Supersymmetry – From Quantum Mechanics to Lattice Field Theories

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1 Introduction

The Standard Model (SM) of particle physics has been extraordinarily successful for more than three decades. It gives an astonishingly good description of known phenomena in high-energy physics up to energies of 200-300 GeV. The recent measurement of the neutrino mass, however, represents a first experimental deviation. The SM consists of various fermions, scalar fields and gauge bosons of the strong $SU(3)_c$ and electroweak $SU(2)_L \times U(1)_Y$ gauge group. The electroweak gauge symmetry is spontaneously broken to $U(1)_Q$ of Quantum Electrodynamics (QED), such that one obtains massive W and Z gauge bosons. Their mass is parametrized by the vacuum expectation value of a complex scalar field, the Higgs field. All particles, except for the Higgs, have been observed in particle accelerators and are listed in Table 1.1 with their corresponding charges. Additional faith in the SM is provided by the renormalizability of the model, which was not rigorously proven until 1997 [1].

Still, it seems quite clear that the SM should be viewed as an effective field theory that will have to be extended to describe physics at arbitarily high energies. Certainly, for energies close to the Planck scale of 10^{19} GeV, gravity has to be incorporated into the SM to obtain a consistent theory. Furthermore, there are many free parameters in the SM, and there is no guide to the origin of flavour, charge quantization and quark-lepton distinction. Another challenge is the hierarchy problem. The lightness of the Higgs is not protected by a gauge or chiral symmetry. In contrast to fermions, where one obtains logarithmic divergences for the masses, the mass of the spin-zero Higgs is quadratic in the cutoff Λ in one-loop perturbation theory. One needs to fine-tune the mass of the Higgs at the Planck scale to obtain the rather small mass of the Higgs at the weak scale of 100 GeV. Finally, fermions and bosons are treated quite differently. Fermions correspond to matter fields, while bosons are mediators of interaction.

In the 1960s, there have been several attemps to unify particles of different spin. There are approximate symmetries in non-relativistic quark models, where for example the

Particles	Description	Spin	Charges
Gauge bosons:		1	
g	$\mathrm{SU}(3)_c$ gauge bosons (gluons)		$(8,1)_0$
W	$\mathrm{SU}(2)_L$ gauge bosons		$(1,3)_0$
B	$\mathrm{U}(1)_Y$ gauge bosons		$(1,1)_0$
Chiral matter (three families):		$\frac{1}{2}$	
$q_a = \begin{pmatrix} u_a \\ d_a \end{pmatrix}$	quarks		$(3,2)_{\frac{1}{6}}$
u_a^c	antiquarks		$(\bar{3},1)_{-\frac{2}{3}}$
d_a^c	antiquarks		$(\bar{3},1)_{\frac{1}{3}}$
$L_a = \begin{pmatrix} \nu_a \\ e_a^- \end{pmatrix}$	Leptons		$(1,2)_{-\frac{1}{2}}$
$(e_a^-)^c$	anti-Leptons		$(1,1)_1$
$H = \begin{pmatrix} H^0 \\ H^- \end{pmatrix}$	Higgs	0	$(1,2)_{-\frac{1}{2}}$

Table 1.1: The SM field content. $(Q_c, Q_L)_{Q_{Y/2}}$ lists colour, weak isospin and hypercharge of a given particle and $Q_c = 1$, $Q_L = 1$ or $Q_{Y/2} = 0$ indicate a singlet under the respective group. The T_3 isospin operator is $+1/2$ $(-1/2)$ when acting on the upper (lower) component of an isospin doublet and zero otherwise. The electric charge of QED is given by $Q = T_3 + Y/2$.

$\mathrm{SU}(3)_c$ symmetry is replaced by a larger SU(6) symmetry. The attempt to generalize these non-relativistic quark models to relativistic ones failed, and it was even proved that such a model cannot exist under the following assumptions [2]. If there is only a finite number of particles below any given mass and if the S-matrix is nontrivial and analytic, the most general Lie algebra of symmetry operators which commute with the S-matrix is a direct product of some internal symmetry group and the Poincaré group. One possibility to circumvent this Coleman-Mandula theorem is to replace the Lie algebra by a graded Lie algebra. This leads us to supersymmetry.

The first paper on supersymmetry in four spacetime dimensions was published by Gol'fand and Likhtman in 1971 [3] followed by a paper of Volkov and Akulov [4] with nonlinear realization of supersymmetry. Their theories are not renormalizable. A few years later, Wess and Zumino constructed a renormalizable supersymmetric model [5] and pointed out the way to circumvent the Coleman-Mandula theorem [6]. Inspired by these observations, Haag, Łopuszański and Sohnius [7] extended the results of the Coleman-Mandula theorem to symmetry operators obeying anticommutation relations. In the context of string theory, supersymmetry was first introduced by Ramond [8] and Neveu and

Schwarz [9] in 1971. Also gravity has been supersymmetrized to supergravity by using local supersymmetry parameters instead of rigid ones [10]. Supersymmetry gives the possibility to unify the SM with gravity. The low-energy effective action of such theories leads to supersymmetric field theories in four dimensions.

There have been several attempts to solve the hierarchy problem. One proposal suggests that the Higgs is no fundamental particle but a meson, composed of fermions [11]. A different and more recent proposal is the embedding of the theory into higher dimensions. The most convincing one seems to be supersymmetry, as it can suppress the quadratically divergent terms and therefore solve the hierarchy problem.

In the literature many supersymmetric extensions of the SM have been discussed. The supersymmetric extension of the SM with minimal particle content is called the minimal supersymmetric standard model (MSSM), see [12, 13]. Nice introductions to the MSSM can be found in [14, 15]. In this model, the hierarchy problem is solved by construction and, rather amazingly, the running couplings of the different interactions unify at a scale of 10^{16} GeV (see [16] and references therein). However, for each particle contained in the SM, there is an associated superpartner of the same mass in the MSSM which is not observed in nature. One possibility to overcome this problem is to introduce soft sypersymmetry-breaking terms. This introduces lots of new parameters and makes the theory not so elegant. Another possibility, which will be discussed further below, is spontaneous breaking of supersymmetry.

At this point one should stress that despite all hopes, supersymmetry has not been observed in nature yet. But even if supersymmetry is not realized in nature, we would like to emphasize its mathematical beauty which was used to prove several theorems in mathematics or even to find and define new invariants of manifolds, as for example the Seiberg-Witten invariants of a four-manifold. Also, as supersymmetric theories can be solved more easily than non-supersymmetric counterparts, one can better understand several non-perturbative effects by studying supersymmetric theories. For example, the ingenious work by Seiberg and Witten [17, 18] gave further insight in the mechanism of confinement and dualities. One should also mention the Maldacena conjecture, stating that $\mathcal{N} = 4$ super-conformal SU(N_c)-gauge theories arising on parallel D3-branes are dual to supergravity theories on an AdS_5-background [19] (in the limit of large 't Hooft coupling and large N_c).

In spite of these striking results there is still a long way to go towards a better understanding of non-perturbative effects in supersymmetric theories. In particular, since low-energy physics is manifestly non-supersymmetric, it is necessary that supersymmetry is broken below some energy scale. As supersymmetry breaking is difficult to address in perturbation theory, one is motivated to study supersymmetric models on a lattice. Unfortunately, supersymmetry is explicitly broken by most discretization procedures, and it is a nontrivial problem to recover supersymmetry in the continuum limit. Note that also Poincaré invariance is broken by a lattice, but the hypercubic crystal symmetry forbids relevant operators which could spoil the Poincaré invariance in the continuum limit.

There are many supersymmetric lattice models circulating in literature, all of them with their own advantages and disadvantages. Let us give a short survey of some of them.

One attempt is to realize the full algebra on the lattice. Nicolai and Dondi [20] were the first to point out that one needs to introduce nonlocal interaction terms. They were only able to obtain lattice theories for infinitely extended lattices. Bartels and Bronzan [21] continued along this line. They formulated Lagrangian and Hamiltonian Wess-Zumino models which preserve the superalgebra. Their construction is based on Fourier transformation and they considered both, the continuum and the strong-coupling limit. In [22, 23] Nojiri introduced translation operators on the lattice and used them to construct lattice theories which incorporate the full superalgebra.

As the nonlocal interaction terms are difficult to treat (for example in the strong-coupling limit or in computer simulations), several people proceeded along another way. Banks and Windey [24] and later Rittenberg and Yankielowicz [25] tried to preserve not the full algebra, but only a part of it by keeping time continuous. Unfortunately, their lattice version does not fully recover the Lorentz symmetry in the continuum limit. Elitzur, Rabinovici and Schwimmer [26] further continued in this direction. They were considering subalgebras of the full algebra which contain only the Hamiltonian, and put this theories on a spatial lattice. Their method only works for $\mathcal{N} \geq 1$ models in two dimensions and for models with $\mathcal{N} \geq 2$ supersymmetry in four dimensions. In [27], Elitzur and Schwimmer investigated the $\mathcal{N} = 2$ Wess-Zumino model in two dimensions and observed the rather strange result of a infinitely-degenerate ground state in the continuum limit.

Every supersymmetric field theory on a spatial lattice may be reinterpreted as a high-dimensional supersymmetric quantum mechanical system. The first studies of such systems go back to Nicolai [28] and have been extended by Witten in his work on supersymmetry breaking [29, 30, 31]. Soon after that, de Crombrugghe and Rittenberg [32] presented a very general analysis of supersymmetric Hamiltonians. Since then, many supersymmetric Hamiltonians have been investigated. In particular, it has been demonstrated that supersymmetry is a useful technique to construct exact solutions in quantum mechanics [33, 34, 35, 36]. For example, all ordinary Schrödinger equations with shape-invariant potentials can be solved algebraically with the methods of supersymmetry. On the other hand, apparently different quantum systems may be related by supersymmetry, and this relation may shed new light on the physics of the two systems. For example, the hydrogen atom (its Hamiltonian, angular momentum and Runge-Lenz vector) can be supersymmetrized. The corresponding theory contains both the proton-electron and the proton-positron system as subsectors [JDL1].

The aim of this thesis is to further analyze lattice models of supersymmetric field theories. We restrict our attention to Wess-Zumino models in 1+1 dimensions. As mentioned above, the Hamiltonian formulation of lattice models – keeping time continuous and discretizing space – leads to high-dimensional supersymmetric quantum mechanical systems.

Therefore, the first part of this thesis concentrates on supersymmetric quantum mechanics in arbitrary dimensions. In Section 2.1 we give the definition of a supersymmetric quantum mechanical system with $\mathcal{N}$ supercharges and define the Witten index. Then, in Section 2.2, we specify the first supercharge to be the Dirac operator on an arbitrary even dimensional manifold with gauge field background. We call this the *chiral supersymmetry* of the Dirac operator. We investigate under which conditions there are further supercharges. This gives constraints for the gravitational as well as for the gauge field background. For the case of two supercharges, we show the existence of a number operator. Thus, the $\mathbb{Z}_2$-grading into left- and right-handed spinors is extended to a finer grading into a Fock-space. Another important property of the $\mathcal{N} = 2$ case is the existence of a superpotential. This superpotential can be used to determine zero modes of the Dirac operator, and we illustrate this for complex projective spaces. In Section 2.3 we use dimensional reduction to relate the Dirac operator on a flat manifold with Abelian gauge field to multi-dimensional supersymmetric matrix-Schrödinger operators. We also discuss some basic properties of these Hamiltonians. At this point, in Section 2.4, we examine a specific matrix-Hamiltonian describing the supersymmetric hydrogen atom. We determine the super-Laplace-Runge-Lenz vector and the spectrum of the super-Hamiltonian by group theoretical methods in the spirit of Pauli's algebraic approach [37]. We conclude the first part of this thesis with a brief discussion in Section 2.5 about supersymmetry breaking in quantum mechanical systems. In the framework of perturbation theory we show that under certain assumptions zero modes remain zero modes if the problem is appropriated deformed. We give some examples to illustrate this fact and also show that perturbation theory not always leads to the correct result.

The second part of this thesis is dealing with lattice versions of Wess-Zumino models in two-dimensional Minkowski space. We start the discussion in Section 3.1 with details of the $\mathcal{N}=1$ Wess-Zumino model in the continuum. Then, we examine under which conditions the on-shell formulation of the models allows for further supersymmetries. At the end of this section we investigate how a specific extended $\mathcal{N}=2$ model is related to the $\mathcal{N}=1$ model in four dimensions. In Section 3.2 we investigate Wess-Zumino models on a spatial lattice. We give a short discussion about various lattice derivatives and their advantages and disadvantages. Then we discuss the $\mathcal{N}=1$ and $\mathcal{N}=2$ Wess-Zumino models on the lattice, determine their ground states for the massive free theory and in the strong-coupling limit. Finally, we relate our strong-coupling results to the full problem.

The appendices summarize several facts needed in the main part of this thesis. In Appendix A we determine the spectrum and eigenvectors of the Dirac operator on a ball with chiral-bag boundary conditions. In Appendix B we prove that the four-dimensional super-Poincaré algebra with $\mathcal{N}=1$ does not allow for a subalgebra which closes only on the Hamiltonian. The mathematical Appendix C is devoted to the proof of analyticity of specific perturbations.

2 Supersymmetric Quantum Mechanics

In the first part of this thesis we are interested in several aspects of supersymmetric quantum mechanics. In Section 2.1 we give a definition of a supersymmetric quantum mechanical system[1] and introduce the Witten index. Then, in Section 2.2, we consider a specific example for a supercharge, the Dirac operator on an even dimensional manifold with gauge field. In this example, supersymmetry is equivalent to chiral symmetry between left and right handed spinors. We elaborate in great detail under which conditions further supersymmetries exist. This leads us to the introduction of a particle number operator and the notion of a superpotential which contains a gauge part as well as a gravitational part. Finally, we use the developed techniques to determine zero modes of the Dirac operator on complex projective spaces. In Section 2.3 we make the connection of the Dirac operator on flat space with supersymmetric matrix-Schrödinger Hamiltonians and discuss some of their properties.

For a model to be integrable, symmetries play a crucial role. A well known example is the hydrogen atom, where a hidden symmetry allows to solve for the spectrum by algebraic means. For low-dimensional systems supersymmetry may be already sufficient, but often, further symmetries are needed for the model to be integrable. In Section 2.4 we combine supersymmetry with the hydrogen atom in arbitrary dimensions. We show that this system is integrable. The last section in this chapter, Section 2.5, investigates the mechanism of supersymmetry breaking. This section will be important in the second part of the thesis, where we will discuss supersymmetry breaking in field theories on a spatial lattice.

[1] There are different definitions of supersymmetric quantum mechanics in the literature. A recent discussion can be found in [38].

2.1 Supersymmetry

We define supersymmetric quantum mechanics by a set $(\mathcal{H}, H, \Gamma, Q_1, \ldots, Q_{\mathcal{N}})$. H is a self-adjoint Hamiltonian acting on a Hilbert space $\mathcal{H}$. We require the existence of a self-adjoint operator Γ (*grading operator*), which squares to the identity, and self-adjoint operators Q_i (*supercharges*), $i = 1, \ldots, \mathcal{N}$, such that

$$\delta_{ij} H = \frac{1}{2} \{Q_i, Q_j\} \qquad \text{and} \qquad \{Q_i, \Gamma\} = 0. \tag{2.1}$$

The $+1$ and -1 eigenspaces of Γ are called *bosonic* and *fermionic sectors* respectively,

$$\mathcal{H} = \mathcal{H}_{\mathrm{B}} \oplus \mathcal{H}_{\mathrm{F}}, \quad \mathcal{H}_B = \mathcal{P}_+ \mathcal{H}, \quad \mathcal{H}_F = \mathcal{P}_- \mathcal{H}, \quad \mathcal{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma). \tag{2.2}$$

Every linear operator on $\mathcal{H}$ has a 2×2 matrix representation with respect to the decomposition (2.2). We obtain for example

$$\Gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad Q_i = \begin{pmatrix} 0 & L_i^\dagger \\ L_i & 0 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} L_i^\dagger L_i & 0 \\ 0 & L_i L_i^\dagger \end{pmatrix}. \tag{2.3}$$

The algebra of linear operators naturally splits into a bosonic and a fermionic part. Every operator which is diagonal in the 2×2 matrix representation, for example Γ and H, is called bosonic. Off-diagonal ones, for example the Q_i, are called fermionic. Bosonic operators do not mix $\mathcal{H}_{\mathrm{B}}$ with $\mathcal{H}_{\mathrm{F}}$. Contrarily, fermionic operators map $\mathcal{H}_{\mathrm{B}}$ into $\mathcal{H}_{\mathrm{F}}$ and vice versa. The definition of bosonic and fermionic operators is equivalent to the statement that bosonic operators commute with the grading operator Γ while fermionic ones anticommute. Observe that only the bosonic operators form a subalgebra.

The *superalgebra* (2.1) implies that the supercharges Q_i commute with the Hamiltonian,

$$[Q_i, H] = 0, \tag{2.4}$$

and generate symmetries of the system. The simplest models exhibiting this structure are 2×2-matrix Schrödinger operators in one dimension [28, 29, 30]. Let us consider specific $\mathcal{N}$'s in the following.

For one supercharge, every eigenstate of $H = Q_1^2 \geq 0$ with positive energy is paired by the action of Q_1. For example, if $|B\rangle$ is a bosonic eigenstate with positive energy, then

$|F\rangle \sim Q_1|B\rangle$ is a fermionic eigenstate with the same energy. However, a normalizable eigenstate with zero energy is annihilated by the supercharge, $Q_1|0\rangle = 0$, and hence has no superpartner. As the grading operator Γ commutes with Q_1 on the space of zero modes, we can choose the zero modes to be eigenstates of Γ. We denote the number of zero modes in the bosonic (fermionic) subspace by n^0_{B} (n^0_{F}). We say that supersymmetry is spontaneously broken if there is no state which is left invariant by the supercharges or, equivalently, if zero is not in the discrete spectrum of H. A useful tool to investigate whether supersymmetry is broken or not is the Witten index. Assume that L_1 is Fredholm. The index of L_1 (in the context of supersymmmetry called Witten index) is defined as

$$\operatorname{ind} L_1 = \dim\ker L_1 - \dim\ker L_1^\dagger = n^0_{\mathrm{B}} - n^0_{\mathrm{F}}. \tag{2.5}$$

It counts the difference between bosonic and fermionic zero modes of H. Clearly, $\operatorname{ind} L_1 = 0$ is a necessary condition for supersymmetry to be broken. In the case of $n^0_{\mathrm{B}} = 0$ or $n^0_{\mathrm{F}} = 0$, $\operatorname{ind} L_1 = 0$ is a necessary and sufficient condition for broken supersymmetry. A useful expression for evaluating the Witten index, in the case that $\exp(-\beta H)$, $\beta > 0$, is trace-class, is given by

$$\operatorname{ind} L = \mathrm{Tr}(\Gamma \mathrm{e}^{-\beta H}), \tag{2.6}$$

where Tr is the trace in Hilbert space, see for example [39].

In the case of two supercharges, there exist two anticommuting and self-adjoint roots of the Hamiltonian,

$$H = Q_1^2 = Q_2^2, \qquad \{Q_1, Q_2\} = 0. \tag{2.7}$$

Later we shall use the nilpotent *complex supercharge*,

$$Q = \frac{1}{2}(Q_1 + \mathrm{i}Q_2) \tag{2.8}$$

and its adjoint $Q^\dagger$, in terms of which the supersymmetry algebra takes the form

$$H = \{Q, Q^\dagger\}, \quad Q^2 = Q^{\dagger\,2} = 0 \quad \text{and} \quad [Q, H] = 0. \tag{2.9}$$

The number of normalizable zero modes of H is given by a cohomological argument. A zero mode $|\phi\rangle$ has to be annihilated by the complex supercharge Q, that is $|\phi\rangle \in \ker Q$. Assume there exists an eigenstate $|\psi\rangle$ of H with $|\phi\rangle = Q|\psi\rangle$. As Q commutes with H, $|\psi\rangle$ has to be a zero mode too, which contradicts the fact that $|\psi\rangle$ is not annihilated by Q. We conclude that the zero mode $|\phi\rangle$ is not in the image of Q. The same arguments hold for $Q^\dagger$ and therefore we can express the number of zero modes as

$$n^0 = n^0_{\mathrm{B}} + n^0_{\mathrm{F}} = \dim(\ker Q/\mathrm{im}\ Q) = \dim(\ker Q^\dagger/\mathrm{im}\ Q^\dagger). \tag{2.10}$$

2.2 Supersymmetries and the Dirac Operator

There is a fundamental supersymmetric Hamiltonian, the square of the Dirac operator. The *chiral supersymmetry* with one charge exists on all even-dimensional Riemannian spin manifolds, with arbitrary gauge fields. The existence of further supercharges gives constraints for the Riemannian manifold and for the gauge field.

Let us consider a smooth Riemannian spin manifold $\mathcal{M}$ of dimension D with metric g. In local coordinates $\{x^M\}_{M=1,\dots,D}$ the metric is specified by the metric coefficients $g_{MN} = g(\partial_M, \partial_N)$, where $\{\partial_M = \partial/\partial x^M\}$ is the holonomic coordinate basis for the tangent space at each point. We introduce an orthonormal basis with the help of vielbeine e_A^M, $e_A = e_A^M \partial_M$ and obtain

$$\delta_{AB} = g(e_A, e_B) = e_A^M\, e_B^N\, g_{MN}. \tag{2.11}$$

The Lorentz indices $A, B \in \{1, \dots, D\}$ are converted into coordinate indices $M, N \in \{1, \dots, D\}$ (or vice versa) with the help of the vielbein e_A^M or its inverse given by $e_M^A = g_{MN} e_B^N \delta^{BA}$. The Clifford algebra is generated by the Hermitian $2^{D/2} \times 2^{D/2}$-matrices Γ^A satisfying

$$\{\Gamma^A, \Gamma^B\} = 2\delta^{AB} \quad \text{or} \quad \{\Gamma^M, \Gamma^N\} = 2g^{MN}, \tag{2.12}$$

where the $\Gamma^M = \Gamma^A e_A^M$ are the matrices with respect to the holonomic basis ∂_M.

2.2.1 Chiral Supersymmetry

In even dimensions, $D = 2d$, we always have *chiral supersymmetry* generated by the Hermitian Dirac operator viewed as supercharge,

$$Q_1 = \mathrm{i}\nabla\!\!\!\!/ = \mathrm{i}\Gamma^M \nabla_M = \mathrm{i}\Gamma^A \nabla_A, \qquad \nabla_A = e_A^M \nabla_M. \tag{2.13}$$

The covariant derivative acting on spinors,

$$\nabla_M = \partial_M + \omega_M + A_M, \tag{2.14}$$

contains the connection $\omega = \frac{1}{4}\omega_{MAB}\Gamma^{AB}$, $\Gamma^{AB} = \frac{1}{2}[\Gamma^A, \Gamma^B]$ and a gauge potential A which takes values in some Lie algebra $\mathfrak{g}$. In the notation chosen, both ω and A are antihermitian. The Γ-matrices are covariantly constant in the following sense,

$$\nabla_M \Gamma^N = \partial_M \Gamma^N + \Gamma^N_{MP}\Gamma^P + [\omega_M, \Gamma^N] = 0, \tag{2.15}$$

where Γ^P_{MN} are the usual Christoffel symbols. For the involutory operator Γ in (2.1) we take

$$\Gamma = \Gamma_* = \alpha \Gamma^1 \ldots \Gamma^d, \tag{2.16}$$

with the phase α chosen such that Γ is Hermitian and squares to $\mathbb{1}$, $\alpha^2 = (-)^d$. The bosonic and fermionic subspaces consist of spinor fields with positive and negative chiralities, respectively. The index defined in (2.5) is the usual index of a Dirac operator, which can be calculated using the Atiyah-Singer index theorem for compact manifolds without boundaries [40] or the Atiyah-Patodi-Singer index theorem for compact manifolds with boundary [41]. Actually, supersymmetry can be used to derive such index theorems. In [42], for example, a quantum mechanical sigma model is used to determine various index theorems.

The commutator of two covariant derivatives yields the gauge field strength and curvature tensor in the spinor-representation,

$$\begin{aligned}[][\nabla_M, \nabla_N] &= \mathcal{F}_{MN} = F_{MN} + R_{MN}, \\ F_{MN} &= \partial_M A_N - \partial_N A_M + [A_M, A_N], \\ R_{MN} &= \partial_M \omega_N - \partial_N \omega_M + [\omega_M, \omega_N] = \frac{1}{4} R_{MNAB} \Gamma^{AB},\end{aligned} \tag{2.17}$$

where the Riemann curvature tensor is obtained from the connection via

$$R_{MNAB} = \partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{MA}{}^C \omega_{NCB} - \omega_{NA}{}^C \omega_{MCB}. \tag{2.18}$$

We find the squared Dirac operator,

$$H = (Q_1)^2 = -\nabla\!\!\!/^2 = -g^{MN} \nabla_M \nabla_M - \frac{1}{2} \Gamma^{AB} \mathcal{F}_{AB}. \tag{2.19}$$

Here we have used the components of $\mathcal{F}_{MN}$ with respect to an orthonormal basis,

$$\mathcal{F}_{AB} = e_A^M e_B^N \mathcal{F}_{MN} = [\nabla_A, \nabla_B]. \tag{2.20}$$

Note that the two covariant derivatives in (2.19) act on different types of fields. The derivative on the right acts on spinors and is given in (2.14), whereas the derivative on the left acts on spinors with a coordinate index and hence contains an additional term proportional to the Christoffel symbols,

$$\nabla_M \psi_N = \partial_M \psi_N + \omega_M \psi_N - \Gamma^P_{MN} \psi_P + A_M \psi_N. \tag{2.21}$$

2.2.2 Extended Supersymmetries

In the following we investigate under which conditions the Hamiltonian (2.19) has further supersymmetries besides the chiral supersymmetry. We characterize a class of first-order differential operators which square to $H = -\nabla\!\!\!/^2$. Our ansatz is motivated by previous results in [43, 44] and the simple observation that both, the free Dirac operator $\partial\!\!\!/$ on flat space and

$$I^M_N \Gamma^N \partial_M, \tag{2.22}$$

have the same square for any orthogonal matrix I. Thus, we are led to the following ansatz for additional supercharges

$$Q_i = Q(I_i) = \mathrm{i}I_i{}^M{}_N\Gamma^N\nabla_M = \mathrm{i}(I_i\Gamma)^M\nabla_M, \qquad i = 1, \ldots, \mathcal{N} - 1, \tag{2.23}$$

where the I_i are real tensor fields with components $I_i{}^M{}_N$. As the $\mathcal{N}$th supercharge we take $Q_{\mathcal{N}} = Q(\mathbb{1}) = \mathrm{i}\nabla\!\!\!\!/$. Imposing the condition that the supercharges Q_i, $i = 1, \ldots, \mathcal{N}$ obey the algebra (2.1), one obtains the following Lemma, the details of which can be found in [JDL3],

Lemma: The $\mathcal{N}$ charges

$$Q_{\mathcal{N}} = \mathrm{i}\nabla\!\!\!\!/ \quad \text{and} \quad Q_i = \mathrm{i}I_i{}^A{}_B\Gamma^B\nabla_A, \quad i = 1, \ldots, \mathcal{N} - 1, \tag{2.24}$$

are Hermitian and generate an extended superalgebra (2.1), if and only if

$$\{I_i, I_j\} = -\,2\delta_{ij}\mathbb{1}_d, \quad I_i^{\mathrm{T}} = -I_i \tag{2.25}$$

$$\nabla I_i = 0, \quad [I_i, F] = 0. \tag{2.26}$$

These formulae should be read as relations with respect to the orthonormal frame, for which we do not have to distinguish between upper and lower indices. Especially, these conditions mean that the I_i are complex structures. These complex structures form a D-dimensional real representation of the Clifford algebra in $\mathcal{N} - 1$ dimensions. We call a representation irreducible, if only $\mathbb{1}$ commutes with all matrices I_i. In the irreducible cases, only flat space and no gauge field are allowed. Irreducible representations are known to exist for the cases

$\mathcal{N} - 1$	$8k$	$6 + 8k$	$7 + 8k$
D	16^k	$8 \cdot 16^k$	$8 \cdot 16^k$

$$, \ k \in \mathbb{N}_0. \tag{2.27}$$

In the following we investigate the $\mathcal{N} = 2$ case in more detail, as it will be important for further considerations. We will not elaborate on the more special cases of higher $\mathcal{N}$ for which the interested reader is referred to [JDL3].

2.2.3 $\mathcal{N}=2$ and Number Operator

The existence of one covariantly constant complex structure implies in particular, that the manifold is Kähler. Therefore, on any Kähler manifold of dimension $D=2d$, the Dirac operator admits an extended $\mathcal{N}=2$ supersymmetry if the field strength commutes with the complex structure (2.26). With respect to a suitably chosen orthonormal frame this structure has the form $(I^A{}_B)=\mathrm{i}\sigma_2\otimes\mathbb{1}_d$, while the condition that the field strength commutes with the complex structure is equivalent to the condition

$$(F_{AB})=\begin{pmatrix}U & V\\ -V & U\end{pmatrix},\qquad U^{\mathrm{T}}=-U,\ V^{\mathrm{T}}=V. \tag{2.28}$$

The complex nilpotent charge in (2.8) takes the simple form

$$Q=\frac{1}{2}(Q(\mathbb{1})+\mathrm{i}Q(I))=\mathrm{i}\psi^A\nabla_A \tag{2.29}$$

with operators

$$\psi^A=P^A_B\Gamma^B,\qquad P^A_B=\frac{1}{2}(\mathbb{1}+\mathrm{i}I)^A{}_B. \tag{2.30}$$

P projects onto the d-dimensional I-eigenspace corresponding to the eigenvalue $-\mathrm{i}$, its complex conjugate $\bar{P}$ onto the d-dimensional eigenspace $+\mathrm{i}$. The two eigenspaces are complementary and orthogonal, $P+\bar{P}=\mathbb{1}$ and $P\bar{P}=0$. The ψ^A and their adjoints form a fermionic algebra,

$$\{\psi^A,\psi^B\}=\{\psi^{A\dagger},\psi^{B\dagger}\}=0\qquad\text{and}\qquad\{\psi^A,\psi^{B\dagger}\}=2P^{AB}. \tag{2.31}$$

At this point it is natural to introduce the *number operator*,

$$N=\frac{1}{2}\psi^\dagger_A\psi^A=\frac{1}{4}(D\mathbb{1}+\mathrm{i}I_{AB}\Gamma^{AB}), \tag{2.32}$$

the eigenvalues of which are lowered and raised by ψ^A and $\psi^{A\dagger}$, respectively,

$$[N,\psi^{A\dagger}]=\psi^{A\dagger},\qquad[N,\psi^A]=-\psi^A. \tag{2.33}$$

Since only $d=\operatorname{rank}P$ of the D creation operators are linearly independent we have inserted a factor $\frac{1}{2}$ in the definition of the number operator N in (2.32). This operator commutes with the covariant derivative, because covariant constancy of the complex structure is equivalent to

$$[\nabla_M, N] = \partial_M N + [\omega_M, N] = 0. \tag{2.34}$$

It follows with the definition of Q in (2.29) and (2.33) that Q decreases N by one, while its adjoint $Q^\dagger$ increases it by one,

$$[N, Q] = -Q \quad \text{and} \quad [N, Q^\dagger] = Q^\dagger. \tag{2.35}$$

The real supercharges are now given by

$$Q(\mathbb{1}) = Q + Q^\dagger = \mathrm{i}\nabla\!\!\!\!/ \quad \text{and} \quad Q(I) = \mathrm{i}(Q^\dagger - Q) = \mathrm{i}[N, \mathrm{i}\nabla\!\!\!\!/]. \tag{2.36}$$

Let us introduce the *Clifford vacuum* $|0\rangle$, which is annihilated by all operators ψ^A and hence has $N = 0$. The corresponding Clifford space $\mathcal{C}$ is the Fock space built over this vacuum state. Since only d creation operators are linearly independent, we obtain the following grading of the Clifford space,

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_d, \quad \dim \mathcal{C}_p = \binom{d}{p}, \tag{2.37}$$

with subspaces labeled by their particle number

$$N\big|_{\mathcal{C}_p} = p \cdot \mathbb{1}. \tag{2.38}$$

Along with the Clifford space the Hilbert space of all square integrable spinor fields decomposes as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d \qquad \text{with} \qquad N\big|_{\mathcal{H}_p} = p \cdot \mathbb{1}. \tag{2.39}$$

Since the Hamiltonian commutes with the number operator it leaves $\mathcal{H}_p$ invariant. The nilpotent charge Q maps $\mathcal{H}_p$ into $\mathcal{H}_{p-1}$, and its adjoint $Q^\dagger$ maps $\mathcal{H}_p$ into $\mathcal{H}_{p+1}$.

The raising and lowering operators $\psi^{A\dagger}$ and ψ^A are linear combinations of Γ^A and therefore anticommute with Γ in (2.16). Hence they map left- into right-handed spinors and vice versa. Since $\Gamma|0\rangle$ is annihilated by all ψ^A,

$$\psi^A(\Gamma|0\rangle) = -\Gamma\psi^A|0\rangle = 0, \tag{2.40}$$

and since the Clifford vacuum $|0\rangle$ is unique, we conclude that $|0\rangle$ has definite chirality. It follows that all states with even particle number have the same chirality as $|0\rangle$, and all states with odd particle number have opposite chirality,

$$\Gamma = \pm(-)^N. \tag{2.41}$$

The conclusion is that we get a finer grading as the $\mathbb{Z}_2$-grading of left- and right-handed spinors, if we have $\mathcal{N}=2$ superymmetry. Observe that this is similar to the grading of differential forms, where one has the fine grading of p-forms and the coarse $\mathbb{Z}_2$-grading of even and odd forms. The Witten index (2.5) corresponds to the Euler characteristic.

Finally, we observe that the Hermitian matrix

$$\Sigma = N - \frac{D}{4}\mathbb{1} = \frac{\mathrm{i}}{2}I_{AB}\Gamma^{AB} \in \mathfrak{spin}(D) \tag{2.42}$$

generates a U(1) subgroup of Spin(D). This is the R-symmetry of the superalgebra,

$$\begin{pmatrix} Q(\mathbb{1}) \\ Q(I) \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} Q(\mathbb{1}) \\ Q(I) \end{pmatrix}. \tag{2.43}$$

2.2.4 $\mathcal{N}=2$ and Superpotential

In the preceding subsection we have discussed a special feature of $\mathcal{N}=2$ supersymmetry, the number operator. In the following we will show a second important property, the existence of a superpotential. The superpotential can be useful for finding zero modes of a given Dirac operator. We will illustrate this in the subsequent section.

A Kähler manifold is in particular a complex manifold of even dimension $D = 2d$. A nice introduction to complex and Kähler manifolds can be found in [45]. Therefore we may introduce complex coordinates $(z^\mu, \bar{z}^{\bar{\mu}})$ with $\mu, \bar{\mu} = 1, \ldots, d$, which diagonalize the complex structure I,

$$I^{\mu}{}_{\nu} = \frac{\partial z^{\mu}}{\partial x^{M}} \frac{\partial x^{N}}{\partial z^{\nu}} I^{M}{}_{N} = -\mathrm{i}\delta^{\mu}_{\nu}, \qquad I^{\bar{\mu}}{}_{\bar{\nu}} = \frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^{M}} \frac{\partial x^{N}}{\partial \bar{z}^{\bar{\nu}}} I^{M}{}_{N} = \mathrm{i}\delta^{\bar{\mu}}_{\bar{\nu}},$$
$$I^{\mu}{}_{\bar{\nu}} = \frac{\partial z^{\mu}}{\partial x^{M}} \frac{\partial x^{N}}{\partial \bar{z}^{\bar{\nu}}} I^{M}{}_{N} = 0, \qquad I^{\bar{\mu}}{}_{\nu} = \frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^{M}} \frac{\partial x^{N}}{\partial z^{\nu}} I^{M}{}_{N} = 0. \tag{2.44}$$

The real and complex coordinate differentials are related as follows

$$\mathrm{d}z^{\mu} = \frac{\partial z^{\mu}}{\partial x^{M}} \mathrm{d}x^{M}, \qquad \mathrm{d}\bar{z}^{\bar{\mu}} = \frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^{M}} \mathrm{d}x^{M},$$
$$\partial_{\mu} = \frac{\partial x^{M}}{\partial z^{\mu}} \partial_{M}, \qquad \partial_{\bar{\mu}} = \frac{\partial x^{M}}{\partial \bar{z}^{\bar{\mu}}} \partial_{M}. \tag{2.45}$$

The line element in complex coordinates reads

$$\mathrm{d}s^2 = g_{MN} \mathrm{d}x^M \mathrm{d}x^N = 2 g_{\mu\bar{\nu}} \mathrm{d}z^{\mu} \mathrm{d}\bar{z}^{\bar{\nu}}, \qquad g_{\mu\bar{\nu}} = g(\partial_{\mu}, \partial_{\bar{\nu}}) = \frac{\partial x^{M}}{\partial z^{\mu}} \frac{\partial x^{N}}{\partial \bar{z}^{\bar{\nu}}} g_{MN}, \tag{2.46}$$

where $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}$ are the only non-vanishing components of the metric in the complex basis and can be obtained from a real Kähler potential K via

$$g_{\mu\bar{\nu}} = \partial_{\mu} \partial_{\bar{\nu}} K. \tag{2.47}$$

Covariant and exterior derivatives split into holomorphic and antiholomorphic pieces,

$$\nabla = \mathrm{d}z^{\mu} \nabla_{\mu} + \mathrm{d}\bar{z}^{\bar{\mu}} \nabla_{\bar{\mu}},$$
$$\mathrm{d} = \mathrm{d}z^{\mu} \partial_{\mu} + \mathrm{d}\bar{z}^{\bar{\mu}} \partial_{\bar{\mu}} = \partial + \bar{\partial}, \tag{2.48}$$

where we have introduced

$$\nabla_{\mu} = \frac{\partial x^{M}}{\partial z^{\mu}} \nabla_{M} \qquad \text{and} \qquad \nabla_{\bar{\mu}} = \frac{\partial x^{M}}{\partial \bar{z}^{\bar{\mu}}} \nabla_{M}. \tag{2.49}$$

The only non-vanishing components of the Christoffel symbols are

$$\Gamma^{\rho}_{\mu\nu} = g^{\rho\bar{\sigma}} \partial_{\mu} g_{\bar{\sigma}\nu} = g^{\rho\bar{\sigma}} \partial_{\bar{\sigma}\mu\nu} K, \tag{2.50}$$
$$\Gamma^{\bar{\rho}}_{\bar{\mu}\bar{\nu}} = g^{\bar{\rho}\sigma} \partial_{\bar{\mu}} g_{\sigma\bar{\nu}} = g^{\bar{\rho}\sigma} \partial_{\sigma\bar{\mu}\bar{\nu}} K. \tag{2.51}$$

Let us introduce complex vielbeine $e_{\alpha} = e^{\mu}_{\alpha} \partial_{\mu}$ and $e^{\alpha} = e^{\alpha}_{\mu} \mathrm{d}z^{\mu}$, such that $g_{\mu\bar{\nu}} = \frac{1}{2} \delta_{\alpha\bar{\beta}} e^{\alpha}_{\mu} e^{\bar{\beta}}_{\bar{\nu}}$. The components of the complex connection can be related to the metric $g_{\mu\bar{\nu}}$ and the vielbeine with the help of the Leibniz rule and (2.50) as follows,

$$\begin{aligned}
\omega_{\mu\alpha}^{\beta} e_{\beta} &\equiv \nabla_{\mu} e_{\alpha} \\
&= (\partial_{\mu} e_{\alpha}^{\nu})\partial_{\nu} + e_{\alpha}^{\nu}\Gamma_{\mu\nu}^{\rho}\partial_{\rho} \\
&= (\partial_{\mu} e_{\alpha}^{\nu})\partial_{\nu} + e_{\alpha}^{\nu} g^{\rho\bar{\sigma}}(\partial_{\mu} g_{\bar{\sigma}\nu})\partial_{\rho} \\
&= g^{\rho\bar{\sigma}}\partial_{\mu}(e_{\alpha}^{\nu} g_{\bar{\sigma}\nu})\partial_{\rho} \\
&= e_{\rho}^{\beta} g^{\rho\bar{\sigma}}\partial_{\mu}(e_{\alpha}^{\nu} g_{\bar{\sigma}\nu}) e_{\beta}.
\end{aligned} \tag{2.52}$$

Comparing the coefficients of e_{β} yields the connection coefficients $\omega_{\mu\alpha}^{\beta}$. The remaining coefficients are obtained the same way, and one finds altogether,

$$\begin{aligned}
\omega_{\mu\alpha}^{\beta} &= e^{\beta\bar{\sigma}}\partial_{\mu} e_{\bar{\sigma}\alpha}, \qquad & \omega_{\mu\bar{\alpha}}^{\bar{\beta}} &= e_{\bar{\sigma}}^{\bar{\beta}}\partial_{\mu} e_{\bar{\alpha}}^{\bar{\sigma}}, \\
\omega_{\bar{\mu}\bar{\alpha}}^{\bar{\beta}} &= e^{\bar{\beta}\sigma}\bar{\partial}_{\bar{\mu}} e_{\sigma\bar{\alpha}}, \qquad & \omega_{\bar{\mu}\alpha}^{\beta} &= e_{\sigma}^{\beta}\bar{\partial}_{\bar{\mu}} e_{\alpha}^{\sigma},
\end{aligned} \tag{2.53}$$

where for example $e^{\beta\bar{\sigma}} = g^{\bar{\sigma}\rho} e_{\rho}^{\beta}$.

Now we are ready to rewrite the Dirac operator in complex coordinates,

$$\mathrm{i}\nabla\!\!\!/ = Q + Q^{\dagger} = 2\mathrm{i}\psi^{\mu}\nabla_{\mu} + 2\mathrm{i}\psi^{\bar{\mu}\dagger}\nabla_{\bar{\mu}}. \tag{2.54}$$

We are led to the independent fermionic lowering and raising operators,

$$\psi^{\mu} = \frac{1}{2}\frac{\partial z^{\mu}}{\partial x^{M}}\Gamma^{M} = \frac{1}{2}\frac{\partial z^{\mu}}{\partial x^{M}}\psi^{M} \qquad \text{and} \qquad \psi^{\bar{\mu}\dagger} = \frac{1}{2}\frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^{M}}\Gamma^{M} = \frac{1}{2}\frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^{M}}\psi^{M\dagger}. \tag{2.55}$$

Of course, the supercharge Q in (2.54) is just the supercharge in (2.29) rewritten in complex coordinates. The fermionic operators introduced in (2.55) fulfill the anticommutation relations

$$\{\psi^{\mu}, \psi^{\nu}\} = \{\psi^{\bar{\mu}\dagger}, \psi^{\bar{\nu}\dagger}\} = 0, \qquad \{\psi^{\mu}, \psi^{\bar{\nu}\dagger}\} = \frac{1}{2} g^{\mu\bar{\nu}}, \tag{2.56}$$

where $g^{\mu\bar{\nu}}$ is the inverse of $g_{\mu\bar{\nu}}$ in (2.46). The operators ψ^{μ} lower the value of the Hermitian number operator N in (2.32) by one, while the $\psi^{\bar{\mu}\dagger}$ raise it by one. N reads in complex coordinates

$$N = 2 g_{\mu\bar{\nu}}\psi^{\bar{\nu}\dagger}\psi^{\mu}. \tag{2.57}$$

Now we can prove that, in cases where $\not\nabla$ admits an extended supersymmetry, there exists a superpotential for the spin and gauge connections. Indeed, if space is Kähler and the gauge field strength commutes with the complex structure, then the complex covariant derivatives commute

$$[\nabla_\mu, \nabla_\nu] = \mathcal{F}_{\mu\nu} = \frac{\partial x^M}{\partial z^\mu}\frac{\partial x^N}{\partial z^\nu}\mathcal{F}_{MN} = 0. \qquad (2.58)$$

But equation (2.58) is just the integrability condition (cf. Yang's equation [46]) for the existence of a superpotential h such that the complex covariant derivative can be written as

$$\nabla_\mu = h\partial_\mu h^{-1} = \partial_\mu + h(\partial_\mu h^{-1}) = \partial_\mu + \omega_\mu + A_\mu. \qquad (2.59)$$

This useful property is true for any (possibly charged) tensor field on a Kähler manifold provided that the field strength commutes with the complex structure. If the Kähler manifold admits a spin structure, as for example $\mathbb{C}P^d$ for odd d, then (2.59) also holds true for a (possibly charged) spinor field.

Of course, the superpotential h depends on the representation according to which the fields transform under the gauge and Lorentz group. One of the more severe technical problems in applications is to obtain h in the representation of interest. It consists of two factors, $h = h_A h_\omega$. The first factor h_A is the path-ordered integral of the gauge potential. According to (2.53) and (2.59), the matrix h_ω in the vector representation is just the vielbein $e^{\beta\bar{\sigma}}$. If one succeeds in rewriting this h_ω as the exponential of a matrix, then the transition to any other representation is straightforward: one contracts the matrix in the exponent with the generators in the given representation. This will be done for the complex projective spaces in the following subsection. Now let us assume that we have found the superpotential h. Then we can rewrite the complex supercharge in (2.54) as follows,

$$Q = 2\mathrm{i}\psi^\mu\nabla_\mu = hQ_0h^{-1}, \qquad Q_0 = 2\mathrm{i}\psi_0^\mu\partial_\mu, \qquad \psi_0^\mu = h^{-1}\psi^\mu h. \qquad (2.60)$$

The annihilation operators ψ^μ are covariantly constant,

$$\nabla_\mu\psi^\nu = \partial_\mu\psi^\nu + \Gamma^\nu_{\mu\rho}\psi^\rho + [\omega_\mu, \psi^\nu] = 0, \qquad (2.61)$$

and this translates into

$$
\begin{aligned}
\partial_\mu \psi_0^\nu &= h^{-1}\left(\partial_\mu \psi^\nu + [h\partial_\mu h^{-1}, \psi^\nu]\right) h \overset{(2.59)}{=} h^{-1}\left(\partial_\mu \psi^\nu + [\omega_\mu, \psi^\nu]\right) h \\
&= -\Gamma^\nu_{\mu\rho} h^{-1} \psi^\rho h \overset{(2.50)}{=} -g^{\nu\bar\sigma}\left(\partial_\mu g_{\bar\sigma\rho}\right)\psi_0^\rho.
\end{aligned}
\tag{2.62}
$$

This implies the following simple equation,

$$
\partial_\mu \left(g_{\bar\sigma\rho}\psi_0^\rho\right) = 0, \tag{2.63}
$$

stating that the transformed annihilation operators $\psi_{0\bar\sigma}$ are antiholomorphic. Indeed, one can show that they are even constant.
The relation (2.60) between the free supercharge Q_0 and the h-dependent supercharge Q is the main result of this subsection. It can be used to determine zero modes of the Dirac operator. With (2.36) we find

$$
\mathrm{i}\nabla\!\!\!\!/\,\chi = 0 \quad \Longleftrightarrow \quad Q\chi = 0,\ Q^\dagger\chi = 0. \tag{2.64}
$$

In sectors with particle number $N = 0$ or $N = d$ one can easily solve for all zero modes. For example, $Q^\dagger$ annihilates all states in the sector with $N = d$, such that zero modes only need to satisfy $Q\chi = 0$ in this sector. Because of (2.60), the general solution of this equation reads

$$
\chi = \bar f(\bar z) h \psi^{\dagger 1}\cdots\psi^{\dagger\bar d}|0\rangle, \tag{2.65}
$$

where $\bar f(\bar z)$ is some antiholomorphic function. Of course, the number of *normalizable* solutions depends on the gauge and gravitational background fields encoded in the superpotential h. With the help of the novel result (2.65) we shall find the explicit form of zero modes on $\mathbb{C}P^d$ in the following.

2.2.5 The Dirac Operator on Complex Projective Spaces

The ubiquitous two-dimensional $\mathbb{C}P^d$ models possess remarkable similarities with non-Abelian gauge theories in 3+1 dimensions [47]. They are frequently used as toy models displaying interesting physics like fermion-number violation analogous to the electroweak theory [48] or spin excitations in quantum Hall systems [49, 50]. Their instanton solutions have been studied in [51], and their supersymmetric extensions describe quantum integrable systems with known scattering matrices [52].

It would be desirable to construct manifestly supersymmetric extensions of these models on a spatial lattice. In the second part of this thesis, we will see how field theories on a spatial lattice can be related to Dirac operators. To this end we reconsider the Dirac operator on the symmetric Kähler manifolds $\mathbb{C}P^d$. We shall calculate the superpotential h in (2.59) and the explicit zero modes of the Dirac operator.

First we briefly recall those properties of the complex projective spaces $\mathbb{C}P^d$ which are relevant for our purposes. The space $\mathbb{C}P^d$ consists of complex lines in $\mathbb{C}^{d+1}$ through the origin. Its elements are identified with the following equivalence classes of points $u = (u^0, \dots, u^d) \in \mathbb{C}^{d+1} \backslash \{0\}$,

$$[u] = \{v = \lambda u | \lambda \in \mathbb{C}^*\}, \tag{2.66}$$

such that $\mathbb{C}P^d$ is identified with $(\mathbb{C}^{d+1}\backslash\{0\})/\mathbb{C}^*$. In each class there are elements with unit norm, $\bar{u} \cdot u = \sum \bar{u}^j u^j = 1$, and thus there is an equivalent characterization as a coset space of spheres, $\mathbb{C}P^d = S^{d+1}/S^1$. The u are homogeneous coordinates of $\mathbb{C}P^d$. We define the $d+1$ open sets

$$U_k = \left\{ u \in \mathbb{C}^{d+1} | u^k \neq 0 \right\} \subset \mathbb{C}^{d+1} \backslash \{0\}, \tag{2.67}$$

where the kth coordinate is fixed to one. This defines a complex analytic structure. The line element on $\mathbb{C}^{d+1}$,

$$\mathrm{d}s^2 = \sum_{j=0}^{d} \mathrm{d}u^j \mathrm{d}\bar{u}^j = \mathrm{d}u \cdot \mathrm{d}\bar{u}, \tag{2.68}$$

can be restricted to S^{d+1}/S^1 and has the following representation on the kth chart,

$$\mathrm{d}s^2 = \left(\frac{\partial u}{\partial z^\mu} \mathrm{d}z^\mu + \frac{\partial u}{\partial \bar{z}^{\bar{\mu}}} \mathrm{d}\bar{z}^{\bar{\mu}} \right) \cdot \left(\frac{\partial \bar{u}}{\partial z^\mu} \mathrm{d}z^\mu + \frac{\partial \bar{u}}{\partial \bar{z}^{\bar{\mu}}} \mathrm{d}\bar{z}^{\bar{\mu}} \right). \tag{2.69}$$

We shall use the (local) coordinates

$$u = \varphi_0(z) = \frac{1}{\rho}(1, z) \in U_0, \quad \text{where} \quad \rho^2 = 1 + \bar{z} \cdot z = 1 + r^2, \tag{2.70}$$

for representatives with non-vanishing u^0. With these coordinates the line element takes the form

$$\mathrm{d}s^2 = \frac{1}{\rho^2}\mathrm{d}z \cdot \mathrm{d}\bar{z} - \frac{1}{\rho^4}(\bar{z} \cdot \mathrm{d}z)(z \cdot \mathrm{d}\bar{z}), \tag{2.71}$$

and is derived from a Kähler potential $K = \ln \rho^2$. This concludes our summary of $\mathbb{C}P^d$.

To apply the results of the preceding subsection, we have to find complex orthonormal vielbeine $\mathrm{d}s^2 = e^\alpha \delta_{\alpha\bar{\beta}} e^{\bar{\beta}}$. The complex vielbeine may be found from the Maurer-Cartan form on $\mathrm{SU}(d+1)$ by considering $\mathbb{C}P^d$ as the symmetric space $\mathrm{SU}(d+1)/\mathrm{U}(d)$. A nice introduction to symmetric spaces can be found in [53]. We do not give the details of this calculation, but in this way, we obtained the following representation for the vielbeine of the Fubini-Study metric (2.71),

$$\begin{aligned} e^\alpha &= e^\alpha_\mu \mathrm{d}z^\mu = \rho^{-1}\left(\mathbb{P}^\alpha{}_\mu + \rho^{-1}\mathbb{Q}^\alpha{}_\mu\right)\mathrm{d}z^\mu \quad \text{and} \\ e_\alpha &= e^\mu_\alpha \partial_\mu = \rho\left(\mathbb{P}^\mu{}_\alpha + \rho\mathbb{Q}^\mu{}_\alpha\right)\partial_\mu. \end{aligned} \tag{2.72}$$

Here, we have introduced the matrices

$$\mathbb{P} = \mathbb{1} - \frac{\boldsymbol{z}\boldsymbol{z}^\dagger}{r^2} \quad \text{and} \quad \mathbb{Q} = \frac{\boldsymbol{z}\boldsymbol{z}^\dagger}{r^2}, \qquad \boldsymbol{z} = \left(z^1 \ldots z^d\right)^{\mathrm{T}}, \tag{2.73}$$

which satisfy

$$\mathbb{P}^2 = \mathbb{P}, \qquad \mathbb{Q}^2 = \mathbb{Q}, \qquad \mathbb{P}\mathbb{Q} = \mathbb{Q}\mathbb{P} = 0, \qquad \mathbb{P}^\dagger = \mathbb{P}, \qquad \mathbb{Q}^\dagger = \mathbb{Q}, \tag{2.74}$$

and hence are orthogonal projection operators. For the particular space $\mathbb{C}P^2$, the vielbeine are known, and can be found in [54]. They are related to those of (2.72) by a local Lorentz transformation. We are not aware of explicit formulae for the vielbeine for $d > 2$ in the literature. Expressing the vielbeine in terms of projection operators as in (2.72) allows us to relate the superpotentials in different representations. From (2.53) and (2.72) we obtain the connection (1,0)-form,

$$\omega^\alpha_{\mu\beta} = -\frac{\bar{z}_\mu}{\rho^2}\left(\frac{1}{2}\mathbb{P}^\alpha{}_\beta + \mathbb{Q}^\alpha{}_\beta\right) + \frac{1-\rho}{\rho r^2}\mathbb{P}^\alpha{}_\mu \bar{z}_\beta. \tag{2.75}$$

In the following we calculate zero modes of the Dirac operator on $\mathbb{C}P^d$. Actually, only for odd values of d a spin bundle S exists on $\mathbb{C}P^d$. We can tensor S with $L^{k/2}$, where L

is the generating line bundle, and k takes on even values. In the language of field theory this means that we couple fermions to a U(1) gauge potential A. For even values of d, there is no spin structure, so S does not exist globally. Similarly, for odd values of k, $L^{k/2}$ is not globally defined. There is, however, the possibility to define a generalized spin bundle S_c which is the formal tensor product of S and $L^{k/2}$, k odd [55]. Again, in the language of field theory, we couple fermions to a suitably chosen U(1) gauge potential with half-integer instanton number. In both cases, the gauge potential reads

$$A = \frac{k}{2}\bar{u} \cdot \mathrm{d}u = \frac{k}{4}(\partial - \bar{\partial})K = h_A \partial h_A^{-1} + h_A^{\dagger -1}\bar{\partial} h_A^{\dagger}, \quad h_A = \mathrm{e}^{-kK/4} = (1+r^2)^{-\frac{k}{4}}, \quad (2.76)$$

with corresponding field strength

$$F = \mathrm{d}A = (\partial + \bar{\partial})A = \frac{k}{2}\bar{\partial}\partial K. \quad (2.77)$$

h_A is the part of the superpontential h that gives rise to the gauge connection A. It remains to determine the spin connection part h_ω of $h \equiv h_\omega h_A$.

Uupon using (2.72), (2.73), the identity (2.53) can be written in matrix notation as $(\omega_\mu)^\alpha{}_\beta = (S\partial_\mu S^{-1})^\alpha{}_\beta$, where

$$S = \rho(\mathbb{P} + \rho\mathbb{Q}) \overset{(2.74)}{=} \exp\left(\mathbb{P}\ln\rho + \mathbb{Q}\ln\rho^2\right) = \exp\Big((\mathbb{1} + \mathbb{Q})\ln\rho\Big), \quad (2.78)$$

and we have succeeded in finding an exponential form of S. From the matrix form of S in (2.78) we read off the superpotential h_ω in the *spinor representation*,

$$h_\omega = \exp\left(\frac{1}{4}(\delta_{\bar{\alpha}\beta} + \mathbb{Q}_{\bar{\alpha}\beta})\Gamma^{\bar{\alpha}\beta}\ln\rho\right), \quad (2.79)$$

where we have introduced

$$\Gamma^{\bar{\alpha}\beta} \equiv \frac{1}{2}[\Gamma^{\bar{\alpha}}, \Gamma^{\beta}] = 2[\psi^{\dagger\bar{\alpha}}, \psi^{\beta}], \qquad \Gamma^{\bar{\alpha}} = 2\psi^{\dagger\bar{\alpha}}, \qquad \Gamma^{\beta} = 2\psi^{\beta}. \quad (2.80)$$

Next, we study zero modes of Q and $Q^\dagger$ in the gauge field background (2.76). In the sector of interest ($N = d$), the superpotential h_ω in the spinor representation simplifies to

$$h_\omega\big|_{N=d} = (1+r^2)^{\frac{d+1}{4}}, \quad \text{since} \quad \Gamma^{\bar{\alpha}\beta}\big|_{N=d} = 2\delta^{\bar{\alpha}\beta}. \quad (2.81)$$

All states in the $N = d$ sector are annihilated by $Q^\dagger$. Zero modes χ satisfy in addition

$$0 = Q\chi = 2\mathrm{i}\psi^\mu \nabla_\mu \chi = 2\mathrm{i}\psi^\mu h \partial_\mu h^{-1}\chi, \qquad h = h_A h_\omega. \tag{2.82}$$

Using (2.76) and (2.81) we conclude that

$$\chi = h\bar{f}(\bar{z})\psi^{\dagger\bar{1}} \cdots \psi^{\dagger\bar{d}}|0\rangle = (1+r^2)^{\frac{d+1-k}{4}} \bar{f}(\bar{z})\psi^{\dagger\bar{1}} \cdots \psi^{\dagger\bar{d}}|0\rangle, \tag{2.83}$$

with some antiholomorphic function $\bar{f}$. Normalizability of χ will put restrictions on the admissible functions $\bar{f}$. Since the operators $\bar{z}^{\bar{\mu}}\partial_{\bar{\mu}}$ (no sum) commute with ∂_μ and with each other, we can diagonalize them simultaneously on the kernel of ∂_μ. Thus, we are let to the following most general ansatz,

$$\bar{f}_m = (\bar{z}^1)^{m_1} \cdots (\bar{z}^d)^{m_d}, \qquad \sum_{i=1}^{d} m_i = m. \tag{2.84}$$

There are $\binom{m+d-1}{d-1}$ independent functions of this form. The solution χ in (2.83) is square-integrable if and only if

$$\begin{aligned} \|\chi\|^2 &= \int \mathrm{dvol}\ (\det h)\ \chi^\dagger\chi \\ &\overset{(2.83)}{\propto} \int \mathrm{d}\Omega \int \mathrm{d}r\ r^{2m+2d-1}(1+r^2)^{-\frac{d+k+1}{2}} < \infty, \end{aligned} \tag{2.85}$$

so normalizable zero modes in the $N = d$ sector exist for

$$m = 0, 1, 2, \ldots, q \equiv \frac{1}{2}(k-d-1). \tag{2.86}$$

Note that q is always integer-valued, since k is odd (even) if d is even (odd). Note further that there are no zero modes in this sector for $k < d+1$ or equivalently $q < 0$. In particular, for odd d and vanishing gauge potential there are no zero modes, in agreement with [56].

For $q \geq 0$, the total number of zero modes in the $N = d$ sector is

$$\sum_{m=0}^{q} \binom{m+d-1}{d-1} = \frac{1}{d!}(q+1)(q+2)\ldots(q+d). \tag{2.87}$$

Similar considerations show that there are no normalizable zero modes in the $N = 0$

sector for $q' < 0$, where $q' = \frac{1}{2}(-k-d-1)$. For $q' \geq 0$ there are zero modes in the $N = 0$ sector, and their number is given by (2.87) with q replaced by q'.

Observe, that the states in the $N = 0$ sector are of the same (opposite) chirality as the states in the $N = d$ sector for even (odd) d. The contribution of the zero modes in those sectors to the index of $\mathrm{i}\nabla\!\!\!\!/$ is given by

$$\frac{1}{d!}(q+1)(q+2)\dots(q+d), \qquad q = \frac{1}{2}(k-d-1), \tag{2.88}$$

for all $q \in \mathbb{Z}$.

On the other hand, the index theorem on $\mathbb{C}P^d$ reads [57]

$$\text{ind } \mathrm{i}\nabla\!\!\!\!/ = \int_{\mathbb{C}P^d} \mathrm{ch}(L^{-k/2})\hat{A}(\mathbb{C}P^d) = \frac{1}{d!}(q+1)(q+2)\dots(q+d), \tag{2.89}$$

where ch and $\hat{A}$ are the Chern character and the $\hat{A}$-genus, respectively. Note, that this index coincides with (2.88). This leads us to conjecture that for positive (negative) k *all normalizable zero modes* of the Dirac operator on the complex projective spaces $\mathbb{C}P^d$ with Abelian gauge potential (2.76) reside in the sector with $N = d$ ($N = 0$) and have the form (2.83).

We can prove this conjecture in the particular cases $d = 1$ and $d = 2$. For $\mathbb{C}P^1$ we have constructed all zero modes. The same holds true for $\mathbb{C}P^2$ for the following reason: Let us assume that there are zero modes in the $N = 1$ sector. According to (2.41) they have opposite chirality as compared to the states in the $N = 0$ and $N = 2$ sectors. Hence, the index would be less than the number of zero modes in the extreme sectors. On the other hand, according to the index theorem, the index (2.89) is equal to this number. We conclude that there can be no zero modes in the $N = 1$ sector.

2.3 Dimensional Reduction to Matrix-Schrödinger Hamiltonians

In this section we show how Dirac operators are related to multi-dimensional supersymmetric matrix-Schrödinger operators [28, 29, 30]. We proceed in two steps. First, let us consider a special example of the results obtained in Section 2.2 and then dimensionally reduce it to obtain a matrix-Schrödinger Hamiltonian.

We consider the Dirac operator on flat space in even dimension $D = 2d$ with Abelian gauge field,

$$\mathrm{i}\nabla\!\!\!/ = \mathrm{i}\Gamma^A \nabla_A, \qquad \nabla_A = \frac{\partial}{\partial x^A} - \mathrm{i}A_A, \tag{2.90}$$

where $\{x^A\}_{A=1,\dots,D}$ are coordinates. The $2^d \times 2^d$-dimensional Γ-matrices obey the Clifford algebra in D dimensions

$$\{\Gamma^A, \Gamma^B\} = 2\delta^{AB}. \tag{2.91}$$

Upper-case indices like A run from 1 to D. We will also use lower case and Greek indices like a and α, which run from 1 to d only. Observe that we introduced a Hermitian gauge field here in contrast to (2.14). The corresponding Hamiltonian reads

$$\begin{aligned} H = -\nabla\!\!\!/^2 &= -\nabla_A \nabla_A + \frac{\mathrm{i}}{4}[\Gamma^A, \Gamma^B] F_{AB} \\ &= -\Delta + (A_A)^2 + \mathrm{i}(\partial_A A_A) + 2\mathrm{i}A_A \partial_A + \frac{\mathrm{i}}{4}[\Gamma^A, \Gamma^B] F_{AB}, \end{aligned} \tag{2.92}$$

with $F_{AB} = \partial_A A_B - \partial_B A_A$.

In this section we restrict to the case of $\mathcal{N} = 2$ supersymmetry, that is we require the field strength to commute with a complex structure chosen to be of the standard form $(I^A{}_B) = \begin{pmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{pmatrix}$. We introduce complex coordinates $z^\alpha = x^\alpha + \mathrm{i}x^{d+\alpha}$, $\alpha = 1, \dots, d$, such that $I^\alpha{}_\beta = -\mathrm{i}\delta^\alpha_\beta$ holds in line with the conventions of (2.44). Furthermore, for the fomulae given in (2.45) and (2.55), we obtain the explicit expressions

$$\begin{aligned} \partial_\mu = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - \mathrm{i}\frac{\partial}{\partial x^{d+\mu}}\right), &\qquad \partial_{\bar\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x^{\bar\mu}} + \mathrm{i}\frac{\partial}{\partial x^{d+\bar\mu}}\right) \\ \psi^\mu = \frac{1}{2}(\gamma^\mu + \mathrm{i}\gamma^{d+\mu}), &\qquad \psi^{\mu\dagger} = \frac{1}{2}(\gamma^\mu - \mathrm{i}\gamma^{d+\mu}). \end{aligned} \tag{2.93}$$

As we have seen in 2.2.4, $\mathcal{N} = 2$ allows for a superpotential h, which for flat space only contains the gauge field part h_A, such that the complex supercharge is given by $Q = hQ_0h^{-1}$ with $Q_0 = 2\mathrm{i}\psi^\mu \nabla_\mu$, see (2.60). The complex covariant derivative $\nabla_\mu = \partial_\mu - \mathrm{i}\alpha_\mu$ contains the complex gauge field $\alpha_\mu = \frac{1}{2}(A_\mu - \mathrm{i}A_{d+\mu}) = \mathrm{i}h(\partial_\mu h^{-1})$. For any h there exists a polar decomposition $h = UR$ with $U^\dagger U = 1$ and a positive, real function $R = \exp(\chi)$. We may eliminate the U part by a gauge transformation, such that Q becomes

$$Q = RQ_0R^{-1} \quad \text{with} \quad R = \exp(-\chi) \quad \text{real.} \tag{2.94}$$

The Hamiltonian (2.92) contains a term which is first-order in the derivatives. As we want to obtain a matrix-Schrödinger Hamiltonian, which is only second-order in derivatives, the term $A_A\partial_A$ must vanish. If all ∂_{d+a}, $a = 1, \dots, d$, commute with the Dirac operator, we can consider the subspace of spinors which only depend on the x^a coordinates. On this subspace, half of the disturbing terms vanish. This subspace of spinors can be of reasonable interes if one considers for example the product space of $\mathbb{R}^d$ with the d-Torus T^d with small radius. In this case, one can Fourier expand the spinors in the x^{d+a}-directions, and x^{d+a}-independent spinors decouple from the others. If we demand in addition that $F_{ab} = 0$, the A_a can be gauged to zero and the term $A_A\partial_A$ vanishes.

Let us find out what these assumptions imply for the complex supercharge Q. As the gauge field is independent of the last d coordinates, the superpotential χ should only depend on the x^a. On the space of x^{d+a}-independent spinors, the complex partial derivatives reduce to real derivatives again, and we obtain

$$\begin{aligned} Q &= \mathrm{e}^{-\chi}Q_0\mathrm{e}^{\chi} = \mathrm{i}\psi_a(\partial_a + \chi_{,a}) \quad \text{with} \quad Q_0 = \mathrm{i}\psi_a\partial_a \\ Q^\dagger &= \mathrm{e}^{\chi}Q_0^\dagger\mathrm{e}^{-\chi} = \mathrm{i}\psi_a^\dagger(\partial_a - \chi_{,a}) \quad \text{with} \quad Q_0^\dagger = \mathrm{i}\psi_a^\dagger\partial_a, \end{aligned} \tag{2.95}$$

where $\partial_a = \frac{\partial}{\partial x^a}$, $\chi_{,a} = (\partial_a\chi)$ and $\psi_a = \psi^{\mu=a}$, $\psi_a^\dagger = \psi^{\mu=a\dagger}$. The supersymmetric Hamiltonian in (2.92) reduces to a $2^d \times 2^d$-dimensional matrix-Schrödinger Hamiltonian,

$$H = \{-\Delta + \chi_{,a}\chi_{,a} + (\Delta\chi)\}\mathbb{1}_{2^d} - 2\psi_a^\dagger\chi_{,ab}\psi_b, \quad \chi_{,ab} = \frac{\partial^2\chi}{\partial x^a\partial x^b}, \tag{2.96}$$

which – by construction – does not contain any first-order derivative term. Unlike the supercharge and its adjoint, see (2.35), the supersymmetric Hamiltonian H commutes with the number operator $N = \psi_a^\dagger\psi_a$ defined in (2.32) and hence leaves each subspace $\mathcal{H}_p$ in the decomposition (2.39) invariant,

$$H : \mathcal{H}_p \to \mathcal{H}_p. \tag{2.97}$$

On each subspace $\mathcal{H}_p$ the supersymmetric Hamiltonian is still a matrix Schrödinger operator,

$$H\big|_{\mathcal{H}_p} = -\Delta \mathbb{1} + V^{(p)}, \qquad \mathrm{tr}\,\mathbb{1} = \binom{d}{p}. \tag{2.98}$$

Only in the extreme sectors $\mathcal{H}_0$ and $\mathcal{H}_p$ do we get ordinary Schrödinger operators acting on one-component wave functions. The nilpotent supercharges give rise to the following Hodge-type decomposition of the Hilbert space,

$$\mathcal{H} = Q\mathcal{H} \oplus Q^\dagger \mathcal{H} \oplus \ker H, \tag{2.99}$$

where the subspace $\ker H$ is spanned by the zero modes of H. Indeed, on the orthogonal complement of $\ker H$ we may invert H and write

$$\begin{aligned}(\ker H)^\perp &= (QQ^\dagger + Q^\dagger Q)H^{-1}(\ker H)^\perp \\ &= Q\left(\frac{Q^\dagger}{H}(\ker H)^\perp\right) + Q^\dagger\left(\frac{Q}{H}(\ker H)^\perp\right),\end{aligned} \tag{2.100}$$

which proves (2.99). The supercharge Q maps every energy-eigenstate in $Q^\dagger\mathcal{H} \cap \mathcal{H}_p$ with positive energy into an eigenstate in $Q\mathcal{H} \cap \mathcal{H}_{p-1}$ with the same energy. Its adjoint maps eigenstates in $Q\mathcal{H} \cap \mathcal{H}_p$ into those in $Q^\dagger\mathcal{H} \cap \mathcal{H}_{p+1}$ with the same energy. With the exception of the zero-energy states there is an *exact pairing* between the eigenstates and energies in the bosonic and in the fermionic sector as depicted in Figure 2.1.

The supersymmetric system with supercharges (2.95) admits a duality relating $\mathcal{H}_p$ with $\mathcal{H}_{d-p}$. This can be seen as follows: exchanging $(\psi_a, \psi_a^\dagger, \chi)$ by $(\psi_a^\dagger, \psi_a, -\chi)$, Q and $Q^\dagger$ are interchanged while the Hamiltonian H stays the same. But this is the same as to interpret the ψ_a as creation operators, the $\psi_a^\dagger$ as anihilation operators and, in addition to flip the sign of χ. We obtain the important duality

$$H^\chi\Big|_{\mathcal{H}_p} = H^{-\chi}\Big|_{\mathcal{H}_{d-p}}. \tag{2.101}$$

This concludes our general discussion. In the next section we will discuss a particularly beautiful example of a matrix-Schrödinger Hamiltonian.

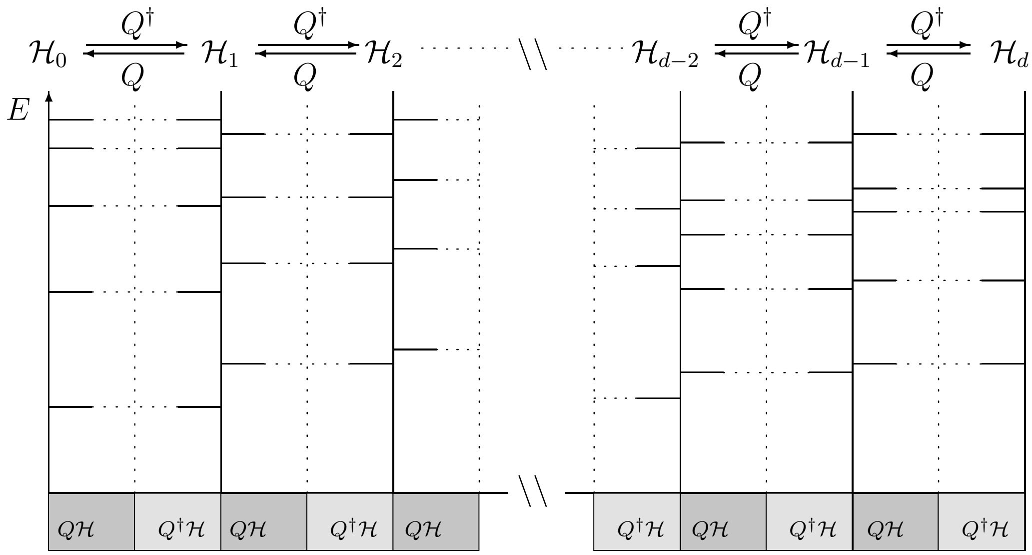

Figure 2.1: Pairing of eigenstates of the Hamiltonian.

2.4 The Supersymmetric Hydrogen Atom

For a closed system of two non-relativistic point particles interacting via a central force the angular momentum $\boldsymbol{L}$ of the relative motion is conserved and the motion is always in the plane perpendicular to $\boldsymbol{L}$. If the force is derived from the Newton or Coulomb potential, there is an additional conserved quantity: the Laplace-Runge-Lenz[2] vector [60]. For the hydrogen atom this vector has the form

$$\boldsymbol{C} = \frac{1}{m}\boldsymbol{p} \times \boldsymbol{L} - \frac{e^2}{r}\boldsymbol{r}, \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{2.102}$$

where m denotes the reduced mass of the proton-electron system. The Laplace-Runge-Lenz vector is perpendicular to $\boldsymbol{L}$ and hence is a vector in the plane of the orbit. It points in the direction of the semi-major axis.

Quantum mechanically, one defines the Hermitian Laplace-Runge-Lenz vector

$$\boldsymbol{C} = \frac{1}{2m}\big(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}\big) - \frac{e^2}{r}\boldsymbol{r}. \tag{2.103}$$

By exploiting the existence of this conserved vector operator, Pauli calculated the spec-

[2] A more suitable name for this constant of motion would be Hermann-Bernoulli-Laplace vector, see [58, 59].

trum of the hydrogen atom by purely algebraic means [37]. He noticed that the angular momentum $\boldsymbol{L}$ together with the vector

$$\boldsymbol{K} = \sqrt{\frac{-m}{2H}}\boldsymbol{C}, \tag{2.104}$$

which is well-defined and Hermitian on bound states with negative energies, generates a hidden $\mathfrak{so}(4)$ symmetry algebra,

$$[L_a, L_b] = \mathrm{i}\hbar\epsilon_{abc}L_c, \quad [L_a, K_b] = \mathrm{i}\hbar\epsilon_{abc}K_c, \quad [K_a, K_b] = \mathrm{i}\hbar\epsilon_{abc}L_c. \tag{2.105}$$

The Hamiltonian can be expressed in terms of $\boldsymbol{K}^2 + \boldsymbol{L}^2$, one of the two second-order Casimir operators of this algebra, acording to

$$H = -\frac{me^4}{2}\frac{1}{\boldsymbol{K}^2 + \boldsymbol{L}^2 + \hbar^2}. \tag{2.106}$$

One further observes that the other Casimir operator $\boldsymbol{K} \cdot \boldsymbol{L}$ vanishes and finally arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector $\boldsymbol{K}$ also explains the accidental degeneracy of the hydrogen spectrum.

In this section we sketch the generalization of these results in two directions: to the hydrogen atom in arbitrary dimensions[3] and to the corresponding supersymmetric extensions. A more detailed treatment can be found in [JDL1].

First we consider d-dimensional supersymmetric systems (2.96) with spherically symmetric superpotential $\chi(r)$. In the following we set $\hbar = 1$. The supercharges (2.95) simplify to

$$Q = \mathrm{i}\psi_a\big(\partial_a + x_a f\big) \quad \text{and} \quad Q^\dagger = \mathrm{i}\psi_a^\dagger\big(\partial_a - x_a f\big), \quad \text{where} \quad f = \frac{\chi'}{r}, \tag{2.107}$$

with the short-hand notation $\chi' = \frac{\mathrm{d}}{\mathrm{d}r}\chi$. We define total angular momenta J_{ab}, which

[3] When speaking of the d-dimensional hydrogen atom, we always mean the $1/r$-potential, although this potential permits the application of Gauss' law in three dimensions only.

contain an orbital part L_{ab} and a fermionic part S_{ab},

$$
\begin{aligned}
J_{ab} &= L_{ab} + S_{ab}, \\
L_{ab} &= x_a p_b - x_b p_a, \qquad p_a = -\mathrm{i}\partial_a, \\
S_{ab} &= -\mathrm{i}(\psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a).
\end{aligned} \tag{2.108}
$$

They obey the usual commutation relations of the $\mathfrak{so}(d)$ algebra

$$
[J_{ab}, J_{cd}] = \mathrm{i}(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}). \tag{2.109}
$$

x_a, p_a as well as ψ_a, $\psi_a^\dagger$ are vectors with respect to J_{ab}, as x_a, p_a are vectors with respect to L_{ab} and ψ_a, $\psi_a^\dagger$ are vectors with respect to S_{ab}. It follows that J_{ab} commutes with Q, $Q^\dagger$ and therefore also with H.

We have shown in [JDL1] that for $\chi = -\lambda r$ the symmetry group is even larger. There exists a supersymmetric conserved Laplace-Runge-Lenz vector, given by

$$
C_a = J_{ab}p_b + p_b J_{ab} - \lambda \hat{x}_a A, \qquad \lambda > 0, \tag{2.110}
$$

with

$$
A = (d-1)\mathbb{1} - 2N + 2S^\dagger S, \quad S = \hat{x}^a \psi_a, \quad \hat{x}^a = \frac{x^a}{r}. \tag{2.111}
$$

The C_a not only commute with the Hamiltonian, but also with the supercharges Q and $Q^\dagger$. This choice of the superpotential leads to the following supersymmetric extension of the Coulomb Hamiltonian,

$$
H = -\Delta + \lambda^2 - \frac{\lambda A}{r}, \tag{2.112}
$$

with supercharges

$$
Q = Q_0 - \mathrm{i}\lambda S \qquad \text{and} \qquad Q^\dagger = Q_0^\dagger + \mathrm{i}\lambda S^\dagger, \tag{2.113}
$$

where the free supercharge has been defined in (2.95). Restricted to the zero-particle sector this is the Hamiltonian of the hydrogen atom and restricted to the d-particle sector it corresponds to the electron-antiproton system.

A straightforward calculation shows that total angular momentum J_{ab} supplemented by the vector operator

$$K_a = \frac{C_a}{4(\lambda^2 - H)}, \tag{2.114}$$

restricted to the subspace of bound states ($E < \lambda^2$), form a $\mathfrak{so}(d+1)$ algebra.

Similarly to (2.106) one can write

$$\begin{aligned} H|_{\ker Q} &= QQ^\dagger = \lambda^2 - \frac{(d-2N-1)^2\lambda^2}{(d-2N-1)^2 + 4\mathcal{C}_{(2)}}, \\ H|_{\ker Q^\dagger} &= Q^\dagger Q = \lambda^2 - \frac{(d-2N+1)^2\lambda^2}{(d-2N+1)^2 + 4\mathcal{C}_{(2)}}, \end{aligned} \tag{2.115}$$

where $\mathcal{C}_{(2)}$ is the second-order Casimir of the dynamical symmetry algebra $\mathfrak{so}(d+1)$,

$$\mathcal{C}_{(2)} = \frac{1}{2} J_{ab} J_{ab} + K_a K_a. \tag{2.116}$$

All zero modes of H are annihilated by both Q and $Q^\dagger$, and according to (2.115) the second-order Casimir must vanish on these modes. We conclude that every normalizable zero mode Ψ of H must transform trivially under the dynamical symmetry group.

To obtain the bound state energies we need to determine those irreducible representations of the dynamical symmetry group which are realized in $\mathcal{H}$ and the corresponding values of the second-order Casimir operator. The degeneracy of an energy level is then equal to the dimension of the corresponding representation.

We use the abbreviation $\mathcal{D}^\ell_\wp$ to denote multiplets of the orthogonal groups corresponding to Young tableaux of the form

$$\begin{array}{|c|c|c|c|} \hline 1 & \cdot & \cdot & \ell \\ \hline \cdot & \multicolumn{3}{c}{} \\ \cline{1-1} \cdot & \multicolumn{3}{c}{} \\ \cline{1-1} \wp & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad , \tag{2.117}$$

since in the following only those representations will appear.

Each Fock space $\mathcal{C}_p$ forms an irreducible representation of $\mathfrak{so}(d)$, the totally antisymmetric representation $\mathcal{D}^1_{\wp=p}$. The homogeneous polynomials of degree l form a totally symmetric representation of $\mathfrak{so}(d)$, $\mathcal{D}_1^{\ell=l}$. It follows that the tensor-product representations,

$$\mathcal{D}_p^1 \otimes \mathcal{D}_1^l = \mathcal{D}_{p-1}^l \oplus \mathcal{D}_p^{l-1} \oplus \mathcal{D}_p^{l+1} \oplus \mathcal{D}_{p+1}^l, \tag{2.118}$$

are realized in $\mathcal{H}_p$. Observe that for special p and l some of the representations may be absent. When using the results (2.118), one should take into account that the representations with $\wp$ and $d-\wp$ are equivalent, therefore we consider only $\wp \leq d/2$. Furthermore, for even dimensions the representations $\mathcal{D}_{d/2}^\ell$ are reducible and contain one selfdual and one antiselfdual multiplet.

As the symmetry algebra $\mathfrak{so}(d)$ is a subalgebra of $\mathfrak{so}(d+1)$ with given embedding, every representation of $\mathfrak{so}(d+1)$ in the p-particle sector has to branch into representations of $\mathfrak{so}(d)$ given in (2.118). The only possibilities[4] are given by

$$\mathcal{D}_\wp^\ell\big|_{\mathfrak{so}(d+1)} \to \left\{\mathcal{D}_\wp^\ell \oplus \mathcal{D}_\wp^{\ell-1} \oplus \cdots \oplus \mathcal{D}_\wp^1 \oplus \mathcal{D}_{\wp-1}^\ell \oplus \mathcal{D}_{\wp-1}^{\ell-1} \oplus \cdots \oplus \mathcal{D}_{\wp-1}^1\right\}\big|_{\mathfrak{so}(d)} \tag{2.119}$$

with $\wp = p, p+1$. Of course, for $\wp = 1$ the last representations of the rotation group are absent. There is one notable exception for even d: in the middle sector $\mathcal{H}_{d/2}$ two representations of the dynamical symmetry group with $\wp = d/2$ appear, as can be seen from (2.118) and the fact mentioned above that $\mathcal{D}_\wp^\ell \sim \mathcal{D}_{d-\wp}^\ell$.

As all representations in (2.118) are realized, both representations of the dynamical symmetry group in each particle sector are needed. In the last section and in the introductory section 2.1 we have seen that all states except for zero modes are paired via the action of Q and $Q^\dagger$. As Q and $Q^\dagger$ commute with the generators of the Lie algebra $\mathfrak{so}(d+1)$, they map a nontrivial representation in the particle sector p to the same representation in particle sector $p-1$ or $p+1$. From these facts, one can derive how the different representations are realized in the various sectors.

In odd dimensions, $d = 2n+1$, the following representations of $\mathfrak{so}(d+1)$ arise for $\ell \geq 1$

[4]They can be obtained from [61] or by using the program LiE.

$$
\begin{array}{c|c|c|c|c|c}
\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \xrightarrow{Q^\dagger} \quad \xleftarrow{Q} & \mathcal{H}_{n-1} & \mathcal{H}_n \\
\hline
\mathcal{D}_1^\ell \rightleftarrows & \mathcal{D}_1^\ell & & & & \\
 & \mathcal{D}_2^\ell \rightleftarrows & \mathcal{D}_2^\ell & & & \\
 & & \mathcal{D}_3^\ell \rightleftarrows & & & \\
 & & & \rightleftarrows & \mathcal{D}_{n-1}^\ell & \\
 & & & & \mathcal{D}_n^\ell \rightleftarrows & \mathcal{D}_n^\ell
\end{array}
$$

It is sufficient to consider only the sectors presented here, as the others are obtained by the duality (2.101). In the subspace $\mathcal{H}_0$ we have in addition the trivial representation (zero mode). In even dimensions, $d = 2n$, the following representations of the dynamical symmetry algebra $\mathfrak{so}(d+1)$ arise for $\ell \geq 1$:

$$
\begin{array}{c|c|c|c|c|c}
\mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 & \xrightarrow{Q^\dagger} \quad \xleftarrow{Q} & \mathcal{H}_{n-1} & \mathcal{H}_n \\
\hline
\mathcal{D}_1^\ell \rightleftarrows & \mathcal{D}_1^\ell & & & & \\
 & \mathcal{D}_2^\ell \rightleftarrows & \mathcal{D}_2^\ell & & & \\
 & & \mathcal{D}_3^\ell \rightleftarrows & & & \\
 & & & \rightleftarrows & \mathcal{D}_{n-1}^\ell & \\
 & & & & \mathcal{D}_n^\ell \rightleftarrows & \mathcal{D}_n^\ell (\oplus \mathcal{D}_n^\ell)
\end{array}
$$

The energy eigenvalue (2.115) and its degeneracy for a given represention $\mathcal{D}_\wp^\ell$ can be obtained from the value of the quadratic Casimir $\mathcal{C}_{(2)}$ and the dimension of the representation. The interested reader may consult the group theory literature or [JDL1].

So far we have not investigated normalizability of the states. For this purpose we consider the Hamiltonian (2.112). It is easy to see that the Hermitian operator $S^\dagger S$ is an orthogonal projector, and hence has eigenvalues 0 and 1. It follows at once that for $p > d/2$ the operator A is negative and hence $H > \lambda^2$ for $\lambda > 0$. We conclude that H has no bound states in the sectors $\mathcal{H}_{p>d/2}$. In particular, there can exist exactly one normalizable zero mode in $\mathcal{H}_0$, which corresponds to a trivial representation of the dynamical symmetry group. In $\mathcal{H}_n$, for $d = 2n$, the operator A has both positive and negative eigenvalues. Only one of the two representations (for each ℓ) of the dynamical symmetry group contains bound states, as the other representation would give via $Q^\dagger$ a representation in $\mathcal{H}_{n+1}$. That the remaining representations are realized on normalizable states was shown in [JDL1] by an explicit construction of the representations. We will not elaborate on this construction of bound states here. Related techniques, however, are illustrated in Appendix A, where we determine the spectrum of the Dirac operator on a ball with chiral-bag boundary conditions.

2.5 Supersymmetry Breaking

In Section 2.1 we have introduced a helpful criterion for supersymmetry breaking. In this section we will further elaborate on this point. In the first subsection we will see that in perturbation theory, under certain assumptions, zero modes remain zero modes. Then we will consider two illuminating examples which will become important in the second part of this thesis.

2.5.1 Perturbation Theory and Zero Modes

Let us recall a well known result for perturbation theory of zero modes in supersymmetric quantum mechanics, e.g. [62]. We consider the $\mathcal{N}=1$ case of Section 2.1 and denote the single Hermitian supercharge by Q_0. In the following we assume that $n_{\mathrm{F}}^0 = 0$ and choose a zero mode out of the bosonic sector denoted by Ψ_0. We perturb the operator Q_0 by an operator ϵQ_1 with real parameter ϵ, $Q(\epsilon) = Q_0 + \epsilon Q_1$, where $\{Q_1, \Gamma\} = 0$. We want to solve the eigenvalue equation

$$Q(\epsilon)\Psi(\epsilon) = \lambda(\epsilon)\Psi(\epsilon), \tag{2.120}$$

with $\lambda(0) = 0$ and $\Psi(0) = \Psi_0$. We are considering the following formal power series in ϵ,

$$\begin{aligned} \Psi(\epsilon) &= \Psi_0 + \textstyle\sum_{k=1}^{\infty} \epsilon^k \Psi_k, \\ \lambda(\epsilon) &= \textstyle\sum_{k=1}^{\infty} \epsilon^k \lambda_k. \end{aligned} \tag{2.121}$$

Proposition: Under the assumptions above one has $\lambda(\epsilon) = 0$ and $\Gamma\Psi(\epsilon) = \Psi(\epsilon)$ in the sense of formal power series.

Proof by induction: To order ϵ^0 the proposition holds. Let us assume that the proposition holds for order $\leq j-1$. To order ϵ^j we obtain the equation

$$Q_0\Psi_j + Q_1\Psi_{j-1} = \lambda_j\Psi_0. \tag{2.122}$$

Taking the scalar product in Hilbert space (denoted by $\langle\cdot,\cdot\rangle$) with Ψ_0, (2.122) becomes

$$\lambda_j = \langle\Psi_0, Q_1\Psi_{j-1}\rangle. \tag{2.123}$$

Rewriting

$$\lambda_j = \langle\Gamma^2\Psi_0, Q_1\Psi_{j-1}\rangle = -\langle\Gamma\Psi_0, Q_1\Gamma\Psi_{j-1}\rangle = -\langle\Psi_0, Q_1\Psi_{j-1}\rangle = -\lambda_j, \tag{2.124}$$

we obtain $\lambda_j = 0$. Furthemore, with

$$Q_0\Gamma\Psi_j = -\Gamma Q_0\Psi_j = \Gamma Q_1\Psi_{j-1} = -Q_1\Psi_{j-1} = Q_0\Psi_j, \tag{2.125}$$

we conclude

$$Q_0\mathcal{P}_-\Psi_j = 0, \tag{2.126}$$

where we used the projection operator $\mathcal{P}_-$ introduced in (2.2). As $\mathcal{P}_-\Psi_j$ is a zero mode of Q_0 we obtain by assumption $\mathcal{P}_-\Psi_j \in \mathcal{H}_\mathrm{B}$. But as $\mathcal{P}_-$ projects onto $\mathcal{H}_\mathrm{F}$, we find $\mathcal{P}_-\Psi_j \in \mathcal{H}_\mathrm{B} \cap \mathcal{H}_\mathrm{F} = \{0\}$ and therefore $\Psi_j \in \mathcal{H}_\mathrm{B}$ which proves the statement ■

Observe that the statement holds for formal power series. It may happen that $\lambda(\epsilon)$ is not analytic for $\epsilon = 0$ and the result above is misleading. Non-perturbative effects can be important which we will illustrate in what follows.

2.5.2 Analyticity of Perturbations

First we consider the Hamiltonian (2.96) for $d = 1$, that is

$$H = -\partial_x\partial_x + (\chi')^2 - [\psi^\dagger, \psi]\chi'', \tag{2.127}$$

where $\chi' = \partial_x\chi$ and $\chi'' = \partial_x\partial_x\chi$. The Fock space consists only of two states, $|0\rangle$, which is annihilated by ψ, and $|1\rangle = \psi^\dagger|0\rangle$. To solve for zero modes in this case is rather simple. In the zero-sector all states are annihilated by Q. We have to find the most general state in this sector which is also annihilated by $Q^\dagger$. The only solution is given by $\exp(\chi)|0\rangle$. Similarly, in the one-sector all states are automatically annihilated by $Q^\dagger$. The only state which is also annihilated by Q is $\exp(-\chi)|1\rangle$. The number of normalizable zero modes depends on the chosen superpotential χ. Assume that $\chi = \lambda x^p + q(x)$, $\lambda \neq 0$, where the degree of the polynomial q is less than p. If p is odd, no normalizable zero mode exists. If p is even, for $\lambda > 0$ ($\lambda < 0$), there is one normalizable solution in $\mathcal{H}_1$ ($\mathcal{H}_0$). Observe that the zero modes are not analytic for $\lambda = 0$, but with respect to the parameters in q they are. Let us consider for example the superpotential $\chi = \lambda x^3 + \mu x^2$. For $\lambda \neq 0$ we do not have any zero mode, but for $\lambda = 0$ we have one zero mode, say in the one-sector for $\mu > 0$. In perturbation theory, one would obtain a zero mode also for $\lambda \neq 0$ which, however, contradicts the exact result. In this case perturbation theory is not a convergent series, the radius of convergence is zero.

As a second example let us consider the Hamiltonian (2.96) with $d = 2$ and assume in addition that the superpotential is harmonic. Actually, in this case, one can show that we have $\mathcal{N} = 4$ supersymmetry. It is easy to see that in the extreme sectors, $\mathcal{H}_0$ and $\mathcal{H}_2$, there are no normalizable zero modes. The same considerations as above show that the only solutions for zero modes in these sectors are given by $\exp(\chi)|0\rangle \in \mathcal{H}_0$ and $\exp(-\chi)|12\rangle \in \mathcal{H}_2$, where we defined $|12\rangle = \psi_1^\dagger\psi_2^\dagger|0\rangle$. As χ is harmonic, it is neither bounded from above nor from below and both of them are not normalizable. There may be zero modes in the middle sector $\mathcal{H}_1$. For the investigation of zero modes in the middle sector, we specialize to the harmonic superpotential $\chi = \lambda\Re z^p/p$ with $z = x_1 + \mathrm{i}x_2$, $\lambda > 0$. The zero modes of this particular problem were already constructed in [63]. To obtain a harmonic superpotential χ, we take the real part $\Re$ of a holomorphic function. Using polar coordinates $z = r\mathrm{e}^{\mathrm{i}\varphi}$ we obtain $\chi = \lambda r^p\cos(p\varphi)/p$. A short calculation gives

$$\begin{aligned} \chi_{,1} &= \lambda r^{p-1}\cos((p-1)\varphi), \qquad & \chi_{,2} &= -\lambda r^{p-1}\sin((p-1)\varphi), \\ \chi_{,11} &= \lambda(p-1)r^{p-2}\cos((p-2)\varphi), \qquad & \chi_{,12} &= -\lambda(p-1)r^{p-2}\sin((p-2)\varphi). \end{aligned} \tag{2.128}$$

The supercharges (2.95) read

$$\begin{aligned} Q &= \mathrm{i}\psi_1(\cos\varphi\partial_r - \frac{\sin\varphi}{r}\partial_\varphi + \chi_{,1}) + \mathrm{i}\psi_2(\sin\varphi\partial_r + \frac{\cos\varphi}{r}\partial_\varphi + \chi_{,2}), \\ Q^\dagger &= \mathrm{i}\psi_1^\dagger(\cos\varphi\partial_r - \frac{\sin\varphi}{r}\partial_\varphi - \chi_{,1}) + \mathrm{i}\psi_2^\dagger(\sin\varphi\partial_r + \frac{\cos\varphi}{r}\partial_\varphi - \chi_{,2}). \end{aligned} \tag{2.129}$$

Comparing the commutators $[-\mathrm{i}\partial_\varphi, H]$ and $[\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1, H]$ one finds that

$$J \equiv -\mathrm{i}\partial_\varphi + \mathrm{i}\frac{p-2}{2}(\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1) \tag{2.130}$$

commutes with the Hamiltonian. The eigenstates of $\mathrm{i}(\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1))$ are given by $|\uparrow\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}|1\rangle + |2\rangle)$ and $|\downarrow\rangle = \frac{1}{\sqrt{2}}(-\mathrm{i}|1\rangle + |2\rangle)$ with eigenvalues $+1$ and -1 respectively, where we defined the states $|a\rangle = \psi_a^\dagger|0\rangle$. For the zero modes of the Hamiltonian we consider the most general eigenstate of J with eigenvalue j,

$$\Psi_j = R_+(r)\,\mathrm{e}^{\mathrm{i}(j+1-p/2)\varphi}|\uparrow\rangle + R_-(r)\,\mathrm{e}^{\mathrm{i}(j-1+p/2)\varphi}|\downarrow\rangle. \tag{2.131}$$

Observe that for the functions to be single valued, j has to be integer for p even and half-integer for p odd. For Ψ_j to be a zero mode we have to demand that $Q\Psi_j = Q^\dagger\psi_j = 0$. We obtain two coupled first-order differential equations,

$$R'_\pm - \frac{(p-2)/2 \mp j}{r}R_\pm - \lambda r^{p-1}R_\mp = 0, \tag{2.132}$$

where $R'_\pm(r) = \partial_r R_\pm(r)$. The solutions are given in terms of Bessel functions,

$$R_\pm(r) = r^{p-1}\left(C_1 I_{|1/2\pm j/n|}(\lambda r^p/p) + C_2 K_{|1/2\pm j/p|}(\lambda r^p/p)\right). \tag{2.133}$$

For $r \to \infty$ only the K-functions are normalizable. Furthermore, as $K_{-\nu} = K_\nu$ we can omit the absolute value. We obtain

$$R_\pm(r) = Cr^{p-1}K_{\frac{1}{2}\pm j/p}(\lambda r^p/p). \tag{2.134}$$

These functions are normalizable at the origin for all $|\frac{1}{2} \pm j/p| < 1$ which results in the condition $|j| < p/2$. There are $p-1$ solutions (remember that for p even, j is integer and for p odd, j is half-integer). For the case $p = 3$, for example, we obtain two normalizable zero modes given by

$$\begin{aligned}
\Psi_{\frac{1}{2}} &= \frac{\lambda r^2}{2^{1/2}\pi 3^{1/4}}\left(K_{\frac{2}{3}}(\frac{\lambda}{3}r^3)|\uparrow\rangle + K_{\frac{1}{3}}(\frac{\lambda}{3}r^3)\mathrm{e}^{\mathrm{i}\varphi}|\downarrow\rangle\right), \\
\Psi_{-\frac{1}{2}} &= \frac{\lambda r^2}{2^{1/2}\pi 3^{1/4}}\left(K_{\frac{1}{3}}(\frac{\lambda}{3}r^3)\mathrm{e}^{-\mathrm{i}\varphi}|\uparrow\rangle + K_{\frac{2}{3}}(\frac{\lambda}{3}r^3)|\downarrow\rangle\right).
\end{aligned} \tag{2.135}$$

A useful formula for calculating the norm of theses states or expectation values of operators is given by

$$\int_0^\infty \mathrm{d}t\; t^\lambda K_\alpha(t)K_\beta(t) \;=\; \frac{2^{\lambda-2}}{\Gamma(1+\lambda)}\Gamma(\tfrac{1+\lambda}{2}+\tfrac{\alpha+\beta}{2})\Gamma(\tfrac{1+\lambda}{2}+\tfrac{\alpha-\beta}{2})\Gamma(\tfrac{1+\lambda}{2}-\tfrac{\alpha-\beta}{2})\Gamma(\tfrac{1+\lambda}{2}-\tfrac{\alpha+\beta}{2}) \tag{2.136}$$

for $\lambda > |\alpha|+|\beta|-1$ (see fomula (6.576.4) in [64]), where Γ is the usual Gamma function. In [62] it was shown that one can add an arbitrary potential $\Re q(z)$ with deg $q < p$ without changing the index. The unperturbed problem has only zero modes in the middle sector, we call them bosonic zero modes. Observe that we can never loose any zero mode without changing the index (2.5). Hence, all zero modes remain zero modes and perturbation theory gives the correct result. If the interaction generates new zero modes, they have to come in pairs.

3 Supersymmetry on a Spatial Lattice

In the second part of this thesis we consider Wess-Zumino models on a spatial lattice. A general introduction to supersymmetric field theories can be found in various lecture notes [65] and in the books [66]. In Section 3.1 we recall various Wess-Zumino models in two-dimensional Minkowski space. We start with the most general $\mathcal{N} = 1$ Wess-Zumino model and discuss its superalgebra. Then, we determine the conditions under which this model possesses $\mathcal{N} = 2$ or higher supersymmetry. Finally, we investigate how these models are related to Wess-Zumino models in higher dimensions. In Section 3.2 we formulate lattice versions of the Wess-Zumino models we obtained so far. We give a short discussion of lattice derivatives, e.g. the left- or right-derivative and the SLAC derivative, and their implications for fermions on the lattice. For both, the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Wess-Zumino model on the lattice, we determine the ground state of the massive free model and zero modes in the strong-coupling limit. Finally we compare the strong-coupling result with results from perturbation theory. Some mathematical proofs needed in this discussion can be found in Appendix C.

3.1 Wess-Zumino Models

Let us start with the most general superalgebra in two-dimensional Minkowski space with metric $\eta_{\mu\nu} = \mathrm{diag}(+,-)$. The algebra consists of N right-handed Hermitian supercharges Q_+^I, N' left-handed Hermitian supercharges $Q_-^{I'}$ and Hermitian momentum operators $P_\pm$ and reads

$$\begin{aligned}
\{Q_+^J, Q_+^K\} &= 2\delta^{JK} P_+, \quad J,K = 1,\dots,N,\\
\{Q_-^{J'}, Q_-^{K'}\} &= 2\delta^{J'K'} P_-, \quad J',K' = 1,\dots,N',\\
\{Q_+^J, Q_-^{K'}\} &= 2\mathcal{Z}^{JK'}.
\end{aligned} \tag{3.1}$$

The mass dimensions of $(Q_+^J, Q_-^{J'}, P_\pm)$ are $(\frac{1}{2}, \frac{1}{2}, 1)$, respectively. Under Lorentz boosts Q_+^J aquire a prefactor $\mathrm{e}^{\theta/2}$, the $Q_-^{J'}$ a factor $\mathrm{e}^{-\theta/2}$, whereas the $P_\pm$ aquire a factor $\mathrm{e}^{\pm\theta}$. $\mathcal{Z}^{JK'}$ are central charges, they do not transform under Lorentz transformations and commute with all supercharges.

In this thesis we consider chiral invariant theories only, i.e. $\mathcal{N} \equiv N = N'$. Let us introduce γ-matrices

$$\gamma^0 = \sigma^2, \quad \gamma^1 = \mathrm{i}\sigma^1, \quad \gamma_* = \gamma^0\gamma^1 = \sigma^3. \tag{3.2}$$

They form both, a Majorana and a chiral representation for the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \text{with} \quad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu. \tag{3.3}$$

Now, the algebra (3.1) can be written in a compact way,

$$\{Q_\alpha^J, Q_\beta^K\} = 2\left(\delta^{JK}(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu + \mathrm{i}(\gamma^0)_{\alpha\beta}\mathcal{Z}_\mathrm{A}^{JK} + \mathrm{i}(\gamma_*\gamma^0)_{\alpha\beta}\mathcal{Z}_\mathrm{S}^{JK}\right), \quad J, K = 1, \ldots, \mathcal{N}, \tag{3.4}$$

where $P_\pm = P_0 \mp P_1$ and $\mathcal{Z}_\mathrm{S}^{JK}$ ($\mathcal{Z}_\mathrm{A}^{JK}$) is the (anti)symmetric part of $\mathcal{Z}^{JK}$. Here we introduced spinor indices $\alpha, \beta \in \{1, 2\}$, such that in the chosen basis (3.2) 1 corresponds to $+$ and 2 to $-$. Introducing the Dirac conjugate spinors $\bar{Q}^J = (Q^J)^\dagger\gamma^0$, the algebra (3.4) reads

$$\{Q_\alpha^J, \bar{Q}_\beta^K\} = 2\left(\delta^{JK}\not{P} + \mathrm{i}\delta_{\alpha\beta}\mathcal{Z}_\mathrm{A}^{JK} + \mathrm{i}(\gamma_*)_{\alpha\beta}\mathcal{Z}_\mathrm{S}^{JK}\right) \quad \text{with} \quad \not{P} = \gamma^\mu P_\mu. \tag{3.5}$$

3.1.1 The $\mathcal{N} = 1$ Wess-Zumino Model

We derive a representation of the algebra (3.4) for $\mathcal{N} = 1$ without central charges on fields with minimal spin by the procedure of "seven easy steps" [67]. For the realization of infinitesimal translations with real parameters a^μ generated by $-\mathrm{i}a^\mu P_\mu$ and for infinitesimal supersymmetry transformations with Grassmann-valued parameters ϵ generated by $-\mathrm{i}\bar{\epsilon}Q$ we write δ_a and δ_ϵ, respectively. The corresponding mass dimensions of the parameters (a^μ, ϵ) are $(-1, -\frac{1}{2})$.

Let us start with a real scalar field ϕ of mass dimension 0. Translations on this field, as for any other field, are given by

$$\delta_a \phi = -\mathrm{i} a^\mu \delta_{P_\mu} \phi = -a^\mu \partial_\mu \phi. \tag{3.6}$$

The supersymmetry transformation acting on ϕ is realized as

$$\delta_\epsilon \phi = -\mathrm{i} \bar{\epsilon}_\alpha \delta_{Q_\alpha} \phi \equiv \bar{\epsilon} \psi, \tag{3.7}$$

where we introduced the Majorana spinor field $\psi_\alpha = -\mathrm{i} \delta_{Q_\alpha} \phi$ of mass dimension $\frac{1}{2}$. We write for the supersymmetry transformation of this field

$$\delta_\epsilon \psi_\alpha = -\mathrm{i} \bar{\epsilon}_\beta \delta_{Q_\beta} \psi_\alpha \equiv -\mathrm{i} \bar{\epsilon}_\beta F_{\beta\alpha}. \tag{3.8}$$

The symmetric part of $F_{\alpha\beta}$ is fixed by imposing the algebra (3.4) on ϕ. The antisymmetric part is arbitrary and can be parametrized by a single real scalar field F of mass dimension 1. We obtain

$$\delta_\epsilon \psi = (-\mathrm{i} \partial\!\!\!/ \phi + F) \epsilon. \tag{3.9}$$

Next, imposing the algebra on ψ results in the supersymmetry transformation

$$\delta_\epsilon F = -\mathrm{i} \bar{\epsilon} \partial\!\!\!/ \psi, \tag{3.10}$$

in particular, we do not obtain further new fields. At last, the algebra is automatically realized on the real scalar field F.

Let us summarize the results. We can realize the algebra (3.4) with $\mathcal{N} = 1$ and without central charges on the fields (ϕ, ψ, F), where ϕ, F are real scalar fields and ψ is a Majorana spinor, by the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon \phi &= \bar{\epsilon} \psi, \\ \delta_\epsilon \psi &= (-\mathrm{i} \partial\!\!\!/ \phi + F) \epsilon \quad \text{and} \\ \delta_\epsilon F &= -\mathrm{i} \bar{\epsilon} \partial\!\!\!/ \psi. \end{aligned} \tag{3.11}$$

In the following we construct a model with field content (ϕ, ψ, F) which is invariant under these transformations by using superspace formalism. The two-dimensional Minkowski space with coordinates $(x^\mu) = (t, x)$ can be extended by two real Grassmann variables θ_α, $\alpha = 1, 2$ to the superspace $\mathbb{R}^{2|2}$. We arange the fields (ϕ, ψ, F) into one real superfield

$$\Phi(x, \theta) = \phi(x) + \bar{\theta}\psi(x) + \frac{1}{2}\bar{\theta}\theta F(x), \tag{3.12}$$

where $\bar{\theta} = \theta^\dagger\gamma^0$ again denotes the Dirac conjugate spinor. The coordinate transformations

$$\theta_\alpha \to \theta_\alpha + \epsilon_\alpha, \qquad x^\mu \to x^\mu - \mathrm{i}\bar{\theta}\gamma^\mu\epsilon \tag{3.13}$$

generate the supersymmetry transformations in (3.11). Therefore, we may introduce vector fields on superspace generating these supersymmetry transformations,

$$\delta_a\Phi = -\mathrm{i}a^\mu\mathcal{P}_\mu\Phi, \qquad \delta_\epsilon\Phi = -\mathrm{i}\bar{\epsilon}\mathcal{Q}\Phi = -\mathrm{i}\bar{\mathcal{Q}}\epsilon\Phi, \tag{3.14}$$

with

$$\mathcal{P}_\mu = -\mathrm{i}\partial_\mu, \quad \mathcal{Q}_\alpha = \mathrm{i}\frac{\partial}{\partial\bar{\theta}_\alpha} - (\gamma^\mu\theta)_\alpha\partial_\mu \quad \text{and} \quad \bar{\mathcal{Q}}_\alpha = -\mathrm{i}\frac{\partial}{\partial\theta_\alpha} + (\bar{\theta}\gamma^\mu)_\alpha\partial_\mu. \tag{3.15}$$

We can check explicitly, that the map $Q_\alpha \to \mathcal{Q}_\alpha$, $P_\mu \to \mathcal{P}_\mu$ from the algebra into the vector fields is an antihomomorphism, i.e.

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = -2(\gamma^\mu)_{\alpha\beta}\mathcal{P}_\mu. \tag{3.16}$$

Observe that the F-term in the superfield (3.12) transforms into a total derivative under supersymmetry transformations. Furthermore, products of superfields are again superfields and the covariant derivatives

$$\mathcal{D}_\alpha = \mathrm{i}\frac{\partial}{\partial\bar{\theta}_\alpha} + (\gamma^\mu\theta)_\alpha\partial_\mu, \qquad \bar{\mathcal{D}}_\alpha = -\mathrm{i}\frac{\partial}{\partial\theta_\alpha} - (\bar{\theta}\gamma^\mu)_\alpha\partial_\mu \tag{3.17}$$

with

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}\mathcal{P}_\mu \tag{3.18}$$

anticommute with $\mathcal{Q}$ and $\bar{\mathcal{Q}}$. It follows that the action

$$S = \mathrm{i} \int \mathrm{d}^2\theta \, \mathrm{d}^2 x \, \left(-\frac{1}{4} \bar{\mathcal{D}}_\alpha \Phi \mathcal{D}_\alpha \Phi + W(\Phi) \right) \tag{3.19}$$

is invariant under the transformations (3.13). In (3.19) we introduced the superpotential W, which is an arbitrary function of the superfield Φ. Integrating over the Grassmann variables θ_α, we obtain the component Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\mathrm{i}}{2} \bar{\psi} \partial\!\!\!/ \psi + \frac{1}{2} F^2 + W'(\phi) F - \frac{1}{2} W''(\phi) \bar{\psi} \psi. \tag{3.20}$$

Here W' and W'' denote the first and second derivative of the superpotential with respect to the scalar field ϕ. One may further generalize this Lagrangian to the case of several multiplets (ϕ^a, ψ^a, F^a), $a = 1, \ldots, d$, now the Lagrangian reads

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi_a + \frac{\mathrm{i}}{2} \bar{\psi}^a \partial\!\!\!/ \psi_a + \frac{1}{2} F^a F_a + F^a W_{,a} - \frac{1}{2} W_{,ab} \bar{\psi}^a \psi^b, \tag{3.21}$$

and the superpotential W depends on the fields $\phi^1, \ldots, \phi^d$. We denote the derivative with respect to ϕ^a by $W_{,a}$. For Wess-Zumino models the target space is $\mathbb{R}^d$ with Euclidean metric δ_{ab}. Hence we would not have to distinguish between upper and lower indices a, b, as they are lowered (raised) with δ_{ab} (δ^{ab}). Nevertheless, we will keep track of the position of indices, as the generalization to nonlinear sigma models is more obvious and we have to be careful with the position of indices after introducing complex coordinates anyhow.

Let us check invariance of the action under the transformations

$$\begin{aligned} \delta_\epsilon \phi^a &= \bar{\epsilon} \psi^a, \\ \delta_\epsilon \psi^a &= (-\mathrm{i} \partial\!\!\!/ \phi^a + F^a) \epsilon, \\ \delta_\epsilon F^a &= -\mathrm{i} \bar{\epsilon} \partial\!\!\!/ \psi^a, \end{aligned} \tag{3.22}$$

explicitly and determine the corresponding Noether charges. A straightforward calculation gives

$$\delta\mathcal{L} = \partial_\mu \left(-\frac{\mathrm{i}}{2} F^a \bar{\epsilon}\gamma^\mu \psi_a - \frac{1}{2}\partial_\nu \phi^a \bar{\epsilon}\gamma^\nu\gamma^\mu\psi_a + \partial^\mu\phi^a\bar{\epsilon}\psi_a - \mathrm{i}W_{,a}\bar{\epsilon}\gamma^\mu\psi^a \right)$$
$$- \frac{1}{2}W_{,abc}(\bar{\epsilon}\psi^a)(\bar{\psi}^b\psi^c). \tag{3.23}$$

The last term in (3.23) vanishes because of the Fierz identity

$$(\bar{\epsilon}\psi^a)(\bar{\psi}^b\psi^c) + (\bar{\epsilon}\psi^b)(\bar{\psi}^c\psi^a) + (\bar{\epsilon}\psi^c)(\bar{\psi}^a\psi^b) = 0, \tag{3.24}$$

and the action is indeed invariant under the transformations (3.22). The corresponding conserved Noether current reads

$$J^\mu = \partial_\nu\phi^a\bar{\epsilon}\gamma^\nu\gamma^\mu\psi_a + \mathrm{i}W_{,a}\bar{\epsilon}\gamma^\mu\psi^a \tag{3.25}$$

and we obtain the Noether charge

$$Q = \int \mathrm{d}x \; \left(\pi_a - \gamma_*\partial_x\phi_a + \mathrm{i}W_{,a}\gamma^0\right)\psi^a, \tag{3.26}$$

where we introduced the conjugate momentum of ϕ^a, $\pi^a = \dot{\phi}^a$.

The canonical structure is more transparent in the on-shell formulation. This is obtained from the off-shell version by replacing the auxiliary fields F_a via their equations of motion by $-W_{,a}$. The on-shell Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi_a + \frac{\mathrm{i}}{2}\bar{\psi}^a\partial\!\!\!/\psi_a - \frac{1}{2}\delta^{ab}W_{,a}W_{,b} - \frac{1}{2}W_{,ab}\bar{\psi}^a\psi^b, \tag{3.27}$$

whereas the on-shell supersymmetry transformations are

$$\begin{aligned} \delta_\epsilon\phi^a &= \bar{\epsilon}\psi^a, \\ \delta_\epsilon\psi_a &= (-\mathrm{i}\partial\!\!\!/\phi_a - W_{,a})\epsilon. \end{aligned} \tag{3.28}$$

The nontrivial equal time (anti)commutators read

$$\{\psi^a_\alpha(x), \psi^b_\beta(y)\} = \delta_{\alpha\beta}\delta^{ab}\delta(x-y) \quad \text{and} \quad [\phi^a(x), \pi^b(y)] = \mathrm{i}\delta^{ab}\delta(x-y). \tag{3.29}$$

The Hamiltonian

$$H = \int \mathrm{d}x \, \left(\frac{1}{2}\pi^a \pi_a + \frac{1}{2}\partial_x \phi^a \partial_x \phi_a + \frac{1}{2}\delta^{ab} W_{,a} W_{,b} + \frac{1}{2}\psi^{a\dagger} (h_\mathrm{F})_{ab} \psi^b \right) \tag{3.30}$$

with

$$(h_\mathrm{F})_{ab} = (h_\mathrm{F}^0)_{ab} + \gamma^0 W_{,ab}, \quad (h_\mathrm{F}^0)_{ab} = -\mathrm{i}\gamma_* \delta_{ab} \partial_x \tag{3.31}$$

is the Legendre transform of the Lagrangian. The action is also invariant under spacetime translations generated by the Noether charges

$$P_0 = H \quad \text{and} \quad P_1 = \int \mathrm{d}x \, \left(\partial_x \phi_a \pi^a + \frac{\mathrm{i}}{2} \bar{\psi}^a \gamma^0 \partial_x \psi_a \right). \tag{3.32}$$

Using (3.29), one proves that Q and P_μ satisfy the superalgebra (3.5) for $\mathcal{N} = 1$ with central charges

$$\mathcal{Z}_\mathrm{A} = 0 \quad \text{and} \quad \mathcal{Z}_\mathrm{S} = \int \mathrm{d}x \, \partial_x W. \tag{3.33}$$

The central charge $\mathcal{Z}_\mathrm{S}$ is a surface term and therefore a topological quantity. The appearance of this central charge was first observed by Witten and Olive in [68].

For $d = 1$, the energy of a pure bosonic, static field configuration is given by

$$E = \frac{1}{2} \int \mathrm{d}x \, \left(\partial_x \phi \partial_x \phi + W' W' \right) = \frac{1}{2} \int \mathrm{d}x \, \left(\partial_x \phi \pm W' \right)^2 \mp \mathcal{Z}_\mathrm{S}. \tag{3.34}$$

We obtain the Bogomolny-Prasad-Sommerfield (BPS) bound

$$E \geq |\mathcal{Z}_\mathrm{S}|. \tag{3.35}$$

The bound is saturated if and only if the first order differential equation

$$\partial_x \phi = \mp W'(\phi) \tag{3.36}$$

holds. Solutions of this first order differential equation are solutions of the equations of motion.

In the last few years there has been an active discussion about one-loop corrections to

E and $\mathcal{Z}_{\mathrm{S}}$ and whether the bound is still saturated or not (see [69, 70] and references therein). At first sight, it is surprising that the central charge gets any quantum corrections at all, because classically the central charge is a surface term only. But in [69], the classical central charge is amended by an anomalous term proportional to the second derivative of the superpotential W. With this anomalous term the bound is saturated to one-loop, too.

3.1.2 Extended On-shell Wess-Zumino Models

Let us investigate under which conditions the on-shell $\mathcal{N}=1$ Wess-Zumino model (3.27) allows for further supersymmetries. Similar considerations for the more general nonlinear sigma models can be found in [71].

In most explicit calculations we take the Majorana representation

$$\gamma^0 = \sigma^2, \quad \gamma^1 = \mathrm{i}\sigma^3 \quad \text{and} \quad \gamma_* = \gamma^0\gamma^1 = -\sigma^1, \tag{3.37}$$

which can be obtained from the represention in (3.2) by a unitary transformation, such that the superalgebra (3.4) takes the simple form

$$\begin{aligned} \{Q_1^I, Q_1^J\} &= 2(\delta^{IJ}H + \mathcal{Z}_{\mathrm{S}}^{IJ}), \\ \{Q_2^I, Q_2^J\} &= 2(\delta^{IJ}H - \mathcal{Z}_{\mathrm{S}}^{IJ}), \\ \{Q_1^I, Q_2^J\} &= 2(\delta^{IJ}P_1 + \mathcal{Z}_{\mathrm{A}}^{IJ}). \end{aligned} \tag{3.38}$$

We start with the investigation of the free theory, that is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a\partial^\mu\phi_a + \mathrm{i}\bar{\psi}^a\partial\!\!\!/\psi_a). \tag{3.39}$$

We make the most general ansatz for supersymmetry transformations respecting Lorentz structure, dimensionality of the fields, parity and fermion-boson rule,

$$\begin{aligned} \delta^J\phi_a &= I^J_{ab}\bar{\epsilon}\psi^b, \\ \delta^J\psi_a &= A^J_{ab}\partial\!\!\!/\phi^b\epsilon + B^J_{abc}(\bar{\epsilon}\psi^b)\psi^c + C^J_{abc}(\bar{\epsilon}\gamma^\mu\psi^b)\gamma_\mu\psi^c + D^J_{abc}(\bar{\epsilon}\gamma_*\psi^b)\gamma_*\psi^c. \end{aligned} \tag{3.40}$$

J runs from 1 to $\mathcal{N}$, $\mathcal{N}$ being the number of independent supersymmetry transformations. Invariance of the action constraints this ansatz to

$$\delta^J \phi_a = I^J_{ab} \bar{\epsilon} \psi^b, \qquad \delta^J \psi_a = -\mathrm{i} I^J_{ba} \partial\!\!\!/ \phi^b \epsilon \tag{3.41}$$

with

$$\frac{\partial}{\partial \phi^a} I^J_{bc} = 0. \tag{3.42}$$

Further conditions for the I^J are obtained by demanding that the supersymmetry algebra

$$[\delta^J_{\epsilon_1}, \delta^K_{\epsilon_2}] = 2\mathrm{i}\delta^{JK} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu \tag{3.43}$$

holds. Imposing the algebra on the fields yields

$$I^J (I^K)^{\mathrm{T}} + I^K (I^J)^{\mathrm{T}} = 2\delta^{JK}, \tag{3.44}$$

in particular the matrices I^J have to be orthogonal. For $J = 1$ we set $I^1 = \mathbb{1}$, such that the first supersymmetry transformation coincides with the one in (3.28) with $W = 0$. The remaining conditions read

$$\begin{aligned} I^J + (I^J)^{\mathrm{T}} &= 0 \quad \text{and} \\ I^J I^K + I^K I^J &= -2\delta^{JK} \quad \text{for} \quad J, K = 2, \ldots, \mathcal{N}. \end{aligned} \tag{3.45}$$

We obtain a similar result as we got for the Dirac operator in subsection 2.2.2. The I^J, $J > 1$, have to be real, antisymmetric, constant matrices forming a Clifford algebra in $\mathcal{N} - 1$ dimensions, especially each I^J, $J > 1$, is a complex structure on the flat target space $\mathbb{R}^d$.

Next, we introduce an interaction term into the Lagrangian such that the model is given by (3.27). As the potential W has mass dimension 1, the most general ansatz for the supersymmetry transformations contains an additional term,

$$\begin{aligned} \delta^J \phi_a &= I^J_{ab} \bar{\epsilon} \psi^b \\ \delta^J \psi_a &= -\mathrm{i} I^J_{ba} \partial\!\!\!/ \phi^b \epsilon + G^J_a \epsilon, \end{aligned} \tag{3.46}$$

where G^J_a is linear in W or its derivatives such that G^J_a is of mass dimension 1, too.

Invariance of the action to first-order in W gives the condition

$$G^J_{a,c} = -W_{,ab}(I^J)_c{}^b, \tag{3.47}$$

whereas invariance to second-order in W gives

$$\delta^{ab}W_{,a}W_{,bc}(I^J)^c{}_d = -W_{,da}G^{J^a}. \tag{3.48}$$

For $J=1$, that is $I^1 = \mathbb{1}$, we have to choose

$$G^1_a = -W_{,a} \tag{3.49}$$

and reobtain the original supersymmetry transformations (3.28).

To find the most general solution of the equations (3.47) and (3.48) for $J>1$ is a more involved calculation. As we consider in the following one fixed index $J>1$, we omit this index in the calculation. G^a can be written as

$$G^a = I^{ab}(2K_b - W_{,b}) \tag{3.50}$$

with a set of arbitrary functions K_a. The condition (3.47) reads

$$2K_{c,d} = (\delta^a{}_c\delta^b{}_d + I^a{}_c I^b{}_d)W_{,ab}. \tag{3.51}$$

As the right hand side is symmetric in the indices c and d we get

$$K_{[c,d]} = 0, \tag{3.52}$$

which is the integrability condition for the existence of a potential K, such that

$$K_a = K_{,a}. \tag{3.53}$$

To get further insight into the equations, we introduce complex coordinates on the target space $\mathbb{R}^d$ as in subsection 2.2.4, such that the complex structure is diagonal, $I^\alpha{}_\beta = \mathrm{i}\delta^\alpha{}_\beta$,

$I^{\bar\alpha}{}_{\bar\beta} = -\mathrm{i}\delta^{\bar\alpha}{}_{\bar\beta}$ and $I^{\alpha}{}_{\bar\beta} = I^{\bar\alpha}{}_{\beta} = 0$. In this coordinates eqn. (3.51) looks like

$$K_{,\alpha\beta} = K_{,\bar\alpha\bar\beta} = 0, \tag{3.54}$$

$$(K - W)_{,\alpha\bar\beta} = 0. \tag{3.55}$$

The most general solution of the first equation is given by

$$K = a_\alpha \phi^\alpha + \bar a_{\bar\alpha}\bar\phi^{\bar\alpha} + a_{\alpha\bar\beta}\phi^\alpha\bar\phi^{\bar\beta} \quad \text{with} \quad \overline{a_{\alpha\bar\beta}} = \bar a_{\bar\alpha\beta} \stackrel{!}{=} a_{\beta\bar\alpha}, \tag{3.56}$$

where we fixed the coefficients, such that K is real. The solution of the second equation is

$$W(\phi^\alpha, \bar\phi^{\bar\alpha}) = K(\phi^\alpha, \bar\phi^{\bar\alpha}) + h(\phi^\alpha) + \bar h(\bar\phi^{\bar\alpha}), \tag{3.57}$$

where h is an arbitrary holomorphic function of the complex fields ϕ^α. It remains to solve eqn. (3.48), which is equivalent to

$$(W_{,a}G^a)_{,c} = 0 \tag{3.58}$$

by using eqn. (3.47). With (3.50) and (3.57) this condition reads

$$I^{ab}K_{,b}(h + \bar h)_{,a} = \text{const}, \tag{3.59}$$

or equivalently

$$\mathcal{L}_V(h + \bar h) = \text{const}, \tag{3.60}$$

where we have introduced the vector field $V = V^a\partial_a$ with $V^a = K_{,b}I^{ba}$. In complex coordinates we have $V_\alpha = \mathrm{i}K_{,\alpha}$ and $\bar V_{\bar\alpha} = -\mathrm{i}K_{,\bar\alpha}$. V is a holomorphic vector field since (3.54) implies $0 = K_{,\bar\alpha\bar\beta} = \mathrm{i}\partial_{\bar\alpha}\bar V_{\bar\beta} = \frac{\mathrm{i}}{2}\delta_{\bar\beta\gamma}\partial_{\bar\alpha}V^\gamma$. Eqn. (3.60) can be written in complex coordinates as follows:

$$V^\alpha\partial_\alpha h + \bar V^{\bar\alpha}\partial_{\bar\alpha}\bar h = \text{const}, \tag{3.61}$$

that is the real part of $V^\alpha\partial_\alpha h$ has to be constant. But as $V^\alpha\partial_\alpha h$ is holomorphic, it has to be constant itself, so

$$V^\alpha \partial_\alpha h = \text{const.} \tag{3.62}$$

Finally we want to mention that V is a Killing vectorfield. The Killing equations in complex coordinates read

$$\begin{aligned} \partial_\alpha V_\beta + \partial_\beta V_\alpha &= 0 \quad \text{and} \\ \partial_\alpha \bar{V}_{\bar{\beta}} + \partial_{\bar{\beta}} V_\alpha &= 0. \end{aligned} \tag{3.63}$$

They are satisfied because of eqn. (3.54) and the definition of V.

Let us summarize the result. With respect to each complex structure I, the superpotential W is expressed in eqn. (3.57) via the Killing potential K (3.56) and a holomorphic function h. Furthermore, the Lie derivative of h along the holomorphic Killing vector V corresponding to K has to be constant (3.62).

Two particular solutions should be mentioned at this point. First, if we choose $K = 0$ (the constant in (3.62) has to be zero for consistency), there is no further condition on h. We obtain

$$W = h + \bar{h}. \tag{3.64}$$

W is the solution of the differential equation $W_{,\alpha\bar{\beta}} = 0$ which is in real corrdinates equivalent to the fact that the Hessian of W anticommutes with the complex structure,

$$I_a{}^c W_{,cb} + W_{,ac} I^c{}_b = 0. \tag{3.65}$$

Second we consider the case where h is constant and we may choose this constant to be zero, $h = 0$. Then we get

$$W = K = a_\alpha \phi^\alpha + \bar{a}_{\bar{\beta}} \bar{\phi}^{\bar{\beta}} + a_{\alpha\bar{\beta}} \phi^\alpha \bar{\phi}^{\bar{\beta}}, \tag{3.66}$$

which corresponds to a massive free theory. This is the solution of the equations $W_{,\alpha\beta} = W_{,\bar{\alpha}\bar{\beta}} = 0$ which is in real coordinates equivalent to the fact that the Hessian of W

commutes with the complex structure,

$$I_a{}^c W_{,cb} - W_{,ac} I^c{}_b = 0. \tag{3.67}$$

Let us calculate the supersymmetry algebra, realized on the fields. It reads

$$\begin{aligned} [\delta^I_{\epsilon_1}, \delta^I_{\epsilon_2}] &= 2\mathrm{i}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu, \\ [\delta^I_{\epsilon_1}, \delta^J_{\epsilon_2}] &= 2\bar{\epsilon}_1 \epsilon_2 (\mathcal{L}_{I^J V^I} - \mathcal{L}_{I^I V^J}) \quad \text{for} \quad I \neq J, \end{aligned} \tag{3.68}$$

where the Lie derivatives act on the fields as

$$\begin{aligned} \mathcal{L}_V \phi^a &= V^a, \\ \mathcal{L}_V \psi^a &= (\partial_b V^a) \psi^b. \end{aligned} \tag{3.69}$$

Now we are ready to consider special values for $\mathcal{N}$.

Example: $\mathcal{N} = 2$

The smallest target space dimension which allows for the existence of one complex structure is $d = 2$. For the complex structure we choose

$$I_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{ab}. \tag{3.70}$$

In (3.56) the most general Killing potential K was already given for any dimension. The corresponding Killing vector reads

$$\begin{aligned} \bar{V}_{\bar{\phi}} &= -\mathrm{i}(\bar{a} + b\phi), \\ V^\phi &= 2\bar{V}_{\bar{\phi}} = -2\mathrm{i}(\bar{a} + b\phi) \quad \text{with} \quad a \in \mathbb{C}, b \in \mathbb{R}. \end{aligned} \tag{3.71}$$

We obtain the following differential equation for $h(\phi)$,

$$(\bar{a} + b\phi)\partial_\phi h = c, \tag{3.72}$$

for some constant c. The solution of this differential equation is

$$h = \begin{cases} \frac{c}{b}\log(\bar{a}+b\phi) & \text{for} \quad b \neq 0, \\ \frac{c}{\bar{a}}\phi & \text{for} \quad b=0, a\neq 0 \end{cases} \tag{3.73}$$

and h arbitrary for $a=b=c=0$. In the first case h is not holomorphic except for $c=0$. The results are summarized in Table 3.1. We only get an interacting theory

$b\neq 0,\ c=0$	$h=0$	$W=a\phi+\bar{a}\bar{\phi}+b\phi\bar{\phi}$	massive free theory
$b=0,\ a\neq 0$	$h=\frac{c}{\bar{a}}\phi$	$W=(a+\frac{c}{\bar{a}})\phi+(\bar{a}+\frac{\bar{c}}{a})\bar{\phi}$	free theory
$a=b=c=0$	h holomorphic	$W=h+\bar{h}$	interacting theory

Table 3.1: Possible superpotentials for $\mathcal{N}=2$ and $d=2$.

for $a=b=c=0$ such that the superpotential W is harmonic. Then, the second supersymmetry has the form

$$\begin{aligned} \delta^2_\epsilon\phi^a &= I^{ab}\bar{\epsilon}\psi_b, \\ \delta^2_\epsilon\psi^a &= I^{ab}(\mathrm{i}\partial\!\!\!/\phi_b - W_{,a}). \end{aligned} \tag{3.74}$$

The corresponding Noether charge reads

$$Q^2 = \int \mathrm{d}x\ (\pi_a - \partial_x\phi_a\gamma_* - \mathrm{i}W_{,a}\gamma^0)(I\psi)^a. \tag{3.75}$$

The central charges take the form

$$\mathcal{Z}^{JK}_{\mathrm{A}} = 0 \quad \text{and} \quad \mathcal{Z}^{JK}_{\mathrm{S}} = (\sigma^3)^{JK}\int \mathrm{d}x\ \frac{\mathrm{d}W}{\mathrm{d}x} - (\sigma^1)^{JK}\int \mathrm{d}x\ \frac{\mathrm{d}U}{\mathrm{d}x}, \tag{3.76}$$

where U is the imaginary part of the analytic funtion $h=\frac{1}{2}(W+\mathrm{i}U)$ with real part W. Observe that the central charge $\mathcal{Z}^{JK}_{\mathrm{S}}$ is again a surface term and therefore a topological quantity.

Let us conclude this example by writing the Lagrangian and supercharges in terms of the complex scalar field and by composing the two Majorana spinors to a Dirac spinor, i.e.

$$\phi = \frac{1}{\sqrt{2}}(\phi^1+\mathrm{i}\phi^2) \quad \text{and} \quad \psi = \frac{1}{\sqrt{2}}(\psi^1+\mathrm{i}\gamma_*\psi^2). \tag{3.77}$$

The on-shell Lagrangian takes the form

$$\mathcal{L} = \partial_\mu\phi(\partial^\mu\phi)^* + \mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}|h'|^2 - h''\bar{\psi}P_+\psi - \bar{h}''\bar{\psi}P_-\psi, \tag{3.78}$$

where h' is the derivative of h with respect to the complex field ϕ and we have introduced the chiral projectors

$$\mathbb{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_*). \tag{3.79}$$

Along with the real scalar fields one combines the corresponding conjugate momenta to the complex momentum $\pi = \frac{1}{\sqrt{2}}(\pi^1 - \mathrm{i}\pi^2)$ such that

$$[\phi(x), \pi(y)] = \mathrm{i}\delta(x-y) \quad \text{and} \quad \{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y). \tag{3.80}$$

The complex supercharges take the form

$$Q = \frac{1}{2}(Q^1 + \mathrm{i}\gamma_* Q^2) = \int \mathrm{d}x \ \big((\pi - \partial_x\bar{\phi} + \mathrm{i}h'\gamma^0)\mathbb{P}_+\psi + (\bar{\pi} + \partial_x\phi + \mathrm{i}\bar{h}'\gamma^0)\mathbb{P}_-\psi\big) \tag{3.81}$$

and satisfy the anticommutation relations

$$\{Q, Q\} = 0 \quad \text{and} \quad \{Q, \bar{Q}\} = P\!\!\!/ + \mathrm{i}\gamma_*\mathcal{Z}^{11}_{\mathrm{S}} - \mathcal{Z}^{12}_{\mathrm{S}}. \tag{3.82}$$

Let us point out, that γ_* in the definition of Q (3.81) and the vanishing of the antisymmetric central charges $\mathcal{Z}^{IJ}_{\mathrm{A}}$ are crucial for $Q^2 = 0$.

Example: $\mathcal{N} = 4$

The smallest realization of the Clifford algebra (3.45) is given in target space dimension $d = 4$. We choose for the complex structures

$$\begin{aligned} {I^2}^a{}_b &= ((-\mathrm{i}\sigma_2) \otimes \sigma_0)^a{}_b, \\ {I^3}^a{}_b &= (\sigma_3 \otimes (-\mathrm{i}\sigma_2))^a{}_b, \\ {I^4}^a{}_b &= (\sigma_1 \otimes (-\mathrm{i}\sigma_2))^a{}_b. \end{aligned} \tag{3.83}$$

The superpotential is given by

$$W = K^J + H^J, \quad \forall\, J = 2, 3, 4, \tag{3.84}$$

where H^J is the real part of an I^J-holomorphic function h^J. As for flat target space the K^J are at most quadratic in the coordinates, we split

$$H^J = H^J_{\text{quad}} + H^J_{\text{rest}} \tag{3.85}$$

and get

$$W = W_{\text{quad}} + W_{\text{rest}} = K^J + H^J_{\text{quad}} + H^J_{\text{rest}}. \tag{3.86}$$

Considering terms of third or higher-order we get

$$H^J_{\text{rest}} = W_{\text{rest}} \quad \text{for} \quad J = 2, 3, 4. \tag{3.87}$$

But that means that W_{rest} is the real part of a triholomorphic function which is of order three or higher. In the following we prove, that this implies $W_{\text{rest}} = 0$.

Proposition: Any triholomorphic function W (holomorphic with respect to the three complex structures (3.83)) is at most linear in the coordinates.

Proof: In real coordinates, W being holomorphic with respect to the complex structure I means that

$$W_{,ab} + I^c{}_a I^d{}_b W_{,cd} = 0, \tag{3.88}$$

which is equivalent to

$$I_a{}^c W_{,cb} + W_{,ac} I^c{}_b = 0, \tag{3.89}$$

such that the anticommutator of the complex structure with the Hessian of W has to be zero. This has to hold for all three complex structures. But as $I^4 = I^2 I^3$ and since the Hessian of W anticommutes with I^2 and I^3 we conclude that the Hessian of W commutes with I^4. As the Hessian of W commutes and anticommutes with I^4 it has to vanish. The most general solution for W is therefore linear in ϕ. ■

The conclusion is, that there exists only a massive free $\mathcal{N} = 4$ Wess-Zumino model with target space dimension four. From this conclusion it follows, that there are no nontrivial $\mathcal{N}=2$ Wess-Zumino models in four-dimensional Minkowski space with two complex scalar fields, as by dimensional reduction one would obtain from this model a $\mathcal{N}=4$ model in two dimensions with $d=4$. For a proof of this statement, but only for renormalizable theories in four spacetime dimensions, the reader may consult [72].

One way out would be to consider models with higher target space dimensions, such that one can choose $I^4 \neq I^2 I^3$ and the proposition above does not hold. But let us stop the discussion here and investigate in the next subsection, how the obtained models are related to the $\mathcal{N}=1$ Wess-Zumino model in four dimensions by dimensional reduction.

3.1.3 Dimensional Reduction of the $\mathcal{N}=1$ Wess-Zumino Model in Four Dimensions

In this subsection, we start with the $\mathcal{N}=1$ Wess-Zumino model in four-dimensional Minkowski space and reduce it to two dimensions. For Lorentz indices in four dimensions we will use Latin indices like $m,n=0,\dots,3$. For the Γ-matrices in four dimensions, we choose a Majorana representation, such that Majorana spinors are real. We denote the Γ-matrices by Γ^m,

$$\{\Gamma^m,\Gamma^n\}=2\eta^{mn},\quad \Gamma_*=\mathrm{i}\Gamma^0\Gamma^1\Gamma^2\Gamma^3,\quad \mathbb{P}_\pm=\frac{1}{2}(\mathbb{1}\pm\Gamma_*). \tag{3.90}$$

The Lagrangian of the model is given by

$$\mathcal{L}_{(4)}=(\partial^m\phi)^*(\partial_m\phi)+\frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi-\left|\frac{\partial h}{\partial\phi}\right|^2-\frac{1}{2}\frac{\partial^2 h}{\partial\phi^2}\bar{\psi}\mathbb{P}_-\psi-\frac{1}{2}\left(\frac{\partial^2 h}{\partial\phi^2}\right)^*\bar{\psi}\mathbb{P}_+\psi, \tag{3.91}$$

where h is any holomorphic function depending on the complex scalar field ϕ and ψ is a four-component Majorana spinor. The fields (ϕ,ψ) have mass dimensions $(1,\frac{3}{2})$, respectively. As we reduce this model to two dimensions, we do not care about renormalizability in four dimensions. Otherwise we would have to assume, that h is a polynomial in ϕ of degree less than four. In the following we change to real scalar fields by defining

$$\phi=\frac{1}{\sqrt{2}}(\phi_1+\mathrm{i}\phi_2),\quad W=h+h^*, \tag{3.92}$$

such that the Lagrangian reads

$$\mathcal{L}_{(4)} = \frac{1}{2}\partial_m\phi_a\partial^m\phi_a + \frac{\mathrm{i}}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}(W_{,a})^2 - \frac{1}{2}W_{,11}\bar{\psi}\psi - \frac{\mathrm{i}}{2}W_{,12}\bar{\psi}\Gamma_*\psi. \tag{3.93}$$

We denoted the derivative of W with respect to ϕ_a by $W_{,a}$. W is a harmonic function of the ϕ_a, as it is the real part of the holomorphic function h. The supersymmetry transformations leaving the action corresponding to (3.93) invariant are given by

$$\begin{aligned}
\delta_\epsilon\phi_1 &= \bar{\epsilon}\psi, \\
\delta_\epsilon\phi_2 &= \mathrm{i}\bar{\epsilon}\Gamma_*\psi, \\
\delta_\epsilon\psi &= -\mathrm{i}\left((\partial\!\!\!/\phi_1 - \mathrm{i}W_{,1}) + \mathrm{i}(\partial\!\!\!/\phi_2 + \mathrm{i}W_{,2})\Gamma_*\right)\epsilon.
\end{aligned} \tag{3.94}$$

For the dimensional reduction, we fix a specific representation of the Γ-matrices, such that we obtain the Majorana representation (3.37) in two dimensions. This can be realized by the choice,

$$\Gamma^\mu = \mathbb{1}_2 \otimes \gamma^\mu, \quad \Gamma^{a+1} = \Delta^a \otimes \gamma_*, \quad \Delta^1 = \mathrm{i}\sigma^1, \quad \Delta^2 = \mathrm{i}\sigma^3. \tag{3.95}$$

In the following, Greek indices like $\mu \in \{0,1\}$ are Lorentz indices in two dimensions and Latin indices like a run from one to two. We are led to the relations

$$\mathcal{C}_{(4)} = \mathbb{1}_2 \otimes \mathcal{C}_{(2)}, \quad \mathcal{C}_{(2)} = -\gamma^0, \quad \Gamma_* = -\sigma^2 \otimes \gamma_*. \tag{3.96}$$

The Majorana condition now reads

$$\psi = \xi \otimes \chi = \psi_c = \psi^* \quad \Leftrightarrow \quad \xi \in \mathbb{R}^2, \chi = \chi_c = \chi^*, \tag{3.97}$$

and an arbitrary Majorana spinor in four dimensions can be written as

$$\psi = e_a \otimes \chi_a, \tag{3.98}$$

where we choose for $\{e_a\}$ the canonical basis in $\mathbb{R}^2$.

Dimensional Reduction of the Action

Let us now reduce the Lagrangian (3.93) to two dimensions. We assume, space is $\mathbb{R}^2 \times S_1 \times S_1$ and we write $V_{(2)}$ for the volume of the two-dimensional torus $S_1 \times S_1$. We may Fourier expand all fields in the compact directions. If $V_{(2)}$ gets small, the constant modes decouple from the others, i.e. in the limit of small $V_{(2)}$ we can restrict the theory to fields, which are independend of the compact coordinates. Next, we have to verify the dimensions of the various fields. The fields (ϕ_a, ψ) have mass dimensions $(1, \frac{3}{2})$ in four dimensions. But scalar fields in two dimensions have zero mass dimension and spinors are of mass dimension $\frac{1}{2}$. Therefore we have to rescale the fields,

$$\phi_a \to \frac{1}{\sqrt{V_{(2)}}}\phi_a, \qquad \psi = \frac{1}{\sqrt{V_{(2)}}} e_a \otimes \chi_a. \tag{3.99}$$

Now we reduce the different terms. As the fields do not depend on the compact coordinates, the corresponding derivatives drop out. The reduction of the bosonic term is then rather obvious. For the fermionic terms we obtain the following expressions,

$$\begin{aligned} V_{(2)}\bar{\psi}\Gamma^m\partial_m\psi &= \bar{\chi}_a\gamma^\mu\partial_\mu\chi_a, \\ V_{(2)}\bar{\psi}\psi &= \bar{\chi}_a\chi_a, \\ V_{(2)}\bar{\psi}\Gamma_*\psi &= -(\sigma^2)_{ab}\bar{\chi}_a\gamma_*\chi_b. \end{aligned} \tag{3.100}$$

After redefining the second spinor,

$$\chi_2 \to -\gamma_*\chi_2, \tag{3.101}$$

we obtain the Lagrangian

$$\mathcal{L}_{(2)} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \frac{\mathrm{i}}{2}\bar{\psi}_a\not{\partial}\psi_a - \frac{1}{2}(W_{,a})^2 - \frac{1}{2}W_{,ab}\bar{\chi}_a\chi_b, \tag{3.102}$$

which coincides with (3.27).

Dimensional Reduction of Supersymmetry Transformations

Finally we determine the supersymmetry transformations in two dimensions. We write for the supersymmetry parameter

$$\epsilon = e_a \otimes \epsilon_a. \tag{3.103}$$

As the e_a are linear independent, we obtain two independent supersymmetry transformations, both generated by one of the two Majorana spinors ϵ_a. After changing the supersymmetry parameter, $\epsilon_2 \to \gamma_* \epsilon_2$, the supersymmetry transformations read

$$\begin{aligned} &\delta^1_{\epsilon_1}\phi_a = \bar{\epsilon}_1\chi_a, \qquad \delta^1_{\epsilon_1}\chi_a = (-\mathrm{i}\partial\!\!\!/\phi_a - W_{,a})\epsilon_1, \\ &\delta^2_{\epsilon_2}\phi_a = I_{ab}\bar{\epsilon}_2\chi_b, \qquad \delta^2_{\epsilon_2}\chi_a = I_{ab}(\mathrm{i}\partial\!\!\!/\phi_b - W_{,b})\epsilon_2, \end{aligned} \tag{3.104}$$

which coincide with (3.28) and (3.74).

3.1.4 Short Summary

Let us summarize the discussion of the Wess-Zumino model in two dimensions. We derived the $\mathcal{N} = 1$ off-shell Wess-Zumino model from $\mathbb{R}^{2|2}$ superspace. Observe that superspace formulations always lead to off-shell formulations, as there are as many bosonic degrees of freedom as fermionic ones. This can be seen from the expansion of the superfield.

Furthermore, one can obtain a specific $\mathcal{N} = 2$ off-shell Wess-Zumino model from the superspace $\mathbb{R}^{2|4}$, which results from dimensional reduction from four dimensions [73]. We did this dimensional reduction explicitly for the corresponding on-shell model. The resulting Lagrangian is the one of the $\mathcal{N} = 1$ model with harmonic superpotential.

Remember (Table 3.1), that there are further $\mathcal{N} = 2$ on-shell models, which can't be obtained by straighforward dimensional reduction. But these models are anyhow only massive free field theories and therefore of minor interest. It would be interesting to investigate, whether one can get these models by twisting the theory. In the superspace formulation of the four-dimensional theory one imposes the so called chiral constraint. From the point of view of the two-dimensional theory one has the possibility to demand other constraints, the so called twisted chiral constraints [73].

Let us finally mention, that for nonlinear sigma models in the on-shell formulation there may be more interesting extended theories, as the Killing potential K needs not to be only quadratic in the coordinates (3.56).

3.2 Wess-Zumino Models on the Spatial Lattice

In this section we formulate lattice versions of Wess-Zumino models in two-dimensional Minkowski space. The results presented here are part of [JDL4].

It is known that the $\mathcal{N}=1$ Wess-Zumino model does not need wave function renormalization and the $\mathcal{N}=2$ model is actually ultraviolet finite [74]. Anyhow, we introduce a lattice version for both of them, as the lattice provides us not only an ultraviolet-cutoff, but also allows for non-perturbative calculations, e.g. numerical simulations, mean field approximation or strong-coupling expansion.

We choose the Hamiltonian approach, discretizing space and keeping time continuous, such that time translations remain symmetries generated by the Hamiltonian. Following [26] we try to preserve at least a subalgebra of (3.4) which involves the Hamiltonian. For example for the $\mathcal{N}=1$ model, with the choice (3.37) for the γ-matrices, we choose

$$\{Q_1, Q_1\} = 2(P_0 + \mathcal{Z}_\mathrm{S}) \tag{3.105}$$

as the subalgebra. Observe that the Q_1 do not close on H but on H and $\mathcal{Z}_\mathrm{S}$. This will become important for the lattice versions.

We will not consider lattice models of supersymmetric field theories in higher dimensions in this thesis. But let us mention that in four dimensions, there does not exist such a subalgebra for the $\mathcal{N}=1$ model. This No-go theorem is proven in appendix B. Therefore, the requirement to find such a subalgebra gives a restriction on the type of models.

The fields of the supersymmetric model in the Hamiltonian formulation are discretized as follows,

$$(\phi^a(x), \pi^a(x), \psi^a(x)) \to (\phi^a(n), \pi^a(n), \psi^a(n)), \quad n = 1, \dots, N, \quad a = 1, \dots, d, \tag{3.106}$$

where the lattice spacing has been set to one and we choose periodic boundary conditions. Observe that we have not only introduced a lattice but we have also adopted a finite volume and obtain a $N \times d$-dimensional quantum mechanical system. The Hermitian scalar fields $\phi^a(n)$ with corresponding canoncial conjugate momenta $\pi^a(n)$ and the Majorana spinors $\psi^a(n)$ obey the canonical (anti)commutation relations

$$[\phi^a(m), \pi^b(n)] = \mathrm{i}\delta^{ab}\delta(m,n) \quad \text{and} \quad \{\psi^a_\alpha(m), \psi^b_\beta(n)\} = \delta^{ab}\delta_{\alpha\beta}\delta(m,n) \tag{3.107}$$

and the others are trivial.

On a space-lattice the derivative becomes a difference operator. Before we start with the Wess-Zumino models on the lattice, let us investigate various lattice derivatives and consider their consequences for fermion doubling and chiral symmetry on the lattice. But let us already mention at this point that there is no lattice derivative which obey the Leibniz rule. This makes it rather difficult to preserve supersymmetry [20].

3.2.1 Lattice Derivatives

As we consider first of all real scalar fields and Majorana spinors which are also real with our choice of γ-matrices (3.37), we will consider only real lattice derivatives ∂, but the lattice derivatives need not to be antihermitian. We denote the Hermitian conjugate lattice derivative, with respcect to the ℓ_2-scalar product

$$(f, g) = \sum_{n=1}^{N} \bar{f}(n) g(n) \tag{3.108}$$

and with periodic boundary conditions $f(n+N) = f(n)$, by $\partial^\dagger$. The kinetic and mass term for fermions read

$$H_\mathrm{F} = \frac{1}{2}\int \mathrm{d}x\, \psi^\dagger h_\mathrm{F} \psi, \qquad h_\mathrm{F} = h^0_\mathrm{F} + m\gamma^0, \qquad h^0_\mathrm{F} = -\mathrm{i}\gamma_* \partial_x. \tag{3.109}$$

The corresponding lattice version is given by

$$H_\mathrm{F} = \frac{1}{2}\sum_{m=1}^{N} \psi^\dagger(m)(h_\mathrm{F}\psi)(m) \quad \text{with} \quad h^0_\mathrm{F} = \mathrm{i}\begin{pmatrix} 0 & \partial \\ -\partial^\dagger & 0 \end{pmatrix}, \tag{3.110}$$

such that $h_{\rm F}$ is Hermitian. We demand that the left- and right-handed part of the fermions obey the same second-order equation and therefore we only consider normal lattice derivatives, i.e.

$$\partial\partial^\dagger = \partial^\dagger\partial = -\Delta. \tag{3.111}$$

We may define the symmetric and antisymmetric part of the lattice derivative by

$$\partial_{\rm S} = \frac{1}{2}(\partial + \partial^\dagger), \quad \partial_{\rm A} = \frac{1}{2}(\partial - \partial^\dagger) \quad \text{with} \quad [\partial_{\rm A}, \partial_{\rm S}] = 0, \quad \partial_{\rm A}^2 - \partial_{\rm S}^2 = \Delta. \tag{3.112}$$

The last two porperties follow from the assumption $[\partial, \partial^\dagger] = 0$ in (3.111). Since

$$h_{\rm F} = -{\rm i}\gamma_*\partial_{\rm A} + \gamma^0(m - \partial_{\rm S}), \tag{3.113}$$

chirality is preserved for massless fermions if $\partial = \partial_{\rm A}$ is antisymmetric. Thus, if ∂ is antisymmetric and local then, according to a general theorem [75], there is fermion doubling. The theorem can be circumvented by using a nonlocal and antisymmetric derivative. For a general lattice derivative, $h_{\rm F}^0$ contains a momentum dependent mass term $-\gamma^0\partial_{\rm S}$. Such a term has been introduced by Wilson [76] to raise the masses of the unwanted doublers to values of order of the cutoff, thereby decoupling them from continuum physics. Let us start with the investigation of specific lattice derivatives and discuss their advantages and disadvantages.

Left- and Right-Derivative

One very common lattice derivative is the left- and right-derivative or some linear combination of them. They are defined as

$$(\partial_{\rm R} f)(n) = f(n+1) - f(n) \quad \text{and} \quad (\partial_{\rm L} f)(n) = f(n) - f(n-1), \tag{3.114}$$

and the adjoint of the left-derivative is minus the right-derivative

$$(f, \partial_{\rm L} g) = -(\partial_{\rm R} f, g). \tag{3.115}$$

Both derivatives share the property that $(1, \partial_{\rm R} f) = (1, \partial_{\rm L} f) = 0$, but the corresponding momentum operators $\hat{p}_{\rm L} = -{\rm i}\partial_{\rm L}$ and $\hat{p}_{\rm R} = -{\rm i}\partial_{\rm R}$ are not Hermitian and have the following complex eigenvalues,

$$\lambda_k(\hat{p}_{\rm L}) = 2{\rm e}^{-{\rm i}p_k/2}\sin\frac{p_k}{2}, \quad \lambda_k(\hat{p}_{\rm R}) = 2{\rm e}^{{\rm i}p_k/2}\sin\frac{p_k}{2} \quad \text{with} \quad p_k = 2\pi k/N \tag{3.116}$$

and $k = 1, \ldots, N$. The advantage of these kinds of lattice derivatives is that they are ultralocal. In order to better understand the dependency of the spectrum and doubling phenomena of the lattice derivative, we consider the following one-parameter interpolating family of local difference operators

$$\partial_\alpha = \frac{1}{2}(1+\alpha)\partial_{\rm R} + \frac{1}{2}(1-\alpha)\partial_{\rm L} = \partial_{\rm S} + \partial_{\rm A} \tag{3.117}$$

with symmetric and antisymmetric parts

$$\partial_{\rm S} = \frac{1}{2}\alpha(\partial_{\rm R} - \partial_{\rm L}) \quad \text{and} \quad \partial_{\rm A} = \frac{1}{2}(\partial_{\rm R} + \partial_{\rm L}) \equiv \partial_{\rm R+L}. \tag{3.118}$$

For $\alpha = -1$ ($\alpha = 1$) we obtain the left(right)-derivative back. For $\alpha = 0$ we get the antisymmetric operator $\partial_{\rm R+L}$ for which the corresponding Hermitian momentum operator $\hat{p}_{\rm R+L}$ has the spectrum

$$\lambda_k(\hat{p}_{\rm R+L}) = \sin(p_k) \quad \text{with} \quad p_k = 2\pi k/N, \quad k = 1, \ldots, N. \tag{3.119}$$

The $2N$ eigenvalues of the Hermitian Dirac Hamiltonian (3.113) depend on the deformation parameter as follows,

$$\lambda_k(\alpha) = \pm\sqrt{m^2 + 4\alpha(\alpha+m)\sin^2(\frac{p_k}{2}) + (1-\alpha^2)\sin^2(p_k)}, \tag{3.120}$$

where again $p_k = 2\pi k/N$ with $k = 1, \ldots, N$. For $\alpha = \pm 1$ all eigenvalues with p_k in the interior of the first Brioullin zone have multiplicity two and for $\alpha = 0$ they have multiplicity four. One can show that for α greater than α_+ or less than α_-, where

$$4\alpha_\pm = \pm(\sqrt{m^2+8} \mp m), \tag{3.121}$$

all eigenvalues have multiplicity two. However, for $\alpha \in [\alpha_-, \alpha_+]$ some eigenvalues have multiplicity four. This should be compared with the eigenvalues of the derivative operator on the continuous interval of length N,

$$\lambda_k = \pm\sqrt{m^2 + p_k^2}, \quad p_k = 2\pi k/N, \qquad k \in \mathbb{Z}, \tag{3.122}$$

with multiplicity two. The conclusion is that for $\alpha > \alpha_+$ or $\alpha < \alpha_-$ we do not have fermion doublers and for $\alpha \neq 0$ chiral symmetry is explicitly broken.

The nonlocal SLAC Derivative

Let us consider a nonlocal difference operator called the SLAC derivative operator [77]. For the definition of the SLAC derivative let us remember some facts of quantum mechanics. In position space, the position operator is defined by multiplication with the coordinate. Contrarily, the momentum operator is a differential operator. But in momentum space their respective roles are interchanged. There, the position operator is a differential operator and momentum operator is a multiplication operator. On the finite lattice, we want to treat them on the same footing, that is we define both as multiplication operators, the position operator in position space and momentum operator in momentum space. To obtain the momentum operator in position space we have to make a Fourier transformation. The result will be a nonlocal operator.

For the Fourier transformation we choose,

$$\begin{aligned}\phi(x_m) &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}p_n x_m}\tilde{\phi}(p_n), \qquad p_n = \frac{2\pi}{N}(n+\alpha), \qquad x_m = m+\beta,\\ \tilde{\phi}(p_n) &= \frac{1}{\sqrt{N}}\sum_{m=1}^{N} \mathrm{e}^{-\mathrm{i}p_n x_m}\phi(x_m),\end{aligned} \tag{3.123}$$

where α and β are at this point arbitrary parameters. A simple calculation shows, that

$$\phi(x_m + N) = \mathrm{e}^{2\pi\mathrm{i}\alpha}\phi(x_m), \tag{3.124}$$

i.e. the fields are periodic for $\alpha \in \mathbb{Z}$. As discussed above, the momentum operator in position space is defined as

$$
\begin{aligned}
(\hat{p}\phi)(x_m) &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}p_n x_m} p_n \tilde{\phi}(p_n) \\
&= \sum_{k=1}^{N} \underbrace{\left(\frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}p_n (x_m - x_k)} p_n \right)}_{(\partial_{\mathrm{SLAC}})_{mk}} \phi(x_k).
\end{aligned}
\tag{3.125}
$$

To evaluate this sum, we calculate the generating functional

$$
f(x) \equiv \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}p_n x} \tag{3.126}
$$

which must be real, such that the SLAC derivative

$$
(\partial_{\mathrm{SLAC}})_{mk} = \frac{1}{N} \frac{\partial}{\partial x} f(x)\big|_{x = x_m - x_k}, \quad \text{for} \quad m \neq k \tag{3.127}
$$

is real. We treat the case $m = k$ separately. This can be achieved by a symmetric summation, that is we choose

$$
\alpha = -\frac{N+1}{2}, \tag{3.128}
$$

and the generating functional (3.126) reads

$$
f(x) = \frac{\sin(\pi x)}{\sin(\pi x / N)}. \tag{3.129}
$$

As we showed already, for the functions to be periodic, we have to choose $\alpha \in \mathbb{Z}$, therefore we find a real SLAC derivative with periodic boundary conditions only for an odd number N of lattice sites. Evaluating expression (3.127) leads to

$$
(\partial_{\mathrm{SLAC}})_{mk} = (-)^{m-k} \frac{\pi / N}{\sin(\pi(x_m - x_k)/N)}, \quad \text{for} \quad m \neq k. \tag{3.130}
$$

As the summation in (3.125) over p_n with our choice of α is symmetric, the diagonal elements vanish.

By this construction we obtain, as stated already, a nonlocal operator. The main advantages of the SLAC derivative are that it is antisymmetric and hence preserves chirality and that the N eigenvalues coincide with the lowest eigenvalues of the continuum operator on the interval of length N.

In Figure 3.1 we have plotted the positive eigenvalues of h_{F}^{0} for the interpolating operator for the values $\alpha = 0, 1, \alpha_+$ and the SLAC derivative operator.

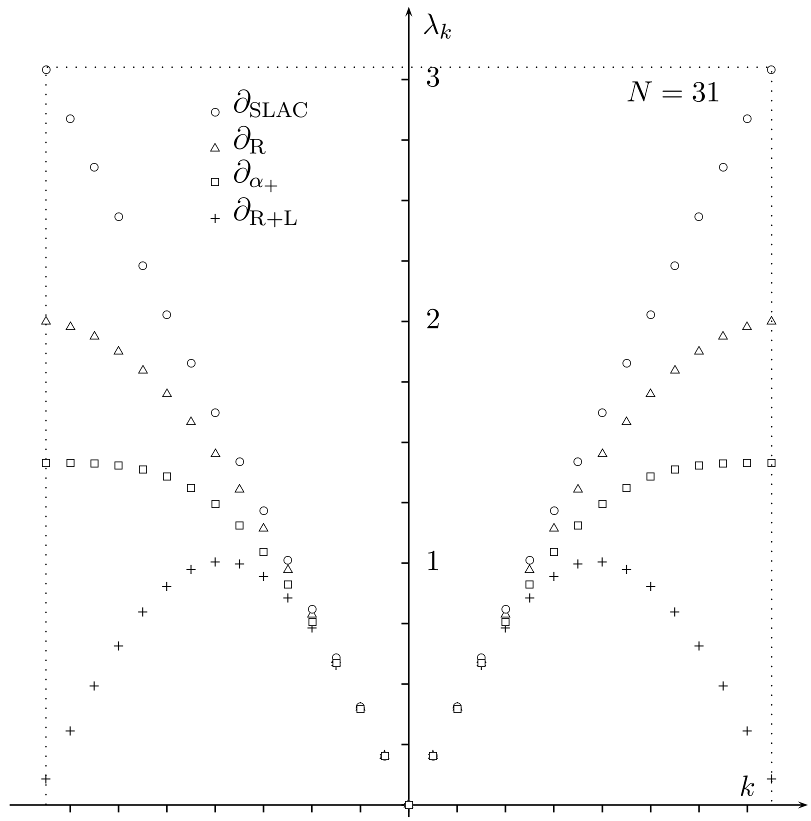

Figure 3.1: Positive eigenvalues of h_{F}^{0} for 3 different α and for the SLAC derivative operator.

The question arises if this nonlocal lattice derivative can be used in numerical simulations. In (supersymmetric) quantum mechanics the result is affirmative. One can discretize the stationary Schrödinger equation by different methods and obtain finite matrices. The approximative spectrum is given by the eigenvalues of these matrices. The use of the SLAC derivative, in contrast to other local derivatives, increases the time for the numerical calculation at most by a factor of two [78]. But the spectrum obtained with the SLAC derivative has an incredibly better accuracy [JDL4]. Unfortunately, the result in the context of field theories is not so clear yet. As the lattice derivative is nonlocal, Monte-Carlo calculations are getting involved. Furthermore, there has been discussions in the literature about problems of the SLAC derivative in the continuum limit. For example in [79], the vacuum polarization in the one-loop approximation in QED on the lattice was calculated with the use of the SLAC derivative operator. The vacuum polarization appears non-covariant and nonlocal in the continuum limit. But in [80] it was pointed out that the problem is due to the way in which the gauge field coupling to fermions is introduced. They could avoid these difficulties, however they payed the price in that the resulting theory is considerably more complicated. We conclude that further investigations are needed and we hope to comment on this point in near future.

Interlude: Derivative Operator for Curved Target Space

In nonlinear sigma models [81, 82] the scalar fields ϕ^k, $k = 1, \dots, d$, are coordinates of a manifold and the considered Lagrangians are invariant under a change of coordinates. $\partial_x \phi^k(x)$ is a tangent vector of the target space at the point $\phi^k(x)$. If we want to find a lattice formulation of this theory which preserves this invariance under coordinate transformations, we have to careful think about which definition we use for the lattice derivative $(\partial\phi)^k(m)$. Let us illustrate one possibility. $\phi^k(m)$ and $\phi^k(m+1)$ are two points on target space. For small distances there is a unique geodesic between these two points. We define $(\partial\phi)^k(m)$ as the tangent vector to this geodesic at the point $\phi^k(m)$. If the target space is flat, this definition corresponds to the right derivative ∂_{R}. We can define a similar lattice derivative corresponding to the left derivative. Unfortunately, it is not so clear whether this lattice derivative is of any use for numerical simulations, but it may be important for analytic investigations of nonlinear sigma models.

3.2.2 The $\mathcal{N} = 1$ Wess-Zumino Model on the Lattice

Let us now discretize the continuum $\mathcal{N} = 1$ Wess-Zumino model, for which details were given in Subsection 3.1.1. For the discretized version of the supercharge Q (3.26) we choose

$$Q = (\pi, \psi) + \mathrm{i}(\phi, h_{\mathrm{F}}^0 \psi) + \mathrm{i}(W', \gamma^0 \psi) \quad \text{with} \quad (h_{\mathrm{F}}^0)_{ab} = \mathrm{i}\delta_{ab} \begin{pmatrix} 0 & \partial \\ -\partial^\dagger & 0 \end{pmatrix}. \tag{3.131}$$

We used for example the short hand notation $(W', \gamma^0\psi) = \sum_{m,a} W_{,a}(\phi^b(m))\gamma^0\psi^a(m)$. A straightforward calculation using (3.107) leads to the anticommutation relations,

$$\frac{1}{2}\{Q_\alpha, \bar{Q}_\beta\} = \not{P}_{\alpha\beta} + \mathrm{i}(\gamma_*)_{\alpha\beta}(W', \partial_\mathrm{A}\phi) - (\gamma^0)_{\alpha\beta}(W', \partial_\mathrm{S}\phi) \tag{3.132}$$

with energy and momentum

$$\begin{aligned} 2P_0 &= (\pi, \pi) - (\phi, \Delta\phi) + (W', W') + (\psi, h_\mathrm{F}\psi) \\ 2P_1 &= 2(\partial_\mathrm{A}\phi, \pi) - (\psi, \gamma_* h_\mathrm{F}^0\psi). \end{aligned} \tag{3.133}$$

The last term in (3.132) is not present in the superalgebra (3.5) and breaks Lorentz covariance. This term is due to the improvement term $\gamma^0\partial_\mathrm{S}$ which we introduced to avoid fermion doublers. For the SLAC derivative we do not have this term and we do not have fermion doublers neither. This is one further advantage of the SLAC derivative operator. But still, choosing the SLAC derivative, there is another poblem. The term $(W', \partial_\mathrm{A}\phi)$ is no central charge anymore and none of the supercharges Q_α commutes with the Hamiltonian P_0. We do not have any supersymmetry left. But let us recall that as we disretize space, spatial translations are not supposed to be good symmetries of the theory anymore. We are actually looking for the subalgebra (3.105) including the Hamiltonian only. Let us *define* the Hamiltonian for any lattice derivative to be the square of the first supercharge Q_1,

$$H \equiv (Q_1)^2 = P_0 + (W', \partial^\dagger\phi). \tag{3.134}$$

This Hamiltonian is manifest supersymmetric, but contains lattice artefacts as well as the central charge.

From now on we only consider the $N = 1$ Wess-Zumino model with one scalar field ϕ, i.e. we set $d = 1$. This makes notation more simple. We remark that it is not difficult to generalize the results to several fields.

From the Dirac Operator to the $\mathcal{N} = 1$ Wess-Zumino Model

Let us relate our discretized version of the $\mathcal{N} = 1$ Wess-Zumino model with N lattice points to the square of a Dirac operator on $2N$ flat dimensions. In Section 2.3 we have already considered the dimensional reduction to matrix-Schrödinger Hamiltonians. The considerations here are similar, but we do not demand $\mathcal{N}=2$. We assume the flat manifold to be $\mathbb{R}^N \times T^N$. In the discretized Wess-Zumino model there are no linear terms in momentum in contrast to the square of the Dirac operator (2.92). Therefore we have to make the same assumptions as in Section 2.3. The gauge fields do not depend on the last N coordinates x_{N+n} and we set $A_n = 0$ for $n = 1, \ldots, N$. Next we make the following identification with the Wess-Zumino model,

$$x_n = \phi(n), \quad -\mathrm{i}\frac{\partial}{\partial x_n} = \pi(n), \quad \begin{pmatrix} \Gamma^n \\ \Gamma^{N+n} \end{pmatrix} = \sqrt{2}\psi(n) \tag{3.135}$$

and the non-vanishing components of the gauge fields have the form

$$A_{N+n} = (\partial^\dagger \phi)(n) - W'(\phi(n)). \tag{3.136}$$

With this choice we find

$$-\frac{1}{\sqrt{2}}\mathrm{i}\nabla\!\!\!/ = (\pi, \psi_1) - (\phi, \partial\psi_2) + (W', \psi_2) = Q_1. \tag{3.137}$$

Ground State of the Free Model

With $2W = m\phi^2$ we obtain the massive non-interacting $\mathcal{N}=1$ Wess-Zumino model with $d=1$. The corresponding Hamiltonian is the sum of two commuting operators, of the bosonic part

$$H_{\mathrm{B}} = \frac{1}{2}(\pi, \pi) + \frac{1}{2}(\phi, A^2\phi), \qquad A^2 = -\triangle + m\partial_{\mathrm{S}} + m^2, \tag{3.138}$$

and the fermionic one

$$H_{\mathrm{F}} = \frac{1}{2}(\psi, h_{\mathrm{F}}\psi), \quad h_{\mathrm{F}} = -\mathrm{i}\gamma_*\partial_{\mathrm{A}} + \gamma^0(m - \partial_{\mathrm{S}}). \tag{3.139}$$

We assume that the parameters are such that A^2 is positive. Near the continuum limit this is always the case if the physical mass is positive. The ground state wave function of the supersymmetric Hamiltonian factorizes,

$$\Psi_0 = \Psi_{\mathrm{B}}\Psi_{\mathrm{F}} \quad \text{with} \quad H_{\mathrm{B}}\Psi_{\mathrm{B}} = E_{\mathrm{B}}\Psi_{\mathrm{B}} \quad \text{and} \quad H_{\mathrm{F}}\Psi_{\mathrm{F}} = E_{\mathrm{F}}\Psi_{\mathrm{F}}. \tag{3.140}$$

The bosonic factor $\Psi_{\rm B}$ ist Gaussian

$$\Psi_{\rm B} = c \cdot \exp\left(-\frac{1}{2}(\phi, A\phi)\right) \quad \text{and} \quad E_{\rm B} = \frac{1}{2}\text{tr } A. \tag{3.141}$$

Here A is the positive root of the positive and Hermitian A^2 in (3.138). For the family of operators in (3.117) the trace of A is just half the sum of the positive eigenvalues in (3.120). For $\partial_{\rm SLAC}$ with eigenvalues $p_k = 2\pi k/N$ we obtain

$$E_{\rm B} = \frac{\pi}{N} \sum_{k=-N'}^{N'} \sqrt{m^2 + p_k^2} \quad \stackrel{m\to 0}{\longrightarrow} \quad \frac{(N-1)(N+1)}{4N}\pi. \tag{3.142}$$

To find $\Psi_{\rm F}$ we introduce the (2-component) eigenfunctions v_k of $h_{\rm F}$ with positive eigenvalues. Since the Hermitian matrix $h_{\rm F}$ is imaginary the v_k cannot be real and we have

$$h_{\rm F} v_k = \lambda_k v_k \Longleftrightarrow h_{\rm F} \bar{v}_k = -\lambda_k \bar{v}_k \quad (\lambda_k > 0). \tag{3.143}$$

The eigenvectors are orthogonal with respect to the Hermitian scalar product,

$$(v_k, v_{k'}) = \sum_{n,\alpha=1,2} \bar{v}_{k\alpha}(n) v_{k'\alpha}(n) = \delta_{kk'} \quad \text{and} \quad (\bar{v}_k, v_{k'}) = 0. \tag{3.144}$$

Now we expand the Majorana spinors in terms of this orthonormal basis,

$$\psi(n) = \sum_{k=1}^{N} \left(\chi_k v_k(n) + \chi_k^\dagger \bar{v}_k(n)\right), \quad \text{where} \quad \chi_k = (v_k, \psi), \quad \chi_k^\dagger = (\bar{v}_k, \psi) \tag{3.145}$$

are one-component complex objects with anticommutation relations

$$\{\chi_k, \chi_{k'}\} = 0 \quad \text{and} \quad \{\chi_k, \chi_{k'}^\dagger\} = \delta_{kk'}. \tag{3.146}$$

Inserting the expansion (3.145) into $H_{\rm F}$ yields

$$H_{\rm F} = \frac{1}{2} \sum_{k:\lambda_k > 0} \lambda_k \left(\chi_k^\dagger \chi_k - \chi_k \chi_k^\dagger\right). \tag{3.147}$$

It follows that the ground state of H_{F} is the Fock vacuum which is annihilated by all annihilation operators χ_k,

$$\chi_k \Psi_{\mathrm{F}} = 0, \quad k = 1, \ldots, N, \quad \text{and} \quad E_{\mathrm{F}} = -\frac{1}{2} \sum_{k:\lambda_k > 0} \lambda_k. \tag{3.148}$$

Since $h_{\mathrm{F}}^2 = \mathbb{1}_2 \otimes A^2$ we conclude, that the positive eigenvalues of h_{F} are identical to the eigenvalues of A such that

$$E = E_{\mathrm{B}} + E_{\mathrm{F}} = 0. \tag{3.149}$$

Since Ψ_0 is normalizable for $A > 0$ we conclude that the Hamiltonian admits a bound supersymmetric ground state for all definitions of the lattice derivative ∂ provided A is positive.

Ground State for Strong-Coupling Limit

Elitzur et al. [26] where the first who investigated the strong-coupling limit of supersymmetric theories. Effectively, the strong-coupling limit corresponds to introducing a coupling constant in front of each lattice derivative and let this constant going to zero. In the following we introduce the parameter λ in the supercharge (3.137),

$$Q_1(\lambda) = \underbrace{(\pi, \psi_1) + (W', \psi_2)}_{A_0} + \lambda \underbrace{\left(-(\phi, \partial\psi_2)\right)}_{A_1}. \tag{3.150}$$

We recall that $\psi_1(m)$ and $\psi_2(m)$ are Hermitian $2^N \times 2^N$-matices. As in our models the only interaction term between different lattice points is in A_1, in the limit $\lambda \to 0$ fields at different lattice points decouple. The strong-coupling limit is not so simple if one introduces nonlocal interaction terms [21] to realize the full superalgebra. In that case the theory does not decouple for different lattice sites.

In our considerations the supercharge Q_1 and the Hamiltonian H are – in the strong-coupling limit – the sum of N identical and commuting operators, each defined on a given lattice site. The ground state is therefore a product state. The operators on a fixed lattice site read

$$Q_1 = -\mathrm{i}\psi_1 \frac{\partial}{\partial\phi} + \psi_2 W'(\phi) \quad \text{and} \quad H = -\frac{\partial^2}{\partial\phi^2} + W'^2 - \mathrm{i}\psi_1\psi_2 W''. \tag{3.151}$$

After introducing complex anihilation operators, $\psi = -\frac{1}{\sqrt{2}}(\psi_1 + \mathrm{i}\psi_2)$, we obtain the complex supercharge

$$Q = \mathrm{i}\psi(\frac{\partial}{\partial\phi} + W'), \qquad Q_1 = \frac{1}{\sqrt{2}}(Q + Q^\dagger) \tag{3.152}$$

and the Hamiltonain reads

$$2H = \{Q, Q^\dagger\} = -\partial_\phi\partial_\phi + (W')^2 - [\psi^\dagger, \psi]W''. \tag{3.153}$$

We have discussed this supersymmetric quantum mechnical model and its ground state already in 2.5.2. Supersymmetry is broken, if the degree of the polynomial W is odd, otherwise it is unbroken. As this holds for each lattice site, we conclude that in the strong-coupling limit the $\mathcal{N} = 1$ Wess-Zumino model on the spatial lattice has always one normalizable zero mode if the degree of the polynomial is even.

From Strong to Weak Coupling

In what follows, we compare the strong-coupling results with the usual perturbation theory in the vicinity of minima of the potential.

In the case $\deg(W) = p$ even, supersymmetry is never broken, neither in the strong-coupling limit nor in perturbation theory. For even W there is at least one minimum of the potential $V = \frac{1}{2}(W')^2$ with $V = 0$. The quadratic approximation of the potential at the critical points yields for each minimum one normalizable zero mode similar to the ground state of the free model. In contrast to the strong-coupling limit there may be more than one perturbative zero mode, but they come in an odd number. Thus the difference of bosonic and fermionic zero modes is ± 1 as in the strong-coupling limit.

In the case $\deg(W) = p$ odd, the difference between the strong-coupling limit and perturbation theory is more severe. Supersymmetry is broken in the strong-coupling limit but it may be unbroken in perturbation theory. Let us consider an explicit example,

$$W(\phi) = \frac{g_2}{2}\phi^3 + g_0\phi. \tag{3.154}$$

Perturbation theory for $g_0 < 0$ predicts one bosonic and one fermionic zero mode (unbroken supersymmetry) and broken supersymmetry and thus no zero mode for $g_0 > 0$. The strong-coupling limit states that supersymmetry is broken for all g_0.

In Appendix C.1 we provide the mathematically nontrivial proof that λA_1 (3.150) is an analytic perturbation of A_0 (3.150) for $\lambda \in \mathbb{R}$. This means that all eigenvalues are analytic functions of the parameter λ. Assume now that in a finite range of the parameter λ there is a ground state with energy exactly equal to zero. As an analytic function which vanishes in some finite range is identically zero, the number of zero modes changes at most at isolated points of the parameter space of λ. Furthermore, in the strong-coupling limit, we have either bosonic or fermionic zero modes. In Section 2.5 we have proven that under this assumption a zero mode always remains a zero mode. We conclude that, generically, the number of zero modes is given by the number of zero modes in the strong-coupling limit. Moreover, as the index also depends analytically on the parameter λ, we are able to calculate the index in the strong-coupling limit.

In the continuum and infinite-volume limit these arguments may break down, as the estimates necessary for proving analyticity (Appendix C.1) may not be valid anymore. In the unbroken case we can definitely conclude that supersymmetry is still unbroken in the continuum and infinite-volume limit. Suppose we know for any finite lattice that there is at least one ground state with energy zero. As the limit of zero is again zero this mode survives in the limit. In the case of broken supersymmetry a non-zero energy eigenstate may become a zero mode in the continuum and infinite-volume limit, and supersymmetry may get restored in this limit although it is broken for all finite lattices.

Indeed, for negative g_0 in our example above, the scalar field has a vacuum expectation value and therefore the fermionic field ψ has a non-zero mass. As there is no massless Goldstone fermion, supersymmetry has to be unbroken in this case [29].

Let us summarize. On a finite lattice, the strong-coupling limit gives the correct number of zero modes of the full problem. There is only one zero mode in the case where $\deg(W) = p$ is even, and otherwise there is no zero mode. Variations of the parameters in the superpotential of power less than p do neither change the number of zero modes nor the index. For example, in the model with superpotential given in (3.154), on a finite lattice it is impossible to have two phases of broken and unbroken supersymmetry depending on the parameter g_0. The numerical simulations in [83], however, may be interpreted as a hint for such a phase transition in the continuum theory.

3.2.3 The $\mathcal{N} = 2$ Wess-Zumino Model on the Lattice

In this subsection we consider the $\mathcal{N} = 2$ Wess-Zumino model with $d = 2$ and harmonic superpotential W, see Subsection 3.1.2. First, we specify which subalgebra of (3.4) we realize on the lattice. Using the Majorana representation (3.37) for the γ-matrices and the expressions (3.76) for the central charges we read off

$$\{Q_1^1, Q_1^1\} = \{Q_2^2, Q_2^2\} = 2(P_0 + \mathcal{Z}) \quad \text{and} \quad \{Q_1^1, Q_2^2\} = 0, \quad \mathcal{Z} = \mathcal{Z}_{\rm S}^{11} = -\mathcal{Z}_{\rm S}^{22}. \tag{3.155}$$

As the central charge appears in the same way for the two supercharges, there is a good chance to maintain this part of the superalgebra on the lattice. Indeed, if we discretize the supercharges Q_1^1 (3.26) and Q_2^2 (3.75) by

$$Q_1^1 = \underbrace{(\pi, \psi_1) + (W', \psi_2)}_{B_0} + \underbrace{(\partial\phi^1, \psi_2^1) - (\partial^\dagger\phi^2, \psi_2^2)}_{B_1}, \tag{3.156}$$

$$Q_2^2 = (\pi, I\psi_2) + (W', I\psi_1) - (\partial\phi^1, \psi_1^2) - (\partial^\dagger\phi^2, \psi_1^1), \tag{3.157}$$

the algebra (3.155) is realized with $H = P_0 + \mathcal{Z}$,

$$\begin{aligned} H &= H_{\rm B} + H_{\rm F}, \\ H_{\rm B} &= \frac{1}{2}\left((\pi,\pi) + (W', W') + (\partial\phi, \partial\phi)\right) + (W_{,1}, \partial\phi^1) - (W_{,2}, \partial^\dagger\phi^2), \\ H_{\rm F} &= \frac{1}{2}\left((\bar\psi, W''\psi) + (\psi^\dagger, (h_{\rm F}^0\psi)\right), \end{aligned} \tag{3.158}$$

where we have introduced

$$(h_{\rm F}^0)^{12}_{\alpha\beta} = (h_{\rm F}^0)^{21}_{\alpha\beta} = 0, \quad (h_{\rm F}^0)^{11}_{\alpha\beta} = \mathrm{i}\begin{pmatrix} 0 & -\partial^\dagger \\ \partial & 0 \end{pmatrix}_{\alpha\beta}, \quad (h_{\rm F}^0)^{22}_{\alpha\beta} = \mathrm{i}\begin{pmatrix} 0 & \partial \\ -\partial^\dagger & 0 \end{pmatrix}_{\alpha\beta}. \tag{3.159}$$

Observe that we had no other choice for the lattice derivatives such that the algebra holds. Furthermore, we have introduced as before the bracket for short hand notation, for example $(\pi,\pi)=\sum_{m,a}\pi_a(m)\pi_a(m)$.

We may also introduce the first component of the complex supercharge in (3.81),

$$Q=\frac{1}{2}(Q_1^1-\mathrm{i}Q_2^2)=(\tilde{\psi},\pi-\mathrm{i}\chi')\quad\text{with}$$
$$\chi=-\sum_m U(\phi^1(m),\phi^2(m))-(\phi^1,\partial^\dagger\phi^2). \tag{3.160}$$

As before, U is the imaginary part of the holomorphic superpotential h and we defined

$$\tilde{\psi}_1(m)=\frac{1}{2}(\psi_1^1(m)+\mathrm{i}\psi_2^2(m)),\quad \tilde{\psi}_2(m)=\frac{1}{2}(\psi_1^2(m)-\mathrm{i}\psi_2^1(m)). \tag{3.161}$$

In Section 2.3 we dimensionally reduced the Dirac operator with $\mathcal{N}=2$ supersymmetry on flat space to a supersymmetric quantum mechanical system with complex supercharges given in (2.95) and corresponding Hamiltonian in (2.96). With $\pi_a(m)=-\mathrm{i}\frac{\partial}{\partial\phi^a(m)}$ it follows that $-Q$ in (3.160) coincides with this supercharge. We conclude that the $\mathcal{N}=2$ Wess-Zumino model can be obtained by dimensional reduction of the Dirac operator in $4N$ dimensions.

Ground State of the Free Model

For the massive free $\mathcal{N}=2$ Wess-Zumino model we have to choose $h=\frac{m}{2}z^2$ and obtain $U=mxy$, with $z=x+\mathrm{i}y$. It follows that the complex supercharge Q can be written as

$$Q=-\mathrm{i}\left(\tilde{\psi},\frac{\partial}{\partial\phi}+\chi''\phi\right) \tag{3.162}$$

with constant, real, symmetric $2N\times 2N$-matrix χ'',

$$\chi''=-\begin{pmatrix}0 & \partial^\dagger+m\mathbb{1}_N\\ \partial+m\mathbb{1}_N & 0\end{pmatrix}. \tag{3.163}$$

We diagonalize χ'' by an orthogonal transformation S,

$$\chi''=S^{-1}DS,\qquad D=\mathrm{diag}(d_1,d_2,\dots,d_{2N}). \tag{3.164}$$

Correspondingly, we rotate the fields with S,

$$\xi = S\phi, \quad \eta = S\psi \quad \text{and} \quad \eta^\dagger = S\psi^\dagger. \tag{3.165}$$

The new fields still obey the standard anticommutation relations, e.g.

$$\{\eta^\dagger_\alpha(m), \eta_\beta(n)\} = \delta_{\alpha\beta}\delta_{mn}, \tag{3.166}$$

and the transformed supercharges read

$$Q = -\mathrm{i}\left(\eta, \frac{\partial}{\partial\xi} + D\xi\right) \qquad \text{and} \qquad Q^\dagger = -\mathrm{i}\left(\eta^\dagger, \frac{\partial}{\partial\xi} - D\xi\right) \tag{3.167}$$

and show, that the new degrees of freedom decouple. Hence the groundstate must have the product form

$$\Psi_0 = \exp\left(-\frac{1}{2}\sum |d_j|\xi_j^2\right)|\Omega\rangle \tag{3.168}$$

and the action of the supercharges on this state reads,

$$Q\Psi_0 = 2\mathrm{i}\sum_{j:\, d_j>0} d_j\eta_j\xi_j\Psi_0, \qquad Q^\dagger\Psi_0 = -2\mathrm{i}\sum_{j:\, d_j<0} d_j\eta^\dagger_j\xi_j\Psi_0. \tag{3.169}$$

This way we arrive at the following conditions for this state to be invariant,

$$d_j > 0 \Longrightarrow \eta_j|\Omega\rangle = 0 \quad \text{and} \quad d_j < 0 \Longrightarrow \eta^\dagger_j|\Omega\rangle = 0. \tag{3.170}$$

This leads to the unique normalizable groundstate (3.168) with

$$\Omega = \prod_{d_j<0}\eta^\dagger_j|0\rangle, \tag{3.171}$$

which is annihilated by the supercharges and hence has vanishing energy. There are N positive and N negative eigenvalues of χ'' such that the invariant vacuum state lies in the middle sector $\mathcal{H}_N$ in the decomposition (2.39) of the Hilbert space with $d = 2N$. All fermionic states with negative eigenvalues of χ'' are filled. This is analogue to the Dirac-sea filling prescription.

Ground States for Strong-Coupling Limit

In the strong-coupling limit we may neglect the spatial derivatives such that both, the supercharges and the Hamiltonian, become the sum of N commuting operators, each defined on one lattice site. The operators on one site take the form

$$Q = -\mathrm{i}\tilde{\psi}_a \left(\frac{\partial}{\partial \phi_a} - \frac{\partial U}{\partial \phi_a} \right) \tag{3.172}$$

with harmonic superpotential $U(\phi_1, \phi_2)$. We have discussed this model in the second part of Subsection 2.5.2. There we concluded that for a polyomial U with $\deg(U) = p$ there are $p-1$ normalizable zero modes.

Since the eigenstates of H are product states, and since we have for each lattice site the choice of $p-1$ zero modes, there are $(p-1)^N$ zero modes for the lattice model in the strong-coupling limit. After we had finished our calculations we realized that Elitzur and Schwimmer already came to the same surprising conclusion [27].

From Strong to Weak Coupling

In the following we want to investigate the implications of the strong-coupling limit for the full problem. We consider one of the two Hermitian supercharges, say Q_1^1 which squares to the Hamiltonian H. We introduce as for the $\mathcal{N}=1$ case a parameter λ,

$$Q_1^1 = B_0 + \lambda B_1, \tag{3.173}$$

where B_0 and B_1 are defined in (3.156). Similar to the $\mathcal{N}=1$ case, we prove in Appendix C.2 that the index is analytic in the parameter λ. This implies that we have $(p-1)^N$ zero modes for the theory on finite lattices. For the continuum theory in a finite volume, it was shown using methods of constructive field theory that the $\mathcal{N}=2$ Wess-Zumino model is ultraviolet finite and that the index is given by $p-1$ [74]. This seems to be in contradiction with our result, as the $(p-1)^N$ zero modes exist for all finite lattices and, by the same arguments as for the $\mathcal{N}=1$ model, remain zero modes in the continuum limit.

We suggest the following solution for this problem. Remember that our lattice Hamiltonian H contains not only the discretized version of the continuum Hamiltonian P_0 but also the central charge $\mathcal{Z}$, i.e.

$$H = P_0 + \mathcal{Z}. \tag{3.174}$$

Furthermore, both P_0 and $\mathcal{Z}$ contain the lattice derivative which couples the various lattice sites to each other. If we choose a zero mode in the strong-coupling limit which varies from lattice point to lattice point, both P_0 and $\mathcal{Z}$ may get very large but will, nevertheless, add up to zero. In the continuum limit the energy P_0 may tend to infinity in which case this rapidly varying zero mode is only a lattice artefact. On the other hand, if we choose the zero mode to be the same for each lattice site, P_0 as well as $\mathcal{Z}$ remain zero in the continuum limit. There are exactly $p-1$ such modes. We are planing to test this conjecture in a perturbative calculation of P_0 or $\mathcal{Z}$, and our results will be presented elsewhere.

4 Summary

Supersymmetric field theories on a spatial lattice result in high-dimensional quantum mechanical systems. Our lattice approach preserves part of supersymmetry such that we do not need fine-tuning of parameters to recover the full supersymmetry algebra in the continuum limit. We refrain from preserving the full algebra on the lattice, as this requires nonlocal interaction terms. These terms would make it difficult to use numerical simulations and to investigate these theories for example in the strong-coupling limit.

This thesis is divided into two parts. The first one begins with a preparatory step where we consider high-dimensional supersymmetric quantum mechanical systems (not necessarily related to lattice theories). Demanding the existence of $\mathcal{N}$ self-adjoint supercharges together with a self-adjoint grading operator Γ which anticommutes with all supercharges and thus commutes with the Hamiltonian, the Hilbert space splits into two sectors, a bosonic and a fermionic subspace. The first supercharge is chosen to be the Dirac operator on an even-dimensional manifold equipped with Riemannian metric and gauge field. The existence of further supercharges yields severe restrictions on both the manifold and the gauge field under consideration. For $\mathcal{N}=2$, we have shown that the manifold has to be Kähler while the field strength has to commute with the complex structure. $\mathcal{N}=3$ automatically implies $\mathcal{N}=4$ and restricts the manifold to be hyper-Kähler while the field strength has to commute with all three complex structures. In the minimal target space dimension having $\mathcal{N}=4$, $d=4$, this implies that the field strength has to be either selfdual or anti-selfdual. After this general considerations we have investigated the $\mathcal{N}=2$ case in more detail. We have proven the existence of a particle-number operator and of a superpotential which contains the gravitational as well as the gauge degree of freedom. One consequence of the existence of the particle-number operator is that the bosonic and fermionic subspaces further split into subspaces with definite particle-number. This is similar to differential forms where one has the coarse grading into even and odd forms in contrast to the finer grading into p-forms. The existence of the superpotential can be used to determine zero modes of the Dirac operator. We have developed a general procedure how to determine this superpotential and to obtain all zero modes in the sectors of maximal and minimal particle number. As an explicit example, we have considered the Dirac operator on complex projective spaces with Abelian gauge fields. With our new method we have constructed for this example the zero modes for the first time. We believe that all zero modes can be obtained in this way. Unfortunately, we could prove this conjecture only for low dimensions. Besides these interesting questions concerning the Dirac operator as a supercharge itself, we wanted to connect our results to well-known results in the literature. We have suceeded to relate the Dirac operator on a flat manifold with Abelian gauge field to matrix-Schrödinger Hamiltonians, which have been investigated by Nicolai, Witten and others before, by dimensional reduction. At this point we could not refrain from discussing a beautiful example of a matrix-Schrödinger Hamiltonian describing the supersymmetric hydrogen atom. We have determined the super-Laplace-Runge-Lenz vector and the spectrum of the super-Hamiltonian by group theoretical methods in the spirit of Pauli's algebraic approach. The first part of this thesis concluc des with a discussion of spontaneous supersymmetry breaking. We have proven a beautiful theorem, which was already known in the literature, which states the following: If a supercharge has only zero modes of one kind, say bosonic ones, zero modes remain zero modes under deformations of the supercharge in perturbation theory to all orders. It may happen that the deformation of the supercharge is non-analytic and perturbation theory is misleading. We have illustrated these facts by investigating some examples.

In the second part of the thesis, we have investigated Wess-Zumino models in two-dimensional Minkowski space on a spatial lattice. To set the stage, we have first discussed the Wess-Zumino model in the continuum in great detail. We have considered the $\mathcal{N}=1$ Wess-Zumino model for the target space $\mathbb{R}^d$. Furthermore, we have determined the various Noether charges and their algebra including central charges. In analogy with the investigations of the Dirac operator, we have clarified under which conditions the on-shell model allows for additional supersymmetries. The restrictions have been further analyzed for explicit examples. One particular solution for $\mathcal{N}=2$ yields a superpotential being a harmonic function of the scalar fields. Furthermore, we have shown that this model can be obtained by dimensional reduction of a $\mathcal{N}=1$ Wess-Zumino model in four-dimensional Minkowski space. Leaving continuum in favor of lattices, we have discussed $\mathcal{N}=1$ and $\mathcal{N}=2$ Wess-Zumino models in the Hamiltonian formulation on a spatial lattice. We first have observed that we have to include central charges in the Hamiltonian to preserve part of the supersymmetry algebra. Second, we have to choose a lattice derivative for our model. As the implications of the chosen lattice derivative for the lattice model are important, we have disussed various lattice derivatives and their advantages and disadvantages. There are the right- and left-derivatives which are local and do not have fermion doublers but break chiral symmetry. In order to better understand the dependency of the spectrum and doubling phenomena of the lattice derivative, we have interpolated between the left- and right-derivative. One important case is the antisymmetric lattice derivative. For the latter, chiral symmetry is not broken but we have fermion doubling. Finally, we have considered the SLAC derivative which is nonlocal and preserves chiral symmetry. Its spectrum coincides with the restriction of the spectrum of the derivative operator on a continuous interval to the first Brioullin zone. In quantum mechanical simulations this nonlocal derivative is of great use. Even though numerical simulations may last longer than with local lattice derivatives, the accuracy of the result increases substantially. Having discussed various lattice derivatives in great detail, we have turned back to the Wess-Zumino models on the lattice. We have determined the ground states of the massive free theories in the $\mathcal{N}=1$ and $\mathcal{N}=2$ case. For both theories there is exactly one zero mode. In the strong-coupling limit, the number of zero modes is different for the $\mathcal{N}=1$ and $\mathcal{N}=2$ model. For the $\mathcal{N}=1$ case, the number of zero modes is zero or one, depending on whether the polynomial degree of the superpotential is odd or even. In the appendix we have given the mathematical proof that the strong-coupling limit gives the correct result for the number of zero modes on finite lattices. This may drastically change in the continuum and infinite-volume limit. We have illustrated this for a specific example where supersymmetry is unbroken at strong coupling and broken at weak coupling. The situation for the $\mathcal{N}=2$ case is quite different. We have shown that in the strong-coupling limit there are $(p-1)^N$ zero modes, where p is the polynomial degree of the superpotential and N is the number of lattice points. Similar to the $\mathcal{N}=1$ case, we have proven in the appendix that again, the strong-coupling limit gives the correct number of zero modes on finite lattices. This is a rather surprising result, as in the literature it was shown by methods of constructive field theories that there are only $p-1$ zero modes in the continuum with finite volume. We have proposed a solution to this paradox. As mentioned before, the Hamiltonian on the lattice conists of the Hamiltonian of the continuum theory and central charges. We have conjectured that for zero modes which vary from lattice point to lattice point drastically, the energy of the continuum Hamiltonian as well as the central charges tend to infinity in the continuum limit and therefore are pure lattice artefacts. Only the zero modes which are constant in the strong-coupling limit, there are $p-1$ of them, survive the continuum limit. We want to test this conjecture by perturbative calculations in future.

One may ask whether one can generalize our considerations to higher dimensions and to other theories. A rather severe restriction is to find a subalgebra which closes on the Hamiltonian only. For example, the $\mathcal{N}=1$ superalgebra in four dimensions does not allow for such a subalgebra. But our approach could be used for the $\mathcal{N}=2$ super-Yang-Mills theory in four dimensions and one could attempt to extend the results obtained by Seiberg and Witten. Encouraged by the impressive results obtained in numerical simulations for quantum mechanical systems by using the SLAC derivative, we would further like to use the SLAC derivative in Monte-Carlo simulations of our lattice models. Furthermore, at present time we are interested in fermions on a lattice coupled to gauge fields formulated with the SLAC derivative. We are confident to comment on these points in future.

A Dirac Operator on a Ball

Possible mechanisms of spontaneous chiral symmetry breaking in QCD habe been discussed for many years. Instead of probing the different phases by a chiral symmetry breaking mass term and remove it after the infinite-volume limit, in [84] it was proposed to investigate the system in a finite box and to impose chiral symmetry breaking boundary conditions and then perform the infinite-volume limit.

Along these lines, we consider in this appendix the free Dirac operator on a $D = 2d$ dimensional ball $\mathcal{B}$ of radius R. The Dirac operator is given by

$$\mathrm{i}\nabla\!\!\!/ = \mathrm{i}\Gamma_A \frac{\partial}{\partial x_A}, \tag{A.1}$$

where we used the Einstein summation convention for the index $A = 1, \ldots, D$, and the Γ-matrices fulfill the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}. \tag{A.2}$$

We have to choose boundary conditions for the spinors Ψ, such that the Dirac operator is Hermitian. In the following we will consider chiral-bag boundary conditions, which are defined with the help of the boundary operator

$$\Pi = \frac{1}{2}(\mathbb{1} - \mathrm{i}\Gamma_* \mathrm{e}^{\Gamma_*\theta} S) \tag{A.3}$$

with free parameter $\theta \in \mathbb{R}$ and

$$\Gamma_* = (-\mathrm{i})^d \Gamma_1 \cdots \Gamma_D, \quad S = x_A \Gamma_A / r, \quad r = \sqrt{x_A x_A}. \tag{A.4}$$

We demand that

$$\Pi\Psi|_{\partial\mathcal{B}} = 0. \tag{A.5}$$

In contrast to Atiyah-Patodi-Singer boundary conditions, the chiral-bag boundary conditions are local. One can further show, that this boundary conditions do not allow for zero modes. More details can be found in [84]. Only recently, strong ellipticity of these boundary conditions has been shown [85].

First we observe that the Dirac operator and the boundary operator commute with total angular momentum

$$J_{AB} = L_{AB} + \Sigma_{AB}, \qquad A, B = 1, \ldots, D, \tag{A.6}$$

where

$$L_{AB} = -\mathrm{i}(x_A \frac{\partial}{\partial x_B} - x_B \frac{\partial}{\partial x_B}) \qquad \text{and} \qquad \Sigma_{AB} = \frac{1}{4\mathrm{i}}[\Gamma_A, \Gamma_B]. \tag{A.7}$$

Therefore, we first diagonalize total angular momentum, i.e. we determine the *spin spherical harmonics* in $D = 2d$ dimensions by group theoretical methods. In the following we construct the highest weight states: these states are eigenstates of the Cartan operators of the $\mathfrak{so}(D)$-algebra and are annihilated by the raising operators. All other states can be obtained by applying lowering operators on the highest weight states.

We choose a Cartan-Weyl basis for the $\mathfrak{so}(D)$-algebra. The Cartan operators, raising and lowering operators corresponding to simple positive roots in the complex basis

$$z_a = x_{2a-1} + \mathrm{i}x_{2a}, \quad \partial_a = \frac{1}{2}(\frac{\partial}{\partial x_{2a-1}} - \mathrm{i}\frac{\partial}{\partial x_{2a}}), \quad \psi_a = \frac{1}{2}(\Gamma_{2a-1} - \mathrm{i}\Gamma_{2i}), \quad a = 1, \ldots, d, \tag{A.8}$$

read

$$\begin{aligned} H_a &= z_a\partial_a - \bar{z}_a\bar{\partial}_a + \frac{1}{2}(\psi_a^\dagger\psi_a - \psi_a\psi_a^\dagger), \quad a = 1, \ldots, d \\ E_a &= -\mathrm{i}(z_a\partial_{a+1} - \bar{z}_{a+1}\bar{\partial}_a + \psi_a^\dagger\psi_{a+1}), \quad a = 1, \ldots, d-1 \\ E_d &= -\mathrm{i}(z_{d-1}\bar{\partial}_d - z_d\bar{\partial}_{d-1} + \psi_{d-1}^\dagger\psi_d^\dagger). \end{aligned} \tag{A.9}$$

We define the state $|0\rangle$ by demanding, that it is annihilated by all operators ψ_a. As the generators in (A.9) act trivially on the radial part of spinor wavefunctions, we consider in the following the angular part. For that we introduce the unit normal vectors

$$\hat{x}_A = x_A/r, \quad \hat{z}_a = z_A/r. \tag{A.10}$$

Let us now first determine the highest weight states for the bosonic part. They are given by

$$\phi_l = (\hat{z}_1)^l. \tag{A.11}$$

One easily verifies that these states are annihilated by all simple positive roots and the eigenvalues with respect to the Cartans are given by $(H_1, \ldots, H_d) = (l, 0, \ldots, 0)$.

For the fermionic part there are only two highest weight states given by

$$\chi^+ = \psi_1^\dagger \cdots \psi_d^\dagger |0\rangle \quad \text{and} \quad \chi^- = \psi_1^\dagger \cdots \psi_{d-1}^\dagger |0\rangle. \tag{A.12}$$

The corresponding eigenvalues of the Cartan operators are given by $(\frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$, respectively.

Next, we determine the highest weight states of fermionic and bosonic degrees of freedom together. Two highest weight states are found easily, they are given by

$$\phi_l^+ = \phi_l \chi^+ \quad \text{and} \quad \phi_l^- = \phi_l \chi^- \tag{A.13}$$

with eigenvalues of the Cartan operators $(l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$, respectively. Furthermore, observe that the operator

$$S = \hat{x}_A \Gamma_A = (\psi_a^\dagger \bar{z}_a + \psi_a z_a)/r = S^\dagger, \qquad S^2 = \mathbb{1}, \tag{A.14}$$

commutes with the total angular momentum and therefore maps highest weight states into highest weight states. We obtain two further highest weight states

$$\psi_l^+ = S\phi_l^+ \quad \text{and} \quad \psi_l^- = S\phi_l^- \tag{A.15}$$

with eigenvalues $(l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$, and we will prove that these, together with the states in (A.13), are all highest weight states. We claim that the following tensor product rule with the corresponding highest weight states holds.

Tensor product rule: The tensor product of the bosonic representation $(l, 0, \ldots, 0)$ with the fermionic representation $(\frac{1}{2}, \ldots, \frac{1}{2})$ is given by

$$\begin{array}{ccccccc} (l, 0, \ldots, 0) & \otimes & (\frac{1}{2}, \ldots, \frac{1}{2}) & = & (l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) & \oplus & (l - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \\ \phi_l & \otimes & \chi^+ & \longrightarrow & \phi_l^+ & \oplus & \psi_{l-1}^- \end{array} \tag{A.16}$$

and the tensor product of the bosonic representation $(l, 0, \ldots, 0)$ with the fermionic representation $(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$ is given by

$$\begin{array}{ccccccc} (l, 0, \ldots, 0) & \otimes & (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) & = & (l + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) & \oplus & (l - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \\ \phi_l & \otimes & \chi^- & \longrightarrow & \phi_l^- & \oplus & \psi_{l-1}^+. \end{array} \tag{A.17}$$

We proved already, that the given states are indeed highest weight states. That they appear at the corresponding places is determined by the degree of the polynomials in x_A and the chirality of the corresponding states. By counting the dimensions of the representations we show that there can't be further representations in the tensor product rule.

Using Weyl's dimension formula for the D_d groups

$$\dim(\ell_1, \ldots, \ell_d) = \prod_{1 \le r < s \le d} \frac{\ell_r + \ell_s + 2d - r - s}{2d - r - s} \frac{\ell_r - \ell_s + s - r}{s - r}, \tag{A.18}$$

we obtain

$$\begin{aligned} \dim(l + \tfrac{1}{2}, \tfrac{1}{2}, \ldots \tfrac{1}{2}, \pm\tfrac{1}{2}) &= 2^{d-1} \binom{l + 2d - 2}{l}, \quad \text{and} \\ \dim(l, 0, \ldots, 0) &= \binom{l + 2d - 1}{l} - \binom{l + 2d - 3}{l - 2}. \end{aligned} \tag{A.19}$$

One can easily check – using the properties of binomial coefficients – that

$$\dim(l,0,\dots,0)\times\dim(\frac{1}{2},\dots\frac{1}{2},\pm\frac{1}{2})=\dim(l+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2}) + \dim(l-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2}). \tag{A.20}$$

Let us summarize the first main result of this appendix: we determined four types of spin spherical harmonics denoted by $\phi_l^\pm$ and $\psi_l^\pm$.

Next, we investigate the chiral-bag boundary conditions. We may express Γ_* with the help of (A.8) by

$$\Gamma_*=\prod_{a=1}^{d}(\psi_a^\dagger\psi_a-\psi_a\psi_a^\dagger). \tag{A.21}$$

As ϕ_l^+ and ψ_l^+ (and likewise ϕ_l^- and ψ_l^-) have the same eigenvalues, with respect to the Cartan operators H_a, we allow for linear combinations

$$\Psi_l^\pm=f_l^\pm(r)\phi_l^\pm+g_l^\pm(r)\psi_l^\pm. \tag{A.22}$$

Imposing the boundary condition $\Pi\Psi_l^\pm=0$ leads to the following equations for the values of the radial functions at the boundary,

$$f_l^\pm(R)\mp\mathrm{i}e^{\theta}g_l^\pm(R)=0. \tag{A.23}$$

We obtained this simple result, because total angular momentum commutes with the boundary operator, and we have chosen an adapted frame.

Finally, we want to solve for the spectrum of the Dirac operator, which reads in complex coordinates

$$\mathrm{i}\nabla\!\!\!/ =2\mathrm{i}\psi_a\bar\partial_a+2\mathrm{i}\psi_a^\dagger\partial_a. \tag{A.24}$$

The eigenvalue equation is given by

$$\mathrm{i}\nabla\!\!\!/\,\Psi_l^\pm=\lambda\Psi_l^\pm. \tag{A.25}$$

One obtains a system of coupled first-order differential equations, which can be easily solved,

$$
\begin{aligned}
f_l^{\pm}(r) &= r^{1-d-l}(c_1 J_{-1+d+l}(|\lambda| r) + c_2 N_{-1+d+l}(|\lambda| r)) \\
g_l^{\pm}(r) &= \mathrm{sign}(\lambda) \mathrm{i} r^{1-d-l}(c_1 J_{d+l}(|\lambda| r) + c_2 N_{d+l}(|\lambda| r)).
\end{aligned} \tag{A.26}
$$

Finally, we impose the boundary condition (A.23). With $k \equiv |\lambda| R$ we obtain

$$
\begin{aligned}
J_{-1+d+l}(k) + \mathrm{sign}(\lambda)\mathrm{e}^{\theta} J_{d+l}(k) = 0 \quad \text{for} \ + \ \text{case} \\
J_{-1+d+l}(k) - \mathrm{sign}(\lambda)\mathrm{e}^{-\theta} J_{d+l}(k) = 0 \quad \text{for} \ - \ \text{case}.
\end{aligned} \tag{A.27}
$$

The degeneracy of the eigenvalues can be determined by Weyl's dimension formula. For each l in $D = 2d$ dimensions and for each case in eq. (A.27) the degeneracy is given by

$$
\dim = 2^{d-1} \begin{pmatrix} 2d + l - 2 \\ l \end{pmatrix}. \tag{A.28}
$$

In this appendix, we have determined the spin spherical harmonics in even dimensions and have calculated the eigenfunctions of the free Dirac equation on a ball, imposing chiral-bag boundary conditions. The method is similar to the method developed in [JDL1]. We use these results in [JDL5] to determine the spectral asymmetry and compare it with the invariants of an associated boundary operator.

B No-go Theorem

In this appendix we prove that, as stated already in [26], the $\mathcal{N} = 1$ superalgebra in four spacetime dimensions does not allow for a subalgebra closing on H only. In the following we use a Majorana representation in four dimensions, such that the Majorana spinor components Q_α, $\alpha = 1, \ldots, 4$, are Hermitian operators. The superalgebra without central charges reads

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^\mu \gamma^0)_{\alpha\beta} P_\mu, \tag{B.1}$$

where P_μ is the four-momentum and $P_0 = H$. We try to find a linear combination of supercharges, $Q = \sum_\alpha b_\alpha Q_\alpha$, $b_\alpha \in \mathbb{R}$, such that

$$\{Q, Q\} = 2H. \tag{B.2}$$

Eqn. (B.2) is equivalent to

$$b_\alpha b_\alpha = 1, \qquad (\gamma^i \gamma^0)_{\alpha\beta} b_\alpha b_\beta = 0, \quad i = 1, 2, 3. \tag{B.3}$$

We are free to choose a particular realization of the γ-matrices. With the realization

$$\gamma^0 = \sigma_0 \otimes \sigma_2, \quad \gamma^1 = \mathrm{i}\sigma_0 \otimes \sigma_3, \quad \gamma^2 = \mathrm{i}\sigma_1 \otimes \sigma_1 \quad \text{and} \quad \gamma^3 = \mathrm{i}\sigma_3 \otimes \sigma_1, \tag{B.4}$$

we obtain the conditions

$$\begin{aligned} \sum_\alpha b_\alpha b_\alpha &= 1, \\ b_1 b_2 + b_3 b_4 &= 0, \\ -b_1 b_3 + b_2 b_4 &= 0, \\ ({b_2}^2 + {b_3}^2) - ({b_1}^2 + {b_4}^2) &= 0. \end{aligned} \tag{B.5}$$

Let us define the two real vectors $u = (b_1, b_4)^{\mathrm{T}}$ and $v = (b_2, b_3)^{\mathrm{T}}$, such that we can rewrite these conditions as

$$\|u\|^2 + \|v\|^2 = 1, \quad \|u\| = \|v\|, \quad (u, v) = 0, \quad (u, Mv) = 0, \quad M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.6}$$

Here we introduced the standard scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$ on $\mathbb{R}^2$. Observe that M is a rotation matrix with angle of rotation equal to $\frac{\pi}{2}$. That means that u has to be both, orthogonal and parallel to v and therefore has to be zero. As $\|u\| = \|v\|$ also v has to be zero and we get no solution.

C Analyticity of Perturbations

In the main part of this thesis we need some rigorous analyticity properties of specific perturbations. We need several theorems in the context of functional analysis which can be found in the literature. Let us mention here the series of books by M. Reed and B. Simon [86] and the book of H. Triebel [87]. For topics concerning perturbation theory the standard reference is Kato [88].

In the following we consider operators on the Hilbert space

$$\mathcal{H} = \mathrm{L}_2(\mathbb{R}^d, \mathrm{d}^d x) \otimes \mathbb{C}^D \tag{C.1}$$

with $D \in \mathbb{N}$ arbitrary and norm

$$\|f\|^2 = \sum_{i=1}^{D} \|f_i\|_{\mathrm{L}_2}^2, \quad f = (f_1, \dots, f_D) \in \mathcal{H}. \tag{C.2}$$

Here, $\|\cdot\|_{\mathrm{L}_2}$ denotes the familiar L_2-norm.

C.1 The $\mathcal{N} = 1$ Case

For the $\mathcal{N} = 1$ Wess-Zumino model on the lattice we specify $D = 2^N$, $d = N$ (number of lattice points) and consider the (unperturbed) operator (3.150)

$$A_0 = \sum_{m=1}^{N} \left(-\mathrm{i}\psi_1(m)\partial_m + \psi_2(m)W'(x_m)\right). \tag{C.3}$$

We recall that W is a polynomial of degree $\deg(W) = p > 1$ and $\psi_\alpha(m)$ are Hermitian $D \times D-$matrices obeying the Clifford algebra

$$\{\psi_\alpha(m), \psi_\beta(n)\} = 2\delta_{\alpha\beta}\delta(m,n), \quad m,n = 1,\ldots,N \quad \text{and} \quad \alpha,\beta = 1,2. \tag{C.4}$$

The operator A_0 with domain of definition

$$\mathcal{D}(A_0) = C_{\mathrm{c}}^\infty(\mathbb{R}^N) \otimes \mathbb{C}^D \tag{C.5}$$

is essentially self-adjoint, where $C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ is the space of C^∞-functions with compact support in $\mathbb{R}^N$. A simple calculation, using (C.4), shows

$$(A_0)^2 = \sum_m \left(-\partial_m\partial_m + W'(x_m)W'(x_m) - \mathrm{i}\psi_1(m)\psi_2(m)W''(x_m)\right). \tag{C.6}$$

Closure of the Operator A_0

To determine the closure $\bar{A}_0$ of the operator A_0, we have to find the closure of its domain $\mathcal{D}(A_0)$ with respect to the norm

$$\|f\|^2_{A_0,a} \equiv a\|f\|^2 + \|A_0 f\|^2, a > 0. \tag{C.7}$$

Note that these norms are equivalent for all $a > 0$. Using the abbreviation

$$\rho_p = 1 + |x|^{p-1}, \qquad |x| = \sqrt{\sum_m (x_m)^2}, \tag{C.8}$$

we can prove the following

Lemma: There exist constants $a, b_1, b_2 > 0$ such that

$$\|f'\|^2 + b_1\|\rho_p f\|^2 \leq a\|f\|^2 + \|A_0 f\|^2 \leq \|f'\|^2 + b_2\|\rho_p f\|^2 \tag{C.9}$$

holds for all $f \in \mathcal{D}(A_0)$.

In the Lemma we used the short hand notation $\|f'\|^2 = \sum_m \|\partial_m f\|^2$.

Proof: First, we show that actually only the degree $\deg(W)=p$ is important. We find

$$\sum_m \|W'(x_m)f\|^2 \leq Na_1\|\rho_p f\|^2, \quad a_1 = \left\|\frac{W'(x_m)}{\rho_p}\right\|_\infty^2, \tag{C.10}$$

where $\|\cdot\|_\infty$ denotes the supremum norm. The factor N arises from the sum over m, as a_1 does not depend on x_m. Similarly, we obtain

$$\begin{aligned}\|\rho_p f\|^2 &= \left\|\frac{\rho_p}{\sqrt{1+\sum_n (W'(x_n))^2}}\sqrt{1+\textstyle\sum_m (W'(x_m))^2}\; f\right\|^2 \\ &\leq a_2(\|f\|^2 + \sum_m \|W'(x_m)f\|^2), \quad a_2 = \left\|\frac{\rho_p}{\sqrt{1+\sum_n (W'(x_n))^2}}\right\|_\infty^2 .\end{aligned} \tag{C.11}$$

Now, it is easy to prove the second inequality in (C.9),

$$\begin{aligned} a\|f\|^2 + \|A_0 f\|^2 &\overset{(C.10)}{\leq} \|f'\|^2 + a\|f\|^2 + Na_1\|\rho_p f\|^2 + \sum_m \|f\|\,\|W''(x_m)f\| \\ &\leq \|f'\|^2 + \underbrace{(a+Na_1+Na_3)}_{b_2}\|\rho_p f\|^2, \end{aligned} \tag{C.12}$$

with $a_3 = \left\|\frac{W''(x_m)}{\rho_p}\right\|_\infty$. We used that the matrix-norm of the matrices $\psi_\alpha(m)$ is one, as the eigenvalues of these matrices are ± 1. In the last inequality we made use of $\|f\| \leq \|\rho_p f\|$ which holds for all $f \in \mathcal{D}(A_0)$.

The other inequality in (C.9) is more difficult to prove. With (C.11) we get

$$a\|f\|^2 + \|A_0 f\|^2 \geq \|f'\|^2 + \frac{1}{a_2}\|\rho_p f\|^2 + (a-1)\|f\|^2 - \sum_m \|f\|\,\|W''(x_m)f\|. \tag{C.13}$$

We have to be careful with estimates for the last term in (C.13) as we want to obtain a positive constant b_1 in the Lemma. We introduce a ball of radius R and split $f \in \mathcal{D}(A_0)$ into two parts, $f = f_< + f_>$, where $f_<$ has its support inside the ball and $f_>$ outside the ball. We obtain

$$\sum_m \|f\|\,\|W''(x_m)f\| = \sum_m \left(\|f_<\|\,\|W''(x_m)f_<\| + \|f_>\|\,\|W''(x_m)f_>\|\right), \tag{C.14}$$

where the terms which contain both, $f_<$ and $f_>$, vanishes. Let us now consider the two terms in (C.14) separately. First, we obtain

$$\sum_m \|f_>\| \, \|W''(x_m) f_>\| \leq N a_4(R) \|\rho_p f\|^2, \quad a_4(R) = \left\| \frac{W''(x_m)}{\rho_p} \right\|_{\infty,>}, \tag{C.15}$$

where we have introduced the supremum norm $\|\cdot\|_{\infty,>} = \sup_{|x|>R}\{|\cdot|\}$. For large R we have $a_4(R) \sim \frac{1}{R}$ such that a_4 gets arbitrarily small in that limit. Second, we obtain

$$\sum_m \|f_<\| \, \|W''(x_m) f_<\| \leq N a_5(R) \|f\|^2, \quad a_5(R) = \|W''(x_m)\|_{\infty,<}, \tag{C.16}$$

with $\|\cdot\|_{\infty,<} = \sup_{|x|<R}\{|\cdot|\}$. For $R \to \infty$ we have $a_5(R) \to \infty$. Altogether, we find

$$a\|f\|^2 + \|A_0 f\|^2 \geq \|f'\|^2 + (a - 1 - N a_5(R))\|f\|^2 + \underbrace{(1/a_2 - N a_4(R))}_{b_1} \|\rho_p f\|^2. \tag{C.17}$$

In the first step we must choose R large enough such that $b_1 > 0$. In the second step we must choose a large such that the constant in front of $\|f\|^2$ is also positive. This finishes our proof. ■

Since all norms

$$\|f\|_b^2 \equiv \|f'\|^2 + b\|\rho_p f\|^2 \tag{C.18}$$

are equivalent for $b > 0$, the Lemma implies that these norms are eqivalent to the norms $\|f\|^2_{A_0,a}$. Therefore, the closure of $\mathcal{D}(A_0)$ with respect to the norm $\|\cdot\|_{A_0,a}$ coincides with the closure with respect to $\|\cdot\|_b$, which is given by

$$\mathcal{D}(\bar{A}_0) = \left\{ f \in W_2^1(\mathbb{R}^N) \otimes \mathbb{C}^D : \|\rho_p f\| < \infty \right\} \equiv W_2^1(\mathbb{R}^N, \rho_p^2) \otimes \mathbb{C}^D. \tag{C.19}$$

Here, $W_2^1(\mathbb{R}^N)$ is the Sobolev space with first weak-derivative in L_2.

Perturbation

In the main part of this thesis, the operator A_0 is perturbed by the operator A_1 (3.150),

$$A_1 = -\sum_{m,n=1}^{N} x_m (\partial)_{mn} \psi_2(n). \tag{C.20}$$

The operator A_1 is obviously self-adjoint and $\mathcal{D}(A_1) = \mathrm{L}_2(\mathbb{R}^N, \tilde{\rho}) \otimes \mathbb{C}^D \supset \mathcal{D}(\bar{A}_0)$ with $\tilde{\rho}$-weighted Lebesgue measure, $\tilde{\rho}(x) = (1 + |x|)^2$. From the following Lemma we will obtain many consequences for the quality of the perturbation.

Lemma: For all $\lambda \in \mathbb{R}$ and arbitrarily small $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\|\lambda A_1 f\| \leq \epsilon \|A_0 f\| + C_\epsilon \|f\|, \quad \forall f \in \mathcal{D}(\bar{A}_0) \tag{C.21}$$

holds.

Proof: We prove the inequality for all $f \in \mathcal{D}(A_0)$ and it follows that it also holds for all elements in the closure. As before we split $f = f_< + f_>$. First, we find

$$\|\lambda A_1 f_<\| \leq |\lambda| N^2 a(R) \|f\|, \quad a(R) = \|x_m\|_{\infty,<} \cdot \max\{|\partial_{mn}| : m, n = 1, \ldots, N\}. \tag{C.22}$$

For $R \to \infty$, $a(R) \to \infty$. Next, we have

$$\begin{aligned} \|\lambda A_1 f_>\| &\overset{(C.9)}{\leq} |\lambda| N^2 b(R)(c\|f\| + \|A_0 f\|), \\ b(R) &= \left\| \frac{x_m}{\rho_p} \right\|_{\infty,>} \cdot \max\{|\partial_{mn}| : m, n = 1, \ldots, N\} \end{aligned} \tag{C.23}$$

for some number $c > 0$. For big R the constant $b(R)$ tends to zero. We choose the radius of the ball large enough such that $|\lambda| N^2 b(R) = \epsilon$ and set $C_\epsilon = \epsilon c + |\lambda| N^2 a(R)$. ■

Note that the latter constant may become huge.

Self-adjointness

We proved already that $\bar{A}_0$ is a self-adjoint operator. Clearly, λA_1 is symmetric on $\mathcal{D}(\bar{A}_0)$. Furthermore, (C.21) shows that λA_1 is $\bar{A}_0$-bounded with bound less than one. The famous Kato-Rellich theorem, see Theorem X.12 in [86], states that under these conditions the operator

$$Q_1(\lambda) = A_0 + \lambda A_1 \tag{C.24}$$

is self-adjoint with domain $\mathcal{D}(Q_1(\lambda)) = \mathcal{D}(\bar{A_0})$. We conclude that $Q_1(\lambda)$ is a family of self-adjoint operators with common domain of definition $\mathcal{D}(\bar{A}_0)$.

Analyticity of Eigenvalues

In the following we prove that $Q_1(\lambda)$ is an analytic family in the sense of Kato for all $\lambda \in \mathbb{R}$. We have shown already that for real λ, $Q_1(\lambda)$ is self-adjoint. For a self-adjoint and analytic family it is known that the eigenvalues depend analytically on the parameter λ, see for example Theorem XII.13 in [86].

For an arbitrary real λ_0, $\lambda_0 A_1$ is $\bar{A}_0$-bounded with arbitrary small bound (C.21). Then, it is easy to see that A_1 is $Q_1(\lambda_0)$-bounded. From this fact it follows that for small ϵ, $Q_1(\lambda_0 + \epsilon)$ is an analytic family of type (A) [86] and therefore also an analytic family in the sense of Kato. But as $\lambda_0 \in \mathbb{R}$ is arbitrary, we haven proven that $Q_1(\lambda)$ is analytic for all real λ.

Actually, the cited Theorem XII.13 [86] above is only valid for isolated eigenvalues with finite degeneracy or equivalently for eigenvalues in the discrete spectrum. In the following we prove that the spectrum of $Q_1(\lambda)$ consists only of the discrete spectrum by proving this statement for the square, $H(\lambda) = Q_1(\lambda)^2$. $H(\lambda)$ is self-adjoint with domain of definition given by

$$\begin{aligned} \mathcal{D}(H(\lambda)) &\equiv \{f \in \mathcal{D}(\bar{A}_0) : Q_1(\lambda) f \in \mathcal{D}(\bar{A}_0)\} \\ &= W_2^2(\mathbb{R}^N, \rho') \otimes \mathbb{C}^D, \quad \rho'(x) = (1 + |x|^{2(p-1)})^2 \end{aligned} \tag{C.25}$$

and it is semibounded

$$H(\lambda) \geq 0. \tag{C.26}$$

Such operators possess entirely discrete spectrum if and only if its resolvent is a compact operator, see Theorem XIII.64 in [86]. In the following we prove that $H(\lambda)$ has compact resolvent for all $\lambda \in \mathbb{R}$.

We have to show that the image of a bounded subset of the Hilbert space, say

$$\{f \in \mathcal{H} : \|f\| < 1, \} \tag{C.27}$$

is mapped to a precompact set in Hilbert space under the map $(H - z)^{-1}$ for some z in the resolvent of H. The image is given by

$$\{g \in \mathcal{D}(H) : \|(H - z)g\| < 1\}. \tag{C.28}$$

We split g as before into $g_>$ and $g_<$ and obtain for radius R large enough, $\|g\| \leq \|g_<\| + \epsilon$. For a compact set $\mathcal{B} = \{x \in \mathbb{R}^N : |x| \leq R\}$ we have Sobolev's embedding theorem and there is an ϵ-net $g_j \in W_2^2(\mathcal{B})$, $j = 1, \ldots, N_\epsilon$ with $\|g_< - g_j\| < \epsilon$ for one $j \in \{1, \ldots, N_\epsilon\}$. We extend the g_j by zero to the region outside the ball and obtain

$$\|g - g_j\| \leq 2\epsilon \tag{C.29}$$

for any g in the image of the unit ball under $(H - z)^{-1}$ and a specific $j \in \{1, \ldots, N_\epsilon\}$. We conclude that there is a 2ϵ-net of the image and therefore the image is precompact. This completes our proof.

Stability of the Index

We have shown that the eigenvalues are analytic functions of the parameter λ on the whole real axis. It follows at once that the index (2.5) – the difference of bosonic and fermionic zero modes – is also an analytic function and, as the index only takes on integer values, is constant.

An alternative, elegant proof of this statement can be given with the help of the theorem that a relatively compact perturbation does not change the index [88]. Indeed, inequaltiy (C.21) implies that out perturbation is relatively compact.[1]

C.2 The $\mathcal{N} = 2$ Case

As we have discussed the $\mathcal{N} = 1$ case in great details, we keep the discussion for $\mathcal{N} = 2$ short. We specify $d = 2N$, $D = 2^{2N}$ and consider B_0 defined in (3.156). We choose the

[1] We thank H. Triebel for the proof of this statement.

following domain of definition

$$\mathcal{D}(B_0) = C_c^\infty(\mathbb{R}^{2N}) \otimes \mathbb{C}^D \tag{C.30}$$

such that B_0 is an essentially self-adjoint operator. We introduce the potential

$$K(x,y) = \sqrt{\sum_a (W_{,a}(x,y))^2}. \tag{C.31}$$

For large radii only the leading power of W is relevant. Therefore, we may consider the particular case

$$W(x,y) = \kappa/p\, \Re z^p \tag{C.32}$$

for which we obtain $K(x,y) = \kappa r^{p-1} \to \infty$ in all directions for $r \to \infty$.

The perturbation contains the lattice derivative,

$$B_1 = \sum_{m,n} \left(x_m(\partial)_{mn} \psi_2^2(n) + y_m(\partial^\dagger)_{mn} \psi_2^1(n) \right). \tag{C.33}$$

Replacing in the estimates of the case $\mathcal{N}=1$ supersymmetry the potential $W'(x_m)$ by $K(x_m, y_m)$ leads to analogous results in the $\mathcal{N}=2$ case. Again, all eigenvalues are analytic functions of the parameter λ, and in particular the index does not depend on this parameter.

Own Publications

[JDL1] A. Kirchberg, J.D. Länge, P.A.G. Pisani and A. Wipf, "Algebraic Solution of the Supersymmetric Hydrogen Atom in d Dimensions", Annals Phys. **303**, 359 (2003) hep-th/0208228.

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[JDL3] A. Kirchberg, J.D. Länge, and A. Wipf, "Extended Supersymmetries and the Dirac Operator", (2004) hep-th/0401134, accepted for publication in *Annals of Physics*.

[JDL4] A. Kirchberg, J.D. Länge, and A. Wipf, "From the Dirac Operator to Wess-Zumino Models on Spatial Lattices", (2004) hep-th/0407207, accepted for publication in *Annals of Physics*.

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Zusammenfassung

Betrachtet man supersymmetrische Feldtheorien auf einem räumlichen Gitter, erhält man hochdimensionale quantenmechanische Systeme. In unserem Zugang zur Gittertheorie behalten wir nur einen Teil der Supersymmetrie auf dem Gitter bei, so dass wir ohne Feinabstimmung der Parameter im Kontinuumslimes die volle Supersymmetriealgebra wiederbekommen. Wir versuchen nicht die ganze Algebra auf dem Gitter zu realisieren, da dies nichtlokale Wechselwirkungsterme erfordert. Diese würden numerische Simulationen und Untersuchungsmethoden, wie zum Beispiel den starken Kopplungslims, sehr erschweren.

Die Arbeit ist in zwei Hauptteile aufgeteilt. Im ersten Teil haben wir als Vorbereitung hochdimensionale, supersymmetrische, quantenmechanische Modelle untersucht. Diese Modelle müssen nicht unbedingt mit einer Gitterfeldtheorie verknüpft sein. Mit der Forderung von $\mathcal{N}$ selbstadjungierten Superladungen und eines selbstadjungierten Graduierungsoperators Γ, welcher mit den Superladungen antikommutiert und damit mit dem Hamiltonoperator kommutiert, zerfällt der Hilbertraum in zwei Teile, in den bosonischen und fermionischen Unterraum. Für die erste Superladung haben wir den Diracoperator auf einer geraddimensionalen Mannigfaltigkeit mit Riemannmetrik und Eichfeld gewählt. Die Forderung der Existenz von weiteren Superladungen ergibt starke Einschränkungen, sowohl an die Mannigfaltigkeit als auch an das Eichfeld. Wir haben gezeigt, dass für $\mathcal{N}=2$ die Mannigfaltigkeit eine Kählermannigfaltigeit sein muss und dass die Feldstärke mit der komplexen Struktur kommutieren muss. $\mathcal{N}=3$ impliziert automatisch $\mathcal{N}=4$ und existiert nur für Hyper-Kähler Mannigfaltigkeiten und für Feldstärken, die mit allen drei komplexen Strukturen kommutieren. In der kleinsten Dimension des Zielraumes, $d=4$, ist dies gleichbedeutend damit, dass die Feldstärke selbst- oder antiselbstdual sein muss. Nach dieser allgemeinen Diskussion haben wir uns dem Spezialfall $\mathcal{N}=2$ zugewendet. Wir haben die Existenz eines Teilchenzahlopera-

tors und eines Superpotentials gezeigt, welches sowohl die gravitativen Freiheitsgrade als auch die Eichfreiheitsgrade enthält. Die Existenz des Teilchenzahloperators hat als Konsequenz, dass der bosonische und auch der fermionische Unterraum in weitere Sektoren zerfallen. Dies ist ähnich zu Differentialformen auf einer Mannigfaltigkeit, bei welchen man die grobe Einteilung in geraden und ungeraden Formen, aber auch die feinere Einteilung in p-Formen hat. Die Existenz des Superpotentials erlaubt es, Nullmoden des Diracoperators zu konstruieren. Wir haben uns eine allgemeine Prozedur zur Bestimmung des Superpotentials überlegt und haben gezeigt, wie dieses zur Konstruktion von Nullmoden im maximalen und minimalen Teilchensektor verwendet werden kann. Als explizites Beispiel haben wir den Diracoperator auf komplexen projektiven Räumen mit abelschen Eichfeldern betrachtet. Mit unserer Methode konnten wir zum ersten mal Nullmoden für dieses Problem explizit bestimmen. Wir glauben, dass wir alle Nullmoden für dieses Beispiel bestimmt haben, konnten dies allerdings nur für niedrige Dimensionen beweisen. Neben diesen interessanten Untersuchungen für den Diracoperator als Superladung, waren wir daran interessiert, den Diracoperator mit bekannten Resultaten aus der Literatur in Verbindung zu bringen. Wir haben es dabei geschafft, den Diracoperator auf einem flachen Raum mit abelschen Eichfeld mittels der Methode der dimensionalen Reduktion mit Matrix-Schrödinger Hamiltonoperatoren in Zusammenhang zu bringen. An dieser Stelle haben wir uns einem hochinteressantem Problem zugewendet, dem supersymmetischen Wasserstoffatom. Wir haben den verallgemeinerten Laplace-Runge-Lenz Vektor und das Spektrum des Hamiltonoperators mit Hilfe von gruppentheoretischen Methoden im Sinne von Paulis Zugang bestimmt. Den ersten Teil dieser Arbeit haben wir mit der Betrachtung von spontaner Supersymmetriebrechung in quantenmechanischen Modellen abgeschlossen. Wir haben ein sehr schönes Theorem, das in der Literatur schon bekannt war, bewiesen. Dieses besagt, dass wenn entweder bosonische oder fermionische Nullmoden einer selbstadjungierten Superladung existieren, diese Nullmoden dann – unter einer Störung – Nullmoden in der Störungstheorie bleiben. Ob die Störung wirklich eine analytische Störung ist oder nicht, muss von Fall zu Fall untersucht werden.

Im zweiten Teil dieser Arbeit haben wir zweidimensionale Wess-Zumino Modelle auf dem Gitter betrachtet. Um damit beginnen zu können, haben wir zuerst ausführlich Wess-Zumino Modelle im Kontinuum studiert. Für das $\mathcal{N}=1$ Wess-Zumino Modell haben wir die Noetherladungen und deren Algebra mit zentralen Ladungen berechnet. Ähnlich wie beim Diracoperator haben wir untersucht, unter welchen Bedingungen das Wess-

Zumino Modell erweiterte Supersymmetrie erlaubt. Die Einschränkungen haben wir an Beispielen untersucht. Insbesondere für $\mathcal{N}=2$ gibt es eine Lösung mit harmonischen Superpotential. Dieses Modell haben wir auch mittels dimensionaler Reduktion eines $\mathcal{N}=1$ Wess-Zumino Modells in vier Dimensionen erhalten. Von nun an haben wir uns den Gittertheorien zugewendet. Wir haben das $\mathcal{N}=1$ und $\mathcal{N}=2$ Wess-Zumino Modell auf das Gitter gesetzt. Zunächst wurden wir dazu veranlasst, die zentrale Ladung in der Definiton des Hamiltonoperators aufzunehmen, um überhaupt eine supersymmetrische Theorie zu erhalten. Weiterhin muss eine Gitterableitung gewählt werden. Daher haben wir Vor- und Nachteile verschiedener Gitterableitungen untersucht. Zu nennen sind hierbei die Links- und Rechtsgitterableitung, welche lokal sind und das Problem der Fermionverdopplung nicht haben, allerdings die chirale Symmetrie explizit brechen. Um die Abhängigkeit des Spektrums und der Fermionverdopplung besser zu verstehen, haben wir zwischen der Rechts- und Linkgitterableitung interpoliert. Ein wichtiger Spezialfall ist die antisymmetrische Gitterableitung, für welche die chirale Symmetrie nicht gebrochen ist, dafür das Fermionverdopplungsproblem existiert. Abschließend haben wir die SLAC-Gitterableitung untersucht. Dies ist eine nichtlokale Gitterableitung, welche chiral invariant ist und deren Spektrum mit der Ableitung auf einem koninuierlichen Intervall, auf die erste Brillouinzone eingeschränkt, übereinstimmt. In quantenmechanischen Modellen haben wir diese Gitterableitung getestet. Obwohl die Zeit für numerische Simulationen größer ist als für lokale Gitterableitungen, ist die Genauigkeit der Ergebnisse mit Hilfe der SLAC-Gitterableitung überzeugend. Nachdem wir die verschiedensten Gitterableitungen ausführlichst diskutiert haben, haben wir uns wieder den Wess-Zumino Modellen auf dem Gitter zugewendet. Wir haben die Grundzustände sowohl für das massive, freie $\mathcal{N}=1$ Modell als auch für das entsprechende $\mathcal{N}=2$ Modell bestimmt. Für beide existiert genau eine Nullmode. Die Anzahl der Nullmoden im starken Kopplungslimes ist allerdings verschieden. Für den $\mathcal{N}=1$ Fall existiert kein oder eine Nullmode, abhängig davon ob der polynomiale Grad des Superpotential ungerade oder gerade ist. Im Anhang haben wir bewiesen, dass der starke Kopplungslimes das korrekte Resultat auf dem endlichen Gitter liefert. Dies kann sich allerdings im Kontinuumslimes und unendlichen Volumenlimes ändern. Dies haben wir an einem einfachen Beispiel diskutiert. Im $\mathcal{N}=2$ Fall ist die Situation eine ganz andere. Im starken Kopplungslimes haben wir $(p-1)^N$ Nullmoden gefunden, wobei p der polynomiale Grad des Superpotentials und N die Anzahl der Gitterpunkte sind. Ähnlich zum $\mathcal{N}=1$ Fall haben wir im Anhang gezeigt, dass auch hier die Anzahl der Nullmoden im starken

Kopplungslimes korrekt wiedergegeben wird. Dies ist allerdings ein sehr überraschendes Resultat, da in der Literatur im Rahmen der konstruktiven Feldtheorien, die Existenz von $p-1$ Nullmoden im Kontinuum und im endlichen Volumen gezeigt wurde. Zu diesem Widerspruch haben wir eine Lösung vorgeschlagen. Wie bereits erwähnt, enthält der Hamiltonoperator auf dem Gitter nicht nur den Hamiltonoperatur des Kontinuums, sondern auch die zentrale Ladung. Wir behaupten nun, dass für Nullmoden die sich von Gitterpunkt zu Gitterpunkt im starken Kopplungslimes stark ändern, sowohl die Energie des Kontinuumshamiltonoperators als auch die zentrale Ladung im Kontinuumslimes divergieren. Diese Nullmoden sind folglich reine Gitterartefakte. Nur die Nullmoden die im starken Kopplunglimes konstant über alle Gitterpunkte sind, davon gibt es $p-1$, überstehen den Kontinuumslimes. Wir möchten diese Behauptung mittels störungstheoretischer Methoden in naher Zukunft untersuchen.

Weiterhin kann man sich die Frage stellen, ob man die benutzte Methode zur Formulierung von Gittermodellen auch für andere Modelle und in anderen Dimenisonen anwenden kann. Eine starke Einschränkung gibt die Existenz einer Unteralgebra, die auf den Hamiltonoperator schließen muss. Die $\mathcal{N}=1$ Algebra in vier Dimensionen erlaubt beispielsweise keine solche Unteralgebra. Aber der Zugang könnte zum Beispiel für die $\mathcal{N}=2$ super-Yang-Mills Theorie in vier Dimensionen angewendet werden. Damit sollten sich dann die Resultate von Seiberg und Witten überprüfen lassen. Ermutigt durch die numerischen Resultate, welche wir für quantenmechanische Systeme mit Hilfe der SLAC-Ableitung erhalten haben, wollen wir nun auch die SLAC-Ableitung in Monte-Carlo-Simulationen für unsere Gittermodelle verwenden. Darüber hinaus wollen wir uns mit Fermionen auf dem Gitter, gekoppelt an Eichfelder und unter Benutzung der SLAC Gitterableitung, beschäftigen. Wir sind zuversichtlich, in Zukunft mehr darüber berichten zu können.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.
Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten in Anspruch genommen.
Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
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Ich versichere ehrenwörtlich, dass ich nach bestem Wissen und Gewissen die reine Wahrheit gesagt und nichts verschwiegen habe.

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