Entropy and the approximation of functions on compact metric and topological spaces

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To my teacher and friend Professor Eike Hertel
on the occasion of his sixtieth birthday

Contents

Preface

In 1994 the author was confronted by Irmtraud Stephani with a particular geometric problem concerning so-called controllable coverings of the m-dimensional cube $[0,2]^m$. This was the starting point of a fruitful collaboration on the uniform approximation of real-valued functions, which has led into two directions to be surveyed in the two parts of this report.

In the first part we shall consider different ways of approximating continuous or, more generally, bounded real-valued functions on a compact metric space (X,d). Linear as well as non-linear approximation schemes will appear, the approximating functions being either continuous or piecewise constant. All these procedures have in common that the approximating functions reflect the geometry of (X,d) by fulfilling a uniformity condition called *controllability*. This geometric condition yields estimates of Jackson type and, under certain additional suppositions, of Bernstein type.

The second part is motivated by questions from the above mentioned approximation theories, however, the geometric condition is dropped. The space X can be a more general topological space. A main result is the characterization of all continuous real-valued functions on a normal space X, which can be expressed as linear combinations of partitions of unity whose supports are subject to a given geometric restriction. Besides that, we shall deal with the approximation of so-called quasi-continuous and cliquish functions on arbitrary topological spaces X.

The present habilitation thesis consists of the reprints [Ri5–H] and [Ri6–H] as well as of the preprints [Ri2–H], [Ri4–H], [Ri7–H], [Ri8–H], [R/S2–H], and [R/S3–H], the emphasis being on [Ri5–H], [Ri6–H], [Ri7–H], [Ri8–H], and [R/S3–H]. The report to follow is to introduce the reader to the main ideas of the habilitation. We take the opportunity to look at the results from a more general point of view by describing developements and relations between the single papers. Moreover, we present some additional results and applications. For details we refer the reader to the original articles. Proofs will be given only if they are not presented there. In this case, however, we have carried them out in detail in this report. The "meat" of the habilitation is to be found in the above mentioned papers.

Our special gratitude goes to Professor Irmtraud Stephani. She supported the author's research concerning the approximation of functions from the early beginning in 1994 and was open for many helpful and stimulating discussions during the last years. Moreover, she spent a lot of time and energy for proofreading this manuscript in a very attentive and constructive way. Besides that, the author thanks his superiors, Professor Bernd Carl and Professor Eike Hertel, for unselfishly respecting his scientific freedom.

Jena, August 2000 Christian Richter

Part I

Approximation on compact metric spaces

The first three chapters of the present report are devoted to the uniform approximation of continuous and, more generally, of bounded real-valued functions on a compact metric space (X, d).

Chapter 1 deals with geometric questions. We shall introduce the most important geometric concepts of this report - controllable coverings, controllable partitions, and chains of controllable partitions of X. Controllable coverings underly a strong restriction of local finiteness, which later will give rise to Bernstein type theorems. Last but not least we shall discuss the connection between the metric d on X and the class of all controllable coverings of X: What properties of d are stored in the controllable coverings? How can one manipulate d without changing the controllability?

The second chapter treats the approximation of continuous functions on (X, d) and theorems of Jackson and Bernstein type. Controllable partitions of unity are used in two different ways for approximating continuous functions. The first method, which refers to a chain of controllable partitions of unity on the cube $[0, 2]^m$, is restricted to continuous functions on the space $([0, 2]^m, d_\infty)$ only. It leads to a linear approximation scheme with a particular Bernstein type estimate for Hölder continuous functions. In contrast with that, the second approximation procedure is a non-linear one. This n-term approximation by controllable partitions of unity works on arbitrary compact metric spaces (X, d). A third method of approximation makes use of so-called controllable step functions. Then the approximating functions are no longer continuous, but they reflect the geometry of (X, d) in a particularly strong way. For a large class of compact spaces (X, d) this comes to light by the fact that the corresponding error quantities have the same asymptotics as the modulus of continuity for all functions in $C(X)$.

Chapter 3 concerns the class of bounded real-valued functions on X which can be attained as uniform limits of controllable step functions. We shall see that this class can be represented as a union of linear approximation spaces, so far as certain approximating step functions are excluded. The class itself, however, is non-linear. In this chapter the emphasis will not be on quantitative results, but rather on qualitative features of the approximable functions such as continuity or integrability properties.

Chapter 1

Entropy properties of compact metric spaces

1.1 Controllable coverings

Let M be a subset of a compact metric space (X, d). *Kolmogoroff's entropy function* $\mathcal{N}(M, \cdot)$ *of* M is defined by

$$\mathcal{N}(M, \delta) \;=\; \min\left\{k \geq 1 : \text{ there exist points } x_1, x_2, \ldots, x_k \in X \text{ with } M \subseteq \bigcup_{i=1}^{k} B(x_i, \delta)\right\}$$

for $\delta > 0$ (cf. [Ko/Ti]), $B(x_i, \varepsilon)$ denoting the closed ball of radius ε centered at x_i. Similarly, the k*-th entropy number of* M is given by

$$\varepsilon_k(M) \;=\; \inf\left\{\varepsilon > 0 : \text{ there exist points } x_1, x_2, \ldots, x_k \in X \text{ with } M \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon)\right\}$$

for $k \geq 1$ (cf. [Ca/Ste]). Obviously, $\mathcal{N}(M, \delta) \leq k$ if and only if $\varepsilon_k(M) \leq \delta$, which shows that the two concepts are closely related to each other.

For quantifying the degree of compactness of M one usually asks for the asymptotics of the function $\mathcal{N}(M, \delta)$ as δ approaches 0 and, correspondingly, of the numbers $\varepsilon_k(M)$ for $k \to \infty$. This idea is used for instance in fractal geometry when introducing the covering dimension or in Banach space geometry, where various estimates of entropy numbers of compact operators and of convex hulls of compact or precompact sets appear.

In contrast with that, we shall deal with the particular coverings which are used for defining the numbers $\mathcal{N}(X, \delta)$, $\delta > 0$. For a finite covering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of X by arbitrary subsets $C_i \subseteq X$, the *fineness* $F(\mathcal{C})$ *of* $\mathcal{C}$ is meant to be the largest radius of the covering sets C_i, that is

$$F(\mathcal{C}) \;=\; \max_{1 \leq i \leq k} \varepsilon_1(C_i)\;.$$

Obviously, the definition of $\mathcal{N}(X,\delta)$ yields

$$\mathcal{N}(X,\delta) \;=\; \min\{\mathrm{card}(\mathcal{C}) : \mathcal{C} \text{ is a covering of } X \text{ with } F(\mathcal{C}) \leq \delta\}\ .$$

Coverings realizing the minimum on the right-hand side for some $\delta > 0$ are called *controllable coverings of* X. In other words, the covering $\mathcal{C}$ of X is controllable if there exists some $\delta > 0$ such that $\mathcal{C}$ is of minimal cardinality among all coverings of X whose fineness is bounded by δ.

Proposition 1.1 ([Ri4–H], Proposition 1 and Corollary 2) *Let* $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ *be a covering of a compact metric space* (X,d)*. Then the following are equivalent.*
(i) $\mathcal{C}$ *is controllable.*
(ii) If $\mathcal{C}'$ *is a covering of less cardinality than* $\mathcal{C}$ *then* $F(\mathcal{C}') > F(\mathcal{C})$*.*
(iii) Either $k = 1$*, i.e.* $\mathcal{C} = \{X\}$ *is the trivial covering, or* $k \geq 2$ *and*

$$\varepsilon_1(C_i) \;<\; \varepsilon_{k-1}(X) \qquad \textit{for} \qquad 1 \leq i \leq k\ . \qquad \square$$

The notion of controllability originates from [Ste], where a partition of unity $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ in $C(X)$ has been called controllable if the supports of the partition functions fulfil the uniformity condition (iii) from Proposition 1.1, that is

$$\varepsilon_1(\mathrm{supp}(\varphi_i)) \;<\; \varepsilon_{k-1}(X) \qquad \text{for} \qquad 1 \leq i \leq k\ .$$

Example 1.1 We consider the m-dimensional cube $X = [-1,1]^m$ equipped with the maximum metric d_∞, which is induced by the norm of l_∞^m. The entropy numbers of X are

$$\varepsilon_{l^m}(X) \;=\; \varepsilon_{l^m+1}(X) \;=\; \ldots \;=\; \varepsilon_{(l+1)^m-1}(X) \;=\; \frac{1}{l} \qquad \text{for} \qquad l = 1,2,3,\ldots$$

(cf. [Ba/Pi],[Bö/Ri]), which means that

$$\mathcal{N}(X,\delta) \;=\; l^m \qquad \text{if} \qquad \frac{1}{l} \leq \delta < \frac{1}{l-1}$$

for $l \geq 2$ and $\mathcal{N}(X,\delta) = 1$ if $\delta \geq 1$. Hence there exist controllable coverings of X for the cardinalities l^m, $l = 1,2,3,\ldots$, only. A covering $\mathcal{C} = \{C_1, C_2, \ldots, C_{l^m}\}$ of X with $l \geq 2$ fulfils the controllability condition (iii) if $\varepsilon_1(C_i) < \varepsilon_{l^m-1}(X) = \frac{1}{l-1}$. That is, $\mathcal{C}$ is controllable if and only if there exists a radius $\delta_0 < \frac{1}{l-1}$ and a covering $\{B(x_i,\delta_0) : 1 \leq i \leq l^m\}$ of X by balls of that radius such that $C_i \subseteq B(x_i,\delta_0)$ for $1 \leq i \leq l^m$.

The simplest controllable covering of cardinality l^m is the covering of X by l^m balls of radius $\frac{1}{l}$. However, there exist much more difficult examples even in the one-dimensional case $m = 1$. If the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is arbitrarily covered by two subsets A_1 and A_2 then the covering $\mathcal{C} = \{C_1, C_2\}$ with $C_1 = \left[-1, -\frac{1}{2}\right) \cup A_1$ and $C_2 = A_2 \cup \left(\frac{1}{2}, 1\right]$ of the "cube"

$X = [-1, 1]$ is controllable, since $\varepsilon_1(C_i) \leq \frac{3}{4} < 1 = \varepsilon_1(X)$. This shows that the sets from controllable coverings need not have pleasant topological properties. C_1 and C_2 become non-measurable if A_1 and A_2 are. $\Box$

Example 1.2 Let the Hilbert cube $X = \left\{(\xi_i)_{i=1}^{\infty} : 0 \leq \xi_i \leq 2 \cdot 2^{-i}\right\}$ be equipped with the distance d_∞ from l_∞. We shall see that, for $n = 1, 2, 3, \ldots$, the coverings

$$\mathcal{C}_n = \left\{B\left(\left((2l_1-1)2^{-n}, \ldots, (2l_j-1)2^{-n}, \ldots, (2l_n-1)2^{-n}, 0, 0, 0, \ldots\right), 2^{-n}\right) : 1 \leq l_j \leq 2^{n-j}\right\}$$

of X by balls (subcubes) of radius 2^{-n} are controllable.

$\mathcal{C}_n$ is a covering with $F(\mathcal{C}_n) = 2^{-n}$. It suffices to show that $\mathcal{C}_n$ is of minimal cardinality among all coverings $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ with $F(\mathcal{C}) \leq 2^{-n}$. The cardinality of $\mathcal{C}_n$ is

$$\mathrm{card}(\mathcal{C}_n) = 2^{n-1} \cdot 2^{n-2} \cdot \ldots \cdot 2^1 \cdot 2^0 = 2^{\frac{(n-1)n}{2}} .$$

According to $F(\mathcal{C}) \leq 2^{-n}$, there exist k balls $B_i \subseteq X$ of radius 2^{-n} such that $C_i \subseteq B_i$ for $1 \leq i \leq k$. Hence $\{B_i : 1 \leq i \leq k\}$ covers X. Let π_n be the projection of X into $\mathbf{R}^n$ with $\pi_n\left((\xi_i)_{i=1}^{\infty}\right) = (\xi_1, \xi_2, \ldots, \xi_n)$. Then $\{\pi_n(B_i) : 1 \leq i \leq k\}$ is a covering of $\pi_n(X) = [0, 2 \cdot 2^{-1}] \times [0, 2 \cdot 2^{-2}] \times \ldots \times [0, 2 \cdot 2^{-n}]$. By estimating the volume of the union $\pi_n(X) = \bigcup_{i=1}^{k} \pi_n(B_i)$ in $\mathbf{R}^n$, we obtain

$$2^{-\frac{(n-1)n}{2}} = (2 \cdot 2^{-1})(2 \cdot 2^{-2}) \ldots (2 \cdot 2^{-n}) = \mathrm{vol}_n(\pi_n(X)) = \mathrm{vol}_n\left(\bigcup_{i=1}^{k} \pi_n(B_i)\right) \leq$$
$$\leq k(2 \cdot 2^{-n})^n = k \cdot 2^{-(n-1)n} .$$

Consequently,

$$\mathrm{card}(\mathcal{C}) = k \geq 2^{\frac{(n-1)n}{2}} = \mathrm{card}(\mathcal{C}_n) .$$

This extremality property of $\mathcal{C}_n$ among all coverings $\mathcal{C}$ with fineness $F(\mathcal{C}) \leq 2^{-n}$ proves that $\mathcal{C}_n$ is controllable. $\Box$

1.2 Chains of controllable partitions

An important subclass of controllable coverings of a compact metric space (X, d) is formed by the *controllable partitions of* X, where a partition of X is meant to be a covering by pairwise disjoint sets. Given a controllable covering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$, one can easily obtain a controllable partition $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ by a "cutting process": For every $x \in X$ we choose an index $i(x)$ such that $x \in C_{i(x)}$. Then the sets $P_i = \{x \in X : i(x) = i\} \subseteq C_i$, $1 \leq i \leq k$, form a controllable partition. For instance, if we put $i(x) = \min\{i : x \in C_i\}$ then we get $P_1 = C_1$ and $P_i = C_i \setminus \bigcup_{j=1}^{i-1} C_j$ for $2 \leq i \leq k$.

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ of X, $\mathcal{Q}$ is called a *refinement of* $\mathcal{P}$ if every set $Q \in \mathcal{Q}$ is contained in a suitable set $P \in \mathcal{P}$. Moreover, the refinement is called *strict* if $\mathcal{Q} \neq \mathcal{P}$. A sequence $(\mathcal{P}_i)_{i=1}^{\infty}$ of partitions is called a *chain* if $\mathcal{P}_{i+1}$ is a strict refinement of $\mathcal{P}_i$ for $i = 1, 2, 3, \ldots$.

The geometric question for the existence of chains of controllable partitions of a compact metric space X is motivated by the approximation of bounded real-valued functions on X by step functions. Namely, every controllable partition $\mathcal{P}$ induces the finite-dimensional Banach space of all step functions φ which are constant on the sets $P \in \mathcal{P}$. Hence any chain of controllable partitions gives rise to an approximation scheme formed by an increasing sequence of finite-dimensional spaces of bounded real-valued functions on X.

Theorem 1.1 ([Ri8–H], Corollary 2) *Every infinite compact metric space (X, d) admits a chain of controllable partitions.* □

It is clear that the cube $[-1,1]^m$ considered in Example 1.1 possesses chains. An example is the sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ consisting of partitions $\mathcal{P}_n$ of $[-1,1]^m$ into 2^{nm} disjoint half-open or closed balls of radius 2^{-n}. Of course, common boundary points of two or more balls have to be attached to one of them in a way such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. The coverings $\mathcal{C}_n$ of the Hilbert cube introduced in Example 1.2 give rise to a chain of controllable partitions in a similar way.

However, these chains are closely related to the geometric structure of the cube, which admits a kind of "regular" coverings. The situation becomes dramatically worse if we consider arbitrary compact metric spaces (X, d). Then it is even a very hard and usually an unsolved problem to compute the exact entropy numbers of X.* In the general situation Theorem 1.1 is not a trivial statement, since the controllability condition is surprisingly delicate.

Before considering some examples on the particular space $X = [-1,1]$ we give a necessary topological condition concerning partitions belonging to a chain of controllable partitions. We use the symbols $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ for denoting the closure and the interior, respectively, of subsets in topological spaces.

Proposition 1.2 ([Ri8–H], Proposition 5) *Let $\mathcal{P}$ be a partition from a chain of controllable partitions of a compact metric space (X, d). Then*

$$P \subseteq \mathrm{cl}(\mathrm{int}(P)) \qquad \textit{for all} \qquad P \in \mathcal{P} . \qquad \square$$

*What are the entropy numbers of the unit ball in the Euclidean plane? Similar problems are recently discussed in high-standard geometric journals. An examle is the question for the largest radius r_n such that n circles of radius r_n can be packed into a unit square. The solution for $n \leq 27$ is given in [Nu/Öst].

Accordingly, the controllable partition $\{M_1, M_2\}$ of $[-1,1]$ with

$$M_1 \;=\; \left[-1,-\frac{1}{2}\right] \cup \left\{x \in \left(-\frac{1}{2},\frac{1}{2}\right) : x \text{ is rational}\right\} \qquad \text{and} \qquad M_2 \;=\; [-1,1] \setminus M_1$$

can not belong to a chain of controllable partitions. In [Ri1] it is shown that every (not necessarily controllable) finite partition of $[-1,1]$ into subintervals of positive length admits a controllable strict refinement. Hence every controllable partition $\mathcal{P}_1$ of $[-1,1]$ into intervals of positive length belongs to a suitable chain $(\mathcal{P}_i)_{i=1}^{\infty}$ of controllable partitions. However, not all partitions from a chain of controllable partitions of $[-1,1]$ are of that pleasant shape. The twisted partition $\mathcal{P} = \{P_1, P_2\}$ with

$$P_1 \;=\; \bigcup_{i=0}^{\infty} (-1)^i \left(2 \cdot 2^{-2^{i+1}}, 2 \cdot 2^{-2^i}\right] \qquad \text{and} \qquad P_2 \;=\; [-1,1] \setminus P_1$$

can be extended to a chain of controllable partitions of $[-1,1]$ as well (cf. [Ri1]). The more surprising it is that the controllable partition $\mathcal{Q} = \{Q_1, Q_2\}$ with

$$Q_1 \;=\; \bigcup_{i=0}^{\infty} (-1)^i \left(2^{-i-1}, 2^{-i}\right] \qquad \text{and} \qquad Q_2 \;=\; [-1,1] \setminus Q_1$$

does not belong to a chain of controllable partitions, although it is of the same topological structure as $\mathcal{P}$. In fact, $\mathcal{Q}$ even does not possess any controllable strict refinement. This property of $\mathcal{Q}$ can be shown by a relatively long discussion of different cases leading to a contradiction. We omit this unpleasant proof. The fact will not be used in the following.

1.3 Compact metric subspaces of finite-dimensional Banach spaces

Given a finite covering $\mathcal{C}$ of a compact metric space (X,d), one can ask for a measure of local finiteness of $\mathcal{C}$. The appropriate local quantity at a point $x \in X$ is

$$\kappa(\mathcal{C}, x) \;=\; \min\left\{\mathrm{card}\Big(\{C \in \mathcal{C} : C \cap U \neq \emptyset\}\Big) : U \text{ is a neighbourhood of } x\right\}.$$

The following simple fact illustrates that controllable coverings are relatively "thin". It is closely related to Proposition 2.1 from [Ste] and Proposition 3 from [Ri8–H].

Proposition 1.3 *Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a controllable covering of a compact metric space (X,d). Then there exist points $x_i \in C_i$, $1 \le i \le k$, such that $\kappa(\mathcal{C}, x_i) = 1$.*

Proof. We assume that the set C_k does not contain a point x_k with $\kappa(\mathcal{C}, x_k) = 1$. Then $C_k \subseteq \mathrm{cl}(C_1) \cup \mathrm{cl}(C_2) \cup \ldots \cup \mathrm{cl}(C_{k-1})$. Hence $\mathrm{cl}(C_1), \mathrm{cl}(C_2), \ldots, \mathrm{cl}(C_{k-1})$ form a covering of

X. By claim (iii) of Proposition 1.1, there exists a radius $r_0 < \varepsilon_{k-1}(X)$ such that every set $\mathrm{cl}(C_i)$ is covered by a ball B_i of radius r_0. But then X is covered by $k-1$ balls B_i, $1 \le i \le k-1$, of radius $r_0 < \varepsilon_{k-1}(X)$ in contradiction to the definition of $\varepsilon_{k-1}(X)$. □

Proposition 1.3 shows in particular that every set C_i from a controllable covering $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ contains a point x_i with $x_i \in C_i \setminus \bigcup_{j \neq i} C_j$. Coverings of that type will be called *peaked with peaks* x_i, $1 \le i \le k$ (see Chapter 4).

However, we are mainly interested in the coefficient

$$\kappa(\mathcal{C}) \;=\; \max_{x \in X} \kappa(\mathcal{C}, x)$$

globally describing the local finiteness of the covering $\mathcal{C}$. The coverings $\mathcal{C}_n$ of the Hilbert cube introduced in Example 1.2 yield $\kappa(\mathcal{C}_n) = 2^{n-1}$. Hence the coefficient of local finiteness $\kappa(\mathcal{C})$ of controllable coverings $\mathcal{C}$ of the Hilbert cube can become arbitrarily large.

We shall see that the behaviour of controllable coverings $\mathcal{C}$ of compact metric subspaces of finite-dimensional Banach spaces is different in so far as $\kappa(\mathcal{C})$ is uniformly bounded for all controllable coverings by a constant depending on the dimension m of the underlying space only. This result rests on estimates of the m-dimensional volume. Similar estimates are valid for the surface measure of the m-dimensional Euclidean sphere S^m. We consider S^m as the boundary of the unit ball of l_2^{m+1} equipped with the so-called angular distance $d_\angle$. That is, $d_\angle(x, y)$ is the size of the angle $\angle(x, 0, y)$, where 0 is the center of S^m.

Theorem 1.2 ([Ri5–H], Theorem 2.1) *Let (X, d) be a compact metric subspace of an m-dimensional Banach space or of the m-dimensional Euclidean sphere, $\mathcal{C}$ a controllable covering of cardinality $k \ge 2$, and let $B(x_0, r)$ be a closed ball in (X, d) of radius $r \ge 0$. Then the number of sets $C \in \mathcal{C}$ which intersect $B(x_0, r)$ is bounded by*

$$\mathrm{card}\Big(\{C \in \mathcal{C} : C \cap B(x_0, r) \neq \emptyset\}\Big) \;\le\; \left(\frac{2r}{\varepsilon_{k-1}(X)} + 5\right)^m . \qquad \square$$

If one chooses $r = r(X, m) > 0$ such that $\left(\frac{2r}{\varepsilon_{k-1}(X)} + 5\right)^m < 5^m + 1$ then, for all $x \in X$, the neighbourhood $B(x, r)$ intersects at most 5^m sets from $\mathcal{C}$. This yields the following strong form of local finiteness.

Corollary 1.1 *Let $\mathcal{C}$ be a controllable covering of a compact metric subspace (X, d) of an m-dimensional Banach space or of the m-dimensional Euclidean sphere. Then*

$$\kappa(\mathcal{C}) \;\le\; 5^m . \qquad \square$$

In Example 1.1 we have already seen that controllable coverings of the cube $([-1, 1]^m, d_\infty)$ can have the cardinalities l^m, $l = 1, 2, 3, \ldots$, only. The following lemma shows that there is a close relation between the controllable coverings of cardinality l^m and the lattice $G_{l^m} \subseteq [-1,1]^m$ with

$$G_{l^m} \; = \; \left\{ g[i_1, i_2, \ldots, i_m] : (i_1, i_2, \ldots, i_m) \in \{1, 2, \ldots, l\}^m \right\}$$

where

$$g[i_1, i_2, \ldots, i_m] \; = \; \left(-1 + \frac{2(i_1 - 1)}{l-1}, -1 + \frac{2(i_2 - 1)}{l-1}, \ldots, -1 + \frac{2(i_m - 1)}{l-1} \right) .$$

Lemma 1.1 ([Bö]; [Ri3], Satz 2.6) *Let $\mathcal{C}$ be a controllable covering of the m-dimensional cube $([-1,1]^m, d_\infty)$ of cardinality l^m, $l \geq 2$. Then every point of the lattice G_{l^m} is covered by exactly one of the covering sets $C \in \mathcal{C}$. Conversely, every set $C \in \mathcal{C}$ contains exactly one point from G_{l^m}.* $\square$

We can employ this lemma on the "lattice-like" shape of controllable coverings for sharpening the above corollary for the particular space $([-1,1]^m, d_\infty)$.

Corollary 1.2 *Let $\mathcal{C}$ be a controllable covering of the space $([-1,1]^m, d_\infty)$. Then*

$$\kappa(\mathcal{C}) \; \leq \; 2^m \, .$$

Proof. We know that $\mathcal{C}$ has a cardinality $\mathrm{card}(\mathcal{C}) = l^m$. The case $l = 1$ is trivial. Hence we can assume that $l \geq 2$. Let $x \in [-1,1]^m$. The cube $[-1,1]^m$ is covered by the $(l-1)^m$ balls $B\left(g[i_1, i_2, \ldots, i_m] + \left(\frac{1}{l-1}, \frac{1}{l-1}, \ldots, \frac{1}{l-1}\right), \frac{1}{l-1}\right)$, $(i_1, i_2, \ldots, i_m) \in \{1, 2, \ldots, l-1\}^m$, of radius $\frac{1}{l-1}$ whose vertices belong to G_{l^m}. Let B be one of that balls with $x \in B$ and let $\mathrm{vert}(B)$ consist of the vertices of B. By Lemma 1.1, $\mathcal{C}$ splits into the subsystem $\mathcal{C}_1$, which contains the 2^m sets from $\mathcal{C}$ each covering one of the vertices of B, and the remainder $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$, each of whose sets includes one point from $G_{l^m} \setminus \mathrm{vert}(B)$. For each set $C \in \mathcal{C}_2$, there exists a neighbourhood U_C of x which does not intersect C, since $x \in B$, whereas C contains a point from $G_{l^m} \setminus \mathrm{vert}(B)$, whose distance from B is at least $\frac{2}{l-1}$, and C is contained in a subcube whose edges are shorter than $\frac{2}{l-1}$ according to $\varepsilon_1(C) < \varepsilon_{l^m-1}([-1,1]^m) = \frac{1}{l-1}$. Hence the neighbourhood $U_x = \bigcap_{C \in \mathcal{C}_2} U_C$ of x is disjoint from all sets from $\mathcal{C}_2$ and thus

$$\kappa(\mathcal{C}, x) \; \leq \; \mathrm{card}(\{C \in \mathcal{C} : C \cap U_x \neq \emptyset\}) \; \leq \; \mathrm{card}(\mathcal{C}_1) \; = \; 2^m \, .$$

This proves our claim $\kappa(\mathcal{C}) \leq 2^m$. $\square$

Let us remark that the last estimate is sharp. If $\mathcal{C}$ is the covering of $([-1,1]^m, d_\infty)$ by l^m closed balls of radius $\frac{1}{l}$, $l \geq 2$, then $\kappa(\mathcal{C}) = 2^m$.

1.4 Metric and entropy

Controllable coverings of a compact metric space (X, d) are optimal in a particular metric sense. But what characteristics of the metric d are stored in the controllable coverings of (X, d)? What happens if d is replaced by another distance d'? How do the controllable coverings change? What relations between the two metrics d and d' are necessary or sufficient for the coincidence of the controllable coverings of (X, d) and (X, d')?

We assume that (X, d) and (X, d') are two compact metric spaces with the same supporting set X. $\mathring{B}(x, \varepsilon)$ is to denote the open ball with respect to d of radius ε centered at $x \in X$. Moreover, we write $\mathcal{N}'(X, \delta)$, $\varepsilon'_k(M)$, and $\mathring{B}'(x, \varepsilon)$ for the corresponding values or balls, respectively, with respect to d'. The statement to discuss is:

(C) *d and d' admit the same controllable coverings of X.*

Although we did not find a practicable equivalent statement, we can relate **(C)** to the following geometric conditions:

(D) $\mathcal{N}(X, d(x, y)) = \mathcal{N}'(X, d'(x, y))$ *for all* $x, y \in X$ *with* $x \neq y$.

(D*) $\mathring{B}(x, \varepsilon_k(X)) = \mathring{B}'(x, \varepsilon'_k(X))$ *for all* $x \in X$ *and* $k \in \{1, 2, 3, \ldots\}$.

(E) $\mathcal{N}(X, \varepsilon_1(M)) = \mathcal{N}'(X, \varepsilon'_1(M))$ *for all* $M \subseteq X$ *with* $\mathrm{card}(M) > 1$.

(E*) $\varepsilon_1(M) < \varepsilon_k(X) \iff \varepsilon'_1(M) < \varepsilon'_k(X)$ *for all* $M \subseteq X$ *and* $k \in \{1, 2, 3, \ldots\}$.

(J) *The sequences* $(\varepsilon_k(X))_{k=1}^{\infty}$ *and* $(\varepsilon'_k(X))_{k=1}^{\infty}$ *have the same jumps, i.e.*

$$\varepsilon_{k-1}(X) > \varepsilon_k(X) \iff \varepsilon'_{k-1}(X) > \varepsilon'_k(X) \qquad \text{for all } k = 2, 3, 4, \ldots .$$

(R) *Kolmogoroff's entropy functions* $\mathcal{N}(X, \cdot), \mathcal{N}'(X, \cdot) : (0, \infty) \to \{1, 2, 3, \ldots\}$ *have the same ranges.*

(T) *d and d' generate the same topology on X.*

Theorem 1.3 ([Ri4–H], Theorem 3) *Let (X, d) and (X, d') be compact metric spaces. Then the above claims are related in the following way:*

$$\mathbf{(D)} \iff \mathbf{(D^*)} \;\overset{\Longrightarrow}{\not\Longleftarrow}\; \mathbf{(E)} \iff \mathbf{(E^*)} \;\overset{\Longrightarrow}{\not\Longleftarrow}\; \mathbf{(C)} \begin{array}{l} \overset{\Longrightarrow}{\not\Longleftarrow}\; \mathbf{(J)} \iff \mathbf{(R)} \\ \qquad\; \not\Uparrow\not\Downarrow \\ \overset{\Longrightarrow}{\not\Longleftarrow}\; \mathbf{(T)} . \end{array} \qquad \square$$

Next we want to deal with the following question: How can one construct a new metric d' on a compact metric space (X, d) generating the same controllable coverings as d? Clearly, the controllable coverings agree if d' can be written as $d' = \varphi \circ d$ with a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, since then $B'(x, \varphi(r)) = B(x, r)$, $\varepsilon'_k(M) = \varphi(\varepsilon_k(M))$, and the controllability property $\varepsilon_1(C_i) < \varepsilon_{k-1}(X)$ from Proposition 1.1 (iii) is equivalent to

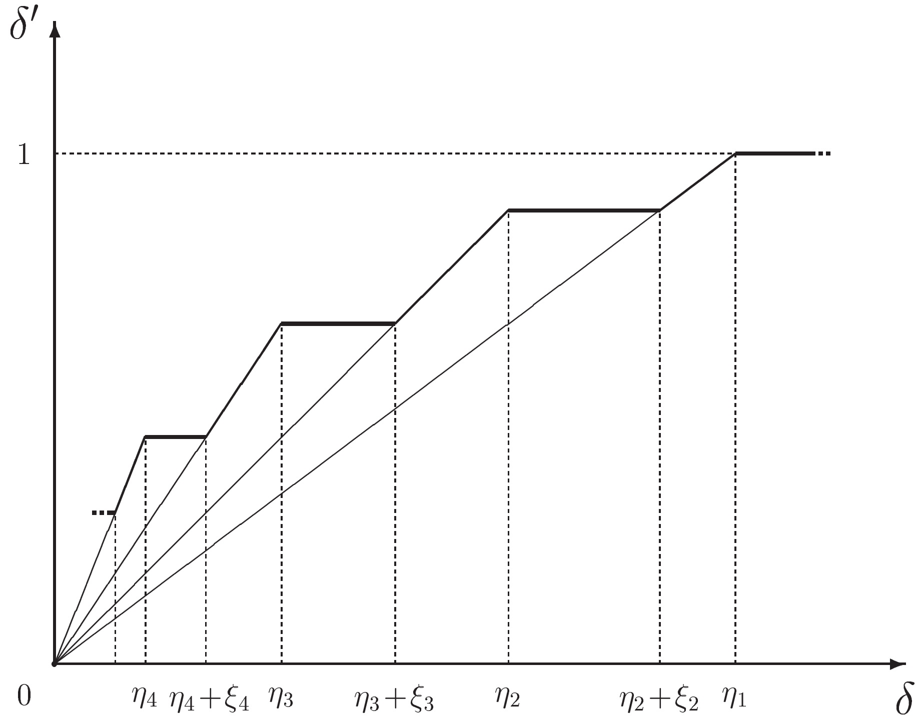

Figure 1.1: $\delta' = \varphi(\delta)$

$\varepsilon'_1(C_i) < \varepsilon'_{k-1}(X)$. However, φ can be chosen more generally. The following result rests on the implication **(D)⇒(C)** from Theorem 1.3. We call a monotonically increasing function $f : \mathbf{R} \to \mathbf{R}$ *strictly increasing from the left in the point* $x_0 \in \mathbf{R}$ if $f(x) < f(x_0)$ for all $x < x_0$.

Proposition 1.4 ([Ri4–H], Theorem 4) *Let (X,d) and (X,d') be compact metric spaces such that $d' = \varphi \circ d$ with a continuous and monotonically increasing function $\varphi : [0,\infty) \to [0,\infty)$ which is strictly increasing from the left in the points $\varepsilon_k(X)$, $k \geq 1$. Then d and d' admit the same controllable coverings of X.* □

Corollary 1.3 ([Ri4–H], Corollary 6) *Let (X,d) be an infinite compact metric space and let $\{\eta_i : i = 1,2,3,\ldots\} = \{\varepsilon_k(X) : k = 1,2,3,\ldots\}$ be the corresponding set of entropy numbers such that $\eta_1 > \eta_2 > \eta_3 > \ldots > 0$. We choose an arbitrary sequence $(\xi_i)_{i=2}^{\infty}$ of real numbers $\xi_i \in [0, \eta_{i-1} - \eta_i)$ fulfilling the condition $\prod_{i=2}^{\infty} \frac{\eta_i + \xi_i}{\eta_{i-1}} = 0$. Finally, we define a function $\varphi : [0,\infty) \to [0,\infty)$ inductively by*

$$\varphi(\delta) = \begin{cases} 1 & \text{for} \quad \delta \in [\eta_1, \infty)\,, \\ \frac{\delta}{\eta_{i-1}}\,\varphi(\eta_{i-1}) & \text{for} \quad \delta \in [\eta_i + \xi_i, \eta_{i-1}]\,, \\ \frac{\eta_i+\xi_i}{\eta_{i-1}}\,\varphi(\eta_{i-1}) & \text{for} \quad \delta \in [\eta_i, \eta_i + \xi_i]\,, \\ 0 & \text{for} \quad \delta = 0 \end{cases}$$

as displayed in Figure 1.1.

Then $(X, \varphi \circ d)$ is a compact metric space as well and the metric $\varphi \circ d$ gives rise to the same controllable coverings as d. $\square$

The metrics d and $d' = \varphi \circ d$ from Corollary 1.3 are essentially different in so far as d can not be expressed as a function $d = \psi \circ d'$ of d'. That is, d can not be reproduced from d'. This shows that the classes of metrics on a compact metrizable space X, which give rise to the same controllable coverings, are large. There even exist examples of metrics d and d' on suitable sets X generating the same controllability such that neither d' is a function of d nor vice versa. A very simple example of that type is given in [Ri4–H]. The question for a practicable and general characterization of "equivalent" metrics in this sense seems to be highly non-trivial and remains open.

Chapter 2

Continuous functions on compact metric spaces

2.1 Controllable partitions of unity

Given a compact metric space (X, d), we use the symbol $C(X)$ for denoting the Banach space of all continuous real-valued functions on X equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$. We define the *support of a function* $f \in C(X)$ by

$$\mathrm{supp}(f) \;=\; \{x \in X : f(x) \neq 0\} \; . \tag{2.1}$$

Note that this differs from the classical notation, where the closure of the open set $\{x \in X : f(x) \neq 0\}$ is defined to be the support of f. This modification of the usual definition will become essential in Chapter 4. Finally, a system of non-negative functions $\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \subseteq C(X)$ is called a *partition of unity on* X if $\sum_{i=1}^{k} \varphi_i(x) = 1$ for all $x \in X$.

Partitions of unity fulfilling geometric restrictions can be used for approximating functions $f \in C(X)$. The following error estimate makes use of the *modulus of continuity of* f, which is defined by

$$\omega(f, \delta) \;=\; \{|f(x) - f(y)| : x, y \in X, d(x, y) \leq \delta\} \tag{2.2}$$

for $\delta \geq 0$. We do not present the straightforward proof of Proposition 2.1. It follows an idea from [Ca/Ste], pp. 178-179.

Proposition 2.1 *Let* $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ *be a partition of unity on a compact metric space* (X, d), $x_1, x_2, \ldots, x_k$ *points from* X, *and* $r \geq 0$ *a radius such that* $\mathrm{supp}(\varphi_i) \subseteq B(x_i, r)$ *for* $1 \leq i \leq k$. *Then*

$$\left\| f - \sum_{i=1}^{k} f(x_i)\varphi_i \right\| \;\leq\; \omega(f, r) \; . \quad \square$$

In [Ste] the concept of the partition of unity has been associated with the geometric controllability condition. A partition of unity $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ on (X, d) is called *controllable* if $\{\mathrm{supp}(\varphi_i) : 1 \le i \le k\}$ is a controllable covering of X. It is shown in [Ste] (but can also be seen by the aid of Proposition 1.3) that controllable partitions of unity are *peaked*. That is, all partition functions φ_i, $1 \le i \le k$, are of norm one (cf. [Mi/Pe1]) or, in other words, there exist *peak points* $t_1, t_2, \ldots, t_k \in X$ with $\varphi_i(t_j) = \delta_{ij}$. Hence the partition functions are linearly independent, i.e. $\dim(\mathrm{span}(\Phi)) = \mathrm{card}(\Phi)$. Moreover, the norm of a function φ from the span of Φ can be computed by

$$\|\varphi\| \;=\; \max\{|\varphi(t_1)|, |\varphi(t_2)|, \ldots, |\varphi(t_k)|\}$$

(cf. [Mi/Pe1]), which shows that $\mathrm{span}(\Phi)$ is isometrically isomorphic to $l_\infty^{\mathrm{card}(\Phi)}$.

When uniformly approximating functions $f \in C(X)$ by linear combinations of the partition functions φ_i, $1 \le i \le k$, we define the corresponding *error quantity* by

$$E(f, \mathrm{span}(\Phi)) \;=\; \inf\{\|f - \varphi\| : \varphi \in \mathrm{span}(\Phi)\}\ . \tag{2.3}$$

In the non-trivial case $\mathrm{card}(\Phi) = k > 1$ there exist points $x_1, x_2, \ldots, x_k \in X$ with $\mathrm{supp}(\varphi_i) \subseteq B(x_i, \varepsilon_{k-1}(X))$ according to the controllability property (iii) from Proposition 1.1. Hence, by Proposition 2.1,

$$E(f, \mathrm{span}(\Phi)) \;\le\; \omega\left(f, \varepsilon_{\mathrm{card}(\Phi)-1}(X)\right)$$

for all $f \in C(X)$ and all controllable partitions of unity Φ on X with $\mathrm{card}(\Phi) > 1$.

It is shown in [Ste] that there exist controllable partitions of unity of cardinality k if and only if $k = 1$ or $\varepsilon_k(X) < \varepsilon_{k-1}(X)$. If X is an infinite space then there are infinitely many jumps $\varepsilon_k(X) < \varepsilon_{k-1}(X)$ in the sequence of the entropy numbers of X, since $\lim\limits_{k\to\infty} \varepsilon_k(X) = 0$. Hence one can find a sequence $(\Phi_n)_{n=1}^\infty$ of controllable partitions of unity with $\lim\limits_{n\to\infty} \mathrm{card}(\Phi_n) = \infty$. By the above estimate, the corresponding error quantities tend to zero, $\lim\limits_{n\to\infty} E(f, \mathrm{span}(\Phi_n)) = 0$. However, if the sequence $(\Phi_n)_{n=1}^\infty$ is subject to the condition $\lim\limits_{n\to\infty} \mathrm{card}(\Phi_n) = \infty$ only, the linear spaces $\mathrm{span}(\Phi_n)$ in general will not form a linear approximation scheme in $C(X)$. Namely, the spaces $\mathrm{span}(\Phi_n)$ need not be ordered by inclusion.

Now the following question arises: Given an infinite compact metric space (X, d), does there exist a *chain* $(\Phi_n)_{n=1}^\infty$ *of controllable partitions of unity on* X? That is, the partitions have to be ordered increasingly in so far as $\mathrm{span}(\Phi_n) \subseteq \mathrm{span}(\Phi_{n+1})$ and $\mathrm{span}(\Phi_n) \ne \mathrm{span}(\Phi_{n+1})$ for $n = 1, 2, 3, \ldots$.

We are not able to solve the problem in general, although we expect a positive answer for all infinite spaces (X, d). A very simple chain of controllable partitions of unity is presented in Example 1 from [Ri8–H], where the space $X = \{2^{-i} : i = 0, 1, 2, \ldots\} \cup \{0\}$ equipped with the usual distance $d(x, y) = |x - y|$ is considered. Stephani has shown that there exist chains for the cantor set $X = 2^\omega$ (private communication). However, these two spaces are totally disconnected and the partition functions are step functions with the values 0 and 1 only. If X is connected (or has at least one infinite connected component) the situation becomes much more difficult, since then we do not have as simple continuous functions as on disconnected spaces.

The following necessary condition for the existence of a chain of controllable partitions of unity on the simple space $[0,2]$ equipped with the usual distance illustrates the problem. The partition functions must be very close to step functions in the following sense. (In the first chapter we have considered the interval $[-1,1]$ and the cube $[-1,1]^m$. Sometimes we shall switch over to $[0,2]$ and $[0,2]^m$ in order to have the opportunity to use particular notations from [Ri6–H] and other papers.)

Proposition 2.2 ([Ri6–H], Proposition 2) *Let $(\Phi_n)_{n=0}^\infty$ be a chain of controllable partitions of unity in $C([0,2])$. Then, for any $n_0 \geq 0$, there exists a subset $D_{n_0} \subseteq [0,2]$ of Lebesgue measure $\nu(D_{n_0}) = 2$ such that $D_{n_0} = \bigcup_{\iota \in \mathcal{I}} I_\iota$ is a countable union of intervals I_ι and any function from Φ_{n_0} is constant on any interval I_ι, $\iota \in \mathcal{I}$.* $\square$

In the following section we shall illustrate how a chain of controllable partitions of unity on $[0,2]$ and on $[0,2]^m$ can be constructed. This particular construction will give rise to certain applications.

2.2 A chain of controllable partitions of unity on the cube and the approximation of Hölder continuous functions

Theorem 2.1 ([Ri6–H], Theorem 1) *There exists a chain $(\Psi_n)_{n=0}^\infty$ of controllable partitions of unity on the cube $([0,2]^m, d_\infty)$.* $\square$

Let us have a look at the shape of the partition functions. The main part of the construction happens in the one-dimensional case, where partitions $\Phi_n = \left\{\varphi_i^{(n)} : 1 \leq i \leq 2^{4^n}\right\}$ are defined for $n \geq 0$. The general structure of a function $\varphi_i^{(n)}$ is illustrated in Figure 2.1. The support $C_i^{(n)}$ of $\varphi_i^{(n)}$ consists of the middle part $M_i^{(n)}$, where the value of $\varphi_i^{(n)}$ is 1, and of two "critical" parts $CL_i^{(n)}$ and $CR_i^{(n)}$, where $\varphi_i^{(n)}$ increases from 0 to 1 and decreases from 1 to 0, respectively. In the left critical part $CL_i^{(n)}$ the support of $\varphi_i^{(n)}$ overlaps with supp $\left(\varphi_{i-1}^{(n)}\right)$, in the right critical part $CR_i^{(n)}$ with supp $\left(\varphi_{i+1}^{(n)}\right)$. Proposition 2.2 demands that $\varphi_i^{(n)}$ is locally constant almost everywhere. This suggests that on $CL_i^{(n)}$ and $CR_i^{(n)}$ the function $\varphi_i^{(n)}$ behaves similarly as the well-known Lebesgue singular function (cf. Figure 2.2). The

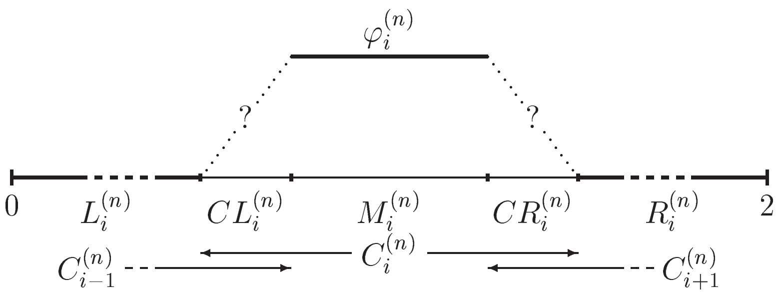

Figure 2.1: General structure of $\varphi_i^{(n)}$

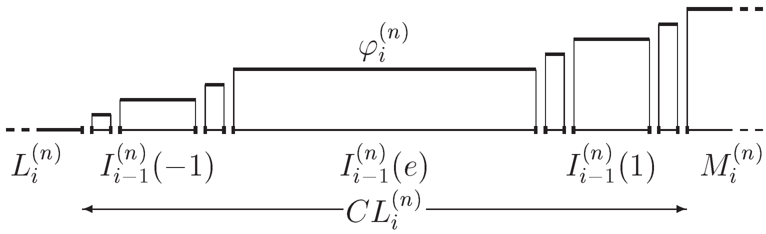

Figure 2.2: Behaviour of $\varphi_i^{(n)}$ on $CL_i^{(n)}$

particular choice of the supports $C_i^{(n)}$ and of the intervals displayed in Figure 2.2 guarantee that every partition Φ_n is controllable and that the chain condition $\Phi_n \subseteq \operatorname{span}(\Phi_{n+1})$ is fulfilled. Figure 2.3 shows how $\varphi_i^{(n)}$ can be attained as a linear combination of functions from Φ_{n+1} on the right critical part $CR_i^{(n)}$. For details we refer the reader to [Ri6–H]. The final step leading from the interval $[0,2]$ to the m-dimensional cube $[0,2]^m$ is simply a tensor product argument, namely

$$\Psi_n \;=\; \left\{\psi^{(n)}_{(i_1,i_2,\ldots,i_m)} : i_1, i_2, \ldots, i_m \in \left\{1, 2, \ldots, 2^{4^n}\right\}\right\}$$

with

$$\psi^{(n)}_{(i_1,i_2,\ldots,i_m)}(\xi_1,\xi_2,\ldots,\xi_m) \;=\; \varphi^{(n)}_{i_1}(\xi_1)\,\varphi^{(n)}_{i_2}(\xi_2)\ldots\varphi^{(n)}_{i_m}(\xi_m)$$

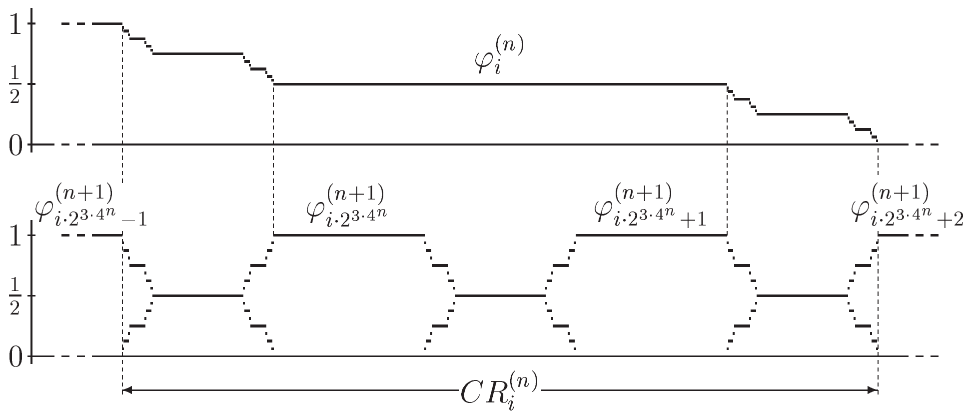

Figure 2.3: $\varphi_i^{(n)}$ on $CR_i^{(n)}$ as a linear combination

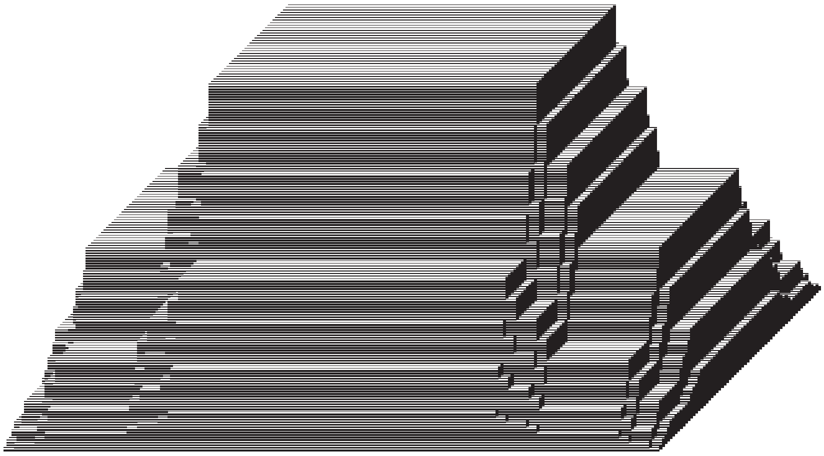

Figure 2.4: Graph of a function $\psi^{(n)}_{(i_1,i_2)}$

(cf. Figure 2.4).

The increasing sequence of linear spaces $\mathrm{span}(\Psi_n)$ can be considered as a linear approximation scheme in $C([0,2]^m)$. By Proposition 2.1, we obtain the following estimate of Jackson type.

Proposition 2.3 ([Ri6–H], Theorem 2) *Let $f \in C([0,2]^m)$ and $n \geq 0$. Then*

$$E(f, \mathrm{span}(\Psi_n)) \;\leq\; \omega\left(f, \frac{1}{2^{4^n}-1}\right) . \qquad \square$$

Moreover, the complicated shape of the partition functions gives rise to a surprising inverse estimate if f is *Hölder continuous of type* α, $0 < \alpha \leq 1$, that is if

$$|f|_\alpha \;=\; \sup_{\delta>0} \frac{\omega(f,\delta)}{\delta^\alpha} \;<\; \infty .$$

Proposition 2.4 ([Ri6–H], Theorem 4) *Let $f \in C([0,2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$. Then*

$$\liminf_{n\to\infty} \frac{E(f, \mathrm{span}(\Psi_n))}{\omega\left(f, \frac{1}{2^{4^n}-1}\right)} \;\geq\; \frac{1}{2\left(\alpha^{-\frac{1}{2}}+1\right)} . \qquad \square$$

Unfortunately, the approximation scheme

$$\mathrm{span}(\Psi_0) \;\subseteq\; \mathrm{span}(\Psi_1) \;\subseteq\; \mathrm{span}(\Psi_2) \;\subseteq\; \ldots$$

is relatively "thin", since $\dim(\mathrm{span}(\Psi_n)) = \mathrm{card}(\Psi_n) = 2^{m4^n}$. However, one can pass to a completed scheme

$$\mathrm{span}\left(\tilde{\Psi}_1\right) \;\subseteq\; \mathrm{span}\left(\tilde{\Psi}_2\right) \;\subseteq\; \mathrm{span}\left(\tilde{\Psi}_3\right) \;\subseteq\; \ldots$$

with $\dim\left(\operatorname{span}\left(\tilde{\Psi}_n\right)\right) = n$ by inserting suitable peaked partitions of unity between the members of the original chain $(\Psi_n)_{n=0}^{\infty}$. Note that the completed chain $\left(\tilde{\Psi}_n\right)_{n=1}^{\infty}$ does not consist of controllable partitions only! In Example 1.1 we have seen that controllable partitions of unity on the cube $[0,2]^m$ must be of cardinality l^m, $l \geq 1$.

In [Ri6–H] the additional partitions have been chosen in a way which preserves the pleasant approximation properties. Before presenting these results we introduce corresponding approximation quantities for bounded linear operators $T \in L(E, C([0,2]^m))$ mapping a Banach space E into $C([0,2]^m)$. We put

$$E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \;=\; \inf\left\{\|T - A\| :\ A \in L(E, C([0,2]^m)) \text{ with } A(E) \subseteq \operatorname{span}\left(\tilde{\Psi}_n\right)\right\} .$$

The *modulus of continuity of* T is defined by

$$\omega(T,\delta) \;=\; \sup_{\|z\|\leq 1} \omega(Tz,\delta)$$

for $\delta > 0$. Let us recall that T is compact if and only if $\lim\limits_{\delta\to 0}\omega(T,\delta) = 0$. The operator T is called *Hölder continuous of type* α, $0 < \alpha \leq 1$, if

$$|T|_\alpha \;=\; \sup_{\delta>0} \frac{\omega(T,\delta)}{\delta^\alpha} \;<\; \infty$$

(cf. [Ca/Ste]).

Theorem 2.2 ([Ri6–H], Theorem 6) *There exist positive operators $\tilde{A}_n \in L(C([0,2]^m))$, $n \geq 1$, mapping $C([0,2]^m)$ into $\operatorname{span}\left(\tilde{\Psi}_n\right)$ such that:*
(a) For any $f \in C([0,2]^m)$,

$$E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \;\leq\; \left\|f - \tilde{A}_n f\right\| \;\leq\; 7\,\omega(f, \varepsilon_n([0,2]^m)) .$$

(b) For any Banach space E and any operator $T \in L(E, C([0,2]^m))$,

$$E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right) \;\leq\; \left\|T - \tilde{A}_n T\right\| \;\leq\; 7\,\omega(T, \varepsilon_n([0,2]^m)) . \qquad \square$$

Theorem 2.3 ([Ri6–H], Theorem 7) *(a) Let $f \in C([0,2]^m)$ be non-constant and Hölder continuous of type α, $0 < \alpha \leq 1$. Then*

$$\liminf_{n\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)}{\omega\left(f, \varepsilon_n([0,2]^m)\right)} \;\geq\; \frac{1}{16\left(2\,\alpha^{-\frac{1}{2}} + 1\right)} .$$

(b) Let E be a Banach space and let $T \in L(E, C([0,2]^m))$ be Hölder continuous of type α, $0 < \alpha \leq 1$, such that the image $T(E)$ does not consist of constant functions only. Then

$$\liminf_{n\to\infty} \frac{E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)}{\omega\left(T, \varepsilon_n([0,2]^m)\right)} \;\geq\; \frac{1}{16\left(2\,\alpha^{-\frac{1}{2}} + 1\right)} . \qquad \square$$

Note that the bounds $\omega(f, \varepsilon_n([0,2]^m))$ and $\omega(T, \varepsilon_n([0,2]^m))$ do not only reflect the smoothness of f and T, respectively, but rather relate it to the degree of compactness of the space $[0,2]^m$ by the aid of the geometric quantity $\varepsilon_n([0,2]^m)$. The theorems show that the asymptotics of the modulus of continuity of a Hölder continuous function $f \in C([0,2]^m)$ and of a Hölder continuous operator $T \in L(E, C([0,2]^m))$ coincides with the asymptotics of the approximation quantities $E\left(f, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)$ and $E\left(T, \operatorname{span}\left(\tilde{\Psi}_n\right)\right)$, respectively. This implies in particular that all non-constant functions from the spaces $\operatorname{span}\left(\tilde{\Psi}_n\right)$ as well as the operators $\tilde{A}_n$ are not Hölder continuous.

Of course, the asymptotics of the entropy numbers of the cube $[0,2]^m$ and of the modulus of continuity of functions f and operators T are not affected if the metric d_∞ is replaced by an *equivalent* metric d', that is, if $c_1 \cdot d'(x,y) \leq d_\infty(x,y) \leq c_2 \cdot d'(x,y)$ for all $x, y \in [0,2]^m$ with universal constants $c_1, c_2 > 0$. Also the property of Hölder continuity of functions $f \in C([0,2]^m)$ and operators $T \in L(E, C([0,2]^m))$ remains untouched under a change from d_∞ to d'. Hence the last two theorems remain true for the space $([0,2]^m, d')$ up to a modification of the constants. For instance, one can consider the Euclidean distance on $[0,2]^m$. But even the m-dimensional Euclidean ball equipped with the Euclidean metric can be considered as a space $([0,2]^m, d')$ of that type, as can be shown by the aid of a suitable homeomorphism mapping the ball onto the cube.

2.3 A basis in $C([0,2]^m)$ consisting of Cantor-like functions

The above mentioned chain $\left(\tilde{\Psi}_n\right)_{n=1}^\infty$ of peaked partitions of unity gives rise to a Schauder basis in $C([0,2]^m)$ as can be seen by a general statement on so-called peaked partition subspaces (cf. e.g. [Se], Section 4.4). We shall construct a monotone and interpolating basis $(\varphi_i)_{i=1}^\infty$ of $C([0,2]^m)$ which essentially preserves the error estimates from the last two theorems. We recall that a basis $(\varphi_i)_{i=1}^\infty$ is called *monotone* if the partial sums $S_n f = \sum_{i=1}^n \alpha_i \varphi_i$ of a function $f = \sum_{i=1}^\infty \alpha_i \varphi_i$ fulfil $\|S_1 f\| \leq \|S_2 f\| \leq \|S_3 f\| \leq \ldots$. The basis is called *interpolating with nodes* $(t_i)_{i=1}^\infty \subseteq [0,2]^m$ if $S_n f(t_i) = f(t_i)$ for all $n \geq 1$ and $1 \leq i \leq n$.

Let us go into some details of the construction of the chain $\left(\tilde{\Psi}_n\right)_{n=1}^\infty$ of peaked partitions of unity on the cube $[0,2]^m$. The partitions $\tilde{\Psi}_{4^{ml}}$ of cardinality 4^{ml} had been defined by

$$
\begin{aligned}
\tilde{\Psi}_{4^{ml}} &= \left\{ \tilde{\psi}^{(4^{ml})}_{(i_1, i_2, \ldots, i_m)} : i_1, i_2, \ldots, i_m \in \left\{1, 2, \ldots, 4^l\right\} \right\} \qquad \text{with} \\
\tilde{\psi}^{(4^{ml})}_{(i_1, i_2, \ldots, i_m)}(\xi_1, \xi_2, \ldots, \xi_m) &= \tilde{\varphi}^{(4^l)}_{i_1}(\xi_1) \tilde{\varphi}^{(4^l)}_{i_2}(\xi_2) \ldots \tilde{\varphi}^{(4^l)}_{i_m}(\xi_m) \,,
\end{aligned}
\tag{2.4}
$$

where the functions $\tilde{\varphi}^{(4^l)}_i$, $1 \leq i \leq 4^l$, form a peaked partition of unity $\tilde{\Phi}_{(4^l)}$ on the interval $[0,2]$. The partition $\tilde{\Phi}_{(4^0)} = \left\{\tilde{\varphi}^{(4^0)}_1\right\}$ is the trivial one. If $l \geq 1$ one makes use of the corresponding exponent $e(l) \in \{0,1,2,\ldots\}$ determined by

$$4^l \in \left\{2^{4^{e(l)}}, 2^{4^{e(l)}}+1, \ldots, 2^{4^{e(l)+1}}-1\right\} \tag{2.5}$$

and of the function

$$H(n) = 2 \sum_{j=n+1}^{\infty} 2^{-4^j} .$$

Then the supports of the functions from $\tilde{\Phi}_{(4^l)}$, $l \geq 1$, are

$$\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right) = \begin{cases} \left[0,\ 2\cdot 4^{-l} + H(e(l))\right) & , \ i = 1 , \\ \left(2(i-1)\cdot 4^{-l} - H(e(l)),\ 2i\cdot 4^{-l} + H(e(l))\right) & , \ 1 < i < 4^l , \\ \left(2(4^l-1)\cdot 4^{-l} - H(e(l)),\ 2\right] & , \ i = 4^l , \end{cases}$$

(cf. [Ri6–H], p. 184). Hence $\tilde{\varphi}_i^{(4^l)}(x) = 1$ if $x \in \left[2(i-1)\cdot 4^{-l} + H(e(l)), 2i\cdot 4^{-l} - H(e(l))\right]$. (Formulas (7), (9), and (40) from [Ri6–H] show that these intervals are non-empty.)

For $l \geq 1$ and $1 \leq i \leq 4^l$, we put $t_i^{(l)} = 2i\cdot 4^{-l} - H(e(l))$. If $l = 0$ we define $t_1^{(0)} = t_1^{(1)}$. Then $\tilde{\varphi}_i^{(4^l)}\left(t_j^{(l)}\right) = \delta_{ij}$ for $i,j \in \{1,2,\ldots,4^l\}$ and $l \geq 0$. Obviously, $\operatorname{supp}\left(\tilde{\varphi}_i^{(4^l)}\right) \subseteq B\left(t_i^{(l)}, 2\cdot 4^{-l}\right)$. Moreover, we have

$$\left\{t_i^{(l)} : 1 \leq i \leq 4^l\right\} \subseteq \left\{t_i^{(l+1)} : 1 \leq i \leq 4^{l+1}\right\} .$$

Indeed, this claim is trivial if $l = 0$. If $l \geq 1$ and $e(l) = e(l+1)$ then $t_i^{(l)} = t_{4i}^{(l+1)}$. In the remaining case $l \geq 1$ and $e(l) \neq e(l+1)$ we obtain $e(l+1) = e(l)+1$ and $2^{4^{e(l)+1}} = 4^{l+1}$ according to (2.5) and hence

$$t_i^{(l)} = 2i\cdot 4^{-l} - H(e(l)) = 2i\cdot 4^{-l} - 2\cdot 2^{-4^{e(l)+1}} - H(e(l)+1) = 2(4i-1)\cdot 4^{-(l+1)} - H(e(l+1)) = t_{4i-1}^{(l+1)}.$$

Let us come back to the m-dimensional case. We put $t_{(i_1,i_2,\ldots,i_m)}^{(l)} = \left(t_{i_1}^{(l)}, t_{i_2}^{(l)}, \ldots, t_{i_m}^{(l)}\right)$ for $l \geq 0$ and $i_1, i_2, \ldots, i_m \in \left\{1,2,\ldots,4^l\right\}$. By definition (2.4), the last claims yield

$$\tilde{\psi}_{(i_1,i_2,\ldots,i_m)}^{(4^{ml})}\left(t_{(j_1,j_2,\ldots,j_m)}^{(l)}\right) = \delta_{(i_1,i_2,\ldots,i_m)\,(j_1,j_2,\ldots,j_m)} , \tag{2.6}$$

$$\operatorname{supp}\left(\tilde{\psi}_{(i_1,i_2,\ldots,i_m)}^{(4^{ml})}\right) \subseteq B\left(t_{(i_1,i_2,\ldots,i_m)}^{(l)}, 2\cdot 4^{-l}\right) , \tag{2.7}$$

$$\left\{t_{(i_1,i_2,\ldots,i_m)}^{(l)} : 1 \leq i_1, i_2, \ldots, i_m \leq 4^l\right\} \subseteq \left\{t_{(i_1,i_2,\ldots,i_m)}^{(l+1)} : 1 \leq i_1, i_2, \ldots, i_m \leq 4^{l+1}\right\} .$$

Let $\{t_i : i = 1,2,3,\ldots\} = \left\{t_{(i_1,i_2,\ldots,i_m)}^{(l)} : l \geq 0,\ 1 \leq i_1, i_2, \ldots, i_m \leq 4^l\right\}$. According to the last inclusion we can choose the indices of the points $t_i \in [0,2]^m$ such that

$$\{t_1, t_2, \ldots, t_{4^{ml}}\} = \left\{t_{(i_1,i_2,\ldots,i_m)}^{(l)} : 1 \leq i_1, i_2, \ldots, i_m \leq 4^l\right\} . \tag{2.8}$$

Now we define the basis sequence $(\varphi_i)_{i=1}^{\infty} \subseteq C([0,2]^m)$ by the aid of the points t_i, $i \geq 1$. We start with $\varphi_1 = \tilde{\psi}_{(1,1,\ldots,1)}^{(4^{m\cdot 0})}$. By (2.6), the functions φ_i, $2 \leq i \leq 4^{m\cdot 1}$, can be taken from

$\tilde{\Psi}_{4^{m\cdot 1}}$ such that $\varphi_i(t_j) = \delta_{ij}$ for $1 \le j \le 4^{m\cdot 1}$. Similarly, we choose φ_i, $4^{m\cdot 1}+1 \le i \le 4^{m\cdot 2}$, from $\tilde{\Psi}_{4^{m\cdot 2}}$ with $\varphi_i(t_j) = \delta_{ij}$ for $1 \le j \le 4^{m\cdot 2}$, etc. In the end we obtain a sequence $(\varphi_i)_{i=1}^{n} \subseteq C([0,2]^m)$ with

$$\varphi_i(t_j) \;=\; \delta_{ij} \qquad \text{for} \qquad 4^{ml}+1 \le i \le 4^{m(l+1)}\,,\; 1 \le j \le 4^{m(l+1)}\;. \tag{2.9}$$

This shows in particular that the functions $\varphi_i \in \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)$, $1 \le i \le 4^{ml}$, are linearly independent. Accordingly,

$$\operatorname{span}\left(\{\varphi_1, \varphi_2, \ldots, \varphi_{4^{ml}}\}\right) \;=\; \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right) \qquad \text{for} \qquad l = 0, 1, 2, \ldots\;. \tag{2.10}$$

Theorem 2.4 *The above functions $(\varphi_i)_{i=1}^{\infty}$ form a monotone and interpolating basis of $C([0,2]^m)$ with nodes $(t_i)_{i=1}^{\infty}$. The n-th partial sum $S_n f$ of the expansion of a function $f \in C([0,2]^m)$ is subject to the error estimate*

$$\|f - S_n f\| \;\le\; 12\,\omega(f, \varepsilon_n([0,2]^m))\;.$$

Moreover, if f is non-constant and Hölder continuous of type α, $0 < \alpha \le 1$, then

$$\liminf_{n\to\infty} \frac{\|f - S_n f\|}{\omega\left(f, \varepsilon_n([0,2]^m)\right)} \;\ge\; \frac{1}{16\left(2\,\alpha^{-\frac{1}{2}}+1\right)}\;.$$

Proof. Before verifying the basis properties and the two error estimates, we begin with a technical claim.

1. Let $(\alpha_i)_{i=1}^{\infty}$ be a sequence of reals such that the function $f = \sum_{i=1}^{\infty} \alpha_i \varphi_i$ exists in $C([0,2]^m)$ and let $f_n = \sum_{i=1}^{n} \alpha_i \varphi_i$ be the n-th partial sum. We shall show that

$$\|f_n\| \;=\; \max\{|f(t_1)|, |f(t_2)|, \ldots, |f(t_n)|\}\;. \tag{2.11}$$

We fix $l \ge 0$ such that $4^{ml} \le n < 4^{m(l+1)}$. By (2.10), we have $f_n \in \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)$. The partition of unity $\tilde{\Psi}_{4^{m(l+1)}}$ has the peak points $\left\{t^{(l+1)}_{(i_1,i_2,\ldots,i_m)} : 1 \le i_1, i_2, \ldots, i_m \le 4^{l+1}\right\} = \{t_1, t_2, \ldots, t_{4^{m(l+1)}}\}$ in accordance with (2.6) and (2.8). Thus the norm of f_n can be computed by

$$\|f_n\| \;=\; \max\{|f_n(t_1)|, |f_n(t_2)|, \ldots, |f_n(t_{4^{m(l+1)}})|\}\;. \tag{2.12}$$

Similar arguments concerning the function $f_{4^{ml}} = \sum_{i=1}^{4^{ml}} \alpha_i \varphi_i \in \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)$ yield

$$\|f_{4^{ml}}\| \;=\; \max\{|f_{4^{ml}}(t_1)|, |f_{4^{ml}}(t_2)|, \ldots, |f_{4^{ml}}(t_{4^{ml}})|\}\;.$$

By (2.9), we obtain $f_n(t_j) - f_{4^{ml}}(t_j) = \sum_{i=4^{ml}+1}^{n} \alpha_i \varphi_i(t_j) = \sum_{i=4^{ml}+1}^{n} \alpha_i \delta_{ij}$ for $1 \le j \le 4^{m(l+1)}$. Hence $|f_n(t_j)| = |f_{4^{ml}}(t_j)|$ for $1 \le j \le 4^{ml}$ as well as for $n+1 \le j \le 4^{m(l+1)}$. The equations with $1 \le j \le 4^{ml}$ show that

$$\|f_{4^{ml}}\| \;=\; \max\{|f_n(t_1)|, |f_n(t_2)|, \ldots, |f_n(t_{4^{ml}})|\}\;, \tag{2.13}$$

in particular $\|f_{4^{ml}}\| \le \|f_n\|$. The remaining equations imply that $|f_n(t_j)| \le \|f_{4^{ml}}\|$ for $n+1 \le j \le 4^{m(l+1)}$. Thus (2.12) can be continued to

$$\|f_n\| \;=\; \max\{|f_n(t_1)|, |f_n(t_2)|, \dots, |f_n(t_n)|, \|f_{4^{ml}}\|, \|f_{4^{ml}}\|, \dots, \|f_{4^{ml}}\|\}\,.$$

Combining this with (2.13) yields

$$\|f_n\| \;=\; \max\{|f_n(t_1)|, |f_n(t_2)|, \dots, |f_n(t_n)|\}\,.$$

Finally, we note that

$$\varphi_i(t_i) \;=\; 1 \quad \text{and} \quad \varphi_i(t_j) \;=\; 0 \quad \text{for} \quad i = 1, 2, 3, \dots\,,\; 1 \le j < i \tag{2.14}$$

according to (2.9). Hence $f_n(t_j) = f(t_j)$ for $1 \le j \le n$ so that $\|f_n\|$ can be written in the desired form (2.11).

2. Equation (2.11) shows in particular that $\left\|\sum\limits_{i=1}^{n} \alpha_i\varphi_i\right\| \;\le\; \left\|\sum\limits_{i=1}^{n+1} \alpha_i\varphi_i\right\|$ for arbitrary $\alpha_i \in \mathbf{R}$ and $n \ge 1$. The linear span of the sequence $(\varphi_i)_{i=1}^{\infty}$ is dense in $C([0,2]^m)$, since $\operatorname{span}(\{\varphi_1, \varphi_2, \varphi_3, \dots\}) = \bigcup\limits_{l=0}^{\infty} \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)$ by (2.10) and $\bigcup\limits_{l=0}^{\infty} \operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)$ is dense as Theorem 2.2 shows. Of course, $\|\varphi_i\| \ne 0$ for $i \ge 1$. These three properties imply that $(\varphi_i)_{i=1}^{\infty}$ is a basis of $C([0,2]^m)$ (cf. [Li/Tza], Proposition 1.a.3). The basis is monotone, for $\left\|\sum\limits_{i=1}^{n} \alpha_i\varphi_i\right\| \le \left\|\sum\limits_{i=1}^{n+1} \alpha_i\varphi_i\right\|$, and interpolating with nodes $(t_i)_{i=1}^{\infty}$ according to (2.14) (cf. [Se], Proposition 1.3.2).

3. Next we shall prove the estimate

$$\|f - S_n f\| \;\le\; 12\,\omega(f, \varepsilon_n([0,2]^m)) \tag{2.15}$$

for arbitrary functions $f = \sum\limits_{i=1}^{\infty} \alpha_i\varphi_i \in C([0,2]^m)$ and $n \ge 1$.

The supports of the partition functions from $\tilde{\Psi}_{4^{ml}}$, $l \ge 0$, are subject to the inclusions (2.7). Hence Proposition 2.1 yields

$$\left\|f - \tilde{\psi}\right\| \;\le\; \omega\left(f, 2\cdot 4^{-l}\right)$$

where $\tilde{\psi} \;=\; \sum\limits_{(i_1,i_2,\dots,i_m)\in\{1,2,\dots,4^l\}^m} f\left(t^{(l)}_{(i_1,i_2,\dots,i_m)}\right) \tilde{\psi}^{(4^{ml})}_{(i_1,i_2,\dots,i_m)}$. We have $\tilde{\psi}\left(t^{(l)}_{(i_1,i_2,\dots,i_m)}\right) = f\left(t^{(l)}_{(i_1,i_2,\dots,i_m)}\right)$, since the points $t^{(l)}_{(i_1,i_2,\dots,i_m)}$ are peaks of the partition of unity $\tilde{\Psi}_{4^{ml}}$. By (2.8), this means that

$$\tilde{\psi}(t_j) \;=\; f(t_j) \qquad \text{for} \qquad 1 \le j \le 4^{ml}\,.$$

On the other hand, the 4^{ml}-th partial sum $S_{4^{ml}}f$ belongs to $\operatorname{span}\left(\tilde{\Psi}_{4^{ml}}\right)$ as well (cf. (2.10)). Moreover, $S_{4^{ml}}f(t_j) = f(t_j) = \tilde{\psi}(t_j)$ for $1 \le j \le 4^{ml}$, since the basis is interpolating. Thus $S_{4^{ml}}f = \tilde{\psi}$. This shows that

$$\|f - S_{4^{ml}}f\| \;\le\; \omega\left(f, 2\cdot 4^{-l}\right) \qquad \text{for} \qquad l = 0, 1, 2, \dots\,. \tag{2.16}$$

Now let $n \geq 1$ be fixed. We choose $l \geq 0$ such that $4^{ml} \leq n < 4^{m(l+1)}$. Of course,

$$\|f - S_n f\| \leq \|f - S_{4^{m(l+1)}} f\| + \|S_{4^{m(l+1)}} f - S_n f\| . \tag{2.17}$$

The function $S_{4^{m(l+1)}} f - S_n f = \sum_{i=n+1}^{4^{m(l+1)}} \alpha_i \varphi_i$ can be considered as the $4^{m(l+1)}$-st partial sum of itself. Hence, by (2.11) and (2.9),

$$\begin{aligned} \|S_{4^{m(l+1)}} f - S_n f\| &= \max_{1 \leq j \leq 4^{m(l+1)}} \left| \sum_{i=n+1}^{4^{m(l+1)}} \alpha_i \varphi_i(t_j) \right| = \max_{1 \leq j \leq 4^{m(l+1)}} \left| \sum_{i=n+1}^{4^{m(l+1)}} \alpha_i \delta_{ij} \right| \\ &= \max\{|\alpha_{n+1}|, |\alpha_{n+2}|, \ldots, |\alpha_{4^{m(l+1)}}|\} . \end{aligned}$$

The same arguments prove that $\|S_{4^{m(l+1)}} f - S_{4^{ml}} f\| \leq \max\{|\alpha_{4^{ml}+1}|, |\alpha_{4^{ml}+2}|, \ldots, |\alpha_{4^{m(l+1)}}|\}$. Thus $\|S_{4^{m(l+1)}} f - S_n f\| \leq \|S_{4^{m(l+1)}} f - S_{4^{ml}} f\|$ and equation (2.17) can be continued, with the additional help of (2.16) and the subadditivity of the modulus of continuity, namely

$$\begin{aligned} \|f - S_n f\| &\leq \|f - S_{4^{m(l+1)}} f\| + \|S_{4^{m(l+1)}} f - S_{4^{ml}} f\| \\ &\leq \|f - S_{4^{m(l+1)}} f\| + \|S_{4^{m(l+1)}} f - f\| + \|f - S_{4^{ml}} f\| \\ &\leq 2\,\omega\left(f, 2 \cdot 4^{-(l+1)}\right) + \omega\left(f, 2 \cdot 4^{-l}\right) \\ &= 2\,\omega\left(f, 2 \cdot 4^{-(l+1)}\right) + \omega\left(f, 8 \cdot 4^{-(l+1)}\right) \\ &\leq 12\,\omega\left(f, 4^{-(l+1)}\right) \\ &= 12\,\omega\left(f, \varepsilon_{4^{m(l+1)}}([0,2]^m)\right) \\ &\leq 12\,\omega\left(f, \varepsilon_n([0,2]^m)\right) . \end{aligned}$$

This proves the estimate (2.15).

4. The remaining inverse inequality rests on the proof of the corresponding inequality from Theorem 2.3 (cf. [Ri6–H], pp. 188-189). There it is shown that

$$\liminf_{n\to\infty} \frac{E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)\right)}{8\,\omega\left(f, 2\left(4^{-(l+1)} - H(e(l+1))\right)\right)} \geq \frac{1}{16\left(2\,\alpha^{-\frac{1}{2}} + 1\right)} \tag{2.18}$$

and

$$\omega\left(f, \varepsilon_n([0,2]^m)\right) \leq 8\,\omega\left(f, 2\left(4^{-(l+1)} - H(e(l+1))\right)\right) , \tag{2.19}$$

where $l \geq 0$ is determined by $4^{ml} \leq n < 4^{m(l+1)}$ as above. By (2.10), we have $S_n f \in \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)$ and hence $\|f - S_n f\| \geq E\left(f, \operatorname{span}\left(\tilde{\Psi}_{4^{m(l+1)}}\right)\right)$. Combining this with (2.18) and (2.19) yields the last inequality from Theorem 2.4 and completes the proof. □

Theorem 2.4 shows that the errors $\|f - S_n f\|$ asymptotically behave as the modulus of continuity of f if f is Hölder continuous of type α, $0 < \alpha \leq 1$. A similar representation can be given for compact operators $T \in L(E, C([0,2]^m))$. If $S_n : C([0,2]^m) \to C([0,2]^m)$ denotes the projection mapping a function f onto the n-th partial sum $S_n f$, then T can be expressed by $T = \lim\limits_{n\to\infty} S_nT = \sum\limits_{i=1}^{\infty} T_i$, where $T_1 = S_1T$ and $T_{i+1} = (S_{i+1} - S_i)T$ are operators of rank one (or vanish). This representation is possible for compact operators, since, by the estimate from Theorem 2.4, $\lim\limits_{n\to\infty} \|T - S_nT\| \le \lim\limits_{n\to\infty} 12\,\omega(T, \varepsilon_n([0,2]^m)) = 0$. As in Theorem 2.4, one can infer an inverse estimate from the proof of Theorem 2.3.

Corollary 2.1 *Let $T \in L(E, C([0,2]^m))$ be a compact operator mapping a Banach space E into $C([0,2]^m)$. Then T can be expressed as an infinite sum $T = \sum\limits_{i=1}^{\infty} T_i$ of rank-one operators T_i such that*

$$\left\| T - \sum_{i=1}^{n} T_i \right\| \;\le\; 12\,\omega(T, \varepsilon_n([0,2]^m)) \,.$$

Moreover, if T is Hölder continuous of type α, $0 < \alpha \le 1$, and $T(E)$ does not consist of constant functions only, then

$$\liminf_{n\to\infty} \frac{\left\| T - \sum\limits_{i=1}^{n} T_i \right\|}{\omega\left(T, \varepsilon_n([0,2]^m)\right)} \;\ge\; \frac{1}{16\left(2\,\alpha^{-\frac{1}{2}} + 1\right)} \,. \qquad \square$$

Let us remark that a similar construction with the functions of the well-known Faber-Schauder system would give rise to a monotone and interpolating basis in $C([0,2]^m)$ fulfilling Jackson type estimates of the same kind as in Theorem 2.4 and Corollary 2.1. The essential advantage of the rather complicated functions considered above lies in the inverse estimates for Hölder continuous functions and operators. These estimates can be valid only if the basis functions are not Hölder continuous themselves. Hence the above Bernstein type theorems do not remain true for a basis resting on the Faber-Schauder system.

2.4 n-Term approximation by controllable partitions of unity

Given an arbitrary compact metric space (X, d), we are not able to present a linear approximation scheme

$$\mathrm{span}(\Phi_1) \;\subseteq\; \mathrm{span}(\Phi_2) \;\subseteq\; \mathrm{span}(\Phi_3) \;\subseteq\; \ldots \;\subseteq C(X)$$

formed by controllable partitions of unity Φ_n, since we are not able to prove the existence of a chain $(\Phi_n)_{n=1}^{\infty}$ of such partitions. Even in the fortunate case $(X, d) = ([0,2]^m, d_\infty)$ the chain $(\Psi_n)_{n=0}^{\infty}$ from [Ri6–H] is disappointing in some sense, since the dimension $\dim(\mathrm{span}(\Psi_n)) = \mathrm{card}(\Psi_n) = 2^{4^n}$ increases rapidly. Although there exist controllable partitions of unity Φ for all cardinalities expressable as $\mathrm{card}(\Phi) = \mathcal{N}([0,2]^m, \delta)$, i.e. $\mathrm{card}(\Phi) \in \{1^m, 2^m, 3^m, \ldots\}$, the chain condition has led us to such strong increase.

However, every compact metric space (X,d) admits the following non-linear approximation scheme in $C(X)$ based on controllable partitions of unity (cf. [Ste]). The n-th set of approximating functions is defined by

$$\Phi_n(X) = \left\{ \sum_{i=1}^{k} \lambda_i \varphi_i : \{\varphi_1, \varphi_2, \ldots, \varphi_k\} \text{ is a controllable partition of unity on } X \text{ with } k \leq n \right\}.$$

The classes $\Phi_n(X)$ form an increasing sequence

$$\Phi_1(X) \subseteq \Phi_2(X) \subseteq \Phi_3(X) \subseteq \ldots \subseteq C(X) ,$$

but are not closed under linear operations in general as the following example shows.

Proposition 2.5 ([Ri/Ste1], Proposition 2) *A function $f \in C([0,2])$ belongs to the closure* $\mathrm{cl}(\Phi_2([0,2]))$ *if and only if*

$$\min\{f(0), f(2)\} \leq f(x) \leq \max\{f(0), f(2)\}$$

for all $x \in [0,2]$. $\square$

In Section 4.3 we shall give a characterization for the corresponding approximation class $\Phi_{2^m}([0,2]^m)$ of the m-dimensional cube, which requires a deeper analysis for $m \geq 2$.

The approximation scheme $(\Phi_n(X))_{n=1}^{\infty}$ induces a decreasing sequence $(a_n(f))_{n=1}^{\infty}$ of approximations quantities

$$a_n(f) = \inf\{\|f - \varphi\| : \varphi \in \Phi_n(X)\} \tag{2.20}$$

for every function $f \in C(X)$. The approximation numbers are subject to the following Jackson type inequality.

Theorem 2.5 ([Ste]) *Let (X,d) be a compact metric space, $f \in C(X)$, and $n \geq 1$. Then*

$$a_n(f) \leq \omega(f, \varepsilon_n(X)) . \qquad \square$$

The estimate is sharp in so far as the function $f_0(x) = |1-x|$ on the space $[0,2]$ gives rise to the inverse inequality $a_n(f_0) \geq \frac{1}{4}\,\omega(f_0, \varepsilon_n([0,2]))$ for $n \geq 1$ (cf. [Ri/Ste1], Proposition 1).

One can consider the above described approximation procedure as a kind of so-called *n-term approximation.* In [DeV] this notion is used for summarizing different methods of non-linear approximation. A typical simple example is the approximation by piecewise constants with free knots on the interval $[0,2]$. In that case the n-th class of approximating functions consists of all piecewise constant functions on partitions of $[0,2]$ into at most n subintervals of positive length. When approximating a function $f \in C([0,2])$, one has to find an appropriate partition of $[0,2]$ on which a "good" approximating function φ can be defined. This corresponds to choosing a suitable partition of unity for the approximation of a function $f \in C(X)$ in n-term approximation by controllable partitions of unity.

Proposition 2.5 shows that already the approximation class $\Phi_2([0,2])$ is unexpectedly large. This motivates the question for spaces (X,d) with the property that every function $f \in C(X)$ is contained in one of the classes $\Phi_n(X)$ (cf. [Ba]). On these spaces n-term approximation by controllable partitions of unity becomes a trivial procedure. Fortunately, this happens for finite spaces only.

Proposition 2.6 ([Ri2–H], Theorem 1) *If (X,d) is an infinite compact metric space then there exists a function $f_0 \in C(X)$ such that $f_0 \notin \bigcup\limits_{n=1}^{\infty} \Phi_n(X)$.* $\square$

Finite spaces (X,d) are not the ones an analysist is usually interested in. However, n-term approximation by controllable partitions of unity on finite spaces leads to combinatorial questions. If X is finite then $C(X) = \Phi_{\mathrm{card}(X)}(X)$, since the characteristic functions $\mathbf{I}_{\{x\}}$, $x \in X$, form a controllable partition of unity on X of cardinality $\mathrm{card}(X)$. But it may happen that $C(X) = \Phi_{n_0}(X)$ with $n_0 < \mathrm{card}(X)$. Given $n_0 \geq 1$, how can one characterize the spaces (X,d) with $C(X) = \Phi_{n_0}(X)$? The preprint [Ri2–H] contains first results of that type.

Proposition 2.7 ([Ri2–H], Theorem 4) *Let (X,d) be a compact metric space containig at least two points. Then $C(X) = \Phi_2(X)$ if and only if $\varepsilon_1(X \setminus \{x_0\}) < \varepsilon_1(X)$ for all $x_0 \in X$.* $\square$

Proposition 2.7 leads to the surprising consequence that there exist spaces (X,d) with $C(X) = \Phi_2(X)$ for all cardinalities $\mathrm{card}(X) \in \{2,4,6,\ldots\}$.

2.5 n-Term approximation by controllable step functions

In the previous section we have seen that the classes $\Phi_n(X)$ formed by the aid of controllable partitions of unity can be relatively large. Controllable partitions of unity reflect the geometry of the underlying space (X,d), but they lead to approximating functions $\varphi \in \Phi_n(X)$ of a difficult type. Let us introduce a new kind of n-term approximation which is related to the geometry of the space X by the controllability property too, but uses a simpler kind of functions. It will give rise to a similar Jackson type inequality as well as to an inverse estimate. We shall use step functions $\sum\limits_{i=1}^{k} \lambda_i \mathbf{I}_{P_i}$ which are piecewise constant on controllable partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ of X into subsets $P_i \subseteq X$. Such functions are called *controllable step functions.* In this section we deal primarily with the approximation of continuous functions $f \in C(X)$, but the approximating functions are no longer continuous. The approximation takes place in the space $M(X)$ of all bounded real-valued functions on X equipped with the norm $\|f\| = \sup_{x \in X} |f(x)|$.

The n-th approximation class of controllable step functions is defined by

$$\hat{\Phi}_n(X) \ = \ \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{I}_{P_i} : \{P_1, P_2, \ldots, P_k\} \text{ is a controllable partition of } X \text{ with } k \leq n \right\}$$

for $n \geq 1$. The corresponding approximation quantities of a function $f \in M(X)$ are given by

$$\hat{a}_n(f) \ = \ \inf \left\{ \|f - \hat{\varphi}\| : \hat{\varphi} \in \hat{\Phi}_n(X) \right\}$$

(cf. [Ri/Ste1]). If one defines $a_n(f)$ and the modulus of continuity $\omega(f, \delta)$ for functions $f \in M(X)$ as it has been done for continuous functions in (2.20) and (2.2), then one can prove the following relation between the approximation quantities $a_n(f)$ and $\hat{a}_n(f)$:

$$a_n(f) \ \leq \ \hat{a}_n(f) + \inf_{\varepsilon > 0} \omega(f, \varepsilon) \qquad \text{for all} \qquad f \in M(X)\,,\ n \geq 1$$

(cf. [Ri/Ste1], Proposition 4). This yields in particular that

$$a_n(f) \ \leq \ \hat{a}_n(f) \qquad \text{for all} \qquad f \in C(X)\ ,\ n \geq 1\ .$$

Again we are able to give a Jackson type inequality in terms of the modulus of continuity.

Theorem 2.6 ([Ri/Ste1], Proposition 5) *Let (X, d) be a compact metric space, $f \in M(X)$, and $n \geq 1$. Then*

$$\hat{a}_n(f) \ \leq \ \omega(f, \varepsilon_n(X))\ . \qquad \square$$

(Note that this estimate does not imply that $\lim_{n \to \infty} \hat{a}_n(f) = 0$ if f is not continuous.)

The corresponding Bernstein type theorem refers to the concept of metric spaces which have a finite coefficient of convex deformation. We use the terminology of [JiP], although spaces of that type seem to have been considered for the first time by Menger (cf. [Me]). We recall that a simple Jordan arc $\Gamma = \Gamma[\tau(0), \tau(1)]$ in a metric space (X, d) with parametrization $\tau : [0,1] \to X$ is said to be *rectifiable with length* $l(\Gamma)$ if

$$l(\Gamma) = \sup \left\{ \sum_{i=1}^{n} d(\tau(t_{i-1}), \tau(t_i)) : \ n \geq 1,\ 0 = t_0 < t_1 < \ldots < t_n = 1 \right\} < \infty\ .$$

The infinite metric space (X, d) is said to have *a finite coefficient ρ_X of convex deformation* if X is rectifiable pathwise connected, i.e. for all $x, y \in X$, $x \neq y$, the set $R(x, y)$ of rectifiable Jordan arcs $\Gamma[x, y]$ connecting x and y is non-empty, and

$$\rho_X = \sup \left\{ \inf \left\{ \frac{l(\Gamma[x,y])}{d(x,y)} : \ \Gamma[x,y] \in R(x,y) \right\} : \ x, y \in X, x \neq y \right\} < \infty\ .$$

Of course, $\rho_X \geq 1$. Convex metric subspaces (X, d) of normed spaces are typical examples with $\rho_X = 1$.

Theorem 2.7 ([Ri5–H], Corollary 3.2) *Let (X,d) be a compact metric subspace of an m-dimensional Banach space or of the m-dimensional Euclidean sphere with a finite coefficient ρ_X of convex deformation. Then*

$$\frac{1}{2(\rho_X+5)^m}\,\omega(f,\varepsilon_n(X)) \;\leq\; \hat{a}_n(f) \;\leq\; \omega(f,\varepsilon_n(X))$$

for all $f \in C(X)$ and $n \geq 1$. □

The lower estimate essentially rests on the definition of ρ_X as well as on the particular property of local finiteness of controllable coverings, which is claimed in Theorem 1.2. In Section 1.3 we have seen that controllable partitions of the cube $([0,2]^m, d_\infty)$ have a particular pleasant structure (cf. Lemma 1.1). This can be used for improving the constant $\frac{1}{2(\rho_{[0,2]^m}+5)^m} = \frac{1}{2\cdot 6^m}$ from the lower estimate to $\frac{1}{4\cdot 2^m}$ (cf. [Ri3], Satz 2.17). Moreover, the particular function $f_0 \in C([0,2]^m)$, $f_0(\xi_1,\xi_2,\ldots,\xi_m)=\xi_1$, fulfils $\hat{a}_n(f_0)=\omega(f_0,\varepsilon_n([0,2]^m))$ for all $n \geq 1$ (cf. [Ri3], Satz 2.18). Hence the Jackson type inequality can not be sharpened.

By Theorem 2.7, the approximation numbers $\hat{a}_n(f)$ give all informations on the smoothness of the function $f \in C(X)$ which depend on the asymptotics of $\omega(f,\cdot)$. Hence the sequence $(\hat{a}_n(f))_{n=1}^\infty$ can be used for describing classes of smooth functions in $C(X)$. Let us demonstrate this for certain Besov spaces $B^s_{p,q}([0,1]^m)$ on the cube, namely for $0 \leq s < 1$, $p=\infty$, $0<q<\infty$ as well as for $0 < s \leq 1$, $p=q=\infty$. Let $f \in C([0,1]^m)$. Then

$$f \in B^s_{\infty,q}([0,1]^m) \quad\Longleftrightarrow\quad \begin{cases} \int\limits_0^1 (t^{-s}\cdot\omega(f,t))^q\cdot t^{-1}\,dt \;<\; \infty, & 0\leq s<1,\ 0<q<\infty,\\ \sup\limits_{0<t\leq 1} t^{-s}\cdot\omega(f,t) \;<\; \infty, & 0<s\leq 1,\ q=\infty \end{cases}$$

(cf. [Os]). This is equivalent to

$$f \in B^s_{\infty,q}([0,1]^m) \quad\Longleftrightarrow\quad \begin{cases} \int\limits_1^\infty \left(y^{\frac{s}{m}}\cdot\omega\left(f,y^{-\frac{1}{m}}\right)\right)^q\cdot y^{-1}\,dy \;<\; \infty, & 0\leq s<1,\ 0<q<\infty,\\ \sup\limits_{y\geq 1} y^{\frac{s}{m}}\cdot\omega\left(f,y^{-\frac{1}{m}}\right) \;<\; \infty, & 0<s\leq 1,\ q=\infty. \end{cases}$$

Now we discretize the formulas by considering $n^{-\frac{1}{m}}$ instead of $y^{-\frac{1}{m}}$. Moreover, we use that $\omega\left(f,n^{-\frac{1}{m}}\right)$ behaves like $\omega(f,\varepsilon_n([0,1]^m))$ and thus like $\hat{a}_n(f)$. Hence

$$f \in B^s_{\infty,q}([0,1]^m) \quad\Longleftrightarrow\quad \begin{cases} \sum\limits_{n=1}^\infty \left(n^{\frac{s}{m}}\cdot\hat{a}_n(f)\right)^q\cdot n^{-1} \;<\; \infty, & 0\leq s<1,\ 0<q<\infty,\\ \sup\limits_{n\geq 1} n^{\frac{s}{m}}\cdot\hat{a}_n(f) \;<\; \infty, & 0<s\leq 1,\ q=\infty. \end{cases}$$

But this means that $(\hat{a}_n(f))_{n=1}^\infty$ belongs to the Lorentz sequence space $l_{\frac{m}{s},q}$ (cf. [Ca/Ste]).

Corollary 2.2 *Let $f \in C([0,1]^m)$ and let $0 \leq s < 1$, $0 < q < \infty$ or $0 < s \leq 1$, $q = \infty$. Then $f \in B^s_{\infty,q}([0,1]^m)$ if and only if $(\hat{a}_n(f))_{n=1}^\infty \in l_{\frac{m}{s},q}$.* □

We obtain in particular that a function $f \in C([0,1]^m)$ is Hölder continuous of type α, $0 < \alpha \leq 1$, that is $f \in B^\alpha_{\infty,\infty}([0,1]^m)$, if and only if $(\hat{a}_n(f))_{n=1}^\infty \in l_{\frac{m}{\alpha},\infty}$.

Corollary 2.2 once more illustrates that the behaviour of the approximation quantities $\hat{a}_n(f)$ does not depend on the smoothness of $f \in C([0,1]^m)$ only, but on the degree of compactness of $X = [0,1]^m$ as well. Functions from $B^s_{\infty,q}([0,1]^m)$ give rise to a faster decrease of the errors than functions from $B^s_{\infty,q}([0,1]^{m'})$ with $m' > m$, since $l_{\frac{m}{s},q}$ is a proper subset of $l_{\frac{m'}{s},q}$.

2.6 An application to $C(X)$-valued operators

Let us recall that the *approximation numbers* $a_n(T)$, $n \geq 1$, of an operator $T \in L(E,F)$ between Banach spaces E and F are defined by

$$a_n(T) \;=\; \inf\{\|T-A\| :\; A \in L(E,F) \text{ with } \operatorname{rank}(A) < n\}\;.$$

The *Gelfand numbers* $c_n(T)$ are given by the infimum

$$c_n(T) \;=\; \inf\left\{\left\|TI_M^E\right\| :\; M \text{ is a linear subspace of } E \text{ with } \operatorname{codim}(M) < n\right\}\;,$$

where I_M^E denotes the natural embedding of M into E (cf. [Ca/Ste]). We are interested in the case $F = C(X)$ with a compact metric space (X,d). By the aid of suitable partitions of unity, one can show that

$$a_{n+1}(T) \;\leq\; \omega(T, \varepsilon_n(X))\;, \qquad n = 1,2,3,\ldots\;,$$

if T is a compact operator (cf [Ca/Ste], Theorem 5.6.1). The method of controllable step functions will give rise to the more general inequality

$$c_{n+1}(T) \;\leq\; \omega(T, \varepsilon_n(X))\;, \qquad n = 1,2,3,\ldots\;,$$

for arbitrary operators $T \in L(E, C(X))$. This includes the previous estimate, since $a_n(T) = c_n(T)$ if T is compact (cf. [Ca/Ste], Theorem 5.3.2).

Let $T \in L(E, C(X))$ and let $J : C(X) \to M(X)$ be the natural embedding. We put

$$\hat{c}_n(T) \;=\; \inf\left\{\|JT - A\| :\; A \in L(E, M(X)) \text{ with } A(E) \subseteq \hat{\Phi}_n(X)\right\}$$

for $n \geq 1$. (In the paper [Ri5–H] we have introduced related quantities $\hat{a}_n\left(\tilde{T}\right) = \inf\left\{\left\|\tilde{T} - A\right\| : A \in L(E, M(X)) \text{ with } A(E) \subseteq \hat{\Phi}_n(X)\right\}$ for operators $\tilde{T} \in L(E, M(X))$ before defining $\hat{c}_n(T) = \hat{a}_n(JT)$ for $T \in L(E, C(X))$.) The claim $A(E) \subseteq \hat{\Phi}_n(X)$ is sharper than $\operatorname{rank}(A) \leq n$. Hence $a_{n+1}(JT) \leq \hat{c}_n(T)$.

We obtain the following inequalities of Jackson and Bernstein type.

Theorem 2.8 ([Ri5–H], Theorem 4.3 and Corollary 4.6) *Let (X, d) be a compact metric space, E a Banach space, $T \in L(E, C(X))$, and $n \geq 1$. Then*

$$\hat{c}_n(T) \;\leq\; \omega(T, \varepsilon_n(X)) \;.$$

Moreover, if (X, d) is a space as in Theorem 2.7, then

$$\frac{1}{2(\rho_X + 5)^m}\, \omega(T, \varepsilon_n(X)) \;\leq\; \hat{c}_n(T) \;. \qquad \square$$

Under the supposition of Theorem 2.8 we again are in the pleasant situation that an approximation quantity $\hat{c}_n(T)$ describes the behaviour of the modulus of continuity $\omega(T, \cdot)$. Hence one can use the numbers $\hat{c}_n(T)$, $n \geq 1$, for characterizing compactness or Hölder continuity of T.

The numbers $\hat{c}_n(T)$ are related to the Gelfand numbers of T by

$$c_{n+1}(T) \;\leq\; \hat{c}_n(T) \;.$$

Indeed, $c_{n+1}(T) = c_{n+1}(JT)$ (cf. [Ca/Ste], injectivity of the Gelfand numbers), $c_{n+1}(JT) = a_{n+1}(JT)$ (cf. [Ca/Ste], Proposition 2.3.3), and $a_{n+1}(JT) \leq \hat{c}_n(T)$ as mentioned above. This yields the inequality between the Gelfand numbers and the modulus of continuity of T, which has been announced above.

Corollary 2.3 ([Ri/Ste1], Proposition 6) *Let E be a Banach space, (X, d) a compact metric space, $T \in L(E, C(X))$, and $n \geq 1$. Then*

$$c_{n+1}(T) \;\leq\; \omega(T, \varepsilon_n(X)) \;. \qquad \square$$

The estimate is sharp as can be shown by the natural embedding $I_\alpha \in L(C^\alpha(X), C(X))$ from the Banach space of Hölder continuous functions $C^\alpha(X)$, $0 < \alpha \leq 1$, equipped with the norm $\|f\|_\alpha = \max\{\|f\|, |f|_\alpha\}$ into $C(X)$. Then all the sequences $\Big((\varepsilon_n(X))^\alpha\Big)_{n=1}^\infty$, $\Big(a_{n+1}(I_\alpha)\Big)_{n=1}^\infty$, $\Big(c_{n+1}(I_\alpha)\Big)_{n=1}^\infty$, $\Big(\hat{c}_n(I_\alpha)\Big)_{n=1}^\infty$, and $\Big(\omega(I_\alpha, \varepsilon_n(X))\Big)_{n=1}^\infty$ have the same asymptotics. This is proved for the first three sequences in [Ca/Ste], Proposition 5.6.2. Besides that, we have already seen that $c_{n+1}(I_\alpha) \leq \hat{c}_n(I_\alpha) \leq \omega(I_\alpha, \varepsilon_n(X))$. The remaining estimate $\omega(I_\alpha, \varepsilon_n(X)) \leq (\varepsilon_n(X))^\alpha$ is trivial.

Chapter 3

Approximable functions on compact metric spaces

3.1 Characterization of approximable functions

In the previous chapter we have dealt with the approximation of continuous real-valued functions on compact metric spaces (X, d). But when approximating by controllable step functions we made use of the larger space $M(X)$ of all bounded real-valued functions on X. The Jackson type estimate has shown that every continuous function $f \in C(X)$ can be expessed as the uniform limit of a suitable sequence of controllable step functions, that is $\lim\limits_{n\to\infty} \hat{a}_n(f) = 0$. However, uniform limits of controllable step functions need not be continuous. The simplest examples of that type are the controllable step functions themselves. In this chapter we shall consider the class $A(X)$ of *all* functions $f \in M(X)$ with $\lim\limits_{n\to\infty} \hat{a}_n(f) = 0$. These functions are to be called *approximable*.

Unfortunately, not all functions from $M(X)$ are approximable so far as X is an infinite space (cf. [Ri8–H], Theorem 2). In fact, if X is infinite then there even exist step functions on X which are not approximable (cf. [Ri8–H], Proposition 4). (In this chapter we use the name *step function* for functions $f \in M(X)$ whose image $f(X)$ is finite.) Moreover, the class $A(X)$ is not a linear subspace of $M(X)$ if X contains an infinite connected component (cf. [Ri8–H], Theorem 1). The more desirable it would be to get an insight into the structure of the whole class $A(X)$ as well as to find typical properties of the functions belonging to $A(X)$.

A function $f \in A(X)$ is said to be *chain-approximable* if it is the uniform limit of a sequence of controllable step functions $\varphi_n = \sum\limits_{i=1}^{l_n} \lambda_i^{(n)} \mathbf{I}_{P_i^{(n)}}$ on an ascending chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of controllable partitions $\mathcal{P}_n = \left\{P_1^{(n)}, P_2^{(n)}, \ldots, P_{l_n}^{(n)}\right\}$ of X. The class of all chain-approximable functions on X will be denoted by $A^{(c)}(X)$.

A chain-approximable function f allows a satisfactory procedure of approximation. The $(n+1)$-st partition $\mathcal{P}_{n+1}$ preserves information of the n-th one $\mathcal{P}_n$, since $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. In other words, f can be gained within an approximation scheme consisting of an increasing sequence $(E_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of $M(X)$, where E_n contains all step functions on the partition $\mathcal{P}_n$.

Given a fixed chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of controllable partitions of X, we denote the set of all uniform limits of step functions on partitions from K by $A_K(X)$. Of course, $A_K(X)$ is a Banach space. Hence $A^{(c)}(X)$ is the union of the Banach spaces $A_K(X)$. In general, $A^{(c)}(X)$ is not a linear space itself: The characteristic function $f = \mathbf{I}_{[0,1]}$ belongs to $A^{(c)}([-1,1])$, since it is chain-approximable with respect to a chain K of controllable partitions $\mathcal{P}_n$ consisting of 2^n subintervals of length 2^{-n+1}. Similarly, $g = \mathbf{I}_{(0,1]} \in A^{(c)}([-1,1])$. But $f - g = \mathbf{I}_{\{0\}}$ is not chain-approximable nor even approximable, because $\{0\}$ can not be a set from a controllable partition of $[-1,1]$ according to Proposition 1.3.

Theorem 3.1 ([Ri8–H], Theorem 3) *Let (X,d) be a compact metric space and let $f \in A(X)$ be an approximable function.*

If f is a step function then f is a controllable step function, that is, f belongs to one of the classes $\hat{\Phi}_n(X)$, $n \geq 1$.

If f has infinitely many values then f is chain-approximable. $\Box$

Theorem 3.1 says that all functions $f \in A(X)$ whose approximation is non-trivial, i.e. $f \notin \bigcup\limits_{n=1}^{\infty} \hat{\Phi}_n(X)$, are chain-approximable. That is, $A(X)$ coincides "essentially" with the pleasant class $A^{(c)}(X) = \bigcup\limits_{K} A_K(X)$. In other words, the non-linear class $A(X)$ is not much larger than the union of the Banach spaces $A_K(X)$. Moreover, the remaining functions $f \in A(X) \setminus A^{(c)}(X) \subseteq \bigcup\limits_{n=1}^{\infty} \hat{\Phi}_n(X)$ are not very interesting in the sense that they belong to the class of approximating functions.

Let us remark that $A(X) \setminus A^{(c)}(X)$ is non-empty in general. Indeed, in Section 1.2 we have seen that there exist controllable partitions of the interval $[-1,1]$ which do not belong to a chain K of controllable partitions. Of course, these partitions give rise to step functions which are not chain-approximable.

Particular topological properties of partitions from chains K of controllable partitions give rise to continuity properties of functions from $A_K(X)$. A function $f : X \to \mathbf{R}$ is said to be *quasi-continuous* if, for every $x \in X$, every neighbourhood U of x, and every $\varepsilon > 0$, there exists a non-empty open set $G \subseteq U$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in G$ (cf. [Ke], [Ble]). By Proposition 1.2, every step function on a partition $\mathcal{P}$ from a chain K of controllable partitions is quasi-continuous. In [Le] Levine proves that the uniform limit of quasi-continuous functions is quasi-continuous as well. This gives the following claim.

Proposition 3.1 ([Ri8–H], Corollary 4) *All chain-approximable functions on a compact metric space are quasi-continuous.* □

The points of discontinuity of a quasi-continuous function are of the first Baire category (cf. [Ble]). Hence the points of continuity are dense, since (X, d) is a Baire space. Accordingly, the points of continuity of any chain-approximable function are dense in X. This shows that chain-approximable functions are quite close to continuous functions.

Besides that there exists another satisfactory relation between chain-approximable functions and continuous functions, which in particular implies that $C(X)$ is contained in all the classes $A_K(X)$.

Proposition 3.2 ([Ri8–H], Corollary 3) *Let (X, d) be an infinite compact metric space. Then*

$$C(X) \;=\; \bigcap_K A_K(X)$$

where the intersection is taken over all chains K of controllable partitions of X. □

3.2 The spaces $A_K(X)$

Next we shall deal with the structure of a single class $A_K(X)$, K denoting a fixed chain of controllable partitions of (X, d).

When computing an approximation number $\hat{a}_n(f)$ of a function $f \in M(X)$, one has to consider step functions on all partitions from $\hat{\Phi}_n(X)$. These are uncountably many in general. (For instance, all partitions $\{[-1, \alpha], (\alpha, 1]\}$ of $[-1, 1]$ with $-1 < \alpha < 1$ are controllable.) But the approximation procedure can be discretized if f belongs to $A_K(X)$.

Theorem 3.2 ([Ri8–H], Theorem 4) *Let $K = (\mathcal{P}_m)_{m=1}^{\infty}$ be a chain of controllable partitions of a compact metric space (X, d), $f \in A_K(X)$, and $n \geq 1$. Then*

$$\hat{a}_n(f) \;=\; \inf\left\{\left\|f - \varphi^{(K)}\right\| : \varphi^{(K)} \in \hat{\Phi}_n(X) \text{ and } \varphi^{(K)} \text{ is defined on a partition from } K\right\}. \;\square$$

(Note that a step function $\varphi^{(K)} \in \hat{\Phi}_n(X)$ can be defined on a partition from K of a cardinality larger than n.)

In the previous section we have seen that the points of continuity of a function from $A_K(X)$ are dense in X. This can be sharpened in so far as there exists a universal dense set of points of continuity for all functions from $A_K(X)$.

Proposition 3.3 ([Ri8–H], Corollary 5) *Let (X, d) be a compact metric space and $K = (\mathcal{P}_n)_{n=1}^{\infty}$ a chain of controllable partitions of X. Then there exists a dense set $C_K \subseteq X$ of the second Baire category such that every function $f \in A_K(X)$ is continuous at all the points from C_K.* □

However, the significant property of the space $A_K(X)$ is the way it has been gained from the chain K. In the following we shall use K for constructing a Schauder basis in $A_K(X)$. Let $K = (\mathcal{P}_n)_{n=1}^{\infty}$ with $\mathcal{P}_n = \left\{P_1^{(n)}, P_2^{(n)}, \ldots, P_{k_n}^{(n)}\right\}$. Without loss of generality we may assume that $\mathcal{P}_1 = \left\{P_1^{(1)}\right\} = \{X\}$ is the trivial partition. We fix a point $t_1 \in P_1^{(1)}$. We can assume that $t_1 \in P_1^{(2)}$. Now we choose points $t_i \in P_i^{(2)}$, $2 \le i \le k_2$. Next we assume that $t_i \in P_i^{(3)}$ for $1 \le i \le k_2$ and determine points $t_i \in P_i^{(3)}$, $k_2 + 1 \le i \le k_3$. We continue this procedure. Hence we obtain a sequence $(P_i)_{i=1}^{\infty} = \left(P_1^{(1)}, P_2^{(2)}, P_3^{(2)}, \ldots, P_{k_2}^{(2)}, P_{k_2+1}^{(3)}, P_{k_2+2}^{(3)}, \ldots, P_{k_3}^{(3)}, P_{k_3+1}^{(4)}, P_{k_3+2}^{(4)}, \ldots, P_{k_4}^{(4)}, \ldots\right)$ of partition sets with $t_i \in P_i$ for $i \ge 1$.

Proposition 3.4 *The sequence $(\mathbf{I}_{P_i})_{i=1}^{\infty}$ is a monotone and interpolating basis in $A_K(X)$ with nodes $(t_i)_{i=1}^{\infty}$. Moreover, the l-th partial sum $S_l f$, $l \ge 1$, of $f \in A_K(X)$ gives rise to the estimate*

$$\|f - S_l f\| \;\le\; \max_{P^{(n)} \in \mathcal{P}_n} \sup_{s,t \in P^{(n)}} |f(s) - f(t)| \;\le\; \omega(f, 2\varepsilon_{k_n - 1}(X)) \tag{3.1}$$

where n is determined by $k_n \le l \le k_{n+1} - 1$. (The second estimate can be given for $l \ge k_2$ only, since $\varepsilon_{k_1 - 1}(X) = \varepsilon_0(X)$ does not make sense.)

Proof. 1. We follow similar arguments as in the proof of Theorem 2.4. Let $(\alpha_i)_{i=1}^{\infty}$ be a sequence of reals such that $f = \sum\limits_{i=1}^{\infty} \alpha_i \mathbf{I}_{P_i}$ exists in $M(X)$ and let $f_l = \sum\limits_{i=1}^{l} \alpha_i \mathbf{I}_{P_i}$. Then

$$\|f_l\| \;=\; \max\{|f(t_1)|, |f(t_2)|, \ldots, |f(t_l)|\} \,. \tag{3.2}$$

We shall prove this by showing inductively that

$$f_l(X) \;=\; \{f(t_1), f(t_2), \ldots, f(t_l)\} \,. \tag{3.3}$$

But first we note that the points t_i and the sets P_i are chosen such that $t_i \in P_i$ and $t_j \notin P_i$ for $1 \le j < i$. Hence

$$\mathbf{I}_{P_i}(t_i) \;=\; 1 \quad \text{and} \quad \mathbf{I}_{P_i}(t_j) \;=\; 0 \quad \text{for} \quad i = 1, 2, 3, \ldots \,,\; 1 \le j < i \,. \tag{3.4}$$

This yields $f_l(t_j) = f(t_j)$ for $1 \le j \le l$.

Now let us prove (3.3). The claim is trivial for $l = 1$, since $f_1(X) = \{f_1(t_1)\} = \{f(t_1)\}$. Let us consider f_{l+1}, $l \ge 1$. We obtain $f_{l+1}(X \backslash P_{l+1}) = f_l(X \backslash P_{l+1}) \subseteq \{f(t_1), f(t_2), \ldots, f(t_l)\}$ by the induction hypothesis, and $f_{l+1}(X \setminus P_{l+1}) \supseteq \{f_{l+1}(t_1), f_{l+1}(t_2), \ldots, f_{l+1}(t_l)\} = \{f(t_1), f(t_2), \ldots, f(t_l)\}$, since $t_1, t_2, \ldots, t_l \in X \setminus P_{l+1}$. Consequently, $f_{l+1}(X \setminus P_{l+1}) = \{f(t_1), f(t_2), \ldots, f(t_l)\}$. The chain property of the sequence K of controllable partitions implies that, for $i \le l$, $P_{l+1} \cap P_i = \emptyset$ or $P_{l+1} \subseteq P_i$. Accordingly, f_{l+1} is constant on P_{l+1}. Thus $f_{l+1}(P_{l+1}) = \{f_{l+1}(t_{l+1})\} = \{f(t_{l+1})\}$. This proves (3.3) for

$$f_{l+1}(X) \;=\; f_{l+1}(X \setminus P_{l+1}) \cup f_{l+1}(P_{l+1}) \;=\; \{f(t_1), f(t_2), \ldots, f(t_l)\,,\, f(t_{l+1})\} \,.$$

2. Equation (3.2) implies the inequality $\left\| \sum_{i=1}^{l} \alpha_i \mathbf{I}_{P_i} \right\| \leq \left\| \sum_{i=1}^{l+1} \alpha_i \mathbf{I}_{P_i} \right\|$ for arbitray $\alpha_i \in \mathbf{R}$ and $l \geq 1$. The linear span of $(\mathbf{I}_{P_i})_{i=1}^{\infty}$ is dense in $A_K(X)$, since $\operatorname{span}(\{\mathbf{I}_{P_1}, \mathbf{I}_{P_2}, \ldots, \mathbf{I}_{P_{k_n}}\}) = \operatorname{span}\left(\left\{\mathbf{I}_{P_1^{(n)}}, \mathbf{I}_{P_2^{(n)}}, \ldots, \mathbf{I}_{P_{k_n}^{(n)}}\right\}\right)$ and $\bigcup_{n=1}^{\infty} \operatorname{span}\left(\left\{\mathbf{I}_{P_1^{(n)}}, \mathbf{I}_{P_2^{(n)}}, \ldots, \mathbf{I}_{P_{k_n}^{(n)}}\right\}\right)$ is dense. Clearly, $\|\mathbf{I}_{P_i}\| \neq 0$ for all $i \geq 1$. Hence $(\mathbf{I}_{P_i})_{i=1}^{\infty}$ is a basis of $A_K(X)$ (cf. [Li/Tza], Proposition 1.a.3). The estimate $\left\| \sum_{i=1}^{l} \alpha_i \mathbf{I}_{P_i} \right\| \leq \left\| \sum_{i=1}^{l+1} \alpha_i \mathbf{I}_{P_i} \right\|$ and property (3.4) imply that the basis is monotone and interpolating with nodes $(t_i)_{i=1}^{\infty}$ (cf. [Se], Proposition 1.3.2).

3. Now let us prove the error estimate (3.1). Let $f = \sum_{i=1}^{\infty} \alpha_i \mathbf{I}_{P_i} \in A_K(X)$. Then $S_l f = f_l = f_{k_n} + \sum_{i=k_n+1}^{l} \alpha_i \mathbf{I}_{P_i}$. f_{k_n} is a step function on the partition $\mathcal{P}_n = \left\{P_1^{(n)}, P_2^{(n)}, \ldots, P_{k_n}^{(n)}\right\}$, since $\operatorname{span}(\{\mathbf{I}_{P_1}, \mathbf{I}_{P_2}, \ldots, \mathbf{I}_{P_{k_n}}\}) = \operatorname{span}\left(\left\{\mathbf{I}_{P_1^{(n)}}, \mathbf{I}_{P_2^{(n)}}, \ldots, \mathbf{I}_{P_{k_n}^{(n)}}\right\}\right)$. The sets $P_i = P_i^{(n+1)}$, $k_n + 1 \leq i \leq l$, belong to the refinement $\mathcal{P}_{n+1}$ of $\mathcal{P}_n$. Hence f_l can be considered as a step function on the partition $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_l\}$ with $Q_i = \mathcal{P}_i^{(n)} \setminus \bigcup_{m=k_n+1}^{l} P_m^{(n+1)}$ for $1 \leq i \leq k_n$ and $Q_i = P_i^{(n+1)}$ for $k_n + 1 \leq i \leq l$. Every set $Q_i \in \mathcal{Q}$ is contained in a uniquely determined set $P_{j(i)}^{(n)} \in \mathcal{P}_n$. Moreover, we have $t_i \in Q_i$. Hence $f_l(Q_i) = \{f_l(t_i)\} = \{f(t_i)\}$ for $1 \leq i \leq l$. If x is an arbitrary point from X, say $x \in Q_i$, then

$$|f(x) - f_l(x)| \;=\; |f(x) - f(t_i)| \;\leq\; \sup_{s,t \in Q_i} |f(s) - f(t)| \;\leq\; \sup_{s,t \in P_{j(i)}^{(n)}} |f(s) - f(t)| \,.$$

This yields the first inequality from (3.1), because of

$$\|f - S_l f\| \;=\; \sup_{x \in X} |f(x) - f_l(x)| \;\leq\; \max_{P^{(n)} \in \mathcal{P}_n} \sup_{s,t \in P^{(n)}} |f(s) - f(t)| \,.$$

The remaining inequality is a simple consequence of the controllability property. Every set $P^{(n)} \in \mathcal{P}_n$ is contained in a suitable ball of radius $\varepsilon_{k_n-1}(X)$. Hence $d(s,t) \leq 2\varepsilon_{k_n-1}(X)$ and $|f(s) - f(t)| \leq \omega(f, 2\varepsilon_{k_n-1}(X))$ for $s, t \in P^{(n)}$. Accordingly,

$$\max_{P^{(n)} \in \mathcal{P}_n} \sup_{s,t \in P^{(n)}} |f(s) - f(t)| \;\leq\; \omega(f, 2\varepsilon_{k_n-1}(X)) \,.$$

This completes the proof of Proposition 3.4. □

Note that the error estimate

$$\|f - S_l f\| \;\leq\; \max_{P^{(n)} \in \mathcal{P}_n} \sup_{s,t \in P^{(n)}} |f(s) - f(t)|$$

obtained first is the predestinate one with respect to the space $A_K(X)$ induced by the chain K. Indeed, the bound $\max_{P^{(n)} \in \mathcal{P}_n} \sup_{s,t \in P^{(n)}} |f(s) - f(t)|$ tends to zero for increasing n if and only if f belongs to $A_K(X)$. In contrast with that, the modulus of continuity $\omega(f, 2\varepsilon_{k_n-1}(X))$ approaches zero for $f \in C(X)$ only.

3.3 Chain-approximable functions on cubes and intervals

The particular geometric structure of controllable partitions of the cube $([-1,1]^m, d_\infty)$ gives rise to further results. Example 1.1 shows that controllable step functions on the cube need not be measurable. The more satisfactory it is that chain-approximable functions behave much simpler.

Theorem 3.3 ([Ri8–H], Theorem 5) *All chain-approximable functions on the m-dimensional cube $([-1,1]^m, d_\infty)$ are Riemann integrable.* □

By the aid of Proposition 1.2 we have seen that chain-approximable functions are quasi-continuous (cf. Proposition 3.1). Let us remark that quasi-continuity itself does not imply Riemann integrability. One can construct step functions $\varphi : [-1,1] \to \{0,1\}$ which are quasi-continuous but not Riemann integrable (cf. [Ri8–H], Example 2). The construction gives rise to functions φ which are Lebesgue measurable such that the measure of the set of points of discontinuity of φ can be chosen arbitrarily close to 2. A slightly modified construction yields non-measurable quasi-continuous functions $\psi : [-1,1] \to \{0,1\}$.

Now let us discuss a relation between so-called regulated functions on the interval $[-1,1]$ and chain-approximable functions. A function $f \in M([-1,1])$ is called *regulated* if it possesses the limits from the left $f(x-)$ for all $x \in (-1,1]$ and the limits from the right $f(x+)$ for all $x \in [-1,1)$. An important subclass of the class of regulated functions is formed by the *simple step functions*. A step function φ belongs to this subclass if there exist points $-1 = x_0 < x_1 < \ldots < x_k = 1$ such that φ is constant on each open subinterval (x_{i-1}, x_i), $1 \le i \le k$. The regulated functions can be characterized as the uniform limits of simple step functions (cf. [Au], [Di]).

Since the values $\varphi(x_i)$ of a simple step function φ of the above form can be chosen arbitrarily, φ need not be quasi-continuous and thus not chain-approximable. Of course, φ becomes quasi-continuous if, for $0 \le i \le k$, $\varphi(x_i)$ coincides with the value of φ on one of the intervals (x_{i-1}, x_i) or (x_i, x_{i+1}). In other words, a simple step function φ is quasi-continuous if and only if, for every $x \in [-1,1]$, φ is continuous from the left or from the right at x. Then φ is a step function on a finite partition of $[-1,1]$ into intervals of positive length. In [Ri1] it is shown that every partition of that type admits a controllable strict refinement again consisting of intervals of positive length. Repeated application of the procedure of refining yields a chain K of controllable partitions of $[-1,1]$, such that φ is chain-approximable with respect to K.

Obviously, a function $f \in M([-1,1])$ is a quasi-continuous regulated function if and only if it is continuous at the points -1 and 1 and if, for every $x \in (-1,1)$, the limits $f(x-)$ and

$f(x+)$ exist and the value $f(x)$ coincides with one of these limits. The following theorem characterizes this class of functions. It shows in particular that every quasi-continuous regulated function is chain-approximable.

Theorem 3.4 ([Ri8–H], Theorem 6) *Let $f \in M([-1,1])$. Then the following are equivalent.*
(i) f is a quasi-continuous regulated function.
(ii) f is a uniform limit of quasi-continuous simple step functions.
(iii) f is a uniform limit of quasi-continuous simple step functions which are defined on a chain of controllable partitions of $[-1,1]$ into intervals of positive length. (In particular, $f \in A^{(c)}([-1,1])$.) $\square$

Note that $A^{(c)}([-1,1])$ is strictly larger than the class of quasi-continuous regulated functions. Indeed, in Section 1.2 a partition $\mathcal{P} = \{P_1, P_2\}$ has been presented such that $\mathcal{P}$ can be extended to a chain of controllable partitions, but the partition sets P_1 and P_2 are twisted at the point 0. Hence the function $\mathbf{I}_{P_1}$ is chain-approximable, but the property of being regulated fails at 0.

The non-linear class $A^{(c)}([-1,1])$ is the union $A^{(c)}([-1,1]) = \bigcup\limits_{K} A_K([-1,1])$ of the separable Banach spaces $A_K([-1,1])$, where all chains K of controllable partitions of $[-1,1]$ are considered. But do there exist larger Banach spaces in $A^{(c)}([-1,1])$? Let $E \subseteq M([-1,1])$ be the set of all quasi-continuous regulated functions $f \in M([-1,1])$ which are continuous from the left at all points $x \in (-1,1)$. Clearly, E is a Banach space and $E \subseteq A^{(c)}([-1,1])$ by Theorem 3.4. But E is not separable, since it contains the uncountable subset $\left\{\mathbf{I}_{[-1,\alpha]} : -1 < \alpha < 1\right\}$ with $\left\|\mathbf{I}_{[-1,\alpha_1]} - \mathbf{I}_{[-1,\alpha_2]}\right\| = 1$ for $\alpha_1 \neq \alpha_2$.

Corollary 3.1 *There exist inseparable Banach spaces in $A^{(c)}([-1,1])$. In particular, the spaces $A_K([-1,1])$ are not the largest Banach spaces in $A^{(c)}([-1,1])$.* $\square$

Part II

Approximation on topological spaces

The previous chapters concerned the uniform approximation of real-valued functions on a compact metric space (X, d) with reference to the geometric structure of (X, d). The results to be pesented in the following are motivated by the preceding investigations, but do not make use of metric conditions such as controllability. From now on X is supposed to be a more general topological space.

The fourth chapter deals with the approximation of continuous functions. For normal spaces X we shall give a characterization of all continuous functions which can be attained as linear combinations of partitions of unity subordinated to a fixed open covering. This result can be applied to an optimization problem on polyhedral complexes and, in the end, for describing the approximation class $\Phi_{2^m}([0,2]^m)$ on the cube $([0,2]^m, d_\infty)$. The second problem in Chapter 4 concerns the approximation by step functions. We shall see that the existence of approximation schemes defined by so-called approximating sequences of partitions is a characteristic property of compact metrizable spaces within the class of completey regular spaces.

The last chapter is devoted to quasi-continuous and cliquish functions on arbitrary topological spaces X. When considering n-term approximation by controllable step functions on a compact metric space (X, d), we saw that every chain-approximable function is quasi-continuous (cf. Proposition 3.1). But quasi-continuous functions on a compact metric space need not be chain-approximable in the sense of controllability. However, it will turn out that every quasi-continuous function f on an arbitrary topological space X can be expressed as a uniform limit $f = \lim_{n\to\infty} \varphi_n$ of a sequence of quasi-continuous step functions defined on a chain of so-called semi-open partitions. A similar representation can be obtained for the related class of cliquish functions. Besides these approximation results we shall discuss further properties of quasi-continuous and cliquish functions: Baire spaces can be characterized by the aid of cliquish functions. Cliquish functions can be transformed into quasi-continuous ones by a particular "small" modification. Finally, continuity properties of an associated multifunction of quasi-continuous and cliquish functions will be investigated.

Chapter 4

Covering and partition properties of topological spaces related to continuous functions

4.1 Partitions of unity restricted by open coverings

A partition of unity $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ on a compact metric space (X, d) had been called controllable if and only if there exists a controllable covering $\{C_1, C_2, \ldots, C_k\}$ of X such that $\mathrm{supp}(\varphi_i) \subseteq C_i$, $1 \le i \le k$. Hence one can impose geometrical constraints upon partitions of unity by coverings of the underlying space. Clearly, this can be done on arbitrary topological spaces X.

Now let X be an arbitrary topological space. Let us recall that a (not necessarily finite) covering $\mathcal{C}$ is called *locally finite* if, for every $x \in X$, there exists a neighbourhood U of x intersecting at most finitely many of the sets from $\mathcal{C}$. In this chapter partitions of unity are no longer required to be finite. Now a system $\{\varphi_\iota : \iota \in \mathcal{I}\} \subseteq C(X)$ of non-negative continuous functions is called a *partition of unity on* X if the system of the supports $\{\mathrm{supp}(\varphi_\iota) : \iota \in \mathcal{I}\}$ is a locally finite covering of X and $\sum\limits_{\iota \in \mathcal{I}} \varphi_\iota = 1$. Note that we have defined the *support* of a continuous function by formula (2.1). Hence $\{\mathrm{supp}(\varphi_\iota) : \iota \in \mathcal{I}\}$ is an open covering of X. The local finiteness yields that every function $\sum\limits_{\iota \in \mathcal{I}} \lambda_\iota \varphi_\iota$ with arbitrary $\lambda_\iota \in \mathbf{R}$ belongs to $C(X)$, since for every $x \in X$ the sum is a finite one in some neighbourhood of x. Given a locally finite open covering $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ of X, we say that the partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\} \subseteq C(X)$ is *subordinated to* $\mathcal{C}$ if $\mathrm{supp}(\varphi_\iota) \subseteq C_\iota$ for all $\iota \in \mathcal{I}$ (cf. [Mi/Pe1]).

In n-term approximation by controllable partitions of unity on a compact metric space (X, d) the n-th class $\Phi_n(X)$ of approximating functions consists of all linear combinations of partitions of unity $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ with $k \le n$ which are subject to the condition of controllability. If a controllable open partition $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is fixed, there still is a variety of partitions of unity $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ subordinated to this covering with a variety of approximating functions $\sum_{i=1}^{k} \lambda_i \varphi_i$. This motivates a similar question for general topological spaces X: Given a locally finite open covering $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ of X, we want to characterize all linear combinations $\sum_{\iota \in \mathcal{I}} \lambda_\iota \varphi_\iota$ of partitions of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ subordinated to $\mathcal{C}$. We can give a characterization provided that X is a T_4-space. This means that any two disjoint closed subsets of X can be separated by two disjoint open subsets. In the theorem we use the symbol $\operatorname{conv}(\cdot)$ for denoting the convex hull of a set of reals.

Theorem 4.1 ([Ri7–H], Theorem 1) *Let X be a T_4-space, $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ a locally finite open covering of X, $f \in C(X)$, and $\lambda_\iota \in \mathbf{R}$ for $\iota \in \mathcal{I}$. Then the following are equivalent.*

(i) There exists a partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ subordinated to $\mathcal{C}$ such that

$$f \; = \; \sum_{\iota \in \mathcal{I}} \lambda_\iota \varphi_\iota \, .$$

(ii) For all $x \in X$,

$$f(x) \; \in \; \operatorname{conv}\{\lambda_\iota : x \in C_\iota\} \, . \qquad \square$$

If one wants to find out whether f can be expressed as a linear combination of a partition of unity subordinated to $\mathcal{C}$, the characterization given by Theorem 4.1 is difficult in so far as one has to consider property (ii) for all collections $(\lambda_\iota)_{\iota \in \mathcal{I}}$ of real coefficients. We can simplify the characterization if $\mathcal{C}$ has a property closely related to peaked partitions of unity. The covering $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ is to be called *peaked with peaks $x_\iota \in X$* if $x_\iota \in C_\iota \setminus \bigcup_{\kappa \in \mathcal{I} \setminus \{\iota\}} C_\kappa$ for all $\iota \in \mathcal{I}$ (cf. [Mi/Pe1]).

Corollary 4.1 ([Ri7–H], Corollary 1) *Let X be a T_4-space, $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ a locally finite peaked open covering of X with peaks $x_\iota \in C_\iota$, and $f \in C(X)$. Then the following are equivalent.*

(i) There exists a partition of unity $\{\varphi_\iota : \iota \in \mathcal{I}\}$ subordinated to $\mathcal{C}$ and coefficients $\lambda_\iota \in \mathbf{R}$ such that

$$f \; = \; \sum_{\iota \in \mathcal{I}} \lambda_\iota \varphi_\iota \, .$$

(ii) For all $x \in X$,

$$f(x) \; \in \; \operatorname{conv}\{f(x_\iota) : x \in C_\iota\} \, . \qquad \square$$

In Theorem 4.1 and Corollary 4.1 the implications (i)$\Rightarrow$(ii) are simple and can be shown for arbitrary topological spaces X. But the converse step from the local property (ii) to the global one (i) deeply involves the T_4 separation property of X in terms of the Tietze-Urysohn extension theorem. In fact, we can use the implications (ii)$\Rightarrow$(i) for characterizing T_4-spaces.

Theorem 4.2 ([Ri7–H], Theorem 2) *Let X be a topological space. Then the following are equivalent.*

(α) X is a T_4-space.

(β) The implication (ii)$\Rightarrow$(i) from Theorem 4.1 holds true for all locally finite open coverings $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ of X, all functions $f \in C(X)$, and all real coefficients λ_ι, $\iota \in \mathcal{I}$.

(γ) The implication (ii)$\Rightarrow$(i) from Corollary 4.1 holds true for all locally finite peaked open coverings $\mathcal{C} = \{C_\iota : \iota \in \mathcal{I}\}$ of X with peaks $x_\iota \in C_\iota$ and all functions $f \in C(X)$. $\square$

4.2 An application to polyhedral complexes

We want to apply Theorem 4.1 in the context of an optimization problem on polyhedral complexes. Let us recall that a *polytope* P in $\mathbf{R}^m$ is meant to be the convex hull of a finite set of points. We assume that P is m-dimensional, that is, the affine hull of P is $\mathbf{R}^m$. Then the *face lattice* $\mathcal{F}(P)$ of P is the set of all k-faces of P, $-1 \leq k \leq m$. A *k-face* of P with $0 \leq k \leq m-1$ is a k-dimensional intersection of P with a supporting hyperplane of P. Besides that the empty set $\emptyset$ and the polytope P itself are considered as the (-1)-face and the m-face of P, respectively. For example, the face lattice $\mathcal{F}(S)$ of a square $S \subseteq \mathbf{R}^2$ consists of $\emptyset$ (the (-1)-face), the four singletons $\{v\}$ formed by the *vertices* $v \in \mathrm{vert}(S)$ (the 0-faces), the four edges (the 1-faces), and S (the 2-face). Note that all faces of a polytope P are polytopes themselves.

Face lattices of polytopes are the simplest examples of so-called polyhedral complexes. A *polyhedral complex $\mathcal{P}$ in $\mathbf{R}^m$* is a non-empty collection of polytopes $P \subseteq \mathbf{R}^m$ fulfilling the following three conditions (cf. e.g. [Rin]):

(C1) For all $P \in \mathcal{P}$, all faces of P belong to $\mathcal{P}$:

$$P \in \mathcal{P} \quad \Longrightarrow \quad \mathcal{F}(P) \subseteq \mathcal{P} \, .$$

(C2) For all $P, Q \in \mathcal{P}$, the intersection $P \cap Q$ is a face of both P and Q:

$$P, Q \in \mathcal{P} \quad \Longrightarrow \quad P \cap Q \in \mathcal{F}(P) \cap \mathcal{F}(Q) \, .$$

(C3) The system $\mathcal{P}$ is locally finite in the topological space formed by the *underlying set*

$$|\mathcal{P}| = \bigcup_{P \in \mathcal{P}} P$$

of $\mathcal{P}$. (Clearly, the topology of $|\mathcal{P}|$ is induced by the usual topology of $\mathbf{R}^m$.)

Figure 4.1 illustrates some examples. The first one is just the face lattice of a square. The next one shows an infinite complex consisting of the unit squares corresponding to the integer points in $\mathbf{R}^2$ and of all faces of these squares. The third complex is given by

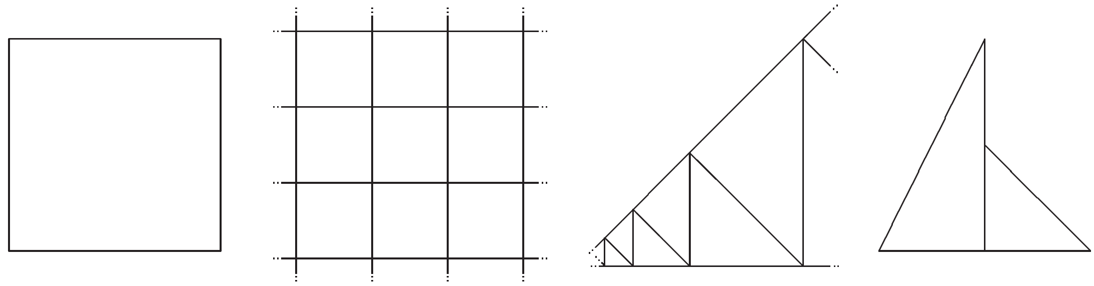

Figure 4.1: Three polyhedral complexes and a counterexample

infinitely many triangles, which cover an angle as displayed in the figure, and, of course, by their faces. (Note that the vertex of the angle does not belong to the complex. Otherwise, condition **(C3)** would be violated.) The fourth example consisting of two triangles and their faces does not fulfil **(C2)** and therefore is not a polyhedral complex.

The following fact is widely used in optimization theory. Given a polyhedral complex $\mathcal{P} \subseteq \mathbf{R}^m$ and a linear function $f : \mathbf{R}^m \to \mathbf{R}$, then

$$\min f(P), \max f(P) \;\in\; f(\mathrm{vert}(P)) \qquad \text{for all} \qquad P \in \mathcal{P}\ . \tag{4.1}$$

We want to characterize all real-valued continuous functions $f \in C(|\mathcal{P}|)$ possessing the extremality property (4.1). These functions will turn out to be linear combinations of partitions of unity subordinated to a particular covering of the space $|\mathcal{P}|$. We define the set of *vertices of the complex* $\mathcal{P}$ by

$$\mathrm{vert}(\mathcal{P}) \;=\; \bigcup_{P \in \mathcal{P}} \mathrm{vert}(P)\ .$$

The *relative interior* $\mathrm{relint}(P)$ of a polytope P is meant to be the interior of P with respect to the topology induced on the affine space spanned by P. (For instance, the relative interior of a triangle T consists of all points of T which do not belong to the edges of T even if T is a subset of $\mathbf{R}^m$ with $m \geq 3$, where the usual interior of T would be empty.) For all vertices $v \in \mathrm{vert}(\mathcal{P})$ of the complex $\mathcal{P}$, we define a corresponding set C_v by

$$C_v \;=\; \bigcup_{P \in \mathcal{P}, v \in \mathrm{vert}(P)} \mathrm{relint}(P)\ .$$

Figure 4.2 illustrates the situation for the three complexes given in Figure 4.1. One can show that $\{C_v : v \in \mathrm{vert}(\mathcal{P})\}$ is a peaked open covering of $|\mathcal{P}|$ with peaks $v \in C_v$ (cf. [Ri7–H]). Corollary 4.1 can be employed for proving the following conclusion.

Theorem 4.3 ([Ri7–H], Theorem 3) *Let $\mathcal{P}$ be a polyhedral complex in $\mathbf{R}^m$ and let $f \in C(|\mathcal{P}|)$. Then the following are equivalent.*
(i) For all polytopes $P \in \mathcal{P}$,

$$\min f(P), \max f(P) \;\in\; f(\mathrm{vert}(P))\ .$$

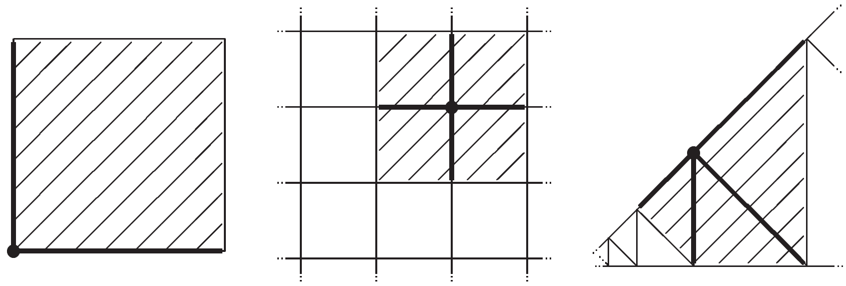

Figure 4.2: The covering sets C_v

(ii) The function f can be represented as

$$f \;=\; \sum_{v \in \mathrm{vert}(\mathcal{P})} \lambda_v \varphi_v$$

with coefficients $\lambda_v \in \mathbf{R}$ and a partition of unity $\{\varphi_v : v \in \mathrm{vert}(\mathcal{P})\}$ on $|\mathcal{P}|$ subordinated to the covering $\{C_v : v \in \mathrm{vert}(\mathcal{P})\}$. $\square$

4.3 An approximation class on the cube

A particular application of Theorem 4.3 brings us back to the n-term approximation by controllable partitions of unity. Proposition 2.5 describes all functions $f \in C([0,2])$ belonging to the closure $\mathrm{cl}(\Phi_2([0,2]))$ of the approximation class $\Phi_2([0,2])$, that is the class of the functions f with $a_2(f) = 0$. We want to generalize it in so far as we want to consider the m-dimensional cube $([0,2]^m, d_\infty)$ instead of the interval $[0,2]$. As we have seen in Example 1.1, there exist controllable coverings of $[0,2]^m$ for the cardinalities $1^m, 2^m, 3^m, \ldots$ only. Hence the approximation classes $\Phi_n([0,2]^m)$, $1 \le n \le 2^m - 1$, contain the constant functions only. Accordingly, the first interesting class is $\Phi_{2^m}([0,2]^m)$.

The problem of characterizing $\mathrm{cl}(\Phi_{2^m}([0,2]^m))$ has led us to the results of the last two sections, since the partitions of unity of cardinality 2^m are closely related to the polyhedral structure of the cube $[0,2]^m$. The functions $f \in C([0,2]^m)$ with $a_{2^m}(f) = 0$ can be characterized as follows.

Theorem 4.4 ([Ri7–H], Theorem 4) *Let f be a continuous real-valued function on the compact metric space $([0,2]^m, d_\infty)$. Then the following are equivalent.*
(i) $f \in \mathrm{cl}(\Phi_{2^m}([0,2]^m))$.
(ii) For all faces $Q \in \mathcal{F}([0,2]^m)$,

$$\min f(Q), \max f(Q) \;\in\; f(\mathrm{vert}(Q)) \,. \qquad \square$$

Although a first lenghty proof of Theorem 4.4 has been given in the Ph.D. thesis [Ri3], the approach via polyhedral complexes is more elegant and could give rise to generalizations concerning arbitrary polytopes. We expect that every polytope P with n vertices can be equipped with a metric d_P such that the corresponding approximation classes $\Phi_i(P)$, $1 \le i \le n-1$, consist of the constants only, while $\mathrm{cl}(\Phi_n(P))$ admits a characterization by an extremality condition as that in Theorem 4.4.

4.4 Approximating sequences of partitions

Now we shall deal with a problem concerning the approximation of continuous functions by step functions. When considering n-term approximation by controllable step functions, we saw that every infinite compact metric space (X, d) admits a chain K of controllable partitions, every continuous function $f \in C(X)$ being approximable with respect to K (cf. Theorem 1.1 and Proposition 3.2).

This motivates the following generalization to arbitrary topological spaces X. Does there exist a sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ of finite partitions of X such that every real-valued bounded continuous function f on X can be expressed as a uniform limit $f = \lim\limits_{n\to\infty} \varphi_n$ of step functions φ_n defined on the partitions $\mathcal{P}_n$, $n \ge 1$? A sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ of that type is to be called an *approximating sequence* of X. (Clearly, we could ask also for a chain of partitions with the above mentioned property. But this would be an equivalent question, since every approximating sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ on X gives rise to an approximating chain $\left(\tilde{\mathcal{P}}_n\right)_{n=1}^{\infty}$, where $\tilde{\mathcal{P}}_n$ can be defined to be the smallest common refinement of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$.)

We shall confine ourselves to completely regular spaces X, characterized by the separation axioms T_1 and $\mathrm{T}_{3\frac{1}{2}}$, which are Hausdorff spaces such that, for every closed subset $A \subseteq X$ and every point $x \in X \setminus A$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in A$. This property guarantees the existence of non-trivial continuous functions in $C(X)$. On the other hand, there exist infinite regular spaces X (defined by T_1 and T_3) such that $C(X)$ consists of constant functions only (cf. [St/See]). In that trivial case every sequence of finite partitions is approximating, of course. In the case of a completely regular space, however, it is non-trivial to ask for an approximating sequence.

Theorem 4.5 ([R/S2–H]) *Let X be a completely regular space admitting an approximating sequence of partitions. Then X is a compact metrizable space.* $\square$[†]

Now we see that the concept of an approximating sequence does not give rise to a generalization of n-term approximation by step functions from compact metric spaces to

[†]Let us remark that the proof of Theorem 4.5 given in [R/S2–H] can essentially be shortened by using the separability of $C(X)$. This was told us by Engelking (private communication).

more general spaces. But we can interpret Theorem 4.5 as a characterization of compact metrizable spaces by approximation theoretical properties.

Let us add an equivalent geometric condition. A sequence $(\mathcal{P}_n)_{n=1}^\infty$ of finite partitions of a topological space X is said to be *globally contracting* if, for every finite open covering $\mathcal{C}$ of X, there exists a number n_0 such that $\mathcal{P}_n$ is a refinement of $\mathcal{C}$ if $n \geq n_0$. That is, for all $n \geq n_0$ and all $P^{(n)} \in \mathcal{P}_n$, there is a covering set $C \in \mathcal{C}$ with $P^{(n)} \subseteq C$.

Theorem 4.6 ([R/S2–H]) *Let X be a completely regular space. Then the following are equivalent.*
(i) X is compact and metrizable.
(ii) X admits an approximating sequence of partitions.
(iii) X admits a globally contracting sequence of partitions. □

We want to remark that a sequence of partitions of a completely regular space is approximating if and only if it is globally contracting. This is not explicitely claimed in Theorem 4.6, but is a simple consequence of the following metric characterization.

Proposition 4.1 *Let $(\mathcal{P}_n)_{n=1}^\infty$ be a sequence of finite partitions of a compact metric space (X, d). Then the following are equivalent.*
(i) $(\mathcal{P}_n)_{n=1}^\infty$ is globally contracting.
(ii) $(\mathcal{P}_n)_{n=1}^\infty$ is approximating.
(iii) The fineness of the partitions $\mathcal{P}_n$ tends to zero, $\lim\limits_{n\to\infty} F(\mathcal{P}_n) = 0$.

Proof. The implication (i)⇒(ii) is shown in [R/S2–H] even for arbitrary topological spaces X. The implication (iii)⇒(i) rests on Lebesgue's covering lemma, which says essentially that, for every open covering $\mathcal{C}$ of a compact space X, there exists an $\varepsilon > 0$ such that every partition $\mathcal{P}$ with $F(\mathcal{P}) \leq \varepsilon$ is a refinement of $\mathcal{C}$ (cf. [Eng]).

For proving (ii)⇒(iii) we assume that (iii) fails. Then there exist some $\delta > 0$ and a subsequence $(\mathcal{P}_{n_k})_{k=1}^\infty$ of partitions with $F(\mathcal{P}_{n_k}) > \delta$, say $\varepsilon_1\left(P^{(n_k)}\right) > \delta$ with $P^{(n_k)} \in \mathcal{P}_{n_k}$. We fix points $x_{n_k}, y_{n_k} \in P^{(n_k)}$ with $d(x_{n_k}, y_{n_k}) \geq \delta$. Now we choose a subsequence $(n_{k_l})_{l=1}^\infty$ such that the limits $x_0 = \lim\limits_{l\to\infty} x_{n_{k_l}}$ and $y_0 = \lim\limits_{l\to\infty} y_{n_{k_l}}$ exist. Moreover, we can assume that the sets $A = \left\{x_{n_{k_l}} : l \geq 1\right\} \cup \{x_0\}$ and $B = \left\{y_{n_{k_l}} : l \geq 1\right\} \cup \{y_0\}$ are disjoint, because of $d(x_0, y_0) \geq \delta$. The compact space X is normal. Hence the disjoint closed sets $A, B \subseteq X$ can be separated by a bounded continuous function $f : X \to [0, 1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$. But f can not be expressed as a uniform limit $f = \lim\limits_{n\to\infty} \varphi_n$ of step functions φ_n defined on the partitions $\mathcal{P}_n$. Indeed, every function $\varphi_{n_{k_l}}$, $l \geq 1$, would be constant on $P^{(n_{k_l})}$, in particular $\varphi_{n_{k_l}}\left(x_{n_{k_l}}\right) = \varphi_{n_{k_l}}\left(y_{n_{k_l}}\right)$, whereas $f\left(x_{n_{k_l}}\right) = 0$ and $f\left(y_{n_{k_l}}\right) = 1$. This contradiction to the approximation property (ii) of the sequence $(\mathcal{P}_n)_{n=1}^\infty$ proves the remaining implication (ii)⇒(iii). □

Chapter 5

Local and global properties of quasi-continuous and cliquish functions

5.1 Quasi-continuous and cliquish functions

The notion of a quasi-continuous function goes back to Kempisty, who introduced this concept for real-valued functions of several variables in 1932 (cf. [Ke]). The survey in the first section of [R/S3–H] shows that the history of quasi-continuity is full of independent approaches to the same concept. In the most general setting quasi-continuity can be defined for mappings between arbitrary topological spaces X and Y. The slightly more general concept of a cliquish function $f : X \to Y$ requires a uniform structure on Y. However, we shall restrict our considerations to functions with values in $\mathbf{R}$.

A real-valued function f on a topological space X is called *quasi-continuous at the point* $x_0 \in X$ if, for every $\varepsilon > 0$ and for every neighbourhood $U \in \mathcal{U}(x_0)$ of x_0, there exists a non-empty open set $G \subseteq U$ such that

$$|f(x) - f(x_0)| < \varepsilon \qquad \text{for all} \qquad x \in G .$$

The function f is called *cliquish at the point* $x_0 \in X$ if under the same conditions as above

$$|f(x) - f(x')| < \varepsilon \qquad \text{for all} \qquad x, x' \in G .$$

The function f is called *quasi-continuous* or *cliquish on* X if it is quasi-continuous or cliquish, respectively, at each point of X.

The property of quasi-continuity on X is closely related to Levine's geometric concept of a semi-open set. A set $S \subseteq X$ is called *semi-open* if $S \subseteq \mathrm{cl}(\mathrm{int}(S))$ (cf. [Le]). Then a function

$f : X \to \mathbf{R}$ turns out to be quasi-continuous on X if the inverse image $f^{-1}(V)$ is semi-open for every open set $V \subseteq \mathbf{R}$. Originally, Levine called this property *semi-continuity* of f. But in [Neu1] Nebrunnova proved the equivalence of semi-continuity and quasi-continuity.

To illuminate the difference between continuity and quasi-continuity, let us consider how the value $f(x_0)$ depends on the values of f in the surroundings of x_0. Clearly, if f is continuous at x_0 then $f(x_0)$ is uniquely determined by the behaviour of f close to x_0. This need not be the case if f is quasi-continuous at x_0. However, also in the case of quasi-continuity $f(x_0)$ underlies certain local restrictions, which can be described by the concept of the *open hull* $HO_f(x_0)$ *of* f *at* x_0.

$$\begin{aligned} HO_f(x_0) \;&=\; \{\gamma \in \mathbf{R} : \text{ for all } U \in \mathcal{U}(x_0) \text{ and all } \varepsilon > 0 \text{ there exists a non-empty} \\ &\qquad \text{open subset } G \subseteq U \text{ such that } |f(x) - \gamma| < \varepsilon \text{ for all } x \in G\} \end{aligned}$$

(cf. [R/S3–H]). The open hull $HO_f(x_0)$ reflects in some sense the "stable" behaviour of f in the neighbourhood of x_0. Here stability means that $HO_f(x_0)$ is not affected by an arbitrary modification of the values of f on a nowhere dense subset of X. The following is a simple consequence of the definition of quasi-continuity.

Proposition 5.1 ([R/S3–H], Proposition 2.1) *A real-valued function f on a topological space X is quasi-continuous at a point $x_0 \in X$ if and only if $f(x_0) \in HO_f(x_0)$.* $\square$

Let us consider the functions $f_1, f_2, f_3 : [-1, 1] \to \mathbf{R}$ defined by

$$f_1(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0, \\ c_1 & \text{if } x = 0, \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{x}\cdot\sin\frac{1}{x} & \text{if } x \neq 0, \\ c_2 & \text{if } x = 0, \end{cases} \qquad f_3(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0, \\ c_3 & \text{if } x = 0. \end{cases}$$

Then $HO_{f_1}(0) = [-1, 1]$, $HO_{f_2}(0) = \mathbf{R}$, and $HO_{f_3}(0) = \emptyset$. Hence f_1 is quasi-continuous so far as $c_1 \in [-1, 1]$ and f_2 for every c_2, whereas f_3 can not be quasi-continuous at 0. The function f_2 illustrates that a quasi-continuous function on a compact topological space X need not be bounded nor even locally bounded. f_3 is an example of a cliquish function.

Note that the value $f(x_0)$ of a cliquish function f is not restricted by the values of f in the surroundings of x_0. In fact, it can easily be seen that f remains cliquish at x_0 even if its values are arbitrarily changed on a nowhere dense subset of X. However, the open hull can be employed for formulating a sufficient condition for the cliquishness of f at x_0, which is even necessary so far as f is locally bounded at x_0.

Proposition 5.2 ([R/S3–H], Proposition 2.3) *Let f be a real-valued function on a topological space X. Then f is cliquish at $x_0 \in X$ if $HO_f(x_0) \neq \emptyset$. Conversely, if f is cliquish and locally bounded at x_0, then $HO_f(x_0) \neq \emptyset$.* $\square$

In the following we shall deal with quasi-continuity and cliquishness as global properties. It is a simple consequence of the definition that the sum of two cliquish functions is cliquish again. Besides that, the class of cliqish functions on a topological space X is closed with respect to uniform limits. Also the uniform limit of quasi-continuous functions is quasi-continuous (cf. [Ble], [Le]). But note that the sum $f+g$ of two quasi-continuous functions need not be quasi-continuous, which can be illustrated by the functions $f = \mathbf{I}_{(0,1]}$ and $g = \mathbf{I}_{[0,1]}$ on the interval $[-1,1]$.

Although a quasi-continuous or cliquish function f on a topological space X is far from being continuous in general, its points of discontinuity form a set of the first category only (cf. [Thie], [Neu2]). If X is a Baire space, which means that every subset of the first category has a dense complement, then the continuity points of f form a dense subset of X (cf. [Neu2]). We shall complete this well-known result by showing that Baire spaces are the only topological spaces on which all cliquish functions have a dense set of continuity points.

Proposition 5.3 *A topological space X is a Baire space if and only if every cliquish function $f : X \to \mathbf{R}$ has a dense set of continuity points.*

Proof. As explained above, it suffices to infer the Baire property of X under the supposition that all cliquish functions $f : X \to \mathbf{R}$ have a dense set of continuity points. So let $A \subseteq X$ be a set of the first category. That is, $A = \bigcup\limits_{n=1}^{\infty} A_n$ with nowhere dense subsets $A_n \subseteq X$. We have to show that $X \setminus A$ is dense in X.

The step functions $\mathbf{I}_{A_n}$, $n \geq 1$, are cliquish, since the sets A_n are nowhere dense. Indeed, we get $HO_{\mathbf{I}_{A_n}}(x) = \{0\}$ for all $x \in X$, so that $\mathbf{I}_{A_n}$ is cliquish according to Proposition 5.2. Then the function $f = \sum\limits_{n=1}^{\infty} 3^{-n} \cdot \mathbf{I}_{A_n}$ is cliquish as well. We denote the set of discontinuity points of f by D_f. Now we want to show that

$$A \subseteq D_f . \tag{5.1}$$

So let $x_0 \in A$, say $x_0 \in A_{n_0}$. Then $f(x_0) \geq 3^{-n_0}$. But every neighbourhood $U \in \mathcal{U}(x_0)$ contains some point $y_0 \in U \setminus \bigcup\limits_{n=1}^{n_0} A_n$, since $\bigcup\limits_{n=1}^{n_0} A_n$ is nowhere dense. Thus $\mathbf{I}_{A_n}(y_0) = 0$, $1 \leq n \leq n_0$, and hence $f(y_0) = \sum\limits_{n=n_0+1}^{\infty} 3^{-n} \cdot \mathbf{I}_{A_n}(y_0) \leq \sum\limits_{n=n_0+1}^{\infty} 3^{-n} = \frac{1}{2} \cdot 3^{-n_0}$. Consequently, f is discontinuous at x_0, for $f(x_0) - f(y_0) \geq 3^{-n_0} - \frac{1}{2} \cdot 3^{-n_0} = \frac{1}{2} \cdot 3^{-n_0}$. This proves the inclusion (5.1).

By the assumption, the set $X \setminus D_f$ of continuity points of f is dense in X. Now the inclusion (5.1) yields that $X \setminus A$ is dense as well. This completes the proof. □

In the context of the present report quasi-continuity appeared as a property of all chain-approximable functions on a compact metric space (X, d) (see Proposition 3.1). The following claim, however, demonstrates that chain-approximable functions in general form a small subclass of all bounded quasi-continuous functions only.

Proposition 5.4 *Let (X, d) be a compact metric subspace of an m-dimensional Banach space or of the m-dimensional Euclidean sphere and let $f : X \to \mathbf{R}$ be a uniform limit of controllable step functions φ_n, $n \geq 1$, on X. Then*

$$\operatorname{card}(HO_f(x)) \;\leq\; 5^m \qquad \textit{for all} \qquad x \in X\ .$$

In the case $(X, d) = ([-1, 1]^m, d_\infty)$ the bound 5^m can be sharpened to 2^m.

Proof. The step functions φ_n are defined on controllable partitions $\mathcal{P}_n$ of X, which must fulfil the strong form of local finiteness $\kappa(\mathcal{P}_n) \leq 5^m$ according to Corollary 1.1. Hence, for every $x \in X$ and every $n \geq 1$, there exists a neighbourhood U_n of x such that $\operatorname{card}(\varphi_n(U_n)) \leq 5^m$. This immediately implies the estimate $\operatorname{card}(HO_f(x)) \leq 5^m$ concerning the open hull of $f = \lim\limits_{n\to\infty} \varphi_n$. Corollary 1.2 yields the stronger claim for the space $(X, d) = ([-1, 1]^m, d_\infty)$. $\square$

In contrast with that, the above example $f_1 : [-1, 1] \to \mathbf{R}$ illustrates that the open hull of a bounded quasi-continuous function on the simple space $[-1, 1]$ may contain uncountably many elements. The more satisfactory it is that every quasi-continuous or cliquish function on an arbitrary topological space X admits a representation as a uniform limit of appropriate "step functions" defined on a chain of partitions of X (see Section 5.4). Hence quasi-continuous and cliquish functions are "chain-approximable" in a more general sense.

5.2 Semi-open and almost semi-open step functions

In the classical sense a step function is meant to be a function with finite range. We want to introduce appropriate concepts of "step functions" which can be used as basic functions for the approximation of quasi-continuous and cliquish functions, respectively. As we have seen above, quasi-continuous and cliquish functions may be unbounded and thus can not be expressed as a uniform limit of functions with finitely many values only. So we must allow basic functions with infinite range. On the other hand, the approximating "step functions" should be quasi-continuous or cliquish, respectively. This is related to particular topological properties of the underlying partitions.

Proposition 5.5 *Let $\varphi = \sum\limits_{i=1}^{k} \lambda_i \mathbf{I}_{P_i}$ be a piecewise constant real-valued function on a finite partition $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ of a topological space X with $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the following hold true.*
(a) The function φ is quasi-continuous if and only if the sets P_i, $1 \leq i \leq k$, are semi-open.
(b) The function φ is cliquish if and only if the set $\bigcup\limits_{i=1}^{k} \operatorname{int}(P_i)$ is dense in X.

Proof. Part (a) is a direct consequence of the characterization of quasi-continuity by the aid of semi-open sets.

If $\varphi = \sum_{i=1}^{k} \lambda_i \mathbf{I}_{P_i}$ is cliquish then every non-empty open set $U \subseteq X$ must contain a non-empty open subset $G \subseteq U$ such that φ is constant on G, say $\varphi(G) = \{\lambda_{i_0}\}$. Then $G \subseteq \operatorname{int}(P_{i_0})$. This shows that $\bigcup_{i=1}^{k} \operatorname{int}(P_i)$ is dense in X.

Conversely, if $\bigcup_{i=1}^{k} \operatorname{int}(P_i)$ is dense then, for every $x \in X$ and every open neighbourhood $U \in \mathcal{U}(x)$, there exists a partition set P_{i_0} with $\operatorname{int}(P_{i_0}) \cap U \neq \emptyset$. Hence φ is constant on the non-empty open subset $G = \operatorname{int}(P_{i_0}) \cap U$ of U. Thus φ is cliquish at x. □

In the following a partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ of a topological space X is to be called *semi-open* if all the sets P_ι, $\iota \in I$, are semi-open. $\mathcal{P}$ is to be called *almost semi-open* if $\bigcup_{\iota \in I} \operatorname{int}(P_\iota)$ is dense in X. A function $\varphi : X \to \mathbf{R}$ is said to be a *semi-open step function* or an *almost semi-open step function* if it is piecewise constant on a semi-open or an almost semi-open partition of X, respectively. Note that in this definition the partition $\mathcal{P}$ and hence the range of the corresponding piecewise constant function φ are allowed to be infinite.

The notion of a semi-open partition has already been used in the literature (cf. [S/Z/Z2]), whereas the concept of an almost semi-open partition is new. As far as we know, semi-open and almost semi-open step functions have not been studied before. The arguments from the above proof show that semi-open and almost semi-open step functions are quasi-continuous or cliquish, respectively, even if the underlying partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ is infinite.

In the present context the notation "step function" has a topological background. The "steps" P_ι of a semi-open step function are relatively "large", since they are semi-open and thus have a "large" interior. In the case of an almost semi-open step function there exists an almost semi-open underlying partition $\mathcal{P} = \{P_\iota : \iota \in I\}$. Then the partition sets P_ι with $\operatorname{int}(P_\iota) \neq \emptyset$ constitute "large steps". But there possibly exists a large number of partition sets with $\operatorname{int}(P_\iota) = \emptyset$, the union of these sets being nowhere dense.

In the proof of Proposition 5.3 we have constructed semi-open step functions in a very simple way. If A is a non-empty nowhere dense subset of a topological space X then the partition $\mathcal{P} = \{X \setminus A, A\}$ is almost semi-open, for $\operatorname{int}(X \setminus A)$ is dense. Hence every function $\lambda \mathbf{I}_{X\setminus A} + \mu \mathbf{I}_A$ is an almost semi-open step function. Moreover, the sum of two almost semi-open step functions is an almost semi-open step function as well. This is due to the fact that the "mixture" $\mathcal{Q} = \{P^{(1)} \cap P^{(2)} : P^{(1)} \in \mathcal{P}_1, P^{(2)} \in \mathcal{P}_2,\ P^{(1)} \cap P^{(2)} \neq \emptyset\}$ of any two almost semi-open partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ is almost semi-open as well. These simple tools combined with the closedness of cliquish functions with respect to uniform limits give rise to a huge number of examples of cliquish functions.

In the case of semi-open step functions the above construction does not work. One problem is that a semi-open partition of X can not contain a nowhere dense set A as a partition element, since nowhere dense sets are not semi-open. Besides that, the sum of two semi-open step functions need not be semi-open, which has been illustrated by the functions $\mathbf{I}_{(0,1]}$ and $\mathbf{I}_{[0,1]}$ in the last section. This shows in particular that the "mixture" of two semi-open partitions need not be semi-open. Nevertheless, we can show that every infinite Hausdorff space X possesses "many" semi-open step functions.

Proposition 5.6 *Every semi-open subset $S \subseteq X$ of a Hausdorff space X with* $\mathrm{card}(S) \geq 2$ *can be decomposed into two non-empty disjoint semi-open subsets S_1 and S_2.*

Proof. The interior $\mathrm{int}(S)$ contains at least two distinct points x_1, x_2, for $S \subseteq \mathrm{cl}(\mathrm{int}(S))$ and $\mathrm{card}(S) \geq 2$. These points can be separated by two disjoint open sets $O_1, O_2 \subseteq \mathrm{int}(S)$ with $x_1 \in O_1$, $x_2 \in O_2$. We put $S_1 = S \cap \mathrm{cl}(O_1)$ and $S_2 = S \setminus \mathrm{cl}(O_1)$. Then S_1 and S_2 form a partition of S consisting of non-empty subsets, since $O_1 \subseteq \mathrm{int}(S_1)$ and $O_2 \subseteq \mathrm{int}(S_2)$. The set S_1 is semi-open, for $S_1 \subseteq \mathrm{cl}(O_1) \subseteq \mathrm{cl}(\mathrm{int}(S_1))$. Moreover, we have

$$\begin{aligned}
S_2 &= S \setminus \mathrm{cl}(O_1) \\
&\subseteq \mathrm{cl}(\mathrm{int}(S)) \setminus \mathrm{cl}(O_1) \\
&= \big(\mathrm{cl}(\mathrm{int}(S) \cap S_1) \cup \mathrm{cl}(\mathrm{int}(S) \cap S_2)\big) \setminus \mathrm{cl}(O_1) \\
&= \big(\mathrm{cl}(\mathrm{int}(S) \cap \mathrm{cl}(O_1)) \cup \mathrm{cl}(\mathrm{int}(S) \setminus \mathrm{cl}(O_1))\big) \setminus \mathrm{cl}(O_1) \\
&\subseteq \big(\mathrm{cl}(O_1) \cup \mathrm{cl}(\mathrm{int}(S_2))\big) \setminus \mathrm{cl}(O_1) \\
&\subseteq \mathrm{cl}(\mathrm{int}(S_2)) \,.
\end{aligned}$$

Hence S_2 is semi-open as well. This completes the proof. □

Corollary 5.1 *Every infinite Hausdorff space X admits a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n$ with* $\mathrm{card}(\mathcal{P}_n) = n$.

Proof. Starting with the semi-open partition $\mathcal{P}_1 = \{X\}$ we define the chain K inductively. Given the semi-open partition $\mathcal{P}_n$ with $\mathrm{card}(\mathcal{P}_n) = n$, we find an infinite set $P \in \mathcal{P}_n$. We obtain the refinement $\mathcal{P}_{n+1}$ of $\mathcal{P}_n$ by decomposing P into two non-empty semi-open subsets P_1 and P_2 according Proposition 5.6. □

Corollary 5.1 shows that every infinite Hausdorff space X admits a large variety of quasi-continuous functions. All step functions on partitions from the chain K are semi-open. Moreover, the Banach space $A_K(X)$ of all uniform limits of step functions on partitions from K even contains quasi-continuous functions with infinite range. This is remarkable in so far as the class $C(X)$ of all continuous real-valued functions on X may consist of constant functions only. In fact, there exist infinite regular spaces X (Hausdorff spaces additionally fulfilling T_3) such that all functions in $C(X)$ are constant (cf. [St/See]).

Let us remark that the situation changes if one considers T_1-spaces instead of Hausdorff spaces. Here a good example is given by the cofinite topology on an infinite set X. In this topology a subset $O \subseteq X$ is defined to be open if and only if $O = \emptyset$ or $X \setminus O$ is finite (cf. [St/See]). This implies that $\mathcal{P} = \{X\}$ is the only semi-open partition of X. Otherwise there would exist at least two disjoint non-empty semi-open subsets P_1 and P_2 of X. But then X would contain two disjoint non-empty open subsets $\text{int}(P_1)$ and $\text{int}(P_2)$ in contradiction with the definition of the cofinite topology. Hence all semi-open step functions are constant. In the sequel it will turn out that the semi-open step functions are dense within the class of all quasi-continuous functions. Accordingly, all quasi-continuous functions on an infinite space X with the cofinite topology are constant. In contrast with that, every infinite T_1-space admits cliquish functions with infinite range.

Proposition 5.7 *Let $(x_i)_{i=1}^{\infty}$ be a sequence of mutually distinct points of an infinite T_1-space X and let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of reals with $\lim\limits_{i\to\infty} \lambda_i = 0$. Then the function $f = \lambda_0 \mathbf{I}_X + \sum\limits_{i=1}^{\infty} \lambda_i \mathbf{I}_{\{x_i\}}$ is cliquish.*

Proof. Every partition $\mathcal{P} = \{X \setminus \{x_0\}, \{x_0\}\}$ of X with arbitrary $x_0 \in X$ is almost semi-open. Indeed, if x_0 is an isolated point, i.e. $\{x_0\}$ is open, then $\text{int}(X \setminus \{x_0\}) \cup \text{int}(\{x_0\}) = (X \setminus \{x_0\}) \cup \{x_0\}$ is dense in X. If x_0 is not isolated then $\text{int}(X \setminus \{x_0\}) \cup \text{int}(\{x_0\}) = (X \setminus \{x_0\}) \cup \emptyset$ is dense as well. Consequently, every function $\mathbf{I}_{\{x_0\}}$ is an almost semi-open step function. Then the functions $\varphi_n = \lambda_0 + \sum\limits_{i=1}^{n} \lambda_i \mathbf{I}_{\{x_i\}}$, $n \geq 1$, are almost semi-open step functions, too. Hence the uniform limit $f = \lim\limits_{n\to\infty} \varphi_n$ is cliquish. $\square$

5.3 Transforming cliquish functions into quasi-continuous functions

In this section we shall study real-valued functions f on an arbitrary topological space X with $HO_f(x) \neq \emptyset$ for all $x \in X$. These functions form an important subclass of cliquish functions as Proposition 5.2 tells. A function f of that type can be transformed into a quasi-continuous function $\tilde{f}$ by a so-called admissible modification. Denoting the set of continuity points of f and $\tilde{f}$ by C_f and $C_{\tilde{f}}$, respectively, this concept says: The function $\tilde{f}$ is an *admissible modification* of a function f if

$$\tilde{f}(x) = f(x) \quad \text{for all} \quad x \in C_f \qquad \text{and} \qquad C_f \subseteq C_{\tilde{f}}$$

(cf. [Au]). Accordingly, f and $\tilde{f}$ differ in discontinuity points of f only. As we have mentioned above, the discontinuity points of a cliquish function form a set of the first category. Hence an admissible modification of a cliquish function is actually a "small" modification.

Theorem 5.1 ([R/S3–H], Theorem 4.1) *Let f be a real-valued function on a topological space X with $HO_f(x) \neq \emptyset$ for all $x \in X$. Then every function $\tilde{f}$ with*

$$\tilde{f}(x) \in HO_f(x) \qquad \textit{for all} \qquad x \in X$$

is a quasi-continuous admissible modification of f with

$$HO_{\tilde{f}}(x) = HO_f(x) \qquad \textit{for all} \qquad x \in X . \quad \square$$

Now Proposition 5.2 implies that every locally bounded cliquish function can be transformed into a quasi-continuous function. The condition $HO_{\tilde{f}}(x) = HO_f(x)$ for all $x \in X$ even becomes necessary provided that $\tilde{f}$ is an admissible modification of a cliquish function f on a Baire space X.

Theorem 5.2 ([R/S3–H], Theorem 4.2) *Let f be a real-valued cliquish function on a Baire space X. Then every admissible modification $\tilde{f}$ of f satisfies the condition*

$$HO_{\tilde{f}}(x) = HO_f(x) \qquad \textit{for all} \qquad x \in X . \quad \square$$

Hence, by Proposition 5.1, every quasi-continuous admissible modification $\tilde{f}$ of a cliquish function f on a Baire space X fulfils

$$\tilde{f}(x) \in HO_{\tilde{f}}(x) = HO_f(x) \qquad \text{for all} \qquad x \in X .$$

Accordingly, f admits a quasi-continuous admissible modification $\tilde{f}$ only if $HO_f(x) \neq \emptyset$ for all $x \in X$. Moreover, every quasi-continuous admissible modification $\tilde{f}$ of f must be a selection of $HO_f(\cdot)$. The function f_3 from Section 5.1 serves as an example of a cliquish function not possessing a quasi-continuous admissible modification, because of $HO_{f_3}(0) = \emptyset$.

Let us remark that the situation changes if X is not a Baire space. For example, consider the function $f(x) = \sum_{x_i \leq x} 2^{-i}$ on the rational numbers $\mathbf{Q} = \{x_1, x_2, x_3, \ldots\}$ equipped with the usual topology induced by the real line. The function f is quasi-continuous, since it is continuous from the right. But f does not have any continuity point, so that every function $\tilde{f} : \mathbf{Q} \to \mathbf{R}$ is an admissible modification of f. Hence there exist admissible modifications $\tilde{f}$ of f such that

$$HO_{\tilde{f}}(x) \neq HO_f(x) \qquad \text{for all} \qquad x \in X .$$

5.4 Quasi-continuous and cliquish functions as uniform limits of corresponding step functions

Now we come to the main results of this chapter concerning the approximation of quasi-continuous and cliquish functions on an arbitrary topological space X. The functions under consideration will be expressed as uniform limits of semi-continuous or almost semi-continuous step functions φ_n, $n \geq 1$, which are defined on suitable sequences $K = (\mathcal{P}_n)_{n=1}^{\infty}$

of partitions of X. The partitions $\mathcal{P}_n$, $n \geq 1$, can be chosen such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, but not necessarily a strict one. A sequence K of that type is to be called a *weak chain*. (In the paper [R/S3–H] the short term "chain" stands for what is called a "weak chain" in the present report.) Note that a weak chain may be stationary. That is, there may exist an index n_0 such that $\mathcal{P}_n = \mathcal{P}_{n_0}$ for all $n \geq n_0$. This phenomenon, however, can not occur in the case of chains of controllable partitions of a compact metric space (X, d) as they were introduced in Section 1.2.

Theorem 5.3 ([R/S3–H], Theorem 3.1) *Let f be a real-valued quasi-continuous function on a topological space X. Then f can be represented as the uniform limit of a sequence $(\varphi_n)_{n=1}^{\infty}$ of semi-open step functions which are defined on a weak chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n = \left\{P_\iota^{(n)} : \iota \in I_n\right\}$. If f is locally bounded then there exists a weak chain K of locally finite partitions with the above property. If f is bounded then K can be chosen to be a weak chain of finite partitions.* □

One can prove Theorem 5.3 by inductively constructing the partitions $\mathcal{P}_n$ and the corresponding semi-open step functions. The construction rests on Lemma 3.1 from [R/S3–H], which is rather involved. Note that it is not enough to find a sequence of semi-open step functions φ_n defined on semi-open partitions $\mathcal{Q}_n$ such that $\sup\limits_{x \in X} |f(x) - \varphi_n(x)| < \frac{1}{n}$. In fact, then the partition $\mathcal{P}_n$ from the weak chain K would have to be a common refinement of $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$. But already two semi-open partitions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ need not possess a semi-open common refinement. This corresponds to the non-linearity of the class of all quasi-continuous functions on X.

Theorem 5.3 enables us to arrange the quasi-continuous functions on X such that they appear as elements of linear spaces $A_K(X)$, which are closed with respect to uniform limits. Here $A_K(X)$ denotes the space of all uniform limits of semi-open step functions defined on a weak chain K of semi-open partitions.

Theorem 5.4 ([R/S3–H], Theorem 3.2) *Let f be a real-valued quasi-continuous function on a compact metrizable space X. Then there exists a weak chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n = \left\{P_\iota^{(n)} : \iota \in I_n\right\}$ of the space X such that f as well as any real-valued continuous function on X can be attained as the uniform limit of a sequence of semi-open step functions which are defined on K. If f is locally bounded then there exists a weak chain K of locally finite partitions with the above property. If f is bounded then K can be chosen to be a weak chain of finite partitions.* □

Theorem 5.4 sharpens Theorem 5.3 in so far as now the space $A_K(X)$, besides the function f, additionally contains all real-valued continuous functions. If K consists of finite partitions only then it is an approximating sequence in the sense of Section 4.4. Hence every compact metrizable space X admits an approximating sequence K being a weak chain of semi-open partitions. Theorem 4.5 justifies the restriction to compact metrizable spaces in Theorem 5.4.

The main tool for proving analogous approximation results about cliquish functions is the mechanism of transforming cliquish functions into quasi-continuous functions as described in the previous section. Given a locally bounded cliquish function f on X, one can pass to a quasi-continuous admissible modification $\tilde{f}$. Then one can find a semi-open step function $\tilde{\varphi}$ for realizing the estimate $\sup\limits_{x \in X} \left| \tilde{f}(x) - \tilde{\varphi}(x) \right| \leq \frac{\delta}{2}$. This is the starting point for the construction of an almost semi-open step function φ with $\sup\limits_{x \in X} |f(x) - \varphi(x)| \leq \delta$ (cf. [R/S3–H], Lemma 5.2). The proofs of the two theorems to follow rest on these ideas.

Theorem 5.5 ([R/S3–H], Theorem 5.1) *Let f be a real-valued cliquish function on a topological space X. Then f can be represented as the uniform limit of a sequence $(\varphi_n)_{n=1}^{\infty}$ of almost semi-open step functions which are defined on a weak chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions. If f is locally bounded then there exists a weak chain K of locally finite partitions with the above property. If f is bounded then K can be chosen to be a weak chain of finite partitions.* □

Theorem 5.6 ([R/S3–H], Theorem 5.2) *Let f be a real-valued cliquish function on a compact metrizable space X. Then there exists a weak chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions of the space X such that f as well as any real-valued continuous function on X can be attained as the uniform limit of a sequence of almost semi-open step functions defined on K. If f is locally bounded then there exists a weak chain K of locally finite partitions with the above property. If f is bounded then K can be chosen to be a weak chain of finite partitions.* □

5.5 The associated multifunction

When introducing the open hull $HO_f(x)$ of a real-valued function f on a topological space X at a point $x \in X$, we used the set $HO_f(x)$ for describing local properties of f at x. Now we go a step forward by considering the open hull as a multivalued function reflecting the global behaviour of f. The set-valued map F_f on X defined by

$$F_f(x) \;=\; HO_f(x)$$

is to be called the *associated multifunction* of f. Clearly, F_f maps into the closed subsets of $\mathbf{R}$.

First we claim that $F_f(x) \neq \emptyset$ for all $x \in X$ as is often done if one studies continuity properties of multivalued functions (cf. [Ku], [Mi], [Eng]). Then, by Proposition 5.2, f needs to be cliquish. Moreover, Theorem 5.1 yields that F_f necessarily is the associated multifunction of a quasi-continuous function. Hence there is no loss of generality if we restrict our investigations to associated multifunctions of quasi-continuous functions $f : X \to \mathbf{R}$. Associated multifunctions of that type can be characterized among all set-valued functions in the follwing simple way.

Proposition 5.8 ([R/S3–H], Proposition 6.1) *Let F map a topological space X into the non-empty subsets of $\mathbf{R}$ and let f be any selection of F. Then F is the associated multifunction of some quasi-continuous function g on X if and only if $F = F_f$.* $\square$

The "identity theorem" to follow is a most surprising property of associated multifunctions.

Proposition 5.9 ([R/S3–H], Proposition 6.2) *Let F_f and F_g be the associated multifunctions of two real-valued quasi-continuous functions f and g on a topological space X such that $F_f(x) \cap F_g(x) \neq \emptyset$ for all $x \in X$. Then $F_f = F_g$ and, moreover, the two functions f and g are admissible modifications of each other.* $\square$

Now we want to study continuity properties of associated multifunctions. Let us recall that a multivalued function F mapping a topological space X into the (non-empty) subsets of $\mathbf{R}$ is called *lower (upper) semi-continuous at a point $x_0 \in X$* if, for every open subset $V \subseteq \mathbf{R}$ with $F(x_0) \cap V \neq \emptyset$ (or $F(x_0) \subseteq V$, respectively), there exists an open set $U \subseteq X$ with $x_0 \in U$ such that $F(x) \cap V \neq \emptyset$ (or $F(x) \subseteq V$, respectively) for all $x \in U$ (cf. [Ku], [Mi]).

In her paper [Ew] Ewert defined corresponding concepts of quasi-continuity for multivalued maps (see also [Po]). Accordingly, the multifunction F is called *lower (upper) quasi-continuous at a point $x_0 \in X$* if, for every open subset $V \subseteq \mathbf{R}$ with $F(x_0) \cap V \neq \emptyset$ (or $F(x_0) \subseteq V$, respectively), there exists a semi-open set $S \subseteq X$ with $x_0 \in S$ such that $F(x) \cap V \neq \emptyset$ (or $F(x) \subseteq V$, respectively) for all $x \in S$. Of course, F is called *lower (upper) quasi-continuous* if it is lower (upper) quasi-continuous for all $x \in X$.

Theorem 5.7 ([R/S3–H], Theorem 6.1) *Let f be a real-valued quasi-continuous function on a topological space X. Then the associated multifunction F_f is both lower and upper quasi-continuous.* $\square$

Theorem 5.7 makes clear that the notion of quasi-continuity for single-valued functions on one hand and the notions of lower and upper quasi-continuity for set-valued mappings on the other hand fit well together in the global sense. As far as it concerns continuity of a quasi-continuous single-valued function f at a point x_0, the local notion of lower semi-continuity for set-valued mappings turns out to be an adequate concept. Upper semi-continuity of the associated multifunction F_f at a point $x_0 \in X$ is a consequence of local boundedness of the underlying function f at x_0.

Theorem 5.8 ([R/S3–H], Theorem 6.2) *Let f be a real-valued quasi-continuous function on a topological space X. Then the associated multifunction F_f is lower semi-continuous at a point $x_0 \in X$ if and only if f is continuous at x_0.* $\square$

Theorem 5.9 ([R/S3–H], Theorem 6.3) *Let f be a real-valued quasi-continuous function on a topological space X. Then the associated multifunction F_f is upper semi-continuous at a point $x_0 \in X$ if f is locally bounded at x_0.* $\square$

By Proposition 5.3, we can conclude that the associated multifunction F_f of a quasi-continuous function f on a Baire space X has at least a dense set of points of lower semi-continuity.

One can show that the points where a quasi-continuous function f on X is not locally bounded form a nowhere dense set (see the proof of Theorem 5.1 in [R/S3–H]). Hence F_f fails to be upper semi-continuous on a nowhere dense subset of X only. If f is not locally bounded at x_0 then F_f may be upper semi-continuous, but need not be. This is illustrated by two examples in [R/S3–H].

We want to close the investigations of the present section by establishing a relation between the uniform convergence of quasi-continuous or cliquish functions and the uniform convergence of the associated multifunctions with respect to the Hausdorff distance. Let us recall that the *parallel set A_r of $A \subseteq \mathbf{R}$ with distance $r > 0$* is defined by

$$A_r \;=\; \bigcup_{a \in A} [a-r, a+r] \;=\; \{x \in \mathbf{R} : \text{ there exists } a \in A \text{ with } |x-a| \le r\}\,.$$

The *Hausdorff distance* of two sets $A, B \subseteq \mathbf{R}$ is given by

$$d_H(A,B) \;=\; \begin{cases} \inf\{r>0 : A \subseteq B_r \text{ and } B \subseteq A_r\} & \text{if } \{r>0 : A \subseteq B_r \text{ and } B \subseteq A_r\} \neq \emptyset\,, \\ \infty & \text{otherwise.} \end{cases}$$

For example, $d_H(\{x\},\{y\}) = |x-y|$ for $x, y \in \mathbf{R}$, $d_H(A, \mathrm{cl}(A)) = 0$ for every $A \subseteq \mathbf{R}$, $d_H(\emptyset, B) = \infty$ for all non-empty subsets B of $\mathbf{R}$, and $d_H(\emptyset,\emptyset) = 0$. The system

$$\Big\{\{B \subseteq \mathbf{R} : d_H(A,B) < r\} : \; A \subseteq \mathbf{R}, r > 0\Big\}$$

is a base for a non-separated uniform space on the system of all subsets of $\mathbf{R}$, whose restriction to the class of closed subsets of $\mathbf{R}$ even is separated. (A Hausdorff space is uniformizable if and only if it is completely regular; see [Eng].) The functional $d_H(\cdot,\cdot)$ describes a complete metric space on the system of non-empty compact subsets of $\mathbf{R}$ (cf. [Eng]).

Theorem 5.10 *Let f and g be real-valued cliquish functions on a topological space X. Then*

$$\sup_{x\in X} d_H(F_f(x), F_g(x)) \;\le\; \sup_{x\in X} |f(x)-g(x)| \,.$$

Proof. Let $r = \sup_{x\in X} |f(x)-g(x)|$. We assume that $r<\infty$. It suffices to show the inclusion

$$F_f(x_0) \;\subseteq\; (F_g(x_0))_r \tag{5.2}$$

for every $x_0 \in X$. Then the counterpart $F_g(x_0) \subseteq (F_f(x_0))_r$ is true for the same reasons, so that we obtain the claim $d_H(F_f(x_0), F_g(x_0)) \le r$.

First we prove the inclusion

$$HO_f(U) \;\subseteq\; (HO_g(U))_r \qquad \text{for all} \qquad U \in \mathcal{U}(x_0)\,, \tag{5.3}$$

where $HO_f(U)$ are closed sets of reals defined by

$$\begin{aligned} HO_f(U) \;=\; \{\gamma \in \mathbf{R} :\ & \text{for every } \varepsilon > 0 \text{ there exists a non-empty open} \\ & \text{subset } G \subseteq U \text{ such that } |f(x)-\gamma| < \varepsilon \text{ for all } x \in G\} \end{aligned}$$

(cf. [R/S3–H], Section 2). So let $\gamma \in HO_f(U)$ be fixed. Accordingly, for all $n \ge 1$ there exists a non-empty open set $G_n \subseteq U$ such that $|f(x)-\gamma| < \frac{1}{n}$ for all $x \in G_n$. Since g is cliquish, there exist non-empty open subsets $H_n \subseteq G_n$ with $|g(x)-g(x')| < \frac{1}{n}$ for all $x, x' \in H_n$. Now we fix points $x_n \in H_n \subseteq G_n$. Then

$$|g(x_n)-\gamma| \;\le\; |g(x_n)-f(x_n)| + |f(x_n)-\gamma| \;<\; r + \frac{1}{n}\,. \tag{5.4}$$

Hence the sequence $(g(x_n))_{n=1}^{\infty}$ is bounded and thus contains a convergent subsequence $(g(x_{n_k}))_{k=1}^{\infty}$, say

$$\lim_{k\to\infty} g(x_{n_k}) \;=\; \delta\,.$$

Then, for all $x \in H_{n_k}$,

$$|g(x)-\delta| \;\le\; |g(x)-g(x_{n_k})| + |g(x_{n_k})-\delta| \;<\; \frac{1}{n_k} + |g(x_{n_k})-\delta|\,.$$

This yields

$$\delta \;\in\; HO_g(U)\,.$$

Indeed, for every $\varepsilon > 0$ there exists $k \ge 0$ with $\frac{1}{n_k} + |g(x_{n_k})-\delta| \le \varepsilon$, so that all points x from the non-empty open subset $H_{n_k} \subseteq U$ are subject to the estimate $|g(x)-\delta| < \varepsilon$. This shows that $\delta \in HO_g(U)$. Finally, by (5.4),

$$|\delta-\gamma| \;\le\; |\delta - g(x_{n_k})| + |g(x_{n_k})-\gamma| \;<\; |\delta - g(x_{n_k})| + r + \frac{1}{n_k}$$

for all $k \geq 1$. Letting $k \to \infty$, we obtain $|\delta - \gamma| \leq r$. Hence $\gamma \in [\delta - r, \delta + r] \subseteq (HO_g(U))_r$. This completes the verification of inclusion (5.3).

Next we show that

$$\bigcap_{U \in \mathcal{U}(x_0)} (HO_g(U))_r \subseteq \left(\bigcap_{U \in \mathcal{U}(x_0)} HO_g(U) \right)_r . \tag{5.5}$$

Let $\gamma \in \bigcap_{U \in \mathcal{U}(x_0)} (HO_g(U))_r$ be fixed. Then $\gamma \in (HO_g(U))_r = \bigcup_{\alpha \in HO_g(U)} [\alpha - r, \alpha + r]$ for all $U \in \mathcal{U}(x_0)$. Hence

$$HO_g(U) \cap [\gamma - r, \gamma + r] \neq \emptyset$$

is a non-empty compact set for every neighbourhood U of x_0. The intersection of finitely many sets $HO_g(U_i) \cap [\gamma - r, \gamma + r]$, $1 \leq i \leq m$, is non-empty as well, since

$$\bigcap_{i=1}^{m} \Big(HO_g(U_i) \cap [\gamma - r, \gamma + r] \Big) = \left(\bigcap_{i=1}^{m} HO_g(U_i) \right) \cap [\gamma - r, \gamma + r] \supseteq HO_g\left(\bigcap_{i=1}^{m} U_i \right) \cap [\gamma - r, \gamma + r] \neq \emptyset.$$

Consequently, the complete intersection $\bigcap_{U \in \mathcal{U}(x_0)} \Big(HO_g(U) \cap [\gamma - r, \gamma + r] \Big)$ is non-empty, too. Thus

$$\left(\bigcap_{U \in \mathcal{U}(x_0)} HO_g(U) \right) \cap [\gamma - r, \gamma + r] = \bigcap_{U \in \mathcal{U}(x_0)} \Big(HO_g(U) \cap [\gamma - r, \gamma + r] \Big) \neq \emptyset .$$

Accordingly, there exists a number $\delta \in \bigcap_{U \in \mathcal{U}(x_0)} HO_g(U)$ with $|\delta - \gamma| \leq r$. This shows that $\gamma \in \left(\bigcap_{U \in \mathcal{U}(x_0)} HO_g(U) \right)_r$ and thus proves (5.5).

The remainder of the proof of Theorem 5.10 rests on the representation

$$HO_f(x_0) = \bigcap_{U \in \mathcal{U}(x_0)} HO_f(U)$$

of the open hull, which is an immediate consequence of the definition of $HO_f(x_0)$ (cf. [R/S3–H], Section 2). Thus, by (5.3) and (5.5),

$$\begin{aligned} F_f(x_0) &= HO_f(x_0) = \bigcap_{U \in \mathcal{U}(x_0)} HO_f(U) \subseteq \bigcap_{U \in \mathcal{U}(x_0)} (HO_g(U))_r \subseteq \\ &\subseteq \left(\bigcap_{U \in \mathcal{U}(x_0)} HO_g(U) \right)_r = (HO_g(x_0))_r = (F_g(x_0))_r . \end{aligned}$$

This yields the claim (5.2) and completes the proof of Theorem 5.10. □

The Hausdorff distance gives rise to a concept of uniform convergence of multifunctions. We say that *a sequence $(F_n)_{n=1}^{\infty}$ of multivalued functions mapping a topological space X into the subsets of* $\mathbf{R}$ *uniformly tends to the multifunction F* if

$$\lim_{n \to \infty} \sup_{x \in X} d_H(F_n(x), F(x)) = 0 .$$

Note that the limit F of the convergent sequence $(F_n)_{n=1}^{\infty}$ is uniquely determined if one considers maps with values in the closed subsets of $\mathbf{R}$ only. This applies in particular when F_n and F are associated multifunctions of real-valued functions on X. Theorem 5.10 yields the following conclusions.

Corollary 5.2 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued cliquish functions on a topological space X. If the functions f_n uniformly tend to $f : X \to \mathbf{R}$ then the associated multifunctions F_{f_n} uniformly tend to F_f.* $\square$

Clearly, the converse of Corollary 5.2 is false in general, since different quasi-continuous or cliquish functions, such as $\mathbf{I}_{[0,1]}$ and $\mathbf{I}_{(0,1]}$ on the space $[-1, 1]$, may have the same associated multifunction. However, the following claim gives a counterpart to Corollary 5.2 if the functions f and f_n, $n \geq 1$, are quasi-continuous. It says that uniform convergence of the associated multifunctions F_{f_n} to F_f implies uniform convergence of suitable quasi-continuous admissible modifications $\tilde{f}_n$ of f_n, $n \geq 1$, to f.

Theorem 5.11 *Let f and f_n, $n \geq 1$, be real-valued quasi-continuous functions on a topological space X. Then the associated multifunctions F_{f_n} uniformly tend to F_f if and only if there exist selections $\tilde{f}_n$ of F_{f_n}, $n \geq 1$, which uniformly tend to f.*

Proof. First we assume that F_f is the uniform limit of the multifunctions F_{f_n}. By Proposition 5.1, we have $f(x) \in F_f(x)$. Hence, by the definition of $d_H(F_{f_n}(x), F_f(x))$, there exist reals $\lambda_{x,n} \in F_{f_n}(x)$, $x \in X$, $n \geq 1$, such that $|\lambda_{x,n} - f(x)| \leq d_H(F_{f_n}(x), F_f(x)) + \frac{1}{n}$. Then the functions $\tilde{f}_n(x) = \lambda_{x,n}$, $n \geq 1$, are selections of the multifunctions F_{f_n}. Moreover, the functions $\tilde{f}_n$ uniformly approach f, since

$$\lim_{n\to\infty} \sup_{x\in X} \left|\tilde{f}_n(x) - f(x)\right| = \lim_{n\to\infty} \sup_{x\in X} |\lambda_{x,n} - f(x)| \leq \lim_{n\to\infty} \sup_{x\in X} \left(d_H(F_{f_n}(x), F_f(x)) + \frac{1}{n}\right) = 0.$$

Now let us assume that there exist selections $\tilde{f}_n$ of F_{f_n} which uniformly tend to f. By Theorem 5.1, the functions $\tilde{f}_n$ are quasi-continuous with $F_{\tilde{f}_n} = F_{f_n}$. Theorem 5.10 yields

$$\lim_{n\to\infty} \sup_{x\in X} d_H(F_{f_n}(x), F_f(x)) = \lim_{n\to\infty} \sup_{x\in X} d_H\Big(F_{\tilde{f}_n}(x), F_f(x)\Big) \leq \lim_{n\to\infty} \sup_{x\in X} \left|\tilde{f}_n(x) - f(x)\right| = 0.$$

Hence the multifunctions F_{f_n} uniformly tend to F_f. $\square$

Let us finish this chapter with the remark that many of the results concerning real-valued quasi-continuous and cliquish functions can be generalized to functions $f : X \to Y$ with values in more general topological spaces Y. A wide field of research could be devoted to the question what topological properties of the space Y guarantee that the claims of the previous theorems and propositions can be maintained.

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Selbständigkeitserklärung

Ich erkläre, daß ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

Jena, den 25.08.2000 Christian Richter

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