, \varphi(\cdot))\| &\leq M \left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0)}\right] \quad \forall\, t \geq t_0, \\ \lim_{t\to\infty} x(t; t_0, x_0, \varphi(\cdot)) &= 0, \end{aligned}$$

holds for some $M, \omega > 0$, and all initial data $(t_0, x_0, \varphi(\cdot)) \in \mathbb{R}_{\geq 0} \times \mathbb{K}^n \times L^2(0, t_0; \mathbb{K}^p)$, respectively; "globally" refers to all $(t_0, x_0, \varphi(\cdot))$ and "uniformly" refers to the independence of M and ω from $(t_0, x_0, \varphi(\cdot))$.

Remark 4.1. *Let $B(\cdot) \equiv 0$ and $C(\cdot) \equiv I_p$ in (4.10); that means we consider the time-varying linear systems (4.1) only. Then all of the stability concepts in Definition 4.4 coincide; the equivalence between (i) and (iv) is well known (see, e.g., Rugh [16]), and the equivalence between (i) and (ii) is proved by Daleckiĭ and Kreĭn [2, Theorem III 6.2]; it even holds true for L^p-stability, where $p \in [1, \infty)$ is arbitrary.*

Before we prove relationships between the different stability concepts, we present the following simple but useful proposition and some formulae and inequalities for the initial value problem (4.10).

Proposition 4.1. *Consider a time-varying uniformly exponentially stable system (4.1) and $B(\cdot), C(\cdot)$ as in (4.4). Then the so-called perturbation operator*

$$\mathcal{L}_{t_0} : L^2(t_0, \infty; \mathbb{K}^m) \to L^2(t_0, \infty; \mathbb{K}^p), \quad u(\cdot) \mapsto C(\cdot) \int_{t_0}^{\cdot} \Phi(\cdot, \tau) B(\tau) u(\tau)\, d\tau$$

is well defined for any $t_0 \geq 0$. Moreover $t_0 \mapsto \|\mathcal{L}_{t_0}\|$ is nonincreasing on $\mathbb{R}_{\geq 0}$, and

$$\lim_{t \to \infty} \mathcal{L}_{t_0}(u(\cdot))(t) = 0 \qquad \textit{for every } u(\cdot) \in L^2(t_0, \infty; \mathbb{K}^m).$$

Proof: Since $B(\cdot)$ and $C(\cdot)$ are uniformly bounded and $\Phi(\cdot, \cdot)$ satisfies an inequality of the form (4.2), it follows from the general result of Lemma 4.1 in Section 4.5 that the convolution of two L^2-functions is itself an L^2-function and moreover tends to zero as t tends to infinity. Monotonicity of $t_0 \mapsto \|\mathcal{L}_{t_0}\|$ is straightforward; see Hinrichsen et al. [10]. This completes the proof.

Applying Variation-of-Constants to (4.11) yields, for every $t \geq t_0$,

$$x(t) = \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, \tau) B(\tau) \mathcal{D}\left([Cx]^{\varphi} \right)(\tau)\, d\tau, \tag{4.12}$$

and if the unperturbed system (4.1) is uniformly exponentially stable, then taking norms in (4.12), invoking (4.2), and applying the Cauchy–Schwarz inequality gives, for all $t \geq t_0$,

$$\begin{aligned}
&\|x(t)\| \\
&\leq L e^{-\lambda(t-t_0)} \|x_0\| + L \|B(\cdot)\|_{L^\infty} \int_{t_0}^{t} e^{-\lambda(t-\tau)} \|\mathcal{D}\left([Cx]^{\varphi}\right)(\tau)\|\, d\tau \qquad (4.13)\\
&\leq L e^{-\lambda(t-t_0)} \|x_0\| + \frac{L}{\sqrt{2\lambda}} \|B(\cdot)\|_{L^\infty} \|\mathcal{D}([Cx]^{\varphi})(\cdot)\|_{L^2(0,t)}. \qquad (4.14)
\end{aligned}$$

Since $\mathcal{D}(\cdot)$ is causal we have, for all $t \geq t_0$,

$$\|\mathcal{D}([Cx]^{\varphi})(\cdot)\|^2_{L^2(t_0,t)} \leq \|\mathcal{D}\|^2_{\text{dyn}} \int_0^t \|[Cx]^{\varphi}(\tau)\|^2 \, d\tau \tag{4.15}$$

$$\leq \|\mathcal{D}\|^2_{\text{dyn}} \left[\|\varphi(\cdot)\|^2_{L^2(0,t_0)} + \|C(\cdot)x(\cdot)\|^2_{L^2(t_0,t)}\right] \tag{4.16}$$

and thus

$$\begin{aligned}\|x(t)\| \leq\ & Le^{-\lambda(t-t_0)}\|x_0\| \\ & + \frac{L}{\sqrt{2\lambda}}\|B(\cdot)\|_{L^\infty}\|\mathcal{D}\|_{\text{dyn}}\left[\|\varphi(\cdot)\|_{L^2(0,t_0)} + \|C(\cdot)x(\cdot)\|_{L^2(t_0,t)}\right].\end{aligned} \tag{4.17}$$

Taking L^2-norms in (4.13) and applying Cauchy–Schwarz and (4.16) yields, for all $t \geq t_0$,

$$\begin{aligned}\|x(\cdot)\|_{L^2(t_0,t)} \leq\ & \frac{L}{\sqrt{2\lambda}}\|x_0\| + \frac{L\|B(\cdot)\|_{L^\infty}\|\mathcal{D}\|_{\text{dyn}}}{\sqrt{2\lambda}}\left[\|\varphi(\cdot)\|_{L^2(0,t_0)}\right. \\ & \left. + \|C(\cdot)x(\cdot)\|_{L^2(t_0,t)}\right].\end{aligned} \tag{4.18}$$

Multiplying (4.12) by $C(t)$ and using the perturbation operator yields, for all $t \geq t_0$,

$$C(t)x(t) = C(t)\Phi(t,t_0)x_0 + \mathcal{L}_{t_0}\left(\mathcal{D}([Cx]^{\varphi})\right)(t), \tag{4.19}$$

and taking norms and invoking (4.2), Cauchy–Schwarz, and (4.16) again gives, for all $t \geq t_0$,

$$\begin{aligned}\|C(\cdot)x(\cdot)\|_{L^2(t_0,t)} \leq\ & \|C(\cdot)\|_{L^\infty}\frac{L}{\sqrt{2\lambda}}\|x_0\| + \|\mathcal{D}\|_{\text{dyn}}\|\mathcal{L}_{t_0}\|\left[\|\varphi(\cdot)\|_{L^2(0,t_0)}\right. \\ & \left. + \|C(\cdot)x(\cdot)\|_{L^2(t_0,t)}\right].\end{aligned} \tag{4.20}$$

. □

Now we are in a position to prove the following relationships between the stability concepts.

Proposition 4.2. *The stability concepts in Definition 4.4 are related as follows exponentially*

$$\Longrightarrow \quad L^2 \quad \Longleftrightarrow \quad L^2\text{–output} \quad \Longrightarrow \quad \text{asymptotically},$$

where each stability concept holds globally uniformly.

Proof: "exp. $\Rightarrow L^2$" is trivial. "$L^2 \Rightarrow L^2$-output" follows from the boundedness of $C(\cdot)$. "$L^2 \Leftarrow L^2$-output" is a consequence of (4.17) and (4.18). To see "L^2-output $\Rightarrow$ asymptotically", note that $\mathcal{D}\big([Cx]^{\varphi}(\cdot)\big)(\cdot) \in L^2(0,\infty;\mathbb{C}^m)$, and therefore Lemma 4.1 applied to (4.12) yields the result. □

The equivalence between L^2 and L^2-output stability has also been observed by Jacob [13] in a slightly less general context.

Remark 4.2. *It might be worth noticing that the exponentially stable system* (4.10) *can be viewed as an input-to-state stable system in the sense of Sontag. Set* $\omega(\cdot) \equiv C(\cdot)x(\cdot; t_0, x_0, \varphi(\cdot))$ *and consider*

$$\dot{x}(t) = A(t)x(t) + B(t)\,\mathcal{D}\left(\omega(\cdot)\right)(t). \tag{4.21}$$

Then by a little algebraic manipulation (4.18) *yields, for some* $L > 0$,

$$\int_{t_0}^{t} \|x(\tau)\|^2\, d\tau \leq L\left[\|x_0\| + \|\varphi(\cdot)\|_{L^2(0,t_0;\mathbb{K}^p)}\right] + \int_{t_0}^{t} L\|\omega(\tau)\|^2\, d\tau,$$

and hence by Theorem 1 in Sontag [18] the system (4.21) *is input-to-state stable.*

In the following theorem we present sufficient conditions for the different stability concepts.

Theorem 4.2. *Let* $0 < \bar{\rho} < \|\mathcal{L}_0\|^{-1}$ *and consider the restricted perturbation class* $\mathcal{P}_{\rm dyn}(\mathbb{K})$ *with* $\|\mathcal{D}\|_{\rm dyn} < \bar{\rho}$. *Then the zero solution of the initial value problem* (4.10) *is:*

(i) globally uniformly L^2*-stable;*

(ii) globally uniformly exponentially stable, if $\dot{x}(t) = A(t)x(t), y(t) = C(t)x(t)$ *is uniformly observable; that is, there exist* $\beta_0, \beta_1, \sigma > 0$ *such that*

$$\beta_0 I_n \leq \int_{t-\sigma}^{t} \Phi(s,t-\sigma)^*\, C(s)^*\, C(s)\Phi(s,t-\sigma)\, ds \leq \beta_1 I_n \qquad \text{for all } t \geq \sigma;$$

(iii) globally uniformly exponentially stable, if there exists some $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$, $t \geq 0$, *and* $\psi(\cdot) \in L^2_{\rm loc}(\mathbb{R}_{\geq 0}; \mathbb{K}^p)$ *we have*

$$\|e^{\varepsilon\cdot}\mathcal{D}\left(e^{-\varepsilon\cdot}\psi(\cdot)\right)(\cdot)\|^2_{L^2(0,t;\mathbb{K}^m)} \leq \|\mathcal{D}\|^2_{\rm dyn}\|\psi(\cdot)\|^2_{L^2(0,t;\mathbb{K}^p)};$$

(iv) it is globally uniformly exponentially stable if the perturbation class is furthermore restricted to nonlinear time-varying perturbations $\mathcal{P}_{\rm nt}(\mathbb{K})$.

In the assumption of Theorem 4.2 we could have assumed alternatively that $0 < \bar{\rho} < \|\mathcal{L}_{t'}\|^{-1}$, but then the stability concepts would only be considered for $t_0 \geq t'$.

Proof: (*i*): This statement follows from the well-known small gain theorem, but can readily be established in our context as follows. Since $\|\mathcal{D}\|_{\rm dyn} <$

$\bar{\rho} < \|\mathcal{L}_0\|^{-1} \leq \|\mathcal{L}_{t_0}\|^{-1}$, there exists $\alpha \in (0,1)$ such that $\|\mathcal{D}\|_{\mathrm{dyn}}\|\mathcal{L}_{t_0}\| < \alpha$ for all $\|\mathcal{D}\|_{\mathrm{dyn}} < \bar{\rho}$ and all $t_0 \geq 0$. Now (4.20) yields, for every $t \geq t_0$,

$$(1-\alpha)\|C(\cdot)x(\cdot)\|_{L^2(t_0,t)} \leq \|C(\cdot)\|_{L^\infty(0,\infty)}\frac{L}{\sqrt{2\lambda}}\|x_0\| + \alpha\|\varphi(\cdot)\|_{L^2(0,t_0)}$$

and hence L^2-stability follows from Proposition 4.2.

(ii): We proceed in several steps.

Step 1: Let $\hat{\rho} \in (\bar{\rho}, \|\mathcal{L}_0\|^{-1})$.
By Lemma 8.1 in Ilchmann [12, p.145] there exists some $\eta > 0$ such that $\hat{\rho} < \|\mathcal{L}_0^\eta\|^{-1}$, where $\mathcal{L}_{t_0}^\eta$ denotes the input-output operator of the "shifted" system

$$\mathcal{L}_{t_0}^\eta : L^2(t_0,\infty;\mathbb{K}^m) \to L^2(t_0,\infty;\mathbb{K}^p),\;\; u(\cdot) \mapsto C(\cdot)\int_{t_0}^{\cdot} \Phi_{A+\eta I}(\cdot,\tau)B(\tau)u(\tau)\,d\tau,$$

and $\Phi_{A+\eta I}(\cdot,\cdot)$ denotes the transition matrix of the uniformly exponentially stable system $\dot{x}(t) = [A(t)+\eta I]x(t)$. Now by Theorem 5.11 in Hinrichsen et al. [10] there exists a unique stabilizing, positive-definite, continuously differentiable Hermitian solution

$$P(\cdot) = P(\cdot)^* \in L^\infty(0,\infty;\mathbb{C}^{n\times n})$$

of the differential Riccati equation

$$\begin{aligned}&\dot{P}(t) + [A(t)+\eta I_n]^*\; P(t) + P(t)\;[A(t)+\eta I_n]\\ &\qquad = \hat{\rho}^2 C(t)^*C(t) + P(t)B(t)B(t)^*P(t), \quad \text{for all } t \geq 0 \end{aligned} \tag{4.22}$$

and stabilizing means that

$$\dot{x}(t) = [A(t) - B(t)B(t)^*P(t)]\; x(t)$$

is uniformly exponentially stable.

Step 2: The nonpositive definite matrix function $P(\cdot)$ serves as a Lyapunov function candidate

$$V(t,x) \;:=\; -x^*P(t)x$$

in the following. Differentiating V along the solution of the perturbed system (4.10) yields, for all $t \geq t_0$, by invoking (4.22) and omitting the argument t for simplicity,

$$\begin{aligned}&\frac{d}{dt}V(t,x(t))\\ &\quad = -\left[Ax + B\mathcal{D}\left([Cx]^\varphi\right)\right]^*\; Px - x^*P\left[Ax + B\mathcal{D}\left([Cx]^\varphi\right)\right] - x^*\dot{P}x\\ &\quad = -\hat{\rho}^2\|Cx\|^2 - \|B^*Px\|^2\\ &\qquad\quad -\langle \mathcal{D}\left([Cx]^\varphi\right), B^*Px\rangle - \langle B^*Px, \mathcal{D}\left([Cx]^\varphi\right)\rangle + 2\eta x^*Px\\ &\quad = -2\eta V(t,x) - \hat{\rho}^2\|Cx\|^2 + \|\mathcal{D}\left([Cx]^\varphi\right)\|^2 - \|B^*Px + \mathcal{D}\left([Cx]^\varphi\right)\|^2\\ &\quad \leq -2\eta V(t,x) - \hat{\rho}^2\|Cx\|^2 + \|\mathcal{D}\left([Cx]^\varphi\right)\|^2. \end{aligned} \tag{4.23}$$

Step 3: Set $\zeta^2 := \hat{\rho}^2 - \|\mathcal{D}\|^2_{\mathrm{dyn}} > 0$.
Then integration of $(d/d\tau)\, V(\tau, x(\tau))$ over $[t_0, t]$ and invoking (4.23) yields, for all $t \geq t_0$,

$$\begin{aligned} V(t,x(t)) &\leq V(t_0, x(t_0)) - 2\eta \int_{t_0}^{t} V(\tau, x(\tau))\, d\tau \\ &\quad - \zeta^2 \int_{t_0}^{t} \|C(\tau)x(\tau)\|^2\, d\tau + \|\mathcal{D}\|^2_{\mathrm{dyn}} \|\varphi(\cdot)\|^2_{L^2(0,t_0)} \\ &\leq V(t_0, x(t_0)) - 2\eta \int_{t_0}^{t} V(\tau, x(\tau))\, d\tau + \|\mathcal{D}\|^2_{\mathrm{dyn}} \|\varphi(\cdot)\|^2_{L^2(0,t_0)}. \end{aligned} \tag{4.24}$$

Step 4: Applying the Bellman–Gronwall Lemma to (4.24) gives, for all $t \geq t_0$,

$$V(t,x(t)) \leq \left[V(t_0, x(t_0)) + \|\mathcal{D}\|^2_{\mathrm{dyn}} \|\varphi(\cdot)\|^2_{L^2(0,t_0)} \right] e^{-2\eta(t-t_0)}. \tag{4.25}$$

Note that the transition matrix of $\dot{x}(t) = [A(t) + \eta I]x(t)$ is given by $\Phi(t,s)e^{\eta(t-s)}$, and hence it is easy to see that $\dot{x}(t) = [A(t) + \eta I]x(t),\, y(t) = C(t)x(t)$ is uniformly observable, too. Therefore, by Theorem 6.11*(iv)* in Ilchmann [12, page 137] there exist some $p_1 > p_2 > 0$ such that

$$-p_1 I_n < P(\tau) < -p_2 I_n \quad \text{for all} \quad \tau \geq 0. \tag{4.26}$$

Now substitution of the bounds in (4.26) into (4.25) yields, for all $t \geq t_0$,

$$p_2 \|x(t)\|^2 \leq V(t,x(t)) \leq \left[p_1 \|x(t_0)\|^2 + \|\mathcal{D}\|^2_{\mathrm{dyn}} \|\varphi(\cdot)\|^2_{L^2(0,t_0)} \right] e^{-2\eta(t-t_0)}.$$

This proves global uniform exponential stability of (4.10) and completes the proof of *(ii)*.

(iii): First proceed as in Steps 1 and 2 of the proof of *(ii)*. Note that a difference from *(ii)* is that we do not assume uniform observability, and so it is not necessarily true that $P(t)$ satisfies uniform bounds in (4.26). We have to proceed differently. By (4.23) we may conclude that

$$\frac{d}{dt}\left(e^{2\eta t} V(t,x(t)) \right) \leq e^{2\eta t} \left(-\hat{\rho}^2 \|C(t)x(t)\|^2 + \|\mathcal{D}\left([Cx]^{\varphi}\right)(t)\|^2 \right),$$

and hence by integration over $[t_0, t]$,

$$\begin{aligned} e^{2\eta t} V(t,x(t)) &\leq e^{2\eta t_0} V(t_0, x(t_0)) \\ &\quad + \int_{t_0}^{t} e^{2\eta\tau} \left[-\hat{\rho}^2 \|C(\tau)x(\tau)\|^2 + \|\mathcal{D}\left([Cx]^{\varphi}\right)(\tau)\|^2 \right] d\tau. \end{aligned} \tag{4.27}$$

Since $\eta > 0$ may be chosen smaller than ε^*, an application of the assumption in *(iii)* yields, for all $t \geq t_0$,

$$\begin{aligned}
&\int_{t_0}^{t} e^{2\eta\tau} \|\mathcal{D}([Cx]^{\varphi})(\tau)\|^2 \, d\tau \\
&\qquad \leq \|\mathcal{D}\|_{\text{dyn}}^2 \int_0^t \|e^{\eta\tau}[Cx]^{\varphi}(\tau)\|^2 \, d\tau \\
&\qquad \leq \|\mathcal{D}\|_{\text{dyn}}^2 \left[\|e^{\eta\cdot}\varphi(\cdot)\|_{L^2(0,t_0)}^2 + \|e^{\eta\cdot}C(\cdot)x(\cdot)\|_{L^2(t_0,t)}^2\right]
\end{aligned} \tag{4.28}$$

and this applied to (4.27) gives

$$\begin{aligned}
0 \quad \leq \quad & e^{2\eta t_0} V(t_0, x(t_0)) \\
& - \left[\hat{\rho}^2 - \|\mathcal{D}\|_{\text{dyn}}^2\right] \|e^{\eta\cdot}C(\cdot)x(\cdot)\|_{L^2(t_0,t)}^2 + \|\mathcal{D}\|_{\text{dyn}}^2 \|e^{\eta\cdot}\varphi(\cdot)\|_{L^2(0,t_0)}^2,
\end{aligned}$$

whence, since $\hat{\rho}^2 - \|\mathcal{D}\|_{\text{dyn}}^2 > \hat{\rho}^2 - \bar{\rho}^2$, for all $t \geq t_0$,

$$\begin{aligned}
&\|e^{\eta\cdot}C(\cdot)x(\cdot)\|_{L^2(t_0,t)}^2 \\
&\qquad \leq \frac{1}{\hat{\rho}^2 - \bar{\rho}^2} \left[e^{2\eta t_0} \|P(t_0)\| \, \|x_0\|^2 + \|\mathcal{D}\|_{\text{dyn}}^2 \|e^{\eta\cdot}\varphi(\cdot)\|_{L^2(0,t_0)}^2\right].
\end{aligned} \tag{4.29}$$

Without restriction of generality we may assume that $\eta \in (0, \lambda)$. Applying the Cauchy–Schwarz inequality to (4.13) and applying (4.28) and (4.29) we arrive at, for all $t \geq t_0$,

$$\begin{aligned}
& e^{\eta(t-t_0)}\|x(t)\| \\
& \leq L e^{-(\lambda-\eta)(t-t_0)}\|x_0\| + L\|B(\cdot)\|_{L^\infty} \int_{t_0}^{t} e^{-(\lambda-\eta)(t-\tau)} e^{\eta(\tau-t_0)} \mathcal{D}([Cx]^{\varphi})(\tau) \, d\tau \\
& \leq L e^{-(\lambda-\eta)(t-t_0)}\|x_0\| \\
& \quad + L\|B(\cdot)\|_{L^\infty} \left[\int_{t_0}^{t} e^{-2(\lambda-\eta)(t-\tau)} d\tau\right]^{1/2} \|e^{\eta(\cdot - t_0)} \mathcal{D}([Cx]^{\varphi})(\cdot)\|_{L^2(t_0,t)} \\
& \leq L e^{-(\lambda-\eta)(t-t_0)}\|x_0\| \\
& \quad + \frac{L\|B(\cdot)\|_{L^\infty}}{\sqrt{2(\lambda-\eta)}} e^{-\eta t_0} \|\mathcal{D}\|_{\text{dyn}} \left[\|e^{\eta\cdot}\varphi(\cdot)\|_{L^2(0,t_0)} + \|e^{\eta\cdot}C(\cdot)x(\cdot)\|_{L^2(t_0,t)}\right] \\
& \leq L e^{-(\lambda-\eta)(t-t_0)}\|x_0\| \\
& \quad + \frac{L\|B(\cdot)\|_{L^\infty}}{\sqrt{2(\lambda-\eta)}} e^{-\eta t_0} \|\mathcal{D}\|_{\text{dyn}} \left\{ e^{\eta t_0}\|\varphi(\cdot)\|_{L^2(0,t_0)} \right. \\
& \qquad\qquad \left. + \frac{e^{\eta t_0}}{\sqrt{\hat{\rho}^2 - \bar{\rho}^2}} \left[\|P(t_0)\|^{1/2}\|x_0\| + \|\mathcal{D}\|_{\text{dyn}}\|\varphi(\cdot)\|_{L^2(0,t_0)}\right] \right\}.
\end{aligned}$$

Finally multiplication by $e^{-\eta(t-t_0)}$ yields, for all $t \geq t_0$,

$$\|x(t)\| \leq \left[Le^{-\lambda(t-t_0)} + e^{-\eta(t-t_0)} \frac{L\|B(\cdot)\|_{L^\infty}}{\sqrt{2(\lambda-\eta)}} \frac{\|\mathcal{D}\|_{\text{dyn}}}{\sqrt{\hat{\rho}^2 - \bar{\rho}^2}} \|P(t_0)\|^{1/2} \right] \|x_0\|$$

$$+ e^{-\eta(t-t_0)} \frac{L\|B(\cdot)\|_{L^\infty}}{\sqrt{2(\lambda-\eta)}} \|\mathcal{D}\|_{\text{dyn}} \left[1 + \frac{\|\mathcal{D}\|_{\text{dyn}}}{\sqrt{\hat{\rho}^2 - \bar{\rho}^2}} \right] \|\varphi(\cdot)\|_{L^2(0,t_0)},$$

and since $P(\cdot)$ is uniformly bounded from above and $\|\mathcal{D}\|_{\text{dyn}} \leq \bar{\rho}$, the initial value problem (4.10) is globally uniformly exponentially stable. Thus *(iii)* is complete.

(*iv*): This result has already been proved in Hinrichsen et al. [10, Theorem 6.1]. This completes the proof. □

Remark 4.3. *Note that uniformity in Theorem* 4.2 *(i) to (iv) is due to the assumption* $\|\mathcal{D}\|_{\text{dyn}} < \bar{\rho} < \|\mathcal{L}_0\|^{-1}$. *If we assume* $\|\mathcal{D}\|_{\text{dyn}} < \|\mathcal{L}_{t_0}\|^{-1}$ *instead, then by inspection of the proof we readily see that the statements in Theorem 4.2 (i) to (iv) hold true apart from "uniform," that means M and ω in Definition* 4.4 *depend on the initial data.*

Related results as in Theorem 4.2 have been achieved for time-varying infinite-dimensional systems by Jacob et al. [14]. However, in their setup they do not use any Lyapunov functions but use operator-theoretic methods. The proof of Theorem 4.2*(ii)* with the use of a Lyapunov function as presented here is crucial for deriving properties of slowly time-varying systems in Section 4.

In the present chapter we are only interested in L^2-stability since we use Riccati equations later on, but the proof of Theorem 4.2 also goes through for L^p-stability, $p \geq 1$, which is defined analogously. As a consequence, the class of L^p-stable systems considered by Jacob [13] is also globally asymptotically stable.

4.4 Stability Radii of Slowly Time-Varying Systems

We are now in a position to formulate the concept of the stability radius for exponentially stable systems (4.1) with respect to the different perturbations (4.6) to (4.8). Loosely speaking, in the case of dynamic perturbations we are interested in a sharp number $r_{\mathbb{K}}$ such that all perturbations of the class $\mathcal{P}_{\text{dyn}}(\mathbb{K})$ preserve global L^2-stability of the origin of the perturbed system (4.11) as long as $\|\mathcal{D}(\cdot)\|_{\text{dyn}} < r_{\mathbb{K}}$ and there exists some $\mathcal{D}(\cdot) \in \mathcal{P}_{\text{dyn}}(\mathbb{K})$ with $\|\mathcal{D}(\cdot)\|_{\text{dyn}} = r_{\mathbb{K}}$ such that the origin of (4.10) is not globally L^2-stable. More formally we define the stability radii as follows.

Definition 4.5. *The (structured)* stability radius *of the exponentially stable system* (4.1) *with respect to dynamic perturbations of the class* $\mathcal{P}_{\text{dyn}}(\mathbb{K})$ *and the structure* $(B(\cdot), C(\cdot))$ *as given in* (4.4) *is*

$$r_{\mathbb{K},dyn}(A(\cdot); B(\cdot), C(\cdot)) := \inf \left\{ \|\mathcal{D}(\cdot)\|_{\text{dyn}} \;\middle|\; \begin{array}{l} \mathcal{D}(\cdot) \in \mathcal{P}_{\text{dyn}}(\mathbb{K}), \text{ the origin} \\ \text{of (4.10) is not globally} \\ \text{uniformly } L^2\text{-stable} \end{array} \right\}.$$

The stability radii with respect to the perturbation classes $\mathbb{K}^{n\times n}$, $\mathcal{P}_{\text{lt}}(\mathbb{K})$, *and* $\mathcal{P}_{\text{nt}}(\mathbb{K})$ *are defined analogously and are denoted by* $r_{\mathbb{K}}(A(\cdot); B(\cdot), C(\cdot))$, $r_{\mathbb{K},lt}(A(\cdot); B(\cdot), C(\cdot))$, *and* $r_{\mathbb{K},nt}(A(\cdot); B(\cdot), C(\cdot))$, *respectively.*

The stability radius was introduced for time-invariant systems by Hinrichsen and Pritchard [6;7], for time-varying systems by Hinrichsen et al. [10], and for time-varying systems with multiperturbations by Hinrichsen and Pritchard [8].

Theorem 4.3. *Consider the exponentially stable system* (4.1) *and* $(B(\cdot), C(\cdot))$ *as given in* (4.4). *Then for* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *we have*

$$r_{\mathbb{K}} \geq r_{\mathbb{K},lt} \geq r_{\mathbb{K},nt} \geq r_{\mathbb{K},dyn}, \tag{4.30}$$

where for the sake of notation we omitted the arguments $(A(\cdot); B(\cdot), C(\cdot))$, *and*

$$r_{\mathbb{C}}(A(\cdot); B(\cdot), C(\cdot)) \geq r_{\mathbb{C},\text{dyn}}(A(\cdot); B(\cdot), C(\cdot)) = \sup_{t_0 \geq 0} \|\mathcal{L}_{t_0}\|^{-1}, \tag{4.31}$$

and if $(A(\cdot), B(\cdot), C(\cdot)) = (A, B, C)$ *are constant matrices, then*

$$r_{\mathbb{C}}(A; B, C) = r_{\mathbb{C},\text{dyn}}(A; B, C) = \min_{\omega \in \mathbb{R}} \|C(i\omega I_n - A)^{-1} B\|^{-1}. \tag{4.32}$$

The inequality in (4.31) *is strict.*

Proof: (4.30) is immediate from the inclusions (4.9). The inequality in (4.31) is a repetition of (4.30) and strictness is proved by a scalar periodic system in Example 4.4 in Hinrichsen et al. [10]. The left equality in (4.32) is proved in Theorem 3.12 by Hinrichsen and Pritchard [9], while the right equality is proved in Hinrichsen and Pritchard [7]. It remains to prove the equality in (4.31). To this end we need the nice result due to Jacob [13] where she shows that for every $\alpha > \sup_{t_0 \geq 0} \|\mathcal{L}_{t_0}\|^{-1}$ there exists a linear causal operator $\mathcal{D}(\cdot) \in \mathcal{P}_{\text{dyn}}(\mathbb{K})$ such that $\|\mathcal{D}(\cdot)\|_{\text{dyn}} < \alpha$ and the origin of (4.11) is not globally L^2-stable. We therefore have

$$r_{\mathbb{C},\text{dyn}}(A(\cdot); B(\cdot), C(\cdot)) \geq \sup_{t_0 \geq 0} \|\mathcal{L}_{t_0}\|^{-1} \geq r_{\mathbb{C},\text{dyn}}(A(\cdot); B(\cdot), C(\cdot)),$$

where the left inequality follows from Theorem 4.2(i). This completes the proof. □

It is well known (see, e.g., Rugh [16, Theorem 8.7] and the references therein) that a time-varying system $\dot{x}(t) = A(t)x(t)$ with bounded $A(\cdot)$ is uniformly exponentially stable if all real parts of the eigenvalues are uniformly bounded away from the imaginary axis (i.e., $\operatorname{Re}\lambda_i(A(\tau)) \leq -\mu$ for all $\tau \geq 0$ and some $\mu > 0$) and the time-variation of $A(\cdot)$ is sufficiently slow (i.e., $\|\dot{A}(\tau)\|$ is small uniformly in τ).

In the remainder of this section we prove an analogous statement for the stability radius. If the stability radii of the "frozen" systems $r_{\mathbb{C}}(A(\tau) + \eta I_n; B(\tau), C(\tau))$ are uniformly larger than $\rho > 0$ for some arbitrarily small $\eta > 0$, then the stability radius $r_{\mathbb{C},\mathrm{dyn}}(A(\cdot); B(\cdot), C(\cdot))$ of the time-varying system is — provided the time-variation of the matrices $A(\cdot), B(\cdot), C(\cdot)$ is sufficiently small — at least "close" to ρ. More precisely, we have the following theorem.

Theorem 4.4. *Consider* (4.1) *and scaling matrices* (4.4) *with absolutely continuous* $(A(\cdot), B(\cdot), C(\cdot))$. *Assume that there exist* $\rho, \eta > 0$, *such that the stability radii of the "shifted frozen" systems* (4.5) *satisfy*

$$\rho \leq r_{\mathbb{C}}(A(\tau) + \eta I_n; B(\tau), C(\tau)) \qquad \textit{for all} \quad \tau \geq 0. \tag{4.33}$$

Suppose that $(A(\tau), C(\tau))$ *is an observable matrix pair for every* $\tau \geq 0$. *Then there exists a* $\delta > 0$ *such that*

$$\|\dot{A}(\tau)\| + \|\dot{B}(\tau)\| + \|\dot{C}(\tau)\| \; < \; \delta \qquad \textit{for almost all} \quad \tau \geq 0 \tag{4.34}$$

yields

$$\rho \; \leq \; r_{\mathbb{C},\mathrm{dyn}}(A(\cdot); B(\cdot), C(\cdot)). \tag{4.35}$$

Moreover, the zero solution of the initial value problem (4.10) *is globally uniformly exponentially stable, if we restrict the class of dynamical perturbations* $\mathcal{D}(\cdot) \in \mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$ *to* $\|\mathcal{D}(\cdot)\|_{\mathrm{dyn}} < \rho$.

Certainly, the assumptions (4.33) and (4.34) need only be satisfied "at infinity"; that means $\tau \geq 0$ could be replaced by $\tau \geq \tau_0$ for some $\tau_0 \geq 0$.

Before proving Theorem 4.4, we give some intuition on the "frozen" systems. First note that for any bounded set $\mathcal{K} \subset \mathbb{K}^{n \times n}$ we have

$$\inf_{A \in \mathcal{K}} r_{\mathbb{C}}(A) > 0 \quad \Longleftrightarrow \quad \sup_{A \in \mathcal{K}} \operatorname{Re}\lambda_{\max}(A) < 0. \tag{4.36}$$

Sufficiency follows immediately by contradiction. Necessity also follows by contradiction. Assume there exist sequences $\{A_\tau\}_{\tau \in \mathbb{N}} \subset \mathcal{K}$ and $\{D_\tau\}_{\tau \in \mathbb{N}} \subset \mathbb{K}^{n \times n}$ such that

$$\lim_{\tau \to \infty} \operatorname{Re}\lambda_{\max}(A_\tau + D_\tau) = 0 \qquad \text{and} \qquad \lim_{\tau \to \infty} D_\tau = 0.$$

Then boundedness of $\mathcal{K}$ yields $\lim_{\tau\to\infty} \operatorname{Re} \lambda_{\max}(A_\tau) = 0$, and this is a contradiction.

Note that necessity does not hold true in general if $\mathcal{K}$ is bounded. To this end consider

$$A_\tau := -\begin{bmatrix} \tau & \tau^3 \\ 0 & \tau \end{bmatrix}, \qquad D_\tau := -\frac{1}{\tau}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

which satisfy

$$\operatorname{spec}(A_\tau) = \{-\tau\}, \qquad \operatorname{spec}(A_\tau + D_\tau) = \{-2\tau + 1/\tau,\ 1/\tau\}.$$

This shows that the spectrum of $A(\tau)$ might be bounded away from the imaginary axis (in the above example it even tends to $-\infty$), but the stability radius tends to 0.

The following example shows that even if the stability radii of the frozen systems are bounded away from 0, the time-varying systems might not be exponentially stable. Set

$$a(\tau) = \frac{-1}{1+\tau}, \qquad b(\tau) = \frac{1}{1+\tau}, \qquad c(\tau) = 1 \qquad \text{for all } \tau \geq 0.$$

Then a straightforward calculation gives

$$\frac{1}{1+\tau} = -\lambda_{\max}(a(\tau)) \leq r_{\mathbb{C}}(a(\tau); b(\tau), 1) = 1 \qquad \text{for all } \tau \geq 0,$$

and $r_{\mathbb{C}}(a(\cdot)) = 0$ since the solution of $\dot{x}(t) = -(1+t)^{-1}x(t)$ is $x(t) = (1+t)^{-1})$, which is not exponentially decaying.

One might think (as we erroneously did in an earlier version[2]) that (4.33) can be replaced by $\rho \leq r_{\mathbb{C}}(A(\tau); B(\tau), C(\tau))$ for all $\tau \geq 0$, since $\tau \mapsto r_{\mathbb{C}}(A(\tau); B(\tau), C(\tau))$ is continuous and $A(\cdot), B(\cdot), C(\cdot)$ are bounded. However, as the above example shows, the map $\tau \mapsto r_{\mathbb{C}}(A(\tau); B(\tau), C(\tau))$ is not *uniformly* continuous and therefore $\eta > 0$ as in (4.33) does not necessarily exist.

Proof: **(of Theorem 4.4)** We proceed in several steps.

Step 1. By Lemma 4.2 in Section 5, there exists an absolutely continuous map $\tau \mapsto P(\tau)$, where $P(\tau) = P(\tau)^* \leq 0$ is the unique Hermitian solution of the algebraic Riccati equation

$$[A(\tau) + \eta I_n]^* \; P(\tau) + P(\tau) \; [A(\tau) + \eta I_n] - \rho^2 C(\tau)^* C(\tau) - P(\tau)B(\tau)B(\tau)^* P(\tau) = 0 \qquad (4.37)$$

[2]We are indebted to Fabian Wirth (Bremen) for pointing out the example and error to us.

such that the "frozen" closed-loop systems

$$\dot{x}(t) = \hat{A}(\tau)x(t), \quad \text{where } \hat{\mathrm{A}}(\tau) := \mathrm{A}(\tau) + \eta \mathrm{I_n} - \mathrm{B}(\tau)\mathrm{B}(\tau)^*\mathrm{P}(\tau), \tag{4.38}$$

are uniformly exponentially stable for all $\tau \geq 0$.

Step 2. We show that $\dot{P}(\tau)$ is the unique solution of the Lyapunov equation

$$\hat{A}(\tau)^* \dot{P}(\tau) + \dot{P}(\tau)\hat{A}(\tau) = R(\tau) \qquad \text{for almost all } \tau \geq 0, \tag{4.39}$$

where

$$R(\tau) := -\Big[\dot{A}(\tau) - B(\tau)\dot{B}(\tau)^* P(\tau)\Big]^* P(\tau) - P(\tau)\Big[\dot{A}(\tau) - B(\tau)\dot{B}(\tau)^* P(\tau)\Big]$$
$$+ \rho^2 \left[\dot{C}(\tau)^* C(\tau) + C(\tau)^* \dot{C}(\tau)\right]. \tag{4.40}$$

Differentiability of $\tau \mapsto P(\tau)$ for almost all $\tau \geq 0$ follows from Lemma 4.2, and a straightforward differentiation of (4.37) yields (4.39).

Step 3. We show that there exists some $r > 0$, independent of $\tau \geq 0$, such that

$$\|R(\tau)\| \leq r\delta \qquad \text{for almost all } \tau \geq 0. \tag{4.41}$$

$P(\cdot)$ is uniformly bounded by Lemma 4.2. Since $A(\cdot), B(\cdot), C(\cdot)$ are uniformly bounded in τ, the statement readily follows from (4.34) and (4.40).

Step 4. We show that there exists some $p' > 0$, such that

$$\|\dot{P}(\tau)\| \leq p'\,\delta \qquad \text{for almost all } \tau \geq 0. \tag{4.42}$$

Note that $\hat{A}(\cdot)$ is uniformly bounded and hence, by continuity of $A \mapsto r_{\mathbb{C}}(A; 0, 0)$, there exists some $\omega > 0$ such that

$$0 < \omega := \min_{\tau \geq 0} r_{\mathbb{C}}(\hat{A}(\tau); 0, 0) \leq -\min_{\tau \geq 0} \max_{i=1,\dots,n} \mathrm{Re}\, \lambda_i(\hat{\mathrm{A}}(\tau)).$$

Now we are in a position to apply Proposition 3 of Coppel [1, p.4] to conclude that there exists some $M > 0$, independent of τ, such that, for all $\tau \geq 0$,

$$\|e^{\hat{A}(\tau)t}\| \leq M\, e^{-(\omega/2)t} \qquad \text{for all } t \geq 0. \tag{4.43}$$

Since the solution of (4.39) satisfies

$$\dot{P}(\tau) = -\int_0^\infty e^{\hat{A}(\tau)^* s} R(\tau) e^{\hat{A}(\tau)s}\, ds \quad \text{for almost all } \tau \geq 0, \tag{4.44}$$

an application of (4.41) and (4.43) to (4.44) yields (4.42).

Step 5. Since $A(\cdot) + \eta I_n$, $B(\cdot)$, and $C(\cdot)$ are uniformly bounded and $(A(\tau) + \eta I_n, C(\tau))$ is observable for every $\tau \geq 0$, there exist by Lemma 4.2 $p_1 > p_2 > 0$ such that

$$0 < p_2 I_n \leq -P(\tau) \leq p_1 I_n \qquad \text{for all} \quad \tau \geq 0. \tag{4.45}$$

We show that the derivative of the time-varying positive-definite Lyapunov function candidate

$$V(t,x) \; := \; -x^* P(t) x$$

along the solution of the time-varying system (4.10) satisfies, for any $\mathcal{D}(\cdot) \in \mathcal{P}_{\mathrm{dyn}}(\mathbb{K})$ with $\|\mathcal{D}(\cdot)\|_{\mathrm{dyn}} < \rho$ and for almost all $t \geq 0$,

$$\frac{d}{dt} V(t, x(t)) \leq -\eta\, V(t, x(t)) - \rho^2 \|C(t)x(t)\|^2 + \|\mathcal{D}\left([Cx]^{\varphi}\right)(t)\|^2. \tag{4.46}$$

Set δ in (4.34) to $\delta = \eta p_2 / p'$. Now differentiation of V along (4.10) and invoking (4.37) and (4.42) yields, where we omit the argument t for simplicity,

$$\begin{aligned}
\tfrac{d}{dt} V(t,x(t)) \;&=\; -[Ax + B\mathcal{D}\left([Cx]^{\varphi}\right)]^* \, Px - x^* P[Ax + B\mathcal{D}\left([Cx]^{\varphi}\right)] - x^* \dot{P} x \\
&=\; 2\eta x^* P x - x^* \dot{P} x - \rho^2 \|Cx\|^2 + \|\mathcal{D}\left([Cx]^{\varphi}\right)\|^2 \\
&\quad\; - \|B^* P x + \mathcal{D}\left([Cx]^{\varphi}\right)\|^2 \\
&\leq\; -\left[2\eta - \tfrac{p'}{p_2}\delta\right] x^*(-P)x - \rho^2 \|Cx\|^2 + \|\mathcal{D}\left([Cx]^{\varphi}\right)\|^2.
\end{aligned}$$

This proves (4.46).

Step 6. (4.46) is of the type (4.23). Now proceeding exactly as in Steps 3 and 4 of the proof of Theorem 4.2 yields global uniform exponential stability of (4.10). As a consequence, we have (4.35) and the proof is complete. □

4.5 Two Lemmas

Although the following lemma on the asymptotic behavior of the convolution of L^p-functions is a consequence of the much more general result of Gripenberg et al. [5, Theorem 2.2, p.39], we present a simple proof for completeness.

Lemma 4.1. *If $f(\cdot) \in L^p(\mathbb{R}_{\geq 0}, \mathbb{R})$ and $g(\cdot) \in L^q(\mathbb{R}_{\geq 0}, \mathbb{R})$ for $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, then*

$$\lim_{t \to \infty} \int_0^t f(t-\tau) g(\tau)\, d\tau \;=\; 0.$$

Proof:[3] An application of Hölder's inequality yields

$$\begin{aligned}
\int_0^t f(t-\tau)g(\tau)\,d\tau &= \int_0^{t/2} f(t-\tau)g(\tau)\,d\tau + \int_{t/2}^t f(t-\tau)g(\tau)\,d\tau \\
&\le \|f\|_{L^p(t/2,t)}\|g\|_{L^q(0,t/2)} + \|f\|_{L^p(0,t/2)}\|g\|_{L^q(t/2,t)} \\
&\le \|f\|_{L^p(t/2,\infty)}\|g\|_{L^q(0,\infty)} + \|f\|_{L^p(0,\infty)}\|g\|_{L^q(t/2,\infty)}\,.
\end{aligned}$$

This proves the claim, since by assumption the right-hand side of the last inequality converges to 0 as $t \to \infty$. □

The following lemma is on the existence of Hermitian solutions $P = P^*$ of the *nonstandard algebraic Riccati equation*

$$A^*P + PA - \rho^2 C^*C - PBB^*P = 0. \tag{4.47}$$

The result is known, but we are not aware of a comprehensive presentation, and since it is crucial in our analysis, we present a proof.

Lemma 4.2. *Let $\rho > 0$ be fixed, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and define*

$$\Sigma = \left\{ (A,B,C) \in \mathbb{K}^{n\times(n+m+p)} \;\middle|\; \rho < r_{\mathbb{C}}(A;B,C) \right\}.$$

Then the map

$$\begin{array}{lccl}
\varphi : & \Sigma & \longrightarrow & \mathcal{H} = \{P \in \mathbb{K}^{n\times n} \mid P = P^*\} \\
& (A,B,C) & \mapsto & P, \ \text{where } P \text{ solves (4.47) and } \sigma(A - BB^*P) \subset \mathbb{C}_-
\end{array}$$

is well defined; that is, for every (A,B,C) with $\rho < r_{\mathbb{R}}(A;B,C)$ there exists a solution of the algebraic Riccati equation (4.47) which is unique among all exponentially stabilizing solutions, and

(i) *φ is analytic if $\mathbb{K} = \mathbb{C}$, respectively, real analytic if $\mathbb{K} = \mathbb{R}$,*

(ii) *$P = \varphi(A,B,C)$ satisfies*

$$P = -\int_0^\infty e^{A^*s}\left[\rho^2 C^*C + PBB^*P\right]e^{As}\,ds \tag{4.48}$$

and is hence nonpositive. If (A,C) is observable, then P is negative definite.

Proof: Hinrichsen and Pritchard [7] proved the existence and uniqueness of $P = \varphi(A,B,C)$. Their proof is only for $\mathbb{K} = \mathbb{C}$, but carries over if $\mathbb{K} = \mathbb{R}$.

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[3] We are indebted to Hartmut Logemann (Bath) for pointing out to us this simple proof.

We prove analyticity of φ by using the Implicit Function Theorem; this idea is due to Delchamps [3] who used it for the standard Riccati equation; that is, $-\rho^2 C^*C$ in (4.47) is replaced by $+C^*C$.

The map

$$\begin{array}{rcccl} \psi & : & \mathbb{K}^{n\times(n+m+p)} \times \mathcal{H} & \longrightarrow & \mathcal{H} \\ & & (A,B,C,P) & \mapsto & A^*P + PA - \rho^2 C^*C - PBB^*P \end{array}$$

is certainly analytic and its differential with respect to P is given by

$$\psi_P(A,B,C,P) \; : \; H \mapsto \; [A - BB^*P]^*H + H[A - BB^*P].$$

Let $(A,B,C) \in \Sigma$ and $P = \varphi(A,B,C)$. Then $\psi(A,B,C,P) = 0$ and since $\sigma(A - BB^*P) \subset \mathbb{C}_-$ it follows that $\psi_P(A,B,C,P)$ is regular. Therefore, *(i)* is a consequence of the Implicit Function Theorem.

It remains to show *(ii)*. (4.48) follows from (4.47) and thus

$$P \leq -\rho^2 \int_0^\infty e^{A^*s} C^*Ce^{As}\, ds \leq 0. \tag{4.49}$$

If (A,C) is observable, then an application of the observability Gramian to (4.49) yields that P is negative-definite. □

4.6 Conclusions

This chapter deals with the problem of estimating the stability radius of time-varying linear systems with respect to L^2 finite gain perturbations. Sufficient conditions for uniform exponential stability are given. A conservative but workable bound is presented in terms of the stability radii of the frozen systems, provided the time-variation is sufficiently slow. Thus numerical computations of the stability radii of time-invariant systems (see Hinrichsen et al. [11], Filbir [4], and Sreedhar et al. [19]) might be used to compute a bound of the stability radius of time-varying systems.

The result presented here has implications for the study of gain scheduled control systems. It is generally accepted that for sufficiently slowly scheduled controllers the performance of the frozen system is a good indicator of what to expect in the gain scheduled case. See, for example, Shamma and Athans [17] and Lawrence and Rugh [15]. Quantifying "slow" and the extent to which this folklore is acceptable will be investigated in a separate paper.

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