Entropy Numbers and Approximation Numbers in Weighted Function Spaces of Type $B^s_{p,q}$ and $F^s_{p,q}$, Eigenvalue Distributions of Some Degenerate Pseudodifferential Operators

D i s s e r t a t i o n

zur Erlangung des akademischen Grades

doctor rerum naturalium (Dr. rer. nat)

vorgelegt dem Rat der
Fakultät für Mathematik und Informatik
der Friedrich-Schiller-Universität Jena

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geboren am 3. September 1968 in Jena

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Tag des Rigorosums: 02. 05. 1995

Tag der öffentlichen Verteidigung: 12. 05. 1995

To my father

Acknowledgement

First of all I want to thank Prof. Dr. H. Triebel who has already been a very attentive and interested supervisor. I am grateful to him for a lot of suggestions, essential hints and remarks. Moreover, it is a pleasure for me to give my thanks to Prof. Dr. D. E. Edmunds who introduced me to the concept of entropy numbers and gave me opportunity to discuss parts of this work several times. Furthermore I am indebted to some of my colleagues for helpful comments without my emphasizing one of them.

Contents

Introduction

This work has essentially been motivated by some recent papers of EDMUNDS and TRIEBEL concerning entropy numbers and approximation numbers of compact embeddings between function spaces on domains, see [6] and [7]. In detail, there are investigated function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$ on a bounded domain Ω in $I\!R^n$. Recall that these two scales cover many well-known classical spaces such as (fractional) Sobolev spaces, Hölder-Zygmund spaces, Besov spaces and (inhomogeneous) Hardy spaces. In [8] these results have been applied to get sharp assertions for the distribution of eigenvalues of degenerate elliptic differential operators of the prototype

$$A = a(x)\,(-\Delta)\,a(x) \tag{1}$$

defined as the inverse of

$$B = b(x)\,(-\Delta)^{-1}\,b(x), \quad a(x) = b^{-1}(x), \tag{2}$$

b belonging to some Lebesgue space $L_r(\Omega)$ and $-\Delta$ denotes the Laplacian with vanishing Dirichlet data at the boundary of Ω. CARL's famous inequality

$$|\mu_k| \;\leq\; \sqrt{2}\,e_k(B) \tag{3}$$

gives the crucial link between the eigenvalues μ_k of a compact operator B and its related entropy numbers e_k. Moreover, in [8] there is also indicated another kind of application. Let A be a positive and self-adjoint operator in $L_2(\Omega)$, e.g. the above operator A from (1), and let $V(x) \geq 0$ be a singular potential. The "negative" spectrum of

$$H = A - V^2(x) \tag{4}$$

is of some interest. Via the Birman-Schwinger principle this problem can be reduced to operators B as given in (2) and the corresponding distribution of eigenvalues. All these has been done in [8] in the framework of $L_2(\Omega)$. Recalling the quantum mechanical background of the last problem an extension of these considerations from bounded domains to $I\!R^n$ appears desirable. Consequently, aiming at compact embeddings between function spaces on $I\!R^n$ one naturally arrives at the concept of weighted spaces.

This work is based on the ideas developed in these three papers by EDMUNDS and TRIEBEL, thus we will often return to these results. Otherwise we endeavour to involve some other tools and present slightly different approaches, for instance we make often use of the characterization of B- and F-spaces via local means as well as gain from a recently proved localization property of B- and F-spaces, see [26]. In other words the present work may be regarded both as an application and an extension of the above-mentioned papers. Further historical remarks concerning this subject may be found in [A:1] as well as [B:1] and will not be repeated here.

Assume $s \in I\!R, 0 < p \leq \infty$ ($p < \infty$ in the F-case), $0 < q \leq \infty$ and let $w(x)$ be an admissible weight function, i.e. a smooth function of at most polynomial growth with

$$0 < w(x) \;\leq\; c\,w(y)\,\langle x - y\rangle^{\alpha} \tag{5}$$

for some $\alpha \geq 0$, $c > 0$ and all $x, y \in I\!R^n$. As usual, $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$, $x \in I\!R^n$. Then the spaces $B^s_{p,q}(I\!R^n, w(x))$ and $F^s_{p,q}(I\!R^n, w(x))$ can be defined completely analogous to their unweighted counterparts, see [24: Def.2.3.1/2], replacing the unweighted L_p space by its weighted counterpart $L_p(I\!R^n, w(x))$. But these definitions based on Fourier-analytic decomposition techniques almost disguise the influence of the weight function (as given in $L_p(I\!R^n, w(x))$). This leads back to the original problem that the Fourier-analytic definition almost hides the local-global nature of these spaces, in particular of those classical special cases like Sobolev spaces, defined in a direct manner. Fortunately there has been a solution of this problem by rather modern characterizations of B- and F-spaces using local means (or even atomic representations). Having this background in mind it has been reasonable to ask whether there may hold analogues of the L_p-situation

$$\|f|L_p(I\!R^n, w(x))\| \;=\; \|w\,f|L_p(I\!R^n)\|$$

in B- or F-spaces, too. Though this result is already known in weighted spaces of the above and more general type, see [18], we give some new proofs for the following facts, relying not very much on these forerunners:

$$\|f|F^s_{p,q}(I\!R^n, w(x))\| \quad \text{and} \quad \|w\, f|F^s_{p,q}(I\!R^n)\| \tag{6}$$

are equivalent quasi-norms on $F^s_{p,q}(I\!R^n, w(x))$ (likewise in the B-case). Furthermore we prove that the embedding

$$id^F : F^{s_1}_{p_1,q_1}(I\!R^n, w_1(x)) \longrightarrow F^{s_2}_{p_2,q_2}(I\!R^n, w_2(x)) \tag{7}$$

is compact if and only if

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \longrightarrow 0 \quad \text{for} \quad |x| \to \infty \tag{8}$$

where

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le p_2 < \infty, \quad 0 < q_1 \le \infty, \quad 0 < q_2 \le \infty \tag{9}$$

and $w_1(x), w_2(x)$ are admissible weight functions (similarly in the B-case). Following these considerations where comparatively general weight functions are involved we reduce this problem to a so-called standard situation: in case of $w(x) = \langle x\rangle^\alpha$, $\alpha > 0$, we study compact embeddings of type

$$id^B : B^{s_1}_{p_1,q_1}(I\!R^n, \langle x\rangle^\alpha) \longrightarrow B^{s_2}_{p_2,q_2}(I\!R^n) \tag{10}$$

and

$$id^F : F^{s_1}_{p_1,q_1}(I\!R^n, \langle x\rangle^\alpha) \longrightarrow F^{s_2}_{p_2,q_2}(I\!R^n) \tag{11}$$

where

$$\begin{array}{llll} & -\infty < s_2 < s_1 < \infty & , \quad 0 < p_1 \le \infty, & 0 < q_1 \le \infty \\ \alpha > 0, & \dfrac{1}{p_0} = \dfrac{1}{p_1} + \dfrac{\alpha}{n} & , \quad p_0 < p_2 \le \infty, & 0 < q_2 \le \infty \end{array} \tag{12}$$

and

$$\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0\,. \tag{13}$$

In view of these compact embeddings (10) and (11) it is rather natural to ask for a better "measuring" of this compactness according to the given parameters. A very good tool to deal with such questions consists in determining the entropy numbers as well as the approximation numbers of the embeddings (10) and (11). One knows that

$$\lim_{n\to\infty} e_n(T) = 0 \quad \Longleftrightarrow \quad T \quad \text{compact}$$

and

$$\lim_{n\to\infty} a_n(T) = 0 \quad \Longrightarrow \quad T \quad \text{compact}$$

where $e_k(T)$ and $a_k(T)$ denote the kth entropy number and approximation number of some compact linear operator T, resp. The concept of entropy numbers is more geometrically based: one has to count the number of balls of radius ε covering the image of the unit ball under some map T, whereas "approximation" means approximation by finite-rank operators. Recall that entropy, compactness and approximation (in this sense) are closely related. We decided to care about these two quantities (among a larger number of possibilities) because of two reasons mainly: at first we wish to apply our results to eigenvalue distributions via CARL's inequality (3), on the other hand we have already explained that we want to follow those papers [6] and [7] which are concerned with entropy numbers and approximation numbers, too. One of our main results deals with estimates for the entropy numbers of the embeddings (10) or (11) from above and below. It turned out the following: if $\alpha > 0$ is above or below the critical value $\delta > 0$ from (13) the entropy numbers behave like

$$e_k(id) \ \sim \ k^{-\varrho}, \quad k \in I\!N, \tag{14}$$

whereas in the case $\delta = \alpha$ we achieved

$$e_k(id) \ \sim \ k^{-\varkappa}\left(\log\langle k\rangle\right)^{\mu}, \quad k \in I\!N, \tag{15}$$

where id stands for id^B or id^F from (10) or (11), resp., and the positive numbers ϱ, $\varkappa$ and μ depend upon the parameters (12).

This last case (15) might be the most difficult but also the most interesting one. Though we failed to give a complete description of the behaviour of the entropy numbers of either (10) or (11) in all possible cases we could provide a rather surprising contribution to the study of this subject. In contrast to all the cases indicated by (14) the so-called third or q-indices are of importance in (15). In some cases this q-dependence of the exponent μ in (15) really reflects the right behaviour as we could show. Connected with this phenomenon there is another one: apart from that line "$\delta = \alpha$" we always have the same behaviour of the entropy numbers in (14) independent whether (10) or

(11) is concerned. This is simply due to the elementary embeddings

$$B^s_{p,u}(\mathbb{R}^n) \subset F^s_{p,q}(\mathbb{R}^n) \subset B^s_{p,v}(\mathbb{R}^n) \quad \Longleftrightarrow \quad 0 < u \le \min(p,q), \quad \max(p,q) \le v \le \infty \tag{16}$$

and the fact that no q-parameter occurs in the respective assertions of (14). This situation changes completely at the line "$\delta = \alpha$". The behaviour of those entropy numbers also depends upon the involved q-parameters so sensitively that one has to argue very carefully. An immediate transferring of B-results to the F-case by (16) breaks down. Moreover, this complicated situation gives as a by-product an elegant proof of the "only if"-part of (16) concerning the assertion

$$B^s_{p,u}(\mathbb{R}^n) \subset F^s_{p,q}(\mathbb{R}^n) \subset B^s_{p,v}(\mathbb{R}^n) \quad \Longrightarrow \quad u \le p \le v .$$

Studying this subject as above described there are still a lot of open questions left but probably a lot of interesting insights, also into the interplay between B- and F-spaces (from the point of view of entropy numbers), too.
Though the study of approximation numbers requires by far more attention as we have to distinguish more cases, we arrived at similar results to (14) and (15). These additional complications in case of approximation numbers are mainly caused by the curious role played by the number 2 (in relation to both p-parameters) and is well-known also in case of other width numbers.
We now care about possible applications of our results as already announced in the very beginning. In our context (2) has to be replaced by

$$B = b_2(x)\, b(x,D)\, b_1(x) \tag{17}$$

where $b_1(x)$ and $b_2(x)$ are (typically) functions belonging to some weighted space, preferably of L_p or H^s_p type, and $b(x,D)$ is in the Hörmander class $\Psi^{\varkappa}_{1,\gamma}$, $\varkappa < 0$, $0 \le \gamma \le 1$. Under certain assumptions to the parameters this operator B becomes compact in some weighted L_p space (or even H^s_p space) so that we may apply Carl's inequality (3) to obtain estimates from above for the respective eigenvalues of B. Though we get (nearly) sharp results concerning the expected order of the eigenvalue distribution the method itself seems more remarkable in our opinion. There are several inputs of particular results apart from the already mentioned inequality (3). An important contribution comes from extensions of the classical Hölder inequality to H-spaces (in our context), see also [19]. Finally one needs a mapping property of pseudodifferential operators in weighted spaces. Having prepared those necessary tools the determination of the entropy numbers of B may afterwards be reduced to some elegant "travelling around" in suitable $(\frac{1}{p}, s)$-diagrams complemented by our results (14) and (15). In particular, we decompose B in a tricky manner, verify the continuity of the maps $b_1(x)$, $b(x,D)$ and $b_2(x)$ regarded between respective (weighted) function spaces whereas merely some embedding of type (11) really contributes to the rate of decay of the entropy numbers $e_k(B)$ apart from constants. Having all the above "building blocks" available our indicated method actually simplifies the study of eigenvalues of B. Besides our results might be regarded rather as a guide how to manage similar situations than as a complete list of possible applications one could think about.
Another interesting opportunity to gain from our considerations is offered by investigating the "negative" spectrum of an operator. Let H_β be an operator in L_2,

$$H_\beta = a(x,D) - \beta\, a(x)\, p(x,D)\, a(x),$$

where $a(x,D)$ belongs to some $\Psi^{\varkappa}_{1,\gamma}$, $\varkappa > 0$, $0 \le \gamma < 1$, and is assumed to be positive-definite and self-adjoint in L_2. Let $p(x,D)$ be some symmetric perturbation, i.e. some symmetric pseudodifferential operator of lower order and $a(x)$ a real function. Especially in quantum mechanics there is some particular interest to control the number of negative eigenvalues

$$\#\{\, \sigma(H_\beta) \cap (-\infty, 0]\,\} \quad \text{as} \quad \beta \to \infty.$$

This problem can be carefully handled via the Birman-Schwinger principle, our outcomes (14) and (15), resp., and an approach similar to the above one. Likewise we have cared about a related question, that is

$$\#\{\, \sigma(H) \cap (-\infty, -\varepsilon]\,\} \quad \text{as} \quad \varepsilon \downarrow 0$$

where H may stand for the hydrogen operator

$$H = -\Delta - \frac{c}{|x|}, \quad c > 0$$

in the simplest case. Let us repeat again that our main intention in these investigations has been to demonstrate about what possible applications one may think. In our opinion the presented methods are both effective and elegant.

As we aimed at incorporating already published or completed papers written together with Prof. Dr. H. Triebel ([A], [B]) as well as alone ([C], [D]) the form might appear a little bit unusual for a thesis: the present work mainly consists of three parts where the last and most extensive one very differs from the preceding two.
In the first part we introduce the main concepts of this work and prepare some fundamentals. In particular, we define the weighted spaces we have in mind, some of their basic properties and end up with a description of the so-called standard situation. Furthermore we recall the definition of the Hörmander class $\Psi^{\varkappa}_{1,\gamma}$, $\varkappa \in I\!R$, $0 \leq \gamma \leq 1$, and establish a mapping property of pseudodifferential operators, belonging to some $\Psi^{\varkappa}_{1,\gamma}$, in weighted spaces.
The second part is devoted to our main results concerning entropy numbers and approximation numbers as well as applications to the eigenvalue distribution of degenerate pseudodifferential operators and to the consideration of "negative" spectra. This has been done in the first, third and second subsection, respectively. Every subsection starts with those special definitions and remarks which are typically related to the problem we want to cope with in the sequel.
Finally the last part includes all the proofs. What we called "Appendix" consists of the four papers:

[A] D. Haroske, H. Triebel: *Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators I.* Math. Nachr. **167** (1994), 131-156

[B] D. Haroske, H. Triebel: *Entropy numbers in weighted function spaces and eigenvalue distribution of some degenerate pseudodifferential operators II.* Math. Nachr. **168** (1994), 109-137

[C] D. Haroske: *Approximation numbers in some weighted function spaces.* J.Approx. Theory (to appear)

[D] D. Haroske: *Complements.* (preprint, Jena, 1994)

We always refer to these papers as indicated above in order to emphasize that those publications belong to the present work itself, whereas all the other literature in the first two parts is referred to by numbers [1], [2], etc. which can be found in the References at the end of Section 2. Unfortunately cross-references within [A], [B], [C] and [D] could not be by-passed. This might be understandable by the chronological order of those ingredients and our aim that [A], [B], [C] and [D] should be self-contained. Moreover, we endeavoured to avoid redundancy as far as possible. Thus we mention only basic definitions and main results in the first two sections. Lemmata, further remarks and other tools only necessary in the course of proving may then be found in the referred appendices. Hence it is recommendable in our opinion to read Section 2 rather as a guide or a summary, but as far as details are concerned one should rely on the according appendices. We have to apologize to the reader for that.

We want to add a few technical remarks. All unimportant positive constants are denoted by c, occasionally with additional subscripts within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l whereas (j) means formula (j) in the same subsection. Definitions, propositions, remarks etc. are quoted in a similar way.

1 Definitions and preliminaries

1.1 Function spaces

1.1.1 Weighted function spaces

We start presenting a main subject of our investigations : the weighted function spaces we aim at. We introduce the class of weight functions our interest is confined to in the sequel. Let $\mathbb{R}^n$ be the Euclidean n-space and denote by $\langle x\rangle = (1+|x|^2)^{1/2}$ in $\mathbb{R}^n$.

Definition 1. The class of *admissible weight functions* W is the collection of all positive C^∞ functions $w(x)$ on $\mathbb{R}^n$ with the following properties:

(i) for any multi-index γ there exists a positive constant c_γ with

$$|D^\gamma w(x)| \le c_\gamma\, w(x) \quad \text{for all} \quad x\in\mathbb{R}^n\ , \tag{1}$$

(ii) there exist two constants $c>0$ and $\alpha\ge 0$ such that

$$0 \ < \ w(x) \ \le c\, w(y)\, \langle x-y\rangle^\alpha \quad \text{for all} \quad x\in\mathbb{R}^n \quad \text{and all} \quad y\in\mathbb{R}^n\ . \tag{2}$$

Remark 1. We refer to [A: Rem. 2.1/1, Rem. 2.1/2] for some detailed comments subsequent to this definition. In particular, condition (2) forces local boundedness of the functions in question, whereas the C^∞ assumption is not as restrictive as may be supposed at first glance. But we shall not repeat the arguments now.

Preparing and motivating the definition of weighted function spaces we have in mind recall first some fundamentals of their unweighted counterparts. The theory of function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, $-\infty<s<\infty$, $0<p\le\infty$ ($p<\infty$ in the F-case), $0<q\le\infty$, has been developed and systematically extended by H. Triebel in numerous books and papers, for detailed investigations and results see [23], [24], [25] and the references given there. For the sake of brevity we want to outline basic definitions and properties only roughly and mention former results just as far as necessary.
All spaces in this paper are defined on $\mathbb{R}^n$ - unless otherwise stated - and so we omit "$\mathbb{R}^n$" in the sequel. Let S be the Schwartz space of all rapidly decreasing infinitely differentiable functions and S' its dual of all complex-valued tempered distributions, as usual. Moreover, L_p with $0<p\le\infty$ is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot|L_p\|$. Let $\{\varphi_j\}_{j=0}^\infty$ be a smooth dyadic resolution of unity possibly generated by some function $\varphi\in S$ with

$$\operatorname{supp}\varphi\subset\{y\in\mathbb{R}^n : |y|<2\}, \quad \varphi(x)=1 \quad \text{if} \quad |x|\le 1\ . \tag{3}$$

Put $\varphi_0=\varphi$ and $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$, $x\in\mathbb{R}^n$, $j=1,2,\ldots$, then $1=\sum_{j=0}^{\infty}\varphi_j(x)$, $x\in\mathbb{R}^n$ and this special sequence $\{\varphi_j\}$ may serve as a typical example to get an impression what is meant by a "smooth dyadic resolution of unity". Given any $f\in S'$ we denote by $\hat f$ and $f^\vee$ its Fourier transform

$$\mathcal{F}\varphi\,(x) \ = \ \hat\varphi(x) \ = \ (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-ix\xi}\,\varphi(\xi)d\xi, \quad x\in\mathbb{R}^n,\ \varphi\in S \tag{4}$$

(originally defined on S and afterwards extended to S' in the standard way) and its inverse Fourier transform, respectively. Then we know from the Paley-Wiener-Schwartz theorem that $(\varphi_j\hat f)^\vee$ is an entire analytic function on $\mathbb{R}^n$. The (unweighted) B- or F-spaces are defined in the following way.

Definition 2. Let $-\infty<s<\infty$, $0<q\le\infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity.

(i) Let $0<p\le\infty$. The space $B^s_{p,q}$ is the collection of all $f\in S'$ such that

$$\|f|B^s_{p,q}\| \;=\; \left(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_j\hat{f})^{\vee}|L_p\|^q\right)^{\frac{1}{q}} \tag{5}$$

(with the usual modification if $q=\infty$) is finite.

(ii) Let $0<p<\infty$. The space $F^s_{p,q}$ is the collection of all $f\in S'$ such that

$$\|f|F^s_{p,q}\| \;=\; \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\hat{f})^{\vee}(\cdot)|^q\right)^{\frac{1}{q}}|L_p\right\| \tag{6}$$

(with the usual modification if $q=\infty$) is finite.

Remark 2. These spaces have been studied in great detail in [23], [24] and [25]. We merely want to remind the reader to some basic properties. In particular, both $B^s_{p,q}$ and $F^s_{p,q}$ are quasi-Banach spaces which are independent of the function $\varphi\in S$ chosen according to (3), in the sense of equivalent quasi-norms. This justifies the omission of the subscript φ in (5) and (6) in what follows. If $p\ge1$ and $q\ge1$, then both $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces. Among a large number of properties we solely want to mention two of them having particular importance for us later on.

(i) *Embeddings along constant differential dimension*

$B^{s_0}_{p_0,q}$ is continuously embedded in $B^{s_1}_{p_1,q}$, $\quad B^{s_0}_{p_0,q}\subset B^{s_1}_{p_1,q}$,

$$\text{if}\quad s_0-\frac{n}{p_0}=s_1-\frac{n}{p_1},\quad 0<p_0\le p_1\le\infty,\quad 0<q\le\infty,\quad -\infty<s_1\le s_0<\infty \tag{7}$$

and likewise

$F^{s_0}_{p_0,q}$ is continuously embedded in $F^{s_1}_{p_1,r}$, $\quad F^{s_0}_{p_0,q}\subset F^{s_1}_{p_1,r}$,

$$\text{if}\quad s_0-\frac{n}{p_0}=s_1-\frac{n}{p_1},\ 0<p_0\le p_1<\infty,\ 0<q\le\infty,\ 0<r\le\infty,\ -\infty<s_1<s_0<\infty, \tag{8}$$

see [24: (2.7.1/1), (2.7.1/2)];

(ii) *Embeddings between B- and F-spaces*

Let $s\in I\!R$, $0<p<\infty$ and $0<q\le\infty$. Then

$$B^s_{p,u}\subset F^s_{p,q}\subset B^s_{p,v}\quad\text{if and only if}\quad 0\le u\le\min(p,q)\quad\text{and}\quad \max(p,q)\le v\le\infty\ , \tag{9}$$

for the “if“-part see [24: (2.3.2/9)], a proof of the “only if“-part may be found in [19: 5.1].

Note that there cannot exist any set of parameters in (7) or (8) such that the respective natural embedding becomes compact. Moreover, the two scales cover a lot of well-known classical function spaces like

$F^0_{p,2}=L_p$, $\quad 1<p<\infty$ $\quad$ Lebesgue spaces,
$F^k_{p,2}=W^k_p$, $\quad 1<p<\infty,\ k\in I\!N_0$ $\quad$ Sobolev spaces,
$F^s_{p,2}=H^s_p$, $\quad 1<p<\infty,\ s\in I\!R$ $\quad$ fractional Sobolev spaces,
$F^0_{p,2}=h_p$, $\quad 0<p<\infty$ $\quad$ non-homogeneous Hardy spaces,
$B^s_{p,q}$, $\quad 1<p<\infty,\ s>0,\ 1\le q\le\infty$ $\quad$ classical Besov spaces
$B^s_{\infty,\infty}=\mathcal{C}^s$, $\quad s>0$ $\quad$ Hölder-Zygmund spaces.

We now come to define the weighted function spaces in question. In addition to the unweighted spaces L_p on $I\!R^n$ there are known their weighted generalizations $L_p(w(\cdot))$ furnished with the quasi-norm

$$\|f|L_p(w(\cdot))\| = \|wf|L_p\| \tag{10}$$

where $w(x) > 0$ is an admissible weight function and $0 < p \leq \infty$.

Definition 3. Let $-\infty < s < \infty$, $0 < q \leq \infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity. Assume $w(x) \in W$ to be an admissible weight function in the sense of Definition 1.

(i) Let $0 < p \leq \infty$. The space $B^s_{p,q}(w(\cdot))$ is the collection of all $f \in S'$ such that

$$\|f|B^s_{p,q}(w(\cdot))\| \ = \ \left(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_j\hat{f})^\vee|L_p(w(\cdot))\|^q\right)^{\frac{1}{q}} \tag{11}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. The space $F^s_{p,q}(w(\cdot))$ is the collection of all $f \in S'$ such that

$$\|f|F^s_{p,q}(w(\cdot))\| \ = \ \left\|\left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\hat{f})^\vee(\cdot)|^q\right)^{\frac{1}{q}}|L_p(w(\cdot))\right\| \tag{12}$$

(with the usual modification if $q = \infty$) is finite.

(iii) Let $w(x) = \langle x\rangle^\alpha$ for some $\alpha \in \mathbb{R}$. Then we put

$$B^s_{p,q}(\alpha) = B^s_{p,q}(\langle x\rangle^\alpha) \quad \text{with} \quad B^s_{p,q} = B^s_{p,q}(0) \tag{13}$$

and

$$F^s_{p,q}(\alpha) = F^s_{p,q}(\langle x\rangle^\alpha) \quad \text{with} \quad F^s_{p,q} = F^s_{p,q}(0)\ . \tag{14}$$

Remark 3. Weighted function spaces of the above type have been treated in great detail in [18: 5.1] but we do not rely very much on these results. Moreover one could extend the theory for the unweighted case developed in [24] and [25] to the above weighted classes without particular difficulty. We will use former results and tools which can obviously be transferred to our situation as well as give new (shorter) proofs for some relevant facts, see also [A: Rem. 2.1/3].

Remark 4. It is worth mentioning that the above two weighted scales cover weighted (fractional) Sobolev spaces, weighted Hölder-Zygmund spaces and weighted classical Besov spaces, see also [A: Rem. 2.1/4] for further details.

Remark 5. In essence, we are concerned with situations Definition 3(iii) alludes to, i.e. where $w(x) = \langle x\rangle^\alpha, \alpha \in \mathbb{R}$. Thus we introduced the abbreviations (13) and (14) to avoid clumsy notations.

1.1.2 General weight functions

Preceding our main investigations we establish a first, more general result concerning equivalent normings of the weighted function spaces in question. Comparing Definition 2 and Definition 3 one simply recognizes the replacement of the unweighted L_p space by its weighted counterpart. This might be regarded as the natural generalization climbing up from the unweighted to the weighted case. But in view of (10) it also seems reasonable to seek to characterize weighted spaces in a more direct sense than given by (11) and (12), in other words it would be desirable to have counterparts of (10) with L_p substituted by other B- and F-spaces, too. At first glance Definition 3 involving (inverse) Fourier transforms contradicts this goal but fortunately recent developments in the theory of function spaces supplied powerful tools to cope with this problem. The main trick turns out to be the application of local means as described in [25: 2.4.6, 2.5.3] to get equivalent characterizations of unweighted B- or F-spaces apart from Fourier transforms as in Definition 2. Proceeding completely analogous we obtain the following property in a strikingly simple way, for the technique of local means showed itself to be well-adapted to the local boundedness of our weight functions.

Theorem 1. *Let $s \in \mathbb{R}, 0 < q \leq \infty$, $0 < p \leq \infty$ (with $p < \infty$ in the F-case) and $w(x) \in W$.*

(i) *$B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*

(ii) *The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(w(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(w(x))$ onto $F^s_{p,q}$. Especially,*

$$\|wf|B^s_{p,q}\| \quad \textit{is an equivalent quasi-norm in} \quad B^s_{p,q}(w(x)) \tag{15}$$

and

$$\|wf|F^s_{p,q}\| \quad \textit{is an equivalent quasi-norm in} \quad F^s_{p,q}(w(x))\ . \tag{16}$$

Remark 6. Concerning the proof and further comments see [A: 2.2, 5.1].

As already announced in the beginning we are interested in embeddings between different weighted spaces of the above type. Though we will deal with more specialized situations in the sequel we intend to start our considerations with an embedding theorem formulated in as much generality as seems consistent with simplicity. Thus we consider embeddings of type

$$id : B^{s_1}_{p_1,q_1}(w_1(\cdot)) \longrightarrow B^{s_2}_{p_2,q_2}(w_2(\cdot)) \tag{17}$$

and

$$id : F^{s_1}_{p_1,q_1}(w_1(\cdot)) \longrightarrow F^{s_2}_{p_2,q_2}(w_2(\cdot)) \tag{18}$$

with different weight functions. Regarding the above Theorem 1(ii) one could also extend former results concerning embeddings in unweighted spaces to situations where only one weight function is involved. In spite of this we stick at (17) and (18) with different weights at the moment. By Theorem 1 and the elementary embeddings (9) we may formulate the next result for F-spaces only.

Theorem 2. *Let $-\infty < s_2 < s_1 < \infty$, $0 < p_1 \leq p_2 < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $w_1(x)$, $w_2(x) \in W$.*

(i) *Then $F^{s_1}_{p_1,q_1}(w_1(x))$ is continuously embedded in $F^{s_2}_{p_2,q_2}(w_2(x))$,*

$$F^{s_1}_{p_1,q_1}(w_1(x)) \subset F^{s_2}_{p_2,q_2}(w_2(x)) \quad \textit{if and only if} \quad s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2} \ \textit{and} \ \frac{w_2(x)}{w_1(x)} \leq c < \infty \tag{19}$$

for some $c > 0$ and all $x \in \mathbb{R}^n$.

(ii) *The embedding (19) is compact if and only if*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \textit{and} \quad \frac{w_2(x)}{w_1(x)} \longrightarrow 0 \quad \textit{if} \quad |x| \to \infty\ . \tag{20}$$

Remark 7. The proof and further remarks are given in [A: 2.3, 5.2]. Concerning the “if“-parts (19) is a simple consequence of the unweighted result, see also (7), and Theorem 1, whereas in (20) one returns to the compact embeddings of function spaces on bounded domains, e.g. on balls of radius R, cf. [24: 3.2.2] for a definition. Then one uses a cut-off function as well as the characterization of F-spaces via local means to result in the compact embedding if (20) holds, for details see [A: 5.2]. Conversely, to show the “only if“-parts one needs the localization property, see [26], and firstly obtains the necessity of $s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}$ and $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ in (19) and (20), resp., to get a continuous or even compact embedding of function spaces on bounded domains. Afterwards one can show by contradiction that the assumptions for the weight functions in (19) and (20) are also necessary.

1.1.3 The standard situation

We introduce some special situation we are going to study in detail in what follows. As we are preferably interested in measuring the compactness of embeddings of type (17) or (18), resp., via entropy numbers, we may change the above described more general situation in the following way:

$$\begin{tikzcd} F^{s_1}_{p_1,q_1}(w_1(\cdot)) \arrow[r, "id"] \arrow[d, "w_2\cdot"] & F^{s_2}_{p_2,q_2}(w_2(\cdot)) \arrow[d, "w_2\cdot"] \\ F^{s_1}_{p_1,q_1}(\frac{w_1}{w_2}(\cdot)) \arrow[r, "id'"] & F^{s_2}_{p_2,q_2} \end{tikzcd}$$

looking at the left-hand diagram in connection with Theorem 1 and Theorem 2 we observe that it is sufficient to investigate situations where only one weight appears, e.g. in the basic space, for

$$\|f|F^{s_1}_{p_1,q_1}(w_1(\cdot))\| \sim \left\|w_2\, f|F^{s_1}_{p_1,q_1}(\tfrac{w_1}{w_2}(\cdot))\right\|$$

and

$$\|f|F^{s_2}_{p_2,q_2}(w_2(\cdot))\| \sim \|w_2\, f|F^{s_2}_{p_2,q_2}\|$$

by Theorem 1 and thus both natural embeddings $id : F^{s_1}_{p_1,q_1}(w_1(\cdot)) \longrightarrow F^{s_2}_{p_2,q_2}(w_2(\cdot))$ and $id' : F^{s_1}_{p_1,q_1}(\frac{w_1}{w_2}(\cdot)) \longrightarrow F^{s_2}_{p_2,q_2}$ can be transferred to each other by isomorphisms, $id' = w_2 \circ id \circ w_2^{-1}$. From our point of view - involving entropy numbers later on - this difference may be neglected and the situation can be restricted to an unweighted target space. Additionally one recognizes that in Theorem 2 the criteria to decide whether the embeddings in question are continuous or even compact merely rely on the quotient of both weight functions. Henceforth we thus may always assume $w_2(x) = 1$ without loss of generality. Furthermore we specify in what follows $w(x) = \langle x\rangle^\alpha$ for some $\alpha > 0$. Obviously both

$$id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \tag{21}$$

and

$$id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{22}$$

are compact embeddings by Theorem 2, recall (13) and (14), provided that

$$\delta \; := \; s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \; > \; 0 \,, \qquad \alpha \; > \; 0 \tag{23}$$

and the other parameters are as in Theorem 2. Introducing *weak*-spaces one may even extend the range of parameters in this special situation, for definitions and basic properties see [A: 2.4] and the references given there. Taking these spaces for granted at this moment we obtain that

$$B^s_{p,q}(\alpha) \subset weak - B^s_{p_0,q} \quad \text{and} \quad F^s_{p,q}(\alpha) \subset weak - F^s_{p_0,q} \tag{24}$$

where $s \in I\!R$, $0 < q \le \infty$, $0 < p \le \infty$ ($p < \infty$ in the F-case), $\alpha > 0$ and $\dfrac{1}{p_0} = \dfrac{1}{p} + \dfrac{\alpha}{n}$.

So we may finally describe our *s t a n d a r d s i t u a t i o n* as follows: we study natural embeddings of type

$$id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}$$

and

$$id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2}$$

where the weighted spaces have been introduced above and the parameters satisfy

$$\begin{aligned} -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le \infty \quad , \quad 0 < q_1 \le \infty, \quad \alpha > 0, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n}, \quad p_0 < p_2 \le \infty \quad , \quad 0 < q_2 \le \infty \end{aligned} \tag{25}$$

and

$$\delta \; := \; s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \; > \; 0 \tag{26}$$

(we always assume $p_1 < \infty$ and $p_2 < \infty$ in the F-case).

1.2 Pseudodifferential operators

1.2.1 Definition

Recall the definition of the Hörmander class $S^{\varkappa}_{1,\gamma}$.

Definition. Let $\varkappa \in I\!R$ and $0 \le \gamma \le 1$. The *Hörmander class* $S^{\varkappa}_{1,\gamma}$ consists of all complex C^∞ functions $p(x,\xi)$ in $I\!R^n \times I\!R^n$ satisfying

$$|\, D_x^\beta \, D_\xi^\alpha \, p(x,\xi) \,| \;\le\; c_{\alpha,\beta}\, \langle \xi \rangle^{\varkappa - |\alpha| + \gamma|\beta|} \quad , \; x \in I\!R^n ,\; \xi \in I\!R^n \tag{1}$$

for all multi-indices α and β and some constants $c_{\alpha,\beta} > 0$ where D_ξ^α and D_x^β refer to respective derivatives for the $\xi = (\xi_1, \ldots, \xi_n)$ and the $x = (x_1, \ldots, x_n)$ variables as usual.

The related class of pseudodifferential operators is denoted by $\Psi^{\varkappa}_{1,\gamma}$, where $p(x,D)$ is given, at least formally, by

$$p(x,D)f(x) \;=\; \int\limits_{I\!R^n} e^{ix\xi}\, p(x,\xi)\, \hat{f}(\xi)\, d\xi \;=\; (2\pi)^{-\frac{n}{2}} \int\limits_{I\!R^{2n}} e^{i(x-y)\xi}\, p(x,\xi)\, f(y)\, dy\, d\xi \,. \tag{2}$$

The necessary background material may be found in [20], [21], [22] and [9]. Studying mapping and spectral properties of pseudodifferential operators it is sufficient to assume $\varkappa = 0$, since there is a 1-1 relation between $\Psi^{\varkappa}_{1,\gamma}$ and $\Psi^{0}_{1,\gamma}$ given by

$$p(x,D) = p^\circ(x,D)\, (id - \Delta)^{\frac{\varkappa}{2}} \quad , \; \varkappa \in I\!R, \tag{3}$$

where Δ stands for the Laplacian in $I\!R^n$, $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$ and $p^\circ(x,D) \in \Psi^{0}_{1,\gamma}$. Recall that $\Psi^{0}_{1,1}$ is called the exotic class.

1.2.2 Mapping properties

Aiming at estimates for the eigenvalues of degenerate pseudodifferential operators the main trick turns out to be an appropriate composition of mappings between function spaces. Hence a basic tool incorporated into the respective proofs is a detailed investigation of mapping properties of pseudodifferential operators in weighted spaces according to Definition 1.1/3. We arrive at the following assertion.

Theorem. *Let* $0 \le \gamma \le 1$, $p(x,D) \in \Psi^{0}_{1,\gamma}$, $0 < q \le \infty$ *and* $w(x) \in W$.
(i) *Let* $0 < p \le \infty$, $s \in I\!R$ *(with* $s > n\big(\frac{1}{p} - 1\big)_+$ *in the exotic case* $\gamma = 1$*). Then* $p(x,D)$ *is a linear and bounded map*

$$\text{from} \quad B^s_{p,q}(w(x)) \quad \text{into itself}. \tag{4}$$

(ii) *Let* $0 < p < \infty$, $s \in I\!R$ *(with* $s > n\big(\frac{1}{\min(p,q)} - 1\big)_+$ *in the exotic case* $\gamma = 1$*). Then* $p(x,D)$ *is a linear and bounded map*

$$\text{from} \quad F^s_{p,q}(w(x)) \quad \text{into itself}. \tag{5}$$

Remark 1. A proof due to H. Triebel as well as further remarks concerning the situation in the unweighted case may be found in [B: 3.1]. This proof is based on the localization principle for F-spaces as presented in [25: 2.4.7] and atomic representations in the sense of Frazier and Jawerth but also involves pointwise multipliers for F-spaces, see [25: 4.2.2] and real interpolation (to come to the B-spaces). Another proof has been published by P. Dintelmann only recently, see [4]. Furthermore one can immediately extend the above theorem to the case $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$, $\varkappa \in I\!R$, using (3), see also [B: Rem. 3.1/2].

Remark 2. In the course of our later considerations concerning eigenvalues of pseudodifferential operators it becomes necessary to deal simultaneously with several of the above introduced spaces. Hence one has to care about the dependence of the spectral properties especially of operators of the class $\Psi^{\varkappa}_{1,\gamma}$ on the underlying function space. Note that $(id-\Delta)^{\frac{\varkappa}{2}}$ is an isomorphic map from $B^s_{p,q}(w(x))$ onto $B^{s-\varkappa}_{p,q}(w(x))$ and from $F^s_{p,q}(w(x))$ onto $F^{s-\varkappa}_{p,q}(w(x))$. Consequently it is again sufficient to deal with the case $\varkappa = 0$ by (3). Let $0 \leq \gamma < 1$, $P = p(x,D) \in \Psi^0_{1,\gamma}$, $s \in \mathbb{R}$, $0 < q \leq \infty$ and $w(x) \in W$. Denote by $\varrho(P, B^s_{p,q}(w(x)))$ the resolvent set of P in $B^s_{p,q}(w(x))$, defined as usual, see also [B: 3.2]. Let $\varrho(P) = \varrho(P, L_2)$. Then we achieved in [B: 3.2] the following result :

$$\text{(i)} \quad \textit{Let } 0 < p \leq \infty. \textit{ Then} \quad \varrho(P) \quad \subset \quad \varrho(P, B^s_{p,q}(w(x))) \; . \tag{6}$$

$$\text{(ii)} \quad \textit{Let } 0 < p < \infty. \textit{ Then} \quad \varrho(P) \quad \subset \quad \varrho(P, F^s_{p,q}(w(x))) \; . \tag{7}$$

A proof and further comments are included in [B: 3.2].
On the one hand the preceding sections might be regarded as preparations and background material merely mentioned as necessary ingredients for our main investigations. On the other hand these considerations do not only pave the way for our basic results but are of self-contained interest, too. Likewise in this case of spectral properties. Pursuing the argumentation in [B: Rem. 3.2/2] one almost inevitably comes to the conjecture that the inclusions in (6) and (7) should be replaced by equality. Though we shall not stress this point in the sequel we want to hint at [B: Rem. 3.2/3] and [12] where a refined discussion of this problem can be found.

2 Main results

2.1 Entropy numbers

2.1.1 Definition, basic properties

Let A and B be two complex quasi-Banach spaces and let T be a linear and continuous operator from A into B. If T is compact, then for any given $\varepsilon > 0$ there are finitely many balls in B of radius ε which cover the image of the unit ball $U = \{a \in A : \|a|A\| \leq 1\}$.

Definition. Let $k \in \mathbb{N}$ and let $T : A \to B$ be the above continuous operator. Then the *kth entropy number e_k of T* is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in B of radius ε which cover $T\,U$.

Remark 1. For details and properties we refer to [3], [5] and [10], always restricted to the case of Banach spaces. Extending these properties to quasi-Banach spaces causes no problems. Recall that any quasi-Banach space A is also a λ-Banach space for some suitable number λ, $0 < \lambda \leq 1$, i.e. there is an equivalent quasi-norm such that

$$\|a_1 + a_2|A\|^\lambda \;\leq\; \|a_1|A\|^\lambda + \|a_2|A\|^\lambda \quad \text{for all} \quad a_1, a_2 \in A \;,$$

see [11: §15.10].
Among other features we only want to repeat very few of them having importance for us later on.

(i) *Monotonicity*
Let $T : A \to B$ be as above. Then $\qquad \|T\| \;\geq\; e_1(T) \;\geq\; e_2(T) \;\geq \ldots \geq\; 0 \;.$

(ii) *Additivity*
Let $T_1, T_2 : A \to B$ be two operators in the sense of the above definition. Assume B to be a λ-Banach space (with $\lambda = 1$ in the Banach case). Then

$$e^\lambda_{k_1+k_2-1}(T_1 + T_2) \;\leq\; e^\lambda_{k_1}(T_1) \;+\; e^\lambda_{k_2}(T_2) \quad , \; k_1,\; k_2 \in \mathbb{N} \;.$$

(iii) *Multiplicativity*
Let A, B and C be complex quasi-Banach spaces and $T_1 : A \to B$ and $T_2 : B \to C$ are assumed to be operators in the sense of the above definition, resp. Then

$$e_{k_1+k_2-1}(T_2 \circ T_1) \le e_{k_1}(T_1)\, e_{k_2}(T_2) \quad , \; k_1,\; k_2 \in I\!N\ .$$

Furthermore it is known that

$$\lim_{n\to\infty} e_n(T) = 0 \quad \Longleftrightarrow \quad T \quad \text{compact}\ .$$

Moreover, in the course of our later considerations the interpolation property of entropy numbers becomes very important. Therefore we present it separately. Forerunners in the Banach case are referred to in [A: 3.2]. Using standard notations from the real interpolation theory also repeated in [A: 3.2] we may formulate this extension in the following way. The notation of the respective entropy numbers is adapted in an obvious manner.

Proposition 1. *Let A be a quasi-Banach space and let $\{B_0, B_1\}$ be an interpolation couple of λ-Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_0 \cap B_1 \subset B_\theta \subset B_0 + B_1$ and*

$$\|b|B_\theta\| \le \|b|B_0\|^{1-\theta}\, \|b|B_1\|^{\theta} \quad \text{if} \quad b \in B_0 \cap B_1\ . \tag{1}$$

Let T be a linear and continuous operator from A into $B_0 \cap B_1$. Then for all $k_0, k_1 \in I\!N$

$$e_{k_0+k_1-1}(T : A \to B_\theta) \le 2^{\frac{1}{\lambda}}\, e_{k_0}^{1-\theta}(T : A \to B_0)\, e_{k_1}^{\theta}(T : A \to B_1)\ . \tag{2}$$

Remark 2. The proof may be found in [A: 5.3]. It is a less tricky but careful counting of balls additionally including the assumption (1). Besides there is given an extended version of the above proposition in [A: 3.2] including also the case with an interpolation couple of original spaces and the same target space.

2.1.2 The main theorem

We are well-prepared now to present our first essential outcome. In the centre of attention there are the compact natural embeddings

$$id^B : B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2} \tag{3}$$

and

$$id^F : F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2} \tag{4}$$

where the weighted spaces have been introduced in 1.1 and the parameters satisfy

$$\begin{gathered} -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le \infty \quad , \quad 0 < q_1 \le \infty, \quad \alpha > 0, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n}, \quad p_0 < p_2 \le \infty \quad , \quad 0 < q_2 \le \infty \end{gathered} \tag{5}$$

and

$$\delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} > 0 \tag{6}$$

(with $p_1 < \infty$ and $p_2 < \infty$ in the F-case). In particular, we are interested in a refined qualitative measuring of the compactness of the respective embedding operators dependent upon the value of $\alpha > 0$, i.e. the exponent of the weight function $w(x) = \langle x \rangle^\alpha$. Considering the result below one recognizes that there is an essential difference in the behaviour of the respective entropy numbers according to the cases $\alpha < \delta$, $\alpha = \delta$ and $\alpha > \delta$.

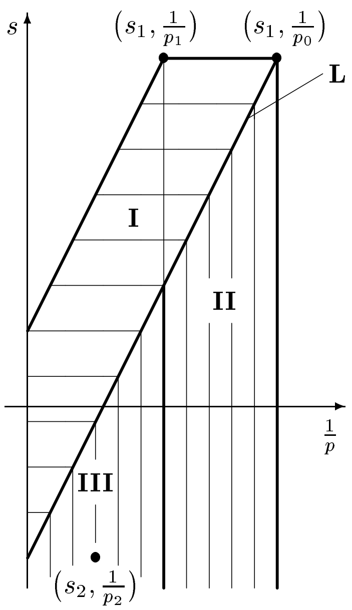


Fig.1

Remember Theorem 1.1/2, (1.1/24) and what we called our standard situation in 1.1. Then one can illustrate the range of parameters s_2 and p_2 (depending upon s_1, p_1 and α) where the embeddings (3) or (4), resp., are compact in a $(\frac{1}{p}, s)$-diagram as we did (to which the outer thick-lined shape refers). Thus a dividing of this region - as symbolized in the left-hand $(\frac{1}{p}, s)$-diagram - turns out to be reasonable. We tried to indicate these 4 cases in this diagram beside.

$$\begin{array}{rcl} \mathbf{I} & : & 0 < \delta < \alpha \\ \mathbf{II} & : & \delta > \alpha, \quad \dfrac{1}{p_1} \le \dfrac{1}{p_2} < \dfrac{1}{p_0} \\ \mathbf{III} & : & \delta > \alpha, \quad \dfrac{1}{p_2} < \dfrac{1}{p_1} \\ \mathbf{L} & : & \delta = \alpha, \quad \dfrac{1}{p_2} < \dfrac{1}{p_0} \end{array}$$

In addition to these notations we put $e_k(id^B)$ or $e_k(id^F)$ for the respective entropy numbers of (3) or (4). Finally we adopt the custom to write $e_k \sim k^{-\varrho}$ in the sense that there exist two positive numbers c_1 and c_2 such that for all $k \in I\!N$

$$c_1\, k^{-\varrho} \;\le\; e_k \;\le\; c_2\, k^{-\varrho}\,. \tag{7}$$

Theorem 1. *Let (5) and (6) be satisfied with $p_1 < \infty$ and $p_2 < \infty$ in the case of F-spaces.*

(i) *In region* **I** *holds*
$$e_k(id^B) \;\sim\; k^{-\frac{s_1-s_2}{n}}\,. \tag{8}$$

(ii) *In region* **II** *holds*
$$e_k(id^B) \;\sim\; k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}\,. \tag{9}$$

(iii) *In region* **III** *there exist a constant $c > 0$ and for any $\varepsilon > 0$ a constant c_ε such that*

$$c\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \;\le\; e_k(id^B) \;\le\; c_\varepsilon\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}\, (\log\langle k\rangle)^{\varepsilon-\frac{1}{p_2}+\frac{1}{p_1}} \quad \textit{for all} \quad k \in I\!N. \tag{10}$$

(iv)$_F$ *Let $\delta = \alpha$ and $\frac{1}{p_2} < \frac{1}{p_0}$, the line* **L**, *then there exists a constant $c > 0$ such that*

$$e_k(id^F) \;\ge\; c\, k^{-\frac{s_1-s_2}{n}}\, (\log\langle k\rangle)^{\frac{\alpha}{n}} \quad \textit{for all} \quad k \in I\!N. \tag{11}$$

(iv)$_B$ *Let $\delta = \alpha$ and $\frac{1}{p_2} < \frac{1}{p_0}$, the line* **L**, *and let in addition*

$$\textit{either} \quad p_2 = q_2 = \infty \tag{12}$$
$$\textit{or} \quad p_2 < \infty \quad \textit{and} \quad q_2 \ge p_2 \frac{q_1}{p_0}\,, \tag{13}$$

then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\, k^{-\frac{s_1-s_2}{n}} \;\le\; e_k(id^B) \;\le\; c_2 \left(\frac{k}{\log\langle k\rangle}\right)^{-\frac{s_1-s_2}{n}} \quad \textit{for all} \quad k \in I\!N. \tag{14}$$

Remark 3. Note that by (1.1/9) (and its weighted counterpart) id^B in (8), (9) and (10) can immediately be replaced by id^F as no q-parameters are involved there. In the figure above we indicated the level lines for the corresponding exponents. Unfortunately we are not able to make the upper estimate in (10) more precise though we would like to improve it. In fact it is not so clear which result will finally turn out in this case, i.e. whether the left-hand side gives the correct behaviour or something in between left- and right-hand estimate.

In [A: Rem.4.2/2] we started some discussion concerning the situation on the line "$\delta = \alpha$", probably the most interesting but difficult part of these investigations. In contrast to this we now omit further considerations because we will return to this subject in the next subsection and thus the detailed study is delayed. Nevertheless we did not completely exclude this complicated situation in the above theorem for one reason. Among all the nasty technicalities there surprisingly appears a fine by-product at least as elegant as strikingly simple to apply. One might call it a first application which suggests itself but as we finally aim at another kind of applications we consider it as a digression. Recall that for $s \in I\!R$, $0 < p < \infty$ and $0 < q \leq \infty$

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v} \tag{15}$$

if and only if

$$0 < u \leq \min(p,q) \qquad \text{and} \qquad \max(p,q) \leq v \leq \infty\,, \tag{16}$$

see (1.1/9). Proving the "if"-part - based on the definitions (1.1/5) and (1.1/6) as well as Hölder's inequality - is well-known and not so difficult, see [24: 2.3.2]. Just to the contrary it is by far more complicated to establish the converse direction. One has again to distinguish whether the assertion concerns the p- or q-parameter, resp. Following [19: 5.1] the proof of $u \leq q \leq v$ is rather easy whereas the case $u \leq p \leq v$ requires some more efforts. It is just this part which can be facilitated using $(\text{iv})_B$ and $(\text{iv})_F$ of the above theorem. The argumentation there is carried out by contradiction and very easy to grasp. But honestly spoken, we have to confess that this advantage offering new and strikingly simple methods of proving now itself leads back to a rather long and less convenient history of complicated investigations. Nevertheless the outcome is convincing by its elegance - at least in our opinion.

The proof of the theorem itself is splitted up into 2 different approaches basically. Concerning estimates from below we endeavour for a more geometrical consideration of the problem. The essential key consists in a localization property only recently proved by H. Triebel in [26]. We may benefit from the fact that it applies both to B- and F-spaces unlike the localization principle which is well-adapted to F-spaces only, see [25: 2.4.7]. Having this available one may transfer the whole argumentation to finite-dimensional l_p-spaces which are by far easier to handle than the original B- or F-spaces. The rest is a tricky counting of balls covering the unit ball of $B^{s_1}_{p_1,q_1}(\alpha)$ in $B^{s_2}_{p_2,q_2}$ (or similarly in the F-case). Only the particular consideration in the case $\delta = \alpha$ is based on the above-mentioned localization principle and hence restricted to the F-case.
The idea to get the upper estimates resembles that one of proving the compactness of embeddings in weighted function spaces as briefly indicated in Remark 1.1/7. In the same spirit we aim at dividing the whole $I\!R^n$ into bounded domains, this time in certain annuli, and a "rest" which can be neglected apart from a well-determined contribution which can be made arbitrarily small. The intention is to cut off the $I\!R^n$ outside a sufficiently large ball where in this outer domain the weight function dominates the norms (applying again the very useful characterization of B- or F-spaces via local means). Within each of these annuli covering the large ball we will regard our weight function as constant (up to general constants independent of the certain annuli). This is justified by its radial structure. Afterwards we apply the results for entropy numbers in function spaces on bounded domains as given in [6] and [7]. One also needs some preparation to cope with the influence of the underlying domain on the entropy numbers estimated in these just mentioned papers, in particular it is sufficient for our investigations to restrict this question to balls of radius $R > 0$ and to handle the dependence of the respective entropy numbers upon R. This has been done in [A: 4.1, 5.4]. Summing up these partly results according to the additivity as presented in Remark 1(ii) one has to care about convergence of sums where the exponent $\delta - \alpha$ appears, see e.g. [A: (5.5/17)]. Hence one recognizes where the different behaviour in the cases $\delta < \alpha$, $\delta = \alpha$ and $\delta > \alpha$ comes from. Furthermore we make use of interpolation arguments according to the above proposition. This proof is carried out in detail in [A: 5.5].

2.1.3 The line '$\delta = \alpha$' revisited, a refined version

As already announced this subsection is devoted to the peculiar situation on the line "$\delta = \alpha$". Already the above outlined digression emphasizes that this case is worth investigating, but at least the result below justifies the special interest in this subject. Unfortunately the applied methods of proving are neither elegant nor surprisingly new apart from involved duality arguments. But in our opinion the outcome is in some sense as extraordinary as the argumentations are long and technically complicated. What is this unusual behaviour announced in such a mysterious way and achieved under some efforts ? The phenomenon is simply the appearance of the q-parameters even in exponents of type (11) or (14). Regarding the assumptions (12) and (13) it seems reasonable to suspect that the third parameters play a more important role in this delicate situation. But at first glance they are only involved in the assumptions. The most surprising but also convincing fact is the following : in some special cases this q-dependence is even the correct behaviour in the exponents. Making the result understandable we spare the reader to learn all the technically-based assumptions here and refer to [D: 5]. Besides it worries us that the complicated description could finally cover the elegance of the result itself. Conversely we prefer to use some more pictures to support the imagination of what happens. We start with a summary of our considerations given in [D: 2,3,4].

Proposition 2. *Let the assumptions* (5) *and* (6) *be satisfied.*

(a) *Let $p_1 = q_1 = \infty$ and $p_2 = q_2$. Then*

$$e_k(\mathcal{C}^{s_1}(\alpha) \longrightarrow B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{17}$$

where $B^{s_1}_{\infty,\infty}(\alpha) = \mathcal{C}^{s_1}(\alpha)$. Moreover, if we additionally have $p_2 = q_2 = \infty$, then

$$e_k(\mathcal{C}^{s_1}(\alpha) \longrightarrow \mathcal{C}^{s_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} . \tag{18}$$

(b) *Let $1 < p_1 = q_1 < \infty$ and let $(\frac{1}{p_2}, \frac{1}{q_2})$ in the hatched area in the right-hand diagram. (The exact description of this domain in Fig.2, depending on $p_1 = q_1$ and $\alpha > 0$, is given in* [D: (5/17)]). *Then it holds*

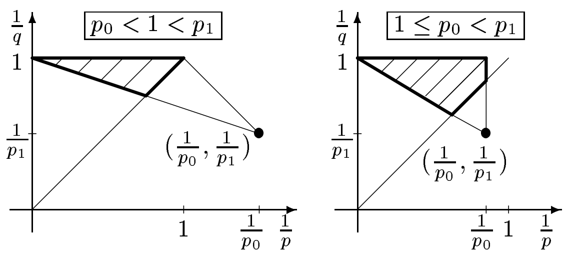


Fig.2

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} . \tag{19}$$

Moreover, if we additionally have $\min(p_0, 1) < p_2 = q_2 < p_0(1+\frac{\alpha}{n})$, *then*

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \longrightarrow B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} . \tag{20}$$

(c) *Let $1 < p_2 = q_2 < \infty$ and let $(\frac{1}{p_1}, \frac{1}{q_1})$ in the hatched area in the right-hand diagram. (The exact assumptions concerning p_1 and q_1 to be admissible in the sense of Fig.3, depending on $p_2 = q_2$ and $\alpha > 0$, may be found in* [D: (5/20)]). *Then it holds*

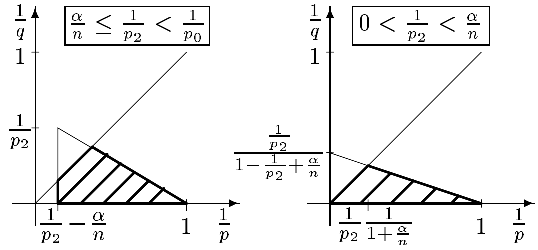


Fig.3

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{q_1}} . \tag{21}$$

Moreover, if we additionally have $(\frac{1}{p_2} - \frac{\alpha}{n})_+ < \frac{1}{p_1} < \frac{1}{p_2}\,\frac{1}{1+\frac{\alpha}{n}}$, *then*

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \longrightarrow B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} . \tag{22}$$

Comparing the above results with Theorem 1, especially (14), and [A: Rem. 4.2/2], assertion (a) seems reasonable because we already had (18) before. In contrast to this one observes quite new exponents in (19) and (21) where some q-index first time appears. But we confess a rather hard and technical way to arrive there. The essential ideas may be outlined by duality arguments, interpolation and tricky compositions of embedding operators. Another approach comes from a refined dealing with approximation numbers. Unlike approximation numbers which will be officially introduced in 2.3 entropy numbers are not compatible with duality in general, that is one would like to have general constants to estimate $e_k(T)$ by $e_k(T^*)$, where T^* denotes the dual operator of T, and vice versa, but this fails to hold in general. Recently this unattainable desire (in case of entropy numbers) could be compensated, at least partly. We used the following assertion proved by Bourgain, Pajor, Szarek and Tomczak-Jaegermann in [2] :

Let A be a uniformly convex Banach space, let B be a Banach space and assume $T \in L(A,B)$ to be compact. Then there is a positive constant $c = c(A)$, such that for all $m \in I\!N$ and $0 < r < \infty$ it holds

$$c^{-1} \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T^*) \;\leq\; \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T) \;\leq\; c \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T^*)\,. \tag{23}$$

To apply this result we had to study which spaces in our investigations are uniformly convex and what about duality of those spaces, for definitions of uniform convexity see [1: 3/II/§1, p.189] or [14: Def.1.e.1, pp. 59/60]. It could be established that B- or F-spaces satisfy the condition of uniform convexity if $1 < p,q < \infty$. Looking for the dual operator of $id : B^{s_1}_{p_1,q_1}(w_1(\cdot)) \to B^{s_2}_{p_2,q_2}(w_2(\cdot))$ it turns out to be $id' = id : B^{-s_2}_{p_2',q_2'}(w_2^{-1}(\cdot)) \to B^{-s_1}_{p_1',q_1'}(w_1^{-1}(\cdot))$, cf [24: (2.11.2/1)] or [18: (5.1.2/6), (5.1.2/7)] for details. One has to care about the different restrictions to the parameters then. The interpolation property of entropy numbers in the form we wish to apply has already been mentioned above. The contribution from the theory of approximation numbers comes from the special structure of the operators we add up. Remember the outline of the proof of the main theorem. One important tool to get the estimates from above has been there a suitable partition of unity according to the annuli which allowed us to regard our problem on bounded domains (annuli) and to combine these separate results afterwards. The somewhat improved estimate for the approximation numbers is based on this fact, see 2.3 for the definition and basic properties of approximation numbers (and the references given there) and [D: 3] for the above mentioned approach. Besides we could benefit from a certain relation between entropy numbers and approximation numbers as proved in [27], namely the e_k's can be estimated from above by the respective a_k's under certain circumstances which fortunately apply to our situation, see Remark 2.3/1. As for the estimates from below we mainly rely on the assertion $(\mathrm{iv})_F$ of Theorem 1 and extend it in the B-case by some tricky compositions of operators and interpolation again. We refer to [D: 2,3,4] for a detailed study of these outlined ideas.

We return to (19) and (21). Note that we have $p_1 = q_1$ and $p_2 = q_2$ in these cases, resp., and the log-exponent can thus also be written as

$$\frac{s_1 - s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} \tag{24}$$

in both cases. Consequently it is natural to ask whether there may even exist cases with $p_1 \neq q_1$ and $p_2 \neq q_2$ but the same log-exponent (24). Fortunately the answer is yes though we have to content ourselves with a complicated set of restrictions to guarantee this phenomenon.

We decided to omit the nasty details this time and present the theorem below rather as a statement about existence.

Theorem 2. *Let the general assumptions* (5) *and* (6) *be satisfied with* $p_1,\ p_2,\ q_1\ q_2 \in (1,\infty)$. *Then there exist conditions for these parameters implying that*

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}}\,. \tag{25}$$

Remark 4. The above mentioned-restrictions to the parameters p_1, p_2, q_1, q_2 are presented in [D: (5/26)-(5/29)]. They are rather nasty and not worth giving them here in full detail. We want to compensate this lack of information in some sense by another picture. Believing in the correctness of these diagrams for the moment they may supply the imagination that such parameters can really exist and cases as described above may appear. There are two different situations basically, see [D: Rem.5/2], thus we sketched two $(\frac{1}{p}, \frac{1}{q})$-diagrams below.

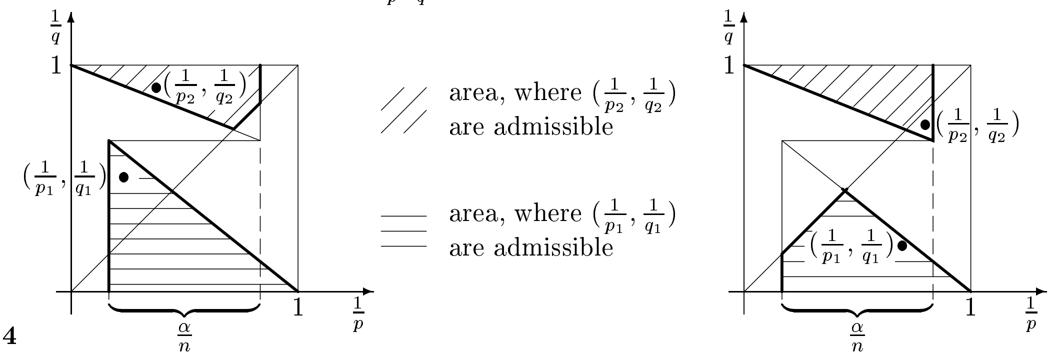


Fig.4

We have tried for a rather extensive study of these results and possible consequences in [D: 5]. We do not plan to repeat all these arguments now. But it is worth mentioning that there are in fact cases with $p_1 \neq q_1$ and $p_2 \neq q_2$ satisfying the above assumptions. Hence the behaviour of the entropy numbers in those cases is really characterized by means of the third indices which is remarkable in our opinion. On the other hand we call the necessity of our restrictions [D: (5/26)-(5/29)] into question but obviously some condition concerning the p-parameters as well as both q-parameters (always in addition to (5) and (6)) is required, for we know by (14) that

$$e_k(id^B) \ \geq \ c\, k^{-\frac{s_1-s_2}{n}} \tag{26}$$

in either q-case (assumptions (12) and (13) are only necessary for the upper estimate, see [A: (5.5/8)]). Consequently one inevitably has for the log-exponent

$$\frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} = \frac{\alpha}{n} - \big(\frac{1}{q_1} - \frac{1}{p_1}\big) - \big(\frac{1}{p_2} - \frac{1}{q_2}\big) > 0\ . \tag{27}$$

Assumptions like this are included in those conditions achieved in [D: (5/26)-(5/29)]. Moreover, we have by (5) that $\frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n}$ and hence (27) would also be satisfied if $q_2 < q_1$,

$$\frac{\alpha}{n} + \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{q_1} > \frac{1}{q_2} - \frac{1}{q_1} > 0\ . \tag{28}$$

At the moment we are neither able to give a complete result nor dare to formulate a conjecture, but a question.

Problem 1 : *Let the general assumptions* (5) *and* (6) *be satisfied. Which additional restrictions one has to impose on the whole range of parameters to achieve*

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}) \ \sim \ k^{-\frac{s_1-s_2}{n}} \ \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}} \quad ?$$

Turning to the F-spaces there is even more uncertainty than in the B-case. We collect our results in a proposition.

Proposition 3. *Let the assumptions* (5) *and* (6) *be satisfied. Let*

$$\begin{array}{ll} \textit{either} & 1 < p_1 < \infty, \quad 0 < q_1 \leq p_1 < \infty, \quad 1 \leq p_2 < p_0(1+\frac{\alpha}{n}), \quad p_2 \leq q_2 \leq \infty \\ \textit{or} & p_2(1+\frac{\alpha}{n}) < p_1 < \infty, \quad 0 < q_1 \leq p_1 < \infty, \quad 1 < p_2 < \infty, \quad p_2 \leq q_2 \leq \infty\ . \end{array} \tag{29}$$

Then

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2}) \ \sim \ k^{-\frac{s_1-s_2}{n}} \ \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\ . \tag{30}$$

Remark 5. We have also discussed these outcomes in [D: 5] in a wider sense, for instance the different role played by the q-parameters in the B- as well as F-case in other frameworks such as embeddings along constant differential dimension, see (1.1/7) and (1.1/8), or within the trace theorem, cf. [D: Rem. 5/6] and the references given there. Furthermore we included a former result of MYNBAEV and OTEL'BAEV in our considerations, see [D: Rem. 5/7]. But for the moment we have to confine ourselves to ask another (open) question.

Problem 2 : *Let the general assumptions* (5) *and* (6) *be satisfied. Which additional restrictions have to be imposed on the whole range of parameters to get*

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2}) \sim k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} \quad ?$$

Though we may not complete our investigations about the entropy numbers of the embeddings on which we have been concentrated in our so-called standard situation we believe in having contributed some more or less interesting ingredients to cope with this problem. We can not content ourselves with this obvious gap in between the upper and lower estimate in (10) but there is no idea at the moment how to manage this task. Otherwise we are convinced that our results at the line "$\delta = \alpha$" are only the beginning of further considerations because this situation promises even more surprises and a completely new insight into the inner correlations at this critical line. Moreover, our outcome in Theorem 1 may be supplied by the achieved results in Proposition 2, Theorem 2, Proposition 3 and further (one-sided) estimates in [D: 5]. There is finally a considerable amount of embedding operators $id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}$ or $id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2}$ where we may provide a satisfying answer about the behaviour of its respective entropy numbers, we gave an example in [D: Rem. 5/9].

2.2 Eigenvalue distribution, negative spectrum

2.2.1 Preparations

Though our results Theorem 2.1/1 and Theorem 2.1/2, resp., are also of self-contained interest our intention to look for possible applications might be understandable. Moreover, we have been influenced by some "forerunner" [8] where similar ideas have been carried out. Our aim now is to study the eigenvalue distribution of some degenerate pseudodifferential operator as well as the "negative spectrum" of related symmetric operators in L_2 based on the Birman-Schwinger principle.

First we present the main tools which enable us to make use of those more abstract results in 2.1 in order to prove rather specialized assertions about eigenvalues of (certain) operators which may also appear in physics. One crucial link is given by CARL's famous inequality.

<u>CARL's inequality</u>

Let X be a complex quasi-Banach space and let $B : X \to X$ be compact and linear. Recall the definition of entropy numbers, see Definition 2.1 and the references given there. Let e_k be the kth entropy number of B and $\{\mu_k\}$ the sequence of its eigenvalues, counted with respect to their algebraic multiplicity and ordered by decreasing modulus

$$|\mu_1| \geq |\mu_2| \geq \dots \tag{1}$$

Proposition 1. *Under the above assumptions,*

$$\left(\prod_{m=1}^{k} |\mu_m|\right)^{\frac{1}{k}} \leq \inf_{n\in \mathbb{N}} 2^{\frac{n}{2k}} e_n \quad , \textit{for all} \quad k \in \mathbb{N}, \tag{2}$$

in particular,

$$|\mu_k| \leq \sqrt{2}\, e_k \quad \textit{for all} \quad k \in \mathbb{N} \,. \tag{3}$$

A proof of this theorem, restricted to Banach cases, may be found in [3: Theorem 4.2.1], for further details and remarks see [B: 2.3].

Another important ingredient to cope with problems we wish to handle in the sequel is some extension of the (classical) Hölder inequality.

Hölder inequalities

This well-known inequality may be written as

$$L_{r_1}\, L_{r_2} \;\subset\; L_r \tag{4}$$

where

$$1 \le r_1 \le \infty, \quad 1 \le r_2 \le \infty \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1\,. \tag{5}$$

One may ask for extensions to other F- as well as B-spaces. Sickel and Triebel followed this way in [19] to get more general results concerning this question. For simplicity we restricted ourselves in [B] to (weighted) spaces of type $H^s_p = F^s_{p,2}$, $s \in I\!R$, $0 < p < \infty$. Likewise we proceed now.

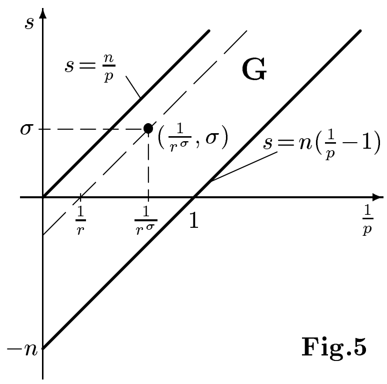


Fig.5

Introduce some strip

$$\mathbf{G} = \Big\{ \big(\frac{1}{p}, s\big) : 0 < p < \infty,\; n\big(\frac{1}{p} - 1\big) < s < \frac{n}{p} \Big\}$$

in the usual $(\frac{1}{p}, s)$-diagram. Then any line of slope n in this diagram is characterized by its "footpoint", i.e. that point at which it meets the axis $s = 0$. Any point on that line in the strip $\mathbf{G}$ has coordinates

$$\big(\frac{1}{r} + \frac{\sigma}{n}, \sigma\big) = \big(\frac{1}{r^\sigma}, \sigma\big) \quad \text{with} \quad \frac{1}{r} + \frac{\sigma}{n} > 0,\; 1 < r < \infty\,. \tag{6}$$

Having these notations in mind we can formulate the needed generalization of (4) as follows.

Proposition 2. *Let*

$$1 < r_1 < \infty, \quad 1 < r_2 < \infty \quad \textit{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} < 1\,. \tag{7}$$

Suppose that

$$s \in I\!R \quad \textit{and} \quad \frac{1}{r_1} + \frac{s}{n} > 0, \tag{8}$$

$w_1(x) \in W, w_2(x) \in W$ and $w(x) = w_1(x)\, w_2(x)$. Then

$$H^s_{r_1^s}(w_1(x))\, H^{|s|}_{r_2^{|s|}}(w_2(x)) \;\subset\; H^s_{r^s}(w(x)) \tag{9}$$

where $\frac{1}{r^s} = \frac{1}{r} + \frac{s}{n}$ and $r_1^s, r_2^{|s|}$ are defined analogously.

Note that (9) coincides with the weighted extension of (4) if $s = 0$, apart from limiting cases. The proof may be found in [8: Theorem 2.4] and [19: Theorem 4.2], the latter one in the case $s > 0$, see also [B: 2.2] and the references given there.

2.2.2 Eigenvalue distribution

We study the map

$$B\, f \;=\; b_2(x)\, b(x, D)\, b_1(x)\, f \tag{10}$$

where b_1 and b_2 belong to some spaces $L_{r_j}(\langle x\rangle^{\alpha_j})$ or $H^s_{r_j^s}(\langle x\rangle^{\alpha_j})$, $b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$ with $\varkappa > 0$, $0 \le \gamma \le 1$. For the moment we stick at the "ground level" $s = 0$, i.e. we look for spaces L_p, $1 < p < \infty$, such that B becomes compact. (As the essential ideas of proving assertions of the type below will turn out even in that simplest case we do not want to complicate the situation more than necessary.) Let $\{\mu_k\}$ be the sequence of its eigenvalues, counted according to their multiplicity and ordered by decreasing modulus. The collection of all associated eigenvectors of a given eigenvalue is denoted as the corresponding root space.

Theorem 1. *Suppose* $\varkappa > 0$, $0 \le \gamma \le 1$, *and* $b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$, (11)

$$\alpha_1 \in I\!R, \quad \alpha_2 \in I\!R, \quad \textit{with} \quad \alpha = \alpha_1 + \alpha_2 > 0, \tag{12}$$

$$1 \le r_1 \le \infty, \quad 1 \le r_2 \le \infty \quad \textit{with} \quad \frac{1}{r_1} + \frac{1}{r_2} < \min\left(1, \frac{\varkappa}{n}\right), \tag{13}$$

$$b_1(x) \in L_{r_1}(\langle x\rangle^{\alpha_1}), \quad b_2(x) \in L_{r_2}(\langle x\rangle^{\alpha_2}). \tag{14}$$

(i) *For any p with $1 < p < \infty$ and*
$$\frac{1}{r_2} < \frac{1}{p} < 1 - \frac{1}{r_1} \tag{15}$$
and any $w(x) \in W$, the operator B, given by (10) is compact in $L_p(w(x))$. Furthermore,
$$|\mu_k| \;\le\; c\,\|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\|\,\|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\|\,k^{-\frac{\varkappa}{n}}, \quad k \in I\!N, \tag{16}$$
$$\textit{if} \quad \frac{1}{r_1} + \frac{1}{r_2} > \frac{\varkappa - \alpha}{n} \tag{17}$$
and
$$|\mu_k| \;\le\; c_\varepsilon\,\|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\|\,\|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\|\,k^{-\frac{\alpha}{n}-(\frac{1}{r_1}+\frac{1}{r_2})}\,(\log\langle k\rangle)^{\varepsilon+\frac{1}{r_1}+\frac{1}{r_2}}, \quad k \in I\!N, \tag{18}$$
for any $\varepsilon > 0$ (with $\varepsilon = 0$ if $r_1 = r_2 = \infty$) and suitably chosen constants c_ε if
$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa - \alpha}{n}\,. \tag{19}$$

(ii) *For fixed p with (15) the root spaces coincide for all $w(x) \in W$.*

The proof of this theorem is detailed in [B: 4.1]. The assertion (ii) means that for different basic spaces in question the eigenvalues coincide and that for any given eigenvalue the corresponding root spaces coincide.

We want to outline the proof of part (i) of the above theorem because it is strikingly simple and effective. Furthermore one may get an impression how the above-mentioned tools are involved into the argumentation. In our opinion the elegance of this procedure is rather obvious.

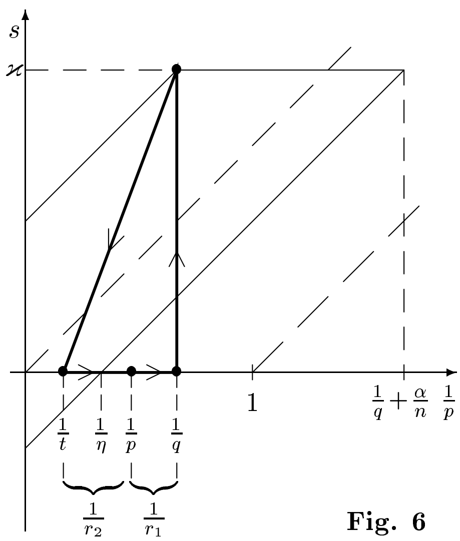


Fig. 6

We decompose B as
$$B = b_2 \circ id \circ b(x,D) \circ b_1\,, \tag{20}$$
see the left-hand diagram, with
$$\left\{\begin{array}{ll} b_1 & : L_p(w(x)) \to L_q(w(x)\langle x\rangle^{\alpha_1}) \\ & \qquad\qquad \text{with} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r_1}, \\ b(x,D) & : L_q(w(x)\langle x\rangle^{\alpha_1}) \to H^\varkappa_q(w(x)\langle x\rangle^{\alpha_1}), \\ id & : H^\varkappa_q(w(x)\langle x\rangle^{\alpha_1}) \to L_t(w(x)\langle x\rangle^{-\alpha_2}) \\ & \qquad\qquad \text{with} \quad \frac{1}{t} = \frac{1}{p} - \frac{1}{r_2}, \\ b_2 & : L_t(w(x)\langle x\rangle^{-\alpha_2}) \to L_p(w(x))\,. \end{array}\right. \tag{21}$$

Using both Proposition 2 and (14) we get the continuity of the first-line embedding as well as the last one, whereas the second line is covered by Theorem 1.2. As for the third line in (21) we have $\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa}{n}$ and $\frac{w_1(x)}{w_2(x)} = \langle x\rangle^\alpha$. Recall our introductory remarks in 1.1.3.

Hence the above embedding $id : H_q^{\varkappa}(w(x)\langle x\rangle^{\alpha_1}) \to L_t(w(x)\langle x\rangle^{-\alpha_2})$ is compact and just of the type we thoroughly investigated in 2.1. Note that

$$\delta = \varkappa - \frac{n}{q} + \frac{n}{t} = \varkappa - \frac{n}{r_1} - \frac{n}{r_2} > 0 \tag{22}$$

in our situation. The application of Theorem 2.1/1 yields for the respective entropy numbers

$$e_k(id) \sim k^{-\frac{\varkappa}{n}} \qquad \text{if} \qquad \frac{1}{t} < \frac{1}{\eta} = \frac{\alpha}{n} + \frac{1}{q} - \frac{\varkappa}{n} \tag{23}$$

and

$$e_k(id) \le c_\varepsilon\, k^{-\frac{\alpha}{n}-(\frac{1}{r_1}+\frac{1}{r_2})} \left(\log\langle k\rangle\right)^{\varepsilon+\frac{1}{r_1}+\frac{1}{r_2}} \qquad \text{if} \qquad \frac{1}{t} > \frac{1}{\eta} \tag{24}$$

according to the different regions **I** or **III** in Fig.4 which may occur in various situations. (For simplicity we excluded the case $\delta = \alpha$, i.e. $\frac{\varkappa-\alpha}{n} = \frac{1}{r_1} + \frac{1}{r_2}$.) Now CARL's inequality, that is (3), finally leads to the assertion.

Remark. One may ask whether a strengthening of the assumptions results in improved information about the image of B as one would hope. Indeed, there is an interplay between improved smoothness and worsened compactness as in classical theories. This has been studied in [B: 4.2]. Furthermore there are also those modifications indicated how to adopt the above theorem to situations where L_p as the basic space is left and replaced by some $H^s_{p^s}$ space, see [B: Theorem 4.2/2].

It is also possible to deal with degenerate pseudodifferential operators of positive order given by, at least formally,

$$A\, f = a_1(x)\, a(x,D)\, a_2(x)\, f \tag{25}$$

with

$$a(x,D) \in \Psi^{\varkappa}_{1,\gamma}, \quad \varkappa > 0, \quad 0 \le \gamma < 1\ . \tag{26}$$

To avoid difficulties we adopt the following point of view (as already indicated in [B: 4.3]). Let B be given by (10) with $0 \le \gamma < 1$, assume that 0 is not an eigenvalue of the compact operator $b(x,D)$ and $b_1(x) \neq 0$, $b_2(x) \neq 0$ a.e. in $\mathbb{R}^n$. Let the assumptions (11)-(15) be satisfied and $w(x) \in W$. Then B is invertible in $L_p(w(x))$ and, at least formally, $A = B^{-1}$ is given by (25) with $a_1 = b_1^{-1}$ and $a_2 = b_2^{-1}$. B is compact, consequently A an unbounded operator in $L_p(w(x))$ with pure point spectrum. Let $\{\lambda_k\}$ be the sequence of its eigenvalues, counted according to algebraic multiplicity and ordered by increasing modulus. Hence $\lambda_k = \mu_k^{-1}$ where $\{\mu_k\}$ is the sequence of eigenvalues of B as above described.

Corollary 1. *Let the above assumptions be satisfied.*

(i) *If* $\frac{1}{r_1} + \frac{1}{r_2} > \frac{\varkappa-\alpha}{n}$, *then*

$$|\lambda_k| \ge c\, \|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\|^{-1}\, \|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\|^{-1}\, k^{\frac{\varkappa}{n}}, \qquad k \in \mathbb{N}\ . \tag{27}$$

(ii) *If* $\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa-\alpha}{n}$, *then*

$$|\lambda_k| \ge c_\varepsilon\, \|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\|^{-1}\, \|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\|^{-1}\, k^{\frac{\alpha}{n}+\frac{1}{r_1}+\frac{1}{r_2}} \left(\log\langle k\rangle\right)^{-\varepsilon-\frac{1}{r_1}-\frac{1}{r_2}},\ k \in \mathbb{N} \tag{28}$$

(*with* $\varepsilon = 0$ *if* $r_1 = r_2 = \infty$).

This assertion and further remarks may be found in [B: 4.3].

2.2.3 The negative spectrum

We begin with a preparatory proposition.

Birman-Schwinger principle

Recall the Birman-Schwinger principle as described in [17: Ch.8, Sect.5, p.193]. Let A be a self-adjoint operator acting in a Hilbert space $\mathcal{H}$ and let A be positive. Let V be a closable operator acting in $\mathcal{H}$ and suppose that $K : \mathcal{H} \to \mathcal{H}$ is a compact linear operator such that

$$K\,u \;=\; V\,A^{-1}\,V^*\,u \qquad \text{for all} \quad u \in \operatorname{dom}(V\,A^{-1}\,V^*) \tag{29}$$

where V^* is the adjoint of V. Assume that $\operatorname{dom}(A) \cap \operatorname{dom}(V^*\,V)$ is dense in $\mathcal{H}$. Then the above-mentioned result provides: $A - V^*\,V$ has a self-adjoint extension H with pure point spectrum in $(-\infty, 0]$ such that

$$\#\,\{\sigma(H) \cap (-\infty, 0]\,\} \;\le\; \#\,\{k \in I\!N : |\lambda_k| \ge 1\} \tag{30}$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of K, counted according to their multiplicity and ordered by decreasing modulus. If M is a finite set, the number of elements of M is denoted by $\#\,M$, as usual. We adapt this fact to our situation, additionally involving CARL's inequality, see Proposition 1. Let $A = a(x, D)$ be a positive-definite self-adjoint pseudodifferential operator in L_2, typically of positive order, and let

$$V \;=\; a(x)\,p(x, D)\,a(x) \tag{31}$$

be a degenerate pseudodifferential operator of lower order than A. Assume that $a(x)$ is a real function belonging to some space L_p (or $H^s_p(w(x))$). We do not want to bother about domains of definition thus we adopt the point of view that all operators in the sequel may be assumed to be defined at least on S, the rest is a matter of completion.

Let $V\,A^{-1}$ be compact in L_2. Hence $A^{-1}\,V$ is also compact (after completion) and $A - V$ self-adjoint on $\operatorname{dom}(A)$. The essential spectra of A and $A - V$ coincide. Thus we may formulate that version of the Birman-Schwinger principle we wish to use.

Proposition 3. *Let the above conditions be satisfied and let $\sigma(A - V)$ be the spectrum of $A - V$. Then*

$$\#\,\{\sigma(A - V) \cap (-\infty, 0]\,\} \;\le\; \#\{\,k \in I\!N : \sqrt{2}\,e_k(V\,A^{-1}) \ge 1\,\} \tag{32}$$

and

$$\#\,\{\sigma(A - V) \cap (-\infty, 0]\,\} \;\le\; \#\{\,k \in I\!N : \sqrt{2}\,e_k(A^{-1}\,V) \ge 1\,\}\,. \tag{33}$$

Regarding (30) as well as (32), (33) one immediately recognizes that CARL's inequality is one reason for the obvious modifications, for further comments see [B: Remark 2.4/1]. In particular, let $p(x, D) = id$ in (31). Then V is a multiplication operator and one may write the above assertion in the form

$$\#\,\{\sigma(A - V) \cap (-\infty, 0]\,\} \;\le\; \#\{\,k \in I\!N : \sqrt{2}\,e_k(a\,A^{-1}\,a) \ge 1\,\}\,. \tag{34}$$

Like in the previous subsection we only want to present a more or less typical example to explain that kind of applications we have in mind in the sequel. Further extensions and generalizations may be found in [B: 5]. We restrict ourselves to the basic space L_2 and want to study the behaviour of the negative spectrum of the self-adjoint unbounded operator

$$H_\beta = a(x, D) - \beta\,b(x)\,p(x, D)\,b(x) \quad \text{as} \quad \beta \to \infty \tag{35}$$

where

$$a(x, D) \in \Psi^{\varkappa}_{1,\gamma} \quad \text{with} \quad \varkappa \ge 0 \quad \text{and} \quad 0 \le \gamma < 1 \tag{36}$$

is assumed to be a positive-definite and self-adjoint operator in L_2 whereas

$$p(x, D) \in \Psi^{\eta}_{1,\gamma} \quad \text{with} \quad -\infty < \eta < \varkappa \quad \text{and} \quad 0 \le \gamma < 1 \tag{37}$$

is symmetric and $b(x)$ is a real-valued function. To simplify the situation from the beginning as far as possible we assume $p(x, D) = id$ in our example. Thus (35) becomes

$$H_\beta = a(x, D) - \beta\,b^2(x) \tag{38}$$

and we get from (34)

$$\#\{\sigma(H_\beta)\cap(-\infty,0]\} \;\le\; \#\{\,k\in I\!N : \sqrt{2}\,e_k \ge \beta^{-1}\,\} \tag{39}$$

with

$$e_k = e_k(b(x)\,b(x,D)\,b(x)), \quad b(x,D) = a^{-1}(x,D) \in \Psi^{-\varkappa}_{1,\gamma}\,. \tag{40}$$

Theorem 2. *Let $\varkappa > 0$, $0 \le \gamma < 1$, $\alpha > 0$,*

$$\max\Big(2, \frac{2n}{\varkappa}\Big) \;<\; r \le \infty \tag{41}$$

and

$$b(x) \in L_r(\langle x\rangle^{\frac{\alpha}{2}}) \quad \text{real}\,. \tag{42}$$

Let H_β be the operator from (38), then

$$\#\{\sigma(H_\beta)\cap(-\infty,0]\} \;\le\; c\,\big(\beta\,\|b|L_r(\langle x\rangle^{\frac{\alpha}{2}})\|^2\big)^{\frac{n}{\varkappa}} \tag{43}$$

if $\varkappa - \alpha < \frac{2n}{r}$ and

$$\#\{\sigma(H_\beta)\cap(-\infty,0]\} \;\le\; c_\varepsilon\,\beta_0^{(\frac{\alpha}{n}+\frac{2}{r})^{-1}}\,\big(\log\langle\beta_0\rangle\big)^{(\varepsilon+\frac{2}{r})(\frac{\alpha}{n}+\frac{2}{r})^{-1}} \tag{44}$$

with $\beta_0 = \beta\,\|b|L_r(\langle x\rangle^{\frac{\alpha}{2}})\|^2$ if $\varkappa - \alpha > \frac{2n}{r}$ (and with $\varepsilon = 0$ if $r_1 = r_2 = \infty$).

The proof is simply a further application of Theorem 1 with $b_1 = b_2 = b$, $p = 2$ and e_k instead of μ_k in (16), (18), see [B: 5.2]. As already announced this theorem can be generalized in some direction. We may admit more general operators $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$ and functions $b(x)$ belonging to some weighted H-spaces. This subject has been treated in [B: 5.2] in some detail. A further modification consists in splitting techniques for the functions $b(x)$. This approach is motivated by its own history: the interest to study the “negative“ spectrum (bound states) comes from quantum mechanics, generalizing the classical hydrogen operator,

$$H = -\Delta - \frac{c}{|x|}, \quad c > 0, \tag{45}$$

in $L_2(I\!R^3)$. Thus “potentials“ $b(x) \sim |x|^{-\alpha}$, $\alpha > 0$, are of peculiar interest, i.e. functions $b(x)$ with local singularities and some decay properties at infinity. Consequently it appears reasonable to split

$$b(x) = d_1(x) + d_2(x) \tag{46}$$

where $d_1(x)$ may be compactly supported (collecting the local singularities) and $\langle x\rangle^\alpha\, d_2(x) \in L_\infty$ for some $\alpha > 0$. As for this approach the investigations in [B: 5.3] are recommended.

Last but not least one faces a rather disagreeable test concerning our considerations : one has to verify already known results by means of our theory developed so far.
Let $a(x,D) \in \Psi^{\varkappa}_{1,\gamma}$, $\varkappa > 0$, $0 \le \gamma < 1$, a self-adjoint positive operator in L_2 with $0 \in \sigma_e$, σ_e is the essential spectrum of $a(x,D)$. Assume that $b(x)$ is a real function and $b^2(x)\,(id + a(x,D))^{-1}$ compact. Then the operator

$$H = a(x,D) - b^2(x) \tag{47}$$

has the same essential spectrum as $a(x,D)$ and it is even more natural to investigate

$$\#\{\sigma(H)\cap(-\infty,-\varepsilon]\} \quad \text{as} \quad \varepsilon \downarrow 0 \tag{48}$$

than

$$\#\{\sigma(H_\beta)\cap(-\infty,0]\} \quad \text{as} \quad \beta \to \infty \tag{49}$$

as we did up to now. This might be understood by its historical background, see [B: 5.3]. In general, it is rather complicated to reduce (48) to (49) but fortunately the situation eases up if $a(x,D)$ as well as $b(x)$ additionally satisfy some homogeneity conditions. We demonstrate at a comparatively simple example what is meant by this. Let

$$a(D) = (-1)^m \sum_{|\gamma|=m} a_\gamma\, D^{2\gamma} \qquad \text{with} \qquad a_\gamma \in I\!R,\; \sum_{|\gamma|=m} a_\gamma\, \xi^{2\gamma} \;\ge\; c\,|\xi|^{2m},\; \xi \in I\!R^n \tag{50}$$

for some $c > 0$, be an elliptic differential operator of order $2m$ with constant coefficients. Let

$$H = a(D) - |x|^{-\eta} \quad \text{with} \quad 0 < \eta < \min(n, 2m)\,. \tag{51}$$

Then we may find some number r with
$$\frac{2n}{\eta} > r > \max(2, \frac{n}{m}) \ . \tag{52}$$

Corollary 2. *Let H be given as above. Then there is some $c > 0$ such that for all $\varepsilon > 0$*
$$\# \{\sigma(H) \cap (-\infty, -\varepsilon]\} \ \leq \ c\, \varepsilon^{-n(\frac{1}{\eta} - \frac{1}{2m})} \ . \tag{53}$$

The proof essentially using already shown results as well as the assumed homogeneity is presented in [B: 5.4]. Hence one may cope with the above-mentioned problems and finally arrive at already known estimates in case of the hydrogen atom, see [B: Remark 5.4/1]. Nevertheless this result covers some more cases than this particular one and may be recognized as a generalization.

Though we could only indicate some typical examples and outline the involved ideas rather roughly we hope to have conveyed some feeling what a large number of problems can be handled by means of our above-described methods. There open up a lot of further possible applications apart from those suggested ones and presented in [B]. One might disagree that the advantages of the above methods are simplicity and effectiveness because the "harvest" of this section has been required some more efforts in the previous one. Nevertheless one can not deny the convincing elegance.

2.3 Approximation numbers

2.3.1 Preliminaries

We return to the study of the compactness of embeddings of type (2.1/3) and (2.1/4), resp. Remember our standard situation as presented in 1.1.3. This time we want to use approximation numbers instead of entropy numbers to measure compactness because entropy and compactness are closely related to approximation properties. 'Approximation' means approximation by finite rank operators. There are more approximation quantities like Kolmogorov and Gelfand numbers, for instance, but we confine ourselves to approximation numbers.
Let A and B be two complex quasi-Banach spaces and let T be a linear and continuous operator from A into B.

Definition. Let $k \in \mathbb{N}$ and assume $T : A \longrightarrow B$ to be the above continuous operator. The *kth approximation number a_k of T* is the infimum of all numbers $\| T - L \|$ where L runs through the collection of all continuous linear maps from A to B with rank $L < k$.

Remark 1. Likewise to the case of entropy numbers recall some basic properties of approximation numbers first, for details see [3] as well as [16]. Approximation numbers also have the properties of monotonicity, additivity and multiplicativity as given in 2.1.1 in case of entropy numbers. But in contrast to the entropy numbers one only has
$$\lim_{n \to \infty} a_n(T) = 0 \quad \Longrightarrow \quad T \quad \text{compact}$$
in general. If the target space B fails to have the approximation property it may happen that the sequence $a_j(T)$ does not tend to zero as j goes to infinity, see [13: Thm.1.e.4, p.32] and [13: Def.1.e.1, p.30] for a definition of the approximation property. Furthermore the approximation numbers do not have such a nice interpolation property for arbitrary operators T, but they behave well under duality, see [3: Prop. 2.5.4] : in case of arbitrary Banach spaces A and B let $T \in L(A, B)$ and T^* its dual operator. Then one has
$$a_n(T^*) \ \leq \ a_n(T) \ \leq \ 5\, a_n(T^*) \ ,$$
for the proof and further details we refer again to [3].

Dealing simultaneously with entropy as well as approximation numbers it is quite natural to ask whether there is some relation between these quantities. At first glance this question has to be refused as there cannot hold some general assertion. Recall some examples which may support this: Let $T \neq 0$ and $e_l(T) \leq c\, 2^{-l\varkappa}$ for some $c > 0$, $\varkappa > 0$ and all $l \in \mathbb{N}$. Then we may apply a result of CARL and STEPHANI as given in [3: p.14] to conclude that T is of finite rank and hence $a_m(T) = 0$ for large $m \in \mathbb{N}$. In [3] one may also find another such example, contradicting the assumption that it might hold $e_k \leq c\, a_k$ for some constant $c > 0$ and all $k \in \mathbb{N}$, this example provides $a_j \sim 2^{-j}$ and $e_j \sim 2^{-\sqrt{2j}}$, see [3: p.106]. Nevertheless we have not to come away from our considerations empty-handed. Assuming a little bit more we may gain a satisfying answer. This has been done in [27] :

(i) *Let* $a_{2^j-1} \leq c\, a_{2^j}$ *for some* $c > 0$ *and all* $j \in \mathbb{N}$. *Then there is some* $C > 0$ *such that for all* $k \in \mathbb{N}$

$$e_k \;\leq\; C\, a_k\ . \tag{1}$$

(ii) *Let* $f(j)$ *be a positive increasing function on* $\mathbb{N}$ *with*

$$f(2^j) \;\leq\; c\, f(2^{j-1}) \tag{2}$$

for some $c > 0$ *and all* $j \in \mathbb{N}$. *Then there is some number* $C > 0$ *such that for all* $k \in \mathbb{N}$

$$\sup_{1 \leq j \leq n} f(j)\, e_j \;\leq\; C \sup_{1 \leq j \leq n} f(j)\, a_j\ . \tag{3}$$

This theorem is very-well adapted to our situation, as both $f(j) = j^{\varkappa}$ and $f(j) = (\log\langle j \rangle)^{\varkappa}$, $\varkappa > 0$, $j \in \mathbb{N}$, satisfy the assumption (2). There is also a forerunner in [3], restricted to Banach spaces, see [3: p.96].
We made use of the above result mainly in [D: 3] where we could achieve some improved estimate in case of approximation numbers and transferred it afterwards to the entropy numbers we dealt with.

2.3.2 The basic assertion

We adopt the standard situation as described in 1.1.3 and repeated at the beginning of 2.1.2. We care for the approximation numbers of the respective embeddings

$$id^B \;:\; B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \tag{4}$$

and

$$id^F \;:\; F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{5}$$

where

$$\begin{gathered} -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \leq \infty \quad , \quad 0 < q_1 \leq \infty, \quad \alpha > 0, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n}, \quad p_0 < p_2 \leq \infty \quad , \quad 0 < q_2 \leq \infty \end{gathered} \tag{6}$$

and

$$\delta \;=\; s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \;>\; 0 \tag{7}$$

(and $p_1 < \infty$, $p_2 < \infty$ in the F-case). Recall that our interest in studying approximation numbers of such embeddings is partly caused by some forerunners concerning entropy numbers and approximation numbers of embeddings of function spaces on bounded domains, i.e. [6] and [7]. At least we made use of the results achieved in those cases several times to prove our estimates. Anyway, one may get a good impression of additional phenomena, appearing only in case of approximation numbers, looking at these papers, cf. [6], [7] or [C: Prop.2.3]. The most striking distinction is the different behaviour of the approximation numbers depending upon the fact whether p_1 and p_2 are both on the same side of 2 or not. In other words one now has to handle the cases $0 < p_1 \leq p_2 \leq 2$ or $2 \leq p_1 \leq p_2 \leq \infty$ and $0 < p_1 \leq 2 \leq p_2 < \infty$ separately. Furthermore there is some quantity $\lambda = \frac{s_1 - s_2}{n} - \max(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2})$ coming in which plays an important role, too. Having in mind these introductory remark it is probably no surprise that also in case of weighted function spaces in which we are interested the situation concerning approximation numbers is much more complicated than that one for the entropy numbers. We postpone a further explanation and present the result first. In the usual $(\frac{1}{p}, s)$-diagram, see also Fig.1 in 2.1.2, we introduce the following regions :

I $: 0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 < \infty$, $\delta < \alpha$

II $: 0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 < \infty$, $\delta > \alpha$

III $: 0 < p_1 < 2 < p_2 < \infty$, $\delta < \alpha$, $\lambda > \frac{1}{2}$

III$_a$ $: 0 < p_1 < 2 < p_1' \le p_2$, $\delta < \alpha$, $\lambda > \frac{1}{2}$

III$_b$ $: 0 < p_1 < 2 < p_2 \le p_1'$, $\delta < \alpha$, $\lambda > \frac{1}{2}$

IV $: 0 < p_1 < 2 < p_2 < \infty$, $\lambda > \frac{1}{2}$,
$\delta > \alpha > n \max(1 - \frac{1}{p_1}, \frac{1}{p_2})$

IV$_a$ $: 0 < p_1 < 2 < p_1' \le p_2$, $\lambda > \frac{1}{2}$
$\delta > \alpha > n(1 - \frac{1}{p_1})$

IV$_b$ $: 0 < p_1 < 2 < p_2 \le p_1'$, $\lambda > \frac{1}{2}$
$\delta > \alpha > \frac{n}{p_2}$

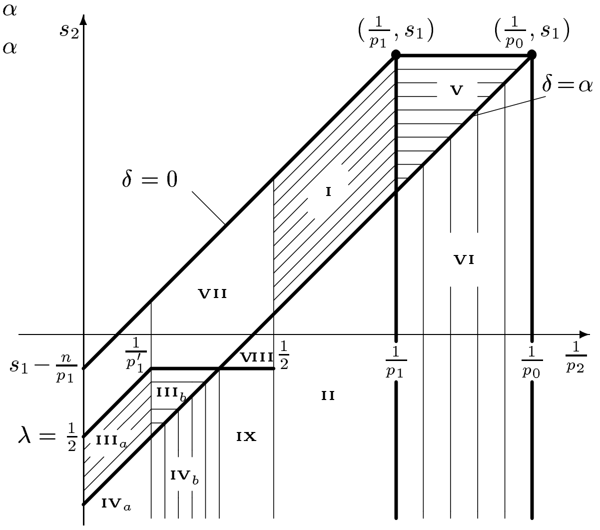
V $: p_0 < p_2 \le p_1$, $\delta < \alpha$

VI $: p_0 < p_2 \le p_1$, $\delta > \alpha$

VII $: 0 < p_1 < 2 < p_2 < \infty$, $\delta < \alpha$, $\lambda \le \frac{1}{2}$

VIII $: 0 < p_1 < 2 < p_2 < \infty$,
$\alpha < \delta \le n \max(1 - \frac{1}{p_1}, \frac{1}{p_2})$

IX $: 0 < p_1 < 2 < p_2 < \infty$, $\alpha \le n \max(1 - \frac{1}{p_1}, \frac{1}{p_2}) < \delta$



$\boxed{0 < p_1 < 2, \quad \alpha > n(1 - \frac{1}{p_1})}$ **Fig.7**

Recall $\lambda = \frac{s_1 - s_2}{n} - \max\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right)$. In the above diagram we assumed the case $0 < p_1 < 2$, $\alpha > n\,(1 - \frac{1}{p_1})$ to illustrate the different regions, whereas in the cases $0 < p_1 < 2$, $\alpha \le n\,(1 - \frac{1}{p_1})$ and $2 \le p_1 < \infty$, resp., this figure degenerates, see [C: Fig.5, Fig.6].

Theorem. *Let a_k be the kth approximation number of the embedding* (4) *and let the assumptions* (6) *and* (7) *be satisfied. Using the above introduced notations we have the following results*:

(i) *in region* **I** $$a_k \sim k^{-\frac{\delta}{n}}, \tag{8}$$

(ii) *in region* **II** $$a_k \sim k^{-\frac{\alpha}{n}}, \tag{9}$$

(iii) *in region* **III**, *i.e.* **III**$_a$ *and* **III**$_b$, $$a_k \sim k^{-\lambda}, \tag{10}$$

(iv) *in region* **IV**, *i.e.* **IV**$_a$ *and* **IV**$_b$,*there exist a positive constant c and for any $\varepsilon > 0$ a positive constant c_ε such that*

$$c\, k^{-\frac{\alpha}{n} - \min(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2})} \le a_k \le c_\varepsilon\, k^{-\frac{\alpha}{n} - \min(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2}) + \varepsilon}, \tag{11}$$

(v) *in region* **V** $$a_k \sim k^{-\frac{s_1 - s_2}{n}}, \tag{12}$$

(vi) *in region* **VI** $$a_k \sim k^{-\frac{\alpha}{n} + \frac{1}{p_2} - \frac{1}{p_1}}, \tag{13}$$

(vii) *in region* **VII** *there exist two positive constants c_1 and c_2 such that*

$$c_1\, k^{-\frac{\delta}{n} - \min(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2})} \le a_k \le c_2\, k^{-\frac{\delta}{n}}, \tag{14}$$

(viii) *in region* **VIII** *there exist two positive constants c_1 and c_2 such that*

$$c_1\, k^{-\frac{\alpha}{n} - \min(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2})} \le a_k \le c_2\, k^{-\frac{\alpha}{n}}, \tag{15}$$

(ix) *in region* **IX** *there exist a positive constant c and for any $\varepsilon > 0$ a positive constant c_ε such that*

$$c\,k^{-\frac{\alpha}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \;\le\; a_k \;\le\; c_\varepsilon\, k^{-\frac{\alpha}{2n\,\max(1-\frac{1}{p_1},\frac{1}{p_2})}+\varepsilon} . \tag{16}$$

Remark 2. Recall the elementary embeddings (1.1/9) (as well as their weighted counterparts). Hence the above theorem is also valid in the F-case as we excluded the line "$\delta = \alpha$" (where a possible influence of the q-parameters might appear). In Fig. 7 we indicated the level lines for the corresponding exponents.

This theorem has been shown in [C: 4.2], including also a preparatory lemma, see [C: Lemma 3.1] and [C: 4.1]. The essential ideas may be outlined in the following way. We wish to apply the already achieved results of EDMUNDS and TRIEBEL as published in [6] and [7] concerning approximation numbers of embeddings of function spaces on bounded domains. Similarly as in the case of entropy numbers we have to care about the dependence of the constants on the underlying domain. These investigations resulted in the above-mentioned Lemma 3.1 in [C]. As for the estimates from below we involved the well-known extension-restriction procedure to use estimates from below in the case EDMUNDS and TRIEBEL studied in [6], [7] as well as another argumentation from those papers: one may transfer the problem we endeavour to handle to some similar one in finite-dimensional l_p-spaces, additionally involving the already cited localization principle for F-spaces, see [25: 2.4.7] and [C: 4.2/Step 2] for the referred procedure. After these preparations one simply applies already proved estimates for the l_p-case, see [7: 3.2.2, 3.2.4] or [C: (4.2/18)-(4.2/20)]. Conversely, aiming at estimates from above, we may reduce our considerations to the F-case first, see [C: 4.2/Step 3]. The justification comes both from tricky embeddings between B- and F-spaces, see [C: (4.2/36)-(4.2/46)], and interpolation arguments, see [C: (4.2/23)-(4.2/34)]. The latter one is based on the constructions in [6] and [7] which guarantee the linearity of the approximating operator and thus permit interpolation. Afterwards, to study estimates from above (now restricted to the F-case) we proceed similarly to the case of entropy numbers. We use a partition of unity, deal with embeddings of function spaces on annuli (i.e. their respective approximation numbers) to apply the results of [6] and [7] complemented by our preparatory lemma and homogeneity estimates. The rest is again a (careful) summation according to the additivity of approximation numbers, this may be found in detail in [C: 4.2/Step 4 - Step 7].

Remark 3. At the stage of our paper [C] we completely excluded the line "$\delta = \alpha$" from our considerations. In fact there are only very few results and a rather vague approach not worth mentioning. Moreover, our intention is not so clear what results should be expected in all the different cases we have to distinguish. Nevertheless we achieved some better results in [D: 3] which should not be concealed.

Corollary. *Let* (6) *and* (7) *be satisfied with $\delta = \alpha$. Let a_k be the kth approximation number of either the embedding*

$$id^F \quad : \quad F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{17}$$

or

$$id^B \quad : \quad B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}, \quad q_2 \ge p_2 \,. \tag{18}$$

(i) $0 < p_1 \le p_2 \le 2$ *or* $2 \le p_1 \le p_2 \le \infty$, *then* $$a_k \le c\,k^{-\frac{\alpha}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}+\frac{1}{p_2}} , \tag{19}$$

(ii) $p_0 < p_2 \le p_1 \le \infty$, *then* $$a_k \le c\,k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} , \tag{20}$$

(iii) $0 < p_1 < 2 < p_2 < \infty$, *then* $$a_k \le c\,k^{-\lambda} \left(\log\langle k\rangle\right)^{\lambda+\frac{1}{p_2}} , \tag{21}$$

where $\lambda = \frac{s_1-s_2}{n} - \max(\frac{1}{2}-\frac{1}{p_2}, \frac{1}{p_1}-\frac{1}{2}) > \frac{1}{2}$.

(Recall that we always assume $p_1 < \infty$, $p_2 < \infty$ in the F-case.)

Remark 4. We do not want to fail to mention a result of MYNBAEV and OTEL'BAEV as given in [15: V, §3, Theorem 9] which reads in terms of our situation as follows. Let a_k be the $k\,th$ approximation number of

$$id : F^{s_1}_{p_1,2}(\alpha) \longrightarrow F^{0}_{p_2,2} \tag{22}$$

where

$$\begin{aligned} s_1 > 0, \quad s_2 = 0 \quad , \quad 1 < p_1 \le p_2 \le 2 \quad \text{or} \quad 2 \le p_1 \le p_2 < \infty, \\ \alpha > 0 \quad , \quad \delta \;=\; s_1 - \frac{n}{p_1} + \frac{n}{p_2} \;>\; 0. \end{aligned} \tag{23}$$

Now MYNBAEV and OTEL'BAEV proved

$$a_k \;=\; a_k(id) \;\sim\; \begin{cases} k^{-\frac{\delta}{n}} & , \quad 0 < \delta < \alpha \\ \left(\frac{k}{\log k}\right)^{-\frac{\alpha}{n}} & , \quad \delta = \alpha, \; k \ge k_0 \\ k^{-\frac{\alpha}{n}} & , \quad \delta > \alpha \;. \end{cases} \tag{24}$$

Comparing (24) with the above theorem and Corollary one recognizes coincidence in the cases $0 < \delta < \alpha$ and $\delta > \alpha$, but a less sharp estimate from above in (19) if $\delta = \alpha$. On the other hand, remembering the situation for entropy numbers, see 2.1.3, a possible influence of q-parameters is rather likely also in these cases. This is not necessarily a contradiction to (24) as there we have $q_1 = q_2 = 2$ and thus a possible influence could have disappeared. Anyway, at least one conjecture seems reasonable: taking into consideration our efforts in 2.1.3 (in case of entropy numbers) as well as the already complicated investigations in case of approximation numbers (apart from the line "$\delta = \alpha$") we dare to assert that a complete study of that subject becomes very involved.

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3 Proofs

3.1 Appendix A

Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators I

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Abstract. In this paper we study weighted function spaces of type $B^s_{p,q}(\mathbb{R}^n, \varrho(x))$ and $F^s_{p,q}(\mathbb{R}^n, \varrho(x))$, where $\varrho(x)$ is a weight function of at most polynomial growth. Of special interest are the weight functions $\varrho(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \in \mathbb{R}$. The main result deals with estimates for the entropy numbers of compact embeddings between spaces of this type.
(AMS classification : 46 E 35)

1. Introduction

In [7, 8] we studied entropy numbers and approximation numbers of compact embeddings between function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$ on a bounded domain Ω in $\mathbb{R}^n$. Recall that these two scales of spaces cover many well-known classical spaces such as (fractional) Sobolev spaces, Hölder-Zygmund spaces, Besov spaces and (inhomogeneous) Hardy spaces. In [9] we employed these results to get sharp assertions for the distribution of eigenvalues of degenerate elliptic differential operators of the prototype

$$A = a(x)(-\Delta)a(x) \tag{1}$$

defined as the inverse of

$$B = b(x)(-\Delta)^{-1}b(x)\ , \qquad a(x) = b^{-1}(x) \tag{2}$$

where b belongs to some Lebesgue space $L_r(\Omega)$ and $-\Delta$ stands for the Laplacian with vanishing Dirichlet data at the boundary of Ω. The crucial link between the eigenvalues $\{\mu_k\}$ of the compact operator B and the related entropy numbers e_k is given by Carl's formula

$$|\mu_k| \le \sqrt{2}\, e_k \qquad (k \in \mathbb{N})\ . \tag{3}$$

Let A be a positive and selfadjoint operator in $L_2(\Omega)$, again the above operator A with vanishing boundary data may serve as the prototype, and let $V(x) \ge 0$ be a singular potential, then the "negative" spectrum (bound states) of

$$H = A - V^2(x) \tag{4}$$

is of interest. Via the Birman-Schwinger principle this question can be reduced to operators of type B in the sense of (2) and the corresponding distribution of eigenvalues. This was done in [9] in the framework of $L_2(\Omega)$. Especially the last problem with its quantum mechanical background asks for an extension of these considerations from bounded domains Ω to $\mathbb{R}^n$. The reduction of these problems to sharp estimates for entropy numbers of compact embeddings between function spaces, now on $\mathbb{R}^n$, requires the introduction of weighted spaces of type $B^s_{p,q}$ and $F^s_{p,q}$ on $\mathbb{R}^n$. It is just this programme which we wish to accomplish in this paper and its planned second part. In this part we concentrate on the relevant weighted function spaces on $\mathbb{R}^n$ and the study of entropy numbers of corresponding compact embeddings. In the second part we apply these results in the outlined way to degenerate elliptic pseudodifferential operators on $\mathbb{R}^n$ to obtain estimates for the distribution of eigenvalues and assertions about the negative spectrum of operators of type (4) on $\mathbb{R}^n$.

The plan of the paper is as follows. In Sect.2 we introduce the spaces $B^s_{p,q}(I\!R^n, \varrho(x))$ and $F^s_{p,q}(I\!R^n, \varrho(x))$, where $s \in I\!R$, $0 < p \le \infty$ ($p < \infty$ in the F-case), $0 < q \le \infty$, and where $\varrho(x)$ is a smooth weight function of at most polynomial growth, that means

$$0 < \varrho(x) \le c\, \varrho(y)\, \langle x - y \rangle^{\alpha} \tag{5}$$

for some $\alpha \ge 0$, some $c > 0$ and all $x \in I\!R^n$ and $y \in I\!R^n$. As usual $\langle x \rangle = (1 + |x|^2)^{1/2}$. This can be done completely parallel to the definition of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ on $I\!R^n$. Weighted spaces of this and more general type have been considered before, especially by H.-J. SCHMEISSER and H. TRIEBEL, see [14: 5.1]. But we do not rely very much on these results and sketch new short proofs for all relevant facts. In particular, we prove that

$$\|f|F^s_{p,q}(I\!R^n, \varrho(x))\| \quad \text{and} \quad \|\varrho f|F^s_{p,q}\| \tag{6}$$

are equivalent quasi-norms. Furthermore, let

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le p_2 < \infty, \quad 0 < q_1 \le \infty \quad \text{and} \quad 0 < q_2 \le \infty\ , \tag{7}$$

then it will be shown that the embedding of

$$F^{s_1}_{p_1,q_1}(I\!R^n, \varrho_1(x)) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(I\!R^n, \varrho_2(x)) \tag{8}$$

and its B-counterpart is compact if and only if

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{\varrho_2(x)}{\varrho_1(x)} \to 0 \quad \text{for} \quad |x| \to \infty\ . \tag{9}$$

Sect.3 deals with entropy numbers of compact mappings in quasi-Banach spaces. Although there is essentially nothing new compared with the situation in Banach spaces we deal carefully with the interpolation properties of these numbers. This is desirable because the treated weighted spaces are often reduced to infinite sequences of unweighted spaces, including interpolation, and then a tight control about the involved constants is indispensable. In Sect.4 we study the entropy numbers of compact embeddings of type (7)-(9). By (6) it is quite clear that we may assume $\varrho_2(x) = 1$ for this purpose. Then it is reasonable to assume that $\varrho_1^{-1}(x)$ belongs to some Lorentz space. This point of view is supported by the applications we have in mind and by what has been done in this direction in literature, see [16, Ch.4], [3], [4]. However we shift this task to a later occasion and simplify our situation by assuming $\varrho_1(x) = \langle x \rangle^{\alpha}$ for some $\alpha > 0$. The main result of the paper is the theorem in 4.2, dealing with the entropy numbers of the compact embedding of

$$F^{s_1}_{p_1,q_1}(I\!R^n, \langle x \rangle^{\alpha}) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(I\!R^n) \tag{10}$$

and its B-counterpart under more general conditions than those ones in (7) and (9). Especially some limiting cases with log-terms are of interest. They will be used to prove as a by-product the sharp (unweighted) inclusion property

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v} \qquad \text{if and only if} \qquad u \le \min(p,q) \quad \text{and} \quad v \ge \max(p,q)\ . \tag{11}$$

Of course, the "if"-part is well-known. The "only if"- part is folklore. A somewhat complicated direct proof has been given recently in [15]. All substantial proofs are presented in Sect.5. Unimportant constants are denoted by c, occasionally with additional subscripts within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l, whereas (j) means formula (j) in the same subsection. In a similar way we quote definitions, propositions and theorems.

2. Weighted function spaces

2.1 Definitions

Let $I\!R^n$ be the Euclidean n-space. Let $\langle x\rangle = (1+|x|^2)^{1/2}$ in $I\!R^n$.

Definition 1. The class of *admissible weight functions* is the collection of all positive C^∞ functions $\varrho(x)$ on $I\!R^n$ with the following properties:
(i) for any multi-index γ there exists a positive constant c_γ with

$$|D^\gamma \varrho(x)| \le c_\gamma \varrho(x) \quad \text{for all} \quad x \in I\!R^n \tag{1}$$

(ii) there exist two constants $c > 0$ and $\alpha \ge 0$ such that

$$0 < \varrho(x) \le c\, \varrho(y)\, \langle x-y\rangle^\alpha \quad \text{for all } x \in I\!R^n \text{ and all } y \in I\!R^n\ . \tag{2}$$

Remark 1. Of course, for appropriate positive constants c_1 and c_2 we have

$$c_1\ \varrho(x) \le \varrho(y) \le c_2\ \varrho(x) \quad \text{for all } x \in I\!R^n,\ y \in I\!R^n \text{ with } |x-y| \le 1\ . \tag{3}$$

If $\varrho(x)$ and $\varrho'(x)$ are admissible weight functions, then $\varrho^{-1}(x)$ and $\varrho(x)\varrho'(x)$ are also admissible weight functions.

Remark 2. Let $\varrho(x)$ be a measurable function in $I\!R^n$ satisfying (2). Let $h(x) \ge 0$ be a C^∞ function in $I\!R^n$, supported by the unit ball, with, say, $\int h(x)dx = 1$. Then

$$(h * \varrho)(x) = \int h(x-y)\varrho(y)dy \tag{4}$$

is an admissible weight function in the sense of the above definition. Furthermore, ϱ and $h*\varrho$ are equivalent to each other. This justifies only to concentrate on C^∞ weight functions without loss of generality.

Next we recall briefly the basic ingredients needed to introduce spaces of type $B^s_{p,q}$ and $F^s_{p,q}$. All spaces in this paper are defined on $I\!R^n$ and so we omit "$I\!R^n$" in the sequel. The Schwartz space S and its dual S' of all complex-valued tempered distributions have the usual meaning here. Furthermore, L_p with $0 < p \le \infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot|L_p\|$. Let $\varphi \in S$ be such that

$$\operatorname{supp}\varphi \subset \{y \in I\!R^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1\ , \tag{5}$$

put $\varphi_0 = \varphi$ and for each $j \in I\!N$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then since $1 = \sum_{j=0}^\infty \varphi_j(x)$ for all $x \in I\!R^n$, the φ_j form a dyadic resolution of unity. Given any $f \in S'$, we denote by $\hat f$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. Then $(\varphi_j\hat f)^\vee$ is an analytic function on $I\!R^n$. Beside the unweighted spaces L_p on $I\!R^n$ we introduce their weighted generalizations $L_p(\varrho(x))$, quasi-normed by

$$\|f|L_p(\varrho(\cdot))\| = \|\varrho f|L_p\| \quad , \tag{6}$$

where $\varrho(x) > 0$ is a weight function on $I\!R^n$ and $0 < p \le \infty$.

Definition 2. Let $\varrho(x)$ be an admissible weight function in the sense of Definition 1. Let $s \in I\!R$, $0 < q \le \infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity.

(i) Let $0 < p \le \infty$. The space $B^s_{p,q}(\varrho(x))$ is the collection of all $f \in S'$ such that

$$\|f|B^s_{p,q}(\varrho(\cdot))\| = \left(\sum_{j=0}^\infty 2^{jsq}\|(\varphi_j\hat f)^\vee|L_p(\varrho(\cdot))\|^q\right)^{1/q} \tag{7}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. The space $F^s_{p,q}(\varrho(x))$ is the collection of all $f \in S'$ such that

$$\|f|F^s_{p,q}(\varrho(\cdot))\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} \Big| L_p(\varrho(\cdot)) \right\| \tag{8}$$

(with the usual modification if $q = \infty$) is finite.

(iii) Let $\varrho(x) = \langle x \rangle^\alpha$ for some $\alpha \in I\!R$. Then we put

$$B^s_{p,q}(\alpha) = B^s_{p,q}(\langle x \rangle^\alpha) \quad \text{with} \quad B^s_{p,q} = B^s_{p,q}(0) \tag{9}$$

and

$$F^s_{p,q}(\alpha) = F^s_{p,q}(\langle x \rangle^\alpha) \quad \text{with} \quad F^s_{p,q} = F^s_{p,q}(0) \quad . \tag{10}$$

Remark 3. The theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ has been developed in detail in [18] and [19]. There is no difficulty to extend this theory to the above weighted classes. Furthermore, in [14: 5.1] we dealt with spaces of type $B^s_{p,q}(\varrho(x))$ and $F^s_{p,q}(\varrho(x))$ in the framework of ultra-distributions for much larger classes of admissible weight functions parallel to [18]. But also the later developments in the theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ as reflected in [19] have their more or less obvious counterparts for the weighted spaces under consideration here. In particular, we shall use equivalent quasi-norms in $B^s_{p,q}(\varrho(x))$ and $F^s_{p,q}(\varrho(x))$ via local means in extension of what has been done in [19: 1.8.4, 2.4.6, 2.5.3] for the unweighted spaces.

Remark 4. As in the unweighted case the above two weighted scales $B^s_{p,q}(\varrho(x))$ and $F^s_{p,q}(\varrho(x))$ cover weighted (fractional) Sobolev spaces, weighted Hölder-Zygmund spaces and weighted classical Besov spaces characterized in the usual way via derivatives and differences. We refer to [14: 5.1] and the literature mentioned there.

2.2 Properties

As we said we feel free to use assertions for the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ also for the above weighted spaces $B^s_{p,q}(\varrho(x))$ and $F^s_{p,q}(\varrho(x))$ if its extension is more or less obvious or, likewise, if it is covered by the more complicated spaces treated in [14: 5.1].

Theorem. *Let $s \in I\!R$, $0 < q \le \infty$ and $0 < p \le \infty$ (with $p < \infty$ in the F-case).*
(i) *$B^s_{p,q}(\varrho(x))$ and $F^s_{p,q}(\varrho(x))$ are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$), and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*

(ii) *The operator $f \mapsto \varrho\, f$ is an isomorphic mapping from $B^s_{p,q}(\varrho(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(\varrho(x))$ onto $F^s_{p,q}$. Especially,*

$$\|\varrho\, f|B^s_{p,q}\| \quad \textit{is an equivalent quasi-norm in } B^s_{p,q}(\varrho(x)) \tag{1}$$

and

$$\|\varrho\, f|F^s_{p,q}\| \quad \textit{is an equivalent quasi-norm in } F^s_{p,q}(\varrho(x)) \quad . \tag{2}$$

Remark. We take part (i) for granted, either parallel to [18] or covered by [14: 5.1]. Also part (ii) is covered by [14: 5.1.3]. But the proof, even reduced to our simpler situation, is rather awkward, also its forerunner by J. Franke [10]. Based on local means we give a very short proof in 5.1.

2.3 Embeddings : general weight functions

The embedding theory for the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ has been developed in [18: 2.3.2 and 2.7.1]. By Theorem 2.2 (ii) it is clear that these assertions can be extended immediately to the above weighted spaces if only one weight function is involved. This will not be done. Just on the contrary we are interested in embeddings of type (1/7) to (1/9) with different weights. By Theorem 2.2 (ii) and the elementary embeddings in [18: p.47] we may restrict ourselves to the F-spaces at this moment. As general conditions for the involved parameters we stick at (1/7) in this subsection. Later on we consider also couples of spaces with $p_2 < p_1$.

Theorem. *Let $\varrho_1(x)$ and $\varrho_2(x)$ be admissible weight functions in the sense of Definition* 2.1/1. *Let*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le p_2 < \infty, \quad 0 < q_1 \le \infty \quad \text{and} \quad 0 < q_2 \le \infty \quad . \tag{1}$$

(i) *Then $F^{s_1}_{p_1,q_1}(\varrho_1(x))$ is continuously embedded in $F^{s_2}_{p_2,q_2}(\varrho_2(x))$,*

$$F^{s_1}_{p_1,q_1}(\varrho_1(x)) \subset F^{s_2}_{p_2,q_2}(\varrho_2(x)) \quad , \tag{2}$$

if and only if

$$s_1 - \frac{n}{p_1} \ge s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{\varrho_2(x)}{\varrho_1(x)} \le c < \infty \tag{3}$$

for some $c > 0$ and all $x \in I\!R^n$.
(ii) *The embedding* (2) *is compact if and only if*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{\varrho_2(x)}{\varrho_1(x)} \to 0 \quad if \quad |x| \to \infty \quad . \tag{4}$$

Remark 1. A proof of this theorem will be given in 5.2.

Remark 2. Let

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le p_2 \le \infty, \quad 0 < q_1 \le \infty, \quad 0 < q_2 \le \infty, \tag{5}$$

then the embedding

$$B^{s_1}_{p_1,q_1}(\varrho_1(x)) \subset B^{s_2}_{p_2,q_2}(\varrho_2(x)) \quad , \tag{6}$$

is compact if (4) holds. This follows immediately from part (ii) of the above theorem, the embeddings for the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$, and Theorem 2.2 (ii). Now p_2 may be infinite and the interesting weighted Hölder-Zygmund spaces $\mathcal{C}^s(\varrho(x)) = B^s_{\infty,\infty}(\varrho(x))$ are now included, see also Remark 2.1/4.

2.4 Embeddings: the weight function $\langle x\rangle^\alpha$

Let ϱ_1 and ϱ_2 be two admissible weight functions in the sense of Definition 2.1/1. Then $\frac{\varrho_1}{\varrho_2}$ is also an admissible weight function and we have by Theorem 2.2 (ii)

$$\|f|F^s_{p,q}(\varrho_1(\cdot))\| \sim \left\|\varrho_2\, f\middle|F^s_{p,q}\big(\frac{\varrho_1}{\varrho_2}(\,\cdot\,)\big)\right\| \tag{1}$$

(equivalent quasi-norms). In other words, $f \mapsto \varrho_2\, f$ is an isomorphic mapping from $F^s_{p,q}(\varrho_1(x))$ onto $F^s_{p,q}\big(\frac{\varrho_1}{\varrho_2}(x)\big)$ for all admissible weight functions ϱ_1. Of course, we have a corresponding assertion for the B-counterparts. In other words, if we wish to study continuous or compact embeddings of type

(2.3/2) we may assume $\varrho_2(x) = 1$ without loss of generality. In this sense we put $\varrho_1(x) = \varrho(x)$ and specify in what follows in this paper $\varrho(x) = \langle x\rangle^\alpha$ for some $\alpha > 0$. Hence we are mainly interested in compact embeddings of type (2.3/2) under these specialized circumstances. In this subsection we formulate a weak type continuous embedding assertion which will be of great help for us later on. Let $L_{p,\infty} = L_{p,\infty}(\mathbb{R}^n)$ with $0 < p < \infty$ be the usual Lorentz space (Marcinkiewicz space) on $\mathbb{R}^n$ with respect to the Lebesgue measure, see [17: 1.18.6] or [1: p.216] for definitions.

Definition. Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Let $\{\varphi_j\}$ be a dyadic resolution of unity in the sense of Subsect.2.1. Then *weak-*$B^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f|weak - B^s_{p,q}\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee | L_{p,\infty}\|^q\right)^{1/q} \tag{2}$$

(with the usual modification if $q = \infty$) is finite. Similarly, *weak-* $F^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f|weak - F^s_{p,q}\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q\right)^{1/q} | L_{p,\infty}\right\| \tag{3}$$

(with the usual modification if $q = \infty$) is finite.

Remark. Of course, there is no problem to replace $L_{p,\infty}$ by the more general Lorentz spaces $L_{p,u}$ with $0 < p \le \infty$ ($p < \infty$ in the case of the F-spaces) and $0 < u \le \infty$. Recall the real interpolation formula

$$\left(L_{p_0,u_0}\ ,\ L_{p_1,u_1}\right)_{\theta,u} = L_{p,u} \quad , \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad , \quad p_0 \ne p_1 \quad , \tag{4}$$

where p_0, p_1, u_0, u_1 and u are positive, possibly infinite numbers, and $0 < \theta < 1$, see [2: 5.3, p.113], [17: 1.18.6, Theorem 2 and Remark 5] and [1: p.300].

Theorem. (i) *Under the restrictions for s, p and q in the above definition both weak-$B^s_{p,q}$ and weak-$F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$) and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*
(ii) *Let $s \in \mathbb{R}, 0 < q \le \infty, 0 < p \le \infty$ ($p < \infty$ in the case of the F-spaces), $\alpha > 0$ and $\frac{1}{p_0} = \frac{1}{p} + \frac{\alpha}{n}$. Then*

$$B^s_{p,q}(\alpha) \subset weak - B^s_{p_0,q} \quad \textit{and} \quad F^s_{p,q}(\alpha) \subset weak - F^s_{p_0,q} \quad . \tag{5}$$

Proof: **Step 1.** Based on (4) and the interpolation property one can easily replace L_p as the basic space in $B^s_{p,q}$ and $F^s_{p,q}$ by $L_{p,\infty}$ or, more general, by $L_{p,u}$. Hence the theory for the spaces $B^s_{p,q}$ and $F^s_{p,q}$ developed in [18] and [19] can be carried over to the corresponding weak spaces. In other words, we take part (i) of the theorem for granted.

Step 2. By (4), the related interpolation property, and Hölder's inequality

$$L_p \cdot L_{\frac{n}{\alpha}} \quad \subset \quad L_{p_0} \quad , \tag{6}$$

we get

$$L_p \cdot L_{\frac{n}{\alpha},\infty} \quad \subset \quad L_{p_0,\infty} \quad . \tag{7}$$

Now $\langle x\rangle^{-\alpha} \in L_{\frac{n}{\alpha},\infty}$ and (7) proves (5).

3. Entropy numbers in quasi-Banach spaces

3.1 Definitions

Let A_1 and A_2 be two complex quasi-Banach spaces and let T be a linear and continuous operator from A_1 into A_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in A_2 of radius ε which cover the image $T\,U_1$ of the unit ball $U_1 = \{a \in A_1 \;:\; \|a|A_1\| \leq 1\}$.

Definition. Let $k \in I\!N$ and let $T : A_1 \to A_2$ be the above continuous operator. Then the *k th entropy number* e_k of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in A_2 of radius ε which cover $T\,U_1$.

Remark 1. For details and properties of entropy numbers we refer to [5], [6] and [11] (always restricted to the case of Banach spaces). The extension of these properties to quasi-Banach spaces causes no problems. We mention only one assertion based on the fact that any quasi-Banach space A is also a λ- Banach space for a suitable number λ with $0 < \lambda \leq 1$, that means that there exists an equivalent quasi-norm such that

$$\|a_1 + a_2|A\|^\lambda \leq \|a_1|A\|^\lambda + \|a_2|A\|^\lambda \quad \text{for all} \quad a_1 \in A \quad \text{and} \quad a_2 \in A \quad . \tag{1}$$

We refer to [12: §15.10]. Let the above space A_2 be a λ-Banach space and let T_1 and T_2 be two operators in the sense of the above definition. Then we have in obvious notations

$$e^\lambda_{k_1+k_2-1}(T_1 + T_2) \leq e^\lambda_{k_1}(T_1) + e^\lambda_{k_2}(T_2)\;, \quad k_1 \in I\!N,\; k_2 \in I\!N\;. \tag{2}$$

The proof is the same as in the case of Banach spaces (then $\lambda = 1$).

Remark 2. In part II of this paper Carl's inequality plays a crucial role. Let $A = A_1 = A_2$ in the above definition and let $\{\mu_k\}$ be the sequence of eigenvalues of the compact operator T, counted with respect to their algebraic multiplicity and ordered by decreasing modulus $|\mu_1| \geq |\mu_2| \geq \dots$. Then

$$|\mu_k| \leq \sqrt{2}\, e_k \quad \text{for all} \quad k \in I\!N \quad . \tag{3}$$

A proof, restricted to Banach spaces, is given in [5: 4.2.1]. It can be extended to quasi-Banach spaces.

3.2 Interpolation properties

The interpolation property for entropy numbers in Banach spaces (originally in terms of entropy functions and entropy ideals) has been developed by J. Peetre and H. Triebel in 1968, 1970 and 1975 and may be found in [17: 1.16.2]. Its reformulation in terms of the entropy numbers has been given in [13: 12.1]. Although the extension of this theory to quasi-Banach spaces causes no trouble we give a careful formulation here and provide a proof in 5.3. This seems to be necessary because later on we handle in this connection infinitely many quasi-Banach spaces and a tight control about the involved constants is indispensable. We use standard notations from real interpolation theory, see [2], [17] or [1]. In particular, if $\{A_0, A_1\}$ is an interpolation couple of quasi-Banach spaces, then

$$K(t,a) = K(t,a,A_0,A_1) = \inf\left(\|a_0|A_0\| + t\,\|a_1|A_1\|\right)\,,\; 0 < t < \infty,\; a \in A_0 + A_1, \tag{1}$$

is Peetre's K-functional, where the infimum is taken over all decompositions $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$.
If T is a linear and continuous operator from a quasi-Banach space A in a quasi-Banach space B then we denote its entropy numbers temporarily by $e_k(A \to B)$.

Theorem. (i) *Let A be a quasi-Banach space and let $\{B_0, B_1\}$ be an interpolation couple of λ-Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_0 \cap B_1 \subset B_\theta \subset B_0 + B_1$ and*

$$\|b|B_\theta\| \le \|b|B_0\|^{1-\theta} \, \|b|B_1\|^\theta \quad if \quad b \in B_0 \cap B_1 \, . \tag{2}$$

Let T be a linear and continuous operator from A into $B_0 \cap B_1$. Then

$$e_{k_0+k_1-1}(A \to B_\theta) \le 2^{\frac{1}{\lambda}} \, e_{k_0}^{1-\theta}(A \to B_0) \, e_{k_1}^\theta(A \to B_1) \, , \quad k_0 \in I\!N, \; k_1 \in I\!N \, . \tag{3}$$

(ii) *Let $\{A_0, A_1\}$ be an interpolation couple of quasi-Banach spaces and let B be a λ-Banach space. Let $0 < \theta < 1$ and let A be a quasi-Banach space such that $A \subset A_0 + A_1$ and*

$$t^{-\theta} K(t, a) \le \|a|\, A\| \quad for \quad a \in A \quad and \quad 0 < t < \infty \, . \tag{4}$$

Let T be a linear operator from $A_0 + A_1$ into B such that its restrictions to A_0 and A_1 are continuous, then its restriction to A is also continuous and

$$e_{k_0+k_1-1}(A \to B) \le 2^{\frac{1}{\lambda}} \, e_{k_0}^{1-\theta}(A_0 \to B) \, e_{k_1}^\theta(A_1 \to B) \, , \quad k_0 \in I\!N, \; k_1 \in I\!N \, . \tag{5}$$

4. Entropy numbers in function spaces

4.1 A preparation: unweighted spaces

If $R \ge 1$, then

$$K_R = \{x \in I\!R^n \; : |x| < R\} \tag{1}$$

is the ball in $I\!R^n$ centered at the origin and of radius R. Under the same conditions for the parameters s, p and q as in Definition 2.1/2 we define in the usual way the spaces $B^s_{p,q}(K_R)$ and $F^s_{p,q}(K_R)$ as the restrictions of $B^s_{p,q}$ and $F^s_{p,q}$, respectively, on K_R, see [18: 3.2.2] for details. Since we wish to study the influence of R on the behaviour of entropy numbers we assume in this subsection that the resolution of unity $\{\varphi_j\}$ in Definition 2.1/2 is fixed. Then also the quasi-norms in $B^s_{p,q}(K_R)$ and $F^s_{p,q}(K_R)$ are fixed. If $t \in I\!R$, then we put $t_+ = \max(0, t)$. We recall one of the main results of [7] and [8]. Let

$$-\infty < s_2 < s_1 < \infty, \quad p_1, p_2, q_1, q_2 \in (0, \infty] \quad \text{and} \tag{2}$$

$$s_1 - s_2 > n\big(\frac{1}{p_1} - \frac{1}{p_2}\big)_+ \, , \tag{3}$$

then the embedding

$$id_R \; : \; B^{s_1}_{p_1,q_1}(K_R) \to B^{s_2}_{p_2,q_2}(K_R) \tag{4}$$

is compact. Let e^R_k be the kth entropy number of id_R, then

$$c_1(R) \, k^{-\frac{s_1-s_2}{n}} \le \; e^R_k \le c_2(R) \, k^{-\frac{s_1-s_2}{n}} \, , \quad k \in I\!N \, , \tag{5}$$

see [8: Theorem 3.1.2]. We are interested in the dependence of $c_2(R)$ on R. By the elementary embedding (1/11) we have immediately corresponding assertions for the F-spaces. In this sense the following proposition covers also (4) and (5) with F instead of B (where $p_1 < \infty$, $p_2 < \infty$).

Proposition. *Suppose* (2) *and* (3) *and*

$$n\big(\frac{1}{p_1} - 1\big)_+ < \; s_1 < \; \frac{n}{p_1} \, . \tag{6}$$

There exists a constant $c > 0$ such that for all $R \ge 1$

$$c_2(R) \le c \, R^\delta \quad with \quad \delta = s_1 - \frac{n}{p_1} - \big(s_2 - \frac{n}{p_2}\big) \, . \tag{7}$$

Remark. A proof will be given in 5.4. Recall that (7) applies both to B-spaces and F-spaces. The proof is based on the interpolation results of the preceding section and on homogeneity properties of the involved spaces. This is the point where the restriction (6) comes in. There is no doubt that (6) can be removed, but we are not interested to do so. Later on we use the above proposition in connection with weighted spaces on $I\!R^n$ and there we have convenient lifts for the s_1-parameters such that (6) may be assumed to hold.

4.2 The main theorem

The main subject of this paper is the study of the entropy numbers of the compact embeddings

$$id^B \ : \ B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2} \tag{1}$$

and

$$id^F \ : \ F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2} \tag{2}$$

where these spaces have been introduced in Definition 2.1/2. By the arguments at the beginning of Subsect.2.4 this covers also the apparently more general case where the unweighted spaces on the right-hand sides of (1) and (2) are replaced by $B^{s_2}_{p_2,q_2}(\beta)$ and $F^{s_2}_{p_2,q_2}(\beta)$, respectively (in that case one has simply to replace α in (1) and (2) by $\alpha - \beta$). Of course, the B-spaces and F-spaces in (1) and (2) can be mixed either simply by (1/11) or by the arguments below. We stick at (1) and (2). Furthermore, with exception of the interesting limiting case "**L**" in the notations below it comes out that the q-indices in (1) and (2) do not play any role. Then we may restrict the formulation to (1), since (2) is covered afterwards via (1/11) and its obvious weighted counterparts. Let

$$-\infty < s_1 < \infty, \quad 0 < p_1 \leq \infty, \quad 0 < q_1 \leq \infty \quad \text{and} \quad \alpha > 0\,. \tag{3}$$

In the sense of Theorem 2.4 we introduce p_0 given by

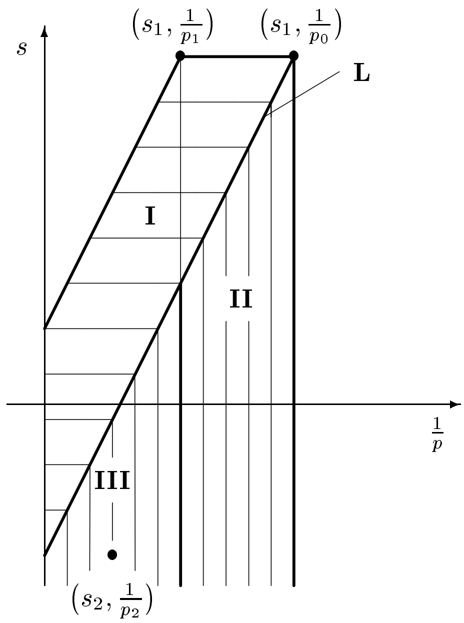
$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n} \quad . \tag{4}$$

As for the target space in (1) we assume in extension of (1/7), Theorem 2.3 and Remark 2.3/2

$$-\infty < s_2 < s_1 < \infty, \quad p_0 < p_2 \leq \infty, \quad 0 < q_2 \leq \infty \tag{5}$$

and

$$\delta = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2}\right) > 0 \quad . \tag{6}$$



I : $0 < \delta < \alpha$

II : $\delta > \alpha, \quad \frac{1}{p_1} \le \frac{1}{p_2} < \frac{1}{p_0}$

III : $\delta > \alpha, \quad \frac{1}{p_2} < \frac{1}{p_1}$

L : $\delta = \alpha, \quad \frac{1}{p_2} < \frac{1}{p_0}$

Fig. 1

In agreement with Theorem 2.3 (ii), Remark 2.3/2 and Theorem 2.4 we divide the region of compact embeddings in the sense of (1) and (2) in the indicated 4 parts. To indicate whether (1) or (2) is considered we put $e_k(id^B)$ and $e_k(id^F)$ for the respective entropy numbers. Furthermore we use $e_k \sim k^{-\varrho}$ in the sense that there exist two positive numbers c_1 and c_2 such that

$$c_1\, k^{-\varrho} \le e_k \le c_2\, k^{-\varrho} \quad \text{for all} \quad k \in I\!N\,. \tag{7}$$

Theorem. *Let* (3), (4) *and* (5) *be satisfied with* $p_1 < \infty$ *and* $p_2 < \infty$ *in the case of the F-spaces.*
(i) *In region* **I** *holds*

$$e_k(id^B) \;\sim\; k^{-\frac{s_1-s_2}{n}}\,. \tag{8}$$

(ii) *In region* **II** *holds*

$$e_k(id^B) \;\sim\; k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}\,. \tag{9}$$

(iii) *In region* **III** *there exist a constant* $c > 0$ *and for any* $\varepsilon > 0$ *a constant* c_ε *such that*

$$c\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \le e_k(id^B) \le c_\varepsilon k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}(1+\log k)^{\varepsilon-\frac{1}{p_2}+\frac{1}{p_1}} \quad \textit{for } k \in I\!N\,. \tag{10}$$

(iv)$_F$ *Let* $\delta = \alpha$ *and* $\frac{1}{p_2} < \frac{1}{p_0}$, *the line* **L**, *then there exists a constant* $c > 0$ *such that*

$$e_k(id^F) \ge c\, k^{-\frac{s_1-s_2}{n}}(1+\log k)^{\frac{\alpha}{n}} \quad \textit{for } k \in I\!N\,. \tag{11}$$

(iv)$_B$ *Let* $\delta = \alpha$ *and* $\frac{1}{p_2} < \frac{1}{p_0}$, *the line* **L**, *and let in addition*

$$\textit{either} \quad p_2 = q_2 = \infty \tag{12}$$

$$\textit{or} \quad p_2 < \infty \quad \textit{and} \quad q_2 \ge p_2\, \frac{q_1}{p_0}\,, \tag{13}$$

then there exist constants $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1\, k^{-\frac{s_1-s_2}{n}} \le e_k(id^B) \le c_2 \left(\frac{k}{1+\log k}\right)^{-\frac{s_1-s_2}{n}} \quad \textit{for } k \in I\!N\,. \tag{14}$$

Remark 1. Recall that id^B in (8), (9) and (10) can be replaced by id^F. In all three cases we indicated in Fig. 1 the level lines for the corresponding exponents. It is not so clear whether the left-hand side or the right-hand side of (10) with $\varepsilon = 0$ is the correct behaviour (or something in between).

Remark 2. By (11) and (1/11) we have on the line **L**

$$e_k(id^B) \geq c\, k^{-\frac{s_1-s_2}{n}}(1+\log k)^{\frac{\alpha}{n}} \quad \text{for all } k \in I\!N \tag{15}$$

if

$$q_1 \geq p_1 \quad \text{and} \quad q_2 \leq p_2 \; . \tag{16}$$

There are no parameters such that (13) and (16) hold simultaneously. On the other hand, (15) holds also if $p_1 = q_1 = \infty$ and/or $p_2 = q_2 = \infty$. This follows from the proof, see Remark 5.5/1. In other words, the only case where we have both (14) and (15) is given by $p_2 = q_2 = \infty$ and $q_1 \geq p_1$, that means for

$$id^B \;:\; B^{s_1}_{p_1,q_1}(\alpha) \to \mathcal{C}^{s_2} \quad \text{with} \quad \mathcal{C}^{s_2} = B^{s_2}_{\infty,\infty} \quad \text{and} \quad q_1 \geq p_1 \; . \tag{17}$$

In particular, in the interesting case

$$id^{\mathcal{C}} \;:\; \mathcal{C}^{s_1}(\alpha) \to \mathcal{C}^{s_2} \quad \text{with} \quad \mathcal{C}^{s_1}(\alpha) = B^{s_1}_{\infty,\infty}(\alpha) \; , \tag{18}$$

we have

$$e_k(id^{\mathcal{C}}) \;\sim\; \left(\frac{k}{1+\log k}\right)^{-\frac{s_1-s_2}{n}} \; , \quad \alpha = s_1 - s_2 \; . \tag{19}$$

4.3 A digression : sharp embeddings

The complicated situation on the line **L** in Theorem 4.2 has also its surprising advantages: It enables us to disprove some embeddings or to prove the "only if"- part of the following assertion.

Corollary. *Let* $s \in I\!R, 0 < p < \infty$ *and* $0 < q \leq \infty$. *Then holds*

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v} \tag{1}$$

(continuous embedding) if and only if

$$0 < u \leq \min(p,q) \quad \textit{and} \quad \max(p,q) \leq v \leq \infty \; . \tag{2}$$

Proof: **Step 1.** The "if"-part is well-known. It follows immediately from (2.1/7) and (2.1/8) and Hölder's inequality, see [18: 2.3.2, p.47]. Assume that we have (1). Then the proof of

$$u \leq q \leq v \tag{3}$$

is easy, see [15: 5.1], and will not be repeated here.

Step 2. It remains to prove

$$u \leq p \leq v \tag{4}$$

if (1) holds. This will be done by contradiction, where we adapt the notations to Theorem 4.2. In this sense we assume

$$B^{s_2}_{p_2,q_2} \subset F^{s_2}_{p_2,q} \quad \text{for some} \quad q_2 > p_2 \quad \text{and some} \quad q \; . \tag{5}$$

We choose $\alpha > 0$ and $p_1 = q_1$ in the sense of (4.2/4) and (4.2/13) such that

$$\frac{q_2}{p_2} \geq q_1 \left(\frac{1}{p_1} + \frac{\alpha}{n}\right) \quad \text{and} \quad \frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n} \, . \tag{6}$$

Next we choose s_1 such that Theorem 4.2(iv)$_B$ can be applied to

$$id^B \ : \ B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2} \, . \tag{7}$$

In particular, we have (4.2/14). On the other hand, by (5) and $B^{s_1}_{p_1,p_1}(\alpha) = F^{s_1}_{p_1,p_1}(\alpha)$ Theorem 4.2(iv)$_F$ can also be applied to (7). But by (6) the estimate (4.2/11) contradicts (4.2/14) since

$$\alpha = \delta = s_1 - s_2 - \frac{n}{p_1} + \frac{n}{p_2} > s_1 - s_2 \, . \tag{8}$$

This disproves (5) and also proves the left-hand side of (4). To prove the right-hand side of (4) by contradiction we assume that

$$F^{s_1}_{p_1,q}(\alpha) \subset B^{s_1}_{p_1,q_1}(\alpha) \quad \text{for some} \quad q_1 < p_1 < \infty \quad \text{and some} \quad q \, . \tag{9}$$

Recall that (9) holds for all $\alpha \in I\!R$ if it holds for some α, see Theorem 2.2(ii). Similarly as in (6) we choose $\alpha > 0$ and $q_2 = p_2$ such that

$$1 = \frac{q_2}{p_2} \geq q_1 \left(\frac{1}{p_1} + \frac{\alpha}{n}\right) \quad \text{and} \quad \frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n} \, . \tag{10}$$

Next we choose s_2 such that Theorem 4.2(iv)$_B$ can be applied to

$$id^B \ : \ B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,p_2} \, . \tag{11}$$

In particular, we have again (4.2/14). On the other hand by (9) and $B^{s_2}_{p_2,p_2} = F^{s_2}_{p_2,p_2}$ Theorem 4.2(iv)$_F$ can be applied to (11). By (10) and (8) we arrive again at a contradiction. This disproves (9) and proves the right-hand side of (4).

Remark. As we said, the easy "if"-part may be found in [18: 2.3.2, p.47]. The "only if"-part has been proved only recently in [15: 3.1 and 5.1]. However the proof of the hard part (4) is rather complicated. Curiously enough it is the one-dimensional case which causes a lot of trouble, see Remark 5 in [15: 5.1]. The above proof seems to be simpler, taking Theorem 4.2(iv)$_B$, (iv)$_F$ for granted.

5. Proofs

5.1 Proof of Theorem 2.2 (ii)

We modify the proof of Theorem 4.2.2 in [19: p.203] where we proved pointwise multiplier assertions for the spaces $F^s_{p,q}$ and $B^s_{p,q}$. We use the characterization of $F^s_{p,q}$ and $B^s_{p,q}$ in terms of local means, see [19: 2.4.6 and 2.5.3], without further explanation of notations. Let $s > \frac{n}{p}$, then by (4.2.2/3) in [19] and (2.1/1) we have

$$|k(t, \varrho f)(x)| \leq c \sum_{|\alpha| \leq m-1} t^{|\alpha|} \varrho(x) \left| \int_{I\!R^n} y^\alpha k(y) f(x+ty)\, dy \right| + c\, t^m \varrho(x) \sup_{|t-y| \leq 1} |f(y)| \, . \tag{1}$$

Let $\varrho_1, \varrho_2, \varrho_3$ and ϱ_4 be admissible weight functions in the sense of Definition 2.1/1 with $\varrho_1 \varrho_2 = \varrho_3 \varrho_4$. We apply (1) to $\varrho_3 f$ and $\varrho_1 \varrho_3^{-1}$ instead of f and ϱ, respectively. Furthermore we replace L_p in (4.2.2/4) in [19] by $L_p(\varrho_2(x))$ and obtain by the same arguments as there

$$\left\| \left(\int_0^1 t^{-sq} |k(t, \varrho_1 f)(\cdot)|^q \frac{dt}{t} \right)^{\frac{1}{q}} | L_p(\varrho_2(\cdot)) \right\| \leq c \, \| \varrho_3 f | F^s_{p,q}(\varrho_4(\cdot)) \| \tag{2}$$

and consequently

$$\|\varrho_1 f|F^s_{p,q}(\varrho_2(\cdot))\| \le c\,\|\varrho_3 f|F^s_{p,q}(\varrho_4(\cdot))\| \,. \tag{3}$$

By the lifting argument on p.204 in [19] we can extend (3) from $s > \frac{n}{p}$ to $s \in I\!R$. Now (2.2/2) follows from (3). The proof for the B-spaces is the same.

5.2 Proof of Theorem 2.3

Step 1. We prove the "if"-parts. The embedding (2.3/2) with $\varrho_1 = \varrho_2$ is an immediate consequence of (2.2/2) and the known embedding theorem for the unweighted spaces, see [18: Theorem 2.7.1(ii)]. Then (2.3/2) with $\varrho_2(x) \le c\,\varrho_1(x)$ follows from (2.1/8). Let $F^s_{p,q}(K_R)$ be the same unweighted space as in Subsect.4.1. Then the embedding

$$F^{s_1}_{p_1,q_1}(K_R) \subset F^{s_2}_{p_2,q_2}(K_R) \quad \text{is compact} \tag{1}$$

if (2.3/1) and the first part of (2.3/4) hold, see [7]. Let $\psi_R(x)$ be a C^∞ function on $I\!R^n$ with, say, $\psi_R(x) = 1$ if $|x| > 2R$ and $\psi_R(x) = 0$ if $|x| < R$. By (2.3/4) and the characterization of the considered F- spaces via local means in the sense of [19: 2.4.6] we have

$$\|\psi_R f|F^{s_2}_{p_2,q_2}(\varrho_2(\cdot))\| \le \varepsilon\,\|\psi_R f|F^{s_1}_{p_1,q_1}(\varrho_1(\cdot))\| \tag{2}$$

where $\varepsilon > 0$ is given and $R = R(\varepsilon)$ is chosen sufficiently large. Now (1) and (2) prove the "if"-part of Theorem 2.3(ii).

Step 2. As for the "only if"-parts of Theorem 2.3 we recall first a localization property recently proved in [20]. Let $f \in F^s_{p,q}$ with a small support concentrated near the origin, satisfying sufficiently many moment conditions. Let

$$f^j(x) = \sum_{k\in Z\!\!Z^n} a_k\, f(2^j x - k)\,, \quad a_k \in \mathbb{C} \quad \text{and} \quad j \in I\!N, \tag{3}$$

then

$$\|f^j|F^s_{p,q}\| \;\sim\; 2^{j(s-\frac{n}{p})}\Big(\sum_{k\in Z\!\!Z^n} |a_k|^p\Big)^{\frac{1}{p}}\, \|f|F^s_{p,q}\|\,, \tag{4}$$

where '$\sim$' means that the equivalence constants are independent of f, j and a_k. We refer again to [20] for the necessary details. Now it follows easily that there is no continuous embedding from $F^{s_1}_{p_1,q_1}(K_R)$ into $F^{s_2}_{p_2,q_2}(K_R)$ if $s_2 - \frac{n}{p_2} > s_1 - \frac{n}{p_1}$, and there is no compact embedding if $s_2 - \frac{n}{p_2} \ge s_1 - \frac{n}{p_1}$. This proves the counterpart of the "only if" assertions of the theorem for spaces on bounded domains.

Step 3. Assume that there exists a sequence $\{x^j\}_{j=1}^\infty \subset I\!R^n$ with $|x^j| \to \infty$ and $\frac{\varrho_2(x^j)}{\varrho_1(x^j)} \to \infty$ if $j \to \infty$. By the localization principle for $F^s_{p,q}$-spaces, see [19: 2.4.7, p.124], and a translation argument it follows that (2.3/2) cannot be true. Similarly if $\varrho_2(x^j) \ge c\,\varrho_1(x^j)$ for some $c > 0$ and $j \in I\!N$, then, if there is a continuous embedding (2.3/2), it cannot be compact.

Remark. Together with the well-known "if"-parts we obtained in step 2 the "only if"-parts of the following assertion.

Corollary. *Let Ω be a bounded C^∞-domain in $I\!R^n$ and let*

$$-\infty < s_2 < s_1 < \infty,\; 0 < p_1 < \infty,\; 0 < p_2 < \infty,\; 0 < q_1 \le \infty,\; 0 < q_2 \le \infty\,. \tag{5}$$

Then holds

$$F^{s_1}_{p_1,q_1}(\Omega) \subset F^{s_2}_{p_2,q_2}(\Omega) \tag{6}$$

(continuous embedding) if and only if $s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}$. *Furthermore the embedding* (6) *is compact if and only if* $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$.

P r o o f: The above arguments cover the case $p_2 \geq p_1$. However Hölder's inequality and the use of local means in the sense of [19: 2.4.6] prove $F^{s_2}_{p_2,q_2}(\Omega) \subset F^{s_2}_{p_3,q_2}(\Omega)$ if $0 < p_3 \leq p_2$.

5.3 Proof of Theorem 3.2

Step 1. We prove part (i). Let U_A be the unit ball in A. Then its image $T\,U_A$ can be covered in B_0 by 2^{k_0-1} balls K_j of radius $(1+\varepsilon)\, e_{k_0}(A \to B_0)$, where $\varepsilon > 0$ can be chosen arbitrarily small. Each of the sets $K_j \cap T\,U_A$ can be covered in B_1 by 2^{k_1-1} balls of radius $(1+\varepsilon)2^{\frac{1}{\lambda}}\, e_{k_1}(A \to B_1)$ with centres belonging to $K_j \cap T\,U_A$. The factor $2^{\frac{1}{\lambda}}$ comes from the additional assumption about the centres, see (3.1/1). We have $2^{k_0+k_1-2}$ such centres b_l. Let $a \in U_A$ be given, then using again (3.1/1) we have

$$\|Ta - b_l|B_r\|^\lambda \leq 2\,(1+\varepsilon)^\lambda\, e^\lambda_{k_r}\,(A \to B_r)\,, \quad r = 0,1\,, \tag{1}$$

for a suitably chosen centre b_l. Now (3.2/3) follows from (3.2/2) and (1).

Step 2. We prove part (ii). Let $a \in U_A$ and $t > 0$, then we choose $a_0 \in A_0$ and $a_1 \in A_1$ such that $a = a_0 + a_1$ and

$$\|a_0|A_0\| + t\,\|a_1|A_1\| \leq (1+\varepsilon)\,K(t,a) \leq (1+\varepsilon)\,t^\theta \tag{2}$$

where again $\varepsilon > 0$ can be chosen arbitrarily small. We used (3.2/4). In particular, we have

$$\|a_0|A_0\| \leq (1+\varepsilon)\,t^\theta \quad \text{and} \quad \|a_1|A_1\| \leq (1+\varepsilon)\,t^{\theta-1}\,. \tag{3}$$

We specify $t = e_{k_1}(A_1 \to B)\, e^{-1}_{k_0}(A_0 \to B)$ and choose afterwards 2^{k_0-1} elements $b_l \in B$ and 2^{k_1-1} elements $b^m \in B$ such that $(1+\varepsilon)t^\theta\, T\,U_{A_0}$ can be covered by 2^{k_0-1} balls of radius

$$(1+\varepsilon)^2\, t^\theta\, e_{k_0}(A_0 \to B) = (1+\varepsilon)^2\, e^{1-\theta}_{k_0}(A_0 \to B)\, e^\theta_{k_1}(A_1 \to B) \tag{4}$$

centered at b_l, and $(1+\varepsilon)\,t^{\theta-1}\,T\,U_{A_1}$ can be covered by 2^{k_1-1} balls of radius

$$(1+\varepsilon)^2\, t^{\theta-1}\, e_{k_1}(A_1 \to B) = (1+\varepsilon)^2\, e^{1-\theta}_{k_0}(A_0 \to B)\, e^\theta_{k_1}(A_1 \to B) \tag{5}$$

centered at b^m. Let $a \in U_A$, then the desired assertion follows from

$$\|Ta - b_l - b^m|B\|^\lambda \leq \|Ta_0 - b_l|B\|^\lambda + \|Ta_1 - b^m|B\|^\lambda\,, \tag{6}$$

an optimal choice of $a = a_0 + a_1, b_l, b^m$, and from (4) and (5).

5.4 Proof of Proposition 4.1

Step 1. Of course

$$\widetilde{B^s_{p,q}}(K_R) = \{f \in B^s_{p,q},\ \mathrm{supp} f \subset \overline{K_R}\,\} \tag{1}$$

is a closed subspace of $B^s_{p,q}$. By the extension property for the spaces $B^s_{p,q}(K_R)$, see [19: 5.1.3], we may replace B in (4.1/4) by $\widetilde{B}$. Since $R \geq 1$ and by the local nature of the construction in [19: 5.1.3] the related extension operators can be estimated uniformly with respect to R.

Step 2. Let

$$D_R\ :\ f(x) \mapsto f(Rx) \quad , \quad R > 0\,, \tag{2}$$

be the dilation operator in $\mathbb{R}^n$. Let $\widetilde{id_R}$ be given by (4.1/4) with $\widetilde{B}$ instead of B. Then we have

$$\widetilde{id_R} = D_{R^{-1}} \circ \widetilde{id_1} \circ D_R \, . \tag{3}$$

Let $R \geq 1$, $s_1 > n\big(\frac{1}{p_1} - 1\big)_+$ and $s_2 < 0$. Then we have

$$\|D_R f | B^{s_1}_{p_1,q_1}\| \leq c\, R^{s_1 - \frac{n}{p_1}} \|f | B^{s_1}_{p_1,q_1}\| \tag{4}$$

and

$$\|D_{R^{-1}} f | B^{s_2}_{p_2,q_2}\| \leq c\, R^{-s_2 + \frac{n}{p_2}} \|f | B^{s_2}_{p_2,q_2}\| \, , \tag{5}$$

see [20: 2.2]. Now (4.1/7) follows from (3), (4), (5) and (4.1/5) with $R = 1$.

Step 3. Let $s_1 > n\big(\frac{1}{p_1} - 1\big)_+$, $s_2 < s_1$ and $p_1 = p_2$. We choose s_3 with $s_3 < \min(s_2, 0)$. Recall the well-known real interpolation formula

$$\big(B^{s_1}_{p_1,q_1} \, , \, B^{s_3}_{p_1,q_1}\big)_{\theta,q_2} = B^{s_2}_{p_1,q_2} \, , \quad s_2 = (1-\theta)s_1 + \theta s_3 \, , \tag{6}$$

see [18: 2.4.2]. Then we have an obvious counterpart of (3.2/2), maybe with an additional constant which is, of course, independent of the above number R. We use the result of step 2 and apply Theorem 3.2(i) with $A = B_0 = \widetilde{B^{s_1}_{p_1,q_1}}(K_R)$ and $B_1 = \widetilde{B^{s_3}_{p_1,q_1}}(K_R)$. Then we obtain (4.1/7) for the special case

$$\widetilde{id_R}^{\,p} \; : \; \widetilde{B^{s_1}_{p_1,q_1}}(K_R) \to \widetilde{B^{s_2}_{p_1,q_2}}(K_R) \, . \tag{7}$$

Step 4. Let now s_1, s_2, p_1 and p_2 be restricted by (4.1/3) and (4.1/6). Then we find numbers $s_3 < 0$ and p_3 such that (as shown in Fig. 2)

$$s_2 = (1-\theta)s_1 + \theta s_3, \quad \frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{p_3} \qquad \text{and} \qquad s_1 - s_3 > n\Big(\frac{1}{p_1} - \frac{1}{p_3}\Big)_+ \tag{8}$$

for some $0 < \theta < 1$.

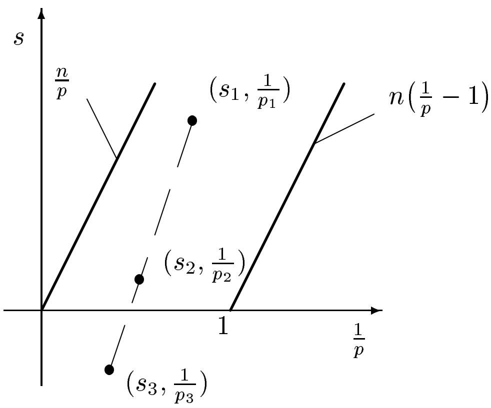


Fig. 2

By the definition of the spaces $B^s_{p,q}$ and Hölder's inequality we have

$$\|f | B^{s_2}_{p_2,q}\| \leq \|f | B^{s_1}_{p_1,q}\|^{1-\theta} \|f | B^{s_3}_{p_3,q}\|^{\theta} \, . \tag{9}$$

Now again by step 2 and Theorem 3.2(i) we obtain (4.1/7) for the special case

$$\widetilde{id_{R;q}} \; : \; \widetilde{B^{s_1}_{p_1,q}}(K_R) \to \widetilde{B^{s_2}_{p_2,q}}(K_R) \, . \tag{10}$$

Step 5. The general case now follows from the special cases (7) and (10) and the multiplicativity

$$e_{k_1+k_2-1}\big(\widetilde{id_R}^{\,p} \circ \widetilde{id_{R;q}}\big) \leq e_{k_1}\big(\widetilde{id_R}^{\,p}\big)\, e_{k_2}\big(\widetilde{id_{R;q}}\big) \tag{11}$$

if the parameters are chosen in a suitable way. As far as (11) is concerned we refer to [5: p.21] or [6: p.47].

5.5 Proof of the main Theorem 4.2

5.5.1 Estimates from below

Step 1. In the annuli

$$A_m = \{x \in \mathbb{R}^n : \; 2^{m-1} \le |x| \le 2^{m+1}\}, \quad m \in \mathbb{N}, \tag{1}$$

we have $\langle x\rangle^\alpha \sim 2^{m\alpha}$. We rely again on the localization property in [20] described in step 2 in Subsect.5.2. We modify (5.2/3) by

$$f_m^j(x) = \sum\nolimits^{j,m} a_k \, f(2^j x - k), \tag{2}$$

where $\sum^{j,m}$ stands for the sum over all lattice points $k \in \mathbb{Z}^n$ with $2^{-j}k \in A_m$ which are the centres of the shifted functions $f(2^j x - k)$. Let $A^s_{p,q}$ be either $B^s_{p,q}$ or $F^s_{p,q}$, then we have

$$\|f_m^j | A^{s_1}_{p_1,q_1}(\alpha)\| \sim 2^{m\alpha} \, 2^{j(s_1 - \frac{n}{p_1})} \big(\sum\nolimits^{j,m} |a_k|^{p_1}\big)^{\frac{1}{p_1}} \tag{3}$$

and

$$\|f_m^j | A^{s_2}_{p_2,q_2}\| \sim 2^{j(s_2 - \frac{n}{p_2})} \big(\sum\nolimits^{j,m} |a_k|^{p_2}\big)^{\frac{1}{p_2}}, \tag{4}$$

see (5.2/4) and for further details [20]. Put $N_l = 2^{nl}$, then we may assume that the number of admitted lattice points in (3) and (4) is N_{j+m} (neglecting constants). Let $span_m^j$ be the subspace spanned by the functions $f(2^j x - k)$ in (2) for fixed j and m. Let the unit ball U in $A^{s_1}_{p_1,q_1}(\alpha)$ be covered by $2^{N_{j+m}}$ balls in $A^{s_2}_{p_2,q_2}$ of radius $2\,e_{N_{j+m}}$. Let K be one of these balls then by (4) we have

$$vol(K \cap span_m^j) \le c^{N_{j+m}} \, e_{N_{j+m}}^{N_{j+m}} \, 2^{-jN_{j+m}(s_2 - \frac{n}{p_2})} \, vol\big(U_{p_2}^{N_{j+m}}\big) \tag{5}$$

where c is an appropriate constant and U_p^N is the unit ball in l_p^N. The sum of all these volumes, which can be estimated from above by the right-hand side of (5) times $2^{N_{j+m}}$ must exceed

$$vol(U \cap span_m^j) \sim 2^{-m\alpha N_{j+m}} \, 2^{-jN_{j+m}(s_1 - \frac{n}{p_1})} \, vol\big(U_{p_1}^{N_{j+m}}\big). \tag{6}$$

Recall $\delta = s_1 - s_2 - \frac{n}{p_1} + \frac{n}{p_2}$. Then we have by (5) and (6)

$$e_{N_{j+m}} \ge c\, 2^{-j\delta} \, 2^{-m\alpha} \left(\frac{vol\big(U_{p_1}^{N_{j+m}}\big)}{vol\big(U_{p_2}^{N_{j+m}}\big)} \right)^{\frac{1}{N_{j+m}}}. \tag{7}$$

By [8: p.162, formula (26)] the last factor in (7) is equivalent to $N_{j+m}^{\frac{1}{p_2} - \frac{1}{p_1}}$, and hence, equivalent to $2^{n(j+m)(\frac{1}{p_2} - \frac{1}{p_1})}$. Then we obtain

$$e_{2^{n(j+m)}} \ge c\, 2^{-j(s_1 - s_2)} \, 2^{-m(\alpha - \frac{n}{p_2} + \frac{n}{p_1})}, \quad j \in \mathbb{N}, \; m \in \mathbb{N}. \tag{8}$$

Choosing $j = 1$ or $m = 1$ this yields the estimates from below in (4.2/8), (4.2/9), (4.2/10) and (4.2/14).

Step 2. As far as estimates from below are concerned it remains to prove (4.2/11). We use the localization principle for the F-spaces which has no counterpart for the B-spaces, see [19: 2.4.7, p.124]. We fix $M \in \mathbb{N}$ and construct in each annulus A_m the functions $f_m^j(x)$ with $j = M - m$ and coefficients a_k^m. Then we have in each annulus $N_M = 2^{nM}$ admitted lattice points, hence all together $M2^{nM} = N^{(M)}$ lattice points (neglecting constants). We have the counterparts of (3) and (4) now with F instead of A. In this limiting case $\alpha = \delta$ the exponent in (3) is given by

$$M\alpha + (M-m)\big(s_1 - \frac{n}{p_1} - \delta\big) = M\alpha + (M-m)\big(s_2 - \frac{n}{p_2}\big) . \tag{9}$$

Hence by the localization principle for the F-spaces and with modified coefficients

$$b_k = 2^{(M-m)\left(s_2 - \frac{n}{p_2}\right)} a_k$$

we have the following counterparts of (3) and (4)

$$\| \sum_{m=1}^{M} f_m^{M-m} | F^{s_1}_{p_1,q_1}(\alpha) \| \sim 2^{M\alpha} \left(\sum |b_k|^{p_1} \right)^{\frac{1}{p_1}} \tag{10}$$

and

$$\| \sum_{m=1}^{M} f_m^{M-m} | F^{s_2}_{p_2,q_2} \| \sim \left(\sum |b_k|^{p_2} \right)^{\frac{1}{p_2}} , \tag{11}$$

where $\sum$ is taken over $N^{(M)} = M2^{nM}$ summands. By the same arguments as in step 1 we have the following counterpart of (7)

$$\begin{aligned} e_{N^{(M)}} \quad &\geq \quad c_1\, 2^{-M\alpha} \left(\frac{vol\left(U^{N^{(M)}}_{p_1}\right)}{vol\left(U^{N^{(M)}}_{p_2}\right)} \right)^{\frac{1}{N^{(M)}}} \geq c_2\, 2^{-M\delta}\, M^{\frac{1}{p_2} - \frac{1}{p_1}}\, 2^{nM\left(\frac{1}{p_2} - \frac{1}{p_1}\right)} \\ &= \quad c_2\, 2^{-M(s_1 - s_2)}\, M^{\frac{1}{p_2} - \frac{1}{p_1}} \end{aligned} \tag{12}$$

where we used $\alpha = \delta$ and again [8: (26) on p.162]. Let $M2^{nM} \sim 2^K$, then $K \sim M$ and (12) can be rewritten as

$$e_{2^K} \geq c\, 2^{-K\frac{s_1 - s_2}{n}}\, K^{\frac{s_1 - s_2}{n} + \frac{1}{p_2} - \frac{1}{p_1}} = c\, 2^{-K\frac{s_1 - s_2}{n}}\, K^{\frac{\delta}{n}} . \tag{13}$$

With $\delta = \alpha$ we obtain (4.2/11). The proof of (iv)$_F$ in Theorem 4.2 is complete.

Remark 1. Compared with step 1 we used in step 2 the localization principle for the F-spaces. There is no counterpart for the B-spaces with exception of $B^s_{p,p}$ with $0 < p \leq \infty$. Especially step 2 can also be applied if either $p_1 = q_1 = \infty$ and/or $p_2 = q_2 = \infty$. This justifies Remark 4.2/2.

5.5.2 Estimates from above

Step 1. Let $0 < \delta < \alpha$ and $\frac{1}{p_2} \leq \frac{1}{p_1}$. Then we have

$$B^{s_1}_{p_1,q_1} \subset B^{s_2}_{p_2,q_2} . \tag{14}$$

We complement the annuli defined in (1) by $A_0 = \{x \in \mathbb{R}^n : |x| \leq 2\}$ and $A^m = \{x \in \mathbb{R}^n : |x| > 2^m\}$. Let $e_{k,j}$ with $j = 0, \ldots, m$, be the entropy numbers of the embeddings

$$id_j \ : \ B^{s_1}_{p_1,q_1}(A_j) \to B^{s_2}_{p_2,q_2}(A_j) , \tag{15}$$

see (4.1/4). Let $\lambda = \min(1, p_2, q_2)$ and let e_k be the entropy numbers of id^B introduced in (4.2/1). Then we obtain by (3.1/2) and the usual technique of the resolution of unity based on $\{A_j\}_{j=0}^m$ and A^m

$$e_k^\lambda(id^B) \leq c \sum_{j=0}^{m} 2^{-j\alpha\lambda}\, e^\lambda_{k_j,j} + c\, 2^{-m\alpha\lambda} , \qquad k = \sum_{j=0}^{m} k_j , \tag{16}$$

where the last term comes from (14). Recall that $I_\sigma\ :\ f \mapsto (<\cdot>^\sigma \hat{f})^\vee$ is a lift for all spaces $B^s_{p,q}(\alpha)$. This shows that we may assume without loss of generality that s_1 is restricted by (4.1/6). Then we can apply Proposition 4.1 and we obtain by (16), (4.1/5) and (4.1/7)

$$e_k^\lambda(id^B) \le c \sum_{j=0}^{m} 2^{j(\delta-\alpha)\lambda}\, k_j^{-\lambda\frac{s_1-s_2}{n}} + c\, 2^{-m\alpha\lambda}\,. \tag{17}$$

We choose $k_j = 2^{b(m-j)}$ for some $b>0$. Then we have $k \sim 2^{bm}$ and since $\delta < \alpha$

$$\begin{aligned} e_k^\lambda(id^B) &\le c_1\, 2^{-\lambda bm\frac{s_1-s_2}{n}} \sum_{j=0}^{m} 2^{\lambda j(\delta-\alpha+b\frac{s_1-s_2}{n})} + c_1\, 2^{-m\alpha\lambda} \\ &\le c_2\, 2^{-\lambda bm\frac{s_1-s_2}{n}} + c_2\, 2^{-m\alpha\lambda} \le c_3\, 2^{-\lambda bm\frac{s_1-s_2}{n}} \end{aligned} \tag{18}$$

if $b>0$ is chosen sufficiently small. Since $k \sim 2^{bm}$ we obtain

$$e_k(id^B) \le c\, k^{-\frac{s_1-s_2}{n}}\,, \quad k \in \mathbb{N}\,. \tag{19}$$

This completes the proof of (4.2/8) provided that $\frac{1}{p_2} \le \frac{1}{p_1}$.

Step 2. Let $0<\delta<\alpha$ and $\frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_0}$. We wish to extend (19) to this case. By the same arguments as in Subsect.2.4 we have

$$B^{s_1}_{p_1,q_1}(\alpha) \subset B^{s_1}_{p,\infty} \qquad \text{if} \quad \frac{1}{p_1} < \frac{1}{p} < \frac{1}{p_0}\,, \tag{20}$$

where the spaces on the right-hand side of (20) correspond to the upper line of the region **I** in Fig. 1 in 4.2. Since $(s_2, \frac{1}{p_2})$ belongs also to the region **I** we find numbers θ, p, q and s with $0<\theta<1$, (20), $0<q\le\infty$ and $s \in \mathbb{R}$ such that

$$s_2 = (1-\theta)s + \theta s_1\,, \quad \frac{1}{p_2} = \frac{\theta}{p}, \quad \frac{1}{q_2} = \frac{1-\theta}{q}, \tag{21}$$

and

$$\delta = s_2 - \frac{n}{p_2} = s = s_1 - \frac{n}{p}\,. \tag{22}$$

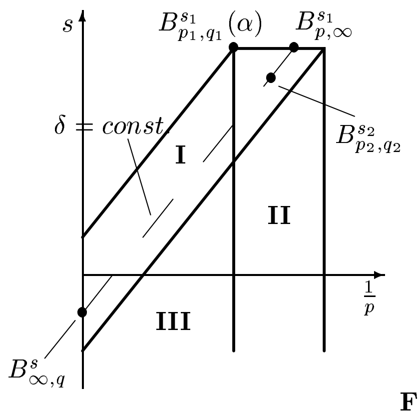


Fig. 3

By Definition 2.1/2 (unweighted case) and Hölder's inequality we have the multiplicative inequality

$$\|f|B^{s_2}_{p_2,q_2}\| \le \|f|B^s_{\infty,q}\|^{1-\theta}\|f|B^{s_1}_{p,\infty}\|^\theta \tag{23}$$

along the line $\delta = const.$ between the indicated endpoints $B^s_{\infty,q}$ and $B^{s_1}_{p,\infty}$. By step 1 we can apply (19) to $B^{s_1}_{p_1,q_1}(\alpha) \to B^s_{\infty,q}$. Then we obtain by Theorem 3.2 with $A = B^{s_1}_{p_1,q_1}(\alpha)$, $B_0 = B^s_{\infty,q}$ and

$B_1 = B^{s_1}_{p,\infty}$

$$\begin{aligned} e_k(id^B \,:\, B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) &\leq c_1\, e_k^{1-\theta}(id^B \,:\, B^{s_1}_{p_1,q_1}(\alpha) \to B^{s}_{\infty,q}) \\ &\leq c_2\, k^{-(1-\theta)\frac{s_1-s}{n}} = c_2\, k^{-\frac{s_1-s_2}{n}} \,. \end{aligned} \tag{24}$$

Now step 1 in 5.5.1 and the above two steps prove part (i) of Theorem 4.2.

Step 3. To deal with region **II** we consider first the special case $p_2 = p_1$. We have (16) and (17) now with

$$\delta = s_1 - s_2 > \alpha \,. \tag{25}$$

Let $0 < \varepsilon < \delta - \alpha$ and $k_j = 2^{m(\alpha+\varepsilon)\frac{n}{\delta}}\, 2^{j\frac{n}{\delta}(\delta-\alpha-\varepsilon)}$. Then we obtain $k \sim 2^{nm}$ and

$$\begin{aligned} e_k^\lambda(id^B) &\leq c_1 \sum_{j=0}^{m} 2^{j(\delta-\alpha)\lambda}\, 2^{-\lambda m(\alpha+\varepsilon)}\, 2^{-\lambda j(\delta-\alpha-\varepsilon)} + c_1\, 2^{-\lambda m\alpha} \\ &\leq c_2\, 2^{-\lambda m\alpha} \,. \end{aligned} \tag{26}$$

Hence, we have the desired estimate

$$e_k(id^B) \leq c\, k^{-\frac{\alpha}{n}} \,, \quad k \in I\!N \,. \tag{27}$$

Step 4. Let $\delta > \alpha$ and $\frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_0}$, the interior of region **II**. On the border line p_0 we have by Theorem 2.4 the continuous embedding

$$B^{s_1}_{p_1,q_1}(\alpha) \subset weak - B^{s}_{p_0,\infty} \quad \text{with} \quad s < s_1 \,. \tag{28}$$

Now we are in the same situation as in step 2. We wish to use multiplicative inequalities similar those ones in (23) along the line $\alpha < \delta = const.$ with the spaces $B^{s_2}_{p_1,q}$, treated in step 3, and the spaces $weak - B^{s}_{p_0,\infty}$ as endpoint spaces. The desired analogue to (23) follows from the real interpolation formula (2.4/4), Definition 2.4, Definition 2.1/2 and Hölder's inequality. Then we get in a similar way as in step 2

$$e_k \leq c\, k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \quad , \quad k \in I\!N \,. \tag{29}$$

Part (ii) of Theorem 4.2 follows now from (29) and the corresponding estimate from below proved in step 1 in 5.5.1.

Step 5. To prove part (iv)$_B$ we consider first

$$id^B \,:\, B^{s_1}_{p_1,q_1}(\alpha) \to \mathcal{C}^{s_2} = B^{s_2}_{\infty,\infty} \,, \quad \alpha = \delta = s_1 - s_2 - \frac{n}{p_1} \,. \tag{30}$$

As in step 1 we may assume, without loss of generality, that s_1 is restricted by (4.1/6) such that Proposition 4.1 can be applied. Furthermore we use the localization principle for the spaces $\mathcal{C}^{s_2}$ proved in [19: 2.4.7, p.124]. Then the counterpart of (16) with $k_j = k$ is given by

$$e_{mk}(id^B) \leq c \sup_{j=0,\ldots,m} 2^{-j\alpha} e_{k,j} + c\, 2^{-m\alpha} \,. \tag{31}$$

By Proposition 4.1 we have $e_{k,j} \leq c\, 2^{j\delta} k^{-\frac{s_1-s_2}{n}}$ and hence, since $\delta = \alpha$,

$$e_{mk}(id^B) \leq c\, k^{-\frac{s_1-s_2}{n}} + c\, 2^{-m\alpha} \,. \tag{32}$$

We choose $k^{\frac{s_1-s_2}{n}} \sim 2^{m\alpha}$, that means $m \sim \frac{s_1-s_2}{n\alpha} \log k$. Then we have

$$e_{c\,k \log k}(id^B) \leq c\, k^{-\frac{s_1-s_2}{n}} \quad , \quad k = 2, 3 \ldots , \tag{33}$$

which finally results in

$$e_k(id^B) \le c \left(\frac{k}{1+\log k}\right)^{-\frac{s_1-s_2}{n}} \quad , \; k \in I\!N \; . \tag{34}$$

Step 6. Now we prove part (iv)$_B$ in the same way as above via interpolation on the basis of Theorem 3.2. One endpoint result is given by (30), the other one by

$$id^B \; : \; B^{s_1}_{p_1,q_1}(\alpha) \to weak - B^{s_1}_{p_0,q_1} \tag{35}$$

where p_0 is again given by (4.2/4). That means, in Fig. 1 in Subsect.4.2 we interpolate between the endpoints of the line **L**. Again in order to get the counterpart of (23) we have to use (2.4/4). In a similar way as in the steps 2 and 4 we obtain

$$e_k(id^B) \le c \left(\frac{k}{1+\log k}\right)^{-\frac{s_1-s_2}{n}} \quad , \quad k \in I\!N \tag{36}$$

for the embedding

$$id^B \; : \; B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2} \tag{37}$$

if

$$\frac{1}{p_2} = \frac{\theta}{p_0} \quad \text{and} \quad \frac{1}{q_2} = \frac{\theta}{q_1} \quad \text{for some} \quad \theta \quad \text{with} \quad 0 < \theta < 1 \; . \tag{38}$$

But this is just the limiting case in (4.2/13). Larger values of q_2 can be incorporated afterwards by the monotonicity of the $B^s_{p,q}$-spaces with respect to the q-index.

Step 7. It remains to prove the right-hand side of (4.2/10). We choose s_3 such that

$$\varepsilon = s_1 - \frac{n}{p_1} - s_3 + \frac{n}{p_2} \; . \tag{39}$$

We can apply (36) to

$$id^B \; : \; B^{s_1}_{p_1,q_1}(\varepsilon) \to B^{s_3}_{p_2,\infty} \tag{40}$$

and by Theorem 2.2(ii) also to

$$id^B_\varepsilon \; : \; B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_3}_{p_2,\infty}(\alpha - \varepsilon) \tag{41}$$

and obtain

$$e_k(id^B_\varepsilon) \le c \left(\frac{k}{1+\log k}\right)^{-\frac{\varepsilon}{n} - \frac{1}{p_1} + \frac{1}{p_2}} \quad , \quad k \in I\!N \; . \tag{42}$$

Furthermore by step 3 we have

$$e_k(id^B_{\alpha-\varepsilon}) \le c\, k^{-\frac{\alpha-\varepsilon}{n}} \quad , \quad k \in I\!N \tag{43}$$

where (in a slight abuse of notation)

$$id^B_{\alpha-\varepsilon} \; : \; B^{s_3}_{p_2,\infty}(\alpha - \varepsilon) \to B^{s_2}_{p_2,q_2} \; . \tag{44}$$

In this case the counterpart of (25) is given by

$$s_3 - s_2 > \alpha - \varepsilon \; . \tag{45}$$

Now the right-hand side of (4.2/10) follows from (42), (43) and the multiplicativity of the entropy numbers

$$e_{2k}(id^B) \le e_k(id^B_{\alpha-\varepsilon})\; e_k(id^B_\varepsilon) \; , \tag{46}$$

see (5.4/11) and the references given there. This finally completes the proof.

Remark 2. The proofs in the steps 4 and 6 depend on the interpolation formula (2.4/4) for the scalar Lorentz spaces. An extension to the vector-valued case with the same quasi-Banach space A, that means $L_{p_0,u_0}(A)$ and $L_{p_1,u_1}(A)$ in (2.4/4), is possible, see [17: 1.18.6, p.134]. However if one replaces A by different spaces A_0 and A_1, respectively, then one cannot expect a reasonable interpolation formula, also not in the weaker version of an inequality of type (3.2/2). If this would be the case then one could extend Theorem 4.2(iv)$_B$ to F-spaces. But this may contradict Theorem (iv)$_F$ similarly as in the proof of Corollary 4.3. In other words, in this way one can disprove wrongly conjectured vector-valued counterparts of the interpolation formula (2.4/4).

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3.2 Appendix B

Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators II

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Abstract. This paper is the continuation of [17]. We investigate mapping and spectral properties of pseudodifferential operators of type $\Psi^{\varkappa}_{1,\gamma}$ with $\varkappa \in I\!R$ and $0 \leq \gamma \leq 1$ in the weighted function spaces $B^s_{p,q}(I\!R^n, w(x))$ and $F^s_{p,q}(I\!R^n, w(x))$ treated in [17]. Furthermore we study the distribution of eigenvalues and the behaviour of corresponding root spaces for degenerate pseudodifferential operators preferably of type $b_2(x)b(x,D)b_1(x)$, where $b_1(x)$ and $b_2(x)$ are appropriate functions and $b(x,D) \in \Psi^{\varkappa}_{1,\gamma}$. Finally, on the basis of the Birman-Schwinger principle, we deal with the "negative spectrum" (bound states) of related symmetric operators in L_2.

(Math. Subject Classification: 46E35, 47G30, 35S05)

1. Introduction

The spaces $B^s_{p,q}$ and $F^s_{p,q}$ with $s \in I\!R, 0 < p \leq \infty$ ($p < \infty$ for the F- spaces) and $0 < q \leq \infty$ on $I\!R^n$ cover many well-known classical spaces such as (fractional) Sobolev spaces, Hölder-Zygmund spaces, Besov spaces and (inhomogeneous) Hardy spaces. In [17] we treated their weighted counterparts $B^s_{p,q}(I\!R^n, w(x))$ and $F^s_{p,q}(I\!R^n, w(x))$ where $w(x)$ is a (smooth) weight function of at most polynomial growth, that means

$$0 < w(x) \leq c\, w(y)\, \langle x - y\rangle^{\alpha} \tag{1}$$

for some $\alpha \geq 0$, some $c > 0$ and all $x \in I\!R^n$ and $y \in I\!R^n$. As usual $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. The main result of [17] deals with the entropy numbers of compact embeddings of

$$F^{s_1}_{p_1,q_1}(I\!R^n, w(x)\langle x\rangle^{\beta}) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(I\!R^n, w(x))\ , \quad \beta > 0\,, \tag{2}$$

and their B-counterparts. In the present paper we wish to use these assertions to estimate the distribution of eigenvalues of degenerate pseudodifferential operators of type

$$B = b_2(x)\; b(x, D)\; b_1(x)\ , \tag{3}$$

where $b_1(x)$ and $b_2(x)$ are typically (singular) functions belonging to some of the spaces of the above type, preferably weighted L_p - spaces or nearby spaces, and $b(x,D)$ is in the Hörmander class $\Psi^{\varkappa}_{1,\gamma}$ with $\varkappa < 0$ and $0 \leq \gamma \leq 1$. We obtain sharp or nearly sharp (up to a *log*-term) results as far as the expected order of the eigenvalue distribution is concerned. This part of the paper should also be considered in the larger context of integral operators in the widest sense of the word. By Schwartz's kernel theorem (3) can be written, at least formally, as

$$Bf = \int B(x,y)f(y)dy \tag{4}$$

with

$$B(x,y) = b_2(x)\; b(x,y)\; b_1(y) \tag{5}$$

where $b \in D'(I\!R^n \times I\!R^n)$ is C^{∞} off the diagonal $x = y$. Hence singularities of the kernel $B(x,y)$ are connected with b_1, b_2, mostly assumed to be functions, and with the behaviour of b near the diagonal $x = y$. In this paper we are exclusively at the ΨDO-side. However one should look at (4) in connection with the method presented in this paper in the larger context of integral operators, in a somewhat extended meaning such that distributional kernels of type (5) are included. The study of mapping properties and of the distribution of eigenvalues of integral operators of type (4), preferably, of course, with functional kernels $B(x,y)$, is one of the great themes of functional analysis since its very beginning and since the beginning of our century with hundreds of papers devoted to this subject. Recent comprehensive treatments about mapping properties and distributions of eigenvalues of integral operators may be found in [28], Ch. 6 and [21], Ch. 3. In particular, PIETSCH gave in his book [28], 7.3.2 and 7.7.5, a very careful account about the historical roots and the state of the art. Here we mention only very few papers which are either closely related to the ΨDO-side of the story and to our results or from which we believe that our method should also be able to shed new light on these previous approaches. As for the first aspect there seems to be a first culmination point at the end of the seventies, see for example [10], [3], [35], Ch.4, and also [4], and the references given there. What has been done afterwards may be found in the recent survey [2]. As for the second aspect PIETSCH's approach should be mentioned. Generalizing the classical Hille-Tamarkin set-up, where the kernel $B(x,y)$ belongs to some mixed L_p-space, $L_p[L_q]$, he expressed smoothness and decay properties of kernels by assuming that $B(x,y)$ belongs to some mixed Besov space $B^s_{p,q}[B^\sigma_{u,v}]$, see [28], [21] and the underlying papers [26] and [27], where the latter one includes weighted spaces and is closely related to our results. H.-J. SCHMEISSER and the second-named author of the present paper suggested an alternative way nearer to Fourier-analytical techniques, see [32], 2.5.1, and the references given there. The results by PIETSCH seem to be the most complete ones in Banach spaces. The set-up is similar as ours: reduction of basic assertions to compact embeddings between Besov spaces, in that case expressed via Weyl numbers. In Hilbert spaces sometimes better results can be obtained by using specific techniques, see the above references to the work by CWIKEL, SIMON and BIRMAN-SOLOMJAK. In particular, looking at our method it seems to be possible that these different approaches can be combined in a kind of "genetic engineering" (implantation of suitable elements in the strings shown in related figures in this paper and in [14]) and they should generate new results.

Via the Birman-Schwinger principle our results will be used to get sharp in order, or nearly sharp, estimates for the negative spectrum of operators of type

$$H_\beta = a(x,D) - \beta\, a(x)p(x,D)a(x) \quad \text{as} \quad \beta \to \infty\,, \tag{6}$$

in L_2, where $a(x,D) \in \Psi^\varkappa_{1,\gamma}$ with $\varkappa > 0$, $0 \le \gamma < 1$, is assumed to be positive-definite and self-adjoint in L_2, whereas $p(x,D)$ is a perturbing symmetric pseudodifferential operator of lower order and $a(x)$ is a real (singular) function.

Our aim in this paper can be described as follows. We hope to add a few new interesting results to this flourishing field of research. But mainly we wish to present a new method which, by our opinion, is beautiful, strikingly simple (after all the hard work of providing building blocks, which should be mathematics in their own right, has been done) and which may be able to shed new light on what is known and what can be expected.

The plan of the paper is as follows. Sect.2 contains background material: the above mentioned function spaces, including related Hölder inequalities; Carl's inequality which links entropy numbers and eigenvalues; and a non-symmetric entropy version of the Birman-Schwinger principle. Sect.3 deals mostly with pseudodifferential operators of type $\Psi^0_{1,\gamma}$ with $0 \le \gamma \le 1$ of order zero in the above weighted function spaces: boundedness and spectral properties. In some sense we complement herewith the existing building block "mapping properties of ΨDO's in $B^s_{p,q}$ and $F^s_{p,q}$" by its weighted extension. Sect.4 contains our main results: the distribution of eigenvalues of degenerate pseudodifferential operators, including a smoothness theory and assertions about the independence of the root spaces of the chosen basic space. This restriction may be considered as the counterpart of the main results in [14], where that paper concentrates mostly on corresponding assertions for fractional powers of general regular elliptic differential operators in bounded smooth domains. In Sect.5 we apply the results of Sect.4 in order to study the negative spectrum of operators of type (6). We present several approaches generalizing also the techniques developed in Sect.4.

Unimportant positive constants are denoted by c, occasionally with additional subscripts within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l, whereas (j) means formula (j) in the same subsection. In a similar way we quote definitions, propositions and theorems.

2. Definitions and preliminaries

2.1 Function spaces

We collect the needed basic notations and basic properties of the underlying weighted function spaces. We refer to [17] for more details, explanations, and references to the literature.
Let $\mathbb{R}^n$ be the Euclidean n-space. Let $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ in $\mathbb{R}^n$. All spaces in this paper are defined on $\mathbb{R}^n$ and so we omit "$\mathbb{R}^n$" in the sequel. The Schwartz space S and its dual S' of all complex-valued tempered distributions have the usual meaning here. Furthermore, L_p with $0<p\le\infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot|L_p\|$. Let $L_p(w(x))$ be its weighted generalization, quasi-normed by

$$\|f|L_p(w(\cdot))\| = \|wf|L_p\| \;, \tag{1}$$

where $w(x)>0$ is a weight function on $\mathbb{R}^n$ and $0<p\le\infty$. Let $\varphi\in S$ be such that

$$\operatorname{supp}\varphi \subset \{y\in\mathbb{R}^n : |y|<2\} \quad\text{and}\quad \varphi(x)=1 \quad\text{if}\quad |x|\le 1 \;, \tag{2}$$

put $\varphi_0=\varphi$ and for each $j\in\mathbb{N}$ let $\varphi_j(x)=\varphi(2^{-j}x)-\varphi(2^{-j+1}x)$. Then since $1=\sum_{j=0}^{\infty}\varphi_j(x)$ for all $x\in\mathbb{R}^n$, the φ_j form a dyadic resolution of unity. Given any $f\in S'$, we denote by $\hat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. Then $(\varphi_j\hat{f})^\vee$ is an analytic function on $\mathbb{R}^n$.

Definition 1. The class W of *admissible weight functions* is the collection of all positive C^∞ functions $w(x)$ on $\mathbb{R}^n$ with the following properties:
(i) for any multi-index γ there exists a positive constant c_γ with

$$|D^\gamma w(x)| \le c_\gamma w(x) \quad\text{for all}\quad x\in\mathbb{R}^n \;, \tag{3}$$

(ii) there exist two constants $c>0$ and $\alpha\ge 0$ such that

$$0<w(x)\le c\,w(y)\,\langle x-y\rangle^\alpha \quad\text{for all}\quad x\in\mathbb{R}^n \quad\text{and all}\quad y\in\mathbb{R}^n \;. \tag{4}$$

Definition 2. Let $w(x)\in W$. Let $s\in\mathbb{R}, 0<q\le\infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity.
(i) Let $0<p\le\infty$. The space $B^s_{p,q}(w(x))$ is the collection of all $f\in S'$ such that

$$\|f|B^s_{p,q}(w(\cdot))\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\big\|(\varphi_j\hat{f})^\vee|L_p(w(\cdot))\big\|^q\Big)^{\frac{1}{q}} \tag{5}$$

(with the usual modification if $q=\infty$) is finite.
(ii) Let $0<p<\infty$. The space $F^s_{p,q}(w(x))$ is the collection of all $f\in S'$ such that

$$\|f|F^s_{p,q}(w(\cdot))\| = \Big\|\big(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\hat{f})^\vee(\cdot)|^q\big)^{\frac{1}{q}}\,|L_p(w(\cdot))\Big\| \tag{6}$$

(with the usual modification if $q=\infty$) is finite.

Remark 1. These two definitions coincide essentially with the corresponding definitions in [17]. The theory of the corresponding unweighted spaces, that means $w(x)=1$ in the above definition, denoted by $B^s_{p,q}$ and $F^s_{p,q}$, has been developed in detail in [41] and [42]. Weighted spaces of the above type even in the more general context of ultra-distributions and for larger classes of weight functions have been treated in [32: 5.1]. We refer to that book and to [17] for further comments and references to the literature. In particular, the above spaces are quasi-Banach spaces (Banach spaces if $p \ge 1$ and $q \ge 1$) and they are independent of the resolution of unity $\{\varphi_j\}$. We collect two properties of the above spaces which are important for us in the sequel.

Proposition. (i) *Let $s \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p \le \infty$ (with $p < \infty$ in the F-case), and let $w(x) \in W$. The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(w(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(w(x))$ onto $F^s_{p,q}$. Especially,*

$$\|wf|B^s_{p,q}\| \quad \text{is an equivalent quasi-norm in} \quad B^s_{p,q}(w(x)) \tag{7}$$

and

$$\|wf|F^s_{p,q}\| \quad \text{is an equivalent quasi-norm in} \quad F^s_{p,q}(w(x)) \,. \tag{8}$$

(ii) *Let $w_1(x) \in W$, $w_2(x) \in W$,*

$$-\infty < s_2 < s_1 < \infty,\ 0 < p_1 \le p_2 < \infty,\ 0 < q_1 \le \infty \quad \text{and} \quad 0 < q_2 \le \infty \,. \tag{9}$$

Then the embedding

$$F^{s_1}_{p_1,q_1}(w_1(x)) \subset F^{s_2}_{p_2,q_2}(w_2(x)) \tag{10}$$

is compact if and only if

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \to 0 \quad \text{as} \quad |x| \to \infty \,. \tag{11}$$

Remark 2. We refer for explanations and proofs to [17]. Of course part (ii) has a B-counterpart. It was the main aim of [17] to study the compactness of embeddings of type (10) expressed in terms of related entropy numbers, see [17:4.2].

2.2 Hölder inequalities

The classical Hölder inequality may be written as

$$L_{r_1}\, L_{r_2} \subset L_r \tag{1}$$

where

$$1 \le r_1 \le \infty,\ 1 \le r_2 \le \infty \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \le 1 \,. \tag{2}$$

It is convenient for us to regard this as a statement that any $g \in L_{r_2}$ is a pointwise multiplier

$$f \mapsto gf \quad \text{from} \quad L_{r_1} \quad \text{to} \quad L_r \,. \tag{3}$$

With the aim of generalizing (1) we carried out in [34] a detailed study of problems of this type, and related ones, for the spaces $B^s_{p,q}$ and $F^s_{p,q}$. Here for the sake of simplicity we shall restrict that part of our consideration to the (fractional) Sobolev (-Hardy) spaces defined by

$$H^s_p(w(x)) = F^s_{p,2}(w(x)) \quad \text{with} \quad s \in \mathbb{R},\ 0 < p < \infty \quad \text{and} \quad w(x) \in W \,. \tag{4}$$

Following [34] and [14] we introduce the strip (see Fig.1)

$$G_1 = \left\{ \left(\frac{1}{p}, s\right) : 0 < p < \infty, n\left(\frac{1}{p} - 1\right)_+ < s < \frac{n}{p} \right\} \tag{5}$$

and the extended strip

$$G_2 = \left\{ \left(\frac{1}{p}, s\right) : 0 < p < \infty, n\left(\frac{1}{p} - 1\right) < s < \frac{n}{p} \right\} \,. \tag{6}$$

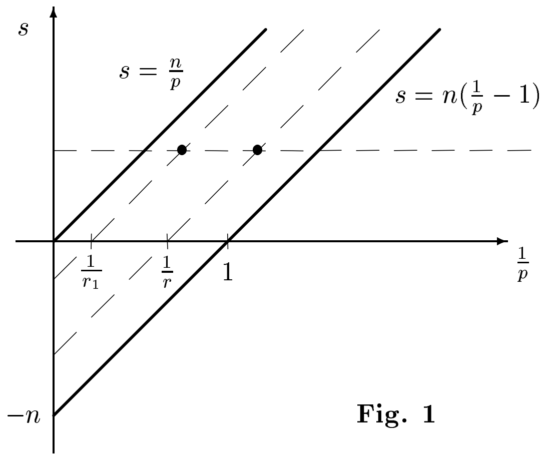

Fig. 1

For convenience we introduce a little terminology at this point. Any line of slope n in the $(\frac{1}{p}, s)$-diagram is characterized by the point at which it meets the axis $s = 0$: we shall refer to this point, $(\frac{1}{r}, 0)$, say, as the "footpoint" of the line. Thus any point on that line in G_2 has coordinates

$$\left(\frac{1}{r} + \frac{s}{n}, s\right) = \left(\frac{1}{r^s}, s\right) \quad \text{say, with} \quad \frac{1}{r} + \frac{s}{n} > 0,\ 1 < r < \infty \ . \tag{7}$$

The needed generalization of (1) can now be formulated as follows.

Theorem. *Let*

$$1 < r_1 < \infty,\ 1 < r_2 < \infty \quad \textit{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} < 1 \ . \tag{8}$$

Suppose that

$$s \in I\!R \quad \textit{and} \quad \frac{1}{r_1} + \frac{s}{n} > 0 \ , \tag{9}$$

$w_1(x) \in W,\ w_2(x) \in W$ *and* $w(x) = w_1(x)w_2(x)$. *Then*

$$H^s_{r_1^s}(w_1(x)) \ \ H^{|s|}_{r_2^{|s|}}(w_2(x)) \subset H^s_{r^s}(w(x)) \tag{10}$$

where $\dfrac{1}{r^s} = \dfrac{1}{r} + \dfrac{s}{n}$ *and* $r_1^s,\ r_2^{|s|}$ *are defined analogously.*

Remark. If $s = 0$, then (10) coincides with the weighted extension of (1), apart from limiting cases. If $s > 0$, (10) gives what we describe as a Hölder inequality at the level s: in the sense of (3), any $g \in H^s_{r_2^s}(w_2(x))$ gives a mapping $f \mapsto gf$ from $H^s_{r_1^s}(w_1(x))$ to $H^s_{r^s}(w(x))$, characterized by the large dots in Fig.1. The unweighted version of the above theorem coincides with Theorem 2.4 in [14], which in turn, at least for $s > 0$, is a special case of Theorem 4.2 in [34]. The extension to $s < 0$ comes from duality arguments, see [14]. The incorporation of the weights follows from the unweighted case and Proposition 2.1(i).

2.3 Carl's inequality

We follow closely the presentation given in [14], 2.1. Let X be a complex quasi-Banach space and let $B : X \to X$ be compact and linear.

Definition. Let $k \in I\!N$. The k^{th} entropy number $e_k = e_k(B)$ of B is the infimum of all $\varepsilon > 0$ such that there exist 2^{k-1} balls in X of radius ε which cover the image of the unit ball in X under B, $\{y \in X : y = Bx, \|x|X\| \le 1\}$.

For details and main properties of entropy numbers we refer to [8], [11] and [25]. Now let $\{\mu_k\}$ be the sequence of eigenvalues of B, counted with respect to their algebraic multiplicity and ordered by decreasing modulus:

$$|\mu_1| \geq |\mu_2| \geq \dots \tag{1}$$

Theorem. *Under the above assumptions,*

$$\left(\prod_{m=1}^{k} |\mu_m| \right)^{\frac{1}{k}} \leq \inf_{n \in \mathbb{N}} 2^{\frac{n}{2k}} e_n \qquad \textit{for all} \quad k \in \mathbb{N} , \tag{2}$$

especially,

$$|\mu_k| \leq \sqrt{2}\, e_k \quad \textit{for all} \quad k \in \mathbb{N} . \tag{3}$$

Remark. A proof of this theorem, restricted to Banach spaces, is given in [8], 4.2.1. The result in this context goes back to [7] and [9]. Proofs may also be found in [11] and [29]. As we need (3) in this paper mostly for Banach spaces, details of the modifications needed to deal with the quasi-Banach space case will be given elsewhere.

The method of the paper. The method of the paper can now be described very easily. To apply (3) to compact operators of type B in (1/3) we reduce the calculation of the entropy numbers $e_k(B)$ to the calculation of the entropy numbers of appropriate compact embeddings of type (2.1/10) and apply our results obtained in [17] to the latter question.

2.4 The Birman-Schwinger principle

We adapt the Birman-Schwinger principle as described in [31] and [35] to our concrete situation and to the method of this paper, see also [14], 2.2. Let $A = a(x, D)$ be a positive-definite self-adjoint pseudodifferential operator in L_2, typically of positive order, and

$$V = a(x)\, p(x, D)\, a(x) \tag{1}$$

be a degenerate pseudodifferential operator, symmetric in L_2, where $p(x, D)$ is a symmetric pseudodifferential operator of lower order than A and $a(x)$ is a real function typically belonging to some spaces L_p or $H^s_p(w(x))$ in the sense of (2.2/4). More precise conditions will be given in Sect.5. In any case all operators in this subsection may be assumed to be defined at least on S, and the rest is always a matter of completion. So we do not care in the formulations below about the respective domains of definitions. We assume that

$$VA^{-1} \quad \text{is compact (in } \ L_2) . \tag{2}$$

Then, after completion, $A^{-1}V$ is also compact. Furthermore, $A - V$ is self-adjoint on $dom(A)$ and the essential spectra of A and $A - V$ coincide. Of interest is the number of eigenvalues of $A - V$ which are smaller than or equal to 0. If M is a finite set, denote by $\# M$ the number of elements of M.

Theorem. *Let the above conditions be satisfied and let $\sigma(A - V)$ be the spectrum of $A - V$. Then*

$$\#\{\sigma(A - V) \cap (-\infty, 0]\} \leq \#\{k \in \mathbb{N} : \sqrt{2}\, e_k(VA^{-1}) \geq 1\} \tag{3}$$

and

$$\#\{\sigma(A - V) \cap (-\infty, 0]\} \leq \#\{k \in \mathbb{N} : \sqrt{2}\, e_k(A^{-1}V) \geq 1\} . \tag{4}$$

Remark 1. Of course, eigenvalues are counted according to their multiplicities. By (3) and (4), $\sigma(A-V) \cap (-\infty, 0]$ consists solely of a finite number of eigenvalues of finite multiplicity (if there are any). Furthermore, $e_k(VA^{-1})$ and $e_k(A^{-1}V)$ are the entropy numbers of the compact operators VA^{-1} and $A^{-1}V$, respectively. Compared with the usual formulation there are two modifications. Firstly one has usually the eigenvalues of VA^{-1} and $A^{-1}V$ in (3) or (4) instead of $\sqrt{2}e_k$. But this replacement of the eigenvalues by $\sqrt{2}e_k$ comes simply from (2.3/3). Secondly, compared with the usual formulation, the above formulation is somewhat unsymmetric in A and V. However, SIMON's beautiful proof in [35], pp 86/87, covers also the above version.

Remark 2. Let $p(x,D) = id$ in (1). Then V is a multiplication operator and the theorem can be complemented by

$$\#\{\sigma(A-V)\cap(-\infty,0]\} \le \#\{k\in \mathbb{N} : \sqrt{2}e_k(a\,A^{-1}\,a)\ge 1\}\ , \tag{5}$$

which is nearer to the formulations in [31] and [35]. Further references may be found in these two books, [5], and in [14], 2.2.

3. Pseudodifferential operators

3.1 Mapping properties

Assuming $\varkappa \in \mathbb{R}$ and $0 \le \gamma \le 1$, the Hörmander class $S^{\varkappa}_{1,\gamma}$ consists of all complex C^∞ functions $p(x,\xi)$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le c_{\alpha,\beta}\, \langle\xi\rangle^{\varkappa-|\alpha|+\gamma|\beta|}\quad ,\ x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n\ , \tag{1}$$

for all α and β and some constants $c_{\alpha,\beta} > 0$, where, of course, D^α_ξ and D^β_x refer to derivatives for the $\xi = (\xi_1,\ldots,\xi_n)$ and the $x = (x_1,\ldots,x_n)$ variables, respectively. The related class of pseudodifferential operators will be denoted by $\Psi^{\varkappa}_{1,\gamma}$, where $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$ is given, at least formally, by

$$\begin{aligned} p(x,D)f(x) &= \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{f}(\xi)d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} p(x,\xi) f(y)dy\, d\xi\ . \end{aligned} \tag{2}$$

As for the necessary background material we refer to [36], [37], [40] or [18]. To study mapping and spectral properties of pseudodifferential operators we assume $\varkappa = 0$, since there is a 1-1 relation between $\Psi^{\varkappa}_{1,\gamma}$ and $\Psi^0_{1,\gamma}$ given by

$$p(x,D) = p^0(x,D)\ (id-\Delta)^{\frac{\varkappa}{2}}\ ,\quad \varkappa\in\mathbb{R}\ , \tag{3}$$

where Δ stands for the Laplacian in $\mathbb{R}^n$, $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$ and $p^0(x,D) \in \Psi^0_{1,\gamma}$ (this can be seen easily by (1) and (2) since $\langle\xi\rangle^{\varkappa}$ is the symbol of $(id-\Delta)^{\frac{\varkappa}{2}}$). Recall that $\Psi^0_{1,1}$ is called the exotic class.

Theorem. *Let $0 \le \gamma \le 1$, $p(x,D) \in \Psi^0_{1,\gamma}$, $0 < q \le \infty$ and $w(x) \in W$.*
(i) *Let $0 < p \le \infty$, $s \in \mathbb{R}$ (with $s > n\left(\frac{1}{p}-1\right)_+$ in the exotic case $\gamma = 1$). Then $p(x,D)$ is a linear and bounded map*

$$\text{from}\quad B^s_{p,q}(w(x))\quad \text{into itself}. \tag{4}$$

(ii) *Let $0 < p < \infty$, $s \in \mathbb{R}$ (with $s > n\left(\frac{1}{\min(p,q)}-1\right)_+$ in the exotic case $\gamma = 1$). Then $p(x,D)$ is a linear and bounded map*

$$\text{from}\quad F^s_{p,q}(w(x))\quad \text{into itself}. \tag{5}$$

P r o o f: **Step 1.** First we recall that the theorem is known for $B^s_{p,q}$ and $F^s_{p,q}$, that means in the unweighted case $w(x) = 1$. We refer to [24] and [42], 6.2.2, in the case $\gamma < 1$, and to [30], [38], [39] in the exotic case $\gamma = 1$. In other words, we have to extend these results from the unweighted to the weighted case.

Step 2. We start with a preparation. Let $f \in F^s_{p,q}$ and

$$\operatorname{supp} f \subset K = \{y \in \mathbb{R}^n : |y| < \sqrt{n}\ \} . \tag{6}$$

Let $k \in \mathbb{N}$, then we have, at least formally, by partial integration

$$\begin{aligned} \langle x\rangle^{2k} p(x,D)f(x) &= \int \big[(id-\Delta)^k_\xi e^{ix\xi}\big] p(x,\xi)\hat{f}(\xi)d\xi \qquad (7)\\ &= \int e^{ix\xi} \sum_{|\alpha|+|\beta|\le 2k} b_{\alpha\beta} D^\alpha_\xi p(x,\xi)\big(y^\beta f(y)\big)^\wedge(\xi)d\xi , \end{aligned}$$

where we used $\big(y^\beta f(y)\big)^\wedge(\xi) = i^{|\beta|}D^\beta\hat{f}(\xi)$ with $y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n}$. The application of the partial integration will be justified afterwards. Then by $D^\alpha_\xi p(x,\xi) \in S^{-|\alpha|}_{1,\gamma} \subset S^0_{1,\gamma}$ and Step 1 we have

$$\big\|\langle x\rangle^{2k} p(x,D)f|F^s_{p,q}\big\| \le c \sum_{|\beta|\le 2k} \|x^\beta f|F^s_{p,q}\| \le c'\ \|f|F^s_{p,q}\| , \tag{8}$$

where the last estimate comes from (6) and the pointwise multiplier properties for $F^s_{p,q}$, see [42], 4.2.2. To justify (7) we assume that f with (6) is given by an atomic representation in the sense of FRAZIER and JAWERTH, see [16] or [38], where all atoms are smooth and located near the origin. For these atoms and their finite linear combinations f^l we may assume that we have (8) uniformly with respect to l. We may also assume $f^l \to f$ in S'. Then (8) with f^l, the independence of c and c' on l, the Fourier-analytic definition of $F^s_{p,q}$ given in (2.1/6) and the lemma of Fatou prove (8) for f, too. An explicit formulation of the just used Fatou property of $F^s_{p,q}$ may be found in [15].

Step 3. Let K_k be a ball in $\mathbb{R}^n$ of radius $\sqrt{n}$ and centered at $k \in \mathbb{Z}^n$. Let $\psi_k(x) = \psi(x-k)$ be a related resolution of unity. We wish to use the localization principle for $F^s_{p,q}$, see [42], 2.4.7. Let $l \in \mathbb{Z}^n$ and $k \in \mathbb{Z}^n$. By (8), (2.1/4), Definition 2.1/2, Proposition 2.1 and, again, by the pointwise multiplier property, it follows for $f \in F^s_{p,q}$

$$\begin{aligned} \|\psi_l\ p(\cdot,D)\ \psi_{k+l}\ f|F^s_{p,q}(w(x))\| &\le c_1\ w(l)\ \|\psi_l\ p(\cdot,D)\ \psi_{k+l}\ f|F^s_{p,q}\| \qquad (9)\\ &\le c_2\ w(l)\langle k\rangle^{-\nu}\|\psi_{k+l}f|F^s_{p,q}\|\\ &\le c_3\ \frac{w(l)}{w(l+k)}\langle k\rangle^{-\nu}\|\psi_{k+l}f|F^s_{p,q}(w(x))\|\\ &\le c_4\ \langle k\rangle^{-\mu}\|\psi_{k+l}f|F^s_{p,q}(w(x))\| , \end{aligned}$$

where $\mu > 0$ is at our disposal. Let $\lambda = \min(1,p,q)$. Then (9), $p < \infty$ and the λ-triangle inequality for $F^s_{p,q}$ yield

$$\begin{aligned} \|\psi_l\ p(\cdot,D)\ f|F^s_{p,q}(w(x))\|^p &= \Big\|\psi_l\ p(\cdot,D)\sum_k \psi_{l+k}f|F^s_{p,q}(w(x))\Big\|^p \qquad (10)\\ &\le \Big(\sum_k \|\psi_l\ p(\cdot,D)\ \psi_{k+l}f|F^s_{p,q}(w(x))\|^\lambda\Big)^{p/\lambda}\\ &\le c_1\ \Big(\sum_k \langle k\rangle^{-\mu}\ \|\psi_{k+l}f|F^s_{p,q}(w(x))\|^\lambda\Big)^{p/\lambda}\\ &\le c_2\ \sum_k \langle k\rangle^{-\nu}\ \|\psi_{k+l}f|F^s_{p,q}(w(x))\|^p , \end{aligned}$$

where again $\nu > 0$ is at our disposal. Summation over $l \in \mathbb{Z}^n$ and the above mentioned localization principle for $F^s_{p,q}$ prove (5),

$$\|p(\cdot,D)f|F^s_{p,q}(w(x))\| \le c\, \|f|F^s_{p,q}(w(x))\| \,. \tag{11}$$

Step 4. By the same arguments we have

$$\|p(\cdot,D)f|B^s_{\infty,\infty}(w(x))\| \le c\, \|f|B^s_{\infty,\infty}(w(x))\| \,. \tag{12}$$

We use the real interpolation formula

$$B^s_{p,q}(w(x)) = \Big(F^{s_0}_{p,q_0}(w(x))\ ,\ F^{s_1}_{p,q_1}(w(x))\Big)_{\theta,q} \tag{13}$$

with $s_0 < s < s_1,\ s = (1-\theta)s_0 + \theta s_1,\ 0 < p \le \infty,\ 0 < q_0 \le \infty,\ 0 < q_1 \le \infty$ and $0 < q \le \infty$ where $F^*_{p,q_0} = F^*_{p,q_1} = B^*_{\infty,\infty}$ if $p = \infty$. The unweighted case of (13) may be found in [41], 2.4.2. Its weighted generalization follows from the unweighted case and the isomorphism properties described in Proposition 2.1. Then (4) follows from (11), complemented by (12), and the interpolation property.

Remark 1. In the unweighted case one needs only finitely many of the assumptions (1) and all estimates depend only on these related numbers $c_{\alpha,\beta}$ in (1). The above proof shows that one has the same assertion for the weighted case. In other words, one can follow the proofs and look for which symbols $p(x,\xi)$ with limited smoothness the theorem remains valid. This may be of interest for non-linear problems, see [37].

Remark 2. Let $p(x,D) \in \Psi^{\varkappa}_{1,\gamma}$ for some $\varkappa \in \mathbb{R}$ and $0 \le \gamma \le 1$. By the theorem and by what had been said in front of the theorem, for all $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ (with $s - \varkappa > n\big(\frac{1}{\min(p,q)} - 1\big)_+$ in the exotic case $\gamma = 1$) $p(x,D)$ is a linear and bounded map

$$\text{from}\quad F^s_{p,q}(w(x))\quad \text{into}\quad F^{s-\varkappa}_{p,q}(w(x))\,. \tag{14}$$

Similarly for the B-spaces.

3.2 Spectral properties

Our method in Sect.4 of this paper and also in [14] to study degenerate pseudodifferential operators of type (1/3) is characterized by a composition of mapping properties of $b_1(x)$, $b_2(x)$ and $b(x,D)$ between function spaces of the above type, preferably $H^s_p(w(x))$, see (2.2/4). Hence we have to deal simultaneously with several of these spaces. This makes it desirable to have a closer look at the dependence of the spectral properties especially of operators of the class $\Psi^{\varkappa}_{1,\gamma}$ on the underlying function space. By (3.1/3) and the fact that $(id - \Delta)^{\frac{\varkappa}{2}}$ is an isomorphic map from $B^s_{p,q}(w(x))$ onto $B^{s-\varkappa}_{p,q}(w(x))$ and from $F^s_{p,q}(w(x))$ onto $F^{s-\varkappa}_{p,q}(w(x))$ it follows that it is completely sufficient to look at the spectral properties of the zero class $\Psi^0_{1,\gamma}$. Here we have Theorem 3.1. Let $P = p(x,D) \in \Psi^0_{1,\gamma}$. Then, as usual, the resolvent set $\varrho(P, B^s_{p,q}(w(x)))$ of P, considered as a linear and bounded map in $B^s_{p,q}(w(x))$ in the sense of Theorem 3.1, is defined to be $\{\lambda \in \mathbb{C} : (P - \lambda\, id)^{-1}$ exists as a bounded map in $B^s_{p,q}(w(x))\}$. Similarly $\varrho(P, F^s_{p,q}(w(x)))$ is defined. Let $\varrho(P) = \varrho(P, L_2)$.

Theorem. *Let $0 \le \gamma < 1$, $P = p(x,D) \in \Psi^0_{1,\gamma}$, $s \in \mathbb{R}$, $0 < q \le \infty$ and $w(x) \in W$.*
(i) *Let $0 < p \le \infty$. Then*

$$\varrho(P) \subset \varrho(P, B^s_{p,q}(w(x)))\,. \tag{1}$$

(ii) *Let $0 < p < \infty$. Then*

$$\varrho(P) \subset \varrho(P, F^s_{p,q}(w(x)))\,. \tag{2}$$

P r o o f: Let $\lambda \in \varrho(P)$. It has been shown in [1] and [45] that

$$(P - \lambda\, id)^{-1} \in \Psi^0_{1,\gamma}\ . \tag{3}$$

Then $(P - \lambda\, id)^{-1}$ is also the inverse on S. Since the exotic case $\gamma = 1$ is excluded it follows by duality arguments that $(P - \lambda\, id)^{-1}$ is also the inverse on S'. Now the above theorem follows from Theorem 3.1.

Remark 1. The exotic case $\gamma = 1$ is now excluded.

Remark 2. One would expect that L_2 is in some sense the best space, in particular the space with the largest resolvent set. Hence, by (1) and (2) the following conjecture seems to be quite natural.

Conjecture. Under the assumptions of the theorem holds

$$\varrho(P) = \varrho(P, B^s_{p,q}(w(x))) \quad , \ 0 < p \le \infty \tag{4}$$

and

$$\varrho(P) = \varrho(P, F^s_{p,q}(w(x))) \quad , \ 0 < p < \infty\ . \tag{5}$$

Remark 3. As usual, the complement of the resolvent set in $\mathbb{C}$ is called the spectrum. Hence the above conjecture is the problem of the spectral invariance. Some partial affirmative answers are known: (5) for $H^s = F^s_{2,2}$ may be found in [1]. This was extended in [45] to $H^s_p = F^s_{p,2}$ with $1 < p < \infty$. A further extension to $B^s_{p,q}$ and $F^s_{p,q}$ with $1 < p < \infty$, $0 < q < \infty$ was given in [22]. In [33] some (anisotropic) weights are included. We return to this subject in [23], where we prove the above conjecture for those $P \in \Psi^0_{1,\gamma}$ which are self-dual and self-adjoint, that means $P = P' = P^*$. This subclass of $\Psi^0_{1,\gamma}$ is also of interest in connection with Sect.5 of this paper. But we do not go into detail here.

4. Eigenvalue distributions of degenerate pseudodifferential operators

4.1 The basic assertion

In [14] we dealt with eigenvalue distributions of degenerate elliptic operators in bounded smooth domains. We concentrated in that paper mostly on fractional powers of regular elliptic operators. The basis was Carl's inequality from Theorem 2.3, an analogue of the Hölder inequalities from Theorem 2.2, and the results from [12], [13] and [44] about entropy numbers of compact embeddings between function spaces on bounded domains. All this was explained by travelling around in $\left(\frac{1}{p}, s\right)$-diagrams. We adapt this pattern here, now dealing exclusively with pseudodifferential operators on $\mathbb{R}^n$, considered in weighted spaces, and based on what had been done in [17] and in this paper so far.
The object is to study the map B,

$$Bf = b_2(x)\, b(x,D)\, b_1(x)\, f\ , \tag{1}$$

where b_1 and b_2 belong to some spaces $L_{r_j}(\langle x\rangle^{\alpha_j})$ or, later on, $H^s_{r^s_j}(\langle x\rangle^{\alpha_j})$ in the sense of 2.2 with $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and $b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$, introduced in 3.1, with $\varkappa > 0$ and $0 \le \gamma \le 1$. To simplify the situation we stick in this subsection to the "ground level" $s = 0$, i.e. we are looking for spaces L_p with $1 < p < \infty$ for which B becomes compact. In that case we denote by $\{\mu_k\}$ the sequence of eigenvalues of B, counted with respect to their algebraic multiplicity and ordered by decreasing modulus, so that $|\mu_1| \ge |\mu_2| \ge \ldots$. The collection of all associated eigenvectors of a given eigenvalue is denoted as the corresponding root space. The phrase that "the root spaces coincide" means that for the different basic spaces in question the eigenvalues coincide and that for any given eigenvalue the corresponding root spaces coincide.

Theorem. *Suppose* $\varkappa > 0$, $0 \le \gamma \le 1$, *and*

$$b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}\ , \tag{2}$$

$$\alpha_1 \in \mathbb{R},\ \alpha_2 \in \mathbb{R}, \quad \textit{with} \quad \alpha = \alpha_1 + \alpha_2 > 0\ , \tag{3}$$

$$1 \le r_1 \le \infty,\ 1 \le r_2 \le \infty \quad \textit{with} \quad \frac{1}{r_1} + \frac{1}{r_2} < \min\,(1, \frac{\varkappa}{n})\ , \tag{4}$$

$$b_1(x) \in L_{r_1}(\langle x\rangle^{\alpha_1}), \quad b_2(x) \in L_{r_2}(\langle x\rangle^{\alpha_2})\ . \tag{5}$$

(i) *For any* p *with* $1 < p < \infty$ *and*

$$\frac{1}{r_2} < \frac{1}{p} < 1 - \frac{1}{r_1} \tag{6}$$

and any $w(x) \in W$, *the operator* B, *given by* (1), *is compact in* $L_p(w(x))$. *Furthermore,*

$$|\mu_k| \le c\,\big\|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\big\|\ \big\|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\big\|\ k^{-\frac{\varkappa}{n}} \quad , \ k \in \mathbb{N}\ , \tag{7}$$

if

$$\frac{1}{r_1} + \frac{1}{r_2} > \frac{\varkappa - \alpha}{n} \tag{8}$$

and

$$|\mu_k| \le c_\varepsilon\big\|b_1|L_{r_1}(\langle x\rangle^{\alpha_1})\big\|\ \big\|b_2|L_{r_2}(\langle x\rangle^{\alpha_2})\big\|\ k^{-\frac{\alpha}{n}-\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}(\log\langle k\rangle)^{\varepsilon+\frac{1}{r_1}+\frac{1}{r_2}},\ k \in \mathbb{N}, \tag{9}$$

for any $\varepsilon > 0$ *(with* $\varepsilon = 0$ *if* $r_1 = r_2 = \infty$*) and suitably chosen constants* c_ε *if*

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa - \alpha}{n}\ . \tag{10}$$

(ii) *For fixed* p *with* (6) *the root spaces coincide for all* $w(x) \in W$.

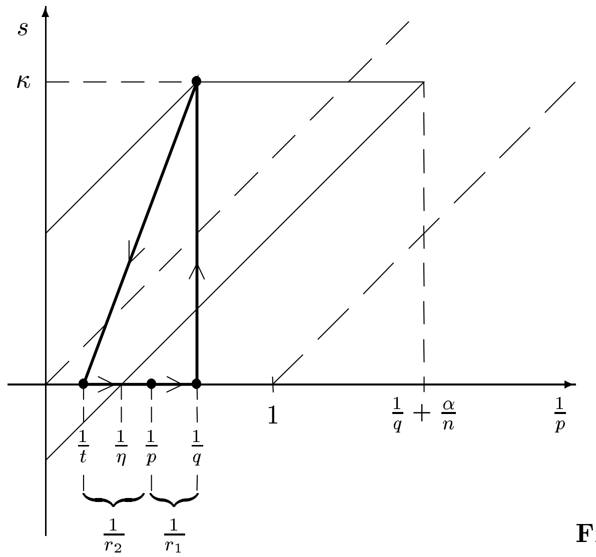

Fig. 2

Proof: **Step 1.** We decompose B as $B = b_2 \circ id \circ b(x,D) \circ b_1$, see Fig.2, where

$$\left\{\begin{array}{lcl} b_1 & : & L_p(w(x)) \to L_q(w(x)\langle x\rangle^{\alpha_1}) \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{r_1} \\ b(x,D) & : & L_q(w(x)\langle x\rangle^{\alpha_1}) \to H^{\varkappa}_q(w(x)\langle x\rangle^{\alpha_1}) \\ id & : & H^{\varkappa}_q(w(x)\langle x\rangle^{\alpha_1}) \to L_t(w(x)\langle x\rangle^{-\alpha_2}) \quad \text{with} \quad \frac{1}{t} = \frac{1}{p} - \frac{1}{r_2} \\ b_2 & : & L_t(w(x)\langle x\rangle^{-\alpha_2}) \to L_p(w(x))\ . \end{array}\right. \tag{11}$$

The first line is simply Hölder's inequality, see (6). The second line is covered by (2.2/4) and Remark 3.1/2. As for the third line in (11) we have

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa}{n}\,, \tag{12}$$

and hence the corresponding line in Fig.2 has a slope steeper than n. The quotient of the two involved weights is $\langle x\rangle^\alpha$. By our remarks at the beginning of 2.4 in [17] we can apply Theorem 4.2 in [17]. Hence id is a compact embedding and we have for the corresponding entropy numbers

$$e_k(id) \sim k^{-\frac{\varkappa}{n}} \quad \text{if} \quad \frac{1}{t} < \frac{1}{\eta} \tag{13}$$

and

$$e_k(id) \le c_\varepsilon\, k^{-\frac{\alpha}{n} - \left(\frac{1}{r_1}+\frac{1}{r_2}\right)} \left(\log\langle k\rangle\right)^{\varepsilon + \frac{1}{r_1} + \frac{1}{r_2}} \quad \text{if} \quad \frac{1}{t} > \frac{1}{\eta} \tag{14}$$

where

$$\frac{1}{\eta} = \frac{\alpha}{n} + \frac{1}{q} - \frac{\varkappa}{n} \tag{15}$$

has the indicated meaning. Here (13) corresponds to region **I** and (14) to region **III** in Fig.1 in [17]. The last line in (11) is again simply Hölder's inequality. However (15) and the conditions in (13) and (14) coincide with (8) and (9), respectively. Then part (i) of the theorem follows (13) and (14), respectively, and Theorem 2.3.

Step 2. id maps also $H^\varkappa_q(w(x)\langle x\rangle^{\alpha_1})$ in $L_t(w(x)\langle x\rangle^{\alpha_1})$. This follows from the corresponding unweighted assertion, see [41], 2.7.1, and the above Proposition 2.1. Application of b_2 shows that B maps also $L_p(w(x))$ in $L_p(w(x)\langle x\rangle^\alpha)$ with $\alpha > 0$. Now one can start again with $w(x)\langle x\rangle^\alpha$ instead of $w(x)$. By iteration the root spaces for all basic spaces $L_p(w(x)\langle x\rangle^\beta)$, $\beta > 0$ arbitrary, coincide. But any weight can be dominated by $w(x)\langle x\rangle^\beta$ for some $\beta > 0$. This completes the proof of part (ii).

Remark 1. Let

$$\frac{1}{r_2} < \frac{1}{p} \le \frac{1}{u} < 1 - \frac{1}{r_1} \tag{16}$$

in the sense of (6). By part (ii) of the above theorem the root spaces of B with respect to, say, L_p, are the same as for $L_p(w(x))$ with $w(x) \in W$. But then it follows from Hölder's inequality that they are also the same for all $L_u(w(x))$ with (16) and $w(x) \in W$. A corresponding assertion for values of u with

$$\frac{1}{r_2} < \frac{1}{u} < \frac{1}{p} < 1 - \frac{1}{r_1} \tag{17}$$

will be considered in the next subsection in connection with a smoothness theory.

Remark 2. By Remark 3.1/1 and the above proof the constants c and c_ε in (7) and (9) depend only on finitely many constants $c_{\alpha,\beta}$ in the sense of (3.1/1).

4.2 Smoothness theory

The intention is quite clear. One strengthens the assumptions (4.1/5) for $b_1(x)$ and $b_2(x)$ and hopes to obtain improved informations about the image of B. It seems to be natural to reinforce the assumptions for $b_1(x)$ and $b_2(x)$ only very slightly by using the limiting embeddings discussed in 2.2, that means to substitute, say, $b_2 \in L_{r_2}(\langle x\rangle^{\alpha_2})$, by the seemingly very nearby hypothesis $b_2 \in H^\nu_{r_2^\nu}(\langle x\rangle^{\alpha_2})$ for some $\nu > 0$, see (2.2/7) for notations. But it comes out that one gains quite a lot on that way. As in related classical theories there is an interplay between improved smoothness and worsened compactness if one looks at B as a mapping between different spaces. However we stick in Theorem 1 at our set-up in 4.1 that B acts in $L_p(w(x))$.

Theorem 1. *Let B be given by (4.1/1) under the same general assumptions as in Theorem 4.1, especially $w(x) \in W$ and (4.1/2)-(4.1/4) and (4.1/6) are assumed to be satisfied. Let (4.1/5) be strengthened by*

$$b_1(x) \in L_{r_1}(\langle x \rangle^{\alpha_1}) \quad \text{and} \quad b_2(x) \in H^{\nu}_{r_2^{\nu}}(\langle x \rangle^{\alpha_2}) \tag{1}$$

for some $\nu > 0$ with

$$\nu < n\left(\frac{1}{p} - \frac{1}{r_2}\right) \quad \text{and} \quad \nu \leq \varkappa - n\left(\frac{1}{r_1} + \frac{1}{r_2}\right) . \tag{2}$$

(i) *Then B is compact in $L_p(w(x))$ and*

$$\operatorname{Im} B \subset H^{\nu}_p(w(x)\langle x \rangle^{\alpha}) . \tag{3}$$

(ii) *The root spaces for all admissible p in the sense of* (4.1/6) *and all $w(x) \in W$ coincide.*

Proof: **Step 1.** Of course, the compactness of B is covered by Theorem 4.1. We prove (3) by travelling around in $\left(\frac{1}{p}, s\right)$- diagrams employing our improved knowledge (1) rather than (4.1/5). There are two typical situations, those in Fig.2 and Fig.3. Of course, we have always (4.1/11) which in the sense of Fig.3 with

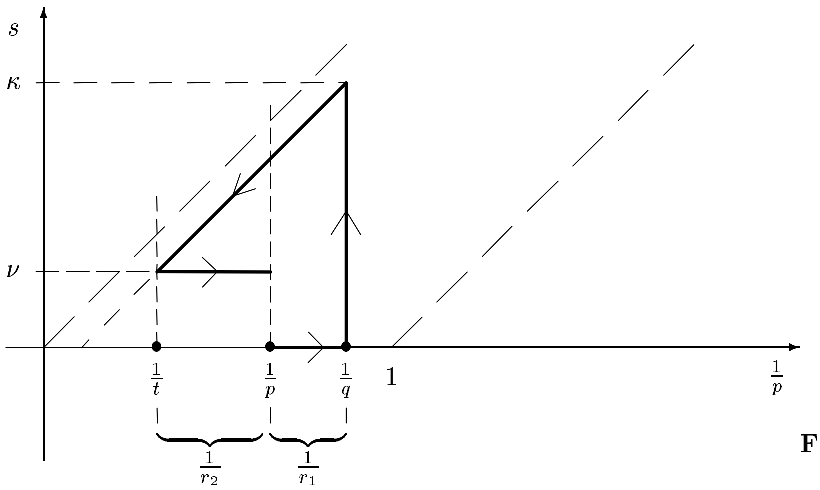

Fig. 3

$$0 < \nu = \varkappa - n\left(\frac{1}{r_1} + \frac{1}{r_2}\right) < \frac{n}{t} = n\left(\frac{1}{p} - \frac{1}{r_2}\right) \tag{4}$$

is complemented by

$$\left\{ \begin{array}{lll} b_1 & : & L_p(w(x)) \to L_q(w(x)\langle x \rangle^{\alpha_1}) \\ b(x,D) & : & L_q(w(x)\langle x \rangle^{\alpha_1}) \to H^{\varkappa}_q(w(x)\langle x \rangle^{\alpha_1}) \\ id & : & H^{\varkappa}_q(w(x)\langle x \rangle^{\alpha_1}) \to H^{\nu}_t(w(x)\langle x \rangle^{\alpha_1}) \\ b_2 & : & H^{\nu}_t(w(x)\langle x \rangle^{\alpha_1}) \to H^{\nu}_p(w(x)\langle x \rangle^{\alpha}) . \end{array} \right. \tag{5}$$

The first two lines are the same as in (4.1/11). The third line is again a limiting embedding along lines of slope n, see Step 2 of the proof of Theorem 4.1, where we use the second inequality in (2). Finally, the fourth line in (5) follows from (1) and Theorem 2.2, where the first inequality in (2) is applied. Obviously, the situation in Fig.3 is even better if $\nu < \varkappa - n\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$. In case of Fig.2 the necessary modifications are clear.

Step 2. To prove (ii) we have only to look at the case described in (4.1/17). However by (3) we have even

$$\operatorname{Im} B \subset H^{\nu}_p(w(x)\langle x \rangle^{\alpha}) \subset L_u(w(x)\langle x \rangle^{\alpha}) \quad \text{if} \quad \nu - \frac{n}{p} = -\frac{n}{u} . \tag{6}$$

Now one can start with $L_u(w(x)\langle x\rangle^\alpha)$. Iteration yields the desired result.

Remark. Both in Theorem 4.1 and in the above theorem we stuck at the ground level $s = 0$, that means our basic space is $L_p(w(x))$. But this is not necessary. By Theorems 2.2 and 3.1 we can start with any space $H^s_p(w(x))$ if the conditions of these two theorems are satisfied. Especially we have to care that we are inside the strips G_1 and G_2 from (2.2/5) and (2.2/6), respectively, if Theorem 2.2 is applied. In particular, the independence of the root spaces can be extended to these more general basic spaces. We refer to [14] for similar considerations. Here we restrict ourselves to a slightly different look at smoothness which is well adapted to what has been said in 2.2 about Hölder inequalities, sailing along lines of slope n.

Theorem 2. *Let B be given by* (4.1/1) *under the same general assumptions as in Theorem* 4.1, *especially $w(x) \in W$, and* (4.1/2)-(4.1/4) *and* (4.1/6) *are assumed to be satisfied. Let* (4.1/5) *be strengthened by*

$$b_1(x) \in H^s_{r_1^s}(\langle x\rangle^{\alpha_1}) \quad \textit{and} \quad b_2(x) \in H^\nu_{r_2^\nu}(\langle x\rangle^{\alpha_2}) \tag{7}$$

for $\nu > s \geq 0$ with

$$\nu - s < n\Big(\frac{1}{p} - \frac{1}{r_2}\Big) \quad \textit{and} \quad \nu - s \leq \varkappa - n\Big(\frac{1}{r_1} + \frac{1}{r_2}\Big)\ . \tag{8}$$

(i) *Then B is compact in $H^s_{p^s}(w(x))$ and*

$$\operatorname{Im} B \subset H^\nu_{p^s}(w(x)\langle x\rangle^\alpha)\ . \tag{9}$$

(ii) *The root spaces for all admissible p in the sense of* (4.1/6), *all basic spaces $H^\lambda_{p^\lambda}(w(x))$ with $0 \leq \lambda \leq s$ and all $w(x) \in W$ coincide.*

P r o o f: We apply Theorem 2.2, where the triangle in Fig.2 is simply pushed along the line with slope n and footpoint $\frac{1}{p}$ as far as allowed by the smoothness of b_1 and b_2 given by (7). Then we have the same situation as in Theorem 1.

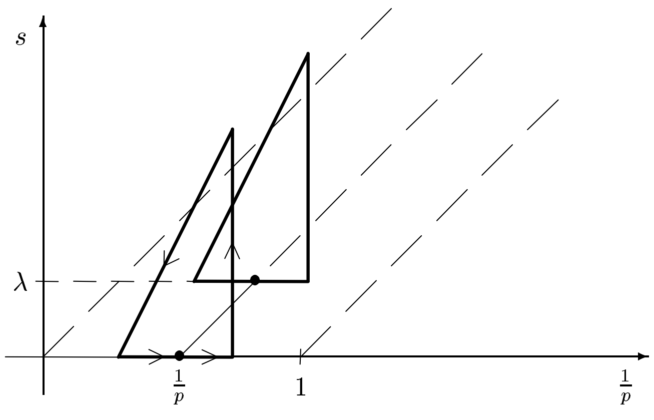

Fig. 4

As for part (ii) we may start with $s = 0$. Assuming $s > 0$, and $\lambda > 0$ sufficiently small, then we obtain by Hölder's inequality

$$\operatorname{Im} B \subset H^\lambda_p(w(x)\langle x\rangle^\alpha) \subset H^\lambda_{p^\lambda}(w(x))\ . \tag{10}$$

The rest is the same as above.

4.3 Degenerate pseudodifferential operators of positive order

Up to now we dealt with the operator

$$Bf = b_2(x)\ b(x,D)\ b_1(x)\ f, \qquad b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}\ ,\ \varkappa > 0\ . \tag{1}$$

Now we are interested in degenerate pseudodifferential operators of positive order given by, at least formally,

$$Af = a_1(x)\, a(x,D)\, a_2(x)\, f \tag{2}$$

with

$$a(x,D) \in \Psi^{\varkappa}_{1,\gamma}\,, \qquad \varkappa > 0,\ 0 \le \gamma < 1\,. \tag{3}$$

One may look at A as a bounded mapping between suitable couples of function spaces or as an unbounded operator in a basic function space, say, of type $H^s_p(w(x))$. We adopt the latter point of view and ask for the spectrum of A and its root spaces. Assume that $a(x,D)$ is invertible in L_2 and

$$0 \in \varrho(a(\cdot, D)) \tag{4}$$

in the sense of 3.2. Since the exotic case $\gamma = 1$ is now excluded, it follows by Theorem 3.2 and what had been said before, see also (3.1/3), that (4) holds in any other space of interest in our context, especially in $H^s_p(w(x))$ with $s \in \mathbb{R}$, $0 < p < \infty$, $w(x) \in W$. Furthermore, again by 3.2 we have

$$b(x,D) = a^{-1}(x,D) \in \Psi^{-\varkappa}_{1,\gamma}\,. \tag{5}$$

If $a_1(x) \ne 0$ and $a_2(x) \ne 0$ a.e. in $\mathbb{R}^n$ are suitably chosen then it makes sense to ask whether $0 \in \varrho(A)$, too, and whether $B = A^{-1}$ can be represented by (1) with $b_1 = a_1^{-1}$ and $b_2 = a_2^{-1}$. To be on the safe side we adopt the reverse point of view.
Let B be given by (1) with $0 \le \gamma < 1$ (the exotic case is excluded now). Assume that 0 is not an eigenvalue of the compact operator $b(x,D)$, that

$$b_1(x) \ne 0 \quad \text{and} \quad b_2(x) \ne 0 \quad \text{a.e. in } \mathbb{R}^n\,, \tag{6}$$

and that the general assumptions of Theorem 4.1, especially $w(x) \in W$ and (4.1/2)-(4.1/6) hold. Then B is invertible in $L_p(w(x))$ and at least formally, $A = B^{-1}$ is given by (2) with $a_1 = b_1^{-1}$ and $a_2 = b_2^{-1}$. Since B is compact it follows by standard arguments that A is an unbounded operator in $L_p(w(x))$ with pure point spectrum. Let $\{\lambda_k\}$ be the sequence of its eigenvalues, counted according to algebraic multiplicity and ordered by increasing modulus, then $\lambda_k = \mu_k^{-1}$ where μ_k are the eigenvalues of B in the sense of Theorem 4.1. This theorem proves the following assertion.

Theorem. *Under the above hypotheses holds*

$$|\lambda_k| \ge c\, \big\| b_1 | L_{r_1}(\langle x\rangle^{\alpha_1}) \big\|^{-1} \big\| b_2 | L_{r_2}(\langle x\rangle^{\alpha_2}) \big\|^{-1} k^{\frac{\varkappa}{n}}\,, \quad k \in \mathbb{N}\,, \tag{7}$$

if

$$\frac{1}{r_1} + \frac{1}{r_2} > \frac{\varkappa - \alpha}{n} \tag{8}$$

and

$$|\lambda_k| \ge c_\varepsilon \big\| b_1 | L_{r_1}(\langle x\rangle^{\alpha_1}) \big\|^{-1} \big\| b_2 | L_{r_2}(\langle x\rangle^{\alpha_2}) \big\|^{-1} k^{\frac{\alpha}{n} + \frac{1}{r_1} + \frac{1}{r_2}} (\log\langle k\rangle)^{-\varepsilon - \frac{1}{r_1} - \frac{1}{r_2}}, k \in \mathbb{N}, \tag{9}$$

if

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa - \alpha}{n} \tag{10}$$

(with $\varepsilon = 0$ if $r_1 = r_2 = \infty$).

Remark. By Theorem 3.2 there is no problem to replace Theorem 4.1 in the above argumentation by Theorem 4.2/2. Then we have a rather satisfactory smoothness theory and the root spaces for all admissible basic spaces coincide.

5. The negative spectrum

5.1 Preliminaries

We wish to apply the results obtained so far, or better the underlying methods, to study the problem of the negative spectrum in the sense of 2.4. To do this we have to generalize some considerations of the previous section. Our basic space in this section is L_2, always on $\mathbb{R}^n$, although any other Hilbert space $H_2^s(w(x))$ can be taken. As announced in (1/6) we study in L_2 the behaviour of the negative spectrum of the self-adjoint unbounded operator

$$H_\beta = a(x,D) - \beta\, b(x)\, p(x,D)\, b(x) \quad \text{as} \quad \beta \to \infty\,, \tag{1}$$

where

$$a(x,D) \in \Psi^{\varkappa}_{1,\gamma} \quad \text{with} \quad \varkappa \geq 0 \quad \text{and} \quad 0 \leq \gamma < 1\,, \tag{2}$$

is assumed to be a positive-definite and self-adjoint operator in L_2, whereas

$$p(x,D) \in \Psi^{\eta}_{1,\gamma} \quad \text{with} \quad -\infty < \eta < \varkappa,\ 0 \leq \gamma < 1\,, \tag{3}$$

is symmetric and $b(x)$ is a real-valued function. Our basic instrument is Theorem 2.4. As we said in 2.4 we will not bother about domains of definition. At least all can be done on S, and the rest may be considered as a matter of completion. In 5.2 we discuss three cases, beside the two ones related to Theorem 2.4, we have also to look at the simple symmetric case when $p(x,D) = id$. We complement these results in 5.3 by a splitting technique and we have in 5.4 a look at examples via homogeneity arguments.

5.2 Basic results

First we assume $p(x,D) = id$ in (5.1/1), hence

$$H_\beta = a(x,D) - \beta\, b^2(x)\,. \tag{1}$$

Thus the symmetric counterpart of (2.4/3) and (2.4/4) is given by

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \leq \#\{k \in \mathbb{N} \ :\ \sqrt{2}\, e_k \geq \beta^{-1}\} \tag{2}$$

with

$$e_k = e_k(b(x)\, b(x,D)\, b(x)), \quad b(x,D) = a^{-1}(x,D) \in \Psi^{-\varkappa}_{1,\gamma}\,, \tag{3}$$

where we used what had been said in 3.2. We always assume that $a(x,D)$ is positive-definite and self-adjoint in L_2.

Theorem 1. *Let $\varkappa > 0,\ 0 \leq \gamma < 1,\ \alpha > 0$,*

$$\infty \geq r > \max\,(2, \frac{2n}{\varkappa}) \tag{4}$$

and

$$b(x) \in L_r(\langle x\rangle^{\frac{\alpha}{2}}) \quad \textit{real}\,. \tag{5}$$

Let H_β be the operator from 5.1, specified by (1), then

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \leq c \left(\beta\, \left\| b | L_r(\langle x\rangle^{\frac{\alpha}{2}})\right\|^2\right)^{\frac{n}{\varkappa}} \tag{6}$$

if $\varkappa - \alpha < \frac{2n}{r}$ and

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \leq c_\varepsilon\ \beta_0^{(\frac{\alpha}{n}+\frac{2}{r})^{-1}} \left(\log\langle\beta_0\rangle\right)^{(\varepsilon+\frac{2}{r})(\frac{\alpha}{n}+\frac{2}{r})^{-1}} \tag{7}$$

with $\beta_0 = \beta\, \| b | L_r(\langle x\rangle^{\frac{\alpha}{2}})\|^2$ if $\varkappa - \alpha > \frac{2n}{r}$ (and with $\varepsilon = 0$ if $r_1 = r_2 = \infty$).

P r o o f: We apply Theorem 4.1 with $b_1 = b_2 = b$, $p = 2$ and e_k instead of μ_k in (4.1/7) and (4.1/9), to (2) and (3). If $\varkappa - \alpha < \frac{2n}{r}$ then (2) and (4.1/7) yield

$$\beta \, \|b|L_r(\langle x\rangle^{\frac{\alpha}{2}})\|^2 \geq c\, k^{\frac{\varkappa}{n}} \tag{8}$$

and (6). In the case of $\varkappa - \alpha > \frac{2n}{r}$ we have

$$\beta_0 = \beta \, \|b|L_r(\langle x\rangle^{\frac{\alpha}{2}})\|^2 \geq c_\varepsilon k^{\frac{\alpha}{n}+\frac{2}{r}} \, (\log\langle k\rangle)^{-\varepsilon-\frac{2}{r}} \,, \tag{9}$$

$$\log\langle k\rangle \ \leq c_\varepsilon \log\beta_0 \,, \tag{10}$$

hence

$$k^{\frac{\alpha}{n}+\frac{2}{r}} \leq c_\varepsilon \; \beta_0 \; (\log\langle\beta_0\rangle)^{\varepsilon+\frac{2}{r}} \tag{11}$$

and (7).

Remark. Of course, c in (6) and c_ε in (7) are independent of b and $k \in \mathbb{N}$. In [5] the exponent $\frac{n}{\varkappa}$ in (6) is called "semiclassical", it is the expected behaviour. In that paper sharper results are obtained based on specific Hilbert space methods where $a(x, D) = (-\Delta)^l$.

In the following theorem we restrict ourselves for sake of simplicity to the counterpart of the above case $\varkappa - \alpha < \frac{2n}{r}$. The method is clear and there is no problem to deal similarly with the counterpart of the other case in the above theorem. Furthermore we excluded in (5.1/2) the exotic case $\gamma = 1$ since we relied on (3). As for $p(x, D)$ in (5.1/3) the situation is somewhat different. One may try to include exotic perturbation operators $p(x, D) \in \Psi^{\eta}_{1,1}$. Then one has to respect the additional restrictions caused by Theorem 3.1. These needed mapping properties are improved decisively if one follows [6], [19] and [20] and assumes that not only $p(x, D)$ but also its dual is an exotic pseudodifferential operator. We do not go into detail here and stick at $\gamma < 1$ also in (5.1/3). As for $\varkappa$ and η in (5.1/2) and (5.1/3) the restrictions are natural. But to avoid awkward formulations we assume in part (ii) of the theorem below in addition $\eta \geq 0$. It will be clear how to deal with the case $\eta < 0$. Finally, as had been said, the restriction $\varkappa - \eta < \alpha + \frac{2n}{r}$ is not necessary. It simply means that we are looking exclusively for the counterpart of (6) and leaving the counterpart of (7) to the reader (if there is any). In other words, besides the treated cases there is a plethora of further possibilities and we managed to resist to deal with all of them in this paper. Recall that we always assume that $a(x, D)$ is positive-definite and self-adjoint in L_2.

Theorem 2. *Let*

$$H_\beta = a(x, D) - \beta\, b(x)\, p(x, D)\, b(x)\,, \quad \beta > 0\,, \tag{12}$$

be the operator from 5.1 *with*

$$a(x, D) \in \Psi^{\varkappa}_{1,\gamma} \quad \textit{and} \quad p(x, D) \in \Psi^{\eta}_{1,\gamma}\,, \quad 0 \leq \gamma < 1, \tag{13}$$

$\varkappa \geq 0$, $\eta \in \mathbb{R}$. *Let* $\infty \geq r > 2$, $\alpha > 0$, *and*

$$\alpha + \frac{2n}{r} > \varkappa - \eta > \frac{2n}{r}\,. \tag{14}$$

(i) *Let in addition*

$$-\frac{1}{2} - \frac{1}{r} < \frac{\eta}{n} < \frac{1}{2} - \frac{2}{r} \tag{15}$$

and

$$b(x) \in H^{|\eta|}_{r^{|\eta|}}(\langle x\rangle^{\frac{\alpha}{2}})\,, \tag{16}$$

then

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \leq c \left(\|b|H^{|\eta|}_{r^{|\eta|}}(\langle x\rangle^{\frac{\alpha}{2}})\| \; \|b|L_r(\langle x\rangle^{\frac{\alpha}{2}})\| \; \beta \right)^{\frac{n}{\varkappa-\eta}}\,. \tag{17}$$

(ii) *Let in addition*

$$0 \le \eta < \varkappa < \frac{n}{2} \tag{18}$$

and

$$b(x) \in H^{\varkappa}_{r^{\varkappa}}(\langle x\rangle^{\frac{\alpha}{2}}) \ , \tag{19}$$

then

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \le c \left(\left\| b|H^{\varkappa}_{r^{\varkappa}}(\langle x\rangle^{\frac{\alpha}{2}})\right\| \ \left\| b|H^{\varkappa-\eta}_{r^{\varkappa-\eta}}(\langle x\rangle^{\frac{\alpha}{2}})\right\| \ \beta\right)^{\frac{n}{\varkappa-\eta}} . \tag{20}$$

Proof: **Step 1.** We prove (i). To apply (2.4/4) we have to estimate the entropy numbers of the compact embedding

$$B = b(x,D)\ b(x)\ p(x,D)\ b(x) \tag{21}$$

with $b(x,D) = a^{-1}(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$, see (2) and (3). We do this by travelling around in the $(\frac{1}{p}, s)$-diagram in analogy to the proof of Theorem 4.1. The typical situation is shown in Fig.5, where the conditions guarantee that we are in G_2, given by (2.2/6), when Theorem 2.2 is applied.

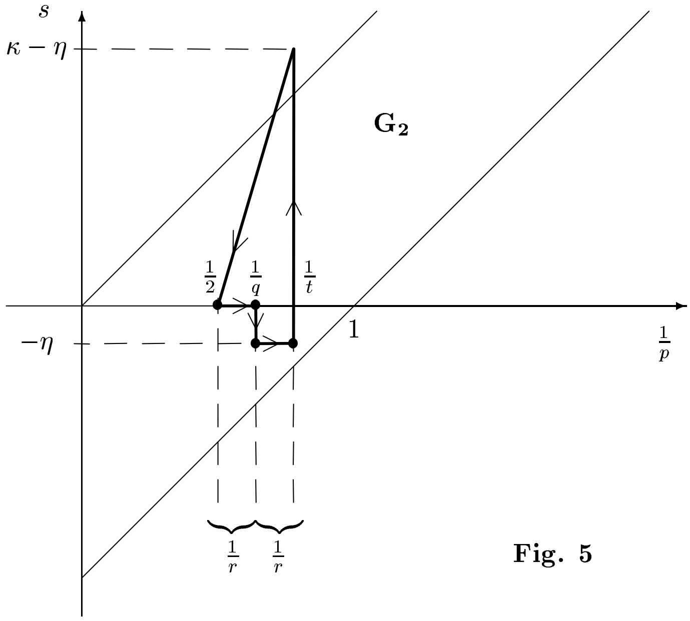

Fig. 5

In analogy to (4.1/11) we decompose B as follows,

$$\left\{ \begin{array}{lll} b & : & L_2 \to L_q(\langle x\rangle^{\frac{\alpha}{2}}) \quad \text{with} \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{r} \\ p(x,D) & : & L_q(\langle x\rangle^{\frac{\alpha}{2}}) \to H^{-\eta}_q(\langle x\rangle^{\frac{\alpha}{2}}) \quad \text{with} \quad (\frac{1}{q}, -\eta) \in G_2 \\ b & : & H^{-\eta}_q(\langle x\rangle^{\frac{\alpha}{2}}) \to H^{-\eta}_t(\langle x\rangle^{\alpha}) \quad \text{with} \quad \frac{1}{t} = \frac{1}{q} + \frac{1}{r} \quad \text{and} \quad (\frac{1}{t}, -\eta) \in G_2 \\ b(x,D) & : & H^{-\eta}_t(\langle x\rangle^{\alpha}) \to H^{\varkappa-\eta}_t(\langle x\rangle^{\alpha}) \\ id & : & H^{\varkappa-\eta}_t(\langle x\rangle^{\alpha}) \to L_2 \ . \end{array} \right. \tag{22}$$

The first line comes from Hölder's inequality and the limiting embedding $H^{|\eta|}_{r^{|\eta|}}(\langle x\rangle^{\frac{\alpha}{2}}) \subset L_r(\langle x\rangle^{\frac{\alpha}{2}})$. The second and the third line are covered by (13), (2.2/4), Remark 3.1/2 and Theorem 2.2, respectively, where $(\frac{1}{q}, -\eta) \in G_2$ and $(\frac{1}{t}, -\eta) \in G_2$ are ensured by (15). The rest is the same as in the proof of Theorem 4.1 and the above Theorem 1.

Step 2. We prove (ii). Now Fig.5 and (22) must be replaced by Fig.6 with G_1 given by (2.2/5),

$$B = b(x)\; p(x,D)\; b(x)\; b(x,D) \tag{23}$$

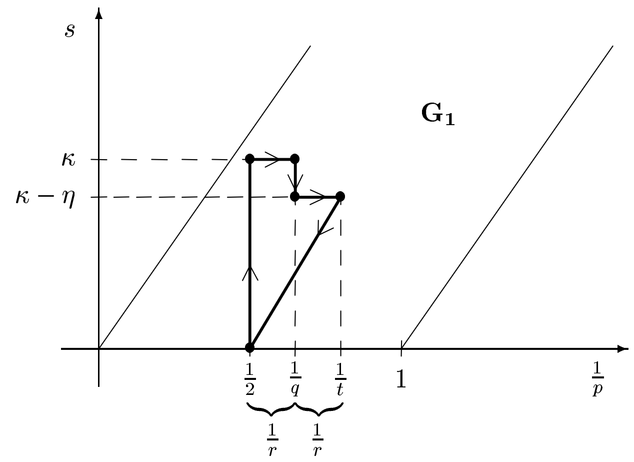

Fig. 6

in the sense of (2.4/3), (2) and (3), and

$$\left\{\begin{array}{lcl} b(x,D) & : & L_2 \to H_2^{\varkappa} \\ b & : & H_2^{\varkappa} \to H_q^{\varkappa}(\langle x\rangle^{\frac{\alpha}{2}}) \\ p(x,D) & : & H_q^{\varkappa}(\langle x\rangle^{\frac{\alpha}{2}}) \to H_q^{\varkappa-\eta}(\langle x\rangle^{\frac{\alpha}{2}}) \\ b & : & H_q^{\varkappa-\eta}(\langle x\rangle^{\frac{\alpha}{2}}) \to H_t^{\varkappa-\eta}(\langle x\rangle^{\alpha}) \\ id & : & H_t^{\varkappa-\eta}(\langle x\rangle^{\alpha}) \to L_2\ . \end{array}\right. \tag{24}$$

The justification is similar as after (22).

5.3 Splitting techniques

The interest in studying the "negative" spectrum (bound states) comes from quantum mechanics, generalizing the classical hydrogen operator,

$$H = -\Delta - \frac{c}{|x|}\ , \quad c > 0\ , \tag{1}$$

in $L_2(\mathbb{R}^3)$. Comparing (1) and (5.2/1) then "potentials" $b(x)$ with $b(x) \sim |x|^{-\alpha}$, $\alpha > 0$, are of peculiar interest, or, more general, "potentials" which have local singularities and some decay properties at infinity. To squeeze both in one condition of type (5.2/5) cannot be optimal in our context; however this should be compared with the results in [5] generalizing the famous Cwikel-Rosenbljum-Lieb-inequality. As a cure one can try to use a splitting of the type

$$b(x) = d_1(x) + d_2(x) \tag{2}$$

where $d_1(x)$ may be compactly supported collecting the local singularities and, say, $\langle x\rangle^{\alpha} d_2(x) \in L_\infty$ for some $\alpha > 0$. Of course one may think about other decompositions but the just recommended one is apparently well adapted to the described classical examples and quite typical for our approach. It is simply a combination of what had been done so far in this paper, [17] and [14]. We do not deal with the general operator (5.2/12) but with the simpler version (5.2/1),

$$H_\beta = a(x,D) - \beta\, b^2(x), \tag{3}$$

where again

$$a(x,D) \in \Psi^{\varkappa}_{1,\gamma} \quad \text{with} \quad \varkappa > 0 \quad \text{and} \quad 0 \le \gamma < 1 \tag{4}$$

is assumed to be a positive-definite self-adjoint operator in L_2, which now is always $L_2(\mathbb{R}^n)$. Then we have (5.2/2) and (5.2/3). In other words, we complement Theorem 5.2/1.

Theorem. *Let* $\varkappa > 0,\ 0 \le \gamma < 1,\ \alpha > 0,\ \varkappa \ne \alpha,$

$$\infty \ge r > \max\,(2, \frac{2n}{\varkappa}) \tag{5}$$

and

$$b(x) = d_1(x) + d_2(x)\,, \tag{6}$$

where $d_1(x)$ *has a compact support, and*

$$d_1(x) \in L_r\,, \quad \langle x\rangle^{\frac{\alpha}{2}}\ d_2(x) \in L_\infty\ . \tag{7}$$

Let H_β *be the above operator, then*

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \le c\ \beta_0^{\overline{\frac{n}{\min\,(\varkappa,\alpha)}}} \tag{8}$$

with $\beta_0 = \beta\ \left(\|d_1|L_r\| + \|\langle x\rangle^{\frac{\alpha}{2}} d_2|L_\infty\|\right)^2$.

P r o o f: In the same way as in the proof of Theorem 5.2/1 we reduce the proof to Theorem 4.1, where we split B in (4.1/1) with $b_2 = b_1 = b$ in four operators

$$B = B^{1,1} + B^{1,2} + B^{2,1} + B^{2,2} \tag{9}$$

where $B^{j,m}$ is given by (4.1/1) with $b_2 = d_j$ and $b_1 = d_m$. The related entropy numbers e_k of the first three operators are covered by [14] and [17], or alternatively by [17] if one takes into account $d_1 \in L_r(\langle x\rangle^\beta)$, where $\beta > 0$ may be chosen arbitrarily large. They can be estimated from above by $c\ k^{-\frac{\varkappa}{n}}$ where the dependence on d_1 and d_2 is the desired one. The entropy numbers of $B^{2,2}$ treated as in the proof of Theorem 4.1 are covered by Theorem 4.2 in [17] and they can be estimated from above by $c\ k^{-\frac{\min\,(\varkappa,\alpha)}{n}}$ where again the dependence on d_2 is the desired one. Hence, the entropy numbers of B can be estimated from above by $c\ k^{-\frac{\min\,(\varkappa,\alpha)}{n}}$. The rest is now the same as in the proof of Theorem 5.2/1.

Remark. There are two types of limiting cases, both avoided by us so far. Firstly, (8) cannot be expected to be correct when $\varkappa = \alpha$. In that case an additional *log*-term should occur, see [17], Theorem 4.2 as far as this weighted limiting embedding is concerned. We refer also to [27] in this context. Secondly, in connection with (5), the limiting cases

$$\frac{2n}{\varkappa} = r \ge 2 \tag{10}$$

are of peculiar interest. In the case of bounded domains these limiting cases are treated in some details in a more general unsymmetric setting, including also application to the just considered problem of the negative spectrum, in [14], 3.4 and 3.5.2. Then one needs limiting embeddings in Orlicz spaces. By the above splitting technique this can be carried over to d_1 in (6). We will not do this here. But we mention again that just these limiting cases are closely connected with the Cwikel-Rosenbljum-Lieb-inequality and its generalizations, see [5]. Dealing with limiting cases of both types it seems to be desirable to generalize also the weights $\langle x\rangle^\alpha$ treated so far in [17] by, say, $\langle x\rangle^\alpha\ (1 + \log\langle x\rangle)^\beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$.

5.4 Homogeneity arguments

Hitherto we adopted the traditional point of view asking for the behaviour of

$$\#\{\sigma(H_\beta) \cap (-\infty, 0]\} \quad \text{as} \quad \beta \to \infty \tag{1}$$

where H_β is given by (5.2/1),

$$H_\beta = a(x, D) - \beta\, b^2(x), \tag{2}$$

or more general by (5.1/1), satisfying the conditions detailed in 5.1 and 5.2. Looking at the classical background briefly described at the beginning of 5.3, then a slightly modified point of view seems to be even more natural. Let

$$H = a(x, D) - b^2(x), \tag{3}$$

where $a(x, D) \in \Psi^{\varkappa}_{1,\gamma}$ with $\varkappa > 0$ and $0 \le \gamma < 1$, is a self-adjoint positive operator in L_2 with, say, $0 \in \sigma_e$, where σ_e is the essential spectrum of $a(x, D)$. (Then, of course, 0 is the bottom point of σ_e). Let, as before, $b(x)$ be a real function and $b^2(x)\big(id + a(x, D)\big)^{-1}$ be compact, then H has the same essential spectrum as $a(x, D)$ and the behaviour of

$$\#\{\sigma(H) \cap (-\infty, -\varepsilon]\} \quad \text{as} \quad \varepsilon \downarrow 0 \tag{4}$$

might be of interest. The attempt to reduce (4) to (1) causes in general some problems. But the situation improves immediately when both $a(x, D)$ and $b(x)$ satisfy some homogeneity conditions. To be as close as possible to the hydrogen operator H in (5.3/1) we restrict ourselves to a comparatively simple but typical example. Let

$$a(D) = (-1)^m \sum_{|\gamma| = m} a_\gamma D^{2\gamma} \tag{5}$$

with

$$a_\gamma \in I\!R \quad \text{and} \quad \sum_{|\gamma| = m} a_\gamma \xi^{2\gamma} \ge c\, |\xi|^{2m}, \quad \xi \in I\!R^n, \tag{6}$$

for some $c > 0$, be an elliptic differential operator of order $2m$ with constant coefficients. Then

$$\sigma(a(D)) = \sigma_e(a(D)) = [0, \infty)\ . \tag{7}$$

See for background material [31] or [11], IX, 6,8. Let

$$H = a(D) - |x|^{-\eta} \quad \text{with} \quad 0 < \eta < \min\,(n, 2m)\ . \tag{8}$$

Then we have $b(x) = |x|^{-\frac{\eta}{2}}$ in the sense of (3). By (8) there is a number r such that

$$\frac{2n}{\eta} > r > \max\,(2, \frac{n}{m})\ . \tag{9}$$

The right-hand side coincides with (5.3/5), whereas the left-hand side covers (5.3/7) with $\alpha = \eta$ and d_1 stands for b near the origin. Hence we can apply Theorem 5.3 to

$$H_\beta = a(D) - \beta\, |x|^{-\eta}\ . \tag{10}$$

What remains is to adapt (5.3/8) to (4).

Theorem. *Let H be given by (8) with (5) and (6). There exists a number $c > 0$ such that for all $\varepsilon > 0$*

$$\#\{\sigma(H) \cap (-\infty, -\varepsilon]\} \le c\, \varepsilon^{-n(\frac{1}{\eta} - \frac{1}{2m})}\ . \tag{11}$$

Proof: The operator H_β in Theorem 5.3 can be identified with $id\ + H_\beta$, where the latter H_β is given by (10). Hence,

$$\#\{\sigma(H_\beta) \cap (-\infty, -1]\} \leq c\, \beta^{\frac{n}{\eta}} \tag{12}$$

since $\eta < \varkappa = 2m$ in our context. Let λ with $\lambda \leq -1$ be an eigenvalue of H_β and let $f(x)$ be a related eigenfunction. Then we have for any $c > 0$ and $f(x) = g(cx)$

$$c^{2m}(-1)^m \sum_{|\gamma|=m} a_\gamma \big(D^{2\gamma} g\big)(cx) - \beta\, |x|^{-\eta}\, g(cx) = \lambda\, g(cx) \tag{13}$$

and

$$(-1)^m \sum_{|\gamma|=m} a_\gamma D^{2\gamma} g(x) - \beta\, c^{\eta-2m} |x|^{-\eta} g(x) = \lambda\, c^{-2m}\, g(x)\ . \tag{14}$$

We choose $c = \beta^{\frac{1}{2m-\eta}}$ and obtain

$$Hg = \lambda\, \beta^{-\frac{2m}{2m-\eta}} g = \lambda\, \varepsilon\, g\ . \tag{15}$$

But the left-hand side of (11) coincides with the left-hand side of (12) if $\beta = \varepsilon^{-\frac{2m-\eta}{2m}}$. Now (12) yields (11).

Remark 1. In case of the hydrogen atom (5.3/1) we have $n = 3$, $\eta = 1$ and $2m = 2$, hence

$$\#\{\sigma(H) \cap (-\infty, -\varepsilon]\} \leq c\, \varepsilon^{-\frac{3}{2}}\ ,$$

as it should be, since the number of eigenvalues less than or equal to $-\varepsilon = -\frac{1}{4} N^{-2}$ with $N \in I\!N$, is $\sum_{j=1}^{N} j^2 \sim N^3$, see e.g. [43], 7.3 and 7.3.4.

Remark 2. One can replace $a(D)$ in (5) by homogeneous pseudodifferential operators. There is also a good chance to extend both the splitting techniques from 5.3 and the homogeneity arguments of this subsection to more general operators of type (5.2/12).

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3.3 Appendix C

Approximation numbers in some weighted function spaces

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In this paper we study weighted function spaces of type $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$ where $w(x)$ is a weight function of at most polynomial growth, preferably $w(x) = (1+|x|^2)^{\alpha/2}$ with $\alpha \in \mathbb{R}$. The main result deals with estimates for the approximation numbers of compact embeddings between spaces of this type. Furthermore we are concerned with the dependence of the approximation numbers a_k of compact embeddings between function spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on an underlying domain Ω.

(Math. Subject Classification: 46E35, 47B06)

1. Introduction

In [4] and [5] entropy and approximation numbers of compact embeddings between function spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, $s \in \mathbb{R}$, $0 < p \leq \infty$ (with $p < \infty$ in the F-case), $0 < q \leq \infty$, on a bounded domain Ω in $\mathbb{R}^n$ were thoroughly investigated. Recall that these two scales of spaces cover many well-known classical spaces such as (fractional) Sobolev spaces, Hölder-Zygmund spaces, Besov spaces and (inhomogeneous) Hardy spaces. In [7] we extended these results in some sense, i.e. we studied weighted function spaces of type $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$ where $w(x)$ is an admissible weight function of at most polynomial growth, that is a smooth function with

$$0 < w(x) \leq c\, w(y)\langle x-y\rangle^{\alpha} \tag{1}$$

for some $\alpha \geq 0$, some $c > 0$ and all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. As usual $\langle x\rangle = (1+|x|^2)^{1/2}$. The main result of [7] dealt with relatively sharp estimates for the entropy numbers of compact embeddings between function spaces of such type

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n, w(x)\langle x\rangle^{\beta}) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(\mathbb{R}^n, w(x)) \quad \text{with } \beta > 0 \tag{2}$$

and their B-counterparts.

We applied these results in [8] to eigenvalue distributions of pseudodifferential operators. In the present paper we return to the study of the compactness of embeddings of type (2) for its own sake estimating the related approximation numbers.

Weighted spaces of the above and more general type have already been treated before, especially by H.-J. Schmeisser and H. Triebel in [12: 5.1]. Nevertheless we sketched new shorter proofs for some relevant facts in [7] relying not very much on former results.

The plan of the paper is as follows. In Sect.2 we introduce the spaces $B^s_{p,q}(\mathbb{R}^n, w(x))$ and $F^s_{p,q}(\mathbb{R}^n, w(x))$. We collect some recently proved results which will be of great service for us later on. In particular, we remind the reader of the equivalence of the quasi-norms

$$\|f|F^s_{p,q}(\mathbb{R}^n, w(x))\| \quad \text{and} \quad \|wf|F^s_{p,q}(\mathbb{R}^n)\| \tag{3}$$

and their B-counterparts. Furthermore recall that for $-\infty < s_2 < s_1 < \infty$, $0 < p_1 \leq p_2 < \infty$, $0 < q_1 \leq \infty$ and $0 < q_2 \leq \infty$, the embedding

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1(x)) \quad \text{into} \quad F^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2(x)) \tag{4}$$

(and its B-counterpart) is compact if and only if

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \longrightarrow 0 \quad \text{as } |x| \to \infty \,. \tag{5}$$

Finally we mention a helpful weak type embedding

$$B^s_{p,q}(\mathbb{R}^n, \langle x\rangle^\alpha) \quad \text{into} \quad weak - B^s_{p_0,q}(\mathbb{R}^n) \tag{6}$$

where $\alpha > 0$ and $\frac{1}{p_0} = \frac{1}{p} + \frac{\alpha}{n}$.
Turning to the entropy and approximation numbers we refer to the respective estimates related to function spaces on domains published in [4] and [5]. In Sect.3 we regard as a preparation the dependence of the approximation numbers on the certain domain Ω on which function spaces $F^s_{p,q}(\Omega)$ and $B^s_{p,q}(\Omega)$ are defined. Afterwards we state our main theorem. Sect.4 contains all the proofs.
Unimportant constants are denoted by c, occasionally with additional subscript within the same formula or the same step of the proof. Furthermore, (k.l/m) refers to formula (m) in subsection k.l, whereas (j) means formula (j) in the same subsection. In a similar way we quote definitions, propositions and theorems.

2. Definitions and Preliminaries
2.1 Weighted Function Spaces

Let $\mathbb{R}^n$ be the Euclidean n-space. We introduce the notation $\langle x\rangle = (1+|x|^2)^{1/2}$ on $\mathbb{R}^n$.

Definition 1. The class of *admissible weight functions* is the collection of all positive C^∞ functions $w(x)$ on $\mathbb{R}^n$ with the following properties:
(i) for any multiindex γ there exists a positive constant c_γ with

$$|D^\gamma w(x)| \le c_\gamma \, w(x) \quad \text{for all} \quad x \in \mathbb{R}^n \,, \tag{1}$$

(ii) there exist two constants $c > 0$ and $\alpha \ge 0$ such that

$$0 < w(x) \le c\, w(y)\langle x-y\rangle^\alpha \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n \,. \tag{2}$$

Remark 1. From (2) it can be easily seen that for suitable constants $c_1 > 0$ and $c_2 > 0$ it holds

$$c_1\, w(y) \le w(x) \le c_2\, w(y) \quad \text{for all } x \in \mathbb{R}^n,\ y \in \mathbb{R}^n \text{ with } |x-y| \le 1 \,. \tag{3}$$

On the other hand we have for admissible weight functions $w_1(x)$ and $w_2(x)$ that both $w_1(x)w_2(x)$ and $w_1^{-1}(x)$ are admissible weight functions, too.

Remark 2. We want to explain briefly that the apparently restrictive assumption for $w(x)$ to be a C^∞ function is in fact almost none. Let $w(x)$ be a measurable function in $\mathbb{R}^n$ satisfying (2) and assume $h(x) \ge 0$ to be a C^∞-function in $\mathbb{R}^n$, supported by the unit ball with, say, $\int h(x)dx = 1$. In other words, $h(x)$ is a so-called mollifier. Then $(h*w)(x)$ defined by

$$(h*w)(x) = \int h(x-y)w(y)dy \tag{4}$$

is an admissible weight function according to the above definition. As w and $h*w$ are equivalent to each other this finally justifies to concentrate only on smooth representatives without loss of generality.

Now we will briefly remind the reader of the well-known spaces $B^s_{p,q}$ and $F^s_{p,q}$ because we want to define their weighted counterparts afterwards. All spaces in this paper are defined on $\mathbb{R}^n$ and

so we omit " $\mathbb{R}^n$ " in the sequel. The Schwartz space S and its dual S' of all complex-valued tempered distributions have the usual meaning here. Furthermore, L_p with $0 < p \leq \infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\| \cdot |L_p\|$.
Let $\varphi \in S$ be such that

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \leq 1, \tag{5}$$

let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and put $\varphi_0 = \varphi$. Then since $1 = \sum_{j=0}^{\infty} \varphi_j(x)$ for all $x \in \mathbb{R}^n$, the $\{\varphi_j\}$ form a dyadic resolution of unity. Given any $f \in S'$, we denote by $\hat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. Thus $(\varphi_j \hat{f})^\vee$ is an analytic function on $\mathbb{R}^n$. Based on the unweighted spaces L_p on $\mathbb{R}^n$ we introduce their weighted generalizations $L_p(w(x))$, quasi-normed by

$$\|f|L_p(w(\cdot))\| = \|wf|L_p\| , \tag{6}$$

where $w(x) > 0$ is an (admissible) weight function on $\mathbb{R}^n$ and $0 < p \leq \infty$.

Definition 2. Let $w(x)$ be an admissible weight function in the sense of Definition 1. Let $s \in \mathbb{R}, 0 < q \leq \infty$ and let $\{\varphi_j\}$ be the above dyadic resolution of unity.
(i) Let $0 < p \leq \infty$. The space $B^s_{p,q}(w(x))$ is the collection of all $f \in S'$ such that

$$\|f|B^s_{p,q}(w(\cdot))\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee |L_p(w(\cdot))\|^q \Big)^{1/q} \tag{7}$$

(with the usual modification if $q = \infty$) is finite.
(ii) Let $0 < p < \infty$. The space $F^s_{p,q}(w(x))$ is the collection of all $f \in S'$ such that

$$\|f|F^s_{p,q}(w(\cdot))\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(w(\cdot))\Big\| \tag{8}$$

(with the usual modification if $q = \infty$) is finite.

(iii) Let $w(x) = \langle x \rangle^\alpha$ for some $\alpha \in \mathbb{R}$. Then we put

$$B^s_{p,q}(\alpha) = B^s_{p,q}(\langle x \rangle^\alpha) \quad \text{with} \quad B^s_{p,q} = B^s_{p,q}(0) \tag{9}$$

and

$$F^s_{p,q}(\alpha) = F^s_{p,q}(\langle x \rangle^\alpha) \quad \text{with} \quad F^s_{p,q} = F^s_{p,q}(0) . \tag{10}$$

Remark 3. The theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$ has been developed in [13] and [14]. Extending this theory to the above weighted classes of function spaces causes no difficulty. Furthermore, in [12: 5.1] spaces of type $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ were investigated in the framework of ultra-distributions for much larger classes of admissible weight functions. Nevertheless also the later developments in the theory of the unweighted spaces $B^s_{p,q}$ and $F^s_{p,q}$, see e.g. [14], have their more or less obvious counterparts for weighted spaces in the above sense.

Remark 4. Likewise to the unweighted case the above two weighted scales $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ cover many other spaces such as weighted (fractional) Sobolev spaces, weighted classical Besov spaces and weighted Hölder-Zygmund spaces. We refer to [12: 5.1] and the literature mentioned there.

2.2 Embeddings

In this section we want to collect some important results associated with our topic which have been proved in recent papers, see [7] and the references given there.

Proposition 1. *Let $s \in \mathbb{R}, 0 < q \leq \infty$ and $0 < p \leq \infty$ (with $p < \infty$ in the F-case).*
(i) *$B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*
(ii) *The operator $f \mapsto wf$ is an isomorphic mapping from $B^s_{p,q}(w(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(w(x))$ onto $F^s_{p,q}$. Especially,*

$$\|wf|B^s_{p,q}\| \quad \text{is an equivalent quasi-norm in} \quad B^s_{p,q}(w(x)) \tag{1}$$

and

$$\|wf|F^s_{p,q}\| \quad \text{is an equivalent quasi-norm in} \quad F^s_{p,q}(w(x))\,. \tag{2}$$

Remark 1. A new short proof of this proposition may be found in [7: 5.1]. Nevertheless there are some other, more complicated proofs and forerunners, e.g. in [12: 5.1] or [6].

Using the above proposition we could immediately extend the embedding theory developed in [13: 2.3.2 and 2.7.1] to the weighted spaces under consideration here if only one weight function is involved. On the other hand we have also regarded in [7] embeddings with different weights. Related to the F-spaces this result reads as follows.

Proposition 2. *Let $w_1(x)$ and $w_2(x)$ be admissible weight functions and*

$$-\infty < s_2 < s_1 < \infty, 0 < p_1 \leq p_2 < \infty, 0 < q_1 \leq \infty \text{ and } 0 < q_2 \leq \infty\,. \tag{3}$$

(i) *Then $F^{s_1}_{p_1,q_1}(w_1(x))$ is continuously embedded in $F^{s_2}_{p_2,q_2}(w_2(x))$,*

$$F^{s_1}_{p_1,q_1}(w_1(x)) \subset F^{s_2}_{p_2,q_2}(w_2(x))\,, \tag{4}$$

if and only if

$$s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \leq c < \infty \tag{5}$$

for some $c > 0$ and all $x \in \mathbb{R}^n$.
(ii) *The embedding* (4) *is compact if and only if*

$$s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2} \quad \text{and} \quad \frac{w_2(x)}{w_1(x)} \longrightarrow 0 \quad \text{if} \quad |x| \to \infty\,. \tag{6}$$

Remark 2. A proof of this theorem is given in [7: 5.2]. Obviously one can extend the above proposition to the B-scale. Then p_2 may be infinite and the interesting weighted Hölder-Zygmund spaces $\mathcal{C}^s(w(x)) = B^s_{\infty,\infty}(w(x))$ are included.

In the following we will specify our situation in some sense. Let w_1 and w_2 be two admissible weight functions in the sense of Definition 2.1/1. Then $\frac{w_1}{w_2}$ is an admissible weight function, too, and Proposition 1 tells us

$$\|f|F^s_{p,q}(w_1(\cdot))\| \;\sim\; \big\|w_2 f|F^s_{p,q}(\frac{w_1}{w_2}(\cdot))\big\| \tag{7}$$

(equivalent quasi-norms), i.e. $f \mapsto w_2 f$ is an isomorphic mapping from $F^s_{p,q}(w_1(x))$ onto $F^s_{p,q}(\frac{w_1}{w_2}(x))$ where $w_1(x)$ is assumed to be an admissible weight function. The same holds in the B-case. Studying continuous or compact embeddings it is sufficient to investigate it, without loss of generality,

for $w_2(x) = 1$. In the sequel we put $w_1(x) = w(x)$ and specify $w(x) = \langle x\rangle^\alpha$ for some $\alpha > 0$.
To finish this subsection we formulate a weak type continuous embedding assertion. Let $L_{p,\infty} = L_{p,\infty}(\mathbb{R}^n)$ with $0 < p < \infty$ be the usual Lorentz space (Marcinkiewicz space) on $\mathbb{R}^n$ with respect to the Lebesgue measure, see [15: 1.18.6] or [1: p.216] for definitions.

Definition 3. Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$. Let $\{\varphi_j\}$ be a dyadic resolution of unity. Then *weak-*$B^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f|weak - B^s_{p,q}\| = \big(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_j \hat{f})^\vee|L_{p,\infty}\|^q\big)^{1/q} \tag{8}$$

(with the usual modification if $q = \infty$) is finite. Similarly, *weak-*$F^s_{p,q}$ is the collection of all $f \in S'$ such that

$$\|f|weak - F^s_{p,q}\| = \|(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j \hat{f})^\vee(\cdot)|^q)^{1/q}|L_{p,\infty}\| \tag{9}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3. It would also be possible to replace $L_{p,\infty}$ by the more general Lorentz spaces $L_{p,u}, 0 < p \leq \infty$ ($p < \infty$ in the F-case) and $0 < u \leq \infty$.

Proposition 3. (i) *Under the restrictions for s, p and q in the above definition both weak-$B^s_{p,q}$ and weak-$F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$) and they are independent of the chosen dyadic resolution of unity $\{\varphi_j\}$.*
(ii) *Let $s \in \mathbb{R}, 0 < q \leq \infty, 0 < p \leq \infty$ ($p < \infty$ in the F-case), $\alpha > 0$ and $\frac{1}{p_0} = \frac{1}{p} + \frac{\alpha}{n}$. Then*

$$B^s_{p,q}(\alpha) \subset weak - B^s_{p_0,q} \quad and \quad F^s_{p,q}(\alpha) \subset weak - F^s_{p_0,q}\,. \tag{10}$$

Remark 4. A very short proof of the above proposition is included in [7: 2.4].

2.3 Entropy and Approximation Numbers

Let B_1 and B_2 be two complex quasi-Banach spaces and let T be a linear and continuous operator from B_1 into B_2. If T is compact then for any given $\varepsilon > 0$ there are finitely many balls in B_2 of radius ε which cover the image TU_1 of the unit ball $U_1 = \{a \in B_1 : \|a|B_1\| \leq 1\}$.

Definition 1. Let $k \in \mathbb{N}$ and assume $T : B_1 \to B_2$ to be the above continuous operator.
(i) The k^{th} *entropy number* e_k of T is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in B_2 of radius ε which cover TU_1.
(ii) The k^{th} *approximation number* a_k of T is the infimum of all numbers $\|T - A\|$ where A runs through the collection of all continuous linear maps from B_1 to B_2 with rank $A < k$.

Remark 1. For details and properties of entropy and approximation numbers we refer to [2], [3], [9] and [11] (always restricted to the case of Banach spaces). There is no difficulty to extend these properties to quasi-Banach spaces.

Similarly to the previous subsection we will collect some recent, already known results which will later on turn out to be the basis for the main result of this paper. We will remind the reader of the papers [4] and [5] concerning entropy and approximation numbers in (unweighted) function spaces on domains.
Before quoting that result we briefly recall the definition of function spaces on domains which are the subject of the succeeding proposition.

DEFINITION 2. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞boundary $\partial\Omega$. Assume $-\infty < s < \infty, 0 < p \le \infty$ ($p < \infty$ in the F-case) and $0 < q \le \infty$. Then $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ are the restrictions of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively, to Ω.

We denote by $a_+ = \max(0, a)$ for $a \in \mathbb{R}$. Furthermore we always use $a_k \sim k^{-\varrho}$ in the sense that there exist two positive numbers c_1 and c_2 such that

$$c_1\, k^{-\varrho} \le a_k \le c_2\, k^{-\varrho} \quad \text{for all} \quad k \in \mathbb{N}. \tag{1}$$

PROPOSITION. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞boundary $\partial\Omega$. Assume*

$$-\infty < s_2 < s_1 < \infty, \quad p_1, p_2, q_1, q_2 \in (0, \infty] \tag{2}$$

and suppose that

$$\delta^+ := s_1 - s_2 - n\big(\frac{1}{p_1} - \frac{1}{p_2}\big)_+ > 0. \tag{3}$$

Let e_k be the k^{th} entropy number of the natural embedding $id : B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega)$ and a_k its k^{th} approximation number.

(i) *Then it holds*

$$e_k \;\sim\; k^{-\frac{s_1 - s_2}{n}}\,. \tag{4}$$

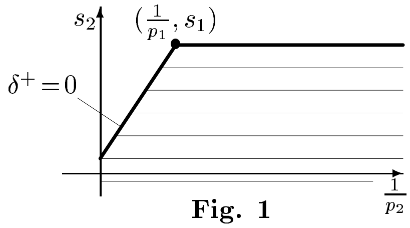


Fig. 1

(ii) *Suppose that in addition to the general hypotheses*

$$\textit{either}\;\; 0 < p_1 \le p_2 \le 2 \;\;\textit{or}\;\; 2 \le p_1 \le p_2 \le \infty \;\;\textit{or}\;\; 0 < p_2 \le p_1 \le \infty \tag{5}$$

is satisfied. Then it holds

$$a_k \;\sim\; k^{-\frac{\delta^+}{n}}\,. \tag{6}$$

(iii) *Suppose that in addition to the general hypotheses*

$$0 < p_1 \le 2 \le p_2 < \infty \;\textit{and}\; \lambda = \frac{s_1 - s_2}{n} - \max\big(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\big) > \frac{1}{2}. \tag{7}$$

Then it holds

$$a_k \;\sim\; k^{-\lambda}\,. \tag{8}$$

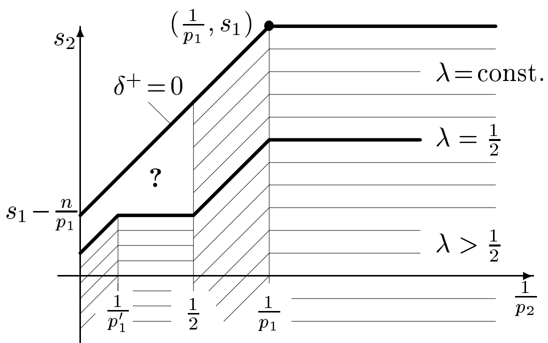
(iv) *Suppose that in addition to the general hypotheses*

$$0 < p_1 \le 2 \le p_2 \le \infty\,. \tag{9}$$

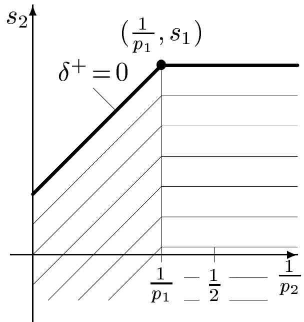
Then there are positive constants c_1 and c_2 such that for all $k \in \mathbb{N}$

$$c_1\, k^{-\lambda} \le a_k \le c_2\, k^{-\frac{\delta^+}{n}} \tag{10}$$

where λ has the same meaning as in (7).



$0 < p_1 < 2$ **Fig. 2**



$2 \leq p_1 < \infty$ **Fig. 3**

Remark 2. The proposition and its proof will be found in [4] and [5]. Obviously, via the elementary embedding

$$B^s_{p,u} \subset F^s_{p,q} \subset B^s_{p,v} \quad \text{if and only if} \quad u \leq \min(p,q) \text{ and } v \geq \max(p,q) \tag{11}$$

the above proposition holds also in the F-case, now with $p_1 < \infty$ and $p_2 < \infty$. (There is a new short proof for the "only if"-part of (11) in [7: 4.3].)

Remark 3. The thin lines in the above diagrams Fig.1 - Fig.3 shall indicate the different level lines on which the exponents of $k \in \mathbb{N}$ are constant. Fig.1 refers to e_k whereas Fig.2 and Fig.3 are related to a_k. In Fig.2 we made use of the convention $p_1' = \infty$ if $p_1 \leq 1$. Then we have for $\frac{1}{p_1'} \leq \frac{1}{p_2} \leq \frac{1}{2}$ there that $\lambda = \frac{1}{2}$ is equivalent to $s_2 = s_1 - \frac{n}{p_1}$.

3. Approximation Numbers in Weighted Function Spaces

3.1 Dependence of the Approximation Numbers on Domains

In this subsection we provide ourselves with a last preparation which may also be regarded as belonging to the proof of the main theorem. But just this proof will already become long enough therefore we prove the following lemma separately and in advance.

Lemma. *Let $K_R = \{x \in \mathbb{R}^n : |x| < R\}$, $R \geq 1$, be a ball in $\mathbb{R}^n$ centered at the origin. Assume*

$$-\infty < s_2 < s_1 < \infty,\ 0 < p_1 \leq p_2 < \infty \quad \text{and} \quad s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}\ . \tag{1}$$

Let a^R_k be the k^{th} approximation number of the compact embedding $id : F^{s_1}_{p_1,q_1}(K_R) \to F^{s_2}_{p_2,q_2}(K_R)$ with $a_k = a^1_k$, $k \in \mathbb{N}$. Then there exist positive constants c_1 and c_2 such that for $k \in \mathbb{N}$ and $R > 1$ we have

$$a^R_{c_1 R^n k} \leq c_2\ a_k\ . \tag{2}$$

Remark 1. The above lemma will be proved in 4.1. We introduced the function spaces on domains in Definition 2.3/2. We always put $a_\lambda = a_{[\lambda]}$ if $\lambda \geq 1$ and $[\lambda]$ is the largest integer with $[\lambda] \leq \lambda$.

Corollary. *Let $A_m = \{x \in \mathbb{R}^n : 2^{m-1} < |x| < 2^{m+1}\}, m \in \mathbb{N}$, be the usual annuli and $a^{(j)}_k$ the respective k^{th} approximation number of the embedding $id^{(j)} : F^{s_1}_{p_1,q_1}(A_j) \to F^{s_2}_{p_2,q_2}(A_j)$ where again (1) is assumed to be satisfied. Then there exist positive constants c_1 and c_2 such that for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$ we get*

$$a^{(j)}_{c_1 2^{jn} k} \leq c_2\ a_k\ . \tag{3}$$

Remark 2. The proof is essentially the same as for the above lemma and will not be repeated here. We have to replace $R > 1$ by $2^j, j \in \mathbb{N}$, then.

3.2 The Main Theorem

As we already announced in the beginning the main subject of this paper is to study the approximation numbers of the compact embeddings

$$id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \tag{1}$$

and

$$id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{2}$$

where the spaces have been introduced in Definition 2.1/2. We also mentioned that this covers the apparently more general cases where the unweighted spaces on the right-hand side of (1) and (2) are replaced by $B^{s_2}_{p_2,q_2}(\beta)$ and $F^{s_2}_{p_2,q_2}(\beta)$, respectively, for some $\beta < \alpha$. One can furthermore imagine to mix B- and F-spaces in (1) and (2) but we give up this possibility. Moreover, it turns out that the third indices never play any role such that we can formulate the theorem for the B-case only and afterwards, via the weighted counterpart of (2.3/11), also the F-case is covered. Let for $1 \le p \le \infty$ the numbers p' be defined by $\frac{1}{p} + \frac{1}{p'} = 1$, for $0 < p < 1$ we put $p' = \infty$. Assume that

$$\begin{aligned} -\infty < s_2 < s_1 < \infty,\ \alpha > 0 \quad &, \quad 0 < p_1 < \infty,\ 0 < q_1 \le \infty, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n} \quad &, \quad p_0 < p_2 < \infty,\ 0 < q_2 \le \infty \end{aligned} \tag{3}$$

and

$$\delta = s_1 - \frac{n}{p_1} - \big(s_2 - \frac{n}{p_2}\big) > 0. \tag{4}$$

In the usual $(\frac{1}{p}, s)$-diagram we introduce the following regions (see Fig.4, 5 and 6) :

I $: 0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 < \infty,\ 0 < \delta < \alpha$

II $: 0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 < \infty,\ \delta > \alpha$

III $: 0 < p_1 < 2 < p_2 < \infty,\ 0 < \delta < \alpha,\ \lambda := \frac{s_1 - s_2}{n} - \max(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}) > \frac{1}{2}$

- **III**$_a$ $: 0 < p_1 < 2 < p_1' \le p_2,\ 0 < \delta < \alpha,\ \lambda = \frac{\delta}{n} + \frac{1}{p_1} - \frac{1}{2} > \frac{1}{2}$
- **III**$_b$ $: 0 < p_1 < 2 < p_2 \le p_1',\ 0 < \delta < \alpha,\ \lambda = \frac{\delta}{n} + \frac{1}{2} - \frac{1}{p_2} > \frac{1}{2}$

IV $: 0 < p_1 < 2 < p_2 < \infty,\ \delta > \alpha > n \max(1 - \frac{1}{p_1}, \frac{1}{p_2}),\ \lambda > \frac{1}{2}$

- **IV**$_a$ $: 0 < p_1 < 2 < p_1' \le p_2,\ \delta > \alpha > n\,(1 - \frac{1}{p_1}),\ \lambda = \frac{\delta}{n} + \frac{1}{p_1} - \frac{1}{2} > \frac{1}{2}$
- **IV**$_b$ $: 0 < p_1 < 2 < p_2 \le p_1',\ \delta > \alpha > \frac{n}{p_2},\ \lambda = \frac{\delta}{n} + \frac{1}{2} - \frac{1}{p_2} > \frac{1}{2}$

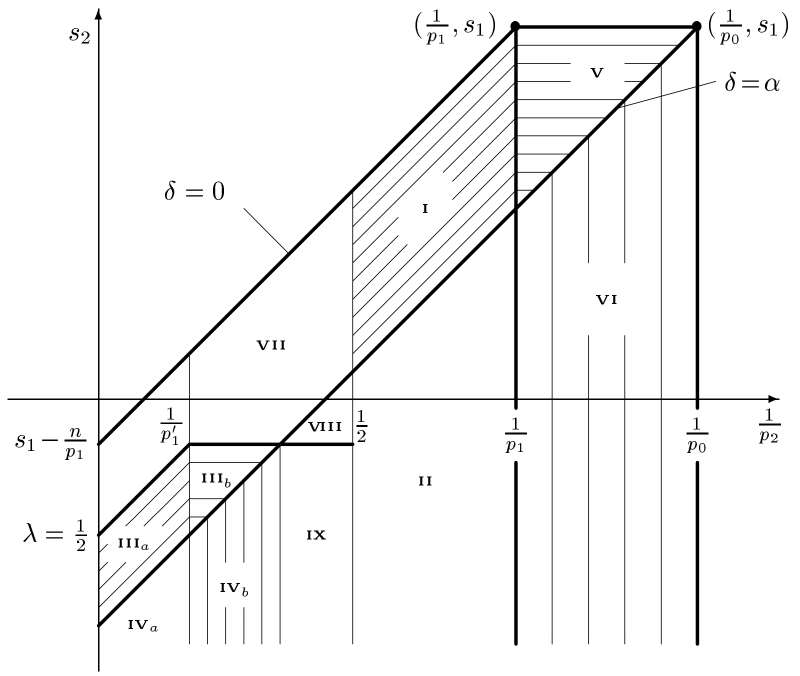
V $: p_0 < p_2 \le p_1,\ 0 < \delta < \alpha$

VI $: p_0 < p_2 \le p_1,\ \delta > \alpha$

VII $: 0 < p_1 < 2 < p_2 < \infty,\ 0 < \delta < \alpha,\ \lambda \le \frac{1}{2}$

VIII $: 0 < p_1 < 2 < p_2 < \infty,\ \alpha < \delta \le n \max(1 - \frac{1}{p_1}, \frac{1}{p_2})$

IX $: 0 < p_1 < 2 < p_2 < \infty,\ \alpha \le n \max(1 - \frac{1}{p_1}, \frac{1}{p_2}) < \delta$



$0 < p_1 < 2, \quad \alpha > n(1 - \frac{1}{p_1})$

Fig. 4

THEOREM. *Let a_k be the k^{th} approximation number of the embedding* (1) *and let the assumptions* (3) *and* (4) *be satisfied. Then using the above notations we have the following results:*

(i) *in region* **I** $\quad a_k \sim k^{-\frac{\delta}{n}}$; (5)

(ii) *in region* **II** $\quad a_k \sim k^{-\frac{\alpha}{n}}$; (6)

(iii) *in region* **III**, *i.e.* $\mathbf{III}_a$ *and* $\mathbf{III}_b$, $\quad a_k \sim k^{-\lambda}$; (7)

(iv) *in region* **IV**, *i.e.* $\mathbf{IV}_a$ *and* $\mathbf{IV}_b$, *there exist a positive constant c and for any $\varepsilon > 0$ a positive constant c_ε such that*

$$c\, k^{-\frac{\alpha}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \le a_k \le c_\varepsilon k^{-\frac{\alpha}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})+\varepsilon}; \tag{8}$$

(v) *in region* **V** $\quad a_k \sim k^{-\frac{s_1-s_2}{n}}$; (9)

(vi) *in region* **VI** $\quad a_k \sim k^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}$; (10)

(vii) *in region* **VII** *there exist two positive constants c_1 and c_2 such that*

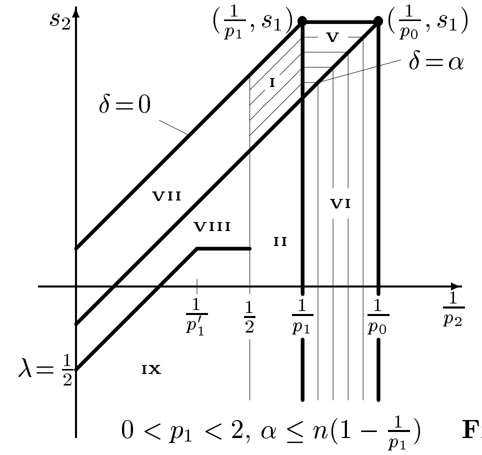
$$c_1 k^{-\frac{\delta}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \le a_k \le c_2 k^{-\frac{\delta}{n}}; \tag{11}$$

(viii) *in region* **VIII** *there exist two positive constants c_1 and c_2 such that*

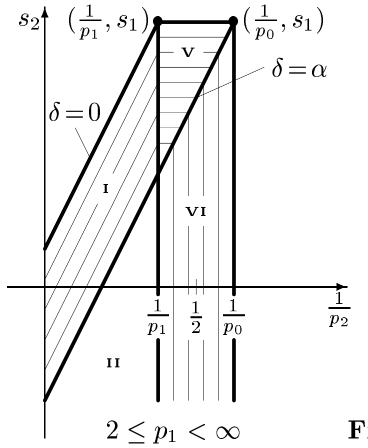
$$c_1 k^{-\frac{\alpha}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \le a_k \le c_2 k^{-\frac{\alpha}{n}}; \tag{12}$$

(ix) *in region* **IX** *there exist a positive constant c and for any $\varepsilon > 0$ a positive constant c_ε such that*

$$c\, k^{-\frac{\alpha}{n}-\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \le a_k \le c_\varepsilon k^{-\frac{\alpha}{2n\max(1-\frac{1}{p_1},\frac{1}{p_2})}+\varepsilon}. \tag{13}$$



$0 < p_1 < 2$, $\alpha \le n(1 - \frac{1}{p_1})$ **Fig. 5**



$2 \le p_1 < \infty$ **Fig. 6**

Remark 1. As we emphasized in front of the theorem the results also hold in the F-case.

Remark 2. Depending on the different values for the parameters p_1 and α we indicated in the diagrams Fig.4 - Fig.6 the level lines for the corresponding exponents. Concerning the above defined regions **VII** - **IX** we omitted this, for looking at (vii)-(ix) in the above theorem the gaps between upper and lower bound appeared too large for having a reasonable intention what the right behaviour of the exponent could be.

Remark 3. Comparing the above theorem with its counterpart (related to entropy numbers) as it is presented in [7: 4.2] we omitted the line '**L**' where $\delta = \alpha$ in our investigations. Up to now we have not succeeded in developing a separate theory there. Nevertheless we could receive upper or lower bounds for a_k via elementary continuous embeddings and the known behaviour for $\delta > \alpha$ and $\delta < \alpha$. On the other hand we can hardly expect to get a nearly sharp result following that way as Remark 4 below will tell us.

Remark 4. We want to hint at a result of Mynbaev and Otel'baev [10: V, §3, Theorem 9] which in terms of our situation for $id : F^{s_1}_{p_1,2}(\alpha) \to F^0_{p_2,2}$ and with

$$\begin{aligned} s_1 > 0,\ s_2 = 0 \quad &, \quad 1 < p_1 \le p_2 \le 2 \quad \text{or} \quad 2 \le p_1 \le p_2 < \infty, \\ \alpha > 0 \quad &, \quad \delta = s_1 - \frac{n}{p_1} + \frac{n}{p_2} > 0, \end{aligned} \tag{14}$$

gives that

$$a_k = a_k(id) \sim \begin{cases} k^{-\frac{\delta}{n}} & , \quad 0 < \delta < \alpha \\ \left(\frac{k}{\log k}\right)^{-\frac{\alpha}{n}} & , \quad \delta = \alpha,\ k \ge k_0 \\ k^{-\frac{\alpha}{n}} & , \quad \delta > \alpha\ . \end{cases} \tag{15}$$

The compatibility of our results and those in the cases $0 < \delta < \alpha$ and $\delta > \alpha$ is the best possible one, namely coincidence. Therefore we should also look for estimates similar to the above ones in the case $\delta = \alpha$. Although the used methods to prove (15) in [10] are completely different from ours we take (15) for granted and try to find a generalization in our sense, i.e. $-\infty < s_2 < s_1 < \infty$, $0 < p_1 \le p_2 \le 2$ or $0 < p_2 \le p_1 < \infty$, $0 < q_1 \le \infty$ and $0 < q_2 \le \infty$. Remembering the situation for the e_k's in [7: 4.2] a dependence on the third indices may well happen. In (15) we have $q_1 = q_2 = 2$ and thus a possible influence could have disappeared.

4. Proofs
4.1 Proof of Lemma 3.1

In the sequel we will denote by $\hat{p} = \min(1,p)$ for any $p, 0 < p < \infty$.

Proof. Step 1 As a preparation we first investigate a special open set $\Omega \subset I\!R^n$, defined as

$$\Omega = \bigcup_{j=1}^{N} K^{(j)}, \quad \overline{K^{(j)}} \cap \overline{K^{(l)}} = \emptyset,\ j \neq l \tag{1}$$

where $N \in I\!N$ is arbitrary and $\{K^{(j)}\}_{j=1}^{N}$ are shifted open unit balls. As usual, $\overline{A}$ means the closure of an open set A. The idea behind is first to handle this simpler case above, i.e. to estimate the respective approximation numbers $a_k^{(\Omega)}$ by a_k and afterwards to cover K_R by finitely many such Ω's from (1).
Let $u \in F^{s_1}_{p_1,q_1}(\Omega)$, then , in a slight abuse of notations,

$$u = \sum_{j=1}^{N} u_j \quad \text{with} \quad u_j \in F^{s_1}_{p_1,q_1}(K^{(j)}) \tag{2}$$

and, by definition,

$$\|u|F^{s_1}_{p_1,q_1}(\Omega)\|^{\widehat{p_1}} = \sum_{j=1}^{N} \|u_j|F^{s_1}_{p_1,q_1}(K^{(j)})\|^{\widehat{p_1}} \tag{3}$$

to adapt it to the localization principle for $F^s_{p,q}$-spaces, see [14: 2.4.7], used in the second step.
Let $\varepsilon > 0$ and choose $T_j \in L(F^{s_1}_{p_1,q_1}(K^{(j)}) \to F^{s_2}_{p_2,q_2}(K^{(j)}))$ such that

$$\text{rank}\, T_j \leq r, \qquad j = 1, \ldots, N, \tag{4}$$

and

$$\|u_j - T_j u_j|F^{s_2}_{p_2,q_2}(K^{(j)})\|^{\widehat{p_2}} \leq (1+\varepsilon)^{\widehat{p_2}}\ a_r^{\widehat{p_2}}\ \|u_j|F^{s_1}_{p_1,q_1}(K^{(j)})\|^{\widehat{p_2}} \tag{5}$$

where we additionally used $a_k^{(K^{(j)})} = a_k, k \in I\!N$, for those (shifted) open unit balls $K^{(j)}$.
Let $T = \sum_{j=1}^{N} T_j$ be such that

$$Tu = \sum_{j=1}^{N} T_j\big(\sum_{l=1}^{N} u_l\big) =: \sum_{j=1}^{N} T_j u_j\ . \tag{6}$$

Then it holds

$$\begin{aligned} \|u - Tu|F^{s_2}_{p_2,q_2}(\Omega)\|^{\widehat{p_2}} &= \sum_{j=1}^{N} \|u_j - T_j u_j|F^{s_2}_{p_2,q_2}(K^{(j)})\|^{\widehat{p_2}} \\ &\leq (1+\varepsilon)^{\widehat{p_2}}\ a_r^{\widehat{p_2}} \sum_{j=1}^{N} \|u_j|F^{s_1}_{p_1,q_1}(K^{(j)})\|^{\widehat{p_2}} \\ &\leq (1+\varepsilon)^{\widehat{p_2}}\ a_r^{\widehat{p_2}}\ \|u|F^{s_1}_{p_1,q_1}(\Omega)\|^{\widehat{p_2}} \end{aligned} \tag{7}$$

where we used (3), (5), $p_1 \leq p_2$ and the special construction of Ω. By (4) and (7) we have for arbitrary small $\varepsilon > 0$

$$\|id_{|\Omega} - T\| \leq (1+\varepsilon)\ a_r \quad , \text{rank}\, T \leq Nr \tag{8}$$

and consequently

$$a_{Nr}^{(\Omega)} \leq a_r\ . \tag{9}$$

Step 2 We consider now the above ball K_R, $R \geq 1$, and look for a suitable covering in the sense of Step 1. Let $\frac{1}{n}\mathbb{Z}^n$ be the lattice such that

$$\theta \in \frac{1}{n}\mathbb{Z}^n \Leftrightarrow \exists\, k \in \mathbb{Z}^n : \theta = \frac{1}{n}k \tag{10}$$

holds for every lattice point θ, which means in terms of its coordinates

$$(\theta_1, \ldots, \theta_n) \in \frac{1}{n}\mathbb{Z}^n \Leftrightarrow \exists\, (k_1, \ldots, k_n) \in \mathbb{Z}^n : \theta_j = \frac{1}{n}k_j,\ j = 1, \ldots, n. \tag{11}$$

Furthermore we have the following sub-lattices $\mathbb{Z}^n_\theta$

$$\mathbb{Z}^n_\theta = \theta + 3\mathbb{Z}^n, \quad \theta \in \frac{1}{n}\mathbb{Z}^n,\ \theta_j \in \{0, \ldots, \frac{3n-1}{n}\}, j = 1, \ldots, n\ . \tag{12}$$

In other words, any sub-lattice $\mathbb{Z}^n_\theta$ is a shifted $3\mathbb{Z}^n$-lattice which is uniquely specified by its "basis point" θ in the cube $\left[0, \frac{3n-1}{n}\right]^n$. Thus

$$\sharp\{\theta \in \frac{1}{n}\mathbb{Z}^n : 0 \leq \theta_j \leq \frac{3n-1}{n}\} = (3n)^n =: L \tag{13}$$

and obviously

$$\bigcup_{r=1}^{L} \mathbb{Z}^n_{\theta_r} = \frac{1}{n}\mathbb{Z}^n \quad , \ \theta_r \in Q, \tag{14}$$

where we introduced the notation

$$Q = \{\theta \in \frac{1}{n}\mathbb{Z}^n : 0 \leq \theta_j \leq \frac{3n-1}{n}, j = 1, \ldots, n\}. \tag{15}$$

Let B^n_r be the following system of translated unit balls

$$B^n_r = \{K(x_l) : x_l \in \mathbb{Z}^n_{\theta_r}\} \tag{16}$$

for $\theta_r \in Q, r = 1, \ldots, L$, and $K(x_l)$ stands for a ball of radius 1 centered at x_l. Consequently (14) and (16) lead to

$$\bigcup_{r=1}^{L} B^n_r = \mathbb{R}^n\ . \tag{17}$$

Consider a resolution of unity $\varphi = \{\varphi^r_l\}_{l \in \mathbb{Z}^n, r=1,\ldots,L}$, assigned to the balls $K(x_l)$ from (16) such that supp $\varphi^r_l \subset K(x_l) \in B^n_r$ and

$$\sum_{r=1}^{L} \sum_{l \in \mathbb{Z}^n} \varphi^r_l(x) = 1, \quad x \in \mathbb{R}^n\ . \tag{18}$$

Setting

$$\varrho_r = \sum_{l \in \mathbb{Z}^n} \varphi^r_l, \quad r = 1, \ldots, L, \tag{19}$$

(18) becomes

$$\sum_{r=1}^{L} \varrho_r(x) = 1, \quad x \in \mathbb{R}^n \tag{20}$$

and

$$\text{supp}\ \varrho_r \subset B^n_r\ . \tag{21}$$

Let $\psi_r \in C^\infty(B_r^n)$ be such that supp $\psi_r \subset B_r^n$ and

$$\psi_r(x) = 1, \quad x \in \text{supp } \varrho_r \ . \tag{22}$$

Let $\varepsilon > 0$ and assume $T_r : F^{s_1}_{p_1,q_1}(K_R \cap B_r^n) \to F^{s_2}_{p_2,q_2}(K_R \cap B_r^n)$ an operator on B_r^n, extended by zero outside $B_r^n \cap K_R$, rank $T_r \le k$ and

$$\|(id - T_r)_{| \ B_r^n \cap K_R}\|^{\widehat{p_2}} \le (1+\varepsilon)^{\widehat{p_2}} \left(a_k^{(B_r^n \cap K_R)}\right)^{\widehat{p_2}} . \tag{23}$$

Caused by the symmetry of our construction we have for large R

$$\|(id - T_r)_{| \ B_r^n \cap K_R}\|^{\widehat{p_2}} \le (1+\varepsilon)^{\widehat{p_2}} \left(a_k^{(B_1^n \cap K_R)}\right)^{\widehat{p_2}} , r = 1, \ldots, L \ . \tag{24}$$

Let $u \in F^{s_1}_{p_1,q_1}(K_R)$. Thus (20), (22), (24), $p_2 \ge p_1$ and the already mentioned localization principle for F-spaces yield

$$\begin{aligned} \|u - \sum_{r=1}^{L} \psi_r T_r \varrho_r u | F^{s_2}_{p_2,q_2}(K_R)\|^{\widehat{p_2}} &\le c_1 \sum_{r=1}^{L} \|\psi_r \varrho_r u - \psi_r T_r \varrho_r u | F^{s_2}_{p_2,q_2}(K_R \cap B_r^n)\|^{\widehat{p_2}} \\ &\le c_2 \sum_{r=1}^{L} \|\varrho_r u - T_r \varrho_r u | F^{s_2}_{p_2,q_2}(K_R \cap B_r^n)\|^{\widehat{p_2}} \\ &\le c_3 \sum_{r=1}^{L} \|(id - T_r)_{| \ B_r^n \cap K_R}\|^{\widehat{p_2}} \cdot \|\varrho_r u | F^{s_1}_{p_1,q_1}(K_R \cap B_r^n)\|^{\widehat{p_2}} \\ &\le c_4 \ (1+\varepsilon)^{\widehat{p_2}} \left(a_k^{(B_1^n \cap K_R)}\right)^{\widehat{p_2}} \|u | F^{s_1}_{p_1,q_1}(K_R)\|^{\widehat{p_2}} \ . \end{aligned} \tag{25}$$

Consequently we have for $T := \sum_{r=1}^{L} T_r$, rank $T \le Lk$,

$$a_{Lk}^R \le c \ a_k^{(B_1^n \cap K_R)} \ . \tag{26}$$

Let N_r be the number of balls $K(x_l)$ belonging to B_r^n which have a non-empty intersection with K_R and put $N := \max\{N_r, r = 1, \ldots, L\}$. Again for large R we get $N \sim N_r, r = 1, \ldots, L$, and after substituting $k \in I\!N$ by Nk, (26) becomes

$$a_{LNk}^R \le c_1 \ a_{Nk}^{(B_1^n \cap K_R)} \le c_2 \ a_k \tag{27}$$

where we used Step 1. Furthermore by usual volume arguments we have $LN \sim c \ R^n$ and so finally

$$a_{c_1 R^n k}^R \le c_2 \ a_k \ . \tag{28}$$

4.2 Proof of the Main Theorem

We divide the long proof into 7 steps. First we prove the estimates from below. Mainly there exist two different methods : to use respective estimates for approximation numbers in function spaces on domains or to shift the problem to the l_p-situation where one already has such estimates. These first two steps will be the same for the B- and F-spaces. Afterwards we show the sufficiency of proving the upper estimates for the F-spaces as we can then reduce the situation of the B-spaces to that one. We have to follow this rather complicated way as we want to make use of Lemma 3.1 which holds in the F-case only. Caring about the estimates from above the main tool will turn out a tricky partition of $I\!R^n$ into annuli in connection with the already investigated situation on domains, see Proposition 2.3 and Corollary 3.1.

Proof. Step 1 Let $0 < \delta < \alpha$ and $0 < p_1 \le p_2 \le 2$ or $2 \le p_1 \le p_2 < \infty$ or $p_0 < p_2 \le p_1 < \infty$, i.e. we handle regions **I** and **V**. By the well-known extension-restriction procedure and Proposition 2.2/1 we have for arbitrary smooth bounded domains $\Omega \subset \mathbb{R}^n$

$$a_k(B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega)) \le c\, a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) = c\, a_k \tag{1}$$

where we additionally used the multiplicativity of approximation numbers. Now recall the already mentioned results for bounded domains, see Proposition 2.3, thus (1) yields

$$a_k \ge c\, k^{-\frac{\delta^+}{n}} \quad , \quad \delta^+ = s_1 - s_2 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ \, . \tag{2}$$

Likewise we handle the situation in the regions **III** and **VII** where (2.3/8) and (2.3/10) provide

$$a_k(B^{s_1}_{p_1,q_1}(\Omega) \to B^{s_2}_{p_2,q_2}(\Omega)) \ge c\, k^{-\lambda}$$

and consequently

$$a_k \ge c\, k^{-\lambda} \tag{3}$$

with $\lambda = \frac{s_1-s_2}{n} - \max(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2}) = \frac{\delta}{n} + \min(\frac{1}{p_1} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_2})$. Hence we have proved the lower estimates in (i), (iii), (v) and (vii).

Step 2 We are now going to prove the lower estimates of (ii), (iv), (vi), (viii) and (ix). Although this could be similarly done for B- and F-spaces we will concentrate on the F-spaces. Regarding the lower estimates in question one observes that no s-parameters are involved in the exponents. It is only $\delta = s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) > \alpha$ assumed to hold. Consequently one can immediately get the estimates in the B-case via the elementary embeddings (2.3/11) and their obvious weighted counterparts

$$B^{s+\varepsilon}_{p,q_0}(\alpha) \subset F^{s}_{p,q_1}(\alpha) \subset B^{s-\varepsilon}_{p,q_2}(\alpha) \tag{4}$$

for $s \in \mathbb{R}, \varepsilon > 0, 0 < p < \infty, 0 < q_0 \le \infty, 0 < q_1 \le \infty, 0 < q_2 \le \infty, \alpha > 0$. In detail, the multiplicativity of approximation numbers then yields

$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ge c\, a_k(F^{s_1+\varepsilon}_{p_1,u_1}(\alpha) \to F^{s_2-\varepsilon}_{p_2,u_2}) \tag{5}$$

where $0 < u_1 \le \infty, 0 < u_2 \le \infty$ and $\varepsilon > 0$ and thus always $(s_1 + \varepsilon) - (s_2 - \varepsilon) - n(\frac{1}{p_1} - \frac{1}{p_2}) > \alpha$ is satisfied.
We now want to make use of an argumentation similar to that one in [4: 4.3.7] and [5: 4.3.1]. We consider the following commutative diagram

$$\begin{tikzcd} F^{s_1}_{p_1,q_1}(\alpha) \arrow[r, "id^F"] & F^{s_2}_{p_2,q_2} \arrow[d, "B"] \\ l^{N_j}_{p_1} \arrow[u, "A"] \arrow[r, "id_l"'] & l^{N_j}_{p_2} \end{tikzcd} \tag{6}$$

where $N_j = 2^{jn}, id_l$ is the identity map from $l^{N_j}_{p_1}$ to $l^{N_j}_{p_2}$ and id^F as in (3.2/2). Recall that l^m_p, $m \in \mathbb{N}$, $0 < p < \infty$, is the linear space of all complex m-tuples $y = (y_j)$, furnished with the quasi-norm

$$\| \, y \mid l^m_p \, \| = \Big(\sum_{j=1}^{m} |y_j|^p \Big)^{\frac{1}{p}} \, .$$

We divide $\mathbb{R}^n$ into the usual annuli $A_j = \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^{j+1}\}$ for $j \in \mathbb{N}$. Let $\Phi \in C^\infty(\mathbb{R}^n)$ with supp $\Phi \subset B_1$, the unit ball, and, say, $\int \Phi(x)dx = 1$. Let A be the following operator

$$A : l^{N_j}_{p_1} \longrightarrow F^{s_1}_{p_1,q_1}(\alpha) \quad , \quad \{\alpha_r\}_{r=1}^{N_j} \longmapsto \sum_{r=1}^{N_j} \alpha_r \Phi(x_r - x) \tag{7}$$

where the x_r are those $k \in \mathbb{Z}^n$ such that $x_r = k \in A_j$. Neglecting constants we thus can assume that there are N_j such points. Applying the localization principle for F-spaces, see [14: 2.4.7], we may assume

$$\|A\{\alpha_r\}|F^{s_1}_{p_1,q_1}(\alpha)\|^{p_1} \sim 2^{j\alpha p_1} \sum_{r=1}^{N_j} |\alpha_r|^{p_1} \tag{8}$$

for $\langle x\rangle^\alpha \sim 2^{j\alpha}$ in A_j. In other words,

$$\|A\| \le 2^{j\alpha} \ . \tag{9}$$

Consider now a map $\Psi \in C^\infty(\mathbb{R}^n)$, supp Ψ concentrated near the origin and $\Psi(x) = 1$ for $x \in$ supp Φ. Denote $\Psi_r(x) := \Psi(x_r - x)$, $r = 1, \ldots, N_j$. Then we put

$$B : F^{s_2}_{p_2,q_2} \longrightarrow l^{N_j}_{p_2}, \quad f \longmapsto \{(f, \Psi_r)\}^{N_j}_{r=1} \ . \tag{10}$$

Estimating the norm of B we get

$$|(f, \Psi_r)| = |\int f(x)\Psi(x_r - x)dx| = |\int f(x)\Psi(x_r - x)\Lambda(x_r - x)dx| \tag{11}$$

where $\Lambda \in C^\infty(\mathbb{R}^n)$, supp Λ concentrated near the origin and $\Lambda(x) = 1$ for $x \in$ supp Ψ. Using $\Lambda_r(x) = \Lambda(x_r - x)$ then (11) becomes

$$\begin{aligned} |(f, \Psi_r)| &= |\int (f\ \Lambda_r)(x)\Psi(x_r - x)dx| = \big|\big((f\Lambda_r) * \Psi\big)(x_r)\big| \\ &\le \sup_{y\in\mathbb{R}^n} \big|\big((f\Lambda_r) * \Psi\big)(y)\big| \le \|f\Lambda_r|B^\sigma_{\infty,\infty}\| \end{aligned} \tag{12}$$

for any $\sigma \in \mathbb{R}$. This follows from the characterization of these spaces via local means, see [14: 2.5.3]. The elementary embedding $F^{s_2}_{p_2,q_2} \subset B^\sigma_{\infty,\infty}$ for $s_2 - \frac{n}{p_2} > \sigma$ yields

$$|(f, \Psi_r)| \le c\ \|f\Lambda_r|F^{s_2}_{p_2,q_2}\| \ . \tag{13}$$

Applying again the above mentioned localization principle for F-spaces to (13) we get

$$\sum_{r=1}^{N_j} |(f, \Psi_r)|^{p_2} \le c\ \|f|F^{s_2}_{p_2,q_2}\|^{p_2} \tag{14}$$

which provides

$$\|B\| \le c \ . \tag{15}$$

By construction we have

$$id_l = B \circ id^F \circ A \ . \tag{16}$$

Hence (9), (15) and the multiplicativity of approximation numbers lead to

$$a_k(id^F) \ge c\ 2^{-j\alpha} a_k(id_l) \ . \tag{17}$$

Concerning $a_k(id_l)$ we make use of [5: 3.2.2 and 3.2.4] which tells us

$$\begin{aligned} a_k(id_l) &\ge c\ 2^{jn(\frac{1}{p_2}-\frac{1}{p_1})} \qquad \text{in region } \mathbf{VI}, && (18)\\ a_k(id_l) &\sim 1 \qquad \text{in region } \mathbf{II}, && (19)\\ a_k(id_l) &\ge c\ 2^{-jn\min(\frac{1}{p_1}-\frac{1}{2},\frac{1}{2}-\frac{1}{p_2})} \qquad \text{in } \mathbf{IV},\ \mathbf{VIII} \text{ and } \mathbf{IX} && (20) \end{aligned}$$

for $k = 2^{jn-1}$. Then (17)-(20) finally result in the estimates from below in (ii), (iv), (vi), (viii) and (ix).

Step 3 We now turn to the estimates from above. First we will show that it is sufficient to deal with the F-case only. Remembering our remark at the beginning of Step 2 this is obvious concerning the regions **II, IV, VI, VIII** and **IX**, whereas the upper estimate in **VII** is a direct consequence of **I** :

$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \le c\, a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_3}_{p_3,q_2}) a_k(B^{s_3}_{p_3,q_2} \to B^{s_2}_{p_2,q_2}) \tag{21}$$

where we choose p_3 such that $0 < p_1 \le p_3 \le 2$ and $s_3 \in I\!R$ such that

$$s_3 - \frac{n}{p_3} = s_2 - \frac{n}{p_2}, \quad s_2 < s_3 < s_1 \ . \tag{22}$$

It remains deriving the cases (i), (iii) and (v) in the B-case from those in the F-case.
We remember again a construction from [4: p.146/147] where $f \in B^s_{p,q}(I\!R^n)$ was divided into $f = \sum_{j=0}^N (\varphi_j \hat{f})^\vee + \sum_{j=N+1}^\infty (\varphi_j \hat{f})^\vee = f_N + f^N$ with $N \in I\!N$ and $\{\varphi_j\}_{j=0}^\infty$ a smooth dyadic partition of unity. Subsequently the above function f_N was splitted up into $f_N = f_{N,1} + f_{N,2}$. We do not want to repeat all the details. We are interested only in the final result that came out : via the above way a linear operator $f \mapsto f - f_{N,1}$ could be constructed approximating the embedding in question in region **I**. The most important point for us is its linearity which allows us to use interpolation arguments even in that case of approximation numbers. Assume the estimates from above in region **I** to be true in the F-case, i.e. we have

$$a_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \le c\, k^{-\frac{\delta}{n}} \tag{23}$$

where $0 < s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) < \alpha, 0 < q_1 \le \infty, 0 < q_2 \le \infty$. We choose now $\sigma_1 < s_1 < \sigma_2$ such that it holds

$$0 < \delta_1 = \sigma_1 - \frac{n}{p_1} - (s_2 - \frac{n}{p_2}) < \alpha, \quad 0 < \delta_2 = \sigma_2 - \frac{n}{p_1} - (s_2 - \frac{n}{p_2}) < \alpha \tag{24}$$

and

$$s_1 = (1-\theta)\sigma_1 + \theta\sigma_2 \tag{25}$$

for some $\theta,\ 0 < \theta < 1$. Then (23) applies also to the embeddings $F^{\sigma_1}_{p_1,u_1}(\alpha) \to F^{s_2}_{p_2,q_2}$ and $F^{\sigma_2}_{p_1,u_2}(\alpha) \to F^{s_2}_{p_2,q_2}$ for arbitrary $0 < u_1 \le \infty, 0 < u_2 \le \infty$. Holding now the target space $F^{s_2}_{p_2,q_2}$ fixed we have for any linear operator T, which maps

$$T : F^{\sigma_1}_{p_1,u_1}(\alpha) \to F^{s_2}_{p_2,q_2}, \quad T : F^{\sigma_2}_{p_1,u_2}(\alpha) \to F^{s_2}_{p_2,q_2}$$

that via real interpolation we get

$$T : \left(F^{\sigma_1}_{p_1,u_1}(\alpha), F^{\sigma_2}_{p_1,u_2}(\alpha)\right)_{\theta,q_1} \longrightarrow \left(F^{s_2}_{p_2,q_2}, F^{s_2}_{p_2,q_2}\right)_{\theta,q_1} , \tag{26}$$

i.e.

$$T : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{27}$$

and

$$\|T|B^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}\| \le c\, \|T\,|\,F^{\sigma_1}_{p_1,u_1}(\alpha) \to F^{s_2}_{p_2,q_2}\|^{1-\theta}\, \|T\,|\,F^{\sigma_2}_{p_1,u_2}(\alpha) \to F^{s_2}_{p_2,q_2}\|^{\theta}. \tag{28}$$

Here it was essential to have the same target space which then, in fact, is not interpolated. For details concerning the real interpolation of B- and F-spaces see [13: 2.4.2] for the unweighted case. The needed extension to weighted spaces then follows from Proposition 2.2/1(ii). Specializing now T by $f \mapsto f - f_{N,1}$ we have from (23), (24), (25) and (28)

$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \le c\, k^{-\frac{\delta}{n}} \ . \tag{29}$$

Afterwards we repeat the same, now fixing the original space $B^{s_1}_{p_1,q_1}(\alpha)$. In other words, (26) and (27) are then replaced by

$$T : \left(B^{s_1}_{p_1,q_1}(\alpha), B^{s_1}_{p_1,q_1}(\alpha)\right)_{\theta,q_2} \longrightarrow \left(F^{\sigma_1}_{p_2,u_1}, F^{\sigma_2}_{p_2,u_2}\right)_{\theta,q_2} \tag{30}$$

and
$$T : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \tag{31}$$
where we choose $\sigma_1 < s_2 < \sigma_2$ such that
$$0 < \delta_1 = s_1 - \frac{n}{p_1} - (\sigma_1 - \frac{n}{p_2}) < \alpha, \quad 0 < \delta_2 = s_1 - \frac{n}{p_1} - (\sigma_2 - \frac{n}{p_2}) < \alpha \tag{32}$$
and
$$s_2 = (1-\theta)\sigma_1 + \theta\sigma_2 \tag{33}$$
are satisfied. Consequently we finally get from $\delta = (1-\theta)\delta_1 + \theta\delta_2$ that
$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \le c\, k^{-\frac{\delta}{n}} \tag{34}$$
in region **I** where always the respective F-result is assumed to hold. In particular we have
$$a_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_1,q_2}) \le c\, k^{-\frac{s_1-s_2}{n}} \quad , \quad 0 < s_1 - s_2 < \alpha \tag{35}$$
and $0 < q_1 \le \infty, 0 < q_2 \le \infty$ and that is just the key to cope with the regions **III** and **V**. The construction is simple but effective. We always have now $0 < \delta = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} < \alpha$ and thus can choose $\sigma_1 \in I\!R$ and $\sigma_2 \in I\!R$ such that for some $\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 < \alpha$ it holds
$$0 < s_1 - \sigma_1 < \alpha_1, \quad 0 < \sigma_1 - \frac{n}{p_1} - \sigma_2 + \frac{n}{p_2} < \alpha - \alpha_1 - \alpha_2, \quad 0 < \sigma_2 - s_2 < \alpha_2 \,. \tag{36}$$
Next we split our embedding $id : B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}$ into five:
$$\begin{aligned}
id_1 &: B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{\sigma_1}_{p_1,\tau_1}(\alpha-\alpha_1) && (37)\\
id_2 &: B^{\sigma_1}_{p_1,\tau_1}(\alpha-\alpha_1) \longrightarrow F^{\sigma_1}_{p_1,u_1}(\alpha-\alpha_1) && (38)\\
id_3 &: F^{\sigma_1}_{p_1,u_1}(\alpha-\alpha_1) \longrightarrow F^{\sigma_2}_{p_2,u_2}(\alpha_2) && (39)\\
id_4 &: F^{\sigma_2}_{p_2,u_2}(\alpha_2) \longrightarrow B^{\sigma_2}_{p_2,\tau_2}(\alpha_2) && (40)\\
id_5 &: B^{\sigma_2}_{p_2,\tau_2}(\alpha_2) \longrightarrow B^{s_2}_{p_2,q_2} && (41)
\end{aligned}$$
where $0 < \tau_1 \le p_1 \le u_1 < \infty, 0 < u_2 \le p_2 \le \tau_2 < \infty$ and σ_1 and σ_2 as in (36). We apply (35) to id_1 and id_5, note the continuity of (38) and (40) and hence the multiplicativity of approximation numbers provides
$$a_k \le c\, k^{-\frac{s_1-\sigma_1}{n} - \frac{\sigma_2 - s_2}{n}} a_k(id_3)\,. \tag{42}$$
Assuming now the respective estimates in the F-case to be true, (42) becomes in region **III**
$$a_k \le c\, k^{-\frac{s_1-\sigma_1}{n} - \frac{\sigma_2-s_2}{n} - \frac{\sigma_1-\sigma_2}{n} - \max(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2})} = c\, k^{-\lambda} \tag{43}$$
and in region **V**
$$a_k \le c\, k^{-\frac{s_1-\sigma_1}{n} - \frac{\sigma_2-s_2}{n} - \frac{\sigma_1-\sigma_2}{n}} = c\, k^{-\frac{s_1-s_2}{n}}\,. \tag{44}$$
Regarding (43) we have only to ensure in region **III** that $\frac{\sigma_1-\sigma_2}{n} > \frac{1}{2} + \max(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2})$ can always be suitably chosen. In other words, by (36) it is necessary to have
$$\frac{\alpha-\alpha_1-\alpha_2}{n} + \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{2} + \max(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2}) \tag{45}$$
which is equivalent to
$$0 < \alpha_1 + \alpha_2 < \alpha - n\, \max(\frac{1}{p_2}, 1 - \frac{1}{p_1})\,. \tag{46}$$
In region **III** we have $\lambda > \frac{1}{2}$ and $\delta < \alpha$ and thus conclude $\alpha > n\,\max(\frac{1}{p_2}, 1-\frac{1}{p_1})$ such that α_1 and α_2 in (46) may be suitably chosen. Consequently the theorem is proved assuming the upper

estimates in the F-case to hold. It remains to verify this supposition.

Step 4 Dealing with the estimates from above in the F-case we rely on a partition of $\mathbb{R}^n$ into annuli up to a certain radius and a simultaneous control of the behaviour outside. For this purpose we make use of Corollary 3.1 several times. Now a_k always means $a_k(id^F)$. Let $l \in \mathbb{N}$ and $a_k^{(l)}$ be again the k^{th} approximation number of the embedding $id^{(l)} : F^{s_1}_{p_1,q_1}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)$, where $A_l = \{x \in \mathbb{R}^n : 2^{l-1} < |x| < 2^{l+1}\}$ for $l = 1, 2, \dots$ and $A_0 = \{x \in \mathbb{R}^n : |x| < 2\}$ are defined as usual. We start considering region **I**. Then Corollary 3.1 and Proposition 2.3 give

$$a_k^{(l)} \le c\, 2^{l\delta} k^{-\frac{\delta}{n}} \, . \tag{47}$$

In the sequel we always investigate suitable unions $\bigcup_{l=0}^{L} A_l$ in $\mathbb{R}^n$ and $L \in \mathbb{N}$ is chosen sufficiently large. We consider operators $B_l : f \mapsto f_{|A_l}, l = 0, 1, \dots, L$, (in the sense of a suitably assigned resolution of unity) and get from the localization principle

$$\|B_l f | F^{s_1}_{p_1,q_1}\| \le c\, 2^{-l\alpha} \|f | F^{s_1}_{p_1,q_1}(\alpha)\| \, . \tag{48}$$

We set

$$B^{L+1} : f \longmapsto \big(id - \sum_{l=0}^{L} B_l\big) f \tag{49}$$

and have

$$\|B^{L+1}\| \le c\, 2^{-\alpha L} \, . \tag{50}$$

Taking the additivity of approximation numbers into consideration (47)-(49) yield for $k = \sum_{l=0}^{L} k_l$

$$\begin{aligned} a_k^{\widehat{p_2}} &= a_k^{\widehat{p_2}}\big(\sum_{l=0}^{L+1} B_l\big) \le c_1 \left(\|B^{L+1}\|^{\widehat{p_2}} + \sum_{l=0}^{L} (a_{k_l}^{(l)})^{\widehat{p_2}} \, \|B_l\|^{\widehat{p_2}} \right) \\ &\le c_2 \left(2^{-L\alpha \widehat{p_2}} + \sum_{l=0}^{L} 2^{l\delta \widehat{p_2}} k_l^{-\frac{\delta}{n}\widehat{p_2}} \, 2^{-l\alpha\widehat{p_2}} \right) \\ &= c_2 \left(2^{-L\alpha \widehat{p_2}} + \sum_{l=0}^{L} 2^{-l(\alpha-\delta)\widehat{p_2}} k_l^{-\frac{\delta}{n}\widehat{p_2}} \right) \end{aligned} \tag{51}$$

where we used again the localization principle for F-spaces and denoted $\widehat{p_2} = \min(1, p_2)$. Let $\varepsilon > 0$ and put $k_l = M\, 2^{-l\varepsilon}$ for some $M > 2^{L\varepsilon}$. (More precisely, we should choose constants $c_l, l = 0, 1, \dots, L$, near 1 such that $k_l = c_l\, M\, 2^{-l\varepsilon} \in \mathbb{N}$, but we neglect this in the following as it causes no trouble.) Then (51) becomes

$$a_{c_1 M}^{\widehat{p_2}} \le c_2 \left(2^{-L\alpha\widehat{p_2}} + M^{-\frac{\delta}{n}\widehat{p_2}} \sum_{l=0}^{L} 2^{-l(\alpha-\delta-\varepsilon\frac{\delta}{n})\widehat{p_2}} \right) \le c_3\, M^{-\frac{\delta}{n}\widehat{p_2}} \tag{52}$$

if L is chosen sufficiently large and $\varepsilon < \frac{n(\alpha-\delta)}{\delta}$. This procedure essentially uses $0 < \delta < \alpha$. Thus (52) is the estimate in question

$$a_k \le c\, k^{-\frac{\delta}{n}} \, .$$

The result for region **VII** now follows similarly as it did in the B-case, see (21). At this point we want to introduce a simplification. Regarding (51) and (52) the number $\widehat{p_2}$ has finally no influence at the result. Therefore we will always assume $\widehat{p_2} = 1$ in the sequel though this is not quite true for $p_2 < 1$. But after all also this exponent cancels itself appearing on both sides.

Step 5 We care about region **III** now. Recall the already known homogeneity estimates, see [7: 5.4/4,5] or [16: 2.2]

$$\|f(R\cdot) | F^{s_1}_{p_1,q_1}\| \le c\, R^{s_1 - \frac{n}{p_1}} \, \|f | F^{s_1}_{p_1,q_1}\| \, , \; s_1 > n\big(\frac{1}{p_1} - 1\big)_+ \, , \; R \ge 1 \tag{53}$$

and

$$\|f(R\cdot)|F^{s_2}_{p_2,q_2}\| \le c\, R^{s_2-\frac{n}{p_2}}\ \|f|F^{s_2}_{p_2,q_2}\|\ ,\ s_2<0\ ,\ R\le 1\ . \tag{54}$$

Applying these results to the annuli A_j we get for $s_1 > n(\frac{1}{p_1}-1)_+$ and $s_2<0$

$$a_k^{(j)}(F^{s_1}_{p_1,q_1}(A_j) \to F^{s_2}_{p_2,q_2}(A_j)) \le c\, 2^{j\delta} k^{-\lambda} \tag{55}$$

where we additionally used Proposition 2.3. Furthermore, we have $\langle x\rangle^\alpha \sim 2^{j\alpha}$ in A_j and hence

$$a_k^{(j)}(F^{s_1}_{p_1,q_1}(A_j,\langle x\rangle^\alpha) \to F^{s_2}_{p_2,q_2}(A_j)) \le c\, 2^{j(\delta-\alpha)} k^{-\lambda}\ . \tag{56}$$

The counterpart of (51) reads then as

$$a_k \le c\left(2^{-\alpha L} + \sum_{j=0}^{L} 2^{j(\delta-\alpha)} k_j^{-\lambda}\right) \tag{57}$$

where we assumed $\widehat{p_2}=1$. Then $k_j = M\ 2^{-j\varepsilon}, \varepsilon>0$, and a suitable choice of $\varepsilon < \frac{\alpha-\delta}{\lambda}$ results in

$$a_{c_1 M} \le c_2\left(2^{-\alpha L} + M^{-\lambda}\right)\ . \tag{58}$$

Assuming $L \ge \frac{\lambda}{\alpha}\log M$ we finally arrive at

$$a_k \le c\, k^{-\lambda} \tag{59}$$

which is the desired result in region **III** under the additional assumptions $s_1 > n(\frac{1}{p_1}-1)_+$ and $s_2<0$. We will remove these restrictions by shifting the problem to an already known situation. The lift operator I_σ on S',

$$I_\sigma f = ((1+|x|^2)^{\frac{\sigma}{2}}\hat{f})^\vee \quad ,\ \sigma\in I\!R, \tag{60}$$

maps $F^s_{p,q}$ isomorphically onto $F^{s-\sigma}_{p,q}$ (for details, see [13: 2.3.8]). This assertion extends to the spaces $F^s_{p,q}(\alpha)$, see [12: Chapter 5] and the references given there.

Suppose first $1\le p_1<2$, i.e. $n(\frac{1}{p_1}-1)_+=0$. We choose s_0 such that $s_2<s_0<s_1$ and $s_1' := s_1-s_0 > n(\frac{1}{p_1}-1)_+$ and $s_2' := s_2-s_0<0$. Then (59) together with $\lambda'=\lambda$ gives

$$a_k(F^{s_1'}_{p_1,q_1}(\alpha) \to F^{s_2'}_{p_2,q_2}) \le c\, k^{-\lambda}$$

and hence (60) guarantees

$$a_k \le c\, k^{-\lambda}\ .$$

The remaining case $0<p_1<1$, i.e. $n(\frac{1}{p_1}-1)>0$, is treated similarly.

Step 6 We handle the cases (iv) and (ix) of the main theorem now where $\lambda = \frac{s_1-s_2}{n} - \max(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2}) = \frac{\delta}{n} + \min(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2}) > \frac{1}{2}, 0<p_1<2<p_2<\infty, \delta>\alpha>0$ and $s_2<s_1$ are assumed to hold. We start dealing with case (iv). We apply the above proved result in region **III** for some $s_3\in I\!R$,

$$s_2<s_3<s_1,\quad \delta_1 = s_1-\frac{n}{p_1}-(s_3-\frac{n}{p_2}) < \alpha\ . \tag{61}$$

In particular, we split up our embedding in question

$$id(F^{s_1}_{p_1,q_1}(\alpha)\to F^{s_2}_{p_2,q_2}) = id(F^{s_3}_{p_2,q_2}\to F^{s_2}_{p_2,q_2})\circ id(F^{s_1}_{p_1,q_1}(\alpha)\to F^{s_3}_{p_2,q_2}) \tag{62}$$

where the embedding $F^{s_3}_{p_2,q_2}\to F^{s_2}_{p_2,q_2}$ is continuous. Then (59) applied to $F^{s_1}_{p_1,q_1}(\alpha)\to F^{s_3}_{p_2,q_2}$ and s_3 chosen such that

$$\lambda_1 = \frac{\delta_1}{n} + \min(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2}) > \frac{1}{2} \tag{63}$$

together with $\delta_1 < \alpha$ finally yields for arbitrary $\varepsilon > 0$

$$a_k \le c_\varepsilon \; k^{-\frac{\alpha}{n}-\min(\frac{1}{2}-\frac{1}{p_2},\frac{1}{p_1}-\frac{1}{2})+\varepsilon} \; , \tag{64}$$

i.e. the desired result in region **IV**. Here the assumption $\delta > \alpha > n \max(1-\frac{1}{p_1},\frac{1}{p_2})$ becomes important for it guarantees the possibility to find $s_3 \in I\!R$ as described in (61) and (63), that is $\delta_1 < \alpha$ and $\lambda_1 > \frac{1}{2}$.
Concerning region (ix) we follow the argumentation of the previous step and arrive at (57) now with $\delta > \alpha$. Choosing $k_j = M\,2^{j\varepsilon}, \varepsilon > 0$, yields (recall $\widehat{p_2} = 1$)

$$a_{c_1\,M\,2^{L\varepsilon}} \le c_2 \; (2^{-\alpha L} + M^{-\lambda} \sum_{j=0}^{L} 2^{j(\delta-\alpha-\lambda\varepsilon)}) \tag{65}$$

which is for $\varepsilon > \frac{\delta-\alpha}{\lambda} > 0$

$$a_{c_1\,M\,2^{L\varepsilon}} \le c_3 \; (2^{-\alpha L} + M^{-\lambda}) \; . \tag{66}$$

Assuming $L \ge \frac{\lambda}{\alpha} \log M$ and afterwards the substitution $k = c\,M^{1+\frac{\lambda}{\alpha}\varepsilon}$ leads to

$$a_k \le c_\varepsilon k^{-\frac{\lambda\alpha}{\alpha+\lambda\varepsilon}} \; . \tag{67}$$

We remember $\varepsilon > \frac{\delta-\alpha}{\lambda}$ and hence

$$a_k \le c_{\varepsilon'} k^{-\frac{\lambda\alpha}{\delta}+\varepsilon'} \tag{68}$$

for any $\varepsilon' > 0$. Looking again for the best possible λ and δ as above (in particular, we introduce again an additional parameter s_3 such that for δ_1 from (61) it holds $\delta_1 > n \max(1-\frac{1}{p_1},\frac{1}{p_2}) \ge \alpha$ and for λ_1 from (63) $\lambda_1 > \frac{1}{2}$) we would have $\delta = n\, \max(1-\frac{1}{p_1},\frac{1}{p_2})$ for $\lambda = \frac{1}{2}$. Consequently (68) becomes then

$$a_k \le c_\varepsilon k^{-\frac{\alpha}{2n\,\max(1-\frac{1}{p_1},\frac{1}{p_2})}+\varepsilon} \tag{69}$$

for arbitrary $\varepsilon > 0$, i.e. the desired upper estimate in region **IX**.

Step 7 We concentrate on the regions **II, V, VI** and **VIII** now. The counterpart of (47) reads for $p_1 = p_2$ now

$$a_k^{(l)} \le c\; 2^{l(s_1-s_2)} k^{-\frac{s_1-s_2}{n}} \; . \tag{70}$$

For $\delta > \alpha$ we determine $k_l, l = 0, \ldots, L$, by

$$k_l^{\frac{s_1-s_2}{n}} = 2^{L(\alpha+\varepsilon)} 2^{l(s_1-s_2-\alpha-\varepsilon)} \tag{71}$$

where $\varepsilon > 0$ satisfies $\varepsilon < s_1 - s_2 - \alpha$. Hence

$$\begin{aligned} \sum_{l=0}^{L} k_l &= 2^{L\frac{n}{s_1-s_2}(s_1-s_2-(s_1-s_2-\alpha-\varepsilon))} \sum_{l=0}^{L} 2^{l(s_1-s_2-\alpha-\varepsilon)\frac{n}{s_1-s_2}} \\ &= 2^{Ln} \sum_{l=0}^{L} 2^{(l-L)(s_1-s_2-\alpha-\varepsilon)\frac{n}{s_1-s_2}} \le c\; 2^{Ln} \end{aligned} \tag{72}$$

and

$$\begin{aligned} \sum_{l=0}^{L} 2^{l(s_1-s_2-\alpha)} k_l^{-\frac{s_1-s_2}{n}} &= \sum_{l=0}^{L} 2^{l(s_1-s_2-\alpha-s_1+s_2+\alpha+\varepsilon)} 2^{-L(\alpha+\varepsilon)} \\ &= 2^{-\alpha L} \sum_{l=0}^{L} 2^{\varepsilon(l-L)} \le c\; 2^{-\alpha L} \end{aligned} \tag{73}$$

and the counterpart of (51) (with $\widehat{p_2} = 1$) obviously results in

$$a_k \leq c\, k^{-\frac{\alpha}{n}} \,. \tag{74}$$

Now (74) leads almost directly to the upper estimates in (ii) and (viii). We choose s_0 as shown in Fig.7 such that

$$s_0 - \tfrac{n}{p_1} - (s_2 - \tfrac{n}{p_2}) = \delta - (s_1 - s_0) > 0$$

and

$$s_1 - s_0 > \alpha \,.$$

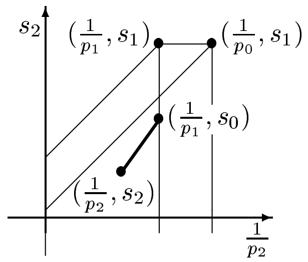


Fig. 7

Then we have $F^{s_0}_{p_1,q_0} \subset F^{s_2}_{p_2,q_2}$ and (74) applied to $F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_0}_{p_1,q_0}$ yield together the upper estimates in region **II** and **VIII**.
We now deal with the regions **V** and **VI**. From (70) we have

$$a^{(l)}_k (F^{s_1}_{p_1,q_1}(A_l) \to F^{s_2}_{p_1,q_2}(A_l)) \leq c\, 2^{l(s_1-s_2)} k^{-\frac{s_1-s_2}{n}} \,.$$

Concerning the remaining embedding $F^{s_2}_{p_1,q_2}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)$ for $p_2 < p_1$ we want to make use of Hölder's inequality. We proceed as in [5: 4.1.1] which is based on local means. Let $\psi_0 \in C^\infty_0(I\!R^n)$ be such that $\int \psi_0(x)dx \neq 0$, let $\psi = \Delta^N \psi_0$ for $N \in I\!N$ and introduce the local means

$$\psi(t,f)(x) = \int \psi(y) f(x+ty)dy \quad , \; x \in I\!R^n, \; t > 0 \tag{75}$$

and define $\psi_0(t,f)(x)$ similarly. Then we have for $2N > \max(s_2, n(\frac{1}{p_1} - 1)_+)$ that for $f \in F^{s_2}_{p_1,q_2}(I\!R^n)$

$$\|\psi_0(1,f)|L_{p_1}(I\!R^n)\| + \big\|(\sum_{j=0}^{\infty} 2^{js_2q_2} |\psi(2^{-j},f)(\cdot)|^{q_2})^{\frac{1}{q_2}} |L_{p_1}(I\!R^n)\big\| \tag{76}$$

is an equivalent quasi-norm in $F^{s_2}_{p_1,q_2}(I\!R^n)$, for details see [14: 2.4.6]. By the usual extension-restriction procedure and Hölder's inequality for $p_2 < p_1$ we consequently get

$$\|id : F^{s_2}_{p_1,q_2}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)\| \leq c\, 2^{nl(\frac{1}{p_2}-\frac{1}{p_1})} \,. \tag{77}$$

Then (70) and (77) give

$$a^{(l)}_k \leq c\, 2^{l\delta} k^{-\frac{s_1-s_2}{n}} \tag{78}$$

both for the regions **V** and **VI**, $s_1 > s_2$ and $\frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n}$. Let first $\delta < \alpha$ and put $k_l = M\, 2^{-l\varepsilon}, l = 0, \ldots, L$, for some $\varepsilon > 0$ and a constant $M > 2^{L\varepsilon}$. The counterpart of (51) (with $\widehat{p_2} = 1$) becomes

$$\begin{aligned} a_{c_1 M} \quad &\leq \quad c_2\, (2^{-\alpha L} + M^{-\frac{s_1-s_2}{n}} \sum_{l=0}^{L} 2^{l(\delta-\alpha+\varepsilon\frac{s_1-s_2}{n})}) \\ &\leq \quad c_3\, (2^{-\alpha L} + M^{-\frac{s_1-s_2}{n}}) \leq c_4\, M^{-\frac{s_1-s_2}{n}} \end{aligned} \tag{79}$$

if $\varepsilon > 0$ is chosen sufficiently small, $\varepsilon < \frac{n(\alpha-\delta)}{s_1-s_2}$, and $L \geq \frac{s_1-s_2}{n\,\alpha} \log M$. Thus (79) gives the result in region **V**,

$$a_k \leq c\, k^{-\frac{s_1-s_2}{n}} \,.$$

It now remains to prove the upper estimate in (vi). Let

$$\varkappa := \frac{(\delta-\alpha)(\frac{1}{p_2}-\frac{1}{p_1})}{(\frac{\alpha}{n}-\frac{1}{p_2}+\frac{1}{p_1})(\frac{\delta}{n}-\frac{1}{p_2}+\frac{1}{p_1})} .$$

Then obviously $\varkappa > 0$ for $\delta > \alpha$ and $\frac{1}{p_1} < \frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n}$, $s_1 > s_2$. Furthermore $\delta - \alpha + \varkappa \frac{s_1 - s_2}{n} > 0$, for

$$\delta - \alpha + \varkappa \frac{s_1 - s_2}{n} = \delta - \alpha + \varkappa(\frac{\delta}{n} - \frac{1}{p_2} + \frac{1}{p_1}) = \frac{\alpha(\delta-\alpha)}{\alpha - \frac{n}{p_2} + \frac{n}{p_1}} > 0 . \tag{80}$$

Let $k_l = M\, 2^{-l\varkappa}$, $l = 0, \ldots, L$, then the counterpart of (51) reads as

$$\begin{aligned} a_{c_1 M} &\leq c_2\,(2^{-\alpha L} + M^{-\frac{s_1-s_2}{n}} \sum_{l=0}^{L} 2^{l(\delta-\alpha+\varkappa\frac{s_1-s_2}{n})}) \\ &= c_2\,(2^{-\alpha L} + M^{-\frac{s_1-s_2}{n}} 2^{L(\delta-\alpha+\varkappa\frac{s_1-s_2}{n})} \sum_{l=0}^{L} 2^{(l-L)(\delta-\alpha+\varkappa\frac{s_1-s_2}{n})}) \\ &\leq c_3\,(2^{-\alpha L} + M^{-\frac{s_1-s_2}{n}} 2^{L(\delta-\alpha+\varkappa\frac{s_1-s_2}{n})}) \end{aligned} \tag{81}$$

where we used the above mentioned properties of $\varkappa$. Substituting the above special $\varkappa$ we get for $L \geq \frac{1}{\alpha}(\frac{\alpha}{n} - \frac{1}{p_2} + \frac{1}{p_1}) \log M$

$$a_{c_1 M} \leq c_2\, M^{-\frac{\alpha}{n}+\frac{1}{p_2}-\frac{1}{p_1}}$$

what we just looked for in the region **VI**. That completes the proof.

Acknowledgements

I want to thank Prof. H. Triebel who suggested writing this paper and who gave me some important hints and helpful remarks.

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3.4 Appendix D

Complements

1. Introduction

We intend to collect some recent outcomes complementing already proved results (see [8], [9], [10]) concerning the situation on the critical line „$\delta = \alpha$“. We use the notation introduced in the just quoted papers. Our general assumptions are the following :
Let

$$\begin{gathered} -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le \infty, \quad 0 < q_1 \le \infty, \quad \alpha > 0, \\ \frac{1}{p_0} = \frac{1}{p_1} + \frac{\alpha}{n}, \quad p_0 < p_2 \le \infty, \quad 0 < q_2 \le \infty, \end{gathered} \tag{1}$$

$$\delta \;=\; s_1 \;-\; \frac{n}{p_1} \;-\; s_2 \;+\; \frac{n}{p_2} \;=\; \alpha \;>\; 0\,. \tag{2}$$

We investigate compact embeddings of type

$$id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{3}$$

and

$$id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}\,. \tag{4}$$

The (weighted) function spaces in question have already been introduced in the beginning, see in particular [9], Fig. 1 in 4.2. We always assume $p_1 < \infty$ and $p_2 < \infty$ in case of the F-spaces, as usual. In particular, we care about the behaviour of the approximation and entropy numbers, resp., of either the embedding (3) or (4). We will mainly concentrate on (4) as afterwards one can easily pursue the idea of the proofs and recognize the necessary modifications. Nevertheless we endeavour to handle both cases (3) and (4) simultaneously as far as it seems to be reasonable.
Furthermore we are preferably interested in the entropy numbers because of their fine link to eigenvalue distributions, given via Carl's inequality, see [9: (1/3)]. This has just been the reason to devote the paper [10] to the applications of our results in [9].
Unfortunately we are not able to give complete results concerning the above described situation, but we may contribute some improvements. Among other things there is a rather surprising outcome : for the first time the third, so-called „q“- indices even occur as exponents, not only in the assumptions. In some cases this „q“-dependence is even the correct exponent, see (5/25).

2. Estimates from above, an approach via duality arguments

Concerning estimates from above on the line $\delta = \alpha$ we proved in [9],

$$e_k(id^B) \le c\, k^{-\frac{s_1-s_2}{n}}\,(\log\langle k\rangle)^{\frac{s_1-s_2}{n}} \tag{1}$$

if

$$\text{either} \quad p_2 = q_2 = \infty \qquad \text{or} \qquad p_2 < \infty, \quad q_2 \;\ge\; \frac{p_2}{p_0}\,q_1\,, \tag{2}$$

see [9: Thm. 4.2 (iv)$_B$]. There is no counterpart for the F-spaces. Thus we will only deal with the B-spaces in this subsection and e_k always means $e_k(id^B)$ (unless otherwise stated). Furthermore we are going to extend (2) slightly, that is we want to show that the inequality (1) is valid for more parameters p and q than admitted by (2). Our result is the following.

Proposition : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Then we have estimate* (1),

$$e_k(id^B) \le c\, k^{-\frac{s_1-s_2}{n}}\, (\log\langle k\rangle)^{\frac{s_1-s_2}{n}}$$

$$\begin{aligned} &\textit{if} \quad \textit{either} \quad p_2 = q_2 = \infty && (3)\\ &\qquad\quad \textit{or} \quad p_2 < \infty \quad \textit{and} \quad \frac{1}{q_2} < \frac{\frac{1}{q_1} - \varkappa}{\frac{1}{p_0}} \cdot \frac{1}{p_2} + \varkappa && (4)\end{aligned}$$

where

$$\varkappa = \begin{cases} \left(1 - \frac{(1-\frac{1}{q_1})_+}{1-\frac{1}{p_1}}\,(1+\frac{\alpha}{n})\right)_+ & , \quad p_1 > 1\,, \\ 1 & , \quad p_1 = q_1 = 1\,, \\ 0 & , \quad \textit{otherwise}\,. \end{cases} \tag{5}$$

Moreover, if $\varkappa \neq 1$ *(i.e. we are not in the case* $p_1 > 1,\ 0 < q_1 \le 1$ *or* $p_1 = q_1 = 1$*, respectively,) then we may even admit*

$$\frac{1}{q_2} = \frac{\frac{1}{q_1} - \varkappa}{\frac{1}{p_0}} \cdot \frac{1}{p_2} + \varkappa\,, \qquad \varkappa \neq 1\,. \tag{6}$$

Remark 1. For some $a \in I\!R$ we denote $a_+ = \max(a, 0)$ as usual.

Remark 2. Taking the above defined parameter $\varkappa$ as a function of p_1 and q_1, i.e. $\varkappa = \varkappa(p_1, q_1)$, one may observe that $\varkappa$ is not continuous in $p_1 = q_1 = 1$. This might also be understood as a hint that the above conditions are only caused by the methods we use in the proof below, but are probably not necessary.

P R O O F : We are going to divide the proof into 5 steps.

<u>Step 1</u>. Obviously one observes from (2) and (3) that we only have to handle the case $p_2 < \infty$. The other assertion has already been proved in [9: 5.5.2/Step 5]. The same argument holds in the case $p_2 < \infty$ whenever $\varkappa = 0$, as (4) together with (6) then simply becomes a reformulation of (2).
Finally, (4) is really an extension of (2) because $\varkappa \ge 0$ and for $\dfrac{1}{p_2} < \dfrac{1}{p_0}$ we have

$$\frac{\frac{1}{q_1}}{\frac{1}{p_0}} \cdot \frac{1}{p_2} \quad \le \quad \frac{\frac{1}{q_1} - \varkappa}{\frac{1}{p_0}} \cdot \frac{1}{p_2} \quad + \quad \varkappa\,.$$

To illustrate those cases, where the above proposition really extends already known results, that is, where $\varkappa > 0$, we sketched three diagrams, one for the case $0 < q_1 \le 1 < p_1$, the second one for $p_1 = q_1 = 1$ and the last one for $1 > \frac{1}{q_1} > \frac{1}{p_0}\frac{1}{1+\frac{\alpha}{n}}$ (which implies $p_1 > q_1$). The hatched areas indicate those domains of parameters $\left(\frac{1}{p_2}, \frac{1}{q_2}\right)$ (in dependence of $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ and α, of course), which have not been covered before.

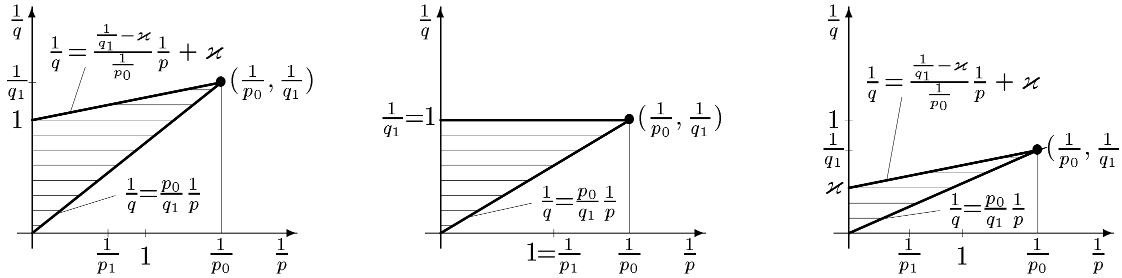


Fig.1 $0<q_1\leq 1<p_1$ $\qquad p_1=q_1=1 \qquad 1>\frac{1}{q_1}>\frac{1}{p_0}\frac{1}{1+\frac{\alpha}{n}}$

<u>Step 2.</u> Assume $p_1>1$ and $q_1>1$. Thus we have to prove (1) for

$$\frac{1}{q_2} \quad\leq\quad \frac{\frac{1}{q_1}-\varkappa}{\frac{1}{p_0}}\cdot\frac{1}{p_2} \quad+\quad \varkappa$$

with

$$\varkappa \;=\; \left(1-\frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}}\left(1+\frac{\alpha}{n}\right)\right)_+ . \tag{7}$$

If $\quad \dfrac{1}{q_1}>\dfrac{1}{p_0}\cdot\dfrac{1}{1+\frac{\alpha}{n}}$, then $\quad \varkappa=\dfrac{\frac{1}{q_1}(1+\frac{\alpha}{n})-\frac{1}{p_0}}{1-\frac{1}{p_1}} > 0 \quad$ and

$$\frac{\frac{1}{q_1}-\varkappa}{\frac{1}{p_0}} \quad=\quad \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}} .$$

In other words, we have to show (1) if

$$\frac{1}{q_2} \quad\leq\quad \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}}\cdot\frac{1}{p_2} \quad+\quad \frac{\frac{1}{q_1}(1+\frac{\alpha}{n})-\frac{1}{p_0}}{1-\frac{1}{p_1}} , \qquad\qquad \frac{1}{q_1}>\frac{1}{p_0}\cdot\frac{1}{1+\frac{\alpha}{n}} . \tag{8}$$

If $\quad \dfrac{1}{q_1}\leq\dfrac{1}{p_0}\cdot\dfrac{1}{1+\frac{\alpha}{n}}$, then (4) becomes

$$\frac{1}{q_2} \quad\leq\quad \frac{\frac{1}{q_1}}{\frac{1}{p_0}}\cdot\frac{1}{p_2} , \qquad\qquad \frac{1}{q_1}\leq\frac{1}{p_0}\cdot\frac{1}{1+\frac{\alpha}{n}} \tag{9}$$

and $\varkappa=0$ as one can easily check. In view of (2) and our remarks in Step 1 it remains to prove (1) under the assumption (8). For this purpose we want to make use of duality arguments and the following lemma, cf. [3: Thm. 1, Rem. 2] or [5: Rem. 1.3.1/5].

L e m m a : *Let A be a uniformly convex Banach space, let B be a Banach space and assume $T\in L(A,B)$ to be compact. Then there is a positive constant $c=c(A)$, such that for all $m\in I\!N$ and $0<r<\infty$ it holds*

$$c^{-1}\cdot\sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T^*) \;\leq\; \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T) \;\leq\; c\cdot\sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(T^*) . \tag{10}$$

The above version uses the symmetric $l_{r,\infty}$ - norm instead of the usual l_p - norm and the monotonicity of entropy numbers. The operator T^* denotes the dual operator of T, as usual.

We want to apply the above lemma to the operator $id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}$. Then we have first to make sure $A = B^{s_1}_{p_1,q_1}(\alpha)$ being a uniformly convex Banach space and $B = B^{s_2}_{p_2,q_2}$ a Banach space. (A Banach space E is said to be uniformly convex if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that for all $x, y \in E$ the conditions

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \qquad \text{imply} \qquad \|\frac{x+y}{2}\| \leq 1-\delta \,,$$

for the definition and details see, for instance, [1: 3/II/§1, p.189] or [11: Def.1.e.1., pp. 59/60].) Studying the above lemma as presented in [3] one recognizes that one only needs one of both spaces A and B to be uniformly convex, that is A or B, and the respective other one is assumed to be a Banach space. Thus we will now restrict ourselves to spaces $B^s_{p,q}(w(\cdot))$ with $1 < p, q < \infty$ in case of the uniformly convex Banach spaces and with $1 \leq p, q \leq \infty$ concerning the Banach spaces. Moreover we may always - without loss of generality - assume the uniformly convex spaces to be unweighted : as we already argued in [9: 2.4] the isomorphic mapping $f \mapsto w_2 f$ from $B^{s_1}_{p_1,q_1}(w_1(\cdot))$ onto $B^{s_2}_{p_2,q_2}\big(\frac{w_1}{w_2}(\cdot)\big)$ admits the restriction to our standard situation $id^B : B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}$ instead of more general problems like $id : B^{s_1}_{p_1,q_1}(w_1(\cdot)) \to B^{s_2}_{p_2,q_2}(w_2(\cdot))$ with $\frac{w_1(x)}{w_2(x)} = \langle x\rangle^\alpha$, w_1 and w_2 being admissible weight functions, because we always neglected the numerical value of constants $c > 0$ and have only been interested in the qualitative behaviour of the entropy numbers in question. Similarly we proceed now. Instead of regarding $id^B : B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}$ we also can consider $id_B : B^{s_1}_{p_1,q_1} \to B^{s_2}_{p_2,q_2}(-\alpha)$ (or, more general, $id : B^{s_1}_{p_1,q_1} \to B^{s_2}_{p_2,q_2}\big(\frac{w_2}{w_1}(\cdot)\big)$) and know $e_k(id^B) \sim e_k(id_B)$. Hence it is always sufficient to investigate situations where the uniformly convex space, either the original or the target one, is unweighted. Furthermore we know $B^s_{p,q}$ is isomorphic to $l_q(l_p)$ for $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, see [19: Thm. 2.11.2/(b), p.237] and these spaces $l_q(l_p)$ have the desired property in the case $1 < p, q < \infty$, for using a result of Mitrinović, Pečarić and Fink it holds

$$\big(\|f+g|\, l_q(l_p)\,\|^r + \|f-g|\, l_q(l_p)\,\|^r\big)^{\frac{1}{r}} \leq 2^{\frac{1}{s'}} \big(\|f|\, l_q(l_p)\,\|^s + \|g|\, l_q(l_p)\,\|^s\big)^{\frac{1}{s}} \,,$$

where $1 < s \leq p, q \leq r$ and $r' \leq s$, see [13: XVIII, Thm.8, p.540]. Choosing now, for instance, $s = \min(p,q) > 1$ and $r = \max(p, q, \frac{s}{s-1}) < \infty$ we get the uniform convexity of $l_q(l_p)$ for $1 < p, q < \infty$. Finally recall Theorem 2.2 in [9] which guarantees the weighted spaces in question to be Banach spaces for $1 \leq p, q \leq \infty$. Summarizing the above observations it causes no problems to apply the lemma to the embedding operator $id : B^{s_1}_{p_1,q_1}(w_1(\cdot)) \longrightarrow B^{s_2}_{p_2,q_2}(w_2(\cdot))$ if

$$\begin{array}{rlcl} \text{either} & 1 < p_1, q_1 < \infty & \text{and} & 1 \leq p_2, q_2 \leq \infty \\ \text{or} & 1 \leq p_1, q_1 \leq \infty & \text{and} & 1 < p_2, q_2 < \infty \,. \end{array} \tag{11}$$

Looking for the dual operator of $id : B^{s_1}_{p_1,q_1}(w_1(\cdot)) \longrightarrow B^{s_2}_{p_2,q_2}(w_2(\cdot))$ it turns out to be $id' = id : B^{-s_2}_{p_2',q_2'}(w_2^{-1}(\cdot)) \longrightarrow B^{-s_1}_{p_1',q_1'}(w_1^{-1}(\cdot))$, where now, additionally to (11), we have to exclude infinity, i.e. $p_1, p_2, q_1, q_2 < \infty$, see [16: (2.11.2/1)] and [15: (5.1.2/6),(5.1.2/7)]. The conjugate indices are defined as usual, e.g. $\frac{1}{p_1} + \frac{1}{p_1'} = 1$. Obviously one has

$$s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) = -s_2 + s_1 - n(\frac{1}{p_2'} - \frac{1}{p_1'}) = \alpha$$

and

$$\frac{w_1(x)}{w_2(x)} \quad = \quad \frac{w_2^{-1}(x)}{w_1^{-1}(x)} \quad = \quad \langle x\rangle^\alpha \,.$$

Again by Theorem 2.2 of [9] or, similarly, [9: (2.4/1)] we have

$$e_k(B^{-s_2}_{p_2',q_2'}(w_2^{-1}(\cdot)) \to B^{-s_1}_{p_1',q_1'}(w_1^{-1}(\cdot))) \quad \sim \quad e_k(B^{-s_2}_{p_2',q_2'}(\alpha) \to B^{-s_1}_{p_1',q_1'}) \,. \tag{12}$$

Then (10) and (12) imply

$$m^{\frac{1}{r}}\, e_m(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c \cdot \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(B^{-s_2}_{p_2',q_2'}(\alpha) \to B^{-s_1}_{p_1',q_1'}) \tag{13}$$

for all $m \in \mathbb{N}$ and $0 < r < \infty$. Let us first consider the case $1 < p_1, q_1 < \infty$ and $1 \le p_2, q_2 < \infty$. Then $p_1' < \infty$ and the entropy numbers on the right-hand side can be estimated from above via (1) and (2),

$$m^{\frac{1}{r}}\, e_m(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c \cdot \sup_{k=1,\ldots,m} k^{\frac{1}{r}-\frac{-s_2+s_1}{n}}\, \big(\log\langle k\rangle\big)^{\frac{-s_2+s_1}{n}} \tag{14}$$

where

$$q_1' \quad \ge \quad p_1' \left(\frac{1}{p_2'} + \frac{\alpha}{n}\right) q_2' \,. \tag{15}$$

Choosing now $\dfrac{1}{r} = \dfrac{s_1 - s_2}{n} > 0$ we arrive at

$$e_m(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c\, m^{-\frac{s_1-s_2}{n}}\, \big(\log\langle m\rangle\big)^{\frac{s_1-s_2}{n}} \tag{16}$$

where

$$p_1,\ q_1 \in (1,\ \infty), \quad p_2,\ q_2 \in [1,\infty), \quad \frac{1}{q_2} \quad \le \quad \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}} \cdot \frac{1}{p_2} \quad + \quad \frac{\frac{1}{q_1}(1+\frac{\alpha}{n}) - \frac{1}{p_0}}{1-\frac{1}{p_1}} \,. \tag{17}$$

Note that the inequality in (17) is nothing else than a reformulation of restriction (15). Moreover, (17) implies $\frac{1}{q_2} < \frac{1}{q_1} < 1$ for $\frac{1}{p_2} < \frac{1}{p_0}$ and the extension to $q_2 = \infty$ comes afterwards from the elementary embedding

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,\infty}) \le c\, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,u_2}) \tag{18}$$

for all $u_2 < \infty$. So we have proved the proposition in the case

$$1 < p_1, q_1 < \infty, \quad 1 \le p_2 < \infty, \quad p_2 > p_0, \quad \frac{1}{q_2} \le \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}} \cdot \frac{1}{p_2} \quad + \quad \frac{\frac{1}{q_1}(1+\frac{\alpha}{n}) - \frac{1}{p_0}}{1-\frac{1}{p_1}} \,. \tag{19}$$

Replacing (13) by the left-hand inequality in (10) (instead of the right-hand one as in (13)) we get in the same way as above the cases $p_1 = \infty$ and $q_1 = \infty$, too (now assuming $B^{s_2}_{p_2,q_2}$ to be uniformly convex). Thus the assertion is shown for

$$p_1 > 1, \quad q_1 > 1, \quad p_2 > \max(1, p_0) \,. \tag{20}$$

<u>Step 3</u>. Next we want to extend (20) to

$$p_1 \;>\; 1, \quad q_1 \;>\; 1, \quad p_2 \;>\; p_0 \,. \tag{21}$$

Thus let $p_0 < 1$. By the same arguments as at the beginning of Step 2 it is sufficient to handle the case $\frac{1}{q_1} > \frac{1}{p_0} \cdot \frac{1}{1+\frac{\alpha}{n}}$ for (1) has already been shown to be valid under the assumption (9) for all $p_2 > p_0$, see [9: Thm. 4.2 (iv)$_B$]. We want to apply interpolation arguments as presented in [9: 3.2].
From [9: Thm. 2.4] we know

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to weak - B^{s_1}_{p_0,q_1}) \;\le\; c \,. \tag{22}$$

On the other hand, Step 2 provides

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma}_{r,u}) \;\le\; c\, k^{-\frac{s_1-\sigma}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-\sigma}{n}} \tag{23}$$

where

$$r > 1, \quad \frac{1}{u} \leq \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}} \cdot \frac{1}{r} + \frac{\frac{1}{q_1}(1+\frac{\alpha}{n})-\frac{1}{p_0}}{1-\frac{1}{p_1}} \tag{24}$$

and

$$\alpha = s_1 - \sigma - \frac{n}{p_1} + \frac{n}{r} > 0 \, . \tag{25}$$

For the given $s_2 < s_1$, p_2 with $p_0 < p_2 \leq 1$ and q_2 with

$$\frac{1}{q_2} \leq \frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}} \cdot \frac{1}{p_2} + \frac{\frac{1}{q_1}(1+\frac{\alpha}{n})-\frac{1}{p_0}}{1-\frac{1}{p_1}}$$

we choose now σ, r and u such that (24), (25) and

$$s_2 = (1-\theta)s_1 + \theta\sigma \, , \quad \frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{r} \, , \quad \frac{1}{q_2} = \frac{1-\theta}{q_1} + \frac{\theta}{u} \tag{26}$$

hold for some θ, $0 < \theta < 1$, as we tried to indicate in the $(\frac{1}{p}, \frac{1}{q})$-diagram below.

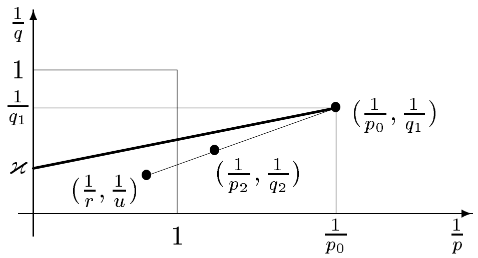


Fig.2

The thick line in Fig.2 refers to the upper bound for $\frac{1}{q_2}$ as given in (17). Using the above-mentioned Theorem 3.2 in [9], the verification of the following inequality

$$\|f|B^{s_2}_{p_2,q_2}\| \leq c \, \|f|weak - B^{s_1}_{p_0,q_1}\|^{1-\theta} \, \|f|B^{\sigma}_{r,u}\|^{\theta} \, , \quad f \in weak - B^{s_1}_{p_0,q_1} \cap B^{\sigma}_{r,u} \tag{27}$$

implies

$$e_{2k}(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \leq c \, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to weak - B^{s_1}_{p_0,q_1})^{1-\theta} \, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma}_{r,u})^{\theta} \, . \tag{28}$$

Taking (27) for granted for a moment, one simply substitutes (22) and (23) into (28) to receive

$$\begin{aligned} e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) &\leq c \, k^{-\frac{s_1-\sigma}{n}\theta} \left(\log\langle k\rangle\right)^{\frac{s_1-\sigma}{n}\theta} \\ &= c \, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} \end{aligned} \tag{29}$$

as desired.
It remains to verify (27). From the real interpolation formula

$$\left(L_{v_0,w_0}, \, L_{v_1,w_1}\right)_{\theta,w} = L_{v,w} \tag{30}$$

where

$$v_0, v_1, w_0, w_1, w \in (0,\infty] \, , \quad \frac{1}{v} = \frac{1-\theta}{v_0} + \frac{\theta}{v_1} \, , \quad v_0 \neq v_1 \tag{31}$$

(cf. [2: 5.3]) we get

$$\|g|L_{p_2}\| \leq c \, \|g|L_{p_0,\infty}\|^{1-\theta} \, \|g|L_r\|^{\theta} \, , \quad g \in L_{p_0,\infty} \cap L_r \, . \tag{32}$$

The rest is a matter of the definition of (*weak*-)B-spaces, Hölder's inequality and (26).

Step 4. We now care about the case $p_1 = q_1 = 1$. We use (13) again, but now for $p_1' = q_1' = \infty$, i.e. $p_1 = q_1 = 1$. Then we get by (2), (11) and (13),

$$m^{\frac{1}{r}}\, e_m(B^{s_1}_{1,1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c \cdot \sup_{k=1,\dots,m} k^{\frac{1}{r}-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} \tag{33}$$

where $0 < r < \infty$, $m \in I\!N$ and

$$\max(p_0, 1) < p_2 < \infty, \quad 1 < q_2 < \infty\ . \tag{34}$$

Concerning q_2 we have proved (1) under the restriction (4), as now $\varkappa = 1$ by definition and (4) means $\frac{1}{q_2} < \frac{1-1}{\frac{\alpha}{n}}\ \frac{1}{p_2} + 1 = 1$, the extension to $q_2 = \infty$ coming again from the elementary embedding (18). The weakening of (34) to $p_0 < p_2 < \infty$ (and $1 < q_2 \le \infty$) is again a matter of interpolation arguments. We proceed completely analogous to Step 3 and won't repeat this here.

Step 5. Finally it remains to handle the case $0 < q_1 \le 1,\ p_1 > 1$. For those q_1 we have $\varkappa = 1$ and shall consequently prove (1) for $p_0 < p_2 < \infty$ and

$$\frac{1}{q_2} \;<\; \frac{\frac{1}{q_1} - 1}{\frac{1}{p_0}} \cdot \frac{1}{p_2} \;+\; 1\ . \tag{35}$$

Assume first $q_2 > 1$. Then we can conclude the existence of some $u_1 > 1$ such that

$$\frac{1}{q_2} \;\le\; \frac{1+\frac{\alpha}{n}-\frac{1}{p_2}}{1-\frac{1}{p_1}} \cdot \frac{1}{u_1} \;-\; \frac{\frac{1}{p_0}-\frac{1}{p_2}}{1-\frac{1}{p_1}} \;<\; 1\ . \tag{36}$$

Now we are in the situation of Step 2 and Step 3, for

$$\frac{1+\frac{\alpha}{n}-\frac{1}{p_2}}{1-\frac{1}{p_1}} \cdot \frac{1}{u_1} \;-\; \frac{\frac{1}{p_0}-\frac{1}{p_2}}{1-\frac{1}{p_1}} \;=\; \frac{1-\frac{1}{u_1}}{1-\frac{1}{p_1}} \cdot \frac{1}{p_2} \;+\; \frac{\frac{1}{u_1}(1+\frac{\alpha}{n})-\frac{1}{p_0}}{1-\frac{1}{p_1}}\ ,$$

and hence

$$e_k(B^{s_1}_{p_1,u_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \le c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}}\ .$$

The multiplicativity of entropy numbers and the elementary embedding

$$B^{s_1}_{p_1,q_1}(\alpha) \;\subset\; B^{s_1}_{p_1,u_1}(\alpha)\ , \quad q_1 \;\le\; u_1\ , \tag{37}$$

then gives (1) for all q_1, $0 < q_1 \le 1$.
The extension to $q_2 \le 1$ is again due to interpolation arguments as in Step 3.
For given $p_1 > 1,\ p_2 > p_0,\ 0 < q_1 \le 1$ and q_2 satisfying (35), choose σ, r and u such that

$$r > 1,\ u > 1,\ \sigma < s_2 < s_1,\ \alpha = s_1 - \sigma - n\left(\frac{1}{p_1} - \frac{1}{r}\right) > 0$$

and

$$s_2 = (1-\theta)s_1 + \theta\sigma, \quad \frac{1}{p_2} = \frac{1-\theta}{p_0} + \frac{\theta}{r}, \quad \frac{1}{q_2} = \frac{1-\theta}{q_1} + \frac{\theta}{u} \tag{38}$$

for some θ, $0 < \theta < 1$.

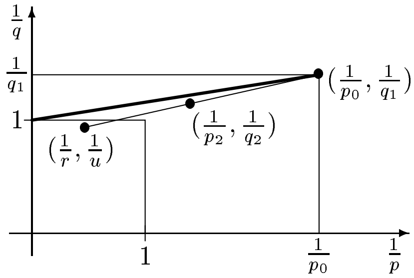


Fig.3

From (22) and (23), now for $0 < q_1 \le 1$ and $u > 1$, we get in the same way as in Step 3

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \le c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}}$$

for

$$0 < q_1 \le 1, \quad \frac{1}{q_2} < \frac{\frac{1}{q_1}-1}{\frac{1}{p_0}} \cdot \frac{1}{p_2} + 1 .$$

This is the very end of the proof.

Remark 3. Concerning the above-mentioned lemma (10) one could also think about applications to estimates from below, that is, extensions to the F-case (where we only have a lower estimate). On the other hand, one gains nothing by doing this way, as duality preserves the exponents in question, i.e. $\frac{s_1-s_2}{n}$ and $\frac{\alpha}{n}$, resp.. One might weaken further restrictions to the parameters as presented above, but there are no such limitations concerning the F-case, see Theorem 4.2(iv)$_F$ in [9]. Consequently the method shown above is only useful in the B-case.

3. Estimates from above, an approach via approximation numbers

In this subsection we want to improve a former outcome concerning approximation numbers as described in [8: (4.2/51)]. Afterwards we want to apply some general results for the relation between entropy and approximation numbers to gain estimates from above for the entropy numbers on the line „$\delta = \alpha$“.

Let A_l be the usual annuli in $\mathbb{R}^n$, $A_l = \{x \in \mathbb{R}^n : 2^{l-1} < |x| < 2^{l+1}\}$, $l = 1,2,\ldots$ and $A_0 = \{x \in \mathbb{R}^n : |x| < 2\}$, consider coverings $\bigcup_{l=0}^{L} A_l \subset \mathbb{R}^n$, $L \in \mathbb{N}$, with a suitably assigned resolution of unity, and let the operator B_l be the following

$$B_l \quad : \quad f \longmapsto f|_{A_l} \quad , \quad l = 0,\ldots,L \tag{1}$$

(in the sense of the above chosen resolution of unity) and

$$B^{L+1} \quad : \quad f \longmapsto \big(id - \sum_{l=0}^{L} B_l\big) f \, . \tag{2}$$

Note, that by construction (1) and the multiplicativity property of approximation numbers we have

$$a_k(B_l) \quad \le \quad c_{\alpha,l}\, a_k^{(l)} \quad , \quad l = 0,\ldots,L \tag{3}$$

where $a_k^{(l)}$ are the respective approximation numbers of the embeddings $id^{(l)} : F^{s_1}_{p_1,q_1}(A_l) \to F^{s_2}_{p_2,q_2}(A_l)$ (or likewise in the B-case).

Proposition : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Consider the embeddings*

$$id^F \ : \ F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{4}$$

and

$$id^B \ : \ B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \ , \quad q_2 \ \geq \ p_2 \ . \tag{5}$$

Denoting the k-th approximation number of the embedding (4) *or* (5) *by* a_k*, we have*

$$a_k \ \leq \ c \left(2^{-L\alpha p_2} + \sum_{l=0}^{L} \left(2^{-l\alpha} \ a^{(l)}_{k_l} \right)^{p_2} \right)^{1/p_2} \ , \qquad k = \sum_{l=0}^{L} k_l \tag{6}$$

(modification if $p_2 = \infty$ *).*

Remark 1. Indeed, (6) improves the already mentioned estimate in [8: (4.2/51)] as there we only had

$$a_k \ \leq \ c \left(2^{-L\alpha\lambda} + \sum_{l=0}^{L} \left(2^{-l\alpha} \ a^{(l)}_{k_l} \right)^{\lambda} \right)^{1/\lambda} \ , \qquad k = \sum_{l=0}^{L} k_l$$

with $\lambda = \min(1, p_2)$ in the F-case (or $\lambda = \min(1, p_2, q_2)$ in the B-case).

Before proving the above proposition we need a lemma.

Lemma : *Let* $-\infty < s < \infty, 0 < p \leq \infty$ *(* $p < \infty$ *in the F-case) and* $0 < q \leq \infty$*. Let* $f_j \in B^s_{p,q}$*, or* $f_j \in F^s_{p,q}$ *, resp.,* $j = 0, \ldots, L$*, and let* $\operatorname{supp} f_j \subset \left(A_{j-1} \cup A_j \cup A_{j+1} \right)$*, i.e. for every* $x \in \mathbb{R}^n$ *there are at most three non-vanishing terms in the sum* $\sum_{j=0}^{L} f_j(x)$*. Then we have the following estimates:*

$$\text{(i)} \qquad \| \sum_{j=0}^{L} \ f_j \mid F^s_{p,q} \|^p \ \leq \ c \ \sum_{j=0}^{L} \| \ f_j \mid F^s_{p,q} \ \|^p \tag{7}$$

$$\text{(ii)} \qquad \| \sum_{j=0}^{L} \ f_j \mid B^s_{p,q} \|^p \ \leq \ c \ \sum_{j=0}^{L} \| \ f_j \mid B^s_{p,q} \ \|^p \qquad \textit{if} \qquad p \ \leq \ q \ . \tag{8}$$

Remark 2. Obviously the certain number of at most three overlapping supports is absolutely arbitrary. Any fixed number $m \in \mathbb{N}$ would do it. Furthermore, the constants in (7) and (8) do not depend on $L \in \mathbb{N}$, but may depend on s, p, q and that $m \in \mathbb{N}$.

P R O O F : Step 1. We prove (i). We use the characterization via local means, see [17: 2.4.6]. Let k_0 and k^0 be C^∞-functions on $\mathbb{R}^n$ with

$$\operatorname{supp} k_0 \ \subset \ B, \quad \operatorname{supp} k^0 \ \subset \ B \tag{9}$$

(where B denotes the unit ball in $\mathbb{R}^n$) and $\widehat{k_0}(0) \neq 0$, $\widehat{k^0}(0) \neq 0$. Put

$$k(y) \ = \ \Delta^N k^0(y), \quad y \in \mathbb{R}^n, \tag{10}$$

for some $N \in \mathbb{N}$. Introduce the local means

$$k(t,g)(x) = \int_{\mathbb{R}^n} k(y) \, g(x+ty) \, dy \ , \quad x \in \mathbb{R}^n, \ t > 0, \tag{11}$$

for $g \in S'$, and likewise for k_0. The above mentioned theorem [17: 2.4.6] then gives

$$\| \, g \mid F^s_{p,q} \, \| \;\sim\; \|k_0(1,g)|L_p\| \,+\, \big\| \big(\sum_{j=0}^{\infty} 2^{jsq} |k(2^{-j},g)(\cdot)|^q \big)^{1/q} \big| L_p \big\| \tag{12}$$

for any $g \in F^s_{p,q}$.
Using our assumptions about the „almost disjoint“ supports we thus can conclude

$$\begin{aligned} \| \sum_{l=0}^{L} f_l \big| F^s_{p,q} \|^p \quad &\le \quad c \left(\big\| k_0(1, \sum_{l=0}^{L} f_l) \big| L_p \big\|^p + \big\| \big(\sum_{j=0}^{\infty} 2^{jsq} |k(2^{-j}, \sum_{l=0}^{L} f_l)(\cdot)|^q \big)^{1/q} \big| L_p \big\|^p \right) \\ &\le \quad c' \cdot \sum_{l=0}^{L} \left(\|k_0(1,f_l)|L_p\|^p + \big\| \big(\sum_{j=0}^{\infty} 2^{jsq} |k(2^{-j}, f_l)(\cdot)|^q \big)^{1/q} \big| L_p \big\|^p \right) \end{aligned}$$

and furthermore, using (12) again,

$$\big\| \sum_{l=0}^{L} f_l \big| F^s_{p,q} \big\|^p \;\le\; c \, \sum_{l=0}^{L} \; \|f_l|F^s_{p,q}\|^p \; . \tag{13}$$

We only used the fact, that - independent of L - there is for any $x \in \mathbb{R}^n$ at most a certain number, say 3, of non-vanishing terms in the sum

$$k_0\big(1, \sum_{l=0}^{L} f_l\big)(x) = \sum_{l=0}^{L} k_0(1,f_l)(x) = \sum_{l=0}^{L} \int_{|y|<1} k_0(y)\, f_l(x+y)\, dy \tag{14}$$

and a similar expression for $k(2^{-j}, \sum\limits_{l=0}^{L} f_l)$.

<u>Step 2</u>. Obviously (ii) is for $p=q$ only a special case of (i). Following the same argumentation as above we have to replace (12) by its B-counterpart, see [17: (2.5.3/2)],

$$\| \, g \mid B^s_{p,q} \, \| \;\sim\; \|k_0(1,g)|L_p\| \,+\, \big(\sum_{j=0}^{\infty} 2^{jsq} \|k(2^{-j},g)|L_p\|^q \big)^{1/q} \; , \tag{15}$$

the notations being the same as above. Applying this to $\sum\limits_{l=0}^{L} f_l$ with the „almost disjoint“ supports, the first term on the right-hand side of (15) causes no trouble. On the other hand, the latter term yields

$$\begin{aligned} \big(\sum_{j=0}^{\infty} 2^{jsq} \|k(2^{-j}, \sum_{l=0}^{L} f_l)|L_p\|^q \big)^{1/q} \quad &\le \quad c \, \big(\sum_{j=0}^{\infty} 2^{jsq} \big(\sum_{l=0}^{L} \|k(2^{-j}, f_l)|L_p\|^p \big)^{q/p} \big)^{1/q} \\ &= \quad c \, \big(\sum_{j=0}^{\infty} \big(\sum_{l=0}^{L} 2^{jsp} \|k(2^{-j}, f_l)|L_p\|^p \big)^{q/p} \big)^{1/q} \; . \end{aligned} \tag{16}$$

By the same arguments as above we arrive at (8), if we can show estimates of type

$$\big(\sum_{j=0}^{\infty} \big(\sum_{l=0}^{L} \; a^p_{jl} \; \big)^{q/p} \big)^{1/q} \;\le\; c \, \big(\sum_{l=0}^{L} \big(\sum_{j=0}^{\infty} \; a^q_{jl} \; \big)^{p/q} \big)^{1/p} \tag{17}$$

to be true for some constant $c > 0$, independent of L, and $0 < p \le q \le \infty$, $a_{jl} \ge 0$. But this is (for $q < \infty$) exactly Jessen's inequality (cf. [14: C.3.10]) and can be derived from Minkowski's inequality for $r = \frac{q}{p} \ge 1$ applied to a_{jl}^p,

$$\Big(\sum_{j=0}^{\infty}\Big(\sum_{l=0}^{L} a_{jl}^p\Big)^{q/p}\Big)^{p/q} \le \sum_{l=0}^{L}\Big(\sum_{j=0}^{\infty} a_{jl}^{pq/p}\Big)^{p/q} . \tag{18}$$

In the case $0 < p < q = \infty$ we have to show

$$\sup_j \Big(\sum_{l=0}^{L} a_{jl}^p\Big)^{1/p} \le \Big(\sum_{l=0}^{L}(\sup_j \; a_{jl})^p\Big)^{1/p} \tag{19}$$

but this is obvious. Likewise if $p = q = \infty$. So we have proved (ii) for $0 < p \le q \le \infty$.

We are now able to prove the proposition.

P r o o f (Proposition) : Let $\sigma > 0$ be small. Then, by definition,

$$\forall\, l = 0, \ldots, L \quad \exists\, T_l,\ \operatorname{rank} T_l < k_l \ :\ \|B_l - T_l\| < (1+\sigma)\, a_{k_l}(B_l) . \tag{20}$$

We may assume that (possibly by another suitable resolution of unity) we additionally have

$$\operatorname{supp} T_j \subset \big(A_{j-1} \cup A_j \cup A_{j+1}\big),\ j = 1, \ldots, L, \quad \operatorname{supp} T_0 \subset A_1 . \tag{21}$$

Applying now the preceding lemma, (3) and (20), we get

$$\begin{aligned}
\Big\|\Big(id - \sum_{l=0}^{L} T_l\Big) f | F^{s_2}_{p_2,q_2}\Big\|^{p_2} &= \Big\|\Big(\sum_{l=0}^{L}(B_l - T_l) + B^{L+1}\Big) f | F^{s_2}_{p_2,q_2}\Big\|^{p_2} \\
&\le c_1 \sum_{l=0}^{L} \|(B_l - T_l) f | F^{s_2}_{p_2,q_2}\|^{p_2} + c_2\, \|B^{L+1} f | F^{s_2}_{p_2,q_2}\|^{p_2} \\
&\le c_3 \Big(\sum_{l=0}^{L} 2^{-l\alpha p_2} \|B_l - T_l\|^{p_2} + 2^{-L\alpha p_2}\Big)\, \|\ f | F^{s_1}_{p_1,q_1}(\alpha)\|^{p_2} \\
&\le c_4 \left(\sum_{l=0}^{L} [2^{-l\alpha}(1+\sigma) a^{(l)}_{k_l}]^{p_2} + 2^{-L\alpha p_2}\right) \|f | F^{s_1}_{p_1,q_1}(\alpha)\|^{p_2} . \qquad (22)
\end{aligned}$$

In other words,

$$\begin{aligned}
a_k &\le c\left((1+\sigma)^{p_2} \sum_{l=0}^{L} [2^{-l\alpha} a^{(l)}_{k_l}]^{p_2} + 2^{-L\alpha p_2}\right)^{1/p_2} \\
&= c\,(1+\sigma)\left(\sum_{l=0}^{L} [2^{-l\alpha} a^{(l)}_{k_l}]^{p_2} + 2^{-L\alpha p_2}\right)^{1/p_2}, \qquad k = \sum_{l=0}^{L} k_l \qquad (23)
\end{aligned}$$

for any $\sigma > 0$. Let now $\sigma \downarrow 0$, we arrive at (6). The proof in the B-case is similar, now using part (ii) of the above lemma, which we only have for $p_2 \le q_2$.

Remark 3. The main reason to improve the usual additivity of approximation numbers in this case is obviously that one, having operators which only act on certain annuli, that is, with a finite number of overlapping supports. Furthermore, the characterization of B- and F-spaces via local means is well-adapted to this behaviour.

Corollary 1 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Let* a_k *be the* k^{th} *approximation number of either the embedding*

$$id^F \ : \ F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2} \tag{24}$$

or

$$id^B \ : \ B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2} \ , \quad q_2 \ \geq \ p_2 \ . \tag{25}$$

(i) $0 < p_1 \leq p_2 \leq 2$ *or* $2 \leq p_1 \leq p_2 \leq \infty$, *then* $a_k \leq c\, k^{-\frac{\alpha}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}+\frac{1}{p_2}}$ (26)

(ii) $p_0 < p_2 \leq p_1 \leq \infty$, *then* $a_k \leq c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}}$ (27)

(iii) $0 < p_1 < 2 < p_2 < \infty$, *then* $a_k \leq c\, k^{-\lambda} \left(\log\langle k\rangle\right)^{\lambda+\frac{1}{p_2}}$ (28)

where $\lambda = \dfrac{s_1-s_2}{n} - \max\left(\dfrac{1}{2}-\dfrac{1}{p_2}, \dfrac{1}{p_1}-\dfrac{1}{2}\right) \ > \ \dfrac{1}{2}$.

(We always assume in the F*-case* $p_1 < \infty$ *and* $p_2 < \infty$ *, as usual.)*

P R O O F : Using a result of [8: (3.1/3)] together with the upper estimates of [7: 3.2.5] we have

$$a^{(l)}_k \ \leq \ c\, 2^{l\delta} \begin{cases} k^{-\frac{\delta}{n}} & , \ 0 < p_1 \leq p_2 \leq 2 \quad \text{or} \quad 2 \leq p_1 \leq p_2 \leq \infty \\ k^{-\frac{s_1-s_2}{n}} & , \ 0 < p_2 \leq p_1 \leq \infty \\ k^{-\lambda} & , \ 0 < p_1 < 2 < p_2 < \infty, \quad \lambda > \frac{1}{2} \end{cases} \tag{29}$$

see [8: (4.2/47), (4.2/55), (4.2/78)]. Next we substitute these estimates in the above proved estimate (6) and get in the first case $0 < p_1 \leq p_2 \leq 2$ or $2 \leq p_1 \leq p_2 \leq \infty$, for instance,

$$a_k \ \leq \ c \left(2^{-L\alpha p_2} + \sum_{l=0}^{L} 2^{-l(\alpha-\delta)p_2}\, k_l^{-\frac{\delta}{n}p_2}\right)^{1/p_2} , \qquad k = \sum_{l=0}^{L} k_l$$

(modification if $p_2 = \infty$), and so, by (1/2),

$$a_k \ \leq \ c \left(2^{-L\alpha p_2} + \sum_{l=0}^{L} k_l^{-\frac{\alpha}{n}p_2}\right)^{1/p_2} , \qquad k = \sum_{l=0}^{L} k_l \ . \tag{30}$$

Now let $k_l = K$, $l = 0, \ldots, L$, for some positive constant $K > 0$ and choose afterwards $L \in I\!N$ such that

$$L\, 2^{L\alpha p_2} \ \sim \ K^{\frac{\alpha}{n}p_2} \ , \tag{31}$$

then (30) becomes

$$a_{K\,L} \ \leq \ c \left(2^{-L\alpha p_2} + L\, K^{-\frac{\alpha}{n}p_2}\right)^{1/p_2} , \tag{32}$$

i.e. by (31)

$$a_{c\,K\,\log\langle K\rangle} \ \leq \ c\, K^{-\frac{\alpha}{n}} \left(\log\langle K\rangle\right)^{1/p_2} , \tag{33}$$

and so

$$a_m \ \leq \ c\, m^{-\frac{\alpha}{n}} \left(\log\langle m\rangle\right)^{\frac{\alpha}{n}+\frac{1}{p_2}}, \quad 0 < p_1 \leq p_2 \leq 2 \quad \text{or} \quad 2 \leq p_1 \leq p_2 \leq \infty \ . \tag{34}$$

(Without loss of generality we will always assume e.g. $KL \in I\!N$, $cK\log\langle K\rangle \in I\!N$ to make expressions as in (32) and (33) reasonable.) In a similar way we handle the other two cases in (29) to obtain

$$a_m \ \leq \ c\, m^{-\frac{s_1-s_2}{n}} \left(\log\langle m\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}}, \quad p_0 < p_2 \leq p_1 < \infty \tag{35}$$

and

$$a_m \ \leq \ c\, m^{-\lambda} \left(\log\langle m\rangle\right)^{\lambda+\frac{1}{p_2}}, \quad 0 < p_1 < 2 < p_2 < \infty, \quad \lambda > \frac{1}{2} \ . \tag{36}$$

Corollary 2 : *Let the situation be as in Corollary* 1 *and denote by* e_k *the* k^{th} *entropy number of the embedding* (25), $id^B : B^{s_1}_{p_1,q_1}(\alpha) \longrightarrow B^{s_2}_{p_2,q_2}$, $q_2 \geq p_2$. *Then*

$$e_k \;\leq\; c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} . \tag{37}$$

Remark 4. As may be seen from the proof below we could get all the estimates (26)-(28) with e_k instead of a_k. On the other hand, using former results in [9: (5.5/17)] we have

$$e_k \;\leq\; c\left(2^{-L\alpha\gamma} + \sum_{l=0}^{L} k_l^{-\frac{s_1-s_2}{n}\gamma}\right)^{1/\gamma} , \qquad k = \sum_{l=0}^{L} k_l \tag{38}$$

with $\gamma = \min(1, p_2, q_2)$. Proceeding in the same way as before we thus get

$$e_k \;\leq\; c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{\gamma}} \tag{39}$$

which is obviously sharper than (26) and (28). To improve (39) in the case $q_2 \geq p_2$ we use again interpolation arguments.

P R O O F : Step 1. First we make use of some general estimate between entropy and approximation numbers,

$$\sup_{j=1,\ldots,m} j^{\varrho}\, e_j \;\leq\; c\cdot \sup_{j=1,\ldots,m} j^{\varrho}\, a_j\ , \quad m \in \mathbb{N}, \tag{40}$$

for arbitrary $\varrho > 0$ and some constant $c > 0$, independent of $m \in \mathbb{N}$, see [18: (1/7)] or [4: p.96], the latter one restricted to Banach spaces only. Combining (27) and (40) yields

$$m^{\varrho}\, e_m \;\leq\; \sup_{j=1,\ldots,m} j^{\varrho}\, e_j \;\leq\; c\cdot \sup_{j=1,\ldots,m} j^{\varrho-\frac{s_1-s_2}{n}} \left(\log\langle j\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{41}$$

provided that $p_0 < p_2 \leq p_1 < \infty,\ q_2 \geq p_2$.
Put $\varrho = \frac{s_1-s_2}{n} > 0$, then (41) becomes

$$m^{\frac{s_1-s_2}{n}}\, e_m \;\leq\; c\left(\log\langle m\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} ,$$

i.e.

$$e_m \;\leq\; c\, m^{-\frac{s_1-s_2}{n}} \left(\log\langle m\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} , \qquad p_0 < p_2 \leq p_1 < \infty,\ q_2 \geq p_2 . \tag{42}$$

In particular,

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma_1}_{p_1,u_1}) \;\leq\; c\, k^{-\frac{s_1-\sigma_1}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-\sigma_1}{n}+\frac{1}{p_1}} \tag{43}$$

if

$$\alpha = s_1 - \sigma_1, \quad u_1 \geq p_1 . \tag{44}$$

Step 2. In view of (42) it remains to prove (37) for $p_2 > p_1$. Secondly we use the already proved estimate

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma_0}_{\infty,\infty}) \;\leq\; c\, k^{-\frac{s_1-\sigma_0}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-\sigma_0}{n}} \tag{45}$$

if

$$\alpha \;=\; s_1 - \sigma_0 - \frac{n}{p_1} , \tag{46}$$

see [9: (4.2/12), (4.2/14)]. Finally, to prove the Corollary, we apply again Theorem 3.2 in [9]. Let $\sigma_0 = s_1 - \frac{n}{p_1} - \alpha$, $\sigma_1 = s_1 - \alpha$ and choose $\theta, 0 < \theta < 1$, such that

$$s_2 \;=\; (1-\theta)\sigma_0 + \theta\sigma_1, \qquad \frac{1}{p_2} \;=\; \frac{\theta}{p_1}, \qquad \frac{1}{q_2} \;=\; \frac{\theta}{u_1} \tag{47}$$

for some $u_1 \geq p_1$. Then the above mentioned theorem tells us that

$$e_{2k}(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\leq\; c\, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma_0}_{\infty,\infty})^{1-\theta}\, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma_1}_{p_1,u_1})^{\theta} \tag{48}$$

if we can guarantee

$$\|f|B^{s_2}_{p_2,q_2}\| \;\leq\; c\, \|f|B^{\sigma_0}_{\infty,\infty}\|^{1-\theta}\, \|f|B^{\sigma_1}_{p_1,u_1}\|^{\theta}\,, \quad f \in B^{\sigma_0}_{\infty,\infty} \cap B^{\sigma_1}_{p_1,u_1}\,. \tag{49}$$

But estimates of the above type have already been shown several times, see (2/30)-(2/32). So (48) together with (43) and (45) gives

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\leq\; c\, k^{-\frac{s_1-\sigma_0}{n}(1-\theta) - \frac{s_1-\sigma_1}{n}\theta}\, \big(\log\langle k\rangle\big)^{\frac{s_1-\sigma_0}{n}(1-\theta) + (\frac{s_1-\sigma_1}{n} + \frac{1}{p_1})\theta}\,. \tag{50}$$

Note, that (47) then yields

$$e_k \;\leq\; c\, k^{-\frac{s_1-s_2}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n} + \frac{1}{p_2}}\,, \qquad q_2 \geq p_2, \quad p_2 > p_1,$$

and that ends the proof.

Remark 5. Applying the same trick as above in Step 2 one could even improve the estimate (37) for $p_0 < p_2 < p_1$. Instead of „interpolating“ $B^{s_2}_{p_2,q_2}$ between $B^{\sigma_0}_{\infty,\infty}$ and $B^{\sigma_1}_{p_1,u_1}$ as in (48) one had to use then $B^{\sigma_0}_{p_1,u_0}$ and $weak - B^{s_1}_{p_0,q_1}$, where we have instead of (43) and (45)

$$\begin{aligned} e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma_0}_{p_1,u_0}) &\;\leq\; c\, k^{-\frac{s_1-\sigma_0}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-\sigma_0}{n} + \frac{1}{p_1}}\,, \\ e_k(B^{s_1}_{p_1,q_1}(\alpha) \to weak - B^{s_1}_{p_0,q_1}) &\;\leq\; c\,, \end{aligned} \tag{51}$$

where

$$s_2 = (1-\theta)\sigma_0 + \theta s_1, \quad \frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{p_0}, \quad \frac{1}{q_2} = \frac{1-\theta}{u_0} + \frac{\theta}{q_1}, \quad u_0 \geq p_1\,. \tag{52}$$

The result is rather technically complicated than beautiful,

$$e_k \;\leq\; c\, k^{-\frac{s_1-s_2}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n} + \frac{1}{p_1}\frac{n}{\alpha}(\frac{1}{p_0} - \frac{1}{p_2})} \tag{53}$$

if

$$\frac{1}{q_2} \;\leq\; \frac{\frac{1}{q_1} - \frac{1}{p_1}}{\frac{\alpha}{n}} \cdot \frac{1}{p_2} + \frac{\frac{1}{p_1}(\frac{1}{p_0} - \frac{1}{q_1})}{\frac{\alpha}{n}}\,, \qquad p_0 < p_2 < p_1\,. \tag{54}$$

Therefore we omit this slight improvement in the exponent which - on the other hand - causes some rather nasty restrictions of kind (54) in some cases.
Likewise we could involve duality arguments as presented in 2. to achieve

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\leq\; c\, k^{-\frac{s_1-s_2}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n} + \frac{1}{p_1'}} \tag{55}$$

where either $1 < p_1, q_1 < \infty, 1 \leq p_2 < \infty, 1 \leq q_2 \leq \infty$ or $1 \leq p_1, q_1 \leq \infty, 1 < p_2 < \infty, 1 < q_2 \leq \infty$ and $q_1' \geq p_1'$. Here we applied (37) to the embedding $id' : e_k(B^{-s_2}_{p_2',q_2'}(\alpha) \to B^{-s_1}_{p_1',q_1'})$ and proceeded similar to (2/13)-(2/16). Thus a possible extension to (37) is

$$\begin{aligned} e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) &\;\leq\; c\, k^{-\frac{s_1-s_2}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n} + 1 - \frac{1}{p_1}}\,, \\ 1 < q_1 \leq p_1 < \infty, 1 \leq p_2 < \infty, 1 \leq q_2 \leq \infty \quad &\text{or} \quad 1 \leq q_1 \leq p_1 \leq \infty, 1 < p_2 < \infty, 1 < q_2 \leq \infty\,. \end{aligned} \tag{56}$$

Obviously (56) is again sharper than (39), for $\frac{1}{\gamma} \geq 1 \geq 1 - \frac{1}{p_1}$. Note that we get for $p_1 = q_1 = 1$ the estimate (2/1) with $\max(p_0, 1) < p_2 < \infty$, $1 < q_2 \leq \infty$. Moreover, (56) gives rise to a quite better estimate for $1 < p_1 = q_1 < \infty$:

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\leq\; c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} \tag{57}$$

$$\text{if} \qquad 1 - \frac{1-\frac{1}{p_1}}{\frac{1}{p_0}}\,\frac{1}{p_2} \;<\; \frac{1}{q_2} \;\leq \begin{cases} 1 & , \quad 1 \leq p_0 < p_2 \quad \text{or} \quad p_0 < 1 \leq p_2 \\ -\frac{1-\frac{1}{p_1}}{\frac{1}{p_0}-1}\,\frac{1}{p_2} + \frac{\frac{\alpha}{n}}{\frac{1}{p_0}-1} & , \quad p_0 < p_2 \leq 1 \end{cases} . \tag{58}$$

This condition is rather nasty and may only be understood by the figure below and the method to prove (57). The hatched areas in these $\big(\frac{1}{p}, \frac{1}{q}\big)$-diagrams indicate those domains where $\big(\frac{1}{p_2}, \frac{1}{q_2}\big)$ are admissible in the sense of (58).

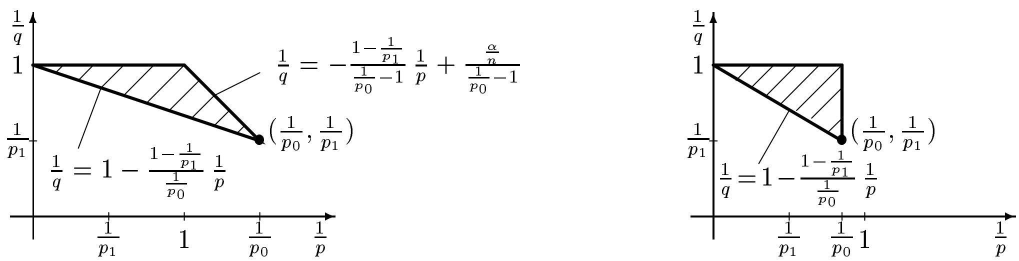


Fig.4a $\qquad p_0 < 1 < p_1 \qquad\qquad 1 \leq p_0 < p_1$

We only want to give the idea of the proof here as it is again based on interpolation arguments. From (56) we know

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{\sigma}_{r,1}) \;\leq\; c\, k^{-\frac{s_1-\sigma}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-\sigma}{n}+1-\frac{1}{p_1}} \tag{59}$$

for $1 < p_1 < \infty$, $1 \leq r < \infty$, and from [9: (2.4/5)]

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to weak - B^{s_1}_{p_0,p_1}) \;\leq\; c\,. \tag{60}$$

Now we choose $r \geq 1$ and σ such that

$$s_2 = (1-\theta)\sigma + \theta s_1, \quad \frac{1}{p_2} = \frac{1-\theta}{r} + \frac{\theta}{p_0}, \quad \frac{1}{q_2} = \frac{1-\theta}{1} + \frac{\theta}{p_1}\,. \tag{61}$$

Thus letting $r \to \infty$ we get the lower bound for $\frac{1}{q_2}$ as given in (58) and likewise for $r \downarrow 1$ the upper one together with (56). We only want to remark that indeed

$$\frac{s_1-\sigma}{n}\,(1-\theta) = \frac{s_1-s_2}{n} \qquad \text{and} \qquad \Big(1 - \frac{1}{p_1}\Big)(1-\theta) = \frac{1}{q_2} - \frac{1}{p_1}\,. \tag{62}$$

Furthermore it holds

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \;\leq\; c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{q_1}} \tag{63}$$

if $1 < p_2 < \infty$ and

$$0 < \frac{1}{q_1} < -\frac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}}\,\frac{1}{p_1} + \frac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}} \quad \text{for} \quad \Big(\frac{1}{p_2} - \frac{\alpha}{n}\Big)_+ < \frac{1}{p_1} \leq 1 \tag{64}$$

$$\text{or} \quad q_1 = \infty \quad \text{and} \quad \Big(\frac{1}{p_2} - \frac{\alpha}{n}\Big)_+ < \frac{1}{p_1} < \infty\,. \tag{65}$$

The result is again based on duality arguments applied to (57) and (58), where we additionally involved (37) in the case $q_1 = \infty$. Probably this may become clearer by the diagrams in Fig. 4b below. Here duality is indicated by a pointwise reflection of the above Fig.4a at the point $(\frac{1}{2}, \frac{1}{2})$ (under the general assumptions (1/1) and (1/2), of course). Now the thick hatched areas are those domains of admissible parameters $(\frac{1}{p_1}, \frac{1}{q_1})$ as given in (64) and (65) depending on $(\frac{1}{p_2}, \frac{1}{q_2})$ and $\alpha > 0$.

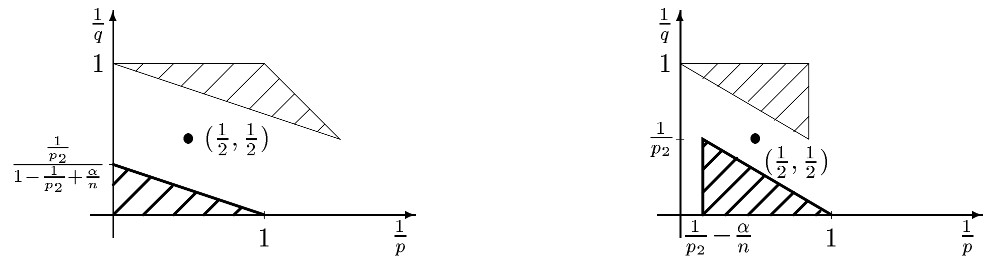


Fig.4b $\quad 0 < \frac{1}{p_2} < \frac{\alpha}{n} \qquad\qquad 0 \le \frac{1}{p_2} - \frac{\alpha}{n} < 1$

The results in (57) and (63) are very interesting in spite of their rather nasty restrictions (58) or (64), (65), resp., as the third indices appear in the exponent, not only in the assumptions, as usual. But we will discuss this point later on in detail.

Corollary 3 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied and denote by* e_k *the* k^{th} *entropy number of the embedding* (24), $id^F : F^{s_1}_{p_1,q_1}(\alpha) \longrightarrow F^{s_2}_{p_2,q_2}$. *Then*

$$e_k \;\le\; c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \; . \tag{66}$$

P r o o f : $\quad$ Step 1. We proceed completely analogous to Step 1 of Corollary 2 and likewise arrive at

$$e_k \;\le\; c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} , \qquad p_0 < p_2 \le p_1 < \infty \; . \tag{67}$$

As an essential difference to (42) we now may admit all parameters q_2, $0 < q_2 \le \infty$, in (67). The counterparts of (43) and (44) then read as

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{\sigma_0}_{p_1,u_0}) \;\le\; c\, k^{-\frac{s_1-\sigma_0}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-\sigma_0}{n}+\frac{1}{p_1}} \tag{68}$$

and

$$\alpha = s_1 - \sigma_0 \; . \tag{69}$$

Step 2. $\quad$ It remains to prove (66) for $p_1 < p_2 < \infty$. Remembering our arguments in Step 2 of the preceding proof one recognizes that they cannot be transferred to the F-case. Nevertheless we can make use of the above Corollary 2 to replace that argument by another one. Note that we have by elementary embeddings

$$F^{s_1}_{p_1,q_1}(\alpha) \subset B^{s_1}_{p_1,q}(\alpha) \; , \qquad B^{\sigma_1}_{r_1,r_1} \subset F^{\sigma_1}_{r_1,u_1} \tag{70}$$

if

$$q \;\ge\; \max(p_1, q_1) \; , \qquad r_1 \;\le\; u_1 \; , \qquad r_1 < \infty \; . \tag{71}$$

Then (37) and the multiplicativity of entropy numbers provide

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{\sigma_1}_{r_1,u_1}) \;\le\; c\, k^{-\frac{s_1-\sigma_1}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-\sigma_1}{n}+\frac{1}{r_1}} \tag{72}$$

if

$$\alpha \;=\; s_1 - \sigma_1 - \frac{n}{p_1} + \frac{n}{r_1} \; , \qquad r_1 \;\le\; u_1 \; , \qquad r_1 < \infty \; . \tag{73}$$

Now (72) and (73) may be considered as counterparts of (45) and (46). The rest is the same, we apply again Theorem 3.2 in [9], choose $\sigma_0 = s_1 - \alpha$, $\sigma_1 = s_1 - \frac{n}{p_1} + \frac{n}{r_1} - \alpha$, $0 < u_0 \le \infty$, $u_1 \ge r_1$ such that

$$s_2 = (1-\theta)\sigma_0 + \theta\sigma_1 , \quad \frac{1}{p_2} = \frac{1-\theta}{p_1} + \frac{\theta}{r_1}, \quad \frac{1}{q_2} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1} \tag{74}$$

holds for some $\theta,\ 0 < \theta < 1$.
To verify

$$\|f|F^{s_2}_{p_2,q_2}\| \ \le\ c\, \|f|F^{\sigma_0}_{p_1,u_0}\|^{1-\theta} \|f|F^{\sigma_1}_{r_1,u_1}\|^{\theta} \tag{75}$$

is again a matter of Hölder's inequality and (74). Consequently [9: Thm. 3.2] yields

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le\ c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n} + \frac{1}{p_2}} \tag{76}$$

where we additionally involved (68), (72) and (74). Although we have assumed $u_1 \ge r_1$ in (73) the final result (76) holds for all $q_2,\ 0 < q_2 \le \infty$ (and $p_2,\ p_0 < p_2 < \infty$), as we may choose u_0 in (74) as we want.

Remark 6. We want to use duality arguments to extend (66). Remembering the discussion in Step 2 of the Proof in Section 2, especially the helpful lemma of [3] cited in (2/10) and the following inequality of Mitrinović, Pečarić and Fink, which remains true replacing $l_q(l_p)$ by $L_p(l_q)$, $1 < p, q < \infty$, see [13: XVIII, Thm.8, p.540], one observes that we likewise may apply Lemma (2/10) to F-spaces. Restricting the parameters to $1 < p, q < \infty$ and $s \in I\!R$ we know $(F^s_{p,q})' = F^{-s}_{p',q'}$, see [16: (2.11.2/2)]. Thus we get in the same way as described concerning the B-case

$$m^{\frac{1}{r}}\, e_m(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le\ c \sup_{k=1,\ldots,m} k^{\frac{1}{r}}\, e_k(F^{-s_2}_{p_2',q_2'}(\alpha) \to F^{-s_1}_{p_1',q_1'}) \tag{77}$$

where

$$p_1,\ p_2,\ q_1,\ q_2 \ \in\ (1,\infty)\ . \tag{78}$$

(The additional restrictions $p_1' < \infty$ and $p_2' < \infty$ in the F-case exclude $p_1 = 1$ or $p_2 = 1$ - unless we have $p_1 = q_1 = 1$ or $p_2 = q_2 = 1$ and are in fact in the B-case.) Then (66) and (77) provide

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le\ c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n} + 1 - \frac{1}{p_1}} \tag{79}$$

where

$$1 < p_1 < \infty, \quad 1 < p_2 < \infty, \quad 0 < q_1 < \infty, \quad 1 < q_2 \le \infty\ . \tag{80}$$

Note that the extension of (78) to (80) is due to elementary embeddings,

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le\ c\, e_k(F^{s_1}_{p_1,u_1}(\alpha) \to F^{s_2}_{p_2,u_2}) \tag{81}$$

if

$$q_1 \ \le\ u_1, \quad q_2 \ \ge\ u_2 \quad . \tag{82}$$

4. Estimates from below

We now turn to investigate estimates from below. As in the last subsections we assume in the sequel again the general assumptions (1/1) and (1/2) to hold.
In our paper [9] we proved one estimate from below on the line „$\delta = \alpha$", originally concerning only F-spaces,

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \ge\ c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}}\ . \tag{1}$$

Obviously this can immediately be extended to the B-case

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ \ge\ c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} \tag{2}$$

if

$$p_1 \le q_1, \quad q_2 \le p_2 \tag{3}$$

(and $p_1 < \infty$, $p_2 < \infty$) via elementary embeddings. Our aim is now to improve (2) in some sense, whereas (3) can unfortunately not be weakened at this moment. This is exactly the reason why there are very few parameters such that we have both upper and lower estimates on the line. But we want to postpone this concluding discussion and separately summarize in the last subsection. Nevertheless we emphasized some interesting aspect in our result already in the very beginning of this chapter, namely the third indices make their appearance not only in restrictions of kind (3), but even in the exponents in question. At first glance this may be surprising, on the other hand it has been to be supposed that the situation on the line „$\delta = \alpha$“ is delicate enough to let even the third indices play a more important role.

Proposition : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Assume additionally* (3) *to hold. Then*

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ge c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} . \tag{4}$$

P R O O F : Note that we have for $\delta = \alpha$

$$\frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{p_1} = \frac{\alpha}{n} + \frac{1}{q_2} - \frac{1}{p_2} \ge \frac{\alpha}{n} \tag{5}$$

by the assumption (3), i.e. (4) really improves (2) as we announced in the beginning.

Step 1. We first handle the case $p_2 = q_2$. But obviously (2) and (5) then cover the assertion.

Step 2. Let $q_2 < p_2$. We consider the following inequality

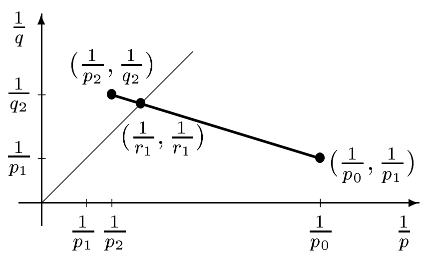
$$e_{2k}(B^{s_1}_{p_1,p_1}(\alpha) \to B^{\sigma_1}_{r_1,r_1}) \le c\, e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2})^{1-\theta}\, e_k(B^{s_1}_{p_1,p_1}(\alpha) \to weak - B^{s_1}_{p_0,p_1})^{\theta} \tag{6}$$

where

$$\sigma_1 = (1-\theta)s_2 + \theta s_1 , \quad \frac{1}{r_1} = \frac{1-\theta}{p_2} + \frac{\theta}{p_0} = \frac{1-\theta}{q_2} + \frac{\theta}{p_1} , \quad \frac{1}{r_1} < \frac{1}{p_0} \tag{7}$$

is assumed to hold for some θ, $0 < \theta < 1$, see also the figure below.

Fig.5



We apply some interpolation property of entropy numbers as described in [9: 3.2]. Thus it is sufficient to verify

$$\|f|B^{\sigma_1}_{r_1,r_1}\| \le c\, \|f|B^{s_2}_{p_2,q_2}\|^{1-\theta}\, \|f|weak - B^{s_1}_{p_0,p_1}\|^{\theta} \tag{8}$$

to have (6). Analogously to Step 3 in the proof of the Proposition in 2., see (2/30)-(2/32), we get (8). Furthermore,

$$\frac{1}{r_1} = \frac{1-\theta}{p_2} + \frac{\theta}{p_0} = \frac{1-\theta}{q_2} + \frac{\theta}{p_1} , \quad \text{i.e.} \quad \frac{1}{q_2} - \frac{1}{p_2} = \frac{\theta}{1-\theta}\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = \frac{\theta}{1-\theta}\,\frac{\alpha}{n} > 0 , \tag{9}$$

as we assumed in this step. Again by a former result of [9: 2.4] we have

$$B^{s_1}_{p_1,p_1}(\alpha) \subset weak - B^{s_1}_{p_0,p_1} ,$$

i.e.

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to weak - B^{s_1}_{p_0,p_1}) \le c . \tag{10}$$

Moreover, (2) leads for $p_1 < \infty$, $r_1 < \infty$ to

$$e_{2k}(B^{s_1}_{p_1,p_1}(\alpha) \to B^{\sigma_1}_{r_1,r_1}) \ge c\, k^{-\frac{s_1-\sigma_1}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} \tag{11}$$

with

$$\alpha = s_1 - \sigma_1 - \frac{n}{p_1} + \frac{n}{r_1} = s_1 - s_2 - \frac{n}{p_1} + \frac{n}{p_2} > 0 . \tag{12}$$

In the case $p_1 = \infty$ or $r_1 = \infty$ (11) remains true, see [9: Rem. 4.2/2].
Combining (6), (10) and (11) yields

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ge c \left(k^{-\frac{s_1-\sigma_1}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}}\right)^{\frac{1}{1-\theta}} \tag{13}$$

and by (7) and (9),

$$\begin{aligned} \frac{s_1-\sigma_1}{n}\cdot\frac{1}{1-\theta} &= \frac{s_1-s_2}{n} , \\ \frac{\alpha}{n}\cdot\frac{1}{1-\theta} &= \left(1+\frac{\theta}{1-\theta}\right)\frac{\alpha}{n} = \frac{\alpha}{n} + \frac{1}{q_2} - \frac{1}{p_2} . \end{aligned}$$

Remembering (5) we have just proved

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ge c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} . \tag{14}$$

The extension to $B^{s_1}_{p_1,q_1}(\alpha)$, $p_1 \le q_1$, as original space is now only a matter of elementary embeddings and the multiplicativity of entropy numbers.

Remark 1. One might have the intention to rescue the above idea, in particular that one of Step 2, to F-spaces as we also have the analogues of (10) and (11) there. But the essential trick to apply estimate (6) was the ability to prove inequality (8). Unfortunately we cannot show such a necessary analogue of (8) for $weak - F$-spaces and have to content ourselves with (1).

Remark 2. The idea additionally to involve duality arguments turns out to be useful again. We only want to give the result here as we have often described the way to apply Theorem 1 in [3] (as cited in (2/10)) to our situation. Thus we get

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ge c\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{q_1}} \tag{15}$$

if

$$1 < p_1 \le q_1 < \infty,\ 1 \le q_2 \le p_2 < \infty \quad \text{or} \quad 1 \le p_1 \le q_1 \le \infty,\ 1 < q_2 \le p_2 < \infty . \tag{16}$$

Obviously one has

$$\frac{1}{q_1'} - \frac{1}{p_2'} = \frac{1}{p_2} - \frac{1}{q_1} \tag{17}$$

and

$$p_1 \le q_1, \quad q_2 \le p_2 \quad \text{iff} \quad p_2' \le q_2', \quad q_1' \le p_1' . \tag{18}$$

5. Some discussion

Before going into detail we have to confess a rather poor harvest after some strong efforts in the preceding subsections. Especially technical difficulties dominated elegance and simplicity of the involved tools. Nevertheless there are very few cases where our investigations resulted in sharp estimates. Apart from a lot of technically complicated conditions we want to encourage the reader to get to the heart of the following considerations, that is estimate (25) in the Proposition. Although the range of restrictions looks rather terrible the content is worth considering :

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}}$$

that is, the correct behaviour of the log-exponent in some cases is characterized via the q-indices, see (26)-(29). So to speak, the general wisdom that appearances are often deceptive is (sometimes) also true in the case of propositions.

Comparing our results in 2. and 4. we can only find a few common cases for the parameters :

(i) Let $p_2 = q_2 = \infty$. Then we have simultaneously the upper estimate (2/1) and the lower one (4/4) if $p_1 \le q_1$,

$$c_1\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}-\frac{1}{p_1}} \;\le\; e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{\infty,\infty}) \;\le\; c_2\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} \tag{1}$$

where c_1, c_2 are positive constants and we assume (1/1), (1/2) and $p_1 \le q_1$ to hold. Obviously (1) leads to a sharp estimate for $p_1 = q_1 = \infty$,

$$e_k(B^{s_1}_{\infty,\infty}(\alpha) \to B^{s_2}_{\infty,\infty}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} \tag{2}$$

where $s_1 - s_2 = \alpha > 0$. Using the notation $B^{s_1}_{\infty,\infty}(\alpha) = \mathcal{C}^{s_1}(\alpha)$ and $B^{s_2}_{\infty,\infty} = \mathcal{C}^{s_2}$ this means

$$e_k(\mathcal{C}^{s_1}(\alpha) \to \mathcal{C}^{s_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} . \tag{3}$$

This has been the only sharp result in [9] concerning the situation on the line $\delta = \alpha$, see [9: (4.2/19)].

(ii) Let $p_1 = q_1 = 1$ and $1 < q_2 \le p_2 < \infty$. Then (2/1) together with (4/4) result in

$$c_1\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-1} \;\le\; e_k(B^{s_1}_{1,1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c_2\, k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}} \tag{4}$$

where $c_1, c_2 > 0$, $1 < q_2 \le p_2 < \infty$. But this fails to provide sharp results as $q_2 > 1$ by (2/4). Using (4/15) instead of (4/4) yields no better estimate.

(iii) It remains to consider the other cases in (2/4) and (2/5) together with assumption (4/3) or (4/16), resp. But any common case is there excluded : if $0 < p_1 \le 1$ (and not $p_1 = q_1 = 1$), then $\varkappa = 0$ in Proposition 2., and if $1 < p_1 \le q_1$, then

$$\frac{1-\frac{1}{q_1}}{1-\frac{1}{p_1}}\,\left(1+\frac{\alpha}{n}\right) \;>\; 1 \tag{5}$$

and hence $\varkappa = 0$ again. On the other hand, $\varkappa = 0$ implies

$$\frac{1}{q_2} \le \frac{p_0}{q_1}\cdot\frac{1}{p_2} < \frac{1}{p_2}, \quad \text{as} \quad \frac{1}{q_1} \le \frac{1}{p_1} < \frac{1}{p_0}\,, \tag{6}$$

consequently we only have upper estimates for either $p_1 \le q_1$, $p_2 < q_2$ or $p_2 \ge q_2$, $p_1 > q_1$. But this contradicts (4/3) and (4/16), resp. Finally there always remains a gap between upper and lower bound concerning admissible q_2, depending on p_1, q_1, α and p_2, as we tried to make clear in the figure below. (We had a similar picture concerning (4/16) and (2/4).)

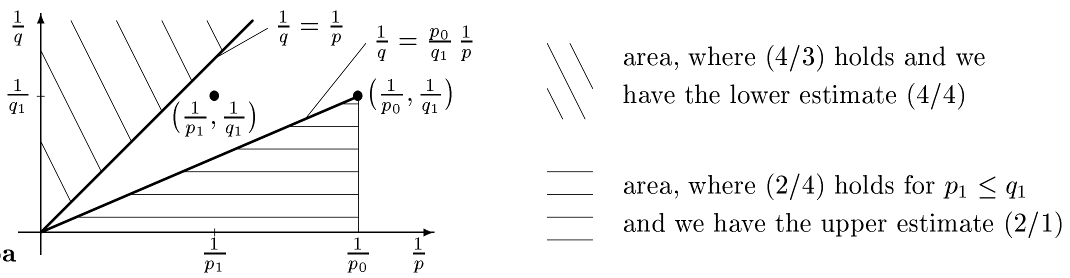


Fig.6a

Next we investigate the compatibility of 3. and 4.

(iv) We start with Corollary 3/2 and the Proposition in 4. Both assumptions (3/25) and (4/3) are satisfied if $p_2 = q_2$ and $p_1 \le q_1$. Then (3/37) and (4/4) yield

$$c_1 k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \le e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \le c_2 k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{7}$$

where $c_1, c_2 > 0$ and $p_1 \le q_1$. Thus we will have the same quantity of the exponents on the left- and right-hand side of (7) if $p_1 = q_1 = \infty$. So we gain from this particular consideration

$$e_k(\mathcal{C}^{s_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \sim k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{8}$$

(using the same notation as in (3)) where $\alpha = s_1 - s_2 + \frac{n}{p_2} > 0$. As one can easily grasp both results (3) and (8) go well together, i.e. (3) is now only a special case of (8) for $p_2 = \infty$.

(v) Involving Remark 3/5 also contributes a little piece of insight into the inner correlations between all the parameters acting in this complicated situation on the line $\delta = \alpha$.
First one can easily show that both (3/54) and (4/3) cannot hold simultaneously. In the figure below we tried to indicate the areas where either (3/54) or (4/3) are true.

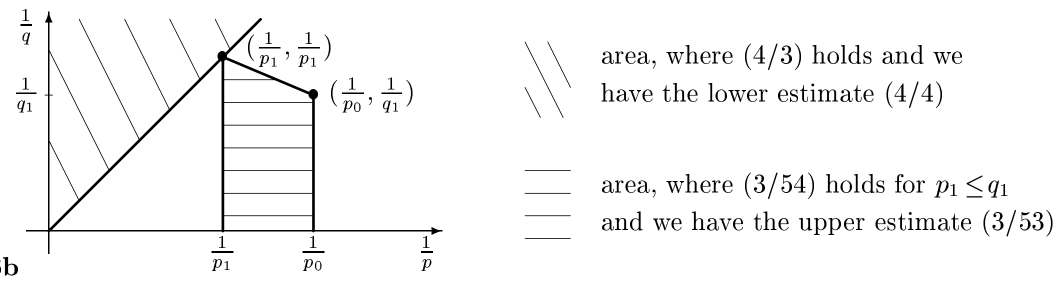


Fig.6b

(vi) In contrast to these considerations we will reap the benefits of combining (3/57) and (4/4). Looking for compatibility of the respective assumptions (3/58) and (4/3) we will use a picture. Let $1 < p_1 = q_1 < \infty$, then we have both estimates (3/57) and (4/4) if $\left(\frac{1}{p_2}, \frac{1}{q_2}\right)$ is in the hatched area (see also Fig.4a),

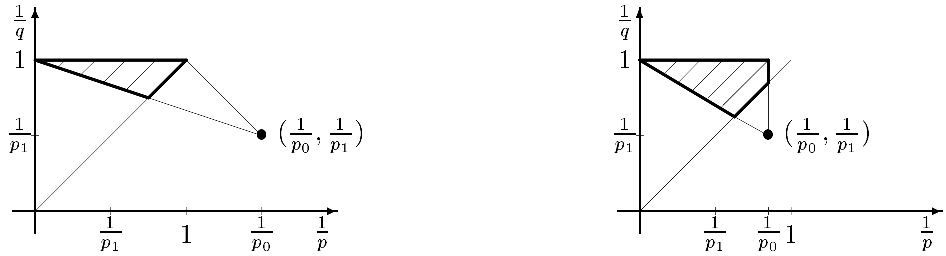


Fig.7a $p_0 < 1 < p_1$ $\qquad$ $1 \le p_0 < p_1$

that is,

$$1 \ge \frac{1}{q_2} \begin{cases} > 1 - \dfrac{1-\frac{1}{p_1}}{\frac{1}{p_0}} \dfrac{1}{p_2} & , \quad \frac{1}{p_2} \le \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}} \\ \ge \frac{1}{p_2} & , \quad \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}} < \frac{1}{p_2} < \min\left(\frac{1}{p_0}, 1\right) . \end{cases} \tag{9}$$

There it holds by (3/57) and (4/4)

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \ \sim \ k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} \tag{10}$$

if

$$1 < p_1 = q_1 < \infty, \quad \max(p_0, 1) < p_2 < \infty \quad \text{and} \quad q_2 \quad \text{bounded by (9)} . \tag{11}$$

Note that (9) always provides $\frac{1}{q_2} > \frac{1}{p_1}$: if $\frac{1}{p_2} \le \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}}$, then $\frac{1}{q_2} - \frac{1}{p_1} > \big(1-\frac{1}{p_1}\big)\big(1-\frac{p_0}{p_2}\big) > 0$, and if $\frac{1}{p_2} > \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}}$, then $\frac{1}{q_2} - \frac{1}{p_1} \ge \frac{1}{p_2} - \frac{1}{p_1} > \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}} - \frac{1}{p_1} = \frac{\frac{\alpha}{n}}{1+\frac{\alpha}{n}} \big(1 - \frac{1}{p_1}\big) > 0$. Hence the log-exponent is always larger than $\frac{s_1-s_2}{n}$ in this case, $\frac{s_1-s_2}{n} + \frac{1}{q_2} - \frac{1}{p_1} > \frac{s_1-s_2}{n}$.

(vii) Finally we combine (3/63) and (4/15). Let $1 < p_2 = q_2 < \infty$ and

$$0 \le \frac{1}{q_1} \begin{cases} \le \frac{1}{p_1} & , \quad \big(\frac{1}{p_2} - \frac{\alpha}{n}\big)_+ < \frac{1}{p_1} < \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}} \\ < -\dfrac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}} \dfrac{1}{p_1} + \dfrac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}} & , \quad \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}} \le \frac{1}{p_1} \le 1 . \end{cases} \tag{12}$$

We use again a diagram, see Fig.7b below, to show the meaning of the above restriction. Both estimates (3/63) and (4/15) are satisfied if $(\frac{1}{p_1}, \frac{1}{q_1})$ is in the thick hatched domain, see also Fig.4b.

Fig.7b $0 < \frac{1}{p_2} < \frac{\alpha}{n}$ $\qquad$ $\frac{\alpha}{n} \le \frac{1}{p_2} < \frac{1}{p_0}$

Then it holds by (3/63) and (4/15)

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \ \sim \ k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{q_1}} . \tag{13}$$

One may observe that we always have $\frac{1}{p_2} > \frac{1}{q_1}$: if $\frac{1}{p_1} < \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}}$, then $\frac{1}{p_2} - \frac{1}{q_1} \ge \frac{1}{p_2} - \frac{1}{p_1} > \frac{1}{p_1} \frac{\frac{\alpha}{n}}{1+\frac{\alpha}{n}} > 0$, otherwise, if $\frac{1}{p_1} \ge \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}}$, then $\frac{1}{p_2} - \frac{1}{q_1} > \frac{1}{p_2} \big(\frac{1}{p_0} - \frac{1}{p_2}\big) \frac{1}{1-\frac{1}{p_2}+\frac{\alpha}{n}} > 0$ and consequently the log-exponent is again larger than $\frac{s_1-s_2}{n}$, i.e. $\frac{s_1-s_2}{n} + \frac{1}{p_2} - \frac{1}{q_1} > \frac{s_1-s_2}{n}$.

Let us now summarize the above considerations.

Corollary 1 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied.*

(*a*) *Let* $p_2 = q_2$ *and* $p_1 \le q_1$. *Then there exist two positive constants* c_1, c_2 *such that for all* $k \in I\!N$

$$c_1\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{p_1}} \;\le\; e_k(B^{s_1}_{p_1,q_1}(\alpha) \;\to\; B^{s_2}_{p_2,p_2}) \;\le\; c_2\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} . \tag{14}$$

In particular, if $p_1 = q_1 = \infty$,

$$e_k(\mathcal{C}^{s_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{15}$$

where $B^{s_1}_{\infty,\infty}(\alpha) = \mathcal{C}^{s_1}(\alpha)$. *Moreover, if we additionally have* $p_2 = q_2 = \infty$, *then*

$$e_k(\mathcal{C}^{s_1}(\alpha) \to \mathcal{C}^{s_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}} . \tag{16}$$

(*b*) *Let* $1 < q_1 = p_1 < \infty$, *and*

$$1 \ge \frac{1}{q_2} \begin{cases} > 1 - \frac{1-\frac{1}{p_1}}{\frac{1}{p_0}} \frac{1}{p_2} & , \quad \frac{1}{p_2} \le \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}} \\ \ge \frac{1}{p_2} & , \quad \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}} < \frac{1}{p_2} < \min\left(\frac{1}{p_0}, 1\right) . \end{cases} \tag{17}$$

Then it holds

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{p_1}} . \tag{18}$$

Moreover, if we additionally have $p_0 < 1 \le p_2 = q_2 < p_0\left(1 + \frac{\alpha}{n}\right)$ *or* $1 \le p_0 < p_2 = q_2 < p_0\left(1 + \frac{\alpha}{n}\right)$, *then by* (1/2)

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}} . \tag{19}$$

(*c*) *Let* $1 < q_2 = p_2 < \infty$, *and*

$$0 \le \frac{1}{q_1} \begin{cases} \le \frac{1}{p_1} & , \quad \left(\frac{1}{p_2} - \frac{\alpha}{n}\right)_+ < \frac{1}{p_1} < \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}} \\ < -\frac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}} \frac{1}{p_1} + \frac{\frac{1}{p_2}}{1-\frac{1}{p_2}+\frac{\alpha}{n}} & , \quad \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}} \le \frac{1}{p_1} \le 1 . \end{cases} \tag{20}$$

Then it holds

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\frac{1}{q_1}} . \tag{21}$$

Moreover, if we additionally have $\left(\frac{1}{p_2} - \frac{\alpha}{n}\right)_+ < \frac{1}{p_1} < \frac{1}{p_2} \frac{1}{1+\frac{\alpha}{n}}$, *then by* (1/2)

$$e_k(B^{s_1}_{p_1,p_1}(\alpha) \to B^{s_2}_{p_2,p_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}} . \tag{22}$$

(*d*) *Let* $p_1 = q_1 = 1$ *and* $1 < q_2 \le p_2 < \infty$. *Then there are two positive constants* $c_1, c_2 > 0$ *such that for all* $k \in I\!N$

$$c_1\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-1} \;\le\; e_k(B^{s_1}_{1,1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\le\; c_2\; k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}} . \tag{23}$$

Remark 1. Naturally one will ask for the right behaviour of the entropy numbers $e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2})$ on the line $\delta = \alpha$ for all possible parameters, but unfortunately we are not able to answer this question at the moment. Nevertheless we may recognize some common behaviour concerning (15), (18) and (21) (and their special cases (16), (19) and (22), resp.,) and we thus come to a rather surprising conclusion.

Proposition: *Let the general assumptions* (1/1) *and* (1/2) *be satisfied with*

$$p_1,\ p_2,\ q_1,\ q_2\ \in (1,\infty)\ . \tag{24}$$

Then we have

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \quad \sim \quad k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}} \tag{25}$$

if the parameters additionally satisfy condition (i) *or* (ii) *below.*

Remark 2. The above-mentioned restrictions (i) and (ii), resp., are the following:

(i)
$$0 < \frac{1}{p_2} \le \frac{1}{q_2} < 1\ , \qquad \frac{\frac{1}{q_1}}{1-(\frac{1}{p_1}-\frac{1}{q_1})_+} < \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}}\ , \tag{26}$$

$$\left(\frac{1}{q_1}-\frac{1}{p_1}\right)_+ < \frac{\alpha}{n} < \begin{cases} \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} - \frac{\frac{1}{q_1}}{1-(\frac{1}{p_1}-\frac{1}{q_1})_+}\left(1+\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}}\right) + \left(\frac{1}{q_1}-\frac{1}{p_1}\right)_+ & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} < \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \\ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} - \frac{1-\frac{1}{q_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}}\left(1+\frac{1-\frac{1}{p_1}}{\frac{1}{q_1}}\right) & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} \ge \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \end{cases} \tag{27}$$

and

(ii)
$$0 < \frac{1}{q_1} \le \frac{1}{p_1} < 1\ , \qquad \frac{\frac{1}{q_1}}{1-\frac{1}{p_1}+\frac{1}{q_1}} < \frac{\frac{1}{p_2}}{1-\left(\frac{1}{q_2}-\frac{1}{p_2}\right)_+} - \left(\frac{1}{p_2}-\frac{1}{q_2}\right)_+\ , \tag{28}$$

$$\left(\frac{1}{p_2}-\frac{1}{q_2}\right)_+ < \frac{\alpha}{n} < \begin{cases} \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} - \frac{\frac{1}{q_1}}{1-\frac{1}{p_1}+\frac{1}{q_1}}\left(1+\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}}\right) & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} < \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \\ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} - \frac{1-\frac{1}{q_2}}{1-(\frac{1}{q_2}-\frac{1}{p_2})_+}\left(1+\frac{1-\frac{1}{p_1}}{\frac{1}{q_1}}\right) + \left(\frac{1}{p_2}-\frac{1}{q_2}\right)_+ & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} \ge \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}}\ . \end{cases} \tag{29}$$

Remark 3. To get a better impression first what the above inequalities mean in terms of our $(\frac{1}{p},\frac{1}{q})$-diagrams (in order to realize, that situations as described above may really appear) we refer to Fig.10.

Remark 4. Unfortunately the above restrictions (i) and (ii) are rather nasty and probably not necessary (in this particular form), but they are due to our methods of proving. Nevertheless the result (25) is really a fine one as we already announced in the beginning. But we will discuss this phenomenon in detail a little bit later having convinced the reader before of the correctness of this assertion.

We will prove the above proposition in two natural steps, at first we show the estimate from above and afterwards that one from below. We only want to mention that we restrict ourselves to those parameters, given by (24) (in addition to (1/1) and (1/2)) from the very beginning as we aim at estimates from both sides. Nevertheless one can extend either Lemma 1 or Lemma 2 below in some special cases even to parameters less than 1, i.e. to the quasi-Banach case. Pursuing the proofs and comparing with the slightly more general results we shall use one can easily recognize possible modifications.

Estimate from above

Lemma 1 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied and assume additionally the following conditions to hold*:

$$p_1,\, p_2,\, q_1,\, q_2 \quad \in \quad (1,\infty) \tag{30}$$

$$\frac{\frac{1}{q_1}}{1-(\frac{1}{p_1}-\frac{1}{q_1})_+} \quad < \quad \frac{\frac{1}{p_2}}{1-(\frac{1}{q_2}-\frac{1}{p_2})_+} - (\frac{1}{p_2}-\frac{1}{q_2})_+ \,, \tag{31}$$

$$(\frac{1}{q_1}-\frac{1}{p_1})_+ \;+\; (\frac{1}{p_2}-\frac{1}{q_2})_+ \;<\; \frac{\alpha}{n}\,, \tag{32}$$

$$\frac{\alpha}{n} < \begin{cases} \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} - \frac{\frac{1}{q_1}}{1-(\frac{1}{p_1}-\frac{1}{q_1})_+}\left(1+\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}}\right) + (\frac{1}{q_1}-\frac{1}{p_1})_+ \,, & \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} < \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \\ \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} - \frac{1-\frac{1}{q_2}}{1-(\frac{1}{q_2}-\frac{1}{p_2})_+}\left(1+\frac{1-\frac{1}{p_1}}{\frac{1}{q_1}}\right) + (\frac{1}{p_2}-\frac{1}{q_2})_+ \,, & \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} \geq \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \,. \end{cases} \tag{33}$$

Then there exists a positive constant c *such that for all* $k \in I\!N$

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \quad \leq \; c\, k^{-\frac{s_1-s_2}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}} \,. \tag{34}$$

P R O O F : We use the multiplicativity of entropy numbers as well as the estimates (3/57) and (3/63) :

$$e_{2k}(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \leq \; c\, e_k(B^{s_1}_{p_1,q_1}(\alpha-\beta) \to B^{\sigma}_{r,r})\; e_k(B^{\sigma}_{r,r}(\beta) \to B^{s_2}_{p_2,q_2}) \tag{35}$$

where

$$s_1 > \sigma > s_2, \quad 0<\beta<\alpha, \quad 1<r<\infty, \quad \frac{1}{p_2} < \frac{1}{r}+\frac{\beta}{n} < \frac{1}{p_1}+\frac{\alpha}{n}, \tag{36}$$

and

$$\alpha-\beta = s_1-\sigma-\frac{n}{p_1}+\frac{n}{r} > 0, \quad \beta = \sigma - s_2 - \frac{n}{r}+\frac{n}{p_2} > 0\,. \tag{37}$$

Note that in (35) we make use of the isomorphism described in [9: (2.4/1)] once more, that is, we may regard $e_k(B^{s_1}_{p_1,q_1}(\alpha-\beta) \to B^{\sigma}_{r,r})$ instead of $e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{\sigma}_{r,r}(\beta))$. We have by (3/63)

$$e_k(B^{s_1}_{p_1,q_1}(\alpha-\beta) \to B^{\sigma}_{r,r}) \;\leq\; c\, k^{-\frac{s_1-\sigma}{n}}\, \big(\log\langle k\rangle\big)^{\frac{s_1-\sigma}{n}+\frac{1}{r}-\frac{1}{q_1}} \tag{38}$$

if, additionally to (30), (36) and (37),

$$0 < \frac{1}{q_1} < \frac{\frac{1}{r}(1-\frac{1}{p_1})}{1-\frac{1}{r}+\frac{\alpha-\beta}{n}}\,, \qquad\qquad (\frac{1}{r}-\frac{\alpha-\beta}{n})_+ < \frac{1}{p_1} < 1\,. \tag{39}$$

Likewise (3/57) provides

$$e_k(B^\sigma_{r,r}(\beta) \to B^{s_2}_{p_2,q_2}) \;\le\; c\, k^{-\frac{\sigma-s_2}{n}} \,\big(\log\langle k\rangle\big)^{\frac{\sigma-s_2}{n}+\frac{1}{q_2}-\frac{1}{r}} \tag{40}$$

if

$$1-\frac{1-\frac{1}{r}}{\frac{1}{r}+\frac{\beta}{n}}\,\frac{1}{p_2} < \frac{1}{q_2} < 1\,, \qquad\qquad \frac{1}{p_2} < \min\,\big(\frac{1}{r}+\frac{\beta}{n},1\big)\,. \tag{41}$$

Then (35), (38) and (40) yield (34) if we can find some β and r simultaneously satisfying (36), (39) and (41). From (39) and (41) we get

$$\frac{1}{p_2}-\frac{\beta}{n} \;<\; \frac{1}{r} \;<\; \frac{\frac{1}{p_2}-\frac{\beta}{n}(1-\frac{1}{q_2})}{1+\frac{1}{p_2}-\frac{1}{q_2}} \;<\; 1 \tag{42}$$

and

$$0 \;<\; \frac{\frac{1}{q_1}(1+\frac{\alpha-\beta}{n})}{1-\frac{1}{p_1}+\frac{1}{q_1}} \;<\; \frac{1}{r} \;<\; \frac{1}{p_1}+\frac{\alpha-\beta}{n}\,. \tag{43}$$

(In fact we had to substitute $\frac{1}{p_2}-\frac{\beta}{n}$ in (42) by $\big(\frac{1}{p_2}-\frac{\beta}{n}\big)_+$ and $\frac{1}{p_1}+\frac{\alpha-\beta}{n}$ in (43) by $\min\big(\frac{1}{p_1}+\frac{\alpha-\beta}{n},1\big)$, but this apparently weaker version is sufficient as we look for r satisfying (42) as well as (43).)
In other words we are first of all interested in conditions which guarantee to find some β, $0<\beta<\alpha$, such that

$$\frac{1}{p_2}-\frac{\beta}{n} \;<\; \frac{\frac{1}{p_2}-\frac{\beta}{n}(1-\frac{1}{q_2})}{1+\frac{1}{p_2}-\frac{1}{q_2}} \tag{44}$$

and

$$\frac{\frac{1}{q_1}(1+\frac{\alpha-\beta}{n})}{1-\frac{1}{p_1}+\frac{1}{q_1}} \;<\; \frac{1}{p_1}+\frac{\alpha-\beta}{n}\,. \tag{45}$$

Concerning (44) and (45) one recognizes that (44) is true for

$$\frac{\beta}{n} > \big(\frac{1}{p_2}-\frac{1}{q_2}\big)_+ \tag{46}$$

and (45) likewise if

$$\frac{\beta}{n} < \frac{\alpha}{n} - \big(\frac{1}{q_1}-\frac{1}{p_1}\big)_+\,. \tag{47}$$

Thus we immediately get (32). Now it remains to look for parameters p_1,p_2,q_1,q_2 and α such that

$$\frac{\frac{1}{q_1}(1+\frac{\alpha-\beta}{n})}{1-\frac{1}{p_1}+\frac{1}{q_1}} \;<\; \frac{\frac{1}{p_2}-\frac{\beta}{n}(1-\frac{1}{q_2})}{1+\frac{1}{p_2}-\frac{1}{q_2}} \tag{48}$$

holds for some β with

$$\big(\frac{1}{p_2}-\frac{1}{q_2}\big)_+ \;<\; \frac{\beta}{n} \;<\; \frac{\alpha}{n}-\big(\frac{1}{q_1}-\frac{1}{p_1}\big)_+\,.$$

Looking at (42) and (43) this is sufficient as we always assume $\frac{1}{p_2}<\frac{1}{p_1}+\frac{\alpha}{n}$ by (1/1) and hence $\frac{1}{p_2}-\frac{\beta}{n}<\frac{1}{p_1}+\frac{\alpha-\beta}{n}$ for all β. Manipulating (48) we arrive at

$$\frac{\beta}{n}\Big(\,(1-\frac{1}{q_2})(1-\frac{1}{p_1})-\frac{1}{p_2}\frac{1}{q_1}\Big) \;<\; \frac{1}{p_2}(1-\frac{1}{p_1})-\frac{1}{q_1}(1-\frac{1}{q_2})-\frac{1}{q_1}\,\frac{\alpha}{n}\,(1+\frac{1}{p_2}-\frac{1}{q_2})\,. \tag{49}$$

If $(1-\frac{1}{q_2})(1-\frac{1}{p_1}) \geq \frac{1}{p_2}\frac{1}{q_1}$ restriction (46) then leads to the lower line in (33) whereas assumption (47) together with the case $(1-\frac{1}{q_2})(1-\frac{1}{p_1}) < \frac{1}{p_2}\frac{1}{q_1}$ in (49) result in the upper line of (33). Finally condition (31) guarantees that it is always possible to find parameters p_1, p_2, q_1 and q_2 according to (32) and (33).

In the diagram below we tried to sketch some typical situation of admissible $(\frac{1}{p_1}, \frac{1}{q_1})$ and $(\frac{1}{p_2}, \frac{1}{q_2})$ in dependence on $\alpha > 0$.

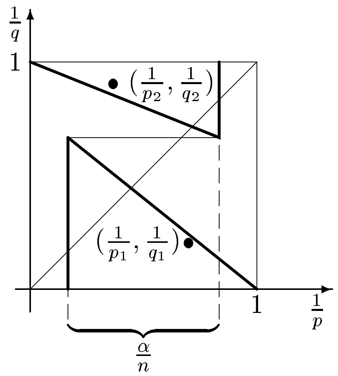


Fig.8

Remark 5. Conversely one could even mention briefly a more geometrically-based argumentation using this $(\frac{1}{p}, \frac{1}{q})$-diagram. Perhaps this also might shed some light on the mysteries happening in the above proof of Lemma 1. Take the points $(\frac{1}{p_1}, \frac{1}{q_1})$ and $(\frac{1}{p_2}, \frac{1}{q_2})$ in the $(\frac{1}{p}, \frac{1}{q})$-diagram (we always assume (24)). Connect now $(\frac{1}{p_1}, \frac{1}{q_1})$ with $(1,0)$ and $(\frac{1}{p_2}, \frac{1}{q_2})$ with $(0,1)$, see Fig.9 below. Look at those points $(\frac{1}{r_1}, \frac{1}{u_1})$ and $(\frac{1}{r_2}, \frac{1}{u_2})$, resp., where these lines meet the diagonal line $\frac{1}{p} = \frac{1}{q}$. In case of $\frac{1}{q_1} > \frac{1}{p_1}$ or $\frac{1}{p_2} > \frac{1}{q_2}$ we put $(\frac{1}{r_1}, \frac{1}{u_1}) = (\frac{1}{p_1}, \frac{1}{q_1})$ or $(\frac{1}{r_2}, \frac{1}{u_2}) = (\frac{1}{p_2}, \frac{1}{q_2})$, resp. (In Fig.9 below we sketched some case where $(\frac{1}{r_1}, \frac{1}{u_1}) = (\frac{1}{p_1}, \frac{1}{q_1})$, but $(\frac{1}{r_2}, \frac{1}{u_2}) \neq (\frac{1}{p_2}, \frac{1}{q_2})$.) Now determine the points $(\frac{1}{\varrho_1}, \frac{1}{u_2})$ and $(\frac{1}{\varrho_2}, \frac{1}{u_1})$ as indicated in the picture, i.e. $(\frac{1}{\varrho_2}, \frac{1}{u_1})$ is the point of intersection of the straight line through $(\frac{1}{p_2}, \frac{1}{q_2})$ and $(0,1)$ with the line $\frac{1}{q} = \frac{1}{u_1}$ whereas $(\frac{1}{\varrho_1}, \frac{1}{u_2})$ is the intersection of the line connecting $(\frac{1}{p_1}, \frac{1}{q_1})$ and $(1,0)$ and the line $\frac{1}{q} = \frac{1}{u_2}$. Now the restrictions (26)-(29) only describe the assumption that $\max(\frac{1}{r_2} - \frac{1}{\varrho_1}, \frac{1}{\varrho_2} - \frac{1}{r_1}) > \frac{\alpha}{n}$. (In the picture below this is indicated by the possibility to find such a broken line with length $\frac{\alpha}{n}$ as we have drawn.) One can easily grasp that this also depends on the angles φ and ψ leading to the different cases

$$\frac{1}{\tan\psi} = \frac{1-\frac{1}{p_1}}{\frac{1}{q_1}} > \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} = \frac{1}{\tan\varphi}$$

or vice versa, in (33). Remember the construction in Fig.4a and Fig.4b as well.
This is the idea of the above proof, the rest is technicality.

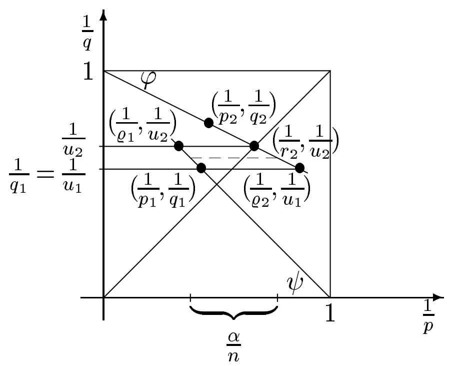


Fig.9

Estimate from below

Lemma 2 : *Let the general assumptions* (1/1) *and* (1/2) *be true and let again* $p_1,\ p_2,\ q_1,\ q_2\ \in (1,\infty)$. *Assume*

$$0<\frac{1}{p_2}\quad\le\quad\frac{1}{q_2}<1 \tag{50}$$

or

$$0<\frac{1}{q_1}\quad\le\quad\frac{1}{p_1}<1\ . \tag{51}$$

Then there is a positive constant c *such that for all* $k\in I\!N$

$$e_k(B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2})\ \ge c\,k^{-\frac{s_1-s_2}{n}}\,\big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}}\quad . \tag{52}$$

P R O O F : We use again the multiplication property of entropy numbers

$$e_{2k}(B^{\sigma}_{r,r}(\alpha+\gamma)\to B^{s_2}_{p_2,q_2})\ \le\ c\,e_k(B^{\sigma}_{r,r}(\gamma)\to B^{s_1}_{p_1,q_1})\,e_k(B^{s_1}_{p_1,q_1}(\alpha)\to B^{s_2}_{p_2,q_2}) \tag{53}$$

where

$$\sigma>s_1>s_2,\quad 1<r<\infty,\quad \gamma>0,\quad \frac{1}{p_2}<\frac{1}{p_1}+\frac{\alpha}{n}<\frac{1}{r}+\frac{\alpha+\gamma}{n} \tag{54}$$

and

$$\alpha+\gamma=\sigma-s_2-\frac{n}{r}+\frac{n}{p_2}>0,\quad \gamma=\sigma-s_1-\frac{n}{r}+\frac{n}{p_1}>0\ . \tag{55}$$

Note that we make use once more of $e_k(B^{\sigma}_{r,r}(\gamma)\to B^{s_1}_{p_1,q_1})\ \sim\ e_k(B^{\sigma}_{r,r}(\alpha+\gamma)\to B^{s_1}_{p_1,q_1}(\alpha))$. We know from (4/3) and (4/4)

$$e_{2k}(B^{\sigma}_{r,r}(\alpha+\gamma)\to B^{s_2}_{p_2,q_2})\ \ge\ c\,k^{-\frac{\sigma-s_2}{n}}\,\big(\log\langle k\rangle\big)^{\frac{\sigma-s_2}{n}+\frac{1}{q_2}-\frac{1}{r}} \tag{56}$$

if

$$0<\frac{1}{p_2}\le\frac{1}{q_2}<1,\quad \frac{1}{p_2}<\frac{1}{r}+\frac{\alpha+\gamma}{n} \tag{57}$$

and likewise from (3/57) and (3/58)

$$e_k(B^{\sigma}_{r,r}(\gamma)\to B^{s_1}_{p_1,q_1})\ \le\ c\,k^{-\frac{\sigma-s_1}{n}}\,\big(\log\langle k\rangle\big)^{\frac{\sigma-s_1}{n}+\frac{1}{q_1}-\frac{1}{r}} \tag{58}$$

if

$$1-\frac{1-\frac{1}{r}}{\frac{1}{r}+\frac{\gamma}{n}}\,\frac{1}{p_1}\ <\ \frac{1}{q_1}\ <\ 1,\qquad \frac{1}{p_1}<\min\,(\frac{1}{r}+\frac{\gamma}{n},1)\ . \tag{59}$$

Then (53) together with (56) and (58) yields (52) if we can find $r,\ 1<r<\infty$, and $\gamma>0$ such that

$$\frac{1}{p_2}-\frac{\alpha+\gamma}{n}\ <\ \frac{1}{p_1}-\frac{\gamma}{n}\ <\ \frac{1}{r}\ <\ \frac{\frac{1}{p_1}-\frac{\gamma}{n}(1-\frac{1}{q_1})}{1+\frac{1}{p_1}-\frac{1}{q_1}}\ <\ 1 \tag{60}$$

holds. We have

$$\frac{1}{p_1}-\frac{\gamma}{n}\ <\ \frac{\frac{1}{p_1}-\frac{\gamma}{n}(1-\frac{1}{q_1})}{1+\frac{1}{p_1}-\frac{1}{q_1}} \tag{61}$$

if

$$\frac{\gamma}{n}>(\frac{1}{p_1}-\frac{1}{q_1})_+\ . \tag{62}$$

On the other hand, to find some $r<\infty$ with (60) we have to assume

$$\frac{1}{p_1}-\frac{\gamma}{n}(1-\frac{1}{q_1})>0\ , \tag{63}$$

i.e.

$$\frac{\gamma}{n} < \frac{\frac{1}{p_1}}{1-\frac{1}{q_1}} \,. \tag{64}$$

We may always find some positive $\gamma > 0$ with (62) and (64) as

$$0 < \frac{1}{q_1}\big(1+\frac{1}{p_1}-\frac{1}{q_1}\big) \quad \Longrightarrow \quad \big(\frac{1}{p_1}-\frac{1}{q_1}\big)_+ < \frac{\frac{1}{p_1}}{1-\frac{1}{q_1}} \,. \tag{65}$$

Hence we get (52) for all $p_1, q_1 \in (1,\infty)$ and p_2, q_2 according to (50) (and $\frac{1}{p_2} < \frac{1}{p_1} + \frac{\alpha}{n}$ as usual).

If we use (3/63) and (4/15) instead of (3/57) and (4/4), resp., then we may (53) replace by

$$e_{2k}(B^{s_1}_{p_1,q_1}(\alpha+\gamma) \to B^{\sigma}_{r,r}) \;\leq\; c\, e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2})\; e_k(B^{s_2}_{p_2,q_2}(\gamma) \to B^{\sigma}_{r,r}) \tag{66}$$

where

$$s_1 > s_2 > \sigma, \quad 1 < r < \infty, \quad \gamma > 0, \quad \frac{1}{r} < \frac{1}{p_2} + \frac{\gamma}{n} < \frac{1}{p_1} + \frac{\alpha+\gamma}{n} \tag{67}$$

and

$$\alpha + \gamma = s_1 - \sigma - \frac{n}{p_1} + \frac{n}{r} > 0, \quad \gamma = s_2 - \sigma - \frac{n}{p_2} + \frac{n}{r} > 0\,. \tag{68}$$

Now (4/15) provides

$$e_{2k}(B^{s_1}_{p_1,q_1}(\alpha+\gamma) \to B^{\sigma}_{r,r}) \;\geq\; c\, k^{-\frac{s_1-\sigma}{n}} \,\big(\log\langle k\rangle\big)^{\frac{s_1-\sigma}{n}+\frac{1}{r}-\frac{1}{q_1}} \tag{69}$$

if

$$0 < \frac{1}{q_1} \leq \frac{1}{p_1} < 1, \qquad\qquad \frac{1}{r} < \frac{1}{p_1} + \frac{\alpha+\gamma}{n} \tag{70}$$

and (3/63) gives

$$e_k(B^{s_2}_{p_2,q_2}(\gamma) \to B^{\sigma}_{r,r}) \;\leq\; c\, k^{-\frac{\sigma-s_2}{n}} \,\big(\log\langle k\rangle\big)^{\frac{\sigma-s_2}{n}+\frac{1}{r}-\frac{1}{q_2}} \tag{71}$$

if

$$0 < \frac{1}{q_2} \;<\; \frac{\frac{1}{r}(1-\frac{1}{p_2})}{1-\frac{1}{r}+\frac{\gamma}{n}}, \qquad\qquad \frac{1}{r} < \min\big(\frac{1}{p_2}+\frac{\gamma}{n}, 1\big)\,. \tag{72}$$

Now the argumentation is completely analogous to (60)-(65) for we may always choose some γ with

$$\big(\frac{1}{q_2}-\frac{1}{p_2}\big)_+ < \frac{\gamma}{n} < \frac{1-\frac{1}{p_2}}{\frac{1}{q_2}} \tag{73}$$

as

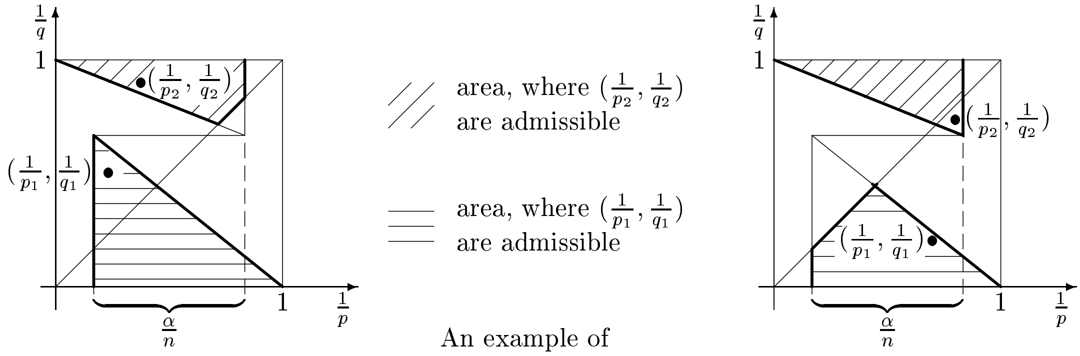
$$\big(1-\frac{1}{q_2}\big)\big(1-\frac{1}{p_2}+\frac{1}{q_2}\big) > 0\,. \tag{74}$$

This ends the proof of Lemma 2.

P r o o f (Proposition) : Having the just proved Lemma 1 and Lemma 2 in mind, to show the above Proposition is only a combination of (34) and (52), where one has to care that the conditions (30)-(33) and (50), (51), resp., are compatible.

Supporting the consideration which parameters may satisfy this complicated set of restrictions (26)-(29) we turn again to the situation in Fig.8 supplemented now by our knowledge of Lemma 2. Thus we arrive at the following diagrams where the different hatched areas indicate where $(\frac{1}{p_1}, \frac{1}{q_1})$

and $(\frac{1}{p_2}, \frac{1}{q_2})$ are admissible according to condition (i) or (ii) in the proposition.



An example of

situation (i) situation (ii)

Fig.10 in the Proposition.

Remark 6. Looking back at part (b) and (c) of Corollary 1, in particular at (18) and (21) we find out that these are special cases of (25) where (17) and (20), resp., imply (26), (27) and (28), (29), resp. (as it should be). As an example we briefly want to demonstrate that (26) and (27) are consequences of (17).

Let $\frac{1}{p_2} \leq \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}}$, then (17) provides

$$\frac{1}{q_2} > 1 - \frac{1-\frac{1}{p_1}}{\frac{1}{p_0}} \frac{1}{p_2} \tag{75}$$

and hence

$$\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}} > \frac{\frac{1}{p_0}}{1+\frac{\alpha}{n}} > \frac{1}{p_1} \,. \tag{76}$$

Likewise, if $\frac{1}{p_2} > \frac{1}{p_0} \frac{1}{1+\frac{\alpha}{n}}$, then $\frac{1}{q_2} \geq \frac{1}{p_2}$ by (17) and

$$\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}} \geq \frac{1}{p_2} > \frac{\frac{1}{p_0}}{1+\frac{\alpha}{n}} > \frac{1}{p_1} \,. \tag{77}$$

Thus (17) yields (26) with $p_1 = q_1$. Concerning (27) we have to show

$$\frac{\alpha}{n} < \begin{cases} \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \left(1-\frac{1}{p_1}\right) - \frac{1}{p_1} & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{p_1}} < \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \\ \frac{1-\frac{1}{p_1}}{\frac{1}{p_1}} - \frac{1-\frac{1}{q_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}} \left(1+\frac{1-\frac{1}{p_1}}{\frac{1}{p_1}}\right) & , \ \frac{1-\frac{1}{p_1}}{\frac{1}{p_1}} \geq \frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \,. \end{cases} \tag{78}$$

One can easily verify that (17) also leads to

$$\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} > \frac{\frac{1}{p_0}}{1-\frac{1}{p_1}} \,, \qquad \frac{1}{p_2} < \frac{1}{p_0} \tag{79}$$

and hence

$$\frac{\frac{1}{p_2}}{1-\frac{1}{q_2}} \left(1-\frac{1}{p_1}\right) - \frac{1}{p_1} > \frac{1}{p_0} - \frac{1}{p_1} = \frac{\alpha}{n} \,, \tag{80}$$

in particular, if $\dfrac{\frac{1}{p_2}}{1-\frac{1}{q_2}} > \dfrac{1-\frac{1}{p_1}}{\frac{1}{p_1}}$. On the other hand, (79) gives

$$\frac{1-\frac{1}{p_1}}{\frac{1}{p_1}} - \frac{1-\frac{1}{q_2}}{1-\frac{1}{q_2}+\frac{1}{p_2}}\left(1+\frac{1-\frac{1}{p_1}}{\frac{1}{p_1}}\right) > \frac{1-\frac{1}{p_1}}{\frac{1}{p_1}\,(1+\frac{\alpha}{n})}\,\frac{\alpha}{n} \tag{81}$$

and in the case $\dfrac{1-\frac{1}{p_1}}{\frac{1}{p_1}} \geq \dfrac{\frac{1}{p_2}}{1-\frac{1}{q_2}} > \dfrac{\frac{1}{p_0}}{1-\frac{1}{p_1}}$ we furthermore know $\dfrac{1-\frac{1}{p_1}}{\frac{1}{p_1}} > 1+\dfrac{\alpha}{n}$, finally arriving at (78) as desired. Similarly we could proceed in case of part (c) of Corollary 1.

We return to our introductory Remark 1 (in front of the Proposition) looking for a more general answer than we could achieve in the Proposition. As we already confessed we call the necessity of our restrictions (26)-(29) into question, whereas it seems to be clear that the third indices cannot be independent of the other parameters. Taking, for instance, the result of [9: (4.2/14)] we know

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\geq\; c\,k^{-\frac{s_1-s_2}{n}} \tag{82}$$

in either „q“-case (note that the restrictions (4.2/12) and (4.2/13) in [9] are only necessary for the estimate from above, see also [9: (5.5/8)] for details). Consequently we always have for the log-exponent

$$\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1} \;=\; \frac{\alpha}{n}-\big(\frac{1}{q_1}-\frac{1}{p_1}\big)-\big(\frac{1}{p_2}-\frac{1}{q_2}\big) \;>\; 0 \tag{83}$$

and condition (32) appears reasonable.
To guarantee (83) it would also be sufficient to assume $\frac{1}{q_2} > \frac{1}{q_1}$, as then by (1/1) it holds

$$\frac{\alpha}{n}+\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{q_2}-\frac{1}{q_1} \;>\; \frac{1}{q_2}-\frac{1}{q_1} > 0\,. \tag{84}$$

Having in mind our already achieved outcomes as presented in Corollary 1 and the above Proposition as well as the just mentioned uncertainty about the necessary conditions which should finally turn out we not dare to formulate a conjecture but a question.

Problem 1 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Which additional assumptions to the whole range of parameters are necessary to achieve*

$$e_k(B^{s_1}_{p_1,q_1}(\alpha) \to B^{s_2}_{p_2,q_2}) \;\sim\; k^{-\frac{s_1-s_2}{n}}\,\big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{q_2}-\frac{1}{q_1}} \qquad ?$$

There should be some condition like

$$\big(\frac{1}{q_1}-\frac{1}{p_1}\big)_+ + \big(\frac{1}{p_2}-\frac{1}{q_2}\big)_+ \;<\; \frac{\alpha}{n} \qquad \text{or} \qquad \frac{1}{q_2} \;>\; \frac{1}{q_1} \tag{85}$$

and probably another restriction replacing (or simplifying) (27) and (29). But without any satisfying answer we may only shift further investigations to the future.

Next we turn to the F-spaces but the discussion is less profitable than in the B-case. We start collecting estimates from above, either originally proved for the F-case or derived from the B-case via elementary embeddings. We have by (3/66) and (3/79)

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \;\leq\; c\,k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} \tag{86}$$

$$\text{if} \qquad 0 < p_1, p_2 < \infty, \quad 0 < q_1, q_2 \le \infty, \tag{87}$$

and

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le \ c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+1-\frac{1}{p_1}} \tag{88}$$

$$\text{if} \qquad 1 < p_1, p_2 < \infty, \quad 0 < q_1 < \infty, \quad 1 < q_2 \le \infty. \tag{89}$$

Via elementary embeddings, duality and (2/1), (2/3) we similarly get

$$e_k(F^{s_1}_{1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le \ c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}} \tag{90}$$

$$\text{if} \qquad 0 < q_1 \le 1 = p_1, \quad 1 < p_2 < \infty, \quad 1 < q_2 \le \infty\,. \tag{91}$$

Likewise we conclude from (3/57)

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le \ c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\max(\frac{1}{p_2},\frac{1}{q_2})-\frac{1}{p_1}} \tag{92}$$

$$\text{if} \qquad 1 < p_1 < \infty, \quad 0 < q_1 \le p_1 < \infty, \quad 1 \le p_2 < p_0\,(1+\frac{\alpha}{n}), \quad 1 \le q_2 \le \infty \tag{93}$$

and from (3/63)

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le \ c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\min(\frac{1}{p_1},\frac{1}{q_1})} \tag{94}$$

$$\text{if} \qquad p_2\,(1+\frac{\alpha}{n}) < p_1 < \infty, \quad 0 < q_1 \le \infty, \quad 1 < p_2 < \infty, \quad p_2 \le q_2 \le \infty\,. \tag{95}$$

Consequently we arrive at

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \le c\, k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}} \tag{96}$$

if

$$\begin{aligned} &\text{either} \quad 1 < p_1 < \infty,\ 0 < q_1 \le p_1,\ 1 \le p_2 < p_0(1+\frac{\alpha}{n}),\ p_2 \le q_2 \le \infty \\ &\text{or} \qquad p_2(1+\frac{\alpha}{n}) < p_1 < \infty,\ 0 < q_1 \le p_1,\ 1 < p_2 < \infty,\ p_2 \le q_2 \le \infty \quad . \end{aligned} \tag{97}$$

Together with (4/1) (or, resp., [9: (4.2/11)] we achieve the following result.

Corollary 2 : *Assume the general assumptions* (1/1) *and* (1/2) *to hold.*

(a) *Let*

$$1 < p_1 < \infty, \quad 0 < q_1 \le p_1 < \infty, \quad 1 \le p_2 < p_0\,(1+\frac{\alpha}{n}), \quad 1 \le q_2 \le \infty\,. \tag{98}$$

Then there are two positive constants c_1, c_2 *such that for all* $k \in I\!N$

$$\begin{aligned} c_1\ k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}} \ \le \ e_k(F^{s_1}_{p_1,q_1}(\alpha) \quad &\to \quad F^{s_2}_{p_2,q_2}) \\ &\le \quad c_2\ k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\max(\frac{1}{p_2},\frac{1}{q_2})-\frac{1}{p_1}}\,. \end{aligned} \tag{99}$$

In particular, if $p_2 \le q_2 \le \infty$,

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \ \sim \ k^{-\frac{s_1-s_2}{n}} \big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\,. \tag{100}$$

(b) *Let*

$$p_2\,(1+\frac{\alpha}{n}) < p_1 < \infty, \quad 0 < q_1 \le \infty, \quad 1 < p_2 < \infty, \quad p_2 \le q_2 \le \infty\ . \tag{101}$$

Then there exist two positive constants c_1, c_2 *such that for all* $k \in I\!N$

$$\begin{aligned} c_1\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}} \ \le\ e_k(F^{s_1}_{p_1,q_1}(\alpha) \ &\to\ F^{s_2}_{p_2,q_2}) \\ &\le\ c_2\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}-\min(\frac{1}{p_1},\frac{1}{q_1})}\ . \end{aligned} \tag{102}$$

In particular, if $0 < q_1 \le p_1$,

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2})\ \sim\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\ . \tag{103}$$

(c) *Let* $0 < p_1,\ p_2 < \infty,\ 0 < q_1,\ q_2 \le \infty$. *Then there are two positive constants* c_1, c_2 *such that for all* $k \in I\!N$

$$c_1\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\ \le\ e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2})\ \le\ c_2\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}+\frac{1}{p_1}}\ . \tag{104}$$

(d) *Let* $1 < p_1,\ p_2 < \infty,\ 0 < q_1 < \infty,\ 1 < q_2 \le \infty$. *Then there are two positive constants* c_1, c_2 *such that for all* $k \in I\!N$

$$c_1\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\ \le\ e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2})\ \le\ c_2\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}+1-\frac{1}{p_2}}\ . \tag{105}$$

(e) *Let* $0 < q_1 \le 1 = p_1,\ 1 < p_2 < \infty,\ 1 < q_2 \le \infty$. *Then there exist two positive constants* c_1, c_2 *such that for all* $k \in I\!N$

$$c_1\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}}\ \le\ e_k(F^{s_1}_{1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2})\ \le\ c_2\ k^{-\frac{s_1-s_2}{n}}\big(\log\langle k\rangle\big)^{\frac{\alpha}{n}+1-\frac{1}{p_2}}\ . \tag{106}$$

Obviously (100) and (103) are extensions of (19) and (22) which are F-results, too.
Looking at (97) one could now suppose the assumptions $q_1 \le p_1$ and $p_2 \le q_2$, resp., to be necessary. Just to the contrary we believe it more likely to get some F-result independent of the third parameters whereas the B-case may depend as discussed above. We want to explain this assumption. The first reason is our method of proving itself as (92) and (94) are derived from B-results and so this p-q-dependence naturally comes in via elementary embeddings,

$$B^s_{p,\min(p,q)}\ \subset\ F^s_{p,q}\ \subset\ B^s_{p,\max(p,q)}\ .$$

Furthermore (104)-(106) may support this supposition, though we fail to achieve sharp results in these cases. Especially in (104) and (105) there is no relation between p_1 and q_1 or p_2 and q_2, resp., assumed to hold, but unfortunately there remains a gap between the lower and upper log-exponent. Our last argument comes from a more abstract point of view, concerning the interplay of B- and F-spaces in other investigations. As an example we want to remind the reader of the continuous embeddings of (unweighted) B- or F-spaces along constant differential dimension, i.e. $\delta = 0$:

$$B^{s_0}_{p_0,q}(I\!R^n)\ \subset\ B^{s_1}_{p_1,q}(I\!R^n) \tag{107}$$

$$\text{if} \qquad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}, \quad 0 < p_0 \le p_1 \le \infty, \quad 0 < q \le \infty, \quad -\infty < s_1 \le s_0 < \infty\ , \tag{108}$$

but

$$F^{s_0}_{p_0,q}(I\!R^n)\ \subset\ F^{s_1}_{p_1,r}(I\!R^n) \tag{109}$$

$$\text{if} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1},\ 0 < p_0 < p_1 < \infty,\ 0 < q \le \infty,\ 0 < r \le \infty,\ -\infty < s_1 < s_0 < \infty\ , \tag{110}$$

see [16: (2.7.1/1), (2.7.1/2)]. Note that we have the same q-parameter in (107), but arbitrary q and r in (109). Another similar phenomenon is the trace theorem for $F^s_{p,q}(\mathbb{R}^n)$ or $B^s_{p,q}(\mathbb{R}^n)$, resp., see [17: Thm. 4.4.2] or [17: Cor. 4.4.2]. Therefore it appears more likely that the B-case depends on the q-parameters than the F-case, but we have no better, more convincing argument and may summarize these considerations in another question.

Problem 2 : *Let the general assumptions* (1/1) *and* (1/2) *be satisfied. Which additional assumptions to the whole range of parameters are necessary to achieve*

$$e_k(F^{s_1}_{p_1,q_1}(\alpha) \to F^{s_2}_{p_2,q_2}) \sim k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} \qquad ?$$

Remark 7. There is also another reason apart from our preceding considerations which may harden the suspicion that the q-indices play a more important role at the line $\delta = \alpha$, at least in the B-case. The only former result (concerning the above described situation) we know is that one of Mynbaev and Otel'baev which deals with approximation numbers and reads in terms of our situation

$$a_k(F^{s_1}_{p_1,2}(\alpha) \to F^0_{p_2,2}) \sim \begin{cases} k^{-\frac{\delta}{n}} & , \quad 0 < \delta < \alpha \\ k^{-\frac{\alpha}{n}}\left(\log\langle k\rangle\right)^{\frac{\alpha}{n}} & , \quad \delta = \alpha \\ k^{-\frac{\alpha}{n}} & , \quad \delta > \alpha\ , \end{cases} \tag{111}$$

where

$$\begin{gathered} s_1 > 0, s_2 = 0 \quad , \quad 1 < p_1 \le p_2 \le 2 \quad \text{or} \quad 2 \le p_1 \le p_2 < \infty, \\ \alpha > 0 \quad , \quad \delta = s_1 - \frac{n}{p_1} + \frac{n}{p_2} > 0\ , \end{gathered} \tag{112}$$

see [12: V, §3, Thm. 9].

Neglecting the different behaviour of the approximation numbers in relation to the entropy numbers in general, we only want to emphasize the role playing both q-indices in the critical case $\delta = \alpha$. Better to say, they do not appear in the exponents in question, but they are equal in the considered situation, $q_1 = q_2 = 2$. Furthermore the approximation numbers could indicate the right behaviour in the exponents while δ increases and passes the critical value α : the exponent $-\frac{\delta}{n}$ for $\delta < \alpha$ changes to $-\frac{\alpha}{n}$ for $\delta > \alpha$ but remains at the same value $-\frac{\delta}{n} = -\frac{\alpha}{n}$ at the line $\delta = \alpha$, but now for $k\left(\log\langle k\rangle\right)^{-1}$ instead of k. Transferring this phenomenon to the case of entropy numbers this means the exponent $-\frac{\delta}{n} + \frac{1}{p_1} - \frac{1}{p_2} = -\frac{s_1-s_2}{n}$ for $\delta < \alpha$, then $-\frac{\alpha}{n} + \frac{1}{p_1} - \frac{1}{p_2}$ for $\delta > \alpha$ and at the line again $-\frac{\delta}{n} + \frac{1}{p_1} - \frac{1}{p_2} = -\frac{s_1-s_2}{n}$, but now for $k\left(\log\langle k\rangle\right)^{-1}$ instead of k. The first assertion is completely proved, see [9: (4.2/8)], whereas the last one has only been shown for $p_0 < p_2 \le p_1$ up to now, see [9: (4.2/9)]. But following this idea it would be reasonable to explain the exponents in (15), (18) and (21) without any help of the p-parameters, but the q-parameters instead. This again strengthens the supposition that it is worth investigating the influence of the q-parameters at the result and refers back to Problem 1 and Problem 2 as well.

Remark 8. Finally we want to point to another aspect, especially concerning (4/4) and (4/15) which seems interesting to us : contrary to former suppositions the exponent $\frac{\alpha}{n}$ at the „log"-term apparently not reflects the right behaviour of the entropy numbers in case of B-spaces. This suspicion had been substantiated by our result concerning lower estimates in the F-case (and might be valid in this case), see [9: (4.2/11)]. Regarding now (4/4) in connection with (4/5) as well as (4/15) and (4/16) this assumption is disproved, at least in the B-case where $q_2 \le p_2$ and $p_1 \le q_1$. This statement is also supported by (25) where the log-exponent is

$$\frac{s_1 - s_2}{n} + \frac{1}{q_2} - \frac{1}{q_1} = \frac{\alpha}{n} + \frac{1}{q_2} - \frac{1}{p_2} + \frac{1}{p_1} - \frac{1}{q_1} \ge \frac{\alpha}{n} \tag{113}$$

if $p_2 \geq q_2$, $q_1 \geq p_1$. Another conclusion we may draw from (16), (19), (22), (25) and (100), (103) is the following: the exponent $\frac{\alpha}{n}$ is possibly the right one if $p_2 = q_2$ and $p_1 = q_1$ and in other F-cases.

Remark 9. In view of (15) we are now able to give a complete result concerning the embedding $id : \mathcal{C}^{s_1}(\alpha) \to B^{s_2}_{p_2,p_2}$, where $\mathcal{C}^{s_1}(\alpha) = B^{s_1}_{\infty,\infty}(\alpha)$.
Let the general assumption (1/1) be satisfied with $p_1 = \infty$ and assume

$$\delta = s_1 - s_2 + \frac{n}{p_2} > 0 . \tag{114}$$

Let e_k be the k^{th} entropy number of the embedding $id : \mathcal{C}^{s_1}(\alpha) \to B^{s_2}_{p_2,p_2}$, where $\mathcal{C}^{s_1}(\alpha) = B^{s_1}_{\infty,\infty}(\alpha)$. Depending on the value of $\alpha > 0$ we have

$$e_k \sim \begin{cases} k^{-\frac{s_1-s_2}{n}} & \text{if} \quad 0 < \delta < \alpha , \\ k^{-\frac{s_1-s_2}{n}} \left(\log\langle k\rangle\right)^{\frac{s_1-s_2}{n}+\frac{1}{p_2}} & \text{if} \quad \delta = \alpha , \\ k^{-\frac{\alpha}{n}+\frac{1}{p_2}} & \text{if} \quad \delta > \alpha . \end{cases} \tag{115}$$

The above version summarizes recent outcomes in [9: (4.2/8), (4.2/9)] and (15).

The original $(\frac{1}{p}, s)$-diagram (see [9: Fig.1]) then degenerates to the following one.

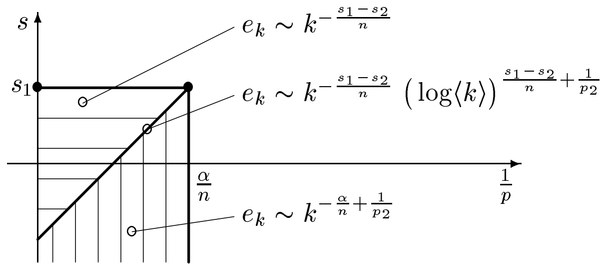


Fig.11

Remark 10. Likewise we might use (18), (21), (25) or (100), (103) to complete former results as in (115). But as the above described restrictions to the parameters are so very complicated and probably not necessary we will omit this here. Nevertheless one could gain complete results similar to (115), that is for $0 < \delta \leq \alpha$ or $\delta > \alpha > 0$, in some special cases if one would like to have them.

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Selbständigkeitserklärung

Ich erkläre, daß ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

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