Macroscopic Quantum Electrodynamics of High-Q Cavities

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Contents

Title Page i

Table of Contents iii

1 Introduction **1**

2 Leaky High-Q Cavities: QNT Approach **9**

2.1 System-Reservoir Approach 9

2.2 Unwanted Losses 12

2.3 Damped Atom-Field Dynamics 14

3 Leaky High-Q Cavities: QED Foundation and Extension of QNT **18**

3.1 Macroscopic QED in Linear Media 19

3.1.1 Medium-Assisted Electromagnetic Field 19

3.1.2 Field Quantization 20

3.1.3 Atom-Field Interaction 22

3.1.4 Operator Input-Output Relations 25

3.1.5 Dynamics of Atom-Field System 33

3.2 Leaky Cavities with Unwanted Losses 35

3.2.1 Nonmonochromatic Modes of the Cavity Field 35

3.2.2 Input-Output Relations 43

3.3 Generalization of QNT Approach: Replacement Schemes 47

3.4 Quantum State of the Outgoing Field 52

3.4.1 Nonmonochromatic Modes of the Outgoing Field 52

3.4.2 Phase-Space Functions 56

3.4.3 Examples of Quantum State Extraction 58

4 Leaky High-Q Cavities: Exact QED beyond QNT 66

4.1 Two-Level Atom in a Cavity 67

4.2 Quantum State of the Outgoing Field 71

4.2.1 Wigner Function 71

4.2.2 Continuing Atom-Field Interaction 74

4.2.3 Short-Term Atom-Field Interaction 81

4.3 Comparison with Quantum Noise Theories 85

4.3.1 Continuing Atom-Field Interaction 85

4.3.2 Short-Term Atom-Field Interaction 87

5 Summary and Outlook 92

Bibliography 97

List of Publications 101

List of Presentations 102

A Field Commutation Relations 103

A.1 Multilayer Field Operators 103

A.2 Cavity Mode Operators 107

A.3 Outgoing Field Operators in Time Domain 107

B One-Dimensional Multilayer Planar Structure 111

C Derivation of the Input-Output Relation in Frequency Domain 115

Zusammenfassung ix

Acknowledgments xiv

Chapter 1

Introduction

The coherent interaction of the electromagnetic field with a single atom, placed in a resonator, is a basic system which has attracted the attention of many physicists over the past decades, due to its conceptual importance to the fundamental aspects of quantum physics. The roots of this interaction go back to the discovery of the spontaneous emission of an atom in free space, which, being genuinely a quantum effect, was phenomenologically described by Einstein in terms of statistical rates [1]. A quantum-mechanical description of the spontaneous emission, for a two-level atom, was later given by Weisskopf and Wigner [2]. Within the framework of quantum electrodynamics (QED), the description of non-commuting field quantities implies non-vanishing moments giving rise to fluctuations of field quantities. The quantum fluctuations of the electromagnetic field induce an interaction with the atom, which among others leads to spontaneous emission. As it was then first proposed by Purcell [3] in the context of nuclear spins, the presence of material bodies changes the structure of the electromagnetic field and therefore may, in particular, influence the emission rate of an atom. In addition, there is a wide variety of phenomena demonstrating the changes in QED effects, for example, dispersion forces (for a review, see Refs. [4, 5]).

On the basis of this intuition, it was first suggested by Bloembergen and Pound [6] that the use of a resonator cavity can enhance the rate of spontaneous emission, and therefore, an atom placed in a resonator cavity may induce generation of radiation. Indeed, a resonator system features sharply peaked electromagnetic field resonances. The presence of the cavity introduces a characteristic time scale related to the coherent exchange of excitation between the atom and the field. In the case, when the atomic transition frequency is in the vicinity of a resonance line of a resonator cavity, the characteristic time of the atom-field interaction can become much shorter than the inverse width of the resonance line and the inverse spontaneous rate of the atom in free space. In other words, the strength of the atom-field interaction can extremely increase, and thus, this regime is commonly referred to as strong atom-field coupling of cavity QED (for a review see [7, 8]).

Since these seminal works, the study of cavity QED has been the subject of intense research, which has led to a number of different implementations of strong atom-field coupling (see the review [9]). In the microwave domain, the strong coupling regime has been achieved by injecting a beam of highly excited Rydberg atoms into superconducting resonators with very long decay times [10]. The strong coupling regime is well-established due to the large magnitude of the relevant electric dipole transition elements. The atomic beam serves not only to manipulate the field inside the resonator cavity, but also to measure it, while the atoms can be detected beyond the interaction region. The major obstacle in this realization is the lack of control over the interaction time of the atoms with the field.

In the context of the optical domain, the strong coupling regime has been achieved by establishing the interaction of neutral atoms with optical cavities. In contrast to microwave resonators, the smaller inverse width of the cavity resonance line, specific for optical resonator cavities, implies the necessity to establish tiny volumes of the resonators to enable the strong coupling. This requirement, obviously, leads here to technical difficulties, related to injecting of the atoms into the resonator cavities and controlling over their exact location [11, 12].[1] On the other hand, the large width of a resonance line leads to various applications related to the input-output coupling. Here, the cavity losses represent not a detrimental decoherence process, but rather the wanted outgoing radiation of the cavity. In particular, it enables further related studies as, for example, quantum feedback and control over quantum dynamics [14, 15].

The cavity QED in the strong coupling regime offers an ideal platform for the realization of protocols and systems related to quantum information science [16]. In this context, the implementation of cooling and trapping techniques in cavity QED has been a milestone in the ideas of quantum communication [17]. The ability to localize an atom and individually address long-lived internal states of the atom by

[1] We omit here the description of a number of other realizations of strong coupling regime in cavity QED, for example, the usage of the linear ion traps in the cavities of bigger size. For further reading, see Ref. [13].

laser fields, makes them one of the leading candidates for accessible preparation and robust storage in quantum information science [18]. The coherent coupling of an atom to the electromagnetic field in a resonator cavity promises a realization of quantum communications, where the atom plays the role a node in a quantum network, and the emitted radiation may be regarded as flying qubits [19] for long-distant quantum communication [20].

Specifically, the basic ingredient in these schemes is the quantum control of the emitted radiation for generation and extraction of single-photon Fock states [17]. The essential requirement of quantum communication protocols is the emission of indistinguishable light pulses of the known quantum state and the known spatio-temporal profile on demand. Single-photon sources on demand have been realized in optical high-Q cavities [21, 22]. Moreover, employing the idea of stimulated Raman adiabatic passage [23], the generation of single-photons with known circular polarization has been realized [24].

The constitutional work of Jaynes and Cummings [25], which provides the first theoretical description of the strong-coupling regime, until today has been widely applied to describe the strong atom-field interaction. In the simplest case of the resonant interaction of a two-level atom with the field, on a time scale sufficiently short in comparison with characteristic dissipation time, the atom-field coupling is modeled by the Jaynes-Cummings interaction term between the atom and a normal mode of a perfect cavity. The main result of this model is the description in terms of the "dressed" eigenstates of the coupled atom-field system, which consists of a ladder of doublet states where the splitting represents the Rabi frequency—the rate of energy exchange between the atom and the field. Clearly, for longer times, the atom-cavity system can no longer be regarded as being a closed system, and the eigenstates of the system will gradually differ from those, given by the Jaynes-Cummings Hamiltonian [26]. For this reason, and in the context of the applications mentioned above, the losses of any realistic cavity must be properly taken into account by means of the study of the input-output problem for the cavity.

It is a widely accepted approach to tackle the input-output problem of a resonator cavity by treating the leaky cavity as an open quantum system [27]. In other words, the fields inside and outside the cavity are regarded as the ones representing independent degrees of freedom. Accordingly, two separate Hilbert spaces for the field are introduced, based on discrete and continuous mode expansions of the fields inside and outside the cavity, respectively. In order to *a posteriori* take into account the input-output coupling, owing to the fractionally transparent mirrors of the resonator cavity, the formalism of the quantum noise theory (QNT) [28] is applied. Namely, the radiation field modes of the idealized lossless cavity are regarded as a "small" system that is linearly coupled to the continuum of the external field modes, which plays the role of a "large" reservoir (dissipative system). The influence of the reservoir on the internal modes is treated in the Markov approximation, i.e., the field variables inside the cavity have a short "memory", in the sense that fluctuations of the system variables are smoothed out over the time scale of the cavity mode decay time. This approximation leads to quantum Langevin equations for the cavity mode variables, where the incoming external field gives rise to operator Langevin forces therein. Equivalently, the effect of dissipation can be described by using the concept of master equations [29, 30, 31], or quantum trajectory methods (see, e.g. Refs. [26, 32, 33]). In order to find the field escaping from the cavity, the quantum Langevin equations are then complemented with input-output relations, which relate the output field to the input field and the intracavity field [34, 35].

We emphasize, that the standard approach to describe the atom-field interaction by means of the Jaynes-Cummings Hamiltonian complemented with QNT approach of a leaky cavity is phenomenological in nature, relies on certain assumptions and does not give a physical description of the involved parameters, and, therefore, the question of whether it has an intrinsic physical origin has to be answered. The true, conceptually fundamental and physically sound theory describing the atom-field interaction in a resonator cavity should be based on the quantum theory of the (macroscopic) electromagnetic field on the basis of the corresponding macroscopic Maxwell equations [36]. The first quantum-theoretical description of a leaky cavity was given by Lang et al. [37], based on the field expansion in terms of the modes of the "universe", i.e., the (orthogonal) normal modes of the closed system formed by the resonator cavity and the region outside the cavity. In an alternative approach [38], the normal modes of a closed resonator cavity are complemented by the penetrating outgoing waves outside the cavity, leading to the expansion in terms of Fox-Li modes [39], where the adjoint modes necessarily should be introduced to achieve completeness. In an analogous way, starting with the normal modes for the region outside the cavity, the external Fox-Li modes and their adjoints are introduced. It is shown that the interaction energy between the cavity modes and the external modes vanishes, and the input-output coupling arises from the non-vanishing commutation relation between the modes inside and outside the cavity. Note, however, that although the expansion in terms of the Fox-Li modes reproduces the fields at the input-output coupling mirror, it does not allow the consideration of only the input or output fields outside the cavity. In a more rigorous approach [40, 41, 42] the electromagnetic field in the presence of non-absorbing linear media is expanded in terms of the ordinary continuous modes, which extend over the whole space. In a coarse-grained approximation, a description of the fields inside and outside a high-Q cavity in terms of quantum Langevin equations and input-output relations, respectively, as used in quantum noise theories, is given. In another version of QED [43, 44], the modes of the universe are separated into two contributions, accounting for the fields inside and outside the cavity by using Feshbach's projection formalism [45]. In this way, the separation of the field Hilbert space is performed, and consequently, by choosing appropriate boundary conditions, the inside and outside modes are defined, which form complete bases in the respective regions, making it possible to establish a consistent relation with the description of the cavity system within the framework of quantum noise theory. In fact, these modes violate the surface boundary conditions of the electromagnetic field, and the (non-pointwise) convergence of the expansion is slow in the vicinity of the coupling mirror even in the limit of the idealized lossless cavity, and, thus, the individual modes defined for the fields inside and outside the cavity have limited physical meaning and applicability.

Despite the tremendous progress in the practical realization of cavity QED, which has led to an explosion in the rapidly growing field of quantum communication, the desire to reduce a complex system to simpler models may create the false impression that the theoretical description of the system is well understood. While such a reduction is often possible to carry out on the classical level, on the quantum level it may fail. We would like to emphasize, that despite its success and appealing simplicity, the traditionally accepted description of the atom-field interaction in a resonator by means of the Jaynes-Cummings Hamiltonian, together with the quantum noise theory, may fail to describe the quantum features crucial for ideas of quantum information science. This refers, specifically, to the separation of the field Hilbert space in the realm of this conventional approach, and, in particular, the division of the time scales that lies beyond—the process of preparation of the quantum state of the field inside the resonator, governed by the Jaynes-Cummings Hamiltonian, is regarded to be separated in time from the further evolution of the field variables described by the input-output relations.

The current work aims to develop an exact theory of the atom-field interaction in resonator cavities based on macroscopic QED and to study the limits and the scope of application of theories based upon the ideas of QNT. Even though the various formalisms developed within macroscopic QED plausibly cover numerous aspects related to the study of leaky cavities, they all suffer from the same limitation. Namely, the absorption losses, necessarily observed in any linear media, are typically disregarded. However, even if from the point of view of classical optics these absorption losses are very small, so that they effectively do not influence classical light, they can lead to a drastic degradation of nonclassical light features (see, e.g., Ref. [46]). In this thesis, we shall give a consistent description of the cavity input-output problem, including the effect of unwanted losses, such as absorption losses, thereby extending standard QNT. Next, we shall study atom-field interaction in a resonator cavity and analyze features of emitted radiation renouncing the approximation of separate Hilbert spaces for the fields inside and outside a cavity.

This thesis is organized as follows. The standard approach to model the medium absorption, presented in Chapter 2, is to introduce a bilinear interaction Hamiltonian, in accordance with the phenomenological quantum noise theory. We shall give a detailed description of the Langevin equation and input-output relation for this case, which reveals, that whereas the dynamics of the cavity mode variables is subject to additional damping term and corresponding fluctuation force, associated with the absorption, the input-output relations effectively remain unchanged. Therefore, the absorption losses necessarily occurring in the processes of the coupling of the fields inside the cavity with the incoming and outgoing fields through the coupling mirrors are disregarded in this approach. This is obviously an important issue, and must be thoroughly investigated in a definitive manner, especially in view of the applications in quantum information science, where in the considered schemes, related to the extraction of the cavity quantum state, the *ad hoc* assumption of nearly perfect extraction is often made [47]. Thus, a more exact theory needs to be established to describe the absorption losses, associated with the coupling mirror.

Chapter 3 is devoted to developing a quantum theory for leaky cavities in dispersive and absorbing media, in the framework of macroscopic QED [MK3]. We start by recalling the quantization scheme [48] of the electromagnetic field in the presence of locally responding, linear dielectric media characterized by a coordinate- and frequency-dependent complex permittivity. The spatial and spectral structure of the electromagnetic field is determined in terms of the Green tensor of the associated Maxwell equations. Applying the quantization scheme to a resonator cavity, we can specify the Green tensor, which determines the characteristic resonant frequencies and the corresponding widths of the cavity. In this way, we shall derive the electric field operators inside and outside the cavity. In particular, as we shall show, the electric field inside the cavity can be expanded in terms of nonmonochromatic modes corresponding to the resonant frequencies of the cavity. The form of the dynamical equations of the cavity mode operators suggests that the Hamiltonian used in the quantum noise theories standard QNT to describe the time evolution of the cavity mode, should be complemented with bilinear interaction energies between the cavity modes and appropriately chosen dissipation channels, in order to model the unwanted absorption losses. The calculations for the electric field outside the cavity allow us to generalize the standard input-output relation to include into the consideration the absorption losses associated with the coupling mirror. This generalization to consistently include absorption losses in the input-output relation cannot be obtained on the basis of interaction energies, commonly introduced in standard QNT to model the absorption losses. However, we shall prove, that to model the absorption losses in the coupling mirror, a more sophisticated scheme with a system of beam splitters can be introduced [MK5, MK6].

In order to study the quantum state of the outgoing field in detail, on the basis of the obtained general operator input-output relation we shall derive the input-output relations for phase-space functions in the case, when the quantum state of the field inside the cavity is known at some initial time [MK2, MK3, MK4]. In the case, when the time of the preparation of the cavity field is sufficiently short on the time scale of the inverse width of the cavity resonance, we introduce an assumption of factorization of the characteristic functions of the cavity field and the input fields, which in fact corresponds to the separation of the field Hilbert space. In this way, we shall calculate the quantum state of the relevant outgoing mode. To illustrate the results, the general condition on the extraction efficiency of realization of a nearly perfect extraction of the quantum state of the cavity mode, will be discussed considering some typical examples.

With the aim to generalize the approach based on macroscopic QED, we devote Chapter 4 to the dynamical description of the atom-field interaction [MK9]. Specifically, to develop an exact theory we renounce the postulate—the major assumption of QNT—to introduce separate Hilbert spaces for the fields inside and outside the cavity, but rather we treat the electromagnetic field in the whole space as an entity. Moreover, to go beyond the regime of short-time preparation of the electromagnetic field, we include in the theory the process of field generation by means of active sources. For the sake of clarity, we consider a simple model system of quantum mechanics, allowing an outright study of the matter-light interaction on a fundamental level—namely, the strong coupling regime of interaction of a two-level atom placed in a high-Q cavity with the cavity-assisted electromagnetic field. We assume that the atom is initially excited to its upper energy eigenstate and study the emitted outgoing radiation in the electric dipole approximation. We shall show, that the excited outgoing mode can be determined, and the Wigner function of the quantum state of this mode can be given. The calculations of the spatio-temporal shape imply, that the wave packet, corresponding to the excited outgoing mode, extends in the regions both inside and outside the cavity. We shall then compare the results with the ones, obtained within QNT in order to study the limits of applicability of QNT [MK10].

The main results of this thesis are summarized in Chapter 5.

Omnia disce, videbis postea nihil esse superfluum.
(Learn everything, you will find nothing superfluous.)
Hugh of St Victor

Chapter 2

Leaky High-Q Cavities: QNT Approach

The main idea of the Quantum (and Classical) Noise Theory, roots of which go back to the study of the Carnot cycle, is to consider the interaction of a rather small system with a large reservoir heath bath. The general assumption is that the reservoir has many degrees of freedom and is large enough, so that in the case when the interaction is sufficiently weak, the reservoir is effectively not influenced by the small system. The influence of the bath on the system can be described by including random noise forces into the equations of motion for the system variables. Within the frame of the Markovian damping theory, when the correlations between the system and the reservoir decay very rapidly on the time scale of the dynamics of the system, the equations of motion will no longer contain time integrals with system variables. In other words, the probability distribution of system variables for future times is determined by its value at the present time.

In this chapter we illustrate that the concept of the Markov process provides a simple description of a leaky optical cavity. We present an intuitive extension of the formalism, which allows one to take into account the unwanted losses of the cavity.

2.1 System-Reservoir Approach

The formalism of quantum noise theory can be applied to describe the radiation field generated in a resonator-like cavity with input and output coupling. In this approach, a single excited mode of the radiation field inside a leaky cavity is treated as a dynamical system, that (weakly) interacts with the continuum of external modes of the radiation field. Consequently, a bilinear coupling Hamiltonian is introduced to describe the interaction between the modes inside and outside the cavity [34].

Let us consider a one-dimensional high-Q cavity bounded with a perfectly reflecting mirror at $x = 0$ and an almost perfectly reflecting mirror at $x = l$. Being interested in resolving times that are large compared with the time of propagation of light through the cavity, we expand the intracavity field in terms of standing waves at frequencies ω_k, where the index k enumerates the modes of the cavity.

Let us focus on a single kth cavity-mode, that plays the role of the system with the free Hamiltonian $\hat{H}_{\mathrm{sys}}$ interacting with the reservoir of the external modes described by the free Hamiltonian $\hat{H}_{\mathrm{res}}$. Assuming the interaction energy between the inside and outside fields to be of a bilinear type, the total Hamiltonian can be given as [35]

$$\hat{H} = \hat{H}_{\mathrm{sys}} + \hat{H}_{\mathrm{res}} + \hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{res}}, \tag{2.1}$$

with

$$\hat{H}_{\mathrm{sys}} = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k, \tag{2.2}$$

$$\hat{H}_{\mathrm{res}} = \sum_k \int_{\Delta_k} \mathrm{d}\omega \hbar\omega \, \hat{b}^\dagger(\omega)\hat{b}(\omega), \tag{2.3}$$

$$\hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{res}} = i\hbar \sum_k \int_{\Delta_k} \mathrm{d}\omega \, \kappa_k(\omega) \left[\hat{b}^\dagger(\omega)\hat{a}_k - \hat{a}_k^\dagger \hat{b}(\omega)\right], \tag{2.4}$$

where $\kappa_k(\omega)$ is the system-reservoir coupling constant, $\hat{a}_k(t)$ is the annihilation operator of the cavity-mode and $\hat{b}(\omega)$ are the annihilation operators of the outside field that satisfy the familiar bosonic (equal time) commutation relations:

$$\left[\hat{a}_k(t), \hat{a}_k^\dagger(t)\right] = 1, \tag{2.5}$$

$$\left[\hat{b}(\omega), \hat{b}^\dagger(\omega')\right] = \delta(\omega - \omega'). \tag{2.6}$$

Note, the assumption that the fields inside and outside the cavity represent independent degrees of freedom yields the following commutation relation:

$$\left[\hat{a}_k(t), \hat{b}^\dagger(\omega)\right] = 0. \tag{2.7}$$

Here and throughout the text, the notation $\int_{\Delta_k} \mathrm{d}\omega \ldots$ is used to indicate the integration over frequencies in the interval $\Delta_k \equiv [\frac{1}{2}(\omega_{k-1} + \omega_k), \frac{1}{2}(\omega_k + \omega_{k+1})]$.

The Markov approximation might be introduced by assuming that $\kappa_k(\omega)$ is independent of ω over a band of frequencies around the characteristic frequency ω_k:

$$\kappa_k(\omega) \cong \kappa_k(\omega_k) \equiv \sqrt{\frac{\Gamma_k}{2\pi}}\,. \tag{2.8}$$

In this case, the evolution of system quantities at an arbitrary time is now determined by system quantities at the same time, and the cavity-mode annihilation operator satisfies the corresponding Langevin equation [35, 40]:

$$\dot{\hat{a}}_k(t) = -i\Omega_k \hat{a}_k(t) + \sqrt{\Gamma_k}\hat{b}_{k\text{in}}(t), \tag{2.9}$$

where

$$\Omega_k = \omega_k - \tfrac{1}{2} i\Gamma_k. \tag{2.10}$$

For a high-Q cavity, the widths Γ_k of the cavity modes at frequencies $\omega_k = k\pi c/l$ are very small compared with their separation $\Delta\omega_k = \frac{1}{2}(\omega_{k+1} - \omega_{k-1}) = \pi c/l$, where c is the velocity of light. The second term in Eq. (2.9) is the Langevin noise force arising from the input radiation field,

$$\hat{b}_{k\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{b}_{k\text{in}}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{b}(\omega, t_0)\, e^{-i\omega(t-t_0)}, \tag{2.11}$$

$[\hat{b}(\omega) \equiv \hat{b}(\omega, t_0)]$. Similarly, the output operator can be introduced as follows:

$$\hat{b}_{k\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{b}_{k\text{in}}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{b}(\omega, t_1)\, e^{-i\omega(t-t_1)} \qquad (t < t_1). \tag{2.12}$$

Notice that on the time scale under consideration the lower and upper integration limits of the frequency integrals can be, without significant error, extended to $-\infty$ and $+\infty$, respectively. Then, using Eq. (2.6) it follows that the commutation relations

$$\left[\hat{b}_{k\text{in}}(t), \hat{b}^\dagger_{k\text{in}}(t')\right] = \delta(t - t') \tag{2.13}$$

and

$$\left[\hat{b}_{k\text{out}}(t), \hat{b}^\dagger_{k\text{out}}(t')\right] = \delta(t - t') \tag{2.14}$$

are valid.

The output operator $\hat{b}_{k\text{out}}(t)$ can be related to the cavity operator $\hat{a}(t)$ and the input operator $\hat{b}_{k\text{in}}(t)$ according to the input-output relation

$$\hat{b}_{k\text{out}}(t) = \sqrt{\Gamma_k}\hat{a}_k(t) + \hat{b}_{k\text{in}}(t). \tag{2.15}$$

Here it is important to note that the term depending on $\hat{b}_{k\text{in}}(t)$ in the Langevin equation Eq. (2.9) can be interpreted as a noise force, provided that the state of the system is initially factorized with respect to the states of the cavity mode and the input field, and that the state of $\hat{b}_{k\text{in}}(t)$ is incoherent. However, independent of the state of the reservoir, the term with $\hat{b}_{k\text{in}}(t)$ in Eq. (2.9) gives rise to a damping for

$\hat{a}_k(t)$, and the operators $\hat{b}_{k\mathrm{in}}(t)$ and $\hat{b}_{k\mathrm{out}}(t)$ can be interpreted as input and output to the cavity mode. Importantly, we assume here, that the (time-dependent) "causality" commutation relations hold true:

$$\left[\hat{a}_k(t), \hat{b}^{\dagger}_{k\mathrm{in}}(t')\right] = 0, \qquad (t' > t). \tag{2.16}$$

The interpretation is straightforward: since the solution of Eq. (2.9) for the cavity-mode operator $\hat{a}_k(t)$ is given in terms of past values of $\hat{b}_{k\mathrm{in}}(t)$, then it is clear that the cavity mode operator $\hat{a}_k(t)$ does not depend on the values of the input in the future $t' > t$. Similarly,

$$\left[\hat{a}_k(t), \hat{b}^{\dagger}_{k\mathrm{out}}(t')\right] = 0, \qquad (t' < t). \tag{2.17}$$

2.2 Unwanted Losses

In the previous section the derivation of the Langevin equation has been given under the assumption that the only damping source for the cavity field is the (wanted) radiative damping due to the input-output coupling. However, in a real physical experiment, there are unwanted losses such as scattering and medium absorption, which lead to additional noise. Let us note that in a realistic situation the unwanted losses, unavoidable for every material system, for high-Q cavities appeared to be of the same order of magnitude as the transmission losses [49, 50].

To include the unwanted losses within the phenomenological quantum noise theory of leaky cavities, an intuitive approach can be applied introducing the coupling of the cavity modes to an additional reservoir [43]. Then, the total Hamiltonian can now be represented by complementing the Hamiltonian (2.1) with an additional interaction energy:[1]

$$\hat{H} = \hat{H}_{\mathrm{sys}} + \hat{H}_{\mathrm{res}} + \hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{res}} + \hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{abs}}, \tag{2.18}$$

where

$$\hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{abs}} = i\hbar \sum_k \int_{\Delta_k} \mathrm{d}\omega\, \kappa_k(\omega) \left[\hat{c}^{\dagger}(\omega)\hat{a}_k - \hat{a}^{\dagger}_k \hat{c}(\omega)\right]. \tag{2.19}$$

A discussion similar to the Sec. 2.1 leads within the Markov approximation to the quantum Langevin equation

$$\dot{\hat{a}}_k(t) = -i\Omega_k \hat{a}_k(t) + \left(\frac{c}{2l}\right)^{\frac{1}{2}} \left[T_k \hat{b}_{k\mathrm{in}}(t) + A_k \hat{c}_k(t)\right], \tag{2.20}$$

[1]The calculations performed in this section are similar to the ones performed for a two-sided cavity [51].

where Ω_k is given by Eq. (2.10). The damping rate of the cavity-mode Γ_k can now be presented as

$$\Gamma_k = \gamma_{k\mathrm{rad}} + \gamma_{k\mathrm{abs}}, \tag{2.21}$$

where

$$\gamma_{k\mathrm{rad}} = \frac{c}{2l}|T_k|^2 \tag{2.22}$$

is the decay rate of the cavity-mode which results from the transmission losses due to the radiative input-output coupling, and

$$\gamma_{k\mathrm{abs}} = \frac{c}{2l}|A_k|^2 \tag{2.23}$$

is the decay rate which results from the unwanted losses, such as the unavoidably existing material absorption and scattering. For a high-Q cavity, both the transmission coefficient T_k and the absorption coefficient A_k are very small compared with unity ($|T_k| \ll 1$, $|A_k| \ll 1$). Note, that T_k and A_k are taken at the cavity mode resonance frequency ω_k. In Eq. (2.20) the term proportional to $T_k\hat{b}_{k\mathrm{in}}(t)$ is the Langevin noise force arising from the input radiation field Eq. (2.11), and the term proportional to $A_k\hat{c}_k(t)$ is the Langevin noise force associated with absorption,

$$\hat{c}_k(t) = \frac{1}{\sqrt{2\pi}}\int_{\Delta_k} \mathrm{d}\omega\, \hat{c}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{\Delta_k} \mathrm{d}\omega\, \hat{c}(\omega,t_0)\, e^{-i\omega(t-t_0)}. \tag{2.24}$$

The operators $\hat{c}(\omega,t)$ satisfy the bosonic equal-time commutation relation

$$\left[\hat{c}(\omega,t), \hat{c}^\dagger(\omega',t)\right] = \delta(\omega-\omega'). \tag{2.25}$$

From Eqs. (2.24) and (2.25), on the time scale under consideration, we derive

$$\left[\hat{c}_k(t), \hat{c}_k^\dagger(t')\right] = \delta(t-t'). \tag{2.26}$$

It is not difficult to see that the solution of Eq. (2.20) can be given in the form of

$$\hat{a}_k(t) = \hat{a}_k(t_0)e^{-i\Omega_k(t-t_0)} + \left(\frac{c}{2l}\right)^{\frac{1}{2}} \int_{t_0}^{t} \mathrm{d}t'\, e^{-i\Omega_k(t-t')} \left[T_k\hat{b}_{k\mathrm{in}}(t') + A_k\hat{c}_k(t')\right]. \tag{2.27}$$

The output operator $\hat{b}_{k\mathrm{out}}(t)$ can be related to the cavity operator $\hat{a}(t)$ and the input operator $\hat{b}_{k\mathrm{in}}(t)$ according to the input-output relation

$$\hat{b}_{k\mathrm{out}}(t) = \left(\frac{c}{2l}\right)^{1/2} T_k\hat{a}_k(t) + R_k\hat{b}_{k\mathrm{in}}(t), \tag{2.28}$$

where

$$R_k = -\frac{T_k}{T_k^*}. \tag{2.29}$$

The derived input-output relation (2.28) can be used to calculate the correlation functions of the outgoing field in terms of those of the cavity and input fields [35].

To conclude, the suggested phenomenological interaction energy Eq. (2.19) introduces an additional damping and Langevin noise term to take into account the effect of the unwanted losses on the cavity field. As the input-output relation (2.28) together with Eq. (2.29) suggests, the intuitive concept disregards the effect of unwanted losses in the coupling mirror, that may influence the outgoing field. A simple example of this kind of effect is the scattering and reflection of the incoming field by the coupling mirror [note, that from Eq. (2.29) $|R_k|^2 = 1$]. Thus, the suggested concept has a limited validity, and can be applied as long as the input port is unused.

2.3 Damped Atom-Field Dynamics

Let us consider the interaction of the two-level atom with the field of a leaky cavity within the framework of the quantum noise theory [MK8]. The temporal evolution of the system can be studied, among others (see, for example, Refs. [52, 53, 54]) by means of the quantum trajectory approach [32, 33, 55].

The dynamical evolution of the reduced density operator $\hat{\rho}(t)$ of the atom and the cavity field is described by the following master equation

$$\frac{d\hat{\rho}(t)}{dt} = \frac{1}{i\hbar}\big[\hat{H}_{\rm int}, \hat{\rho}(t)\big] + \sum_{\sigma} \frac{\gamma_{k\sigma}}{2}\left[2\hat{a}_k\hat{\rho}(t)\hat{a}_k^\dagger - \hat{a}_k^\dagger\hat{a}_k\hat{\rho}(t) - \hat{\rho}(t)\hat{a}_k^\dagger\hat{a}_k\right] \tag{2.30}$$

($\sigma = \text{rad}, \text{abs}$). The Hamiltonian that describes the atom-cavity interaction is given, in the rotating-wave approximation, by

$$\hat{H}_{\rm int} = \hbar g(\hat{a}_k\hat{S}_{12}^\dagger + \hat{a}_k^\dagger\hat{S}_{12}) + \hbar(\omega_{21} - \omega_k)\hat{S}_{22}, \tag{2.31}$$

where g is the atom-field coupling constant, and $\hat{S}_{n'n} = |n'\rangle\langle n|$ $(n, n' = 1, 2)$. We consider the case, when the system is initially (at time $t = 0$) prepared in the state $|2, 0\rangle$, i.e., the atom is in the upper state, and the cavity field is in the vacuum state. Then, we may assume, that the Hilbert space that describes the open quantum system under scrutiny is, in this model, spanned by the three vectors: $|2, 0\rangle$, $|1, 1\rangle$, i.e., the atom in the lower level, one photon in the cavity, and the state $|1, 0\rangle$, the atom in the lower state and no photon in the cavity.

In quantum trajectory approach the temporal evolution of the unnormalized state vector $|\bar{\psi}(t)\rangle$, which describes the system at time t, is governed by the Schrödinger equation with a the Hamiltonian

$$\hat{H}' = \hat{H} - i\hbar\frac{\Gamma_k}{2}\hat{a}_k^\dagger\hat{a}_k. \tag{2.32}$$

The evolution generated by this Schrödinger equation is randomly interrupted, from time to time, by the action of collapse, or jump, operators. If no jump has occurred between the initial time $t=0$ and time t, the system evolves via Eq. (2.32) in the state

$$|\bar{\psi}_{\rm no}(t)\rangle = c_2(t)\,|2,0\rangle + c_1(t)\,|1,1\rangle\,. \tag{2.33}$$

In this case the conditioned density operator for the atom-cavity system is given by

$$\hat{\rho}_{\rm no}(t) = \left(\langle\bar{\psi}_{\rm no}(t)|\bar{\psi}_{\rm no}(t)\rangle\right)^{-1}\,|\bar{\psi}_{\rm no}(t)\rangle\langle\bar{\psi}_{\rm no}(t)|. \tag{2.34}$$

Here, the word "conditioned" is used to emphasize the fact that the density operator at time t is constrained to the condition that no jump has occurred between the initial time $t=0$ and time t. To be more specific, the evolution governed by the Schrödinger equation is randomly interrupted by two kinds of jumps, $\hat{J}_{\rm rad}$ and $\hat{J}_{\rm abs}$,

$$\hat{J}_\sigma = \sqrt{\gamma_{k\sigma}}\,\hat{a}_k, \tag{2.35}$$

related to the radiative input-output coupling and photon absorption, respectively. If a jump has occurred at time $t_{\rm J}$, the wave vector is found collapsed into the state $|c\rangle$ due to the action of one of the jump operators,

$$\hat{J}_\sigma|\bar{\psi}_{\rm no}(t_{\rm J})\rangle = \sqrt{\gamma_{k\sigma}}\,\hat{a}_k|\bar{\psi}_{\rm no}(t_{\rm J})\rangle \longrightarrow |1,0\rangle\,. \tag{2.36}$$

It is clear that in the problem under consideration we can have only one jump. Once the system collapses into the state $|c\rangle$ the system remains in that state for all times, and the conditioned density operator at time t is given by

$$\hat{\rho}_{\rm yes}(t) = |1,0\rangle\,\langle 1,0|\,, \tag{2.37}$$

where the index "yes" indicates the fact that a jump has occurred.

According to the quantum trajectory method, the density operator $\hat{\rho}(t)$ is obtained by performing an ensemble average over the different conditioned density operators at time t. In the present case, starting at the initial time $t=0$ with the density operator $\hat{\rho}_0 = |2,0\rangle\,\langle 2,0|$, the ensemble average is performed over the two possible realizations (histories):

$$\hat{\rho}(t) = p_{\rm no}(t)\hat{\rho}_{\rm no}(t) + p_{\rm yes}(t)\hat{\rho}_{\rm yes}(t). \tag{2.38}$$

Here $p_{\rm no}(t)$ and $p_{\rm yes}(t)$ are the probability that between the initial time t_0 and time t no jump and one jump has occurred, respectively $[p_{\rm no}(t)+p_{\rm yes}(t)=1]$. To evaluate $p_{\rm no}(t)$, according to the the probability method of the delay function [33], the probability $p_{\rm no}(t)$ is given by

$$p_{\rm no}(t) = \langle\bar{\psi}_{\rm no}(t)|\bar{\psi}_{\rm no}(t)\rangle = |c_2(t)|^2 + |c_1(t)|^2. \tag{2.39}$$

Using Eqs. (2.38) and (2.39), it is straightforward to obtain

$$\begin{aligned}\hat{\rho}(t) = |c_2(t)|^2 |2,0\rangle\langle 2,0| + |c_1(t)|^2 |1,1\rangle\langle 1,1| + c_2(t)c_1^*(t)|2,0\rangle\langle 1,1| \\ + c_2^*(t)c_1(t)|1,1\rangle\langle 2,0| + |\gamma(t)|^2 |1,0\rangle\langle 1,0|,\end{aligned} \tag{2.40}$$

where we have defined

$$|c_3(t)|^2 \equiv p_{\rm yes}(t) = 1 - \left[|c_2(t)|^2 + |c_1(t)|^2\right]. \tag{2.41}$$

Clearly, $|c_2(t)|^2$, $|c_1(t)|^2$ and $|c_3(t)|^2$ represent the probabilities that at time t the system can be found in the states $|2,0\rangle$, $|1,1\rangle$ and $|1,0\rangle$, respectively. Moreover, from the master equation (2.30), together with Eq. (2.40), one obtains

$$\frac{d}{dt}|c_3(t)|^2 = \mathrm{Tr}\left[\frac{d}{dt}\hat{\rho}(t)|1,0\rangle\langle 1,0|\right] = \Gamma_k|c_1(t)|^2. \tag{2.42}$$

Notice, that the probability for a jump to occur in the time interval $(t, t+dt]$ is given by

$$p_\sigma(t) = \langle\hat{J}_\sigma^\dagger\hat{J}_\sigma\rangle_t\,dt = \gamma_{k\sigma}\mathrm{Tr}\left[\hat{\rho}(t)\hat{a}_k^\dagger\hat{a}_k\right]dt = \gamma_{k\sigma}|c_1(t)|^2 dt. \tag{2.43}$$

The physical interpretation is clear: the probability of a jump is proportional to the probability $|c_1(t)|^2$ to find the system in the state $|1,1\rangle$, i.e., the atom is in the lower state, the photon is emitted, and [recall Eq. (2.42)] the related jump operator projects the system into $|1,0\rangle$, hence producing an increment of $|c_3(t)|^2$.

By integrating equation (2.42) we obtain

$$p_{\rm yes}(t) = |c_3(t)|^2 = p_{\rm rad}(t) + p_{\rm abs}(t), \tag{2.44}$$

with

$$p_\sigma(t) = \gamma_{k\sigma}\int_0^t dt'|c_1(t')|^2. \tag{2.45}$$

The function $p_{\rm rad}(t)$ represents the probability that a photon has left the cavity in the time interval $[0,t]$, and $p_{\rm abs}(t)$ the probability that a photon is absorbed in the same time interval. From Eq. (2.45) it follows that $p_{\rm rad}(t)$ and $p_{\rm abs}(t)$ have to be monotonically increasing functions: the longer one waits, the larger is the probability that a photon has leaked out of the cavity or has been absorbed. Note that the probability amplitudes $c_1(t)$ and $c_2(t)$ [cf. Eq. (4.49)] can be easily found by solving the Schrödinger equation.

The probability density distribution of measuring the photon outside the cavity can be given by

$$\tilde{I}(z,t) = p_{\mathrm{rad}}(t)|\tilde{\phi}(z,t)|^2, \tag{2.46}$$

where $\tilde{\phi}(z,t)$ is the spatio-temporal shape, that characterizes the photon which has left the cavity. Assume that there is a detector placed at point z, and that the $\tilde{\phi}(z,t)$ does not change significantly in the detection space resolution Δz. The response probability of the detector is then proportional to

$$\int_z^{z+\Delta z} \mathrm{d}z\, \tilde{I}(z,t) = p_{\mathrm{rad}}(t) \int_z^{z+\Delta z} \mathrm{d}z\, |\tilde{\phi}(z,t)|^2 \approx p_{\mathrm{rad}}(t)|\tilde{\phi}(z,t)|^2 \Delta z. \tag{2.47}$$

On the other hand, the probability to detect a signal at time t in the region from z to $z + \Delta z$ is equal to the probability to have a jump $\hat{J}_{\mathrm{rad}}$ in the time interval from $t - z/c - \Delta z/c$ to $t - z/c$, which on using Eq. (2.43) reads

$$\gamma_{k\mathrm{rad}}|c_1(t - z/c)|^2 \Delta z/c. \tag{2.48}$$

Comparison of Eqs.(2.48) and (2.47) yields the relation

$$|\tilde{\phi}(z,t)| = \sqrt{\frac{\gamma_{k\mathrm{rad}}}{c p_{\mathrm{rad}}(t)}}\, |c_1(t - z/c)|. \tag{2.49}$$

The probability of registering of the emitted photon during the time interval $[0, t]$ by a detector placed just outside the cavity is proportional to

$$\tilde{I}^{>}(t) = \int_0^\infty \mathrm{d}z\, \tilde{I}(z,t) = p_{\mathrm{rad}}(t), \tag{2.50}$$

which is on its turn proportional to $p_{\mathrm{rad}}(t)$.

Thus, we have obtained the relation (2.49) between the absolute value of the spatio-temporal shape of the outgoing field outside the cavity to the photodetection probability. In a usual experiment a large number of photodetection events are accumulated to obtain the time-dependent response probability. Note, Eq. (2.49) reveals, that $|\tilde{\phi}(z,t)|$ also depends on the dynamics of the field inside the cavity.

Chapter 3

Leaky High-Q Cavities: QED Foundation and Extension of QNT

As we have discussed in Chapter 2 the conventional approach of a leaky cavity based on QNT provides a simple description of the input-output problem of leaky cavities. In particular, the intuitive description of unwanted losses by coupling the cavity modes to an additional dissipative system and treating this interaction in the Markovian approximation gives rise to the corresponding quantum Langevin noise force leaving the input-output relation unchanged. The applicability of this concept to describe the unwanted losses related to the coupling mirror should be examined.

As we shall discuss in the ensuing chapter, we establish here for the first time a rigorous framework of a leaky high-Q cavity to describe the dynamics of the cavity field and the radiation field outside the cavity within the frame of exact macroscopic quantum electrodynamics in dispersing and absorbing media [MK3]. The theory is based on an exact quantization scheme of the electromagnetic field in causal linear media [36]. We shall obtain the appropriate mode expansion of the field inside the cavity and derive the quantum Langevin equations for the corresponding bosonic operators. We shall derive the input-output relation for the cavity, with special emphasis on the unwanted losses inside the coupling mirror. Based on the correct input-output relation, which takes into account the unwanted losses properly, and assuming that the quantum state of the cavity mode is known at the initial time, we shall calculate, following Refs. [MK3, MK4], the quantum state of the outgoing field, with special emphasis on the mode structure of the field.

3.1 Macroscopic QED in Linear Media

We shall begin with presenting the quantization scheme for the electromagnetic field in the presence of linear media [48], with special emphasis on the interaction of the field with atomic sources. Applying the formalism to the quantized field in presence of a dielectric multilayer system, we shall derive the input-output relations [MK1].

3.1.1 Medium-Assisted Electromagnetic Field

The starting point for the electromagnetic field quantization in the presence of dielectric bodies is the macroscopic Maxwell equations. For the electromagnetic field components in the frequency domain,

$$\mathbf{O}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t}\underline{\mathbf{O}}(\mathbf{r},\omega) = \int_{0}^{\infty} \mathrm{d}\omega\, e^{-i\omega t}\underline{\mathbf{O}}(\mathbf{r},\omega) + \text{H.c.}, \tag{3.1}$$

Maxwell's equations in the absence free charges and currents read [48]

$$\boldsymbol{\nabla}\cdot\underline{\mathbf{B}}(\mathbf{r},\omega) = 0, \tag{3.2}$$

$$\varepsilon_0\boldsymbol{\nabla}\cdot\varepsilon(\mathbf{r},\omega)\underline{\mathbf{E}}(\mathbf{r},\omega) = \underline{\rho}_N(\mathbf{r},\omega), \tag{3.3}$$

$$\boldsymbol{\nabla}\times\underline{\mathbf{E}}(\mathbf{r},\omega) - i\omega\underline{\mathbf{B}}(\mathbf{r},\omega) = 0, \tag{3.4}$$

$$\mu_0^{-1}\boldsymbol{\nabla}\times\underline{\mathbf{B}}(\mathbf{r},\omega) + i\omega\varepsilon_0\varepsilon(\mathbf{r},\omega)\underline{\mathbf{E}}(\mathbf{r},\omega) = \underline{\mathbf{j}}_N(\mathbf{r},\omega), \tag{3.5}$$

where $\underline{\mathbf{E}}(\mathbf{r},\omega)$ and $\underline{\mathbf{B}}(\mathbf{r},\omega)$ denote the electric and magnetic induction fields, respectively, and $\varepsilon(\mathbf{r},\omega)$ denotes the spatially varying relative complex-valued electric permitivity of the media, which satisfies the Kramers-Kronig relation due to causality principles [56]. In the above, it is assumed that the response of the (stationary) medium is linear, local and isotropic, and the charge $\underline{\rho}_N(\mathbf{r},\omega)$ and current density $\underline{\mathbf{j}}_N(\mathbf{r},\omega)$ are attributed to the presence of the medium with unavoidably occurring losses. The noise charge $\underline{\rho}_N(\mathbf{r},\omega)$ and noise current density $\underline{\mathbf{j}}_N(\mathbf{r},\omega)$ are related to the noise polarization $\underline{\mathbf{P}}_N(\mathbf{r},\omega)$,

$$\underline{\rho}_N(\mathbf{r},\omega) = -\boldsymbol{\nabla}\cdot\underline{\mathbf{P}}_N(\mathbf{r},\omega), \tag{3.6}$$

$$\underline{\mathbf{j}}_N(\mathbf{r},\omega) = -i\omega\underline{\mathbf{P}}_N(\mathbf{r},\omega), \tag{3.7}$$

and satisfy the continuity equation

$$-\,i\omega\underline{\rho}_N(\mathbf{r},\omega) + \boldsymbol{\nabla}\cdot\underline{\mathbf{j}}_N(\mathbf{r},\omega) = 0. \tag{3.8}$$

Combining Eqs. (3.4) and (3.5), it is easy to verify that the electric field operator $\underline{\mathbf{E}}(\mathbf{r},\omega)$ satisfies the inhomogeneous Helmholtz equation

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\underline{\mathbf{E}}(\mathbf{r},\omega)-\frac{\omega^2}{c^2}\,\varepsilon(\mathbf{r},\omega)\underline{\mathbf{E}}(\mathbf{r},\omega)=i\omega\mu_0\underline{\mathbf{j}}_N(\mathbf{r},\omega). \tag{3.9}$$

Introducing the retarded Green tensor as the solution to the equation

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)-\frac{\omega^2}{c^2}\,\varepsilon(\mathbf{r},\omega)\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)=\boldsymbol{\delta}^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{3.10}$$

together with the boundary condition at infinity, $\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)\to 0$ if $|\mathbf{r}-\mathbf{r}'|\to\infty$, the solution of Eq. (3.9) can be represented as follows:

$$\underline{\mathbf{E}}(\mathbf{r},\omega)=i\omega\mu_0\int \mathrm{d}^3r'\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)\cdot\underline{\mathbf{j}}_N(\mathbf{r},\omega). \tag{3.11}$$

Note that the Green tensor is an analytic function of ω in the upper complex half-plane and has the following properties:

$$\mathsf{G}^*(\mathbf{r},\mathbf{r}',\omega)=\mathsf{G}(\mathbf{r},\mathbf{r}',-\omega^*), \tag{3.12}$$

$$\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)=\mathsf{G}^T(\mathbf{r}',\mathbf{r},\omega), \tag{3.13}$$

$$\frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)\mathsf{G}(\mathbf{r},\mathbf{s},\omega)\cdot\mathsf{G}^*(\mathbf{r}',\mathbf{s},\omega)=\mathrm{Im}\,\mathsf{G}(\mathbf{r},\mathbf{r}',\omega). \tag{3.14}$$

3.1.2 Field Quantization

Once having specified the solution to Maxwell equations for the electromagnetic field in the presence of medium in terms of noise currents and, therefore, noise polarization, Eq. (3.11), the field quantization is performed by replacing $\underline{\mathbf{j}}_N(\mathbf{r},\omega)$ with the operated-valued quantity $\underline{\mathbf{j}}_N(\mathbf{r},\omega)\to\hat{\underline{\mathbf{j}}}_N(\mathbf{r},\omega)$,

$$\hat{\underline{\mathbf{j}}}_N(\mathbf{r},\omega)=\omega\sqrt{\frac{\hbar\varepsilon_0}{\pi}\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)}\,\hat{\underline{\mathbf{f}}}(\mathbf{r},\omega), \tag{3.15}$$

where $\hat{\underline{\mathbf{f}}}(\mathbf{r},\omega)$ [and $\hat{\underline{\mathbf{f}}}^\dagger(\mathbf{r},\omega)$] are the dynamical variables of the composed system consisting of the electromagnetic field and the linear medium including the dissipative system responsible for absorption:

$$\left[\hat{f}_\mu(\mathbf{r},\omega),\hat{f}^\dagger_{\mu'}(\mathbf{r}',\omega')\right]=\delta_{\mu\mu'}\delta(\omega-\omega')\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{3.16}$$

$$\left[\hat{f}_\mu(\mathbf{r},\omega),\hat{f}_{\mu'}(\mathbf{r}',\omega')\right]=0=\left[\hat{f}^\dagger_\mu(\mathbf{r},\omega),\hat{f}^\dagger_{\mu'}(\mathbf{r}',\omega')\right]. \tag{3.17}$$

Then, the Hamiltonian of the composed system reads

$$\hat{H} = \int \mathrm{d}^3 r \int_0^\infty \mathrm{d}\omega\, \hbar\omega \hat{f}^\dagger(\mathbf{r},\omega) \cdot \hat{f}(\mathbf{r},\omega). \tag{3.18}$$

Thus, by means of Eqs. (3.11) and (3.15) the (medium-assisted) electric field operator can be expressed in terms of the dynamical variables as:

$$\hat{\mathbf{E}}(\mathbf{r}) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}), \tag{3.19}$$

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r}) = \int_0^\infty \mathrm{d}\omega\, \underline{\hat{\mathbf{E}}}(\mathbf{r},\omega), \qquad \hat{\mathbf{E}}^{(-)}(\mathbf{r}) = [\hat{\mathbf{E}}^{(+)}(\mathbf{r})]^\dagger, \tag{3.20}$$

$$\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) = i\sqrt{\frac{\hbar}{\varepsilon_0 \pi}}\, \frac{\omega^2}{c^2} \int \mathrm{d}^3 r' \,\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega)}\, \mathsf{G}(\mathbf{r},\mathbf{r}',\omega) \cdot \hat{\mathbf{f}}(\mathbf{r}',\omega). \tag{3.21}$$

Accordingly, Eq. (3.4) together with (3.15) leads to

$$\hat{\mathbf{B}}(\mathbf{r}) = \int_0^\infty \mathrm{d}\omega\, \underline{\hat{\mathbf{B}}}(\mathbf{r},\omega) + \text{H.c.} \tag{3.22}$$

$$= \mu_0 \int \mathrm{d}^3 r \int_0^\infty \mathrm{d}\omega\, \omega \sqrt{\frac{\hbar\varepsilon_0}{\pi} \mathrm{Im}\,\varepsilon(\mathbf{r},\omega)}\, \boldsymbol{\nabla} \times \mathsf{G}(\mathbf{r},\mathbf{r}',\omega) \cdot \hat{\mathbf{f}}(\mathbf{r}',\omega). \tag{3.23}$$

It can be shown, that the electric field and the magnetic induction field operators satisfy the fundamental equal-time characteristic commutation relation [48]:

$$\left[\hat{E}_\mu(\mathbf{r}), \hat{E}_{\mu'}(\mathbf{r}')\right] = \left[\hat{B}_\mu(\mathbf{r}), \hat{B}_{\mu'}(\mathbf{r}')\right] = 0, \tag{3.24}$$

$$\left[\varepsilon_0 \hat{E}_\mu(\mathbf{r}), \hat{B}_{\mu'}(\mathbf{r}')\right] = -i\hbar \varepsilon_{\mu\mu'\nu} \partial_\nu \delta(\mathbf{r} - \mathbf{r}'). \tag{3.25}$$

It is important to point out, that since the real part $\mathrm{Re}\,\varepsilon(\mathbf{r},\omega)$ and the imaginary part $\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)$ of the permittivity are related to each other through the Kramers-Kronig relations, they are positive-valued for every r and ω. Therefore, the imaginary part of the permittivity in reality cannot vanish identically for existing media. Clearly, $\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega)$ can be very small, so that $\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega)}\, \boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$ in Eq. (3.21) or in Eq. (3.23) is very small in certain areas of space $(\mathbf{r}')$. However, the magnitude of the total integral over the coordinate $\mathbf{r}'$ would still be finite because of the integral relation (3.14) for the Green tensor. Thus, in order to include the areas of the empty space into consideration, all the calculations are to be performed by assuming a permittivity close to unity with a small but finite imaginary part in these areas. Then, after performing the space integrations, the permittivity may be set equal to unity. In practice, an experimental realization of strict (macroscopic) vacuum areas is of course hypothetical.

3.1.3 Atom-Field Interaction

In the previous section we have considered the medium-assisted quantized electromagnetic field in the case, when no additional sources are present. The interaction of the field with external sources can now be included by means of the well-known minimal or multipolar coupling schemes.

Minimal Coupling

Let us consider nonrelativistic point-like charged particles with charges q_α and masses m_α, described in terms of positions $\hat{\mathbf{r}}_\alpha$ and canonically conjugate momenta $\hat{\mathbf{p}}_\alpha$ that interact with the medium-assisted electromagnetic field. The total Hamiltonian in the minimal-coupling scheme reads

$$\hat{H} = \int \mathrm{d}^3 r \int_0^\infty \mathrm{d}\omega\, \hbar\omega \hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)\cdot\hat{\mathbf{f}}(\mathbf{r},\omega) + \sum_\alpha \frac{1}{2m_\alpha}\left[\hat{\mathbf{p}}_\alpha - q_\alpha \hat{\mathbf{A}}(\hat{\mathbf{r}}_\alpha)\right]^2 + \hat{W}, \tag{3.26}$$

where $\hat{\mathbf{A}}(\mathbf{r})$ is the vector potentials of the medium-assisted electromagnetic field in the Coulomb gauge,

$$\hat{\mathbf{A}}(\mathbf{r}) = \int_0^\infty \mathrm{d}\omega\, \underline{\hat{\mathbf{A}}}(\mathbf{r},\omega) + \text{H.c.}, \tag{3.27}$$

and $\underline{\hat{\mathbf{A}}}(\mathbf{r},\omega)$ can be expressed in terms of the transverse part $\hat{\mathbf{E}}^\perp(\mathbf{r})$:

$$\underline{\hat{\mathbf{A}}}(\mathbf{r},\omega) = (i\omega)^{-1}\underline{\hat{\mathbf{E}}}^\perp(\mathbf{r},\omega). \tag{3.28}$$

Here and in the following, the longitudinal and transverse parts of a vector field $\mathbf{O}(\mathbf{r})$ are denoted as $\mathbf{O}^\parallel(\mathbf{r})$ and $\mathbf{O}^\perp(\mathbf{r})$, respectively, e.g.,

$$\mathbf{O}^{\parallel(\perp)}(\mathbf{r}) = \int \mathrm{d}^3 r\, \boldsymbol{\delta}^{\parallel(\perp)}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{O}(\mathbf{r}'), \tag{3.29}$$

with

$$\boldsymbol{\delta}^\parallel(\mathbf{r}) = -\boldsymbol{\nabla}\boldsymbol{\nabla}(4\pi r)^{-1}, \tag{3.30}$$

$$\boldsymbol{\delta}^\perp(\mathbf{r}) = \boldsymbol{\delta}(\mathbf{r}) - \boldsymbol{\delta}^\parallel(\mathbf{r}). \tag{3.31}$$

The scalar vector potentials of the medium-assisted electromagnetic field reads

$$-\boldsymbol{\nabla}\hat{\phi}(\mathbf{r}) = \hat{\mathbf{E}}^\parallel(\mathbf{r}). \tag{3.32}$$

Then, the total Coulomb energy

$$W = \frac{1}{2}\int \mathrm{d}^3 r\, \hat{\rho}_A(\mathbf{r})\hat{\phi}_A(\mathbf{r}) + \int \mathrm{d}^3 r\, \hat{\rho}_A(\mathbf{r})\hat{\phi}(\mathbf{r}) \tag{3.33}$$

is given as the sum of the Coulomb energy of the charge particles and the Coulomb energy of interaction of the charged particles with the medium, where

$$\hat{\rho}_A(\mathbf{r}) = \sum_\alpha q_\alpha \delta(\mathbf{r} - \mathbf{r}'), \tag{3.34}$$

$$\hat{\phi}_A(\mathbf{r}) = \int \mathrm{d}^3 r' \, \frac{\hat{\rho}_A(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \tag{3.35}$$

denoting the charge density and the scalar potential attributed to the charged particles, respectively.

Multipolar Coupling

The interaction of localized atomic systems (atoms, molecules etc., referred below to as atoms for convenience) with the electromagnetic field is commonly useful to present in terms of the field strengths and atomic polarizations. We consider a neutral atomic system localized at position $\mathbf{r}_A$.[1] The atomic polarization can be presented as

$$\hat{\mathbf{P}}_A(\mathbf{r}) = \sum_\alpha q_\alpha (\hat{\mathbf{r}}_\alpha - \mathbf{r}_A) \int_0^1 \mathrm{d}\lambda \, \delta[\mathbf{r} - \mathbf{r}_A - \lambda(\hat{\mathbf{r}}_\alpha - \mathbf{r}_A)]. \tag{3.36}$$

The multipolar coupling Hamiltonian can be obtained from the one of the minimal coupling form by applying the Power-Zienau transformation to the variables [58, 59]:

$$\hat{U} = \exp\left[\frac{i}{\hbar} \int \mathrm{d}^3 r \, \hat{\mathbf{P}}_A(\mathbf{r}) \hat{\mathbf{A}}(\mathbf{r})\right]. \tag{3.37}$$

In particular, using Eq. (3.27) together with Eqs. (3.28), (3.29), (3.30), (3.21) and (3.16), it can be shown that

$$\begin{aligned} \hat{\mathbf{f}}'(\mathbf{r}, \omega) &= \hat{U} \hat{\mathbf{f}}(\mathbf{r}, \omega) \hat{U}^\dagger \\ &= \hat{\mathbf{f}}(\mathbf{r}, \omega) + i\mu_0 \omega \sqrt{\frac{\varepsilon_0}{\pi\hbar} \mathrm{Im}\, \varepsilon(\mathbf{r}, \omega)} \int \mathrm{d}^3 r' \, \hat{\mathbf{P}}_A^\perp \cdot \mathsf{G}^*(\mathbf{r}', \mathbf{r}, \omega). \end{aligned} \tag{3.38}$$

Then, the Hamiltonian Eq. (3.25) can be expressed in terms of the transformed variables. After some calculations, one finds the multipolar Hamiltonian for a neutral

[1] Here, $\mathbf{r}_A$ represents a classical quantity. In the general case, to take into account also moving atomic systems, the center-of-mass coordinate should be considered instead $\mathbf{r}_A \to \hat{\mathbf{r}}_A$, see Ref. [57].

atom in the form (see Ref. [57] for details of calculation):

$$\hat{H} = \int \mathrm{d}^3 r \int_0^\infty \mathrm{d}\omega\, \hbar\omega\, \hat{\mathbf{f}}'^\dagger(\mathbf{r},\omega)\cdot\hat{\mathbf{f}}'(\mathbf{r},\omega)$$
$$+ \sum_\alpha \frac{1}{2m_\alpha}\left\{\hat{\mathbf{p}}'_\alpha + q_\alpha \int_0^1 \mathrm{d}\lambda\, \lambda(\hat{\mathbf{r}}_\alpha - \mathbf{r}_A)\times\hat{\mathbf{B}}'[\mathbf{r}_A + \lambda(\hat{\mathbf{r}}_\alpha - \mathbf{r}_A)]\right\}^2$$
$$+\frac{1}{2\varepsilon_0}\int \mathrm{d}^3 r\, \hat{\mathbf{P}}'^2_A(\mathbf{r}) - \int \mathrm{d}^3 r\, \hat{\mathbf{P}}'_A(\mathbf{r})\cdot\hat{\mathbf{E}}'(\mathbf{r}), \qquad (3.39)$$

where the prime notation for the field operators indicates, that the operator should be thought of as being expressed in terms of $\hat{\mathbf{f}}'(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}'^\dagger(\mathbf{r},\omega)$. The first term in Eq. (3.39)

$$\hat{H}_F = \int \mathrm{d}^3 r \int_0^\infty \mathrm{d}\omega\, \hbar\omega\, \hat{\mathbf{f}}'^\dagger(\mathbf{r},\omega)\cdot\hat{\mathbf{f}}'(\mathbf{r},\omega) \qquad (3.40)$$

is the Hamiltonian of the composed system of the electromagnetic field and linear medium. The further terms can be regrouped to obtain the unperturbed atomic Hamiltonian:

$$\hat{H}'_A = \sum_\alpha \frac{1}{2m_\alpha}\hat{\mathbf{p}}'^2_\alpha + \frac{1}{2\varepsilon_0}\int \mathrm{d}^3 r\, \hat{\mathbf{P}}'^2_A(\mathbf{r}). \qquad (3.41)$$

Let us introduce the electric dipole moment of the atom

$$\hat{\mathbf{d}} = \sum_\alpha q_\alpha \hat{\mathbf{r}}_\alpha. \qquad (3.42)$$

The interaction Hamiltonian in the case of electric-dipole approximation reduces to[2]

$$\hat{H}_{\mathrm{int}} = -\hat{\mathbf{d}}'\cdot\hat{\mathbf{E}}'(\mathbf{r}'_A). \qquad (3.43)$$

Notice, that though the field $\hat{\mathbf{E}}(\mathbf{r})$ in the minimal coupling scheme has the meaning of the medium assisted electric field operator, in the multipolar coupling scheme the operator $\hat{\mathbf{E}}'(\mathbf{r})$ reads

$$\hat{\mathbf{E}}'(\mathbf{r}) = \hat{\mathbf{E}}(\mathbf{r}) + \varepsilon_0^{-1}\hat{\mathbf{P}}^\perp_A(\mathbf{r}) \qquad (3.44)$$

and, therefore, has the meaning of the displacement field with respect to the atomic polarization. It is worth to note that the multipolar coupling scheme presented here can be easily extended to the case of two and more atoms. In this case, the free Hamiltonian of each atom will be given in the form of Eq. (3.41), and each atom interacts individually by means of the multipolar coupling Hamiltonian as described.

[2]Note, in the case of optical radiation long wavelength approximation can be applied, i.e., the field may be regarded as being slowly varying within the atomic volume, leading to the electric-dipole approximation.

3.1.4 Operator Input-Output Relations

The quantization of the electromagnetic field presented in the previous sections is well suited to study the input-output behavior of the fields at macroscopic bodies. Field input-output relations that describe the action of the dispersing and absorbing bodies are useful to study, for example, the statistics of the outgoing fields in terms of the incoming ones. Here, we shall focus on the consideration of the radiation field propagating through a planar multilayer dielectric structure [MK1].

Planar Multilayer Structure

A planar multilayer dielectric structure consist of adjoint layers (Fig. 3.1) and can be characterized by means of a permittivity that changes in a stepwise fashion (let the z-direction be perpendicular to the layers):

$$\varepsilon(\omega, z) = \sum_{j=0}^{n} \lambda_j(z)\varepsilon_j(\omega), \tag{3.45}$$

where

$$\lambda_j(z) = \begin{cases} 1, & \text{if } z \in j\text{th layer}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.46}$$

and $\varepsilon_j(\omega)$ is the complex permittivity of the jth layer. In the above, the index j labels the region on the left of the structure ($j=0$), the region on the right of the structure ($j=n$), and the layers of the planar structure ($j=1,\ldots,n-1$). For simplicity, we express the z-coordinate dependence in shifted coordinate systems introduced in each layer separately, so that the range of the z-coordinate is taken to be $-\infty<z<0$ for the region on the left of the structure ($j=0$), $0<z<\infty$ for the region on the right of the structure ($j=n$), and $0<z<d_j$ for the jth layer of the structure with thickness d_j ($j=1,\ldots,n-1$).

Exploiting the translational symmetry in the (xy)-plane, we may represent the Green tensor as a two-dimensional Fourier integral

$$\mathsf{G}^{(jj')}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\, e^{i\mathbf{k}(\boldsymbol{\rho}-\boldsymbol{\rho}')}\, \mathsf{G}^{(jj')}(z,z',\mathbf{k},\omega), \tag{3.47}$$

where $\boldsymbol{\rho}=(x,y)$, and $\mathbf{k}=(k_x,k_y)$ is the wave vector parallel to the layers. Here and in what follows, the notations $\mathsf{G}^{(jj')}(\mathbf{r},\mathbf{r}',\omega)$ and $\mathsf{G}^{(jj')}(z,z',\mathbf{k},\omega)$ indicate that z varies in the jth layer and z' in the j'th layer. Inserting Eq. (3.47) into Eq. (3.21), we may

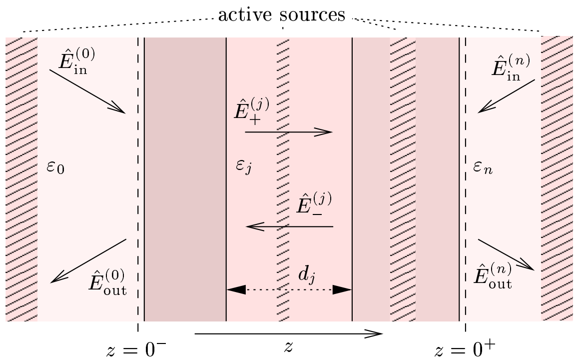

Figure 3.1: Scheme of a planar multilayer dielectric structure. The hatched regions indicate the presence of active light sources.

write the electric-field operator $\hat{\mathbf{e}}^{(j)}(\mathbf{r},\omega)$ as a two-fold Fourier transform,

$$\hat{\mathbf{E}}^{(j)}(\mathbf{r},\omega) = \frac{1}{4\pi^2}\int \mathrm{d}^2k\, e^{i\mathbf{k}\boldsymbol{\rho}}\hat{\mathbf{e}}^{(j)}(z,\mathbf{k},\omega), \tag{3.48}$$

where

$$\hat{\mathbf{E}}^{(j)}(z,\mathbf{k},\omega) = i\omega\mu_0\sum_{j'=0}^{n}\int_{[j']}\mathrm{d}z'\,\mathsf{G}^{(jj')}(z,z',\mathbf{k},\omega)\cdot\hat{\mathbf{j}}^{(j')}(z',\mathbf{k},\omega) \tag{3.49}$$

($[j']$ indicates integration over the j'th region). Here, similarly to Eq. (3.15) the Fourier transformed current operators

$$\hat{\mathbf{j}}^{(j)}(z,\mathbf{k},\omega) = \omega\sqrt{\frac{\hbar\varepsilon_0}{\pi}\mathrm{Im}\,\varepsilon_j(\omega)}\,\hat{\mathbf{f}}^{(j)}(z,\mathbf{k},\omega) \tag{3.50}$$

have been introduced, where the operators $\hat{\mathbf{f}}^{(j)}(z,\mathbf{k},\omega)$ are the Fourier transforms of the original bosonic field operators $\hat{\mathbf{f}}^{(j)}(\mathbf{r},\omega)$,

$$\hat{\mathbf{f}}^{(j)}(\mathbf{r},\omega) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2k\, e^{i\mathbf{k}\boldsymbol{\rho}}\,\hat{\mathbf{f}}^{(j)}(z,\mathbf{k},\omega). \tag{3.51}$$

The Green tensor $\mathsf{G}^{(jj')}(z,z',\mathbf{k},\omega)$ for the planar multilayer structure may be written as [60]

$$\mathsf{G}^{(jj')}(z,z',\mathbf{k},\omega) = -\mathbf{e}_z\,\frac{\delta_{jj'}}{k_j^2}\,\mathbf{e}_z\,\delta(z-z') + \mathsf{g}^{(jj')}(z,z',\mathbf{k},\omega), \tag{3.52}$$

where

$$\mathsf{g}^{(jj')}(z,z',\mathbf{k},\omega) = \frac{i}{2}\sum_{q=p,s}\sigma_q\Big[\boldsymbol{\mathcal{E}}_q^{j>}(z,\mathbf{k},\omega)\,\Xi_q^{jj'}\boldsymbol{\mathcal{E}}_q^{j'<}(z',-\mathbf{k},\omega)\Theta(j-j')$$
$$+\boldsymbol{\mathcal{E}}_q^{j<}(z,\mathbf{k},\omega)\,\Xi_q^{j'j}\boldsymbol{\mathcal{E}}_q^{j'>}(z',-\mathbf{k},\omega)\Theta(j'-j)\Big] \tag{3.53}$$

($\sigma_p = 1$, $\sigma_s = -1$). Note that for $j = j'$ one should write $\Theta(z - z')$ instead of $\Theta(j - j')$ [and $\Theta(z' - z)$ instead of $\Theta(j' - j)$]. In Eq. (3.53), the functions $\boldsymbol{\mathcal{E}}_q^{j>}(\mathbf{k}, \omega, z)$ and $\boldsymbol{\mathcal{E}}_q^{j<}(\mathbf{k}, \omega, z)$ denote waves of unit strength, traveling, respectively, rightward and leftward in the jth layer, and being reflected at the boundary,

$$\boldsymbol{\mathcal{E}}_q^{(j)>}(z, \mathbf{k}, \omega) = \mathbf{e}_{q+}^{(j)}(\mathbf{k}) e^{i\beta_j(z-d_j)} + r_{j/n}^q \mathbf{e}_{q-}^{(j)}(\mathbf{k}) e^{-i\beta_j(z-d_j)}, \tag{3.54}$$

$$\boldsymbol{\mathcal{E}}_q^{(j)<}(z, \mathbf{k}, \omega) = \mathbf{e}_{q-}^{(j)}(\mathbf{k}) e^{-i\beta_j z} + r_{j/0}^q \mathbf{e}_{q+}^{(j)}(\mathbf{k}) e^{i\beta_j z}, \tag{3.55}$$

and

$$\Xi_q^{jj'} = \frac{1}{\beta_n t_{0/n}^q} \frac{t_{0/j}^q e^{i\beta_j d_j}}{D_{qj}} \frac{t_{n/j'}^q e^{i\beta_{j'} d_{j'}}}{D_{qj'}}, \tag{3.56}$$

where

$$D_{qj} = 1 - r_{j/0}^q r_{j/n}^q e^{2i\beta_j d_j} \tag{3.57}$$

($d_0 = d_n = 0$). Here,

$$\beta_j = \sqrt{k_j^2 - k^2} = \beta_j' + i\beta_j'' \qquad (\beta_j', \beta_j'' \geq 0) \tag{3.58}$$

($k = |\mathbf{k}|$), where

$$k_j = \sqrt{\varepsilon_j(\omega)}\, \frac{\omega}{c} = k_j' + ik_j'' \qquad (k_j', k_j'' \geq 0), \tag{3.59}$$

and $t_{j/j'}$ and $r_{j/j'}$ are, respectively, the transmission and reflection coefficients between the layers j' and j (see App. B for recursion formulae). Finally, the unit vectors $\mathbf{e}_{q\pm}^{(j)}(\mathbf{k})$ in Eqs. (3.54) and (3.55) are the polarization unit vectors for transverse electric (TE) ($q = s$) and transverse magnetic (TM) ($q = p$) waves,

$$\mathbf{e}_{s\pm}^{(j)}(\mathbf{k}) = \frac{\mathbf{k}}{k} \times \mathbf{e}_z, \tag{3.60}$$

$$\mathbf{e}_{p\pm}^{(j)}(\mathbf{k}) = \frac{1}{k_j}\left(\mp\beta_j \frac{\mathbf{k}}{k} + k\mathbf{e}_z\right). \tag{3.61}$$

The 'propagation constant' β_j determines the propagation behavior in z-direction of the waves in the jth region. Note that in case of vacuum the waves are propagating only for $\beta_j = \beta_j'$ (i.e., $\omega/c > k$). They are evanescent for $\beta_j = i\beta_j''$ (i.e., $\omega/c \leq k$).

Input-Output Relations

Let us restrict our attention to the electric field, noting that the corresponding expressions can be readily obtained for the magnetic induction field. Substituting the Green tensor (3.52) for $j=0$ and $j=n$ into Eq. (3.49), we may decompose the field operators $\hat{\mathbf{E}}^{(0)}(z,\mathbf{k},\omega)$ and $\hat{\mathbf{E}}^{(n)}(z,\mathbf{k},\omega)$ in the form of

$$\hat{\mathbf{E}}^{(0)}(z,\mathbf{k},\omega) = \sum_{q=p,s}[\mathbf{e}^{(0)}_{q+}(\mathbf{k})\hat{E}^{(0)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) + \mathbf{e}^{(0)}_{q-}(\mathbf{k})\hat{E}^{(0)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega)], \tag{3.62}$$

$$\hat{\mathbf{E}}^{(n)}(z,\mathbf{k},\omega) = \sum_{q=p,s}[\mathbf{e}^{(n)}_{q-}(\mathbf{k})\hat{E}^{(n)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) + \mathbf{e}^{(n)}_{q+}(\mathbf{k})\hat{E}^{(n)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega)]. \tag{3.63}$$

Here, the operators

$$\hat{E}^{(0)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) = -\frac{\mu_0\omega}{2\beta_0}e^{i\beta_0 z}\int_{-\infty}^{z}\mathrm{d}z'\,e^{-i\beta_0 z'}\hat{\mathbf{j}}^{(0)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(0)}_{q+}(\mathbf{k}), \tag{3.64}$$

$$\hat{E}^{(n)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) = -\frac{\mu_0\omega}{2\beta_n}e^{-i\beta_n z}\int_{z}^{\infty}\mathrm{d}z'\,e^{i\beta_n z'}\hat{\mathbf{j}}^{(n)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(n)}_{q-}(\mathbf{k}), \tag{3.65}$$

and

$$\hat{E}^{(0)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega) = e^{-i\beta_0 z}\hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega) + e^{-i\beta_0 z}\int_{z}^{0}\mathrm{d}z'\,e^{i\beta_0 z'}\hat{\mathbf{j}}^{(0)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(0)}_{q-}(\mathbf{k}), \tag{3.66}$$

$$\hat{E}^{(n)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega) = e^{i\beta_n z}\hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) + e^{i\beta_n z}\int_{0}^{z}\mathrm{d}z'\,e^{-i\beta_n z'}\hat{\mathbf{j}}^{(n)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(n)}_{q+}(\mathbf{k}) \tag{3.67}$$

play the role of input and output amplitude operators, respectively. The input and output amplitude operators at the two boundary planes of the structure (i.e., $z=0^-$ for $j=0$ and $z=0^+$ for $j=n$; cf. Fig. 3.1) can be presented as

$$\hat{E}^{(0)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega) = \hat{E}^{(0)}_{q\,\mathrm{in,out}}(z,\mathbf{k},\omega)\Big|_{z=0^-}, \tag{3.68}$$

$$\hat{E}^{(n)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega) = \hat{E}^{(n)}_{q\,\mathrm{in,out}}(z,\mathbf{k},\omega)\Big|_{z=0^+}. \tag{3.69}$$

Then, the input-output relations read

$$\begin{pmatrix}\hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\\ \hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\end{pmatrix} = \begin{pmatrix} r^q_{0/n}(\mathbf{k},\omega) & t^q_{n/0}(\mathbf{k},\omega)\\ t^q_{0/n}(\mathbf{k},\omega) & r^q_{n/0}(\mathbf{k},\omega)\end{pmatrix}\begin{pmatrix}\hat{E}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\\ \hat{E}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\end{pmatrix}$$
$$+\sum_{j=1}^{n-1}\begin{pmatrix}\phi^{(j)}_{q\,0+}(\mathbf{k},\omega) & \phi^{(j)}_{q\,0-}(\mathbf{k},\omega)\\ \phi^{(j)}_{q\,n+}(\mathbf{k},\omega) & \phi^{(j)}_{q\,n-}(\mathbf{k},\omega)\end{pmatrix}\begin{pmatrix}\hat{E}^{(j)}_{q+}(\mathbf{k},\omega)\\ \hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\end{pmatrix}, \tag{3.70}$$

where the amplitude operators, that refer to the layers of the structure, introduced as

$$\hat{E}^{(j)}_{q<\pm}(\mathbf{k},\omega) = -\frac{\mu_0\omega}{2\beta_j}\int_{0}^{z}\mathrm{d}z'\,e^{\mp i\beta_j z'}\hat{\mathbf{j}}^{(j)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(j)}_{q\pm}(\mathbf{k}), \tag{3.71}$$

$$\hat{E}^{(j)}_{q>\pm}(\mathbf{k},\omega) = -\frac{\mu_0\omega}{2\beta_j}\int_{z}^{d_j}\mathrm{d}z'\,e^{\mp i\beta_j z'}\hat{\mathbf{j}}^{(j)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(j)}_{q\pm}(\mathbf{k}), \tag{3.72}$$

$$\hat{E}^{(j)}_{q\pm}(\mathbf{k},\omega) = \hat{E}^{(j)}_{q<\pm}(\mathbf{k},\omega) + \hat{E}^{(j)}_{q>\pm}(\mathbf{k},\omega)$$
$$= -\frac{\mu_0\omega}{2\beta_j}\int_0^{d_j} \mathrm{d}z'\, e^{\mp i\beta_j z'}\hat{\mathbf{j}}^{(j)}(z',\mathbf{k},\omega)\cdot\mathbf{e}^{(j)}_{q\pm}(\mathbf{k}) \tag{3.73}$$

($j=1,2,\ldots,n-1$), are associated with the excitations inside the layers of the planar structure. The ϕ-coefficients in the input-output relations Eq. (3.70) are determined only by the complex permittivities and thicknesses of the layers of the structure and the permittivities of the surrounding media:

$$\phi^{(j)}_{q\,0+} = \frac{t^q_{j/0}e^{2i\beta_j d_j}}{D_{qj}} r^q_{j/n}\,, \qquad \phi^{(j)}_{q\,0-} = \frac{t^q_{j/0}}{D_{qj}}\,, \tag{3.74}$$

$$\phi^{(j)}_{q\,n+} = \frac{t^q_{j/n}e^{i\beta_j d_j}}{D_{qj}}\,, \qquad \phi^{(j)}_{q\,n-} = \frac{t^q_{j/n}e^{i\beta_j d_j}}{D_{qj}} r^q_{j/0}\,. \tag{3.75}$$

It should be pointed out that the first term on the right-hand side in Eq. (3.52), which gives rise to a local contribution to the electric field, has been omitted in Eqs. (3.62) and (3.63). Though this contribution is irrelevant for the incoming and outgoing fields, it must be included in the overall field operator in general, even if there are effectively no sources at the points of observations.

It is not difficult to prove that the z-dependent amplitude operators (3.64)–(3.67) obey quantum Langevin-type equations,

$$\frac{\partial}{\partial z}\hat{E}^{(0)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) = i\beta_0\hat{E}^{(0)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega) - \frac{\mu_0\omega}{2\beta_0}\hat{\mathbf{j}}^{(0)}(z,\mathbf{k},\omega)\cdot\mathbf{e}^{(0)}_{q+}(\mathbf{k}), \tag{3.76}$$

$$\frac{\partial}{\partial z}\hat{E}^{(0)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega) = -i\beta_0\hat{E}^{(0)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega) + \frac{\mu_0\omega}{2\beta_0}\hat{\mathbf{j}}^{(0)}(z,\mathbf{k},\omega)\cdot\mathbf{e}^{(0)}_{q-}(\mathbf{k}). \tag{3.77}$$

Similar equations are also valid for $\hat{E}^{(n)}_{q\,\mathrm{in}}(z,\mathbf{k},\omega)$ and $\hat{E}^{(n)}_{q\,\mathrm{out}}(z,\mathbf{k},\omega)$. These equations, together with Eq. (3.70) make it possible to easily calculate the input and output fields at any position outside the structure. Needless to say that other than the boundary planes $z=0^-$ and $z=0^+$ of the structure can be chosen as reference planes for formulating the input-output relations.

The input-output relations (3.70) enable one to calculate correlation functions of the output field amplitudes in terms of those of the input field amplitudes and the amplitudes of the fields inside a planar structure. The simplest case is the calculation of the expectation values of the field amplitudes. For example, in a typical scattering arrangement, the active light sources are located outside the structure, so that the field inside the structure is the absorption-assisted random field whose thermal-equilibrium expectation value vanishes. Application of Eq. (3.70), thus, leads to the expectation-value relations

$$\big\langle \hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\big\rangle = r^q_{0/n}\big\langle \hat{E}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\big\rangle + t^q_{n/0}\big\langle \hat{E}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\big\rangle, \tag{3.78}$$

$$\big\langle \hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\big\rangle = t^q_{0/n}\big\langle \hat{E}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\big\rangle + r^q_{n/0}\big\langle \hat{E}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega)\big\rangle, \tag{3.79}$$

which exactly correspond to standard results in classical optics. On the other hand, when the active sources are located (in a resonator system) inside the structure, we find that

$$\big\langle \hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\big\rangle = \sum_{j=1}^{n-1} \frac{t^q_{j/0}}{D_{qj}} e^{i\beta_j d_j} \left[e^{-i\beta_j d_j}\big\langle \hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\big\rangle + r^q_{j/n} e^{i\beta_j d_j}\big\langle \hat{E}^{(j)}_{q+}(\mathbf{k},\omega)\big\rangle\right], \tag{3.80}$$

$$\big\langle \hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\big\rangle = \sum_{j=1}^{n-1} \frac{t^q_{j/n}}{D_{qj}} e^{i\beta_j d_j} \left[\big\langle \hat{E}^{(j)}_{q+}(\mathbf{k},\omega)\big\rangle + r^q_{j/0}\big\langle \hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\big\rangle\right]. \tag{3.81}$$

Needless to say, that when there are no active sources inside the structure, then the relation $\big\langle \hat{E}^{(j)}_{q\pm}(\mathbf{k},\omega)\big\rangle = 0$ $(j = 1, 2, \ldots, n-1)$ is valid and, thus, $\big\langle \hat{E}^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\big\rangle = 0$. Clearly, higher-order correlation functions of the outgoing amplitude operators do not necessarily vanish in this case. For example, the spectral intensity (in the **k**-space) of the radiation outgoing from a plane of a multilayer planar structure in thermal equilibrium at temperature T, for chosen polarization, is proportional to $w^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)$, where

$$\big\langle \hat{E}^{(0,n)\dagger}_{q\,\mathrm{out}}(\mathbf{k},\omega)\, \hat{E}^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k}',\omega')\big\rangle = w^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\,\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'). \tag{3.82}$$

Applying Eq. (3.70) together with Eqs. (3.73) and (3.50), and making use of

$$\big\langle \hat{f}^{(j)\dagger}_{\mu}(z,\mathbf{k},\omega)\hat{f}^{(j)}_{\mu'}(z',\mathbf{k}',\omega')\big\rangle = n(\omega,T)\,\delta_{\mu\mu'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'), \qquad (j\neq 0,n), \tag{3.83}$$

we derive

$$w^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) = n(\omega,T)\sum_{j=1}^{n-1}\frac{\left|t^q_{j/n}e^{i\beta_j d_j}\right|^2}{|D_{qj}|^2} \times\left\{\alpha^{(j)}_{q++}(\mathbf{k},\omega) + \left|r^q_{j/0}\right|^2\alpha^{(j)}_{q--}(\mathbf{k},\omega) + [r^q_{j/0}\alpha^{(j)}_{q-+}(\mathbf{k},\omega) + \mathrm{c.c.}]\right\} \tag{3.84}$$

(and $w^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega)$ accordingly). Here, $n(\omega,T)$ denotes the Bose-Einstein distribution function and the coefficients $\alpha^{(j)}_{q\pm\pm}(\mathbf{k},\omega)$ and $\alpha^{(j)}_{q\pm\mp}(\mathbf{k},\omega)$ are given in Eqs. (A.15) and (A.16) in App. A.1.

Input-Output Relations in Terms of Bosonic Operators

In many applications it may be advantageous to express the incoming and the outgoing field operators in terms of appropriately chosen bosonic operators. The commutation relations between different amplitude operators can be obtained by means of the basic commutation relations (3.16) and (3.17), and recalling that the polarization unit vectors (3.60) and (3.61) are orthogonal to each other. In this way, it is easy to verify that (i) input amplitude operators that refer to different sides of the planar structure, commute with each other, (ii) input amplitude operators commute with amplitude operators that refer to the layers of the structure, (iii) and amplitude operators, that refer to different layers, also commute.

To be more specific, the bosonic input and output operators are defined as follows:

$$\hat{b}^{(0,n)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega) = [\alpha^{(0,n)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega)]^{-\frac{1}{2}}\,\hat{E}^{(0,n)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega), \tag{3.85}$$

with

$$\left[\hat{b}^{(0,n)}_{q\,\mathrm{in,out}}(\mathbf{k},\omega), \hat{b}^{(0,n)\dagger}_{q'\,\mathrm{in,out}}(\mathbf{k}',\omega')\right] = \delta_{qq'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'). \tag{3.86}$$

In the above,

$$\alpha^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) = \frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\frac{\beta'_0}{|\beta_0|^2}\mathbf{e}^{(0)}_{q+}(\mathbf{k})\cdot\mathbf{e}^{(0)*}_{q+}(\mathbf{k}), \tag{3.87}$$

$$\alpha^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) = \frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\frac{\beta'_n}{|\beta_n|^2}\mathbf{e}^{(n)}_{q-}(\mathbf{k})\cdot\mathbf{e}^{(n)*}_{q-}(\mathbf{k}), \tag{3.88}$$

and the coefficients $\alpha^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)$ are given by Eqs. (A.12), (A.13), respectively, in App. A.1. The bosonic operators that refer to the layers of the structure are defined according to

$$\hat{c}^{(j)}_{q\pm}(\mathbf{k},\omega) = \frac{1}{\xi^{(j)}_{q\pm}(\mathbf{k},\omega)}\left[e^{i\beta_j d_j}\hat{E}^{(j)}_{q+}(\mathbf{k},\omega) \pm \hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\right], \tag{3.89}$$

so that

$$\left[\hat{c}^{(j)}_{q\pm}(\mathbf{k},\omega), \hat{c}^{(j)\dagger}_{q'\pm}(\mathbf{k}',\omega')\right] = \delta_{qq'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'), \tag{3.90}$$

$$\left[\hat{c}^{(j)}_{q\pm}(\mathbf{k},\omega), \hat{c}^{(j)\dagger}_{q'\mp}(\mathbf{k}',\omega')\right] = 0 \tag{3.91}$$

($j = 1,\ldots,n-1$), where the coefficients $\xi^{(j)}_{q\pm}(\mathbf{k},\omega)$ are given by Eq. (A.18) in App. A.1. Substituting Eqs. (3.85) and (3.89) into Eq. (3.70), we may express the input-output relations in terms of the bosonic operators,

$$\begin{pmatrix} \hat{b}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega) \\ \hat{b}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) \end{pmatrix} = \begin{pmatrix} \tilde{r}^{q}_{0/n}(\mathbf{k},\omega) & \tilde{t}^{q}_{n/0}(\mathbf{k},\omega) \\ \tilde{t}^{q}_{0/n}(\mathbf{k},\omega) & \tilde{r}^{q}_{n/0}(\mathbf{k},\omega) \end{pmatrix} \begin{pmatrix} \hat{b}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) \\ \hat{b}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) \end{pmatrix}$$
$$+ \sum_{j=1}^{n-1} \begin{pmatrix} \tilde{\phi}^{(j)}_{q\,0+}(\mathbf{k},\omega) & \tilde{\phi}^{(j)}_{q\,0-}(\mathbf{k},\omega) \\ \tilde{\phi}^{(j)}_{q\,n+}(\mathbf{k},\omega) & \tilde{\phi}^{(j)}_{q\,n-}(\mathbf{k},\omega) \end{pmatrix} \begin{pmatrix} \hat{b}^{(j)}_{q+}(\mathbf{k},\omega) \\ \hat{b}^{(j)}_{q-}(\mathbf{k},\omega) \end{pmatrix}, \quad (3.92)$$

where now the coefficients [modified in comparison with Eq. (3.70)] read

$$\tilde{r}^{q}_{0/n}(\mathbf{k},\omega) = \sqrt{\frac{\alpha^{(0)}_{q\,\mathrm{in}}}{\alpha^{(0)}_{q\,\mathrm{out}}}}\, r^{q}_{0/n}(\mathbf{k},\omega)\,, \qquad \tilde{t}^{q}_{n/0}(\mathbf{k},\omega) = \sqrt{\frac{\alpha^{(n)}_{q\,\mathrm{in}}}{\alpha^{(0)}_{q\,\mathrm{out}}}}\, t^{q}_{0/n}(\mathbf{k},\omega)\,, \quad (3.93)$$

$$\tilde{t}^{q}_{0/n}(\mathbf{k},\omega) = \sqrt{\frac{\alpha^{(0)}_{q\,\mathrm{in}}}{\alpha^{(n)}_{q\,\mathrm{out}}}}\, t^{q}_{n/0}(\mathbf{k},\omega)\,, \qquad \tilde{r}^{q}_{n/0}(\mathbf{k},\omega) = \sqrt{\frac{\alpha^{(n)}_{q\,\mathrm{in}}}{\alpha^{(n)}_{q\,\mathrm{out}}}}\, r^{q}_{n/0}(\mathbf{k},\omega)\,, \quad (3.94)$$

and

$$\tilde{\phi}^{(j)}_{q\,0,n\,\pm}(\mathbf{k},\omega) = \tfrac{1}{2}\frac{\xi^{(j)}_{q\pm}(\mathbf{k},\omega)}{\sqrt{\alpha^{(0,n)}_{q\,\mathrm{out}}}} \left[e^{-i\beta_j d_j}\phi^{(j)}_{q\,0,n\,+}(\mathbf{k},\omega) \pm \phi^{(j)}_{q\,0,n\,-}(\mathbf{k},\omega)\right]. \quad (3.95)$$

Let us turn to the limiting case where the space outside the structure—except for possible active atomic sources—may be regarded as being vacuum, i.e.,

$$\varepsilon''_{0,n}(\omega) \to 0, \quad \varepsilon'_{0,n}(\omega) \to 1, \quad (3.96)$$

$$\beta'_{0,n}(k,\omega) \to \begin{cases} 0 & \text{if} \quad \omega/c \le k, \\ \sqrt{\omega^2/c^2 - k^2} & \text{if} \quad \omega/c > k. \end{cases} \quad (3.97)$$

For the propagating-field components observed for $\omega/c > k$, the coefficients $\alpha^{(0,n)}_{q\,\mathrm{in}}(\mathbf{k},\omega)$ and $\alpha^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)$, respectively, read

$$\alpha^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) = \alpha^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) = \frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\frac{1}{\beta_0} \quad (3.98)$$

and

$$\alpha^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) = \alpha^{(0,n)}_{q\,\mathrm{in}}(\mathbf{k},\omega), \quad (3.99)$$

as it can be seen from Eqs. (3.87) and (3.88) and Eqs. (A.20) and (A.21) in App. A.1. Moreover, the output amplitude operators that refer to different sides of the structure commute [Eq. (A.19)], which implies that the associated bosonic operators commute as well. Making use of Eqs. (3.98) and (3.99), we see that Eq. (3.92) thus reduces to

$$\begin{pmatrix} \hat{b}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega) \\ \hat{b}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) \end{pmatrix} = \begin{pmatrix} r^q_{0/n}(\mathbf{k},\omega) & t^q_{n/0}(\mathbf{k},\omega) \\ t^q_{0/n}(\mathbf{k},\omega) & r^q_{n/0}(\mathbf{k},\omega) \end{pmatrix} \begin{pmatrix} \hat{b}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) \\ \hat{b}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) \end{pmatrix}$$

$$+\sum_{j=1}^{n-1} \begin{pmatrix} \tilde{\phi}^{(j)}_{q\,0+}(\mathbf{k},\omega) & \tilde{\phi}^{(j)}_{q\,0-}(\mathbf{k},\omega) \\ \tilde{\phi}^{(j)}_{q\,n+}(\mathbf{k},\omega) & \tilde{\phi}^{(j)}_{q\,n-}(\mathbf{k},\omega) \end{pmatrix} \begin{pmatrix} \hat{c}^{(j)}_{q+}(\mathbf{k},\omega) \\ \hat{c}^{(j)}_{q-}(\mathbf{k},\omega) \end{pmatrix}, \quad (3.100)$$

where the transformation matrix connecting the bosonic output operators with the bosonic input operators is exactly the same as that for the corresponding amplitude operators in Eq. (3.70). In this case, the field operators $\hat{\mathbf{E}}^{(0)}(z,\mathbf{k},\omega)$ and $\hat{\mathbf{E}}^{(n)}(z,\mathbf{k},\omega)$ at the boundaries of the multilayer structure [see Eqs. (3.62), (3.63), (3.68), and (3.69)] can be represented as

$$\hat{\mathbf{E}}^{(0)}(0^-,\mathbf{k},\omega) = -\frac{\omega}{c}\sqrt{\frac{\pi\hbar}{\beta_0\varepsilon_0}}\sum_{q=p,s}\left[\mathbf{e}^{(0)}_{q+}(\mathbf{k})\hat{b}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega)+\mathbf{e}^{(0)}_{q-}(\mathbf{k})\hat{b}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\right], \quad (3.101)$$

$$\hat{\mathbf{E}}^{(n)}(0^+,\mathbf{k},\omega) = -\frac{\omega}{c}\sqrt{\frac{\pi\hbar}{\beta_n\varepsilon_0}}\sum_{q=p,s}\left[\mathbf{e}^{(n)}_{q-}(\mathbf{k})\hat{a}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega)+\mathbf{e}^{(n)}_{q+}(\mathbf{k})\hat{a}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)\right] \quad (3.102)$$

($\beta_0=\beta_n=\sqrt{\omega^2/c^2-k^2}$, $\omega/c>k$).

For the evanescent-field components observed for $\omega/c\leq k$, the coefficients $\alpha^{(0)}_{q\,\mathrm{in}}$ [Eq. (3.87)] and $\alpha^{(n)}_{q\,\mathrm{in}}$ [Eq. (3.88)] identically vanish, because of $\beta'_0=\beta'_n=0$. Recalling Eq. (3.85), we see that bosonic input operators cannot be introduced for the evanescent-field components. Hence, it is impossible to extend the validity of Eq. (3.96) to the evanescent-field components. To treat them, one has to, therefore, go back to the generally valid input-output relations (3.70) for the amplitude operators. Clearly, when there are no active light sources at any finite distance from the structure, then the input field can be regarded as being effectively a propagating field [see Eqs. (3.64) and (3.65)], that is to say, a field that does not contain evanescent input components. However, the output field may contain evanescent output components resulting from the field that refers to the layers of the structure.

3.1.5 Dynamics of Atom-Field System

The multipolar coupling Hamiltonian described in Sec. 3.1.3 can be the starting point for investigation of the temporal evolution of the medium-assisted electromagnetic field and the atomic systems. In particular, the dynamical equations for the internal atomic variables can be employed to describe the atomic relaxation rates and the influence of dielectric bodies on them [61] or the dynamical Casimir-Polder forces [62]. In addition, the representation of the field quantities as the solution to the Heisenberg equation of motion will be useful to describe the propagation of the electromagnetic field through various kinds of passive optical systems.

In electric dipole approximation, the Heisenberg equations of motion for the dynamical field variables can be derived by using the multipolar Hamiltonian Eq. (3.39) [or Eqs. (3.40), (3.41), (3.43)], recalling Eq. (3.21) and the commutation relations (3.16), (3.17):[3]

$$\begin{aligned}\dot{\hat{\mathbf{f}}}(\mathbf{r},\omega,t) &= \frac{1}{i\hbar}\big[\hat{\mathbf{f}}(\mathbf{r},\omega,t),\hat{H}\big]\\ &= -i\omega\hat{\mathbf{f}}(\mathbf{r},\omega,t)+\sqrt{\frac{1}{\hbar\pi\varepsilon_0}\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)}\,\frac{\omega^2}{c^2}\sum_A \hat{\mathbf{d}}_A(t)\cdot\mathsf{G}^*(\mathbf{r}_A,\mathbf{r},\omega),\end{aligned} \tag{3.103}$$

the formal solution of which reads

$$\hat{\mathbf{f}}(\mathbf{r},\omega,t) = \hat{\mathbf{f}}_{\mathrm{free}}(\mathbf{r},\omega,t)+\hat{\mathbf{f}}_{\mathrm{s}}(\mathbf{r},\omega,t), \tag{3.104}$$

where

$$\hat{\mathbf{f}}_{\mathrm{free}}(\mathbf{r},\omega,t) = e^{-i\omega(t-t')}\hat{\mathbf{f}}_{\mathrm{free}}(\mathbf{r},\omega,t') \tag{3.105}$$

and

$$\begin{aligned}\hat{\mathbf{f}}_{\mathrm{s}}(\mathbf{r},\omega,t) = &\sqrt{\frac{1}{\hbar\pi\varepsilon_0}\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)}\,\frac{\omega^2}{c^2}\\ &\times\sum_A\int \mathrm{d}t'\,\Theta(t-t')\hat{\mathbf{d}}_A(t')\cdot\mathsf{G}^*(\mathbf{r}_A,\mathbf{r},\omega)e^{-i\omega(t-t')}.\end{aligned} \tag{3.106}$$

Substituting Eq. (3.104) together with Eqs. (3.105) and (3.106) into Eq. (3.21), we obtain the free-field $\underline{\hat{\mathbf{E}}}_{\mathrm{free}}(\mathbf{r},\omega,t)$ and the source-field $\underline{\hat{\mathbf{E}}}_{\mathrm{s}}(\mathbf{r},\omega,t)$ parts of the electric field in the frequency domain:

$$\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega,t) = \underline{\hat{\mathbf{E}}}_{\mathrm{free}}(\mathbf{r},\omega,t)+\underline{\hat{\mathbf{E}}}_{\mathrm{s}}(\mathbf{r},\omega,t), \tag{3.107}$$

where

$$\underline{\hat{\mathbf{E}}}_{\mathrm{free}}(\mathbf{r},\omega,t) = i\sqrt{\frac{\hbar}{\pi\varepsilon_0}}\,\frac{\omega^2}{c^2}\int \mathrm{d}^3r'\,\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega)}\,\mathsf{G}(\mathbf{r},\mathbf{r}',\omega)\cdot\hat{\mathbf{f}}_{\mathrm{free}}(\mathbf{r}',\omega), \tag{3.108}$$

[3] For notational convenience, we shall omit below the prime symbol, introduced earlier to identify the multipolar coupling operators.

and

$$\underline{\hat{\mathbf{E}}}_{\mathrm{s}}(\mathbf{r},\omega,t) = \frac{i}{\pi\varepsilon_0}\frac{\omega^2}{c^2}\sum_A \int \mathrm{d}t'\,\Theta(t-t')\hat{\mathbf{d}}_A(t')\cdot\mathrm{Im}\,\mathsf{G}(\mathbf{r}_A,\mathbf{r},\omega)e^{-i\omega(t-t')}. \tag{3.109}$$

Then, using Eqs. (3.19) and (3.20) it is straightforward to find the source-quantity representation of the electric field operator $\hat{\mathbf{E}}(\mathbf{r},t)$. It is worth to note that Eqs. (3.104)–(3.106) can be used also to find the source-quantity representations of the other electromagnetic field operators.

3.2 Leaky Cavities with Unwanted Losses

In this section we investigate the electromagnetic field of a resonator cavity, employing the quantization scheme described in Sec. 3.1 [MK3]. We may assume, that the active atomic are located inside the cavity and apply the multipolar scheme Hamiltonian in the electric dipole approximation to describe the interaction with the medium-assisted electromagnetic field. We shall show, that the source-quantity representation of the electromagnetic field leads to the expansion of the field inside the cavity into a set of nonmonochromatic standing waves—cavity modes. Accordingly, we shall analyze the dynamics of the mode operators and derive the cavity input-output relations.

3.2.1 Nonmonochromatic Modes of the Cavity Field

Let us consider a modified Ley and Loudon's [63] model of a resonator cavity, i.e., a cavity bounded by a perfectly reflecting mirror and a fractionally transparent mirror. For the sake of transparency, we focus on a one-dimensional case and model the cavity by a four-layer planar dielectric structure (Fig. 3.2, cf. Sec. 3.1.4). The layers $j=0$ and $j=2$ are assumed to correspond, respectively, to mirrors which confine the cavity (layer $j=1$). In particular, the layer $j=0$ corresponds to the perfectly reflecting mirror with $r_{10}=-1$, while $j=2$ corresponds to the fractionally transparent mirror responsible for the input-output coupling. As described in Sec. 3.1.4, we use with respect to z, shifted coordinate systems such that $0<z<l$ for $j=1$, $0<z<d$ for $j=2$, and $0<z<\infty$ for $j=3$. We assign a frequency-dependent complex permittivity $\varepsilon_j(\omega)$ to every layer ($j=1,2,3$).

To use the results of Sec. 3.1.4 here for one-dimensional case, we should perform

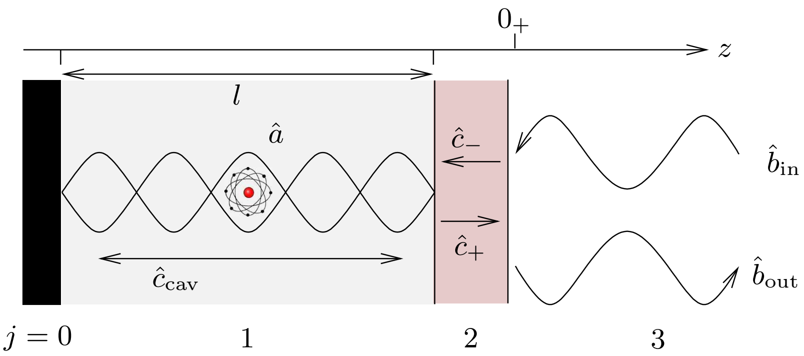

Figure 3.2: Scheme of the cavity. The fractionally transparent mirror [region (2)] is modeled by a dielectric plate. The active sources are in region 1, which can also contain some medium.

the calculations only for the transverse electric component of the field, setting

$$\underline{\hat{\mathbf{E}}}(\mathbf{r},\omega) \longrightarrow \mathcal{A}^{-\frac{1}{2}}\underline{\hat{E}}(z,\omega)\mathbf{e}_s, \tag{3.110}$$

($\mathcal{A}$, mirror area) and keeping only the normally incident waves ($\mathbf{k}=0$). Applying the source-quantity representation (3.107) (one-dimensional case) together with Eqs. (3.108) and (3.109) to the field in the jth layer, we may write for the transverse electric (TE) component of the field (in the Heisenberg picture)

$$\underline{\hat{E}}^{(j)}(z,\omega,t) = \underline{\hat{E}}^{(j)}_{\rm free}(z,\omega,t) + \underline{\hat{E}}^{(j)}_{\rm s}(z,\omega,t), \tag{3.111}$$

with

$$\underline{\hat{E}}^{(j)}_{\rm free}(z,\omega,t) = i\omega\mu_0 \sum_{j'=1}^{3}\int_{[j']} {\rm d}z'\, G^{(jj')}(z,z',\omega)\underline{\hat{j}}^{(j')}_{\rm free}(z',\omega,t) \tag{3.112}$$

and

$$\underline{\hat{E}}^{(j)}_{\rm s}(z,\omega,t) = \frac{i}{\pi\varepsilon_0\mathcal{A}}\frac{\omega^2}{c^2}\sum_A \int {\rm d}t'\,\Theta(t-t')e^{-i\omega(t-t')}\hat{d}_A(t')\,{\rm Im}\,G^{(1j)}(z_A,z,\omega), \tag{3.113}$$

where $[j']$ indicates integration over the j'th layer, and the abbreviating notation

$$\underline{\hat{j}}^{(j)}_{\rm free}(z,\omega,t) = \omega\sqrt{\frac{\hbar\varepsilon_0}{\pi\mathcal{A}}\,{\rm Im}\,\varepsilon_j(\omega)}\,\hat{f}^{(j)}_{\rm free}(z,\omega,t) \tag{3.114}$$

is used.[4]

[4] Here and in the following we omit the index s introduced in Sec. 3.1.4 to distinguish the variables, which refer to TE polarization.

In order to make contact with the familiar standing-wave expansion in the idealized case of a lossless cavity, we evaluate the equations given above for the field inside the cavity ($j=1$) with the aim to obtain a nonmonochromatic mode expansion. To begin, let us first consider the free field in detail. Inserting the Green tensor (B.1) together with Eqs. (B.2) and (B.3) of a one-dimensional planar multilayer structure into Eq. (3.112), $\underline{\hat{E}}^{(1)}_{\rm free}(z,\omega,t)$ can be represented in the form of

$$\begin{aligned}\underline{\hat{E}}^{(1)}_{\rm free}(z,\omega,t) = \frac{1}{D_1}\left[e^{i\beta_1 z} + r_{13}e^{-i\beta_1(z-2l)}\right]\left[\hat{C}^{(1)}_{<+}(z,\omega,t) - \hat{C}^{(1)}_{<-}(z,\omega,t)\right]\\ -\frac{2i\sin(\beta_1 z)}{D_1}\Big\{\hat{C}^{(1)}_{>-}(z,\omega,t) + r_{13}e^{2i\beta_1 l}\hat{C}^{(1)}_{>+}(z,\omega,t)\\ + \frac{t_{21}e^{i\beta_1 l}}{D_2'}\left[\hat{C}^{(2)}_{-}(\omega,t) + r_{23}e^{2i\beta_2 d}\hat{C}^{(2)}_{+}(\omega,t)\right] + t_{31}e^{i\beta_1 l}\hat{C}^{(3)}_{-}(\omega,t)\Big\}\end{aligned} \tag{3.115}$$

($d_1=l,\ d_2=d,\ d_3=0$). Here, $\hat{C}^{(2)}_{\pm}(\omega,t)$ and $\hat{C}^{(3)}_{-}(\omega,t)$ correspond to the intraplate amplitude operator [cf. Eq. (3.73) for $j=2$] and the input amplitude operator at the boundary plane [cf. Eq. (3.69) together with Eq. (3.67) for $n=3$] of the three-dimensional consideration, respectively,

$$\hat{C}^{(2)}_{\pm}(\omega,t) = -\frac{\mu_0 c}{2n_2}\int_{[2]} dz'\, e^{\mp i\beta_2 z'}\underline{\hat{j}}^{(2)}_{\rm free}(z',\omega,t), \tag{3.116}$$

$$\hat{C}^{(3)}_{-}(\omega,t) = -\frac{\mu_0 c}{2n_3}\int_{[3]} dz'\, e^{i\beta_3 z'}\underline{\hat{j}}^{(3)}_{\rm free}(z',\omega,t). \tag{3.117}$$

In Eq. (3.115) D_j is defined by Eq. (3.57), and

$$\beta_j \equiv \beta_j(\omega) = \sqrt{\varepsilon_j(\omega)}\,\frac{\omega}{c} = [n_j'(\omega) + in_j''(\omega)]\,\frac{\omega}{c} = \beta_j' + i\beta_j'' \qquad (\beta_j', \beta_j'' \geq 0) \tag{3.118}$$

according to Eq. (3.58). In the above relations,

$$\hat{C}^{(1)}_{<\pm}(z,\omega,t) = -\frac{\mu_0 c}{2n_1}\int_{[1]} dz'\,\Theta(z-z')e^{\mp i\beta_1 z'}\underline{\hat{j}}^{(1)}_{\rm free}(z',\omega,t), \tag{3.119}$$

$$\hat{C}^{(1)}_{>\pm}(z,\omega,t) = -\frac{\mu_0 c}{2n_1}\int_{[1]} dz'\,\Theta(z'-z)e^{\mp i\beta_1 z'}\underline{\hat{j}}^{(1)}_{\rm free}(z',\omega,t), \tag{3.120}$$

and

$$D_2'(\omega) = 1 - r_{21}r_{23}e^{2i\beta_2 d}. \tag{3.121}$$

Inspection of Eq. (3.115) shows that the function $D_1(\omega)$ characterizes the spectral response of the cavity. In particular, its zeros, which correspond to the singularities of the Green tensor in the lower complex ω half-plane, determine the complex resonance frequencies Ω_k,

$$D_1(\Omega_k) = 1 + r_{13}(\Omega_k) e^{2i\beta_1(\Omega_k)l} = 0. \tag{3.122}$$

Note that in the case when the coupling mirror is not a single plate but a multilayer system, as it is common in practice, then $r_{13}(\omega)$ is the reflection coefficient of the multilayer system and Eq. (3.122) applies as well. Decomposing Ω_k into real and imaginary parts according to

$$\Omega_k = \omega_k - \tfrac{1}{2} i \Gamma_k, \tag{3.123}$$

we can write the formal solution to Eq. (3.122) in the form of

$$\omega_k = \frac{c}{l} \frac{1}{2|n_1|^2} \left\{ n_1' \left[2\pi k + \pi - \tan^{-1}\left(\frac{r_{13}''}{r_{13}'} \right) \right] - n_1'' \ln |r_{13}| \right\} \tag{3.124}$$

and

$$\Gamma_k = \frac{c}{l} \frac{1}{|n_1|^2} \left\{ n_1'' \left[2\pi k + \pi - \tan^{-1}\left(\frac{r_{13}''}{r_{13}'} \right) \right] + n_1' \ln |r_{13}| \right\} \tag{3.125}$$

$[n_1 = n_1(\Omega_k),\ r_{13} = r_{13}' + i r_{13}'' = r_{13}(\Omega_k)]$, from which ω_k and Γ_k may be calculated by iteration, by starting, e.g., with the resonance frequencies of the lossless cavity.

Let $s(t)$ be a function of time whose Fourier transform is given by

$$\underline{s}(\omega) = \frac{S(\omega)}{D_1(\omega)} = \int \mathrm{d}t\, e^{i\omega t} s(t) \tag{3.126}$$

and assume that $S(\omega)$ is analytic in the lower half-plane. Employing the residue theorem, we may write

$$s(t) = \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega t} \frac{S(\omega)}{D_1(\omega)} = \sum_k \frac{c}{2 n_1 l} \Theta(t) e^{-i\Omega_k t} S(\Omega_k). \tag{3.127}$$

Applying Eqs. (3.126) and (3.127) to the c-number functions $[e^{i\beta_1 z} + r_{13} e^{-i\beta_1(z-2l)}] D^{-1}$, $\sin(\beta_1 z) D^{-1}$, and $\sin(\beta_1 z) r_{13} e^{2i\beta_1 l} D^{-1}$ in Eq. (3.115) and disregarding (irrelevant) high-frequency contributions that may arise from poles other than those of $D_1^{-1}(\omega)$, we may rewrite $\underline{\hat{E}}^{(1)}_{\mathrm{free}}(z,\omega,t)$ as

$$\underline{\hat{E}}^{(1)}_{\mathrm{free}}(z,\omega,t) = \sum_k \underline{\hat{E}}^{(1)}_{k\mathrm{free}}(z,\omega,t), \tag{3.128}$$

with

$$\underline{\hat{E}}^{(1)}_{k\mathrm{free}}(z,\omega,t) = i \sqrt{\frac{\mu_0 c \hbar \omega}{\pi \mathcal{A} n_1}} \frac{c}{2 n_1 l} \sin[\beta_1(\Omega_k) z] \times \int \mathrm{d}t'\, e^{-i\Omega_k (t-t')} \Theta(t-t') \Big[T(\omega) \hat{b}_{\mathrm{in}}(\omega,t') + \sum_\lambda A_\lambda(\omega) \hat{c}_\lambda(\omega,t') \Big] \tag{3.129}$$

(λ=cav, +, −), where the operators $\hat{c}_\lambda(\omega,t)$ and $\hat{b}_{\rm in}(\omega,t)$ denote the bosonic amplitude operators for TE waves and can be given according to

$$\hat{c}_{\rm cav}(\omega,t) = -\alpha_{\rm cav}\sqrt{\frac{\pi\mathcal{A}}{\mu_0 c\hbar\omega}}\frac{\mu_0 c}{2n_1}\int_{[1]}{\rm d}z\;\sin(\beta_1 z)\underline{\hat{j}}^{(1)}_{\rm free}(z,\omega,t), \tag{3.130}$$

$$\hat{c}_\pm(\omega,t) = \alpha_\pm\sqrt{\frac{\pi\mathcal{A}}{\mu_0 c\hbar\omega}}\left[e^{i\beta_2 d}\hat{C}^{(2)}_+(\omega,t)\pm\hat{C}^{(2)}_-(\omega,t)\right], \tag{3.131}$$

$$\hat{b}_{\rm in}(\omega,t) = \frac{2|n_3|}{\sqrt{n_3'}}\sqrt{\frac{\pi\mathcal{A}}{\mu_0 c\hbar\omega}}\,\hat{C}^{(3)}_-(\omega,t), \tag{3.132}$$

with

$$\alpha_{\rm cav} = \alpha_{\rm cav}(\omega) = 2\sqrt{2}|n_1|\,[n_1'\sinh(2\beta_1''l) - n_1''\sin(2\beta_1' l)]^{-\frac{1}{2}}, \tag{3.133}$$

$$\alpha_\pm = \alpha_\pm(\omega) = |n_2|e^{\beta_2''d/2}\,[n_2'\sinh(\beta_2''d)\pm n_2''\sin(\beta_2'd)]^{-\frac{1}{2}}, \tag{3.134}$$

$$A_{\rm cav}(\omega) = -4i\,\frac{\sqrt{n_1}}{\alpha_{\rm cav}}, \tag{3.135}$$

$$A_\pm(\omega) = -\frac{t_{21}\sqrt{n_1}}{D_2'\alpha_\pm}\left(r_{23}e^{i\beta_2 d}\pm 1\right)e^{i\beta_1 l}, \tag{3.136}$$

$$T(\omega) = -\frac{t_{31}\sqrt{n_1 n_3'}}{|n_3|}\,e^{i\beta_1 l}. \tag{3.137}$$

It is straightforward to prove that the operators $\hat{c}_\lambda(\omega,t)$ satisfy the Bose commutation relations:

$$\left[\hat{c}_\lambda(\omega,t),\hat{c}^\dagger_{\lambda'}(\omega',t')\right] = \delta_{\lambda\lambda'}\delta(\omega-\omega')e^{-i\omega(t-t')}, \tag{3.138}$$

$$\left[\hat{b}_{\rm in}(\omega,t),\hat{b}^\dagger_{\rm in}(\omega',t')\right] = \delta(\omega-\omega')e^{-i\omega(t-t')}, \tag{3.139}$$

with all other commutators being zero.

To calculate the electric free field

$$\hat{E}^{(1)}_{\rm free}(z,t) = \int_0^\infty {\rm d}\omega\,\underline{\hat{E}}^{(1)}_{\rm free}(z,\omega,t) + {\rm H.c.}, \tag{3.140}$$

we divide the ω axis into intervals $\Delta_k \equiv [\frac{1}{2}(\omega_{k-1}+\omega_k), \frac{1}{2}(\omega_k+\omega_{k+1})]$ and write

$$\hat{E}^{(1)}_{\rm free}(z,t) = \sum_k \hat{E}^{(1)}_{k{\rm free}}(z,t) + {\rm H.c.}, \tag{3.141}$$

where

$$\hat{E}^{(1)}_{k{\rm free}}(z,t) = \int_{\Delta_k}{\rm d}\omega\,\underline{\hat{E}}^{(1)}_{\rm free}(z,\omega,t) \tag{3.142}$$

(recall that the index k is used to numerate the resonances of the cavity). Substitution of Eq. (3.128) together with Eq. (3.129) into Eq. (3.142) yields

$$\hat{E}^{(1)}_{k\mathrm{free}}(z,t) = \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}_{k\mathrm{free}}(z,\omega,t) + \sum_{k'\neq k} \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}_{k'\mathrm{free}}(z,\omega,t). \tag{3.143}$$

In the case when $\Gamma_k \ll \Delta\omega_k$, with

$$\Delta\omega_k = \frac{1}{2}(\omega_{k+1} - \omega_{k-1}), \tag{3.144}$$

the second term in Eq. (3.143) can be regarded as being small compared with the first one and may be omitted in general, leading to the expression

$$\hat{E}^{(1)}_{k\mathrm{free}}(z,t) = \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}_{k\mathrm{free}}(z,\omega,t). \tag{3.145}$$

In this approximation, Eq. (3.141) reduces to

$$\hat{E}^{(1)}_{\mathrm{free}}(z,t) = \sum_k \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}_{k\mathrm{free}}(z,\omega,t). \tag{3.146}$$

Note that within the approximation scheme used, the lower (upper) limit of integration in Eq. (3.146) may be extended to $-\infty$ ($+\infty$).

Further, the source field is given by

$$\hat{E}^{(1)}_{\mathrm{s}}(z,t) = \int_0^\infty \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}_{\mathrm{s}}(z,\omega,t) + \mathrm{H.c.}, \tag{3.147}$$

together with Eq. (3.132). To evaluate the ω-integration we again use the resonance properties of the cavity response function. Performing the Fourier transformation [see Eqs. (3.126), (3.127)] and applying the same approximation that leads from Eq. (3.140) to Eq. (3.146), we obtain

$$\hat{E}^{(1)}_{\mathrm{s}}(z,t) = \sum_k \hat{E}^{(1)}_{k\mathrm{s}}(z,t) + \mathrm{H.c.}, \tag{3.148}$$

where

$$\hat{E}^{(1)}_{k\mathrm{s}}(z,t) = \frac{i\omega_k \sin[\beta_1(\Omega_k)z]}{\varepsilon_0\varepsilon_1(\Omega_k)l\mathcal{A}} \sum_A \int \mathrm{d}t'\, \Theta(t-t') \times e^{-i\Omega_k(t-t')}\hat{d}_A(t') \sin[\beta_1(\Omega_k)z_A] + \mathrm{H.c.}\,. \tag{3.149}$$

Note that in Eq. (3.149) it is assumed that $e^{i\omega_k t}\hat{d}_A(t)$ may be regarded as being an effective slowly varying quantity.

Thus, the full intracavity field, given as the sum of the free (3.146) and the source (3.148) parts,

$$\hat{E}^{(1)}(z,t) = \hat{E}^{(1)}_{\text{free}}(z,t) + \hat{E}^{(1)}_{\text{s}}(z,t), \tag{3.150}$$

yields the desired nonmonochromatic mode expansion. To illustrate this we introduce the operators

$$\hat{c}_{k\lambda}(t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{c}_\lambda(\omega,t), \tag{3.151}$$

$$\hat{b}_{k\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \mathrm{d}\omega\, \hat{b}_{\text{in}}(\omega,t), \tag{3.152}$$

which, on a time scale $\Delta t \gg \Delta\omega_k^{-1}, \Delta\omega_{k'}^{-1}$, obviously obey [recall Eqs. (3.138) and (3.139)] the commutation relations

$$\left[\hat{c}_{k\lambda}(t), \hat{c}^\dagger_{k'\lambda'}(t')\right] = \delta_{kk'}\delta_{\lambda\lambda'}\delta(t-t'), \tag{3.153}$$

$$\left[\hat{b}_{k\text{in}}(t), \hat{b}^\dagger_{k'\text{in}}(t')\right] = \delta_{kk'}\delta(t-t'). \tag{3.154}$$

Further, recalling Eqs. (3.129), (3.146), (3.148), and (3.149), we obtain the nonmonochromatic mode expansion for the intracavity field ($\Gamma_k \ll \Delta\omega_k$),

$$\hat{E}^{(1)}(z,t) = \sum_k E_k(z)\hat{a}_k(t) + \text{H.c.}, \tag{3.155}$$

where the standing wave mode functions are defined as

$$E_k(z) = i\left[\frac{\hbar\omega_k}{\varepsilon_0\varepsilon_1(\omega_k)l\mathcal{A}}\right]^{\frac{1}{2}} \sin[\beta_1(\omega_k)z], \tag{3.156}$$

and

$$\hat{a}_k(t) = \int \mathrm{d}t'\, \Theta(t-t')e^{-i\Omega_k(t-t')} \times \left\{\left[\frac{c}{2n_1(\omega_k)l}\right]^{\frac{1}{2}}\left[T_k\hat{b}_{k\text{in}}(t') + \sum_\lambda A_{k\lambda}\hat{c}_{k\lambda}(t')\right] - \frac{i}{\hbar}\sum_A E_k(z_A)\hat{d}_A(t')\right\} \tag{3.157}$$

$[T_k = T(\omega_k),\ A_{k\lambda} = A_\lambda(\omega_k)]$. From Eq. (3.157) it is not difficult to see that $\hat{a}_k$ obeys the Langevin equation

$$\dot{\hat{a}}_k(t) = -i\left(\omega_k - \tfrac{1}{2}i\Gamma_k\right)\hat{a}_k(t) - \frac{i}{\hbar}\sum_A E_k(z_A)\hat{d}_A(t) + \left[\frac{c}{2n_1(\omega_k)l}\right]^{\frac{1}{2}}\left[T_k\hat{b}_{k\text{in}}(t) + \sum_\lambda A_{k\lambda}\hat{c}_{k\lambda}(t)\right], \tag{3.158}$$

and it can be proved (see App. A.2) that the equal-time commutation relation

$$\left[\hat{a}_k(t), \hat{a}^\dagger_{k'}(t)\right] = \delta_{kk'} \tag{3.159}$$

holds.

The damping rate in the first term on the right-hand side of Eq. (3.158) can be decomposed as follows (see App. B):

$$\Gamma_k = \gamma_{k\mathrm{rad}} + \gamma_{k\mathrm{abs}}\,, \tag{3.160}$$

$$\gamma_{k\mathrm{rad}} = \frac{c}{2|n_1(\omega_k)|l}|T_k|^2, \tag{3.161}$$

$$\gamma_{k\mathrm{abs}} = \sum_\lambda \gamma_{k\lambda} = \frac{c}{2|n_1(\omega_k)|l}\sum_\lambda |A_{k\lambda}|^2. \tag{3.162}$$

Here, $\gamma_{k\mathrm{rad}}$ is the radiative decay rate describing the transmission losses due to the input-output coupling and $\gamma_{k\mathrm{abs}}$ is the (nonradiative) decay rate describing the two kinds of absorption losses: (i) losses inside the cavity[5] (term proportional to $|A_{k\mathrm{cav}}|^2$) (ii) and the losses inside the mirror (terms proportional to $|A_{k\pm}|^2$). Accordingly, the Langevin noise force as given by the third term on the right-hand side of Eq. (3.158) consists of the contributions associated with the losses due to the input-output coupling [term proportional to $T_k\hat{b}_{k\mathrm{in}}(t)$] and the absorption losses inside the cavity [term proportional to $A_{k\mathrm{cav}}\hat{c}_{k\mathrm{cav}}(t)$] and inside the mirror [terms proportional to $A_{k\pm}\hat{c}_{k\pm}(t)$].

At this point it is useful to note that the approximation of $\Gamma_k \ll \Delta\omega_k$ applied in the derivation can be interpreted as the requirement of very small total decay rate of the cavity mode in comparison with resonance frequency line separation Moreover, the field expansion in terms of standing waves is obtained in a coarse-grained approximation, i.e., on a time scale that is large compared with the inverse separation of two neighboring cavity resonance frequencies. In this approximation, bosonic operators associated with the standing waves can be introduced, so that they obey quantum Langevin equations.

It should be pointed out that when $\varepsilon_1(\omega_k)$ can be regarded as being real, then the second term on the right-hand side of Eq. (3.158) is nothing but the familiar commutator term $(i\hbar)^{-1}[\hat{a}_k, \hat{H}_{\mathrm{int}}]$, where

$$\hat{H}_{\mathrm{int}} = -\sum_A\sum_k E_k(z_A)\hat{d}_A\hat{a}_k + \mathrm{H.c.}. \tag{3.163}$$

[5]Note, in a three-dimensional description of the cavity, both absorption and scattering losses give rise to unwanted losses. In this case, the effect of the scattering losses can be thought of as being included in $\gamma_{k\mathrm{abs}}$, effectively increasing the contribution of $\gamma_{k\mathrm{abs}}$ to Γ_k.

Moreover, from Eq. (3.158) together with Eqs. (3.160)–(3.162) it is seen that the effect of absorption losses on the intracavity field may be equivalently described within the framework of Markovian damping theory, with

$$\hat{H}_{\mathrm{sys}\leftrightarrow\mathrm{abs}} = \hbar \sum_{\lambda} \sum_{k} \int_{\Delta_k} \mathrm{d}\omega \left[\frac{c}{2n_1(\omega)l} \right]^{\frac{1}{2}} A_{k\lambda}(\omega) \hat{a}_k^\dagger \hat{c}_{k\lambda}(\omega) + \mathrm{H.c.} \tag{3.164}$$

being the total interaction energy between the cavity modes and the dissipative systems responsible for absorption. Therefore, the result justifies the intuitive concept presented in Sec. 2.2. It is fortunate, that unwanted losses such as absorption losses can be modeled in standard Markovian damping theory by simply complementing the phenomenological Hamiltonian by further bilinear interaction energies of the type (3.164) between the cavity modes and appropriately chosen dissipative channels. In this outline, the radiative losses due to the input-output coupling and the unwanted losses due to absorption can be regarded as representing independent dissipative channels, each giving rise to a damping rate and a corresponding Langevin noise force. Note that Eq. (3.164) implies that each cavity mode is coupled to its own dissipative systems. Needless to say, that in addition to the dissipative effects, considered here, other channels can also be included in the interaction energy. The unwanted losses attributed to the cavity wall, which has been assumed to be perfectly reflecting, is a typical example. As a final note, let us address the following point. Using Eq. (3.157) together with Eqs. (3.151), (3.152) and the commutation relations (3.138) and (3.139), it is not difficult to prove that

$$\left[\hat{a}_k(t), \hat{b}^\dagger_{k'\mathrm{in}}(\omega, t')\right] = \delta_{kk'} \left[\frac{c}{2n_1(\omega_k)l} \right]^{\frac{1}{2}} \frac{T_k}{\sqrt{2\pi}} \frac{ie^{-i\omega(t-t')}}{\omega - \Omega_k}, \tag{3.165}$$

$$\left[\hat{a}_k(t), \hat{c}^\dagger_{k'\lambda}(\omega, t')\right] = \delta_{kk'} \left[\frac{c}{2n_1(\omega_k)l} \right]^{\frac{1}{2}} \frac{A_{k\lambda}}{\sqrt{2\pi}} \frac{ie^{-i\omega(t-t')}}{\omega - \Omega_k}. \tag{3.166}$$

Let us recall, that in quantum noise theories the cavity mode and the input field variables are assumed to commute for all times. Thus, there exist a fundamental difference between the QED approach and the quantum noise theory to the description of a leaky cavity, while in the latter it is *a priori* assumed that the cavity field and the input field belong to two different Hilbert spaces.

3.2.2 Input-Output Relations

In the previous section we have obtained the expansion of the cavity field in terms of nonmonochromatic modes, and the corresponding dynamical equation for the mode operators $\hat{a}_k(t)$ has been derived. To compliment the macroscopic QED analysis of lossy cavities, we turn now to study the field outside the cavity with the aim to discover its relation to the cavity field.

Again, let us first consider the free field. The electric field obtained by inserting the Green tensor as given by Eq. (B.1) into Eq. (3.113) with $j=3$ can be separated into the parts propagating in negative and positive directions along the z axis, representing, respectively, the incoming and the outgoing parts of the field. In particular, the outgoing part reads

$$\hat{\underline{E}}_{\text{out,free}}(z,\omega) = i\sqrt{\frac{\hbar}{\pi\varepsilon_0\mathcal{A}}}\frac{\omega^2}{c^2}\int \mathrm{d}z'\,\sqrt{\operatorname{Im}\varepsilon(z',\omega)}\,G_{\text{out}}(z,z',\omega)\hat{f}_{\text{free}}(z',\omega), \tag{3.167}$$

where the index "out" in $G_{\text{out}}(z,z',\omega)$ indicates the part of the total Green tensor (B.15) that corresponds to the wave propagating in the positive direction along the z axis with respect to the first z variable. Thus, we evaluate the expression on the right-hand side of Eq. (3.167) for $z=0^+$ (cf. Fig. 3.2) using relations (B.15) together with (B.16) to obtain

$$\begin{aligned}\hat{\underline{E}}^{(3)}_{\text{out,free}}(z,\omega,t)\big|_{z=0^+} &= \frac{t_{13}e^{i\beta_1 l}}{D_1}\left[\hat{C}^{(1)}_{<+}(l,\omega,t) - \hat{C}^{(1)}_{<-}(l,\omega,t)\right]\\ &\quad + \frac{t_{23}e^{i\beta_2 d}}{D_2}\left[\hat{C}^{(2)}_{+}(\omega,t) + r_{20}\hat{C}^{(2)}_{-}(\omega,t)\right] + r_{30}\hat{C}^{(3)}_{-}(\omega,t).\end{aligned} \tag{3.168}$$

Then, using Eqs. (B.13), (B.14) for the reflection and transmission coefficients, after straightforward calculations we arrive to the following result:

$$\begin{aligned}\hat{\underline{E}}^{(3)}_{\text{out,free}}(z,\omega,t)\big|_{z=0^+} &= \frac{t_{13}e^{i\beta_1 l}}{D_1}\Bigg\{\left[\hat{C}^{(1)}_{<+}(l,\omega,t) - \hat{C}^{(1)}_{<-}(l,\omega,t)\right]\\ &\quad -\frac{t_{21}e^{i\beta_1 l}}{D_2'}\left[r_{23}e^{2i\beta_2 d}\hat{C}^{(2)}_{+}(\omega,t) + \hat{C}^{(2)}_{-}(\omega,t)\right] - t_{31}e^{i\beta_1 l}\hat{C}^{(3)}_{-}(\omega,t)\Bigg\}\\ &\quad + \frac{t_{23}e^{i\beta_2 d}}{D_2'}\left[\hat{C}^{(2)}_{+}(\omega,t) + r_{21}\hat{C}^{(2)}_{-}(\omega,t)\right] + r_{31}\hat{C}^{(3)}_{-}(\omega,t),\end{aligned} \tag{3.169}$$

where $\hat{C}^{(1)}_{<\pm}(l,\omega,t)$ is defined by Eq. (3.119) (for $z=l$), and $\hat{C}^{(2)}_{\pm}(\omega,t)$ and $\hat{C}^{(3)}_{-}(\omega,t)$ are defined by Eqs. (3.116) and (3.117), respectively. The first term on the right-hand side in Eq. (3.169) is proportional to $D_1^{-1}(\omega)$, that characterizes the spectral response of the cavity. Applying to this term the same procedure as that leading from Eq. (3.115) to Eq. (3.129), we obtain

$$\hat{\underline{E}}^{(3)}_{\text{out,free}}(z,\omega,t)\big|_{z=0^+} = \sum_k \hat{\underline{E}}^{(3)}_{k\text{out,free}}(z,\omega,t)\big|_{z=0^+}, \tag{3.170}$$

where

$$\underline{\hat{E}}^{(3)}_{k\text{out,free}}(z,\omega,t)\big|_{z=0^+} = \frac{1}{2}\sqrt{\frac{\mu_0 c\hbar\omega}{\pi\mathcal{A}}}$$
$$\times\left\{\frac{c}{2n_1(\Omega_k)l}\,\frac{t_{13}}{\sqrt{n_1}}\int \mathrm{d}t'\, e^{-i\Omega_k(t-t')}\Theta(t-t')e^{i\beta_1 l}\left[T(\omega)\hat{b}_{\text{in}}(\omega,t') + \sum_\lambda A_\lambda(\omega)\hat{c}_\lambda(\omega,t')\right]\right.$$
$$\left. + \frac{\sqrt{n_3'}r_{31}}{|n_3|}\,\hat{b}_{k\text{in}}(\omega,t) + \frac{t_{23}}{D_2'}\left[\frac{r_{21}e^{i\beta_2 d}+1}{\alpha_+}\,\hat{c}_{k+}(\omega,t) - \frac{r_{21}e^{i\beta_2 d}-1}{\alpha_-}\,\hat{c}_{k-}(\omega,t)\right]\right\}. \quad (3.171)$$

To give a physical interpretation to this equation, we notice that there are three physically different contributions to the outgoing free field. The first (integral) term is proportional to t_{13} that stands for the transmission coefficient between the layers 1 and 3, i.e., respectively, inside and outside the cavity. As can be seen by comparison with Eq. (3.129), this term represents the fraction of the cavity field transmitted through the mirror. Obviously, the term proportional to $r_{31}\hat{b}_{k\text{in}}(\omega,t)$ represents the reflected part of the incoming field, whereas the terms proportional to $t_{23}\hat{c}_{k\pm}(\omega,t)$ describe the field attributed to the noise sources inside the mirror. To obtain the free-field part of the outgoing electric field in the time domain we integrate Eq. (3.171) with respect to frequency,

$$\hat{E}^{(3)}_{\text{out,free}}(z,t)\big|_{z=0^+} = \sum_k \hat{E}^{(3)}_{k\text{out,free}}(z,t)\big|_{z=0^+} + \text{H.c.}, \quad (3.172)$$

where, for sufficiently high Q value of the cavity, i.e., $\Gamma_k \ll \Delta\omega_k$,

$$\hat{E}^{(3)}_{k\text{out,free}}(z,t)\big|_{z=0^+} = \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(3)}_{k\text{out,free}}(z,\omega,t)\big|_{z=0^+}. \quad (3.173)$$

Starting from Eq. (3.113) with $j=3$, we may rewrite the source-field part of the outgoing electric field to obtain, in close analogy to Eqs. (3.147) and (3.149),

$$\hat{E}^{(3)}_{\text{out,s}}(z,t)\big|_{z=0^+} = \hat{E}^{(3)}_{\text{s}}(z,t)\big|_{z=0^+} = \sum_k \hat{E}^{(3)}_{k,\text{s}}(z,t)\big|_{z=0^+} + \text{H.c.}, \quad (3.174)$$

where

$$\hat{E}^{(3)}_{k\text{s}}(z,t)\big|_{z=0^+} = \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(3)}_{k\text{s}}(z,\omega,t)\big|_{z=0^+} + \text{H.c.} = \frac{\omega_k t_{13}}{2\varepsilon_0\varepsilon_1(\Omega_k)l\mathcal{A}}$$
$$\times \sum_A \int \mathrm{d}t'\,\Theta(t-t')e^{-i\Omega_k(t-t')}e^{i\beta_1(\Omega_k)l}\hat{d}_A(t')\sin[\beta_1(\Omega_k)z_A] + \text{H.c.}. \quad (3.175)$$

Finally, the sum of the free-field part and the source-field part yields the full outgoing field at $z=0^+$,

$$\hat{E}^{(3)}_{\text{out}}(z,t)\big|_{z=0^+} = \hat{E}^{(3)}_{\text{out,free}}(z,t)\big|_{z=0^+} + \hat{E}^{(3)}_{\text{s}}(z,t)\big|_{z=0^+}. \quad (3.176)$$

Further, for simplicity we restrict the considerations to the case of a cavity in free space, i.e., $n_3 \to 1$, and define the operators

$$\hat{b}_{\mathrm{out}}(\omega,t) = 2\sqrt{\frac{\pi\mathcal{A}}{\mu_0 c\hbar\omega}}\,\underline{\hat{E}}^{(3)}_{\mathrm{out}}(z,\omega,t)\big|_{z=0^+}. \tag{3.177}$$

Then, for the corresponding operator in the time domain

$$\hat{b}_{\mathrm{out}}(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\,\hat{b}_{\mathrm{out}}(\omega,t), \tag{3.178}$$

we divide the integration over frequencies in the intervals around the resonances of the cavity, as

$$\hat{b}_{\mathrm{out}}(t) = \sum_k \hat{b}_{k\mathrm{out}}(t), \tag{3.179}$$

with

$$\hat{b}_{k\mathrm{out}}(t) = \frac{1}{\sqrt{2\pi}}\int_{\Delta_k} \mathrm{d}\omega\,\hat{b}_{k\mathrm{out}}(\omega,t), \tag{3.180}$$

where now $\hat{b}_{k\mathrm{out}}(\omega,t)$ is given by

$$\begin{aligned}\hat{b}_{k\mathrm{out}}(\omega,t) &= 2\sqrt{\frac{\pi\mathcal{A}}{\mu_0 c\hbar\omega}}\,\underline{\hat{E}}^{(3)}_{ks}(z,\omega,t)\big|_{z=0^+}\\ &+ \frac{c}{2n_1 l}\,T^{(\mathrm{o})}_k(\omega)\int \mathrm{d}t'\,\Theta(t-t')e^{-i\Omega_k(t-t')}\Big[T_k(\omega)\hat{b}_{k\mathrm{in}}(\omega,t') + \sum_\lambda A_{k\lambda}(\omega)\hat{c}_{k\lambda}(\omega,t')\Big]\\ &+ A^{(\mathrm{o})}_{k+}(\omega)\hat{c}_{k+}(\omega,t) + A^{(\mathrm{o})}_{k-}(\omega)\hat{c}_{k-}(\omega,t) + R^{(\mathrm{o})}_k(\omega)\hat{b}_{k\mathrm{in}}(\omega,t).\end{aligned} \tag{3.181}$$

Here, the functions $A^{(\mathrm{o})}_{k\pm}(\omega)$, $R^{(\mathrm{o})}_k(\omega)$, and $T^{(\mathrm{o})}_k(\omega)$ are defined as follows:

$$A^{(\mathrm{o})}_{k\pm}(\omega) = \frac{t_{23}}{D_2'}\frac{1 \pm r_{21}e^{i\beta_2 d}}{\alpha_\pm}\,, \tag{3.182}$$

$$T^{(\mathrm{o})}_k(\omega) = \frac{t_{13}}{\sqrt{n_1}}\,e^{i\beta_1 l}, \tag{3.183}$$

$$R^{(\mathrm{o})}_k(\omega) = r_{31}. \tag{3.184}$$

To evaluate the ω integration in Eq. (3.180) we recall, that for sufficiently high Q value of the cavity, i.e., $\Gamma_k \ll \Delta\omega_k$, the limits of integration can be extended to $-\infty$ $(+\infty)$. Then, a comparison with Eq. (3.157) reveals that the source term in Eq. (3.181) [cf. Eq. (3.175)] and the second (integral) term in this equation sum up to a term proportional to the cavity-field operator $\hat{a}_k(t)$. Thus, from Eqs. (3.179)–(3.181) it follows that

$$\hat{b}_{k\mathrm{out}}(t) = \left[\frac{c}{2n_1(\omega_k)l}\right]^{\frac{1}{2}} T^{(\mathrm{o})}_k\hat{a}_k(t) + R^{(\mathrm{o})}_k\hat{b}_{k\mathrm{in}}(t) + A^{(\mathrm{o})}_{k+}\hat{c}_{k+}(t) + A^{(\mathrm{o})}_{k-}\hat{c}_{k-}(t) \tag{3.185}$$

$[T_k^{(\mathrm{o})} = T_k^{(\mathrm{o})}(\omega_k),\ A_{k\pm}^{(\mathrm{o})} = A_{k\pm}^{(\mathrm{o})}(\omega_k),\ R_k^{(\mathrm{o})} = R_k^{(\mathrm{o})}(\omega_k)]$. It is shown in App. A.3, that on the time scale considered, i.e., $\Delta t \gg \Delta\omega_k^{-1}, \Delta\omega_{k'}^{-1}$, the commutation relation

$$\left[\hat{b}_{k\mathrm{out}}(t), \hat{b}^{\dagger}_{k'\mathrm{out}}(t')\right] = \delta_{kk'}\delta(t-t') \tag{3.186}$$

holds.

Certain interesting conclusions are immediately apparent from form of the derived input-output relation. Clearly, the third term and the forth term on the right-hand side in Eq. (3.185) result from the absorption losses in the coupling mirror. Omission of these terms would lead to the well-known input-output relation presented in Sec. 2.1 for a leaky cavity whose losses solely result from the wanted radiative input-output coupling (notice that in this case the relation $|R_k^{(\mathrm{o})}(\omega_k)| = 1$ holds). Evidently, an analogous statement holds true also for the intuitive concept to introduce the unwanted losses within QNT discussed in Sec. 2.2. As we have argued in Sec. 3.2.1, the QED approach confirms that indeed the damping of the cavity modes due to unwanted losses can be simply described by introducing into the Hamiltonian an interaction energy of the kind (3.164) and treating its effect in Markov approximation. In contrast, this intuitive concept generally fails with respect to the operator input-output relations. In fact, the above mentioned interaction energies between the cavity modes and the dissipative channels introduced to model the mirror-assisted absorption do not allow inclusion in the theory effects such as the influence of the coupling-mirror-assisted absorption on the outgoing field via the *incoming* field. Therefore, the last two terms in Eq. (3.164) are missing and hence $|R_k^{(\mathrm{o})}| = 1$ is set. As a matter of fact, due to unavoidably occurring unwanted losses, in practice it is always observed that $|R_k^{(\mathrm{o})}| < 1$. Hence, the input-output relations which one would obtain in standard quantum noise theories are incomplete, and, therefore, to include into the consideration the unwanted losses in a general case, more advanced models based on QNT might be needed.

3.3 Generalization of QNT Approach: Replacement Schemes

As we have seen, the input-output relations suggested by the simple intuitive concept within the Markovian damping theory in Sec. 2.2 are incomplete since they do not describe the effect of unwanted losses on the output field which is induced by the

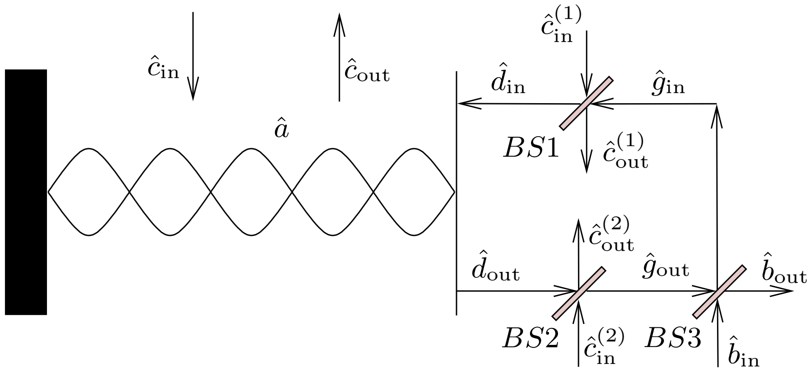

Figure 3.3: Replacement scheme for modeling the unwanted noise in a one-sided cavity. The symmetrical beam splitters BS_1 and BS_2 model the unwanted noise in the coupling mirror, and the asymmetrical beam splitter BS_3 simulates feedback.

reflected input field. The answer to the question of how one should model the effect of the unwanted losses in quantum noise theories is not straightforward. To generalize the quantum noise theory approach with the aim to describe cavity unwanted losses in a complete and consistent way, we make use of the concept of replacement schemes [MK5, MK6]. More precisely, we consider a cavity bounded with a perfectly reflecting mirror and a fractionally transparent mirror, which is responsible for the input-output coupling. Importantly, we assume, that the fractionally transparent mirror does not give rise to the absorption losses. Thus, the Langevin equation for the cavity mode operator reads [cf. Eq. (2.20)]

$$\begin{aligned}\dot{\hat{a}}_k(t) = -i\left[\omega'_k - \tfrac{1}{2}i\frac{c}{2l}\left(|\mathcal{T}^{(c)}_k|^2 + |\mathcal{A}_k|^2\right)\right]\hat{a}_k(t) \\ + \left(\frac{c}{2l}\right)^{\frac{1}{2}}\left[\mathcal{T}^{(c)}_k\hat{d}_{k\rm in}(t) + \mathcal{A}_k\hat{c}_{k\rm in}(t)\right],\end{aligned} \tag{3.187}$$

where for simplicity we have assumed that the cavity tranmission coefficient $\mathcal{T}^{(c)}_k$ is real. In Eq. (3.187) the term proportional to $\hat{d}_{k\rm in}(t)$ denotes the Langevin noise force due to input-output coupling and the term proportional to $\hat{c}_{k\rm in}(t)$ corresponds to the absorption losses inside the cavity. The corresponding input-output relation is given as [cf. Eq. (2.28)]

$$\hat{d}_{k\rm out}(t) = \left(\frac{c}{2l}\right)^{\frac{1}{2}}\mathcal{T}^{(c)}_k\hat{a}_k(t) - \hat{d}_{k\rm in}(t). \tag{3.188}$$

The absorption losses attributed to the coupling mirror can be modeled by an appropriately chosen system of beam splitters in the input and the output channels, as it is shown in Fig. 3.3. Here, BS_1 and BS_2 denote a symmetric beam splitter—a four-port device described by the $SU(2)$ group, where the corresponding input-output relations read

$$\hat{d}_{k\text{in}}(t) = \mathcal{T}_k^{(1)} \hat{g}_{k\text{in}}(t) + \mathcal{R}_k^{(1)} \hat{c}_{k\text{in}}^{(1)}(t), \tag{3.189}$$

$$\hat{g}_{k\text{out}}(t) = \mathcal{T}_k^{(2)} \hat{d}_{k\text{out}}(t) + \mathcal{R}_k^{(2)} \hat{c}_{k\text{in}}^{(2)}(t). \tag{3.190}$$

Further, BS_3 denotes an assymetric $U(2)$ beam splitter with the input-output relations

$$\hat{b}_{k\text{out}}(t) = e^{i\varphi_k^{(3)}} \mathcal{T}_k^{(3)} \hat{g}_{k\text{in}}(t) + e^{i\varphi_k^{(3)}} \mathcal{R}_k^{(3)} \hat{g}_{k\text{in}}(t), \tag{3.191}$$

$$\hat{g}_{k\text{in}}(t) = -\mathcal{R}_k^{(3)*} \hat{g}_{k\text{out}}(t) + \mathcal{T}_k^{(3)*} \hat{b}_{k\text{in}}(t). \tag{3.192}$$

We also assume, that the "input" operators $\hat{b}_{k\text{in}}(t)$, $\hat{c}_{k\text{in}}(t)$ and $\hat{c}_{k\text{in}}^{(1,2)}(t)$ satisfy bosonic commutation rules of the type

$$\left[\hat{o}(t_1), \hat{o}^\dagger(t_2)\right] = \delta(t_1 - t_2). \tag{3.193}$$

Using Eqs. (3.187) and (3.188) and the input-output relations for the beam splitters (3.189)–(3.192), simple algebraic transformations leads to the following equations:

$$\begin{aligned} \dot{\hat{a}}_k(t) &= -i\left(\omega_k - \tfrac{1}{2} i\Gamma_k\right) \hat{a}_k(t) \\ &+ \left(\frac{c}{2l}\right)^{\frac{1}{2}} \left[\mathcal{T}_k \hat{b}_{k\text{in}}(t) + \mathcal{A}_{k(1)} \hat{c}_{k\text{in}}^{(1)}(t) + \mathcal{A}_{k(2)} \hat{c}_{k\text{in}}^{(2)}(t) + \mathcal{A} \hat{c}_{k\text{in}}(t)\right], \end{aligned} \tag{3.194}$$

and

$$\hat{b}_{k\text{out}}(t) = \left(\frac{c}{2l}\right)^{\frac{1}{2}} \mathcal{T}_k^{(\text{o})} \hat{a}_{k\text{cav}}(t) + \mathcal{R}_k^{(\text{o})} \hat{b}_{k\text{in}}(t) + \mathcal{A}_{k(1)}^{(\text{o})} \hat{c}_{k\text{in}}^{(1)}(t) + \mathcal{A}_{k(2)}^{(\text{o})} \hat{c}_{k\text{in}}^{(2)}(t), \tag{3.195}$$

where the cavity total decay rate now reads

$$\Gamma_k = \frac{c}{2l} |\mathcal{T}_k^{(c)}|^2 \left(2 \frac{\operatorname{Re}\Upsilon_k}{|\Upsilon_k|^2} - 1\right) + \frac{c}{2l} |\mathcal{A}_k|^2, \tag{3.196}$$

and

$$\omega_k = \omega_k' + \frac{c}{2l} |\mathcal{T}_k^{(c)}|^2 \frac{\operatorname{Im}\Upsilon_k}{|\Upsilon_k|^2} \tag{3.197}$$

is the shifted resonant frequency of the cavity mode, where

$$\Upsilon_k = 1 - \mathcal{R}_k^{(3)*} \mathcal{T}_k^{(1)} \mathcal{T}_k^{(2)}. \tag{3.198}$$

The other c-number coefficients are defined as follows:

$$\begin{gathered} \mathcal{T}_k = \mathcal{T}_k^{(1)} \mathcal{T}_k^{(3)*} \Upsilon_k^{-1} \mathcal{T}_k^{(c)}, \qquad \mathcal{A}_{k(1)} = \mathcal{R}_k^{(1)} \Upsilon_k^{-1} \mathcal{T}_k^{(c)}, \\ \mathcal{A}_{k(2)} = -\mathcal{T}_k^{(1)} \mathcal{R}_k^{(2)} \mathcal{R}_k^{(3)*} \Upsilon_k^{-1} \mathcal{T}_k^{(c)}, \qquad \mathcal{R}_k^{(\text{o})} = e^{i\varphi^{(3)}} (\mathcal{R}_k^{(3)} - \mathcal{T}_k^{(1)} \mathcal{T}_k^{(2)}) \Upsilon_k^{-1}, \\ \mathcal{A}_{k(1)}^{(\text{o})} = -e^{i\varphi^{(3)}} \mathcal{T}_k^{(2)} \mathcal{R}_k^{(1)} \mathcal{T}_k^{(3)} \Upsilon_k^{-1}, \qquad \mathcal{A}_{k(2)}^{(\text{o})} = e^{i\varphi^{(3)}} \mathcal{R}_k^{(2)} \mathcal{T}_k^{(3)} \Upsilon_k^{-1}, \\ \mathcal{T}_k^{(\text{o})} = \sqrt{\gamma}\, e^{i\varphi^{(3)}} \mathcal{T}_k^{(2)} \mathcal{T}_k^{(3)} \Upsilon_k^{-1}. \end{gathered} \tag{3.199}$$

Note, that the forms of the Langevin equation (3.194) and the input-output relation (3.195) are identical to the Eqs. (3.158) and (3.185), respectively, found in the QED approach. Therefore, we can conclude that the replacement scheme sketched in Fig. 3.3 generates the quantum Langevin equation and the input-output relation, which completely describe the effect of unwanted losses in the coupling mirror.

Apparently, there might be different models of replacement schemes, which would lead to quantum Langevin equations and input-output relations other than those of Eqs. (3.194) and (3.195). In this context, one might ask whether these models are equivalent to each other and whether they completely describe the effect of the unwanted losses. To answer, we begin with very general form of the relations. Namely, one can consider the quantum Langevin equation

$$\dot{\hat{a}}_k = -i(\omega_k - \tfrac{1}{2}i\Gamma_k)\hat{a}_k + \left(\frac{c}{2l}\right)^{\frac{1}{2}} \mathcal{T}_k^{(c)}\hat{b}_{k\mathrm{in}}(t) + \hat{C}_k^{(c)}(t), \tag{3.200}$$

together with the input-output relation

$$\hat{b}_{k\mathrm{out}}(t) = \mathcal{T}_k^{(o)}\hat{a}_k(t) + \mathcal{R}_k^{(o)}\hat{b}_{k\mathrm{in}}(t) + \hat{C}_k^{(o)}(t). \tag{3.201}$$

In these equations, the operators of unwanted noise, $\hat{C}_k^{(c)}(t)$ and $\hat{C}_k^{(o)}(t)$, which commute for $t > 0$ with the input-field operator $\hat{b}_{k\mathrm{in}}(t)$ and the cavity-mode operator at the initial time $t = 0$, $\hat{a}_k(0)$, satisfy the following commutation rules:

$$\left[\hat{C}_k^{(c)}(t_1), \hat{C}_k^{(c)\dagger}(t_2)\right] = \left|\mathcal{A}_k^{(c)}\right|^2 \delta(t_1 - t_2), \tag{3.202}$$

$$\left[\hat{C}_k^{(o)}(t_1), \hat{C}_k^{(o)\dagger}(t_2)\right] = \left|\mathcal{A}_k^{(o)}\right|^2 \delta(t_1 - t_2), \tag{3.203}$$

$$\left[\hat{C}_k^{(c)}(t_1), \hat{C}_k^{(o)\dagger}(t_2)\right] = \Xi_k \delta(t_1 - t_2). \tag{3.204}$$

The set of the coefficients Γ_k, ω_k, $\mathcal{T}_k^{(c)}$, $\mathcal{T}_k^{(o)}$, $\mathcal{R}_k^{(o)}$, $|\mathcal{A}^{(c)}|^2$, $|\mathcal{A}^{(o)}|^2$, and Ξ_k characterizes a cavity with unwanted noise.

Clearly, the above relations (3.200) and (3.201) may correspond to a true physical situation only when the requirement that the cavity-mode operator and the output field operator satisfy the usual commutation relations [cf. Eqs. (2.5) and (2.14), respectively] holds. Imposing this requirement on Eqs. (3.200) and (3.201) yields the following constrains for the coefficients:

$$\Gamma_k = |\mathcal{A}_k^{(c)}|^2 + |\mathcal{T}_k^{(c)}|^2, \tag{3.205}$$

$$|\mathcal{R}_k^{(o)}|^2 + |\mathcal{A}_k^{(o)}|^2 = 1, \tag{3.206}$$

$$\mathcal{T}_k^{(o)} + \mathcal{T}_k^{(c)*}\mathcal{R}_k^{(o)} + \Xi_k = 0. \tag{3.207}$$

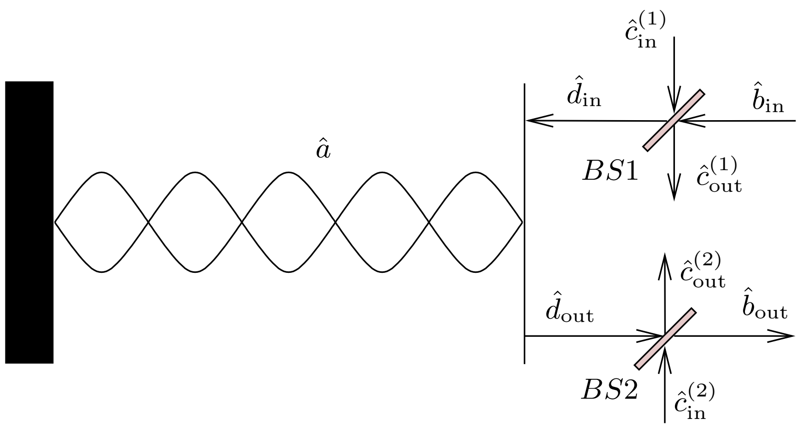

Figure 3.4: An example of a degenerate replacement scheme.

Therefore, the c-number coefficients describing a realistic cavity should belong to the manifold defined by Eqs. (3.205)–(3.207). A replacement scheme that generates the given quantum Langevin equation and the input-output relation represenets, in mathematical sense, a parametrization of the manifold. The paramterization covers the whole manifold in the case when the rank of the Jacobian equals to the dimensionality of the manifold [64]. Otherwise, the model is degenerate since some additional constrains—that do not follow from the requirement of preserving the commutation rules Eqs. (2.5) and (2.14)—hold.

Let us return to the intuitive model suggested in Sec. 2.2. It is easy to check that the quantum Langevin equation and the input-output relation suggested there correspond to the replacement sketched in Fig. 3.3, with excluding the beam spliters BS_1 and BS_2. Obviously, this is an example of the degenerate scheme, since the constrain $|R_k|^2 = 1$ holds, which does not follow from the requirement of preserving the commutation rules Eqs. (2.5) and (2.14).

Yet another example of a degenerate scheme is the replacement scheme sketched in Fig. 3.4. The corresponding quantum Langevin equation and the input-output relation, which are special cases of Eqs. (3.200) and (3.201), read

$$\dot{\hat{a}}_k = -i(\omega_k - \tfrac{1}{2}i\Gamma_k)\hat{a}_k + \left(\frac{c}{2l}\right)^{\frac{1}{2}} \left[\mathcal{T}_k^{(\rm c)}\hat{b}_{k\rm in}(t) + \mathcal{A}_{k(1)}^{(\rm c)}\hat{c}_{k\rm in}^{(1)}(t)\right], \tag{3.208}$$

$$\hat{b}_{k\rm out}(t) = \left(\frac{c}{2l}\right)^{\frac{1}{2}} \mathcal{T}_k^{(\rm o)}\hat{a}_k(t) + \mathcal{R}_k^{(\rm o)}\hat{b}_{k\rm in}(t) + \mathcal{A}_{k(1)}^{(\rm o)}\hat{c}_{k\rm in}^{(1)}(t) + \mathcal{A}_{k(2)}^{(\rm o)}\hat{c}_{k\rm in}^{(2)}(t). \tag{3.209}$$

It is not difficult to prove that for this scheme, the additional constraint

$$\frac{c}{2l}\frac{\mathcal{T}_k^{(\rm o)}\mathcal{T}_k^{(\rm c)}}{\Gamma_k} + \mathcal{R}_k^{(\rm o)} = 0 \tag{3.210}$$

is satisfied, which proves that the replacement scheme is degenerate and does not describe all possible situations.

Let us briefly return to the replacement scheme sketched in Fig. 3.3. The coefficients Γ_k, ω_k, $\mathcal{T}_k$, $\mathcal{A}_{k(1)}$, $\mathcal{A}_{k(2)}$, $\mathcal{A}_k$, $\mathcal{T}_k^{(o)}$, $\mathcal{A}_{k(1)}^{(o)}$, $\mathcal{A}_{k(2)}^{(o)}$ and $\mathcal{R}_k^{(o)}$ represent the set of characteristic parameters. This set represent a manifold with the following constrains, which follow from the commutation relations Eqs. (2.5) and (2.14):

$$\Gamma_k = \frac{c}{2l}\left(|\mathcal{A}_k|^2 + |\mathcal{A}_{k(1)}|^2 + |\mathcal{A}_{k(2)}|^2 + |\mathcal{T}_k|^2\right), \tag{3.211}$$

$$|\mathcal{R}_k^{(o)}|^2 + |\mathcal{A}_{k(1)}^{(o)}|^2 + |\mathcal{A}_{k(2)}^{(o)}|^2 = 1, \tag{3.212}$$

and

$$\mathcal{T}_k^{(o)} + \mathcal{T}_k^{(c)*}\mathcal{R}_k^{(o)} + \mathcal{A}_{k(1)}^{(c)*}\mathcal{A}_{k(1)}^{(o)} + \mathcal{A}_{k(2)}^{(c)*}\mathcal{A}_{k(2)}^{(o)} = 0. \tag{3.213}$$

The relation given by Eqs. (3.196)–(3.199) represent a parameterization of the manifold. It is easy to perform a (numerical) test, which confirms, that indeed the rank of the Jacobian equals to the dimensionality of the manifold, and, therefore, the replacement scheme of Fig. 3.3 is complete and nondegenerate.

3.4 Quantum State of the Outgoing Field

The operator input-output relations allow to calculate, among others, correlation functions of the outgoing field in terms of (generally mixed) correlation functions of the cavity field, the incoming field, and the dissipative channels [41]. In the current section we shall not consider any of the correlation functions, but focus on the quantum state as a whole.

In this context, we would like to note, that Eq. (3.185) as a global input-output relation represents the relation of the electric field operators inside and outside the cavity. However, this relation does not allow one to specify the incoming and outgoing nonmonochromatic modes which, as a matter of fact, connected to the cavity mode in the relevant frequency interval and, hence, carry the quantum state.

3.4.1 Nonmonochromatic Modes of the Outgoing Field

For the sake of transparency we restrict our attention to a single cavity mode and suppose that during the interaction of atomic sources with the cavity the kth cavity mode is prepared in some quantum state. Let us assume that the preparation time is sufficiently short compared with the decay time Γ_k^{-1}, so that the two time scales are well distinguishable. In this case we may assume that at some time t_0 (when the atom effectively leaves the cavity) the cavity mode is prepared in a given quantum state and its evolution in the further course of time (i.e., for times $t \geq t_0$) can be treated as free-field evolution. To specify the relevant modes, it is useful to use Eq. (3.181) and relate therein $\hat{b}_{k\text{out}}(\omega, t)$ to $\hat{a}_k(t_0)$. It can be proved (see App. C) that on the (relevant) time scale $\Delta t \gg \Delta\omega_k^{-1}$, Eq. (3.181) can be rewritten as

$$\hat{b}_{k\text{out}}(\omega, t) = \left[\frac{c}{2n_1(\omega_k)l}\right]^{\frac{1}{2}} T_k^{(\text{o})} \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t+\Delta t} \mathrm{d}t'\, e^{-i\omega(t-t')} \hat{a}_k(t') + R_k^{(\text{o})} \hat{b}_{k\text{in}}(\omega, t_0) e^{-i\omega(t-t_0)}$$
$$+ A_{k+}^{(\text{o})} \hat{c}_{k+}(\omega, t_0) e^{-i\omega(t-t_0)} + A_{k-}^{(\text{o})} \hat{c}_{k-}(\omega, t_0) e^{-i\omega(t-t_0)}. \tag{3.214}$$

Note that integration of both sides of Eq. (3.214) with respect to frequency over the interval Δ_k leads to the input-output relation (3.185) ($n_3 \to 1$). Substituting Eq. (3.157) (for $t \geq t_0$) together with Eqs. (3.151) and (3.152) into Eq. (3.214), we derive

$$\hat{b}_{k\text{out}}(\omega, t) = F_k^*(\omega, t) \hat{a}_k(t_0) + \hat{B}_k(\omega, t), \tag{3.215}$$

where the c-number function $F_k(\omega, t)$ reads[6]

$$F_k(\omega, t) = \frac{i}{\sqrt{2\pi}} \left(\frac{c}{2n_1^* l}\right)^{\frac{1}{2}} T_k^{(\text{o})*} e^{i\omega(t-t_0)} \frac{\exp\left[-i(\omega - \Omega_k^*)(t + \Delta t - t_0)\right] - 1}{\omega - \Omega_k^*}, \tag{3.216}$$

and the operator $\hat{B}_k(\omega, t)$ is a linear functional of the operators $\hat{b}_{k\text{in}}(\omega, t_0)$ and $\hat{c}_{k\lambda}(\omega, t_0)$:

$$\hat{B}_k(\omega, t) = \int_{\Delta_k} \mathrm{d}\omega' \left[G_{k\text{in}}^*(\omega, \omega', t)\, \hat{b}_{k\text{in}}(\omega', t_0) + \sum_\lambda G_{k\lambda}^*(\omega, \omega', t) \hat{c}_{k\lambda}(\omega', t_0)\right]. \tag{3.217}$$

Here,

$$G_{k\text{in}}(\omega, \omega', t) = T_k^{(\text{o})*} T_k^* \upsilon_k(\omega, \omega', t) + R_k^{(\text{o})*} e^{i\omega'(t-t_0)} \delta(\omega - \omega'), \tag{3.218}$$
$$G_{k\text{cav}}(\omega, \omega', t) = T_k^{(\text{o})*} A_{k\text{cav}}^* \upsilon_k(\omega, \omega', t), \tag{3.219}$$
$$G_{k\pm}(\omega, \omega', t) = T_k^{(\text{o})*} A_{k\pm}^* \upsilon_k(\omega, \omega', t) + A_{k\pm}^{(\text{o})*} e^{i\omega'(t-t_0)} \delta(\omega - \omega'), \tag{3.220}$$

with

$$\upsilon_k(\omega, \omega', t) = \frac{1}{2\pi} \frac{c}{2n_1^* l} \frac{e^{-i\omega\Delta t}}{\omega - \omega'}$$
$$\times \left[\frac{e^{i\omega'(t+\Delta t-t_0)} - e^{i\Omega_k^*(t+\Delta t-t_0)}}{\omega' - \Omega_k^*} - \frac{e^{i\omega(t+\Delta t-t_0)} - e^{i\Omega_k^*(t+\Delta t-t_0)}}{\omega - \Omega_k^*}\right]. \tag{3.221}$$

[6]Note that due to the vectorial character of the electromagnetic field a three-dimensional description of the cavity input-output relations in general would be rather complex. The evanescent-field components are necessarily should be taken into account.

Let us introduce a unitary, explicitly time-dependent transformation according to

$$\hat{b}_{k\mathrm{out}}(\omega,t) = \sum_i F_k^{(i)*}(\omega,t)\hat{b}_{k\mathrm{out}}^{(i)}(t), \tag{3.222}$$

$$\hat{b}_{k\mathrm{out}}^{(i)}(t) = \int_{\Delta_k} \mathrm{d}\omega\, F_k^{(i)}(\omega,t)\hat{b}_{k\mathrm{out}}(\omega,t), \tag{3.223}$$

where, for chosen t, the nonmonochromatic mode functions $F_k^{(i)}(\omega,t)$ are a complete set of square integrable orthonormal functions:

$$\sum_i F_k^{(i)}(\omega,t)F_k^{(i)*}(\omega',t) = \delta(\omega-\omega'), \tag{3.224}$$

$$\int_{\Delta_k} \mathrm{d}\omega\, F_k^{(i)}(\omega,t)F_k^{(j)*}(\omega,t) = \delta_{ij}. \tag{3.225}$$

Needless to say that the commutation relation

$$\left[\hat{b}_{k\mathrm{out}}^{(i)}(t), \hat{b}_{k\mathrm{out}}^{(j)\dagger}(t)\right] = \delta_{ij} \tag{3.226}$$

holds.

Let $\hat{b}_{k\mathrm{out}}^{(1)}(t)$ be the operator attributed to the outgoing mode that is associated to the cavity mode. As it clearly follows from the input-output relation (3.215), the nonmonochromatic mode function of the relevant outgoing mode is given by

$$F_k^{(1)}(\omega,t) = \frac{F_k(\omega,t)}{\sqrt{\eta_k(t)}}, \tag{3.227}$$

where the normalization (time-dependent) factor $\eta_k(t)$ reads

$$\eta_k(t) = \int_{\Delta_k} \mathrm{d}\omega\, |F_k(\omega,t)|^2. \tag{3.228}$$

By using Eqs. (3.215), (3.227) and (3.228) we may rewrite Eq. (3.223) as

$$\hat{b}_{k\mathrm{out}}^{(i)}(t) = \begin{cases} \sqrt{\eta_k(t)}\,\hat{a}_k(t_0) + \hat{B}_k^{(i)}(t) & \text{if } i=1, \\ \hat{B}_k^{(i)}(t) & \text{otherwise,} \end{cases} \tag{3.229}$$

where

$$\hat{B}_k^{(i)}(t) = \int_{\Delta_k} \mathrm{d}\omega\, F_k^{(i)}(\omega,t)\hat{B}_k(\omega,t). \tag{3.230}$$

Note, that using Eq. (3.230) together with Eqs. (3.217) and (3.227), from Eqs. (3.165) and (3.166) we find the commutation relation

$$\left[\hat{a}_k(t_0), \hat{B}_k^{(1)\dagger}(t)\right] = 0 \quad (t \geq t_0). \tag{3.231}$$

Inserting Eq. (3.217) in Eq. (3.230), we find

$$\hat{B}_k^{(i)}(t) = \sqrt{\zeta_{k\mathrm{in}}^{(i)}(t)}\,\hat{b}_{k\mathrm{in}}^{(i)}(t) + \sum_\lambda \sqrt{\zeta_{k\lambda}^{(i)}(t)}\,\hat{c}_{k\lambda}^{(i)}(t). \tag{3.232}$$

Here the functions $\zeta_{k\sigma}^{(i)}(t)$ ($\sigma = \mathrm{in}, \lambda$) read

$$\zeta_{k\sigma}^{(i)}(t) = \int_{\Delta_k} \mathrm{d}\omega\, |\chi_{k\sigma}^{(i)}(\omega, t)|^2, \tag{3.233}$$

where

$$\chi_{k\sigma}^{(i)}(\omega, t) = \int_{\Delta_k} \mathrm{d}\omega'\, F_k^{(i)}(\omega', t) G_{k\sigma}^{*}(\omega', \omega, t). \tag{3.234}$$

The operators $\hat{b}_{k\mathrm{in}}^{(i)}(t)$ and $\hat{c}_{k\lambda}^{(i)}(t)$ are defined by

$$\hat{b}_{k\mathrm{in}}^{(i)}(t) = \int_{\Delta_k} \mathrm{d}\omega\, \frac{\chi_{k\mathrm{in}}^{(i)}(\omega, t)}{\sqrt{\zeta_{k\mathrm{in}}^{(i)}(t)}}\, \hat{b}_{k\mathrm{in}}(\omega, t_0), \tag{3.235}$$

$$\hat{c}_{k\lambda}^{(i)}(t) = \int_{\Delta_k} \mathrm{d}\omega\, \frac{\chi_{k\lambda}^{(i)}(\omega, t)}{\sqrt{\zeta_{k\lambda}^{(i)}(t)}}\, \hat{c}_{k\lambda}(\omega, t_0). \tag{3.236}$$

To conclude, starting with the cavity input-output relation (3.181) in the frequency domain, we have derived the input-output relation in terms of the nonmonochromatic modes, Eq. (3.229). The choice of the appropriate orthogonal set of the nonmonochromatic modes of the outgoing field is based on the determination of the relevant outgoing mode. The relevant outgoing mode is the only nonmonochromatic mode in the decomposition that corresponds to the initially prepared cavity mode. As the results of Sec. 3.2.1 establish, the cavity mode operator does not commute with the input field operators in the frequency domain [recall Eq. (3.165)]. Using Eqs. (3.235), (3.234), (3.218) and (3.165), it is straightforward to prove that the commutation relation

$$\left[\hat{a}_k(t_0), \hat{b}_{k\mathrm{in}}^{(1)\dagger}(t)\right] = 0 \quad (t \geq t_0) \tag{3.237}$$

holds [cf. Eq. (3.231)]. Thus, Eq. (3.237) together with Eqs. (3.229) and (3.232) reveals that the relevant outgoing mode is not related to the non-zero commutator of the cavity mode and the incoming field variables, Eq. (3.165).

3.4.2 Phase-Space Functions

To calculate the quantum state of the outgoing field, we start from the multimode characteristic functional

$$C_{\mathrm{out}}[\beta(\omega),t] = \left\langle \exp\left[\int_0^\infty \mathrm{d}\omega\, \beta(\omega)\hat{b}^\dagger_{\mathrm{out}}(\omega,t) - \mathrm{H.c.}\right]\right\rangle, \tag{3.238}$$

i.e., the characteristic functional of the Wigner functional. Dividing again the integration over frequencies in the intervals around the resonance frequencies, the characteristic functional in the frequency interval Δ_k can be presented as

$$C_{k\mathrm{out}}[\beta(\omega),t] = \left\langle \exp\left[\int_{\Delta_k} \mathrm{d}\omega\, \beta(\omega)\hat{b}^\dagger_{k\mathrm{out}}(\omega,t) - \mathrm{H.c.}\right]\right\rangle. \tag{3.239}$$

Introducing the operators $\hat{b}^{(i)}_{k\mathrm{out}}(t)$ according to Eq. (3.222) into the characteristic functional Eq. (3.239) and taking into account the commutation relation (3.226) we find that the operator exponential factorizes as

$$\exp\left[\int_{\Delta_k} \mathrm{d}\omega\, \beta(\omega)\hat{b}^\dagger_{k\mathrm{out}}(\omega,t) - \mathrm{H.c.}\right] = \prod_i \exp\left[\beta^{(i)}_k(t)\hat{b}^{(i)\dagger}_{k\mathrm{out}}(t) - \mathrm{H.c.}\right], \tag{3.240}$$

where

$$\beta^{(i)}_k(t) = \int_{\Delta_k} \mathrm{d}\omega\, F^{(i)}_k(\omega,t)\beta(\omega). \tag{3.241}$$

Let us further consider the case when the nonmonochromatic modes of the incoming field and dissipative channels corresponding to $\hat{B}^{(i)}_k(t)$, $i \neq 1$ are in the vacuum state at the initial time t_0. Then we may assume that the resulting characteristic function factorizes as well, with

$$C_{k\mathrm{out}}[\beta^{(1)}_k(t),t] = \left\langle \exp\left[\beta^{(1)}_k(t)\hat{b}^{(1)\dagger}_{k\mathrm{out}}(t) - \mathrm{H.c.}\right]\right\rangle \tag{3.242}$$

being the characteristic function of the relevant outgoing mode. Using Eqs. (3.229) and (3.231) we may rewrite Eq. (3.242) as

$$C_{k\mathrm{out}}[\beta^{(1)}_k(t),t] = \Big\langle \exp\left[\beta^{(1)}_k(t)\sqrt{\eta_k(t)}\hat{a}^\dagger_k(t_0) - \mathrm{H.c.}\right] \\ \times \exp\left[\beta^{(1)}_k(t)\hat{B}^{(1)\dagger}_k(t) - \mathrm{H.c.}\right]\Big\rangle. \tag{3.243}$$

Noting that in accodance with Eq. (3.232), $\hat{B}^{(1)\dagger}_k(t)$ is a functional of $\hat{b}_{k\mathrm{in}}(\omega,t_0)$ and $\hat{c}_{k\lambda}(\omega,t_0)$ and recalling the assumption that the preparation time of the cavity mode quantum state is sufficiently short comparing with the decay rate, we may assume that the density operator (at the initial time t_0) factorizes with respect to the cavity field, the incoming field and the dissipative channels:

$$C_{k\mathrm{out}}[\beta_k^{(1)}(t),t] = \left\langle \exp\left[\beta_k^{(1)}(t)\sqrt{\eta_k(t)}\hat{a}_k^\dagger(t_0) - \mathrm{H.c.}\right]\right\rangle \times \left\langle \exp\left[\beta_k^{(1)}(t)\hat{B}_k^{(1)\dagger}(t) - \mathrm{H.c.}\right]\right\rangle. \quad (3.244)$$

Inserting Eq. (3.232) into Eq. (3.244), we may express the characteristic function in s order of the quantum state of the relevant outgoing field in terms of the characteristic function of the quantum state of the initially excited cavity mode and the characteristic functions of the quantum states of the incoming field ($\sigma = \mathrm{in}$) and the dissipative channels ($\sigma = \lambda$) as $[\beta \equiv \beta_k^{(1)}(t)]$

$$C_{k\mathrm{out}}(\beta,t;s) = \exp\left[-\tfrac{1}{2}\xi_k(t)|\beta|^2\right] C_k\left[\sqrt{\eta_k(t)}\beta;s'\right] \prod_\sigma C_{k\sigma}\left[\sqrt{\zeta_{k\sigma}(t)}\beta;s_\sigma\right], \quad (3.245)$$

where

$$\xi(t) = \eta_k(t)s' + \sum_\sigma \zeta_\sigma(t)s_\sigma - s. \quad (3.246)$$

From Eq. (3.245) the phase-space function in s order can be then derived to be

$$P_{k\mathrm{out}}(\alpha,t;s) = \frac{2}{\pi}\frac{1}{\xi_k(t)}\int \mathrm{d}^2\alpha'\, P_k(\alpha';s')\prod_\sigma \int \mathrm{d}^2\alpha_\sigma\, P_{k\sigma}(\alpha_\sigma;s_\sigma) \times \exp\left[-\frac{2}{\xi(t)}\left|\sqrt{\eta_k(t)}\alpha' + \sum_\sigma \sqrt{\zeta_{k\sigma}(t)}\alpha_\sigma - \alpha\right|^2\right], \quad (3.247)$$

provided that

$$\xi_k(t) \geq 0, \quad (3.248)$$

where the equality sign must be understood as a limiting process. To calculate $\eta_k(t)$ [Eq. (3.228)] and $\zeta_{k\sigma}(t)$ [Eq. (3.233)], we make use of Eqs. (3.216), (3.218)–(3.221), (3.227), and (3.234). Straightforward calculation yields

$$\eta_k(t) = \frac{\gamma_{k\mathrm{rad}}^{(\mathrm{o})}}{\Gamma_k}\left[1 - e^{-\Gamma_k(t+\Delta t-t_0)}\right], \quad (3.249)$$

$$\zeta_{k\mathrm{in}}(t\to\infty) = \frac{\gamma_{k\mathrm{rad}}^{(\mathrm{o})}\gamma_{k\mathrm{rad}}}{\Gamma_k^2} + |R_k^{(\mathrm{o})}|^2 + \frac{c}{2n_1^* l}\frac{R_k^{(\mathrm{o})}T_k^* T_k^{(\mathrm{o})*}}{\Gamma_k} + \frac{c}{2n_1 l}\frac{R_k^{(\mathrm{o})*}T_k T_k^{(\mathrm{o})}}{\Gamma_k}, \quad (3.250)$$

$$\zeta_{k\pm}(t\to\infty) = \frac{\gamma_{k\mathrm{rad}}^{(\mathrm{o})}\gamma_\pm}{\Gamma_k^2} + |A_{k\pm}^{(\mathrm{o})}|^2 + \frac{c}{2n_1^* l}\frac{A_{k\pm}^{(\mathrm{o})}A_{k\pm}^* T_k^{(\mathrm{o})*}}{\Gamma_k} + \frac{c}{2n_1 l}\frac{A_{k\pm}^{(\mathrm{o})*}A_{k\pm}T_k^{(\mathrm{o})}}{\Gamma_k}, \quad (3.251)$$

$$\zeta_{k\mathrm{cav}}(t\to\infty) = \frac{\gamma_{k\mathrm{rad}}^{(\mathrm{o})}\gamma_{k\mathrm{cav}}}{\Gamma_k^2}, \quad (3.252)$$

where the damping rates $\gamma_{k\mathrm{rad}}$, $\gamma_{k\mathrm{abs}}$, and $\gamma_{k\lambda}$ are defined according to Eqs. (3.161) and Eq. (3.162), and

$$\gamma_{k\mathrm{rad}}^{(\mathrm{o})} = \frac{c}{2|n_1|l}|T_k^{(\mathrm{o})}|^2. \tag{3.253}$$

Note that the functions $\zeta_{k\sigma}(t)$ specify the weights of the quantum states of the input field and the noise sources in the mixed quantum state of the relevant outgoing field. In this context, we notice that the second term on the right-hand side of Eq. (3.250) corresponds to the contribution of the part of the input field which is reflected at the coupling mirror. Further, the third and the fourth terms on the right-hand side of Eqs. (3.250) and (3.251) refer to the input field and noise sources, respectively, which are reflected from the inside of the coupling mirror. The interaction of the input field, the noise sources inside the mirror and inside the cavity with the cavity field give rise to the contributions to the outgoing field described by the first terms on the right-hand side of the Eqs. (3.250), (3.251) and $\zeta_{k\mathrm{cav}}(t)$, respectively.

To conclude, we have used the exact operator input-output relations to explicitly calculate the quantum state of the outgoing field. We have assumed that the preparation process of the quantum state of the cavity field is sufficiently short compared to the decay time of the cavity mode. We have expressed the s-parameterized phase-space function of the quantum state of the relevant outgoing mode in terms of the phase-space functions of the quantum states of the cavity mode, the relevant incoming mode, and the dissipative degrees of freedom responsible for unwanted losses. Most importantly, we note, that the assumption of the factorization of the characteristic functions of the cavity field and the incoming field made above in the derivation of the characteristic function (3.245) is equivalent to the separation of the field Hilbert space into separate Hilbert spaces for the fields inside and outside the cavity. A general description of cavity-assisted radiation field, renouncing this assumption, will be throughly discussed in Chapter 4.

3.4.3 Examples of Quantum State Extraction

Let us consider the typical case of the dissipative channels being in thermal states, i.e.,

$$W_{k\lambda}(\alpha) = \frac{2}{\pi}\frac{1}{1+2\bar{n}_\lambda}\, e^{-2|\alpha|^2/(1+2\bar{n}_\lambda)} \tag{3.254}$$

($\bar{n}_\lambda$, average number of thermal quanta) and calculate the Wigner function of the quantum state of the relevant outgoing mode. Inserting Eq. (3.254) into Eq. (3.247), (with $s_\lambda = 0$), performing the α_λ integrations, and setting $s = s' = s_{\rm in} = 0$, we derive

$$W_{k\rm out}(\alpha, t) = \frac{2}{\pi} \frac{1}{\xi_k^W(t)} \int d^2\alpha' \int d^2\beta \, W_k(\alpha') W_{k\rm in}(\beta) \times \exp\left[-\frac{2|\sqrt{\eta_k(t)}\alpha' + \sqrt{\zeta_{k\rm in}(t)}\beta - \alpha|^2}{\xi_k^W(t)} \right], \tag{3.255}$$

where

$$\xi_k^W(t) = 1 - \eta_k(t) - \zeta_{k\rm in}(t) + 2 \sum_\lambda \bar{n}_\lambda \zeta_{k\lambda}(t). \tag{3.256}$$

Vacuum Noise

When the incoming field is in the vacuum state $[W_{k\rm in}(\beta) = 2\pi^{-1} e^{-2|\beta|^2}]$ and the dissipative channels—in particular, the coupling mirror—are in the vacuum state as well, then the Wigner function of the quantum state of the relevant outgoing mode reads [MK2]

$$W_{k\rm out}(\alpha, t) = \frac{2}{\pi} \frac{1}{1 - \eta_k(t)} \int d^2\alpha' \, W_k(\alpha') \exp\left[-\frac{2|\sqrt{\eta_k(t)}\alpha' - \alpha|^2}{1 - \eta_k(t)} \right]. \tag{3.257}$$

This equation reveals that almost perfect extraction of the quantum state of the cavity mode requires the condition

$$\frac{\eta_k(t)}{1 - \eta_k(t)} \gg 1 \tag{3.258}$$

to be satisfied, i.e., the value of the extraction efficiency $\eta_k(t)$ must be sufficiently close to unity. In other words, recalling Eq. (3.249) together with Eq. (3.160), the nonradiative cavity-field decay rate must be small compared to the radiative one, $\gamma_{k\rm abs}/\gamma_{k\rm rad}^{(o)} \ll 1$—a condition that can hardly be satisfied for high-Q cavities presently available [49, 50]. How small this ratio should be depends on the nonclassical features of the quantum field to be extracted. To illustrate this, let us consider a typical nonclassical state, namely an n-photon Fock state

$$W_k^{(n)}(\alpha) = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} \mathrm{L}_n\big(4|\alpha|^2\big), \tag{3.259}$$

where $\mathrm{L}_n(x)$ is the Laguerre polynomial of order n. Substituting Eq. (3.259) into Eq. (3.257) and employing the integral representation of the Laguerre polynomials [65],

$$\mathrm{L}_n(x) = \frac{1}{2\pi i} \oint_\wp dz \, \frac{e^{-xz/(1-z)}}{z^{n+1}(1-z)}, \tag{3.260}$$

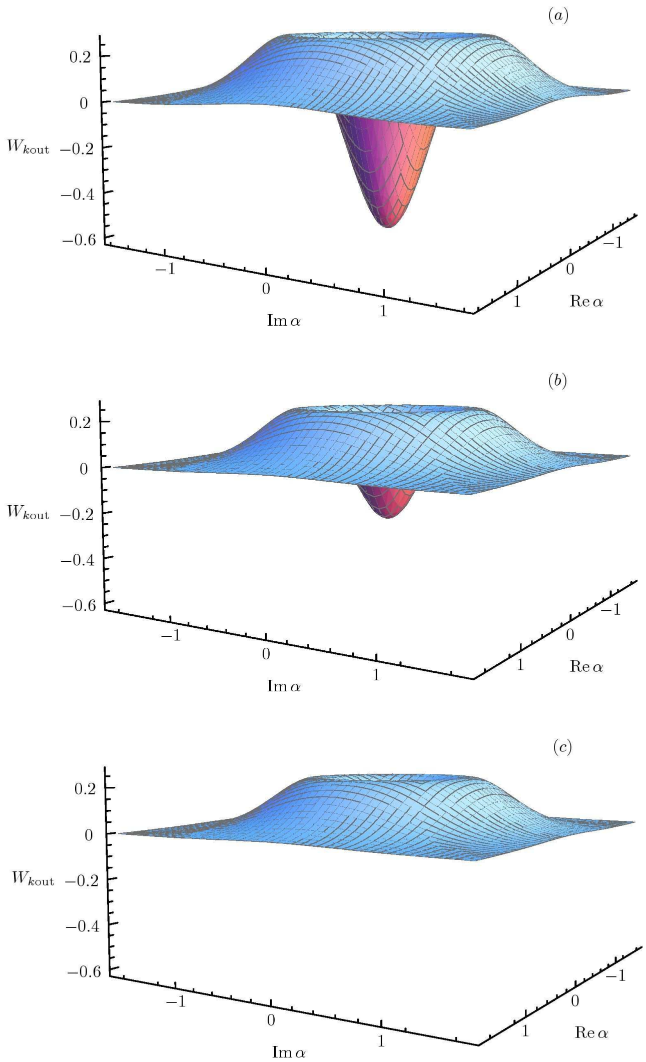

Figure 3.5: Wigner function of the quantum state of the outgoing mode for the case, when the cavity mode is (initially) prepared in a single-photon Fock state. (a) $\eta_k(t) = 0.99$; (b) $\eta_k(t) = 0.71$; (c) $\eta_k(t) = 0.5$.

where the contour $\wp$ encloses the origin but not the point $z = 1$, after straightforward calculations we obtain the Wigner function of the relevant outgoing mode as

$$W^{(n)}_{k\text{out}}(\alpha, t) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2} \left[2\eta_k(t) - 1\right]^n \mathrm{L}_n \left[\frac{4\eta_k(t)}{2\eta_k(t) - 1}|\alpha|^2\right]. \tag{3.261}$$

From Eq. (3.261) it follows that the condition

$$\eta_k(t) > 1 - \frac{1}{2n} \tag{3.262}$$

must be satisfied to guarantee that the n-photon Fock state prevails in the mixed quantum state. In the simplest case of a one-photon Fock state, $n = 1$, the condition reduces to $\eta_k(t) > 0.5$. That is to say, the weight of the one-photon Fock state exceeds the weight of the vacuum state in the mixed state of the outgoing field,

$$W^{(1)}_{k\text{out}}(\alpha, t) = [1 - \eta_k(t)]W^{(0)}(\alpha) + \eta_k(t)W^{(1)}(\alpha), \tag{3.263}$$

only if the extraction efficiency exceeds 50%. The condition (3.262) clearly shows that with increasing value of n the required extraction efficiency rapidly approaches 100%. The dependence on the extraction efficiency of the quantum state of the outgoing field is illustrated in Fig. 3.5 for the case in which a single-photon Fock state is desired to be extracted. Figure 3.5(a) reveals that nearly perfect extraction requires an extraction efficiency that should be not smaller than $\eta_k(t) = 0.99$, which for $t \to \infty$ corresponds to the requirement that $\gamma_{k\text{abs}}/\gamma_{k\text{rad}} \lesssim 0.01$. As long as $\eta_k(t) > 0.5$, the single-photon Fock state is the dominant state in the mixed state, as can be seen from Fig. 3.5(b) [$\eta_k(t) = 0.71$, i.e., $\gamma_{k\text{abs}}/\gamma_{k\text{rad}} = 0.429$ $(t \to \infty)$]. For $\eta_k(t) \leq 0.5$, i.e., $\gamma_{k\text{abs}}/\gamma_{k\text{rad}} \geq 1$ $(t \to \infty)$], the features typical of a single-photon Fock state are lost, Fig. 3.5(c).

Another example of typical nonclassical states are Schrödinger catlike states, e.g.,

$$|\psi\rangle = \mathcal{N}(|\alpha_0\rangle + |-\alpha_0\rangle), \tag{3.264}$$

with α_0 real, and

$$\mathcal{N} = \left[2\left(1 + e^{-4\alpha_0^2}\right)\right]^{-1/2}. \tag{3.265}$$

Let us assume, that the cavity mode is initially prepared in such interference state. Then, the Wigner function reads

$$W_k(\alpha) = \frac{2\mathcal{N}^2}{\pi}\left[e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} + 2e^{-2|\alpha|^2}\cos\left(4\alpha_0 \text{Im}\,\alpha\right)\right]. \tag{3.266}$$

Substitution of Eq. (3.266) into Eq. (3.257) yields the following expression for the Wigner function of the relevant outgoing mode:

$$\begin{aligned} W_{k\text{out}}(\alpha, t) = \frac{2\mathcal{N}^2}{\pi}\Big\{ & e^{-2|\alpha-\sqrt{\eta_k(t)}\alpha_0|^2} + e^{-2|\alpha+\sqrt{\eta_k(t)}\alpha_0|^2} \\ & + 2e^{-2|\alpha|^2}\cos\left[4\sqrt{\eta_k(t)}\alpha_0 \text{Im}\,\alpha\right] e^{-2\alpha_0^2[1-\eta_k(t)]}\Big\}. \end{aligned} \tag{3.267}$$

From Eq. (3.267) it follows that nearly perfect extraction of the state requires the condition

$$1 - \eta_k(t) \ll \frac{1}{2|\alpha_0|^2} \tag{3.268}$$

to be satisfied. Figure 3.6 illustrates the dependence on the extraction efficiency of the quantum state of the outgoing field for a Schrödinger catlike cavity state with

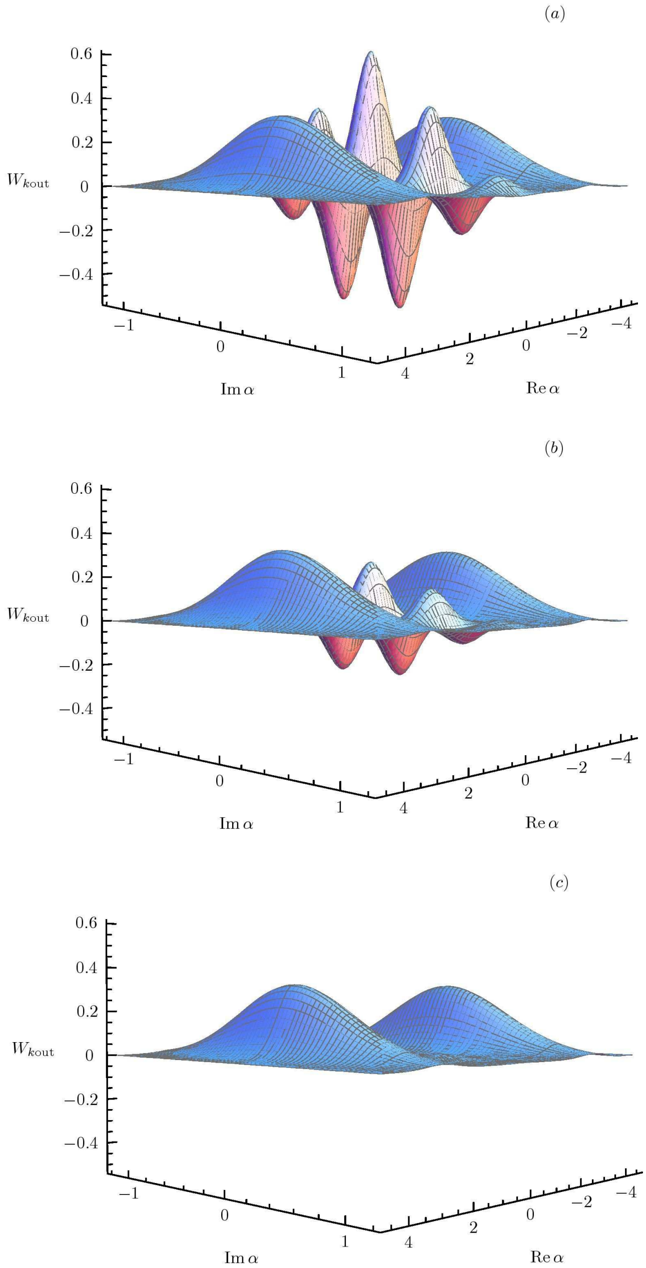

Figure 3.6: Wigner function of the quantum state of the outgoing mode for the case, when the cavity mode is (initially) prepared in a Schrödinger catlike state with $\alpha_0 = 3$. (a) $\eta_k(t) = 0.998$; (b) $\eta_k(t) = 0.952$; (c) $\eta_k(t) = 0.84$.

$\alpha_0 = 3$. Comparing Fig. 3.6 with Fig. 3.5, we see that, as expected, the efficiency for extracting such a Schrödinger catlike state is required to be substantially higher than that for extracting a single-photon Fock state. For a nearly perfect extraction of the chosen Schrödinger catlike state, the efficiency should not be smaller than $\eta_k(t) = 0.998$, i.e., $\gamma_{k\mathrm{abs}}/\gamma_{k\mathrm{rad}} \lesssim 0.002$ for $t \to \infty$ [Fig 3.6(a)]. The nonclassical interference fringes typical of a Schrödinger catlike state can be observed, at least rudimentarily, as long as $\eta_k(t) > 0.84$, i.e., $\gamma_{k\mathrm{abs}}/\gamma_{k\mathrm{rad}} < 0.19$ $(t \to \infty)$ [Fig 3.6(b); $\eta_k(t) = 0.952$, i.e., $\gamma_{k\mathrm{abs}}/\gamma_{k\mathrm{rad}} = 0.05$ $(t \to \infty)$]. For smaller values of the extraction efficiency, the quantum interferences are effectively destroyed [Fig. 3.6(c)].

Role of Thermal Noise

Let us consider Eq. (3.255) together with Eq. (3.256) in the case, when the dissipative channels are thermally excited. At first we assume, that the incoming field mode with the mode function $\chi^{(1)}_{k\mathrm{in}}(\omega, t)$ according to Eq. (3.234) is in the vacuum state. Then, as one can easily see from Eq. (3.256), the condition to ensure nearly perfect extraction of the quantum state of the cavity field is

$$\frac{\eta_k(t)}{1 - \eta_k(t) + 2\sum_\lambda \bar{n}_\lambda \zeta_{k\lambda}(t)} \gg 1\,. \tag{3.269}$$

The condition Eq. (3.269) strengthens even more the requirement of smallness of the nonradiative cavity-field decay rate compared with the radiative one. In addition to the requirement of $\eta_k(t)$ to be close to unity, the value of $\sum_\lambda \bar{n}_\lambda \zeta_{k\lambda}(t)$ should be as small as possible to ensure that the effect of thermal noise effectively does not play a role. This is obviously the case when both $\bar{n}_\lambda$ and $\zeta_{k\lambda}(t)$ are sufficiently small.

As an example, let us suppose that the cavity field is initially prepared in a Schrödinger catlike state $|\psi\rangle = (|\alpha_0\rangle + |-\alpha_0\rangle)/\sqrt{2}$, which in the further course of time is to be extracted from the cavity. A measure of how close the (mixed) quantum state of the outgoing field might be to $|\psi\rangle$ is the extraction fidelity

$$\mathcal{F} = \pi \int \mathrm{d}^2\alpha\, W_{k\mathrm{out}}(\alpha, t) W_k(\alpha, t). \tag{3.270}$$

The result is plotted in Fig. 3.7 for $\alpha_0 = 3$ and mean numbers of thermal photons $\bar{n}_\lambda = \bar{n}_{\lambda'} = 0.02$.The figure clearly reveals that even for small $\bar{n}_\lambda$ the fidelity can noticeably diminish if $\zeta_{k\lambda}(t)$ are not small enough.

Needless to say that small values of $\bar{n}_\lambda$ require sufficiently low temperatures. For cavities with high-quality mirrors [49, 50] with the finesse of several hundred thousands, the second, the third and the forth terms on the right-hand side of Eq. (3.251) are of an order smaller magnitude than the first term on the right-hand side of Eq. (3.251), as well as $\zeta_{k\mathrm{cav}}$, Eq. (3.252), and may be therefore disregarded in the sum $\sum_\lambda \bar{n}_\lambda \zeta_{k\lambda}(t)$. That is to say, in case of unused input port dissipation due to absorption in the coupling mirror can be effectively described by adding appropriate Langevin noise forces in Eq. (3.158).

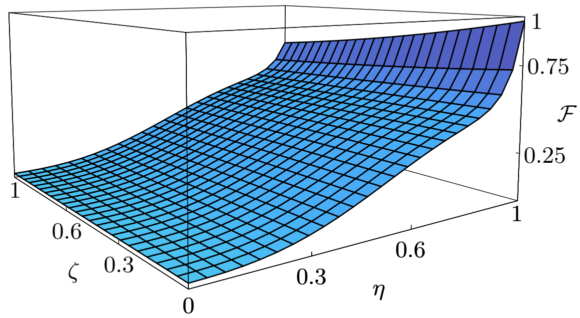

Figure 3.7: Extraction fidelity $\mathcal{F}$ of a Schrödinger catlike state as a function of $\eta = \eta_k(t)$ and $\zeta = \sum_\lambda \zeta_{k\lambda}(t)$ for $\bar{n}_\lambda = \bar{n}_{\lambda'} = 0.02$.

To describe typical problems on engineering of nonclassical states of light [MK4], let us assume, that the dissipative channels are again thermally excited and the incoming field mode with the mode function $\chi^{(1)}_{k\text{in}}(\omega,t)$ according to Eq. (3.234) is prepared in some nonclassical state. Note, that this is the mode of the incoming field, which, together with the cavity mode, contributes to the relevant outgoing mode. Then, the quantum state of the outgoing field mode is the one of the cavity-mode superposed with the reflected incoming field mode as well as the modes of the (thermally excited) dissipative channels. The weights of the modes of the incoming field and the cavity mode field in the resulting superposition are defined respectively by the fractions[7]

$$\frac{\zeta_{k\text{in}}(t)}{1 - \eta_k(t) - \zeta_{k\text{in}}(t) + 2\sum_\lambda \bar{n}_\lambda \zeta_\lambda(t)}\,, \tag{3.271}$$

$$\frac{\eta_k(t)}{1 - \eta_k(t) - \zeta_{k\text{in}}(t) + 2\sum_\lambda \bar{n}_\lambda \zeta_\lambda(t)}\,. \tag{3.272}$$

The additional noise associated with the coupling mirror reduces the fraction of the input field in the resulting superposition, and, therefore, represents the absorption of the incoming field mode in the coupling mirror. For high-Q cavities with the finesse of several hundred thousands [49, 50], the unwanted losses in the coupling mirror reduce the weight of the incoming field mode by about 50%. In this way, the quantum state of the relevant outgoing mode carries additional noise.

The idea of combining the incoming field prepared in the mode $\chi^{(1)}_{k\text{in}}(\omega,t)$ and the cavity mode field in the relevant outgoing mode $F^{(1)}_k(\omega,t)$, can be used to perform the reconstruction of the quantum state of the cavity field [MK5, MK6]. In this case, the

[7] Notice, that dropping the absorption in the coupling mirror, $|R^{(\text{o})}_k| = 1$, and Eq (3.250) reduces to $\zeta_{k\text{in}}(t\to\infty) = [1-\eta_k(t\to\infty)]^2$.

input mode $\chi^{(1)}_{k\mathrm{in}}(\omega,t)$ plays the role of the mode-matched local oscillator field, and the unbalanced homodyne measurement [66] can be realized [MK5]. That is, if the input mode is prepared in a coherent state, measuring the photocounting statistics of the outgoing field the phase-space function of the cavity field can be reconstructed. The scheme can be further extended to determine the parameters of realistic cavities in terms of reflection efficiencies of various modes of the incoming field [MK7].

Notice, to obtain the results presented above, we have assumed, that the non-monochromatic modes of the incoming field and the dissipative channels corresponding to $\hat{B}^{(i)}_k(t)$, $i\neq 1$ [the mode functions $\chi^{(i)}_{k\mathrm{in}}(\omega,t)$, Eq. (3.234), $\sigma=\mathrm{in},\lambda$] are initially prepared in the vacuum state. In practice, this is not necessarily the case, especially with regard to the dissipative channels associated with the coupling mirror, due to the finite number of thermal quanta and the impossibility to prepare the mode of a dissipative channel. As consequence, additional noise is fed into the cavity.

Chapter 4

Leaky High-Q Cavities: Exact QED beyond QNT

As it is shown in the previous chapter, the input-output relation derived for a high-Q cavity within macroscopic QED allows to introduce the relevant outgoing mode, i.e., the nonmonochromatic mode, that is related to the cavity mode. Furthermore, we have expressed the phase-space function of the quantum state of the relevant outgoing mode in terms of the phase-space functions of the quantum states of the cavity mode, the incoming field, and the radiationless dissipative system associated with absorption. In this consideration two major assumptions have been made. First, it is assumed that—in the spirit of the quantum noise theory—the incoming fields and the cavity mode can be regarded as being effectively commuting quantities. Second, the calculations are made explicitely for the case, when the time necessary to prepare the cavity mode in certain quantum state is sufficiently short compared to the decay time of this mode so that the preparation process may be disregarded and instead, an initial condition can be set for the quantum state of the cavity mode. Needless to say that these simplifying assumptions limit the scope of the results in general.

In the ensuing chapter we would like to develop an exact theory based on macroscopic QED in dispersing and absorbing media, with the aim to renounce the approximation that the electromagnetic fields inside and outside a cavity represent independent degrees of freedom [MK9]. To go beyond the regime of short-time preparation we shall include in the theory the preparation process without separating the time scales of the excitation of the field by the (atomic) source and the extraction of field from the cavity.

4.1 Two-Level Atom in a Cavity

Consider a single atom that interacts with the electromagnetic field in the presence of a dielectric medium. The temporal evolution of the system is governed by the multipolar coupling Hamiltonian (3.39). Let us assume that the atom is localized at z_A inside a resonator-like structure and focus on the case when only one atomic transition is involved in the resonant interaction with the medium-assisted electromagnetic field. Then, denoting the bare atomic energy eigenvalues by $E_{1,2}$ and eigenstates by $|1\rangle$ and $|2\rangle$, the atomic Hamiltonian Eq. (3.41) effectively reduces to[1]

$$\hat{H}_A = \sum_n E_n \hat{S}_{nn}, \tag{4.1}$$

and the dipole moment Eq. (3.43) reads

$$\hat{\mathbf{d}} = \sum_{nn'} \mathbf{d}_{nn'} \hat{S}_{nn'}, \tag{4.2}$$

where

$$\hat{S}_{n'n} = |n'\rangle \langle n| \tag{4.3}$$

($\mathbf{d}_{nn'} = \langle n| \hat{\mathbf{d}} |n'\rangle,\ n,\ n' = 1,2$). Further, let us restrict the attention to a one-dimensional cavity in z-direction which is bounded by a perfectly reflecting mirror at the left-hand side and a fractionally transparent mirror at the right-hand side, and assume that electromagnetic field is linearly polarized and propagates along the z axis (Fig. 4.1). The Hamiltonian of the combined system, which consists of the electromagnetic field and the cavity is given by the one-dimensional version of Eq. (3.40). Consequently, the ground state of the combined system $|\{0\}\rangle$ is defined by

$$\hat{f}(z,\omega)|\{0\}\rangle = 0, \qquad \forall z,\, \forall \omega, \tag{4.4}$$

and, therefore, $\hat{f}^\dagger(z,\omega)|\{0\}\rangle$ is a one-excitation quantum state of the combined field-cavity system. Applying the rotating-wave approximation, the multipolar coupling Hamiltonian that governs the temporal evolution of the overall system, which consists of the electromagnetic field, the dielectric medium (including the dissipative degrees of freedom), and the atom coupled to the field, takes the form

$$\hat{H} = \int \mathrm{d}z \int_0^\infty \mathrm{d}\omega\, \hbar\omega \hat{f}^\dagger(z,\omega)\hat{f}(z,\omega) + E_1\hat{S}_{11} + E_2\hat{S}_{22} - \left[d_{21}\hat{S}_{12}^\dagger \hat{E}^{(+)}(z_A) + \mathrm{H.c.}\right], \tag{4.5}$$

[1] Note, for convenience of notation we shall omit the prime sign introduced in Sec. 3.1.3 to identify the multipolar coupling operators.

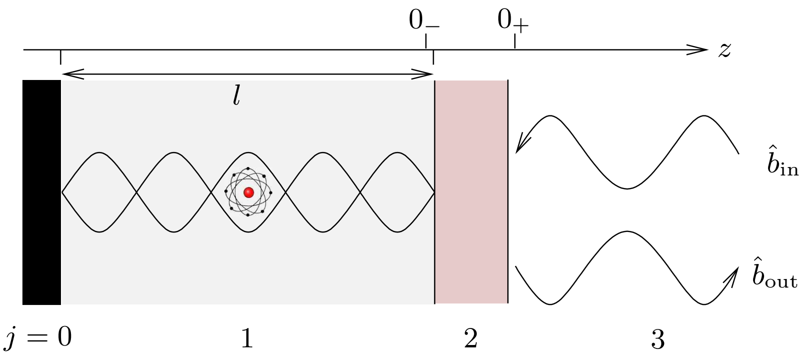

Figure 4.1: Scheme of the cavity. The fractionally transparent mirror of the cavity (region 2) is modeled by a dielectric plate, and the atom inside the cavity (region 1) can be embedded in some dielectric medium.

In what follows we consider, for the sake of transparency, the case when the atom is initially (at time $t=0$) prepared in the upper state $|2\rangle$ and the rest of the system, i.e., the combined system that consists of the electromagnetic field and the cavity, is in the ground state $|\{0\}\rangle$. Clearly, the relevant Hilbert space of the overall system described by the Hamiltonian (4.5) confines just one-excitation states, such as $|2\rangle\,|\{0\}\rangle$ and $|1\rangle\,\hat{f}^\dagger(z,\omega)\,|\{0\}\rangle$. We may therefore expand the state vector of the overall system at a later time t ($t \geq 0$) as

$$|\psi(t)\rangle = C_2(t)e^{-i\omega_{21}t}\,|2\rangle\,|\{0\}\rangle + \int \mathrm{d}z \int_0^\infty \mathrm{d}\omega\, C_1(z,\omega,t)e^{-i\omega t}|1\rangle \hat{f}^\dagger(z,\omega)|\{0\}\rangle, \quad (4.6)$$

where $\omega_{21}=(E_2-E_1)/\hbar$ is the atomic transition frequency, and the phase $\exp(iE_1t/\hbar)$ is introduced in the definition of $|\psi(t)\rangle$. It is not difficult to prove that the Schrödinger equation for $|\psi(t)\rangle$ then leads to the following system of differential equations for the probability amplitudes $C_2(t)$ and $C_1(z,\omega,t)$:

$$\dot{C}_2 = -\frac{d_{21}}{\sqrt{\pi\hbar\varepsilon_0\mathcal{A}}}\int_0^\infty \mathrm{d}\omega \frac{\omega^2}{c^2}\int \mathrm{d}z\, \sqrt{\mathrm{Im}\,\varepsilon(z,\omega)}\, G(z_A,z,\omega)C_1(z,\omega,t)e^{-i(\omega-\omega_{21})t}, \quad (4.7)$$

$$\dot{C}_1(z,\omega,t) = \frac{d_{21}^*}{\sqrt{\pi\hbar\varepsilon_0\mathcal{A}}}\frac{\omega^2}{c^2}\sqrt{\mathrm{Im}\,\varepsilon(z,\omega)}\, G^*(z_A,z,\omega)C_2(t)e^{i(\omega-\omega_{21})t} \quad (4.8)$$

($\mathcal{A}$, mirror area). To eliminate $C_1(z,\omega,t)$ we first formally solve Eq. (4.8) [with the initial condition $C_1(z,\omega,0)=0$],

$$C_1(z,\omega,t) = \frac{d_{21}^*}{\sqrt{\pi\hbar\varepsilon_0\mathcal{A}}}\frac{\omega^2}{c^2}\sqrt{\mathrm{Im}\,\varepsilon(z,\omega)}\, G^*(z_A,z,\omega)\int_0^t \mathrm{d}t'\, C_2(t')e^{i(\omega-\omega_{21})t'}. \quad (4.9)$$

Inserting Eq. (4.9) into Eq. (4.7) we evaluate the z integration therein using the integral relation Eq. (3.14) to obtain the integro-differential equation

$$\dot{C}_2(t) = \int_0^t \mathrm{d}t'\, K(t-t')C_2(t'), \tag{4.10}$$

where the integral kernel $K(t)$ reads

$$K(t) = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \int_0^\infty \mathrm{d}\omega\, \frac{\omega^2}{c^2}\mathrm{Im}\, G(z_A, z_A, \omega)e^{-i(\omega-\omega_{21})t}. \tag{4.11}$$

We now recall that the spectral response of the cavity field is determined by the Green function $G(z, z', \omega)$. As it is thoroughly discussed in Sec. 3.2.1, in the case of a sufficiently high Q value of the cavity the excitation spectrum effectively turns into a quasi-discrete set of lines of mid-frequencies ω_k and widths Γ_k, according to the poles of the Green function at the complex frequencies where the line widths Γ_k are much smaller than the line separations, $\Delta\omega_k$, Eq. (3.144). Therefore, we can again divide the ω axis into intervals $\Delta_k = [\frac{1}{2}(\omega_{k-1}+\omega_k), \frac{1}{2}(\omega_k+\omega_{k+1})]$ and rewrite Eq. (4.11) as

$$K(t) = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \sum_k \int_{\Delta_k} \mathrm{d}\omega\, \frac{\omega^2}{c^2}\mathrm{Im}\, G(z_A, z_A, \omega)e^{-i(\omega-\omega_{21})t}. \tag{4.12}$$

Thus, after inserting Eq. (4.12) into Eq. (4.10), we obtain

$$\dot{C}_2(t) = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \sum_k \int_0^t \mathrm{d}t' \int_{\Delta_k} \mathrm{d}\omega\, \frac{\omega^2}{c^2}\mathrm{Im}\, G(z_A, z_A, \omega)C_2(t')e^{-i(\omega-\omega_{21})(t-t')}. \tag{4.13}$$

To take into account the cavity-induced shift $\delta\omega$ of the atomic transition frequency ω_{21}, we make the ansatz

$$C_2(t) = e^{i\delta\omega t}\tilde{C}_2(t) \tag{4.14}$$

and find from Eq. (4.13)

$$\begin{aligned}\dot{\tilde{C}}_2(t) &= -i\delta\omega\tilde{C}_2(t)\\ &\quad - \frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \sum_k \int_0^t \mathrm{d}t' \int_{\Delta_k} \mathrm{d}\omega\, \frac{\omega^2}{c^2}\mathrm{Im}\, G(z_A, z_A, \omega)\tilde{C}_2(t')e^{-i(\omega-\tilde{\omega}_{21})(t-t')},\end{aligned} \tag{4.15}$$

where

$$\tilde{\omega}_{21} = \omega_{21} - \delta\omega. \tag{4.16}$$

We assume that the atomic transition frequency ω_{21} is in the vicinity of a certain, say the kth, resonance frequency ω_k of the cavity, so that strong atom-field coupling may be realized. In fact, in the off-resonant terms, i.e., all the terms with $k' \neq k$ in the sum in Eq. (4.15), the exponential $\exp[-i(\omega-\tilde{\omega}_{21})(t-t')]$ can be regarded as being rapidly oscillating with respect to time, and the time integrals in these terms can be evaluated in Markov approximation. That is, replacing $\tilde{C}_2(t')$ in the off-resonant terms with $\tilde{C}_2(t)$ and confining ourselves to the times large compared to the inverse separation of the cavity neighboring resonance frequencies, we may identify $\delta\omega$ with

$$\delta\omega = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \sum_{k'\neq k} \int_{\Delta_{k'}} \mathrm{d}\omega\, \frac{\omega^2}{c^2} \frac{\mathrm{Im}\, G(z_A, z_A, \omega)}{\tilde{\omega}_{21}-\omega}. \tag{4.17}$$

Then, from Eq. (4.15) we can see that $\tilde{C}_2(t)$ obeys the integro-differential equation

$$\dot{\tilde{C}}_2(t) = \int_0^t \mathrm{d}t'\, \tilde{K}(t-t')\tilde{C}_2(t'), \tag{4.18}$$

where the kernel function $\tilde{K}(t)$ reads

$$\tilde{K}(t) = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \int_{\Delta_k} \mathrm{d}\omega\, \frac{\omega^2}{c^2} \mathrm{Im}\, G(z_A, z_A, \omega) e^{-i(\omega-\tilde{\omega}_{21})t}. \tag{4.19}$$

In fact, we may use Eq. (4.17) to calculate only the z_A-dependent part of the frequency shift, i.e., the cavity-induced part which arises from the scattering part of the Green tensor [for the decomposition of the Green tensor in bulk and scattering parts, see Eq. (B.19)]. The z_A-independent part of the frequency shift which arises from the bulk part of the Green tensor and which is not related the cavity, would diverge, particularly since the dipole approximation has been made.[2] Since this part can be thought of as being already included in the definition of the transition frequency ω_{21}, we can focus on the z_A-dependent part. Further, the integration in Eq. (4.17) can be approximated by a principal value integration as

$$\delta\omega = -\frac{|d_{21}|^2}{\pi\hbar\varepsilon_0\mathcal{A}} \mathcal{P}\int_0^\infty \mathrm{d}\omega\, \frac{\omega^2}{c^2} \frac{\mathrm{Im}\, G(z_A, z_A, \omega)}{\tilde{\omega}_0-\omega}. \tag{4.20}$$

Then, inserting the scattering part of the Green function as given by Eq. (B.24) in Eq. (4.20), we derive

$$\delta\omega = -\sum_{k'} \frac{\alpha_{k'}}{4|\tilde{\omega}_{21}-\Omega_{k'}|^2} \left[\tilde{\omega}_{21}\omega_{k'} - |\Omega_{k'}|^2 - \frac{\tilde{\omega}_{21}\Gamma_{k'}}{4\pi} \ln\!\left(\frac{\omega_{k'}}{\omega_{21}}\right)\right]. \tag{4.21}$$

Here, Ω_k is the complex resonance frequency, Eq. (3.123), and

$$\alpha_k = \frac{4|d_{21}|^2}{\hbar\varepsilon_0\mathcal{A}|n_1(\Omega_k)|^2 l} \sin^2[\omega_k |n_1(\Omega_k)| z_A/c], \tag{4.22}$$

[2] Note, the translationally invariant part of the Green tensor yields the well-known Lamb shift [4].

where l is the length of the cavity (Fig. 4.1) and $n_1(\omega)$ is the (complex) refractive index of the medium inside the cavity.

To calculate the kernel function $\tilde{K}(t)$, Eq. (4.19), we note that, within the approximation scheme used, the frequency integration can be extended to $\pm\infty$. Employing the Green function with equal argument z_A given by Eq. (B.25), we may exclude the off-resonant terms, which may be regarded as being small comparing to the resonant ones to obtain

$$\tilde{K}(t) = -\tfrac{1}{4}\alpha_k\Omega_k e^{-i(\Omega_k-\tilde{\omega}_{21})t}. \tag{4.23}$$

Having solved Eq. (4.18) and calculated $\tilde{C}_2(t)$, we may eventually calculate $C_1(t)$ according to Eq. (4.9):

$$C_1(z,\omega,t) = \frac{d_{21}^*}{\sqrt{\pi\hbar\varepsilon_0\mathcal{A}}}\frac{\omega^2}{c^2}\sqrt{\mathrm{Im}\,\varepsilon(z,\omega)}\,G^*(z_A,z,\omega)\int_0^t \mathrm{d}t'\,\tilde{C}_2(t')e^{i(\omega-\tilde{\omega}_{21})t'}. \tag{4.24}$$

4.2 Quantum State of the Outgoing Field

In the ensuing section we discuss the properties of the quantum state of the outgoing radiation. Employing the solution of the Schrödinger equation $|\psi(t)\rangle$, Eq. (4.6) directly, we study the multimode phase-space function and obtain inter alia the mode structure of the outgoing field. For the sake of transparency, let us restrict our attention to the case where the cavity is embedded in free space and consider the outgoing field, for example, at the point $z=0^+$ (cf. Fig. 4.1), given by Eq. (3.21). For this purpose, we introduce the bosonic operators [cf. Eqs. (3.85), (3.88), (3.98) and (3.99)]

$$\hat{b}_{\mathrm{out}}(\omega) = 2\sqrt{\frac{\varepsilon_0 c\pi\mathcal{A}}{\hbar\omega}}\;\underline{\hat{E}}_{\mathrm{out}}(z,\omega)\Big|_{z=0^+}, \tag{4.25}$$

where the relation

$$\left[\hat{b}_{\mathrm{out}}(\omega),\hat{b}^\dagger_{\mathrm{out}}(\omega')\right] = \delta(\omega-\omega'). \tag{4.26}$$

holds.

4.2.1 Wigner Function

To calculate the quantum state of the outgoing field, we evaluate the multimode characteristic functional (3.238) now in the Schrödinger picture,

$$C_{\mathrm{out}}[\beta(\omega),t] = \langle\psi(t)|\exp\left[\int_0^\infty \mathrm{d}\omega\,\beta(\omega)\hat{b}^\dagger_{\mathrm{out}}(\omega) - \mathrm{H.c.}\right]|\psi(t)\rangle\,. \tag{4.27}$$

Applying the Baker-Campbell-Hausdorff formula and recalling the commutation relation (4.26), we may rewrite $C_{\rm out}[\beta(\omega),t]$ as

$$C_{\rm out}[\beta(\omega),t]=\exp\left[-\tfrac{1}{2}\int_0^\infty \mathrm{d}\omega\,|\beta(\omega)|^2\right]$$
$$\times\langle\psi(t)|\exp\left[\int_0^\infty \mathrm{d}\omega\,\beta(\omega)\hat{b}^\dagger_{\rm out}(\omega)\right]\exp\left[-\int_0^\infty \mathrm{d}\omega\,\beta^*(\omega)\hat{b}_{\rm out}(\omega)\right]|\psi(t)\rangle. \quad (4.28)$$

To evaluate $C_{\rm out}[\beta(\omega),t]$ for the state $|\psi(t)\rangle$ as given by Eq. (4.6), we first note that from Eq. (4.6) together with the relation (4.4) it follows that

$$\hat{f}(z,\omega)|\psi(t)\rangle=C_1(z,\omega,t)e^{-i\omega t}\,|1\rangle\,|\{0\}\rangle\,. \quad (4.29)$$

Hence, on recalling Eqs. (4.25) and (3.21), it can be seen that

$$\hat{b}_{\rm out}(\omega)|\psi(t)\rangle=F^*(\omega,t)\,|1\rangle\,|\{0\}\rangle\,, \quad (4.30)$$

where

$$F(\omega,t)=-2i\sqrt{\frac{c}{\omega}}\frac{\omega^2}{c^2}\int \mathrm{d}z\,\sqrt{\mathrm{Im}\,\varepsilon(z,\omega)}\,G^*_{\rm out}(0^+,z,\omega)C^*_1(z,\omega,t)e^{i\omega t}, \quad (4.31)$$

with $C_1(z,\omega,t)$ being determined by Eq. (4.24). Then, inserting Eq. (4.30) [together with the corresponding expression for $\hat{b}^\dagger_{\rm out}(\omega)$] into Eq. (4.28), we obtain

$$C_{\rm out}[\beta(\omega),t]=\exp\left[-\tfrac{1}{2}\int_0^\infty \mathrm{d}\omega\,|\beta(\omega)|^2\right]\left[1-\left|\int_0^\infty \mathrm{d}\omega\,\beta(\omega)F(\omega,t)\right|^2\right]. \quad (4.32)$$

To represent $C_{\rm out}[\beta(\omega),t]$ in a more transparent form, we introduce a time-dependent unitary transformation according to

$$\beta(\omega)=\sum_i F^{(i)*}(\omega,t)\beta^{(i)}(t), \quad (4.33)$$
$$\beta^{(i)}(t)=\int_0^\infty \mathrm{d}\omega\,F^{(i)}(\omega,t)\beta(\omega). \quad (4.34)$$

Inserting Eq. (4.33) into Eq. (4.28) and assigning $C_{\rm out}[\beta(\omega),t]\mapsto C_{\rm out}[\beta^{(i)}(t),t]$, we derive

$$C_{\rm out}[\beta^{(i)}(t),t]=\exp\left[-\tfrac{1}{2}\sum_i|\beta^{(i)}(t)|^2\right]$$
$$\times\langle\psi(t)|\exp\left[\sum_i\beta^{(i)}(t)\hat{b}^{(i)\dagger}_{\rm out}(t)\right]\exp\left[-\sum_i\beta^{(i)*}(t)\hat{b}^{(i)}_{\rm out}(t)\right]|\psi(t)\rangle\,, \quad (4.35)$$

where

$$\hat{b}_{\rm out}^{(i)}(t) = \int_0^\infty {\rm d}\omega\, F^{(i)}(\omega,t)\hat{b}_{\rm out}(\omega) \tag{4.36}$$

are the operators associated with nonmonochromatic mode functions $F^{(i)}(\omega,t)$ of the outgoing field, which are not yet specified. Note that

$$\hat{b}_{\rm out}(\omega) = \sum_i F^{(i)*}(\omega,t)\hat{b}_{\rm out}^{(i)}(t). \tag{4.37}$$

Accordingly, we may rewrite Eq. (4.32) as

$$C_{\rm out}[\beta^{(i)}(t),t] = \exp\left[-\tfrac{1}{2}\sum_i |\beta^{(i)}(t)|^2\right] \times \left[1 - \left|\sum_i \beta^{(i)}(t)\int_0^\infty {\rm d}\omega\, F^{(i)}(\omega,t)F(\omega,t)\right|^2\right]. \tag{4.38}$$

We now choose

$$F^{(1)}(\omega,t) = \frac{F(\omega,t)}{\sqrt{\eta(t)}}\,, \tag{4.39}$$

where, within the approximation scheme used,

$$\eta(t) = \int_0^\infty {\rm d}\omega\, |F(\omega,t)|^2 \simeq \int_{-\infty}^\infty {\rm d}\omega\, |F(\omega,t)|^2, \tag{4.40}$$

with $F(\omega,t)$ given by Eq. (4.31). In this way, from Eq. (4.38) we obtain $C_{\rm out}[\beta^{(i)}(t),t]$ in a 'diagonal' form with respect to the nonmonochromatic modes:

$$C_{\rm out}[\beta^{(i)}(t),t] = C_1[\beta^{(1)}(t),t]\prod_{i\neq 1} C_i[\beta^{(i)}(t),t], \tag{4.41}$$

where

$$C_1(\beta,t) = e^{-|\beta|^2/2}\left[1-\eta(t)|\beta|^2\right] \tag{4.42}$$

and

$$C_i(\beta,t) = e^{-|\beta|^2/2} \qquad (i\neq 1). \tag{4.43}$$

Hence, the quantum state of the outgoing field factorizes with respect to the nonmonochromatic modes $F^{(i)}(\omega,t)$.

The Fourier transform of $C_{\rm out}[\beta^{(i)}(t),t]$ with respect to the $\beta^{(i)}(t)$ then yields the desired (multimode) Wigner function $W_{\rm out}(\alpha_i,t)$,

$$W_{\rm out}(\alpha_i,t) = \frac{2}{\pi}\exp\left[-2\sum_i |\alpha_i|^2\right]\left[1-2\eta(t)(1-2|\alpha_1|^2)\right], \tag{4.44}$$

which can be rewritten as

$$W_{\text{out}}(\alpha_i, t) = W_1(\alpha_1, t) \prod_{i \neq 1} W_i^{(0)}(\alpha_i, t), \tag{4.45}$$

where

$$W_1(\alpha, t) = [1 - \eta(t)] W_1^{(0)}(\alpha) + \eta(t) W_1^{(1)}(\alpha), \tag{4.46}$$

with $W_i^{(0)}(\alpha)$ and $W_i^{(1)}(\alpha)$ being, as the notation suggests, the Wigner functions of the vacuum state and the one-photon Fock state, respectively, for the ith nonmonochromatic mode. We see, that the mode labeled by the subscript $i=1$—the excited outgoing mode—is always in a mixed state of a one-photon Fock state and the vacuum state, due to existence of unavoidable unwanted losses. The Wigner function $W_1(\alpha, t)$ reveals that $\eta(t)$ can be regarded as being the efficiency to prepare the excited outgoing mode in a one-photon Fock state. The other nonmonochromatic modes of the outgoing field with $i \neq 1$ are in the vacuum state and, therefore, remain unexcited. Note, that the formulae derived above refer to the case of continuing atom-field interaction. In particular, the efficiency $\eta(t)$ of the excited outgoing mode being prepared in a one-photon Fock state, as given by Eq. (4.40) together with Eq. (4.31), refers to this case. In view of various practical applications, it might be of interest to study also the case of short-term atom-field interaction. Thus, we shall devote the ensuing two sections to the study of features of the excited outgoing mode in two cases of the atom-field interaction: continuing and short-term. Note, that the calculations of Sec. 3.4 are performed in the limiting case of the short-term atom-field interaction, namely, when the interaction time is much smaller that the inverse decay rate of the cavity resonance line. A detailed comparison of the results and, in particular, the relation of the mode functions $F^{(i)}(\omega, t)$ of the outgoing field [cf. Eq. (4.36)] to the mode functions $F_k^{(i)}(\omega, t)$, Eqs. (3.224) and (3.224), introduced in Sec. 3.4 in the context of the outgoing mode relevant to the cavity mode, will be given in Sec. 4.3.2.

4.2.2 Continuing Atom-Field Interaction

To determine the efficiency $\eta(t)$ of the excited outgoing mode being prepared in a one-photon Fock state in the case of the continuing atom-field interaction, Eq. (4.40), it is required first to calculate the probability amplitude $C_1(z, \omega, t)$, which can be obtained from the probability amplitude $\tilde{C}_2(t)$ according to Eq. (4.24). In order to determine the probability amplitude $\tilde{C}_2(t)$ we first substitute Eq. (4.23) into Eq. (4.18) and differentiate both sides of the resulting equation with respect to time. In this way, we derive the following second-order differential equation for $\tilde{C}_2(t)$:

$$\ddot{\tilde{C}}_2 + i(\Omega_k - \tilde{\omega}_{21})\dot{\tilde{C}}_2 + \tfrac{1}{4}\alpha_k\Omega_k\tilde{C}_2(t) = 0, \tag{4.47}$$

where

$$\zeta_k \equiv \rho_k - \tfrac{1}{2}i\gamma_k = \sqrt{(\Omega_k - \tilde{\omega}_{21})^2 + \alpha_k\Omega_k}, \tag{4.48}$$

and α_k is given by Eq. (4.22). The solution to Eq. (4.47) reads

$$\tilde{C}_2(t) = e^{-i(\Omega_k - \tilde{\omega}_{21})t/2}\left[\cos(\zeta_k t/2) + i\frac{\Omega_k - \tilde{\omega}_{21}}{\zeta_k}\sin(\zeta_k t/2)\right]. \tag{4.49}$$

Note that when $\rho_k \gg \frac{1}{2}(\Gamma_k + \gamma_k)$, then damped vacuum Rabi oscillations of the upper-state occupation probability $|\tilde{C}_2(t)|^2 = |C_2(t)|^2$ [recall Eq. (4.14)] are observed (Fig. 4.2), where the vacuum Rabi frequency is given by $R_k = \sqrt{\alpha_k\omega_k}$.

One-Photon Fock State Extraction Efficiency

Substitution of Eq. (4.24) into Eq. (4.31) and use of Eq. (B.18) yields

$$F(\omega,t) = \frac{d_{21}}{\sqrt{\pi\hbar\varepsilon_0\mathcal{A}}}\sqrt{\frac{c}{\omega}}\frac{\omega^2}{c^2}\int_0^t \mathrm{d}t'\, G^*(0^+, z_A, \omega)\tilde{C}_2^*(t')e^{i\omega(t-t')}e^{i\tilde{\omega}_{21}t'}. \tag{4.50}$$

Next, we insert Eq. (4.49) into Eq. (4.50), make use of the Green function as given by Eq. (B.1), and perform the time integration. Omitting off-resonant terms, we obtain

$$F(\omega,t) = \frac{\kappa_k}{2}\frac{1}{(\omega - \tilde{\omega}_{21}/2 - \Omega_k^*/2)^2 - \zeta_k^{*2}/4} \\ \times\left\{e^{i\omega t}\left[1 - e^{-i(\omega-\tilde{\omega}_{21})t}C_2^*(t)\right] - \frac{i\alpha_k^*\Omega_k^*}{2\zeta_k^*}\frac{\sin(\zeta_k^* t/2)}{\omega - \Omega_k^*}e^{i(\tilde{\omega}_{21}+\Omega_k^*)t/2}\right\}, \tag{4.51}$$

where

$$\kappa_k = -\omega_k\frac{d_{21}}{\sqrt{\pi\hbar\varepsilon_0 A}}\sqrt{\frac{c}{\omega_k}}\frac{t_{13}^*(\Omega_k)\,e^{-i\omega_k n_1(\Omega_k)l/c}}{|n_1(\Omega_k)|^2 l}\sin[\omega_k|n_1(\Omega_k)|z_A/c]. \tag{4.52}$$

Combining Eqs. (4.40) and (4.51) and recalling Eq. (4.49), we arrive, after some calculation, at the following expression for the efficiency ($\gamma_k < \Gamma_k$):[3]

$$\eta(t) = \frac{\gamma_{k\mathrm{rad}}\Gamma_k}{\Gamma_k^2 - \gamma_k^2}\frac{R_k^2}{\rho_k^2 + \Gamma_k^2/4}\left[1 - |\tilde{C}_2(t)|^2\right]. \tag{4.53}$$

[3]In what follows we assume that $\gamma_k < \Gamma_k$. Note, that in the opposite case superstrong coupling can be observed (see, e.g., Ref. [67]).

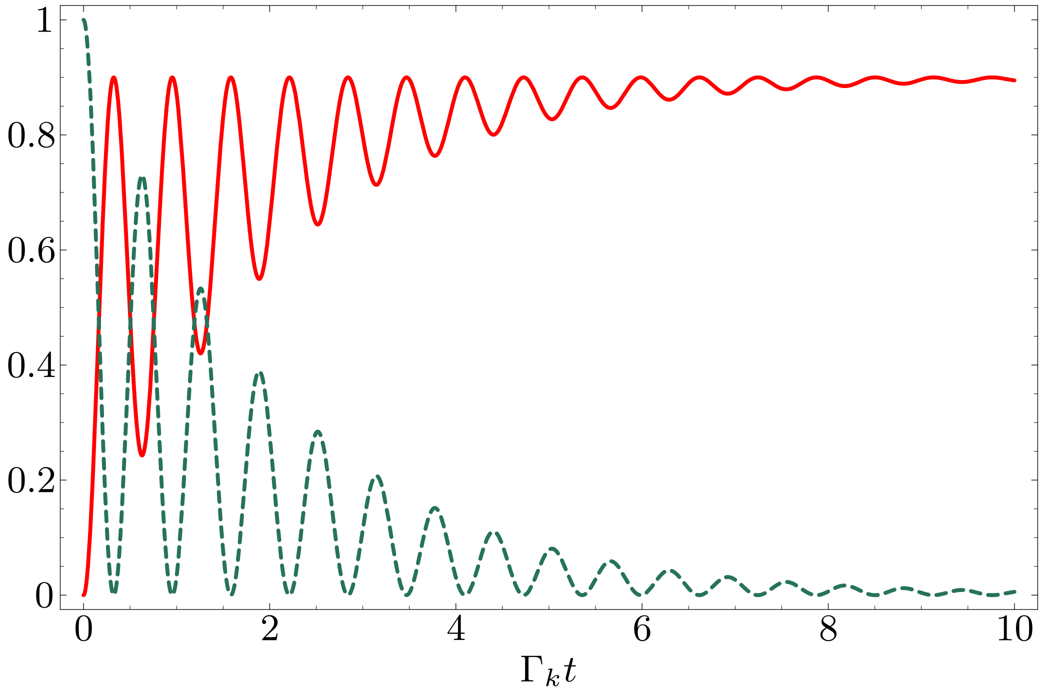

Figure 4.2: The efficiency of one-photon Fock state preparation, $\eta(t)$, Eq. (4.53), (solid curve) and the atomic upper-state occupation probability $|\tilde{C}_2(t)|^2$, Eq. (4.49), (dashed curve) are shown for $\gamma_{k\mathrm{rad}} = 0.9\,\Gamma_k$, $\omega_k - \tilde{\omega}_{21} = 0.1\,\Gamma_k$, $R_k = 10\,\Gamma_k$, $\omega_k = 2 \times 10^8\,\Gamma_k$.

The behavior of the function $\eta(t)$ is illustrated in Fig. 4.2. For comparison, the atomic upper-state occupation probability $|\tilde{C}_2(t)|^2$ is also shown. Figure 4.2 reveals that the efficiency for the excited outgoing mode to be prepared in a one-photon Fock state features Rabi oscillations, where amplitude gradually decreases with time. In particular taking the limit of $t \to \infty$, then $\eta(t) \to [\gamma_{k\mathrm{rad}}\Gamma_k/(\Gamma_k^2 - \gamma_k^2)][R_k^2/(\rho_k^2 + \Gamma_k^2/4)]$, which approximately simplifies to $\eta(t) \to \eta_k(t) = \gamma_{k\mathrm{rad}}/\Gamma_k$, [$\eta_k(t)$ from Eq. (3.249)] for a sufficiently high Q value and almost exact resonance. Recall, that the damping parameter Γ_k, Eq. (3.160), is the sum of the damping rate due to wanted radiative losses $\gamma_{k\mathrm{rad}}$ and the damping rate due to unwanted losses $\gamma_{k\mathrm{abs}}$ [cf. Langevin equation for the cavity mode (3.158) together with Eqs. (3.160)–(3.162)].[4] As it is seen from

[4]In a three-dimensional description of a cavity, both absorption and scattering give rise to unwanted losses. In this case, the efficiency of one-photon Fock state preparation can be expected to be again described by an equation of the type of Eq. (4.53), where the effect of scattering losses may be thought of as being included in $\gamma_{k\mathrm{abs}}$, thereby effectively reducing the contribution of $\gamma_{k\mathrm{rad}}$ to Γ_k as well as the efficiency of one-photon Fock state preparation. Clearly, due to vectorial character of the electromagnetic field in a three-dimensional description, the exact determination of the excited outgoing mode will be rather complex, in general.

Fig. 4.2, the value $\eta(t)=\gamma_{k\mathrm{rad}}/\Gamma_k$ is always observed at the instants when the atom is in the lower state. At first sight, this kind of behavior of the efficiency is unusual and may seem to contradict the result for the extraction efficiency $\eta_k(t)$, given by Eq. (3.249), particularly since $\eta(t)$ has oscillatory and not monotonically increasing behavior. To resolve this issue, let us first briefly consider the field inside the cavity and then study the features of the outgoing field.

Outgoing Field Inside the Cavity

For simplicity, let us restrict our attention to the case where the cavity is not filled with medium ($n_1\equiv 1$) and apply Eq. (3.21) to describe the outgoing part of the field operator, i.e. the part, which travels from the left to the right at position $z=0^-$ inside the cavity (cf. Fig. 4.1). It is straightforward to prove that the operators

$$\hat{\tilde{b}}_{\mathrm{out}}(\omega) = -2i\sqrt{\frac{\varepsilon_0 c\pi\mathcal{A}}{\hbar\omega\Gamma_k\gamma_{k\mathrm{rad}}}}\,\frac{|\omega-\Omega_k|^2}{\omega-\Omega_k^*}t_{31}^*(\Omega_k)e^{-i\omega_k l/c}\,\underline{\hat{E}}_{\mathrm{out}}(z,\omega)\Big|_{z=0_-} \tag{4.54}$$

and $\hat{\tilde{b}}^\dagger_{\mathrm{out}}(\omega)$ satisfy the bosonic commutation relation

$$\big[\hat{\tilde{b}}_{\mathrm{out}}(\omega),\hat{\tilde{b}}^\dagger_{\mathrm{out}}(\omega')\big] = \delta(\omega-\omega'). \tag{4.55}$$

Recalling Eq. (4.29), we derive

$$\hat{\tilde{b}}_{\mathrm{out}}(\omega)|\psi(t)\rangle = \tilde{F}^*(\omega,t)|1\rangle\;|\{0\}\rangle, \tag{4.56}$$

where

$$\tilde{F}(\omega,t) = \sqrt{\frac{\Gamma_k}{\gamma_{k\mathrm{rad}}}}\,F(\omega,t), \tag{4.57}$$

with $F(\omega,t)$ being given by Eq. (4.31). Similarly to Eq. (4.36), we now introduce the unitary transformation

$$\hat{\tilde{b}}^{(i)}_{\mathrm{out}}(t) = \int_0^\infty \mathrm{d}\omega\,\tilde{F}^{(i)}(\omega,t)\hat{\tilde{b}}_{\mathrm{out}}(\omega) \tag{4.58}$$

and make the particular choice of the mode function $\tilde{F}^{(1)}(\omega,t)$, namely

$$\tilde{F}^{(1)}(\omega,t) = \frac{\tilde{F}(\omega,t)}{\sqrt{\tilde{\eta}(t)}}, \tag{4.59}$$

where

$$\tilde{\eta}(t) \equiv \int_0^\infty \mathrm{d}\omega\,|\tilde{F}(\omega,t)|^2 = \frac{\Gamma_k}{\gamma_{k\mathrm{rad}}}\,\eta(t). \tag{4.60}$$

Using Eqs. (4.57) and Eq. (4.60) and recalling Eq. (4.39), from Eq. (4.59) we see that

$$\tilde{F}^{(1)}(\omega,t) = F^{(1)}(\omega,t). \tag{4.61}$$

Performing the calculations leading from Eq. (4.27) to Eq. (4.38) for the outgoing field inside the cavity instead of the outgoing field outside the cavity, we arrive at the characteristic function

$$\begin{aligned}\tilde{C}_{\mathrm{out}}[\beta^{(i)}(t),t] = \exp&\left[-\tfrac{1}{2}\sum_i |\beta^{(i)}(t)|^2\right] \\ &\times\left[1-\left|\sum_i \beta^{(i)}(t)\int_0^\infty \mathrm{d}\omega\, \tilde{F}^{(i)}(\omega,t)\tilde{F}(\omega,t)\right|^2\right],\end{aligned} \tag{4.62}$$

i.e., $F^{(i)}(\omega,t)$ and $F(\omega,t)$ in Eq. (4.38) are simply replaced by $\tilde{F}^{(i)}(\omega,t)$ and $\tilde{F}(\omega,t)$, respectively, where $\tilde{F}(\omega,t)$ is given by Eq. (4.57), and $\tilde{F}^{(i)}(\omega,t)$ for $i\neq 1$ can be chosen to be $F^{(i)}(\omega,t)$.

Thus, we may suppose, that, as follows from Eqs. (4.61) and Eq. (4.62), the excited outgoing mode with the mode function $F^{(1)}(\omega,t)$ describes the outgoing field both inside and outside the cavity. From Eq. (4.58) (with $i=1$) together with Eq. (4.60) one can conclude, that the operators associated with the excited outgoing mode are in general different inside and outside the cavity. Note, in the limit of vanishing absorption, $\gamma_{k\mathrm{rad}} = \Gamma_k$, the operators associated with the excited outgoing mode inside and outside the cavity are equal each other.

Needless to say, that for the operator of the electric field Eq. (3.167) that refers to the points in the region $z<-l$, one finds

$$\underline{\hat{E}}^{(0)}_{\mathrm{out}}(z,\omega)|\psi(t)\rangle = 0. \tag{4.63}$$

As expected, the excited outgoing mode is restricted to the region $z\geq -l$.

Shape of the Excited Outgoing Field

In this section we study in detail the propagation of the excited outgoing field in space and time. Let us consider the outgoing part of the operator of the electric field strength outside the cavity

$$\hat{E}^{(+)}_{\mathrm{out}}(z) = \int_0^\infty \mathrm{d}\omega\, e^{i\omega z/c}\, \underline{\hat{E}}_{\mathrm{out}}(z,\omega)\Big|_{z=0_+} \tag{4.64}$$

($z>0$). On recalling Eq. (4.30), we obtain

$$\hat{E}^{(+)}_{\rm out}(z) = \tfrac{1}{2}\int_0^\infty {\rm d}\omega \sqrt{\frac{\hbar\omega}{\varepsilon_0 c\pi\mathcal{A}}}\, e^{i\omega z/c}\hat{b}_{\rm out}(\omega) \tag{4.65}$$

or equivalently, inserting Eq. (4.37) into Eq. (4.65),

$$\hat{E}^{(+)}_{\rm out}(z) = \sum_i \phi_i^*(z,t)\hat{b}_{{\rm out}\,i}(t), \tag{4.66}$$

where

$$\phi_i(z,t) = \tfrac{1}{2}\int_0^\infty {\rm d}\omega \sqrt{\frac{\hbar\omega}{\varepsilon_0 c\pi\mathcal{A}}}\, e^{-i\omega z/c}F^{(i)}(\omega,t). \tag{4.67}$$

The intensity of the outgoing field at position z is then determined by

$$I(z,t) = \langle\psi(t)|\, \hat{E}^{(-)}_{\rm out}(z)\hat{E}^{(+)}_{\rm out}(z)\,|\psi(t)\rangle \tag{4.68}$$

with $\hat{E}^{(+)}_{\rm out}(z)$ and $\hat{E}^{(-)}_{\rm out}(z) = [\hat{E}^{(+)}_{\rm out}(z)]^\dagger$ from Eq. (4.66). Using Eqs. (4.66), (4.36), (4.30), (4.39), and (4.40), we derive

$$I(z,t) = \eta(t)|\phi_1(z,t)|^2, \tag{4.69}$$

which reveals that $\phi_1(z,t)$ represents the spatio-temporal shape of the outgoing field associated with the excited mode $F^{(1)}(\omega,t)$, and $\eta(t)$ is nothing but the expectation value $\langle\psi(t)|\,\hat{b}^{1\dagger}_{\rm out}(t)\,\hat{b}^{(1)}_{\rm out}(t)\,|\psi(t)\rangle$.

Similarly, performing the calculations leading from Eq. (4.64) to Eq. (4.66) for the outgoing field inside the cavity ($-l\le z<0$),

$$\hat{E}^{(+)}_{\rm out}(z) = \int_0^\infty {\rm d}\omega\, e^{i\omega z/c}\, \underline{\hat{E}}^{(1)}_{\rm out}(z,\omega)\Big|_{z=0_-}, \tag{4.70}$$

instead of the field outside the cavity, we again arrive at an equation of the form of Eq. (4.66), with the same functions $\tilde{\phi}_i(z,t)=\phi_i(z,t)$, but, in general, different associated operators $\tilde{\hat{b}}^{(i)}_{\rm out}(t)\neq\hat{b}^{(i)}_{\rm out}(t)$.

We conclude that $\phi_1(z,t)$ can be regarded as also comprising the part of the excited outgoing wave packet that may be still inside the cavity, i.e., $-l\le z<0$. Hence, assuming that the thickness of the fractionally transparent mirror is small compared with the cavity length, we may write

$$\begin{aligned}\phi_1(z,t) &= \Theta(z+l)\tfrac{1}{2}\int_0^\infty {\rm d}\omega \sqrt{\frac{\hbar\omega}{\varepsilon_0 c\pi\mathcal{A}}}\, e^{-i\omega z/c}F^{(1)}(\omega,t)\\ &\simeq \Theta(z+l)\tfrac{1}{2}\sqrt{\frac{\hbar\omega_k}{\varepsilon_0 c\pi\mathcal{A}}}\int_0^\infty {\rm d}\omega\, e^{-i\omega z/c}F^{(1)}(\omega,t),\end{aligned} \tag{4.71}$$

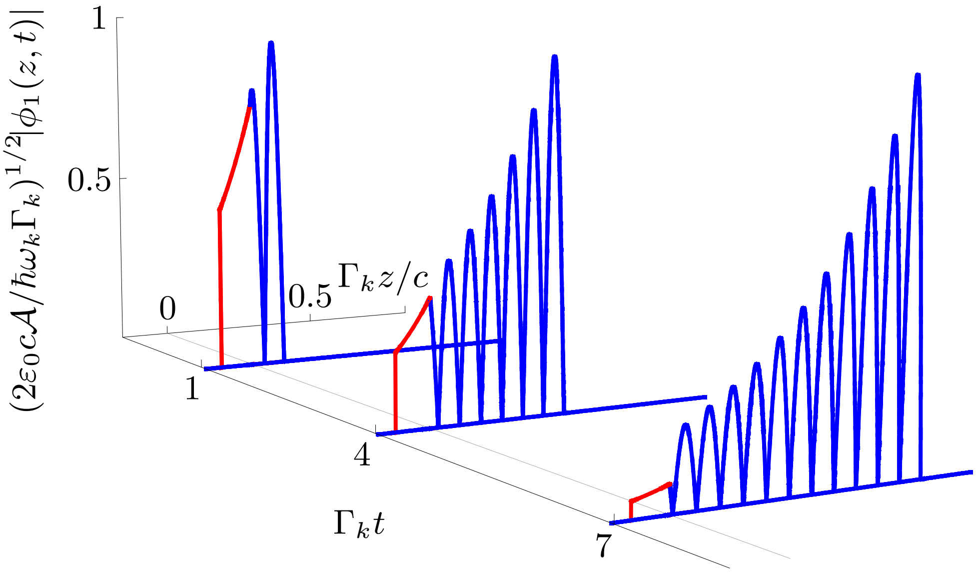

Figure 4.3: The spatio-temporal behavior of the excited outgoing wave packet, $|\phi_1(z,t)|$, Eq. (4.72), in the case of continuing atom-field interaction for $\Gamma_k l/c = 0.7$. The parameters are the same as in Fig. 4.2.

which, by means of Eqs. (4.39) and (4.51), can be evaluated to (approximately) yield

$$\phi_1(z,t) = \Theta(-z)\Theta(z+l)\phi_1^{<}(z,t) - \Theta(z)\Theta(ct-z)\phi_1^{>}(z,t), \tag{4.72}$$

where

$$\phi_1^{>}(z,t) = \sqrt{\frac{\pi\hbar\omega_k}{\varepsilon_0 c\mathcal{A}\eta(t)}}\frac{\kappa_k}{\zeta_k^*}e^{i(\tilde{\omega}_{21}+\Omega_k^*)(t-z/c)/2}\sin[\zeta_k^*(t-z/c)/2] \tag{4.73}$$

and

$$\phi_1^{<}(z,t) = \sqrt{\frac{\pi\hbar\omega_k}{\varepsilon_0 c\mathcal{A}\eta(t)}}\frac{\kappa_k}{\zeta_k^*}e^{i\Omega_k^*(t-z/c)}e^{i(\tilde{\omega}_{21}-\Omega_k^*)t/2}\sin(\zeta_k^* t/2). \tag{4.74}$$

The behavior of the absolute value of $\phi_1(z,t)$ as a function of t and z is illustrated in Fig. 4.3. Comparison with Fig. 4.2 reveals that $|\phi_1^{>}(z,t)|$ oscillates according to the Rabi frequency of the atom-field interaction.

It should be emphasized that in Eq. (4.41) the argument $\beta^{(1)}(t)$ of the characteristic function $C_1[\beta^{(1)}(t),t]$ of the quantum state of the excited outgoing mode $F^{(1)}(\omega,t)$ refers to the wave packet $\phi_1(z,t)$ as a whole, i.e., it does not refer only to its part outside the cavity but also to its part inside the cavity [cf. Eq. (4.62)]. Thus, for not too long times $\Gamma_k t \lessapprox 1$, the corresponding wave packet covers the areas both inside and outside the cavity so that a photon emitted by the atom belongs simultaneously to the two areas. Therefore, this photon can be reabsorbed by the atom leading to the oscillations of the efficiency $\eta(t)$ (cf. Fig. 4.2). In the course of time, the wave packet corresponding to the excited outgoing mode (almost) completely leaves the cavity, and, thus, the oscillations of $\eta(t)$ decrease. It is worth to note, that in the description of the cavity-assisted electromagnetic field in Sec. 3.4, based on the assumption of separate Hilbert spaces for the fields inside and outside the cavity, it is, in particular, not possible to consider (excited) outgoing modes that simultaneously belong to the cavity and the outside world.

4.2.3 Short-Term Atom-Field Interaction

Let us now consider the case where the interaction of the atom with the cavity has a limited duration, so that it terminates at some finite time τ. To analyze this case, we obviously have to split the time evolution of the system into two distinct time intervals: one interval from the time $t=0$ to the time τ, and the second interval for times t greater than τ. For times t in the interval $0 \leq t \leq \tau$, the time evolution is still described as it has been analyzed in Sec. 4.2.2, and the state vector $|\psi(t)\rangle$ is again given by Eq. (4.6). For times $t \geq \tau$, when the interaction of the atom with the medium-assisted electromagnetic field is set to zero, the solution of the Schrödinger equation reads

$$|\psi(t)\rangle = e^{-i(t-\tau)\hat{H}_0/\hbar}|\psi(\tau)\rangle. \tag{4.75}$$

Here, $\hat{H}_0$ is the Hamiltonian of the uncoupled system, i.e., the sum of the first three terms in Eq. (4.5), and $|\psi(\tau)\rangle$ is given by $|\psi(t)\rangle$ from Eq. (4.6) for $t=\tau$. Hence, Eq. (4.75) can be written as ($t \geq \tau$)

$$|\psi(t)\rangle = C_2(\tau)e^{-i\omega_{21}t}|2\rangle|\{0\}\rangle + \int \mathrm{d}z \int_0^\infty \mathrm{d}\omega\, C_1(z,\omega,\tau)e^{-i\omega t}|1\rangle \hat{f}^\dagger(z,\omega)|\{0\}\rangle. \tag{4.76}$$

Note that the condition $\rho_k\tau \gg 1$ is required in order to observe damped vacuum Rabi oscillations.

One-Photon Fock State Extraction Efficiency

To calculate the quantum state of the outgoing field in the case of short-term atom-field interaction, we insert Eq. (4.76) in Eq. (4.27) and use Eq. (4.25) together with Eq. (3.21). In this way, we again arrive at Eq. (4.38), but now with

$$F(\omega,t,\tau) = -2i\sqrt{\frac{c}{\omega}\frac{\omega^2}{c^2}} \int \mathrm{d}z'\, \sqrt{\mathrm{Im}\,\varepsilon(z',\omega)}\, G^*_{\mathrm{out}}(0^+,z',\omega)C^*_1(z',\omega,\tau)e^{i\omega t} \tag{4.77}$$

in place of $F(\omega, t)$. Comparing Eq. (4.77) with Eq. (4.31), we easily see that

$$F(\omega, t, \tau) = e^{i\omega(t-\tau)} F(\omega, \tau). \tag{4.78}$$

Choosing $[F^{(i)}(\omega, t) \mapsto F^{(i)}(\omega, t, \tau)]$

$$F^{(1)}(\omega, t, \tau) = \frac{F(\omega, t, \tau)}{\sqrt{\eta(t, \tau)}}, \tag{4.79}$$

we are again left with an equation of the form of Eq. (4.41) [together with Eqs. (4.42) and (4.43)], where, according to Eq. (4.35) $[\eta(t) \mapsto \eta(t, \tau)]$, the efficiency of preparation of the excited outgoing mode in a one-photon Fock state now reads

$$\eta(t, \tau) = \int_0^\infty \mathrm{d}\omega \, |F(\omega, t, \tau)|^2 = \eta(\tau). \tag{4.80}$$

Hence, we conclude, that the efficiency of the excited outgoing mode being prepared in a one-photon Fock state is constant for all times $t \geq \tau$. It is simply given by the value of the efficiency observed at time $t = \tau$ in the case of continuing atom-field interaction.

Shape of the Excited Outgoing Field

A calculation in line with that leading from Eq. (4.64) to Eq. (4.45) now yields the following expression for the spatio-temporal shape of the excited outgoing mode $[\phi(z, t) \mapsto \phi(z, t, \tau)]$:

$$\begin{aligned} \phi_1(z, t, \tau) = {} & \Theta(ct - c\tau - z)\Theta(z + l)\phi_1^<(z, t, \tau) \\ & + \Theta(z - ct + c\tau)\Theta(ct - z)\phi_1^>(z, t, \tau), \end{aligned} \tag{4.81}$$

where

$$\phi_1^>(z, t, \tau) = \sqrt{\frac{\pi\hbar\omega_k}{\varepsilon_0 c \mathcal{A} \eta(t)}} \frac{\kappa_k}{\zeta_k^*} e^{i(\tilde{\omega}_{21} + \Omega_k^*)(t - z/c)/2} \sin[\zeta_k^*(t - z/c)/2] \tag{4.82}$$

and

$$\phi_1^<(z, t, \tau) = \sqrt{\frac{\pi\hbar\omega_k}{\varepsilon_0 c \mathcal{A} \eta(t)}} \frac{\kappa_k}{\zeta_k^*} e^{i\Omega_k^*(t - z/c)} e^{i(\tilde{\omega}_{21} - \Omega_k^*)\tau/2} \sin(\zeta_k^* \tau/2). \tag{4.83}$$

The behavior of the absolute value of $\phi_1(z, t, \tau)$ is illustrated in Fig. 4.4, where the interaction time τ is chosen in such a way that it is the time at which the atom appears the first time in the lower state (half of Rabi cycle, cf. the upper-state occupation probability shown in Fig. 4.2). As we can see, the excited outgoing field has, for

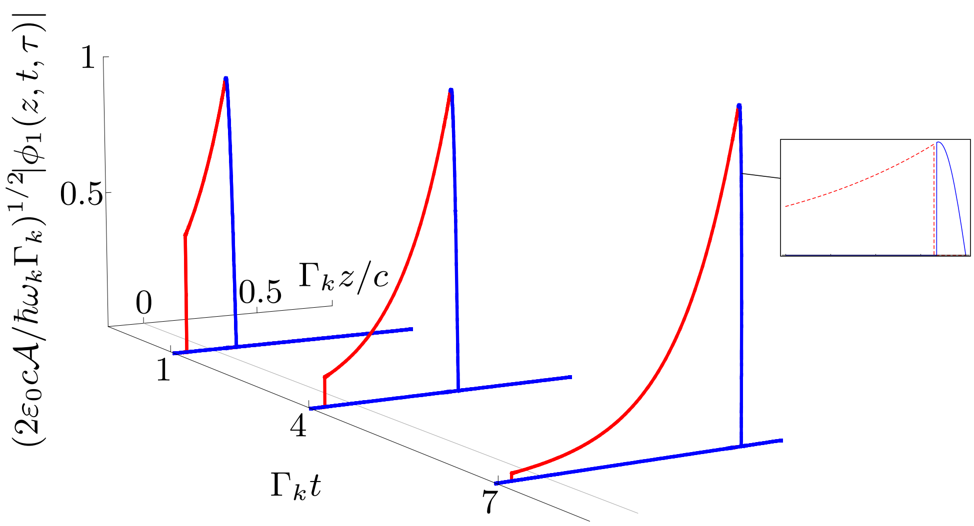

Figure 4.4: The spatio-temporal behavior of the excited outgoing wave packet $|\phi_1(z,t,\tau)|$, Eq. (4.81), in the case of short-term atom-field interaction for $\Gamma_k\tau=0.3$, $\Gamma_k l/c=0.7$. The parameters are the same as in Fig. 4.2. The inset shows the leading edge (solid curve) and the trailing edge (dashed curve).

the chosen interaction times τ, the form of a single-peaked pulse, where the trailing and the leading edges are determined by $\phi_1^<(z,t,\tau)$ and $\phi_1^>(z,t,\tau)$, respectively. Note that the leading edge $\phi_1^>(z,t,\tau)$, Eq. (4.82), coincides with $\phi_1^>(z,t)$, Eq. (4.73), found in the case of the continuing atom-field interaction, which leads to the conclusion that the shape of the leading edge $\phi_1^>(z,t,\tau)$ is directly related to the atom-field interaction in the time interval $0\le t\le\tau$. Particularly, since $\phi_1^>(z,t,\tau)$ approaches zero as τ tends to zero, for some applications the pulse can be regarded as being fully determined by $\phi_1^<(z,t,\tau)$ for sufficiently short interaction times.

For comparison, in Fig. 4.5 we plot the absolute value of $\phi_1(z,t,\tau)$ for the case, where the interaction time is chosen as the time at which the atom returns the first time to the upper state (one Rabi cycle). Here, the leading edge $\phi_1^>(z,t,\tau)$ has a single-peaked form, and the amplitude of the tailoring edge $\phi_1^<(z,t,\tau)$ is practically zero. The situation can drastically change when longer interaction times are considered, so that τ cannot be regarded as being small compared to Γ_k^{-1}. In this case, the contribution of $\phi_1^>(z,t,\tau)$ to $\phi_1(z,t,\tau)$ can become the dominating one, and, if the atom is allowed to undergo Rabi oscillations before it leaves the cavity, a multi-peaked pulse is observed, as can be seen from Fig. 4.6.

The time control of the atom-field interaction analyzed here may be realized in the cases of a neutral atom [22, 68] or a trapped ion [69] in an optical cavity. With the

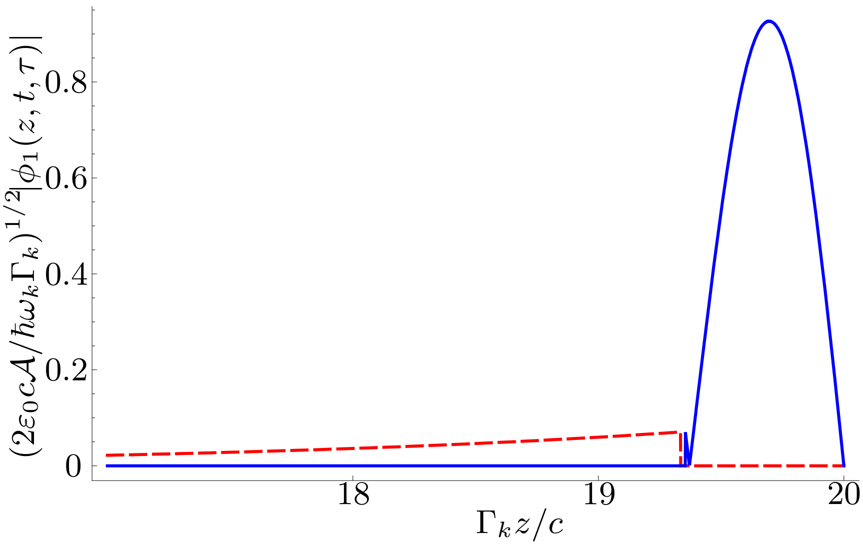

Figure 4.5: The spatio-temporal shape of the excited outgoing mode, $|\phi_1(z,t,\tau)|$, Eq. (4.81), for $\Gamma_k\tau = 0.6$, $\Gamma_k t = 20$, and $\Gamma_k l/c = 0.7$. The parameters are the same as in Fig. 4.2. The solid (dashed) curve shows the leading (trailing) edge.

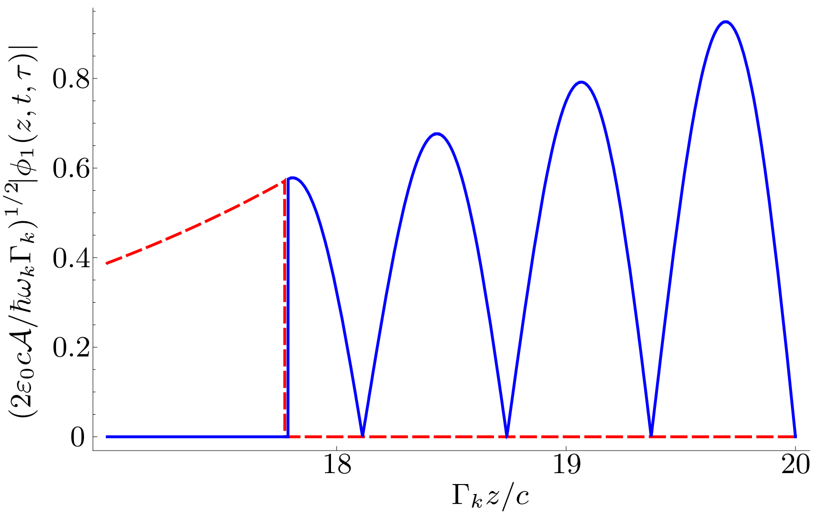

Figure 4.6: The spatio-temporal shape of the excited outgoing mode, $|\phi_1(z,t,\tau)|$, Eq. (4.81), for $\Gamma_k\tau = 2.2$, $\Gamma_k t = 20$, and $\Gamma_k l/c = 0.7$. The parameters are the same as in Fig. 4.2. The solid (dashed) curve shows the leading (trailing) edge.

use of an external laser pulse it is possible to excite the atom to an auxiliary electronic state, to decouple the atom from the cavity mode. In this way the interaction time between the atom and the cavity can be regulated. Moreover, for a continuous time-dependent control of the interaction, a pulsed Raman coupling could be useful, or the atom could be tuned out of resonance by external electric or magnetic fields.

4.3 Comparison with Quantum Noise Theories

Let us compare the results of Sec.4.2 with the ones obtained by means of the methods of the quantum noise theory [MK10]. As we have already mentioned, quantum noise theory is based on the assumption that the electromagnetic fields inside and outside the cavity represent independent degrees of freedom, which give rise to two separate Hilbert spaces. To go into more details, let us first consider the regime of continuing atom-field interaction.

4.3.1 Continuing Atom-Field Interaction

Comparing the spatio-temporal shape of the outgoing mode, Eq. (4.81), with the amplitude of the spatio-temporal shape of the field outside the cavity obtained within QNT, Eq. (2.49), one may conclude, that to describe the excited field outside the cavity from the point of view of QNT, one could consider wave packets of the types of $\phi_1^{>}(z,t)$ [Eq. (4.73)]. For comparison, one could also consider $\phi_1^{<}(z,t)$ [Eq. (4.74)] to characterize the outgoing field inside the cavity. Treating $\phi_1^{>}(z,t)$ and $\phi_1^{<}(z,t)$ as spatio-temporal shapes of modes of the outgoing field one could introduce functions $F_1^{>}(\omega,t)$ and $F_1^{<}(\omega,t)$ according to

$$F_1^{>(<)}(\omega,t) = \frac{1}{\sqrt{2\pi c \mathcal{N}_1^{>(<)}(t)}} \int_{>(<)} \mathrm{d}z\, e^{i\omega z/c} \phi_1^{>(<)}(z,t), \tag{4.84}$$

where

$$\mathcal{N}_1^{>(<)}(t) = \int_{>(<)} \mathrm{d}z \left|\phi_1^{>(<)}(z,t)\right|^2. \tag{4.85}$$

Here, the integral $\int_{>(<)}\mathrm{d}z\ldots$ runs over the interval $0<z<ct$ $(-l<z<0)$. Now, one could identify $F^{(1)}(\omega,t)$ in Eq. (4.38) for $C_{\mathrm{out}}[\beta^{(i)}(t),t]$ with $F_1^{>}(\omega,t)$. Similarly, for the inside field one could also identify $F^{(1)}(\omega,t)$ with $F_1^{<}(\omega,t)$ in Eq. (4.62). Disregarding the 'interference' terms, which prevent $C_{\mathrm{out}}[\beta^{(i)}(t),t]$ from being a product, one may introduce the single-mode characteristic function

$$C_1^{>(<)}(\beta,t) = e^{-|\beta|^2/2}\left[1-\eta^{>(<)}(t)|\beta|^2\right] \tag{4.86}$$

together with

$$\eta^{>(<)}(t) = \left|\int_0^\infty \mathrm{d}\omega\, F_1^{>(<)}(\omega,t)F^{>(<)*}(\omega,t)\right|^2, \tag{4.87}$$

where, according to Eqs. (4.38) and (4.62), respectively,

$$F^{>}(\omega,t) = F(\omega,t) \tag{4.88}$$

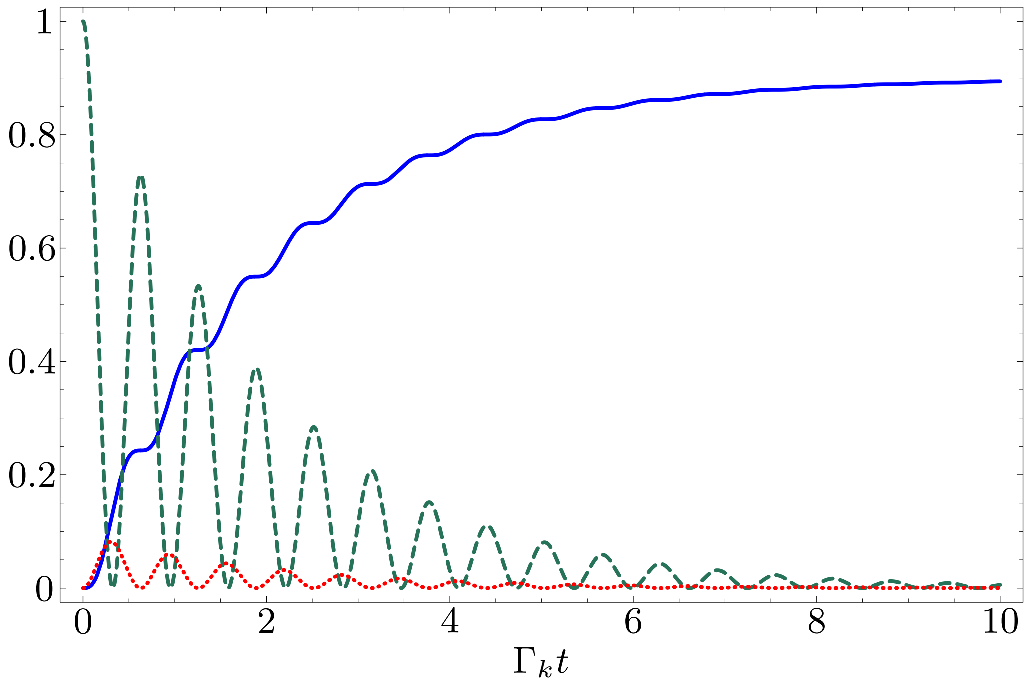

Figure 4.7: The quantities $\eta^{>}(t)$ (solid curve) and $\eta^{<}(t)$ (dotted curve), c.f. Eqs. (4.90) and (4.91) ($\Gamma_k l/c = 0.7$), and the atomic upper-state occupation probability $|\tilde{C}_2(t)|^2$, Eq. (4.49), (dashed curve). The parameters are the same as in Fig. 4.2.

and

$$F^{<}(\omega,t) = \tilde{F}(\omega,t). \tag{4.89}$$

Obviously, Eq. (4.86) together with Eq. (4.87) replaces Eq. (4.42) together with Eq. (4.40). Using Eqs. (4.39), (4.71), (4.84) and (4.85), after some calculations we find that $\eta^{>(<)}(t)$ can be written as

$$\eta^{>}(t) = \frac{2\varepsilon_0\mathcal{A}}{\hbar\omega_k}\eta(t)\mathcal{N}_1^{>}(t), \tag{4.90}$$

$$\eta^{<}(t) = \frac{\Gamma_k}{\gamma_{k\mathrm{rad}}}\frac{2\varepsilon_0\mathcal{A}}{\hbar\omega_k}\eta(t)\mathcal{N}_1^{<}(t). \tag{4.91}$$

In particular, $\eta^{>}(t)$ corresponds to the probability of detection of a photon outside the cavity found by the quantum trajectory method Eq. (2.50). Note, from Eq. (4.69) it follows that

$$I^{>}(t) = \int_{>}\mathrm{d}z\, I(z,t) = \frac{\hbar\omega_k}{2\varepsilon_0\mathcal{A}}\eta^{>}(t). \tag{4.92}$$

From the theory of photodetection of light [36] we know that the probability of registering a photon during the time interval $[0,t]$ by a detector at $z = 0_+$ is proportional to $I^{>}(t)$. Hence, the quantity $\eta^{>}(t)$, which refers to the part of the excited outgoing wave packet that is quite outside the cavity, cannot be regarded, in general, as being the efficiency of preparation of the outgoing field in a one-photon Fock state; it can be merely regarded as being proportional to the probability of registering an emitted photon outside the cavity during the chosen time interval. To clarify the meaning of the quantity $\eta^{<}(t)$, which refers to the part of the excited outgoing wave packet that is quite inside the cavity, we recall that the total excited field inside the cavity consists of a part traveling from the left to the right and a part traveling from the right to the left. Obviously, $\eta^{<}(t)$ refers to the (small) fraction of the former part which is transmitted through the fractionally transparent mirror. It is, therefore, proportional to the probability of registering a photon if this fraction of the part of the excited field inside the cavity, which travels from the left to the right, could be detected. The dependence on t of $\eta^{>}(t)$ and $\eta^{<}(t)$ is illustrated in Fig. 4.7. As we can see, $\eta^{>}(t)$ almost monotonically increases with time, while $\eta^{<}(t)$ decreases with increasing time in an oscillatory manner. As expected, $\eta^{>}(t)$ approaches $\eta(t)$ as $\Gamma_k t$ tends to infinity (cf. Fig. 4.2). Clearly, in the limit when $\Gamma_k t \to \infty$, the wave packet $\phi_1(z,t)$ associated with the excited outgoing mode $F^{(1)}(\omega,t)$ is strictly localized outside the cavity and $C_1^{>}(\beta,t)$ equals the characteristic function $C_1(\beta,t)$ of the quantum state of the excited outgoing mode.

Since the wave packet associated with the excited outgoing mode covers the areas inside and outside the cavity in general, a photon carried by the mode belongs simultaneously to the two areas in general. To model this effect within the framework of QNT, where the fields inside and outside the cavity are considered as belonging to two different Hilbert spaces, one had to introduce (on disregarding absorption losses) entangled states between a photon inside the cavity and a photon outside the cavity. Needless to say that from the point of view of QED, such a concept would be rather artificial.

4.3.2 Short-Term Atom-Field Interaction

Let us compare the results of Sec. 4.2.3 with the ones of Sec. 3.4, where—in analogy to QNT—the outgoing field outside the cavity is considered on the basis of the input-output relations, and the assumption of the separate Hilbert spaces for the fields inside and outside the cavity is made. In Sec. 3.4 it is also assumed that a single excited cavity mode, say the kth mode, at initial time $t=0$ is prepared in an excited state so, that the time of preparation of the mode in the excited state, which corresponds to the time τ of the atom-field interaction in Sec. 3.2, must be much smaller than its decay time Γ_k^{-1}. In this case we have introduced the relevant nonmonochromatic mode of the outgoing field $F_k^{(1)}(\omega,t)$, Eq. (3.227), representing the outgoing field outside the cavity relevant to the excited cavity mode. As it is shown in Sec. 3.4.2 for the case when the cavity mode is initially prepared in a one-photon Fock state and the field outside is in the vacuum state, the efficiency to find the relevant outgoing mode in a one-photon Fock state is $\eta_k(t)$, Eq. (3.249). In contrast to $\eta(t,\tau)$ given by Eq. (4.80), $\eta_k(t)$ is a time-dependent function which is not surprising, since—contrary to the relevant mode, which by construction defines a wave packet located entirely outside the cavity—the outgoing wave packet $\phi_1(z,t,\tau)$ [Eq. (4.81)], which corresponds to the excited mode that in reality carries the photon, covers simultaneously the areas inside and outside the cavity (cf. Fig. 4.4). As in the case of continuing atom-field interaction, it is natural to expect that only in the case when the condition $\Gamma_k t \gg 1$ holds, then $\eta_{k\mathrm{rel}}(t) \simeq \eta(\tau)$ for a value of τ for which the atom is in the ground state.

Performing a similar procedure as in Sec. 4.3.1, let us now consider the leading edge of $\phi_1(z,t,\tau)$ and that part of the trailing edge of $\phi_1(z,t,\tau)$ which is entirely outside the cavity, and, in line with Eq. (4.84), introduce the functions ($t \geq \tau$)

$$F_1^{>(<)}(\omega,t,\tau) = \frac{1}{\sqrt{2\pi c \mathcal{N}_1^{>(<)}(t,\tau)}} \int_{>(<)} \mathrm{d}z\, e^{i\omega z/c} \phi_1^{>(<)}(z,t,\tau), \tag{4.93}$$

where $\int_{>(<)} \mathrm{d}z \ldots$ runs over the interval $c(t-\tau) < z < ct$ $[0 < z < c(t-\tau)]$, and

$$\mathcal{N}_1^{>(<)}(t,\tau) = \int_{>(<)} \mathrm{d}z \left|\phi_1^{>(<)}(z,t,\tau)\right|^2, \tag{4.94}$$

so that, in view of QNT, a characteristic function of the form of Eq. (4.86) could be defined, with $\eta^{>(<)}(t)$ being replaced by

$$\eta^{>(<)}(t,\tau) = \left| \int_0^\infty \mathrm{d}\omega\, F_1^{>(<)}(\omega,t,\tau) F^*(\omega,t,\tau) \right|^2. \tag{4.95}$$

Straightforward calculation, using Eqs. (4.93) and (4.94) together with Eqs. (4.71), (4.79), and (4.81), yields

$$\eta^{>(<)}(t,\tau) = \frac{2\varepsilon_0 \mathcal{A}}{\hbar\omega_k} \eta(\tau) \mathcal{N}_1^{>(<)}(t,\tau). \tag{4.96}$$

In particular,

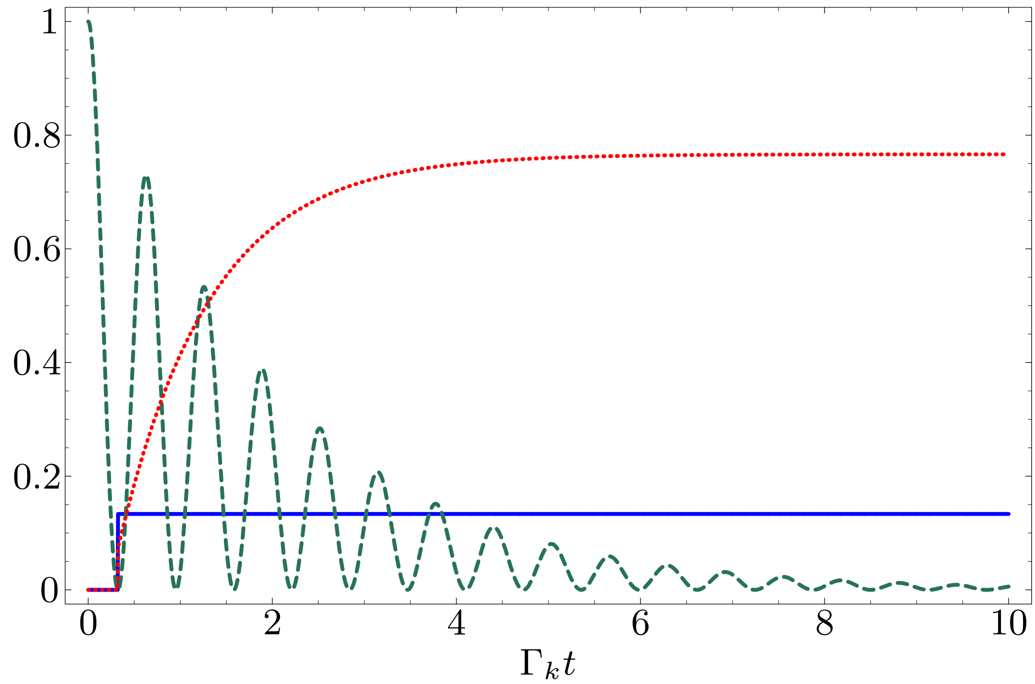

Figure 4.8: The quantities $\eta^{>}(t,\tau)$ (solid curve) and $\eta^{<}(t,\tau)$ (dotted curve), Eq. (4.97), and the atomic upper-state occupation probability $|\tilde{C}_2(t)|^2$, Eq. (4.49), (dashed curve) for $\Gamma_k\tau=0.3$, $\Gamma_k l/c=0.7$. The parameters are the same as in Fig. 4.2.

$$\eta^{<}(t,\tau) = \frac{R_k^2}{|\zeta_k|^2}\,|\sin(\zeta_k\tau/2)|^2\, e^{-\Gamma_k\tau/2}\eta_k(t-\tau), \tag{4.97}$$

with $\eta_k(t)$ according to Eq. (3.249). The behavior of $\eta^{>}(t,\tau)$ and $\eta^{<}(t,\tau)$ as functions of t is illustrated in Fig. 4.8. To compare with the results in Sec. 3.4.2, we assume strong atom-field coupling, almost exact resonance, $R_k^2/|\zeta_k|^2 \simeq 1$, and short interaction time, such that $\Gamma_k\tau \ll 1$ as well as $|\sin(\zeta_k\tau/2)|^2 \simeq 1$, i.e., $|\tilde{C}_2(\tau)|^2 \simeq 0$. In this limiting case we have $\eta^{<}(t,\tau) \simeq \eta_k(t)$. Recalling that when $\Gamma_k\tau \ll 1$, then the contribution of the leading edge $\phi_1^{>}(t,\tau)$ to $\phi_1(t,\tau)$ can be neglected, $\phi_1^{<}(t,\tau) \simeq \phi_1(t,\tau)$, we conclude that $\eta_k(t) \simeq \eta(\tau)$ if $\Gamma_k t \gg 1$. From reasons similar to those applied to the case of continuing atom-field interaction it follows that $\eta^{<}(t,\tau) \simeq \eta_k(t)$ cannot be regarded, in general, as being the efficiency of preparation of the outgoing field in a single-photon Fock state; it may be merely regarded as being the probability of detection of a photon outside the cavity.

Finally, let us address the following point. In quantum noise theory it is commonly assumed that (at equal times) input-field and cavity-field variables commute. On the other hand, the macroscopic QED approach shows [see Eq. (3.165)] that the equal-time commutation relation of the cavity mode operator with the input field operator

$\hat{b}^{\dagger}_{k\text{in}}(\omega,t)$ does not equal to zero:

$$\left[\hat{a}_k(t),\hat{b}^{\dagger}_{k\text{in}}(\omega,t)\right] \equiv \chi^{\text{c}}_{k\text{in}}(\omega) = \delta_{kk'}\left[\frac{c}{2n_1(\omega_k)l}\right]^{\frac{1}{2}}\frac{T_k}{\sqrt{2\pi}}\frac{i}{\omega-\Omega_k}. \tag{4.98}$$

Thus, a question arises of whether or not this commutator can be effectively set equal to zero.

Let us assume, that the incoming field is prepared at time t_0 in the nonmonochromatic mode associated to the commutator Eq. (4.98), i.e.,

$$\hat{b}^{\text{c}}_{k\text{in}}(t) = \int_0^{\infty} \mathrm{d}\omega\, \frac{\chi^{\text{c}}_{k\text{in}}(\omega)}{\sqrt{\zeta^{\text{c}}_{k\text{in}}(t)}} e^{-i\omega(t-t_0)}\hat{b}_{k\text{in}}(\omega), \tag{4.99}$$

with

$$\zeta^{\text{c}}_{k\text{in}}(t) = \int_0^{\infty} \mathrm{d}\omega\, |\chi^{\text{c}}_{k\text{in}}(\omega)|^2. \tag{4.100}$$

Using the formulae in Sec. 3.4.2, it is straightforward to calculate the corresponding nonmonochromatic mode of the output field,

$$F^{\text{c}}_k(\omega,t) = \frac{r^*_{31}(\omega)}{|r_{31}(\omega)|}\frac{\chi^{\text{c}}_{k\text{in}}(\omega)}{\sqrt{\zeta^{\text{c}}_{k\text{in}}(t)}}\, e^{i\omega(t-t_0)}, \tag{4.101}$$

and show that

$$\int_0^{\infty} \mathrm{d}\omega\, F^{(1)*}_k(\omega,t)F^{\text{c}}_k(\omega,t) \simeq \int_{-\infty}^{\infty} \mathrm{d}\omega\, F^{(1)*}_k(\omega,t)F^{\text{c}}_k(\omega,t) = 0. \tag{4.102}$$

Recall that the cavity mode operator commutes with the nonmonochromatic mode of the input field that corresponds to the relevant outgoing mode [see Eq. (3.237)]. Hence, the relevant output mode as defined by Eq. (3.227) is not related to the commutator in question. Applying Eq. (4.71) to the functions $F^{(1)}_k(\omega,t)$, Eq. (3.227), and $F^{\text{c}}_k(\omega,t)$, Eq. (4.101), one can calculate the corresponding spatio-temporal shapes

$$\phi^{(1)}_k(z,t) = \Theta(z)\Theta(ct-ct_0-z)\,\tfrac{1}{2}\sqrt{\frac{\hbar\omega_k}{\varepsilon_0 l\mathcal{A}\eta_k(t)}}\,T^{(o)*}_k\, e^{i\Omega^*_k(t-t_0-z/c)} \tag{4.103}$$

and

$$\phi^{\text{c}}_k(z,t) = \Theta(z-ct+ct_0)\tfrac{1}{2}\sqrt{\frac{\hbar\omega_k}{\varepsilon_0 l\mathcal{A}}}\left(\frac{\Gamma_k}{\gamma_{k\text{rad}}}\right)^{\frac{1}{2}}\frac{r^*_{31}(\Omega_k)}{|r_{31}(\Omega_k)|}T_k e^{i\Omega_k(t-t_0-z/c)}, \tag{4.104}$$

respectively. Recall that the initial time t_0 corresponds to the time τ the atom-field interaction. Comparing Eqs. (4.103) and (4.104) with Eq. (4.81), we see that, as expected, $\phi^{(1)}_k(z,t)$ matches, for $\Gamma_k\tau \ll 1$, that part of $\phi_1(z,t,\tau) \simeq \phi^{<}_1(z,t,\tau)$ which is entirely outside the cavity. Whereas $\phi_k^{c}(z,t)$ contributes only to the leading edge $\phi_1^{>}(z,t,\tau)$. However, the contribution of $\phi_1^{>}(z,t,\tau)$ into $\phi_1(z,t,\tau)$ is negligible in the case when the time τ is sufficiently short. We are thus left with the result that when the interaction time is sufficiently short, then the commutator in question can be effectively set equal to zero in the scheme under consideration.

Chapter 5

Summary and Outlook

In this thesis, the macroscopic quantum electrodynamics in the linear media has been employed to develop a universally valid quantum theory describing the interaction of the electromagnetic field with the atomic sources in high-Q cavities. The theory provides a complete description of the features of the emitted radiation. The theory allows to explore the limits of the applicability, that have to be put on the standard theory.

To study the electromagnetic field we start with the macroscopic Maxwell equations, and include into the consideration the noise current density, associated with the medium absorption. These equations are solved, expressing the electromagnetic field variables in terms of the noise currents, guided by the Green tensor of the macroscopic Maxwell equations. Then, the explicit quantization is performed in terms of the noise current by introducing a diagonal Hamiltonian, which governs the time evolution of the field variables, according to Maxwell's equations and the correct fundamental equal-time commutation relations for the field variables. The dynamics of the free medium-assisted field can be complemented by introducing the atom-field interaction, by means of the canonical minimal or multipolar coupling schemes. The dielectric properties of the media are encoded in the Green tensor of the macroscopic Maxwell equations. Thus, as a preliminary step, by specifying the Green tensor we derive three-dimensional input-output relations for the electromagnetic field operators at a planar multilayer structure. Such general description of the electromagnetic field allows us to include into the consideration the effects of dispersion and absorption in the media, as well as possible active sources inside and outside the multilayer structure, and, therefore, can be used, for example, for the description of the resonator cavities.

We apply the general scheme to the study of a one-dimensional resonator cavity, bounded by a perfectly reflecting and a fractionally transparent mirrors. In fact, to properly study the effect of the unwanted losses, we assume, that both the body of the resonator cavity and the fractionally transparent mirror are composed of linear, locally responding, dielectric, dispersive and absorbing media. It is important to note that the Green tensor of the macroscopic Maxwell equations characterizes the spatial as well as the spectral behavior of the electromagnetic field. In particular, it determines the spectral response of the resonator cavity, providing the resonance frequencies and the corresponding line widths.

At the first stage, aiming to consistently describe the input-output coupling for a resonator cavity, we make use of the source-quantity representation of the electromagnetic field within the macroscopic QED. We assume, that the active atoms are located inside the cavity, and calculated the fields inside and outside the cavity. We show, that on a time scale that is large, compared with the inverse separation of two neighboring cavity resonance frequencies, the field inside the cavity may be expressed in terms of the standing waves, and associated bosonic cavity mode operators can be introduced. We prove, that within the coarse-grained approximation scheme used, the cavity mode operators satisfy quantum Langevin equations, where the radiative losses due to input-output coupling and the absorption losses, play the role of Langevin noise forces, thus, giving rise to the corresponding damping rates. The results lead to an immediate conclusion, that to take into account the absorption losses within QNT description of a leaky high-Q cavity, the intuitive approach can be applied. Namely, the bilinear interaction energy, describing the radiative input-output coupling of a leaky cavity, can be conventionally complimented by similar bilinear terms describing the interaction between the cavity mode and the appropriately chosen dissipative channels, responsible for unwanted losses.

However, the intuitive concept fails to describe the input-output relations in general. As we show, the absorption losses associated with the coupling mirror, give rise to additional force terms in the input-output relations, which cannot be obtained from the standard form of the interaction energy between the cavity modes and the dissipative channels. We explicitly demonstrate, that starting from the description of a leaky cavity by means of QNT, one can model the absorption losses associated with the coupling mirror by introducing an appropriate system of beam-splitters placed in front of the coupling mirror outside the cavity.

It is remarkable, that the equal-time commutator between the cavity mode and the incoming field is non-vanishing. Clearly, this is an indication of the fact, that the radiation field, which could be thought of as expanded in terms of, for example, a continuous set of modes, extends over the whole space. In contrast, in the description of leaky cavities in the framework of QNT, the commutator of the fields inside and outside the cavity vanishes by definition.

The exact input-output relations can be employed to study the problem of quantum state extraction. In particular, we assume that certain cavity mode is prepared in some quantum state, so that the time scale of quantum state preparation is sufficiently short in comparison with the decay time of the cavity mode. Then, we introduce relevant nonmonochromatic mode of the incoming and the relevant nonmonochromatic mode of the outgoing fields outside the cavity, i.e., modes which couple to the cavity mode. Making a factorization assumption with respect to the cavity and incoming fields, which in fact corresponds to the assumption of the separate field Hilbert spaces used in standard QNT, we express the s-parameterized phase-space function of the quantum state of the relevant outgoing mode in terms of the phase-space functions of the quantum states of the cavity mode, the relevant incoming mode, and the dissipative degrees of freedom responsible for unwanted losses. For times sufficiently large in comparison with the cavity decay time, the quantum state extraction efficiency is determined by the ratio of the cavity-mode decay rate, due to unwanted losses, to the cavity-mode decay rate, due to (wanted) radiative losses. This ratio must be sufficiently small in order to realize a nearly perfect extraction. It should be mentioned that to reduce the effect of the unwanted losses it may be useful to deliberately enlarge the transmission, so that the unwanted losses become small compared with the transmission losses. Since, on the other hand, the quality factor of the cavity is reduced, the preparation process must necessarily be included in the calculations.

Thus, to provide a generally applicable theory for high-Q cavities, in the second stage of the thesis we give an exact description of the interaction of the cavity-assisted electromagnetic field with an initially excited two-level atom, localized in a cavity. Importantly, we renounce the approximation of the separate Hilbert spaces for the fields inside and outside the cavity, and consider the electromagnetic field in the whole space as an entity. The strong atom-field coupling may occur in the case when one of the cavity resonance frequencies is nearly resonant to the atomic transition frequency. Then, the quantum state of the excited outgoing mode is always a mixture of the one-photon Fock state and the vacuum state, due to the unavoidably occurring unwanted losses. In the case of the continuing atom-field interaction, the efficiency to prepare the mode in a one-photon Fock state is time-dependent and features Rabi oscillation of the atom-field coupling. This is due to the fact, that for the times, short compared to the inverse width of the cavity resonance line, the wave packet, corresponding to the emitted photon, extends simultaneously in the regions both inside and outside the cavity. Hence, the part of the wave packet, still located inside the cavity, can be re-absorbed by the atom, effectively implying the decrease in the time-dependent efficiency. The probability of registering the emitted photon by a photodetector, placed outside the cavity is determined only by the part of the excited outgoing mode, located in the region outside the cavity. Clearly, this probability monotonously increases with time and asymptotically approaches the efficiency of one-photon Fock-state preparation in the long-time limit.

We also perform the calculations for the case, when the atom-field interaction is interrupted at some time τ. The spatio-temporal shape of the excited outgoing mode indicates, that the mode extends in the regions both inside and outside the cavity. The efficiency to prepare the excited outgoing mode in a one-photon Fock state is constant for all times $t \geq \tau$, and is given by the value of the efficiency observed at time $t=\tau$, in the case of continuing atom-field interaction. The spatio-temporal shape of the mode sensitively depends on the atom-field interaction. In particular, if the atom-field interaction is interrupted at the instant, when the atom is the first time in the ground state, the wave packet can be regarded as a single-peak pulse, the trailing edge of which extends in the two regions, inside and outside the cavity. Moreover, when the interaction time is sufficiently short, in comparison with the inverse width of the cavity resonance line, the pulse can be approximated by its trailing edge, whose part, that is entirely localized outside the cavity, matches the field, determined by the relevant mode, appearing in the input-output relations. Note, that the commutation relation between the cavity mode operator and the incoming field operators effectively contributes only to the leading edge of the excited outgoing mode, and, therefore, can be neglected in the case, when the atom-field interaction time is sufficiently short.

In conclusion, the developed theoretical framework of macroscopic quantum electrodynamics of high-Q cavities provides complete knowledge of the emitted radiation. The theory represents a groundwork that can be used to achieve many goals in quantum information science. In particular, it provides in principle an extensive description of single-photon sources with the full coherent control of the properties of the emitted field. Since the theory is universally applicable to arbitrary atom-field interaction times, it can be extended to describe, e.g., the atom-field interaction in a

Raman adiabatic passage configuration, which would allow to eliminate the dependence of the system dynamics on the position of the atom in the cavity. In addition, the theory also can be used to properly take into consideration excited incoming fields, which allows immediate extension to the description of the interaction of the electromagnetic field with two and more atoms, located in corresponding cavities—a major milestone to the realization of the quantum networks.

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[MK11] *Quantum state extraction from high-Q cavities*, poster, 3rd Workshop on Continuous-Variable Quantum Information Processing, Erlangen, Germany, April 2004.

[MK12] *QED of lossy cavities: Input-output relations and quantum-state extraction*, poster, 333rd WEH seminar New Frontiers in Quantum Theory and Measurement, Reisensburg, Germany, September 2004.

[MK13] *Cavity-assisted spontaneous emission as a single-photon source: Pulse shape and efficiency of one-photon Fock-state preparation*, talk, 9th International Conference on Squeezed States and Uncertainty Relations, Besançon, France, May 2005.

[MK14] *QED of lossy cavities: Generation of one-photon Fock states*, seminar talk, Institute of Physics, Rostock, Germany, November 2006.

[MK15] *Spontaneous emission of a single atom in a high-Q cavity*, talk, 15th Central European Workshop on Quantum Optics, Belgrade, Serbia, May 2008.

[MK16] *Spontaneous emission of a single atom in a high-Q cavity: macroscopic QED vs. quantum noise theory*, invited talk, XIIth International Conference on Quantum Optics and Quantum Information, Vilnius, Lithuania, September 2008.

Appendix A

Field Commutation Relations

A.1 Multilayer Field Operators

Here we present the calculations of the commutation relations of the amplitude operators of the field inside and outside the three-dimensional multilayer plate (see Sec. 3.1.4). Let us begin with the calculation of the commutation relations of the output amplitude operators $\hat{E}^{(0,n)}_{q\,\mathrm{out}}(\mathbf{k},\omega)$. From Eq. (3.70) it follows that

$$\hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega) = r^q_{0/n}\hat{E}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) + t^q_{n/0}\hat{E}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) + \hat{\mathrm{F}}^{(0)}_q(\mathbf{k},\omega) \tag{A.1}$$

and

$$\hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega) = t^q_{0/n}\hat{E}^{(0)}_{q\,\mathrm{in}}(\mathbf{k},\omega) + r^q_{n/0}\hat{E}^{(n)}_{q\,\mathrm{in}}(\mathbf{k},\omega) + \hat{\mathrm{F}}^{(n)}_q(\mathbf{k},\omega), \tag{A.2}$$

where

$$\begin{aligned}\hat{\mathrm{F}}^{(0)}_q(\mathbf{k},\omega) &= \sum_{j=1}^{n-1}\left[\phi^{(j)}_{q\,0+}\hat{E}^{(j)}_{q+}(\mathbf{k},\omega) + \phi^{(j)}_{q\,0-}\hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\right]\\ &= i\omega\mu_0\mathbf{e}^{(0)}_{q-}(\mathbf{k})\cdot\sum_{j=1}^{n-1}\int_{[j]}\mathrm{d}z\,\boldsymbol{g}^{(0j)}(0,z,\mathbf{k},\omega)\cdot\hat{\mathbf{j}}^{(j)}(z,\mathbf{k},\omega),\end{aligned} \tag{A.3}$$

$$\begin{aligned}\hat{\mathrm{F}}^{(n)}_q(\mathbf{k},\omega) &= \sum_{j=1}^{n-1}\left[\phi^{(j)}_{q\,n+}\hat{E}^{(j)}_{q+}(\mathbf{k},\omega) + \phi^{(j)}_{q\,n-}\hat{E}^{(j)}_{q-}(\mathbf{k},\omega)\right]\\ &= i\omega\mu_0\,\mathbf{e}^{(n)}_{q+}(\mathbf{k})\cdot\sum_{j=1}^{n-1}\int_{[j]}\mathrm{d}z\,\boldsymbol{g}^{(nj)}(0,z,\mathbf{k},\omega)\cdot\hat{\mathbf{j}}^{(j)}(z,\mathbf{k},\omega).\end{aligned} \tag{A.4}$$

Hence, the (relevant) commutation relations of the output amplitude operators that refer to different sides of the plate can be given by

$$\big[\hat{E}^{(0)}_{q\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{q'\,\mathrm{out}}(\mathbf{k}',\omega')\big]=\delta_{qq'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\left(\alpha^{(0)}_{q\mathrm{in}}r^{q}_{0/n}t^{q*}_{0/n}+\alpha^{(n)}_{q\mathrm{in}}t^{q}_{n/0}r^{q*}_{n/0}\right)$$
$$+\big[\hat{\mathrm{F}}^{(0)}_{q}(\mathbf{k},\omega),\hat{\mathrm{F}}^{(n)\dagger}_{q'}(\mathbf{k}',\omega')\big],\qquad\text{(A.5)}$$

where Eqs. (3.86)–(3.88) have been used. Making use of Eqs. (3.50) and (3.51) and recalling the basic commutation relations (3.16) and (3.17), we derive

$$\big[\hat{\mathrm{F}}^{(0)}_{q}(\mathbf{k},\omega),\hat{\mathrm{F}}^{(n)\dagger}_{q'}(\mathbf{k}',\omega')\big]=\delta_{qq'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')$$
$$\times\frac{4\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}e^{(0)}_{q-\mu}(\mathbf{k})e^{(n)*}_{q+\mu'}(\mathbf{k})\Bigg\{\sum_{j=0}^{n}\int_{[j]}\mathrm{d}z\,g^{(0j)}_{\mu\nu}(0,z,\mathbf{k},\omega)\frac{\omega^2}{c^2}\mathrm{Im}\,\varepsilon_j g^{(nj)*}_{\mu'\nu}(0,z,\mathbf{k},\omega)$$
$$-\int_{-\infty}^{0}\mathrm{d}z\,g^{(00)}_{\mu\nu}(0,z,\mathbf{k},\omega)\frac{\omega^2}{c^2}\mathrm{Im}\,\varepsilon_0 g^{(n0)*}_{\mu'\nu}(0,z,\mathbf{k},\omega)$$
$$-\int_{0}^{\infty}\mathrm{d}z\,g^{(0n)}_{\mu\nu}(0,z,\mathbf{k},\omega)\frac{\omega^2}{c^2}\mathrm{Im}\,\varepsilon_n g^{(nn)*}_{\mu'\nu}(0,z,\mathbf{k},\omega)\Bigg\}.\qquad\text{(A.6)}$$

It is not difficult to calculate the last two integrals in Eq. (A.6), by using the explicit expression (3.53) for $\boldsymbol{g}^{(jj')}(z,z',\mathbf{k},\omega)$. To calculate the sum of integrals $\sum_{j=0}^{n}\int_{[j]}\mathrm{d}z\,\ldots$, we employ the integral relation (3.14) for the classical Green tensor $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$, rewritten in terms of $\boldsymbol{g}^{(jj')}(z,z',\mathbf{k},\omega)$,

$$\sum_{j''=0}^{n}\int_{[j'']}\mathrm{d}z''\,g^{(jj'')}_{\mu\nu}(z,z'',\mathbf{k},\omega)\frac{\omega^2}{c^2}\mathrm{Im}\,\varepsilon_{j''}g^{(j'j'')*}_{\mu'\nu}(z',z'',\mathbf{k},\omega)$$
$$=\frac{1}{2i}g^{(jj')}_{\mu\mu'}(z,z',\mathbf{k},\omega)-\frac{1}{2i}g^{(j'j)*}_{\mu'\mu}(z',z,\mathbf{k},\omega)$$
$$+g^{(jj')}_{\mu\nu}(z,z',\mathbf{k},\omega)e_{z\,\nu}\frac{\mathrm{Im}\,\varepsilon_{j'}}{\varepsilon^*_{j'}}e_{z\,\mu'}+e_{z\,\mu}\frac{\mathrm{Im}\,\varepsilon_j}{\varepsilon_j}g^{(j'j)*}_{\mu'\nu}(z',z,\mathbf{k},\omega)e_{z\,\nu}.\qquad\text{(A.7)}$$

After lengthy, but straightforward calculations we then derive, on using Eqs. (3.87) and (3.88),

$$\big[\hat{E}^{(0)}_{s\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{p\,\mathrm{out}}(\mathbf{k}',\omega')\big]=\big[\hat{E}^{(0)}_{p\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{s\,\mathrm{out}}(\mathbf{k}',\omega')\big]=0,\qquad\text{(A.8)}$$

$$\big[\hat{E}^{(0)}_{s\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{s\,\mathrm{out}}(\mathbf{k}',\omega')\big]=\delta(\omega-\omega')\,\delta(\mathbf{k}-\mathbf{k}')$$
$$\times\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\left(\frac{i\beta''_0}{|\beta_0|^2}t^{s\,*}_{0/n}-\frac{i\beta''_n}{|\beta_n|^2}t^{s}_{n/0}\right),\qquad\text{(A.9)}$$

$$\left[\hat{E}^{(0)}_{p\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{p\,\mathrm{out}}(\mathbf{k}',\omega')\right]=\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}$$

$$\left\{\frac{t^{p\,*}_{0/n}}{\beta_0^*\left|k_0\right|^2}\left(k^2\frac{k_0^{*\,2}}{k_0^2}-\left|\beta_0\right|^2\right)-\frac{\beta_0'}{\left|\beta_0\right|^2}t^{p*}_{0/n}\left[\mathbf{e}^{(0)*}_{p+}(\mathbf{k})\cdot\mathbf{e}^{(0)}_{p+}(\mathbf{k})\right]\left[\mathbf{e}^{(0)}_{p+}(\mathbf{k})\cdot\mathbf{e}^{(0)}_{p-}(\mathbf{k})\right]\right.$$

$$\left.+\frac{t^{p}_{n/0}}{\beta_n\left|k_n\right|^2}\left(k^2\frac{k_n^2}{k_n^{*\,2}}-\left|\beta_n\right|^2\right)-\frac{\beta_n'}{\left|\beta_n\right|^2}t^{p}_{n/0}\left[\mathbf{e}^{(n)}_{p-}(\mathbf{k})\cdot\mathbf{e}^{(n)*}_{p-}(\mathbf{k})\right]\left[\mathbf{e}^{(n)*}_{p-}(\mathbf{k})\cdot\mathbf{e}^{(n)*}_{p+}(\mathbf{k})\right]\right\}.\tag{A.10}$$

Next, let us consider the commutation relations of the output amplitude operators, that refer to the same sides of the plate. Performing the same steps as before, we now arrive at

$$\left[\hat{E}^{(0)}_{s\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(0)\dagger}_{p\,\mathrm{out}}(\mathbf{k}',\omega')\right]=\left[\hat{E}^{(0)}_{p\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(0)\dagger}_{s\,\mathrm{out}}(\mathbf{k}',\omega')\right]=0,\tag{A.11}$$

$$\begin{aligned}\left[\hat{E}^{(0)}_{s\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(0)\dagger}_{s\,\mathrm{out}}(\mathbf{k}',\omega')\right]&\equiv\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\alpha^{(0)}_{s\,\mathrm{out}}(\mathbf{k},\omega)\\&=\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\left(\frac{\beta_0'}{\left|\beta_0\right|^2}+\frac{2\beta_0''}{\left|\beta_0\right|^2}r^{s\,\prime\prime}_{0/n}\right),\end{aligned}\tag{A.12}$$

$$\begin{aligned}&\left[\hat{E}^{(0)}_{p\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(0)\dagger}_{p\,\mathrm{out}}(\mathbf{k}',\omega')\right]\equiv\alpha^{(0,n)}_{p\,\mathrm{out}}(\mathbf{k},\omega)\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\\&=\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}')\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\left\{\frac{r^{p}_{0/n}}{\beta_0\left|k_0\right|^2}\left(k^2\frac{k_0^2}{k_0^{*\,2}}-\left|\beta_0\right|^2\right)+\frac{r^{p\,*}_{0/n}}{\beta_0^*\left|k_0\right|^2}\left(k^2\frac{k_0^{*\,2}}{k_0^2}-\left|\beta_0\right|^2\right)\right.\\&\quad-\frac{k^2}{\left|k_0\right|^2}\left(\frac{\beta_0}{k_0^{*\,2}}+\frac{\beta_0^*}{k_0^2}\right)+\frac{2\beta_0'}{\left|k_0\right|^2}\left(\frac{k^4}{\left|\beta_0\right|^2\left|k_0\right|^2}+1\right)\\&\quad\left.+\frac{\beta_0'}{\left|\beta_0\right|^2}\mathbf{e}^{(0)}_{p+}(\mathbf{k})\cdot\mathbf{e}^{(0)*}_{p+}(\mathbf{k})\left[\left|r^{q}_{0/n}\right|^2-\left|\mathbf{e}^{(0)}_{p-}(\mathbf{k})\cdot\mathbf{e}^{(0)}_{p+}(\mathbf{k})+r^{p}_{0/n}\right|^2\right]\right\}.\end{aligned}\tag{A.13}$$

The commutation relation $\left[\hat{E}^{(n)}_{q\,\mathrm{out}}(\mathbf{k},\omega),\hat{E}^{(n)\dagger}_{q'\,\mathrm{out}}(\mathbf{k}',\omega')\right]$ are obtained from Eqs. (A.11-A.13), by making the replacements $\beta_0\to\beta_n$, $k_0\to k_n$, $\mathbf{e}^{(0)}_{q\pm}\to\mathbf{e}^{(n)}_{q\mp}$, and $r^{q}_{0/n}\to r^{q}_{n/0}$.

Finally, it can easily be proved that the intraplate amplitude operators (3.73) satisfy the commutation relations ($j=1,\ldots,n-1$; $\lambda=\pm$)

$$\left[\hat{E}^{(j)}_{q\lambda}(\mathbf{k},\omega),\hat{E}^{(j)\dagger}_{q\lambda'}(\mathbf{k}',\omega')\right]=\alpha^{(j)}_{q\lambda\lambda'}(\mathbf{k},\omega)\delta_{qq'}\delta(\omega-\omega')\delta(\mathbf{k}-\mathbf{k}'),\tag{A.14}$$

where

$$\alpha^{(j)}_{q\,\pm\pm}=\pm\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\frac{\beta_j'}{|\beta_j|^2}\left(e^{\pm2\beta_j''d_j}-1\right)\mathbf{e}^{(j)}_{q\pm}(\mathbf{k})\cdot\mathbf{e}^{(j)*}_{q\pm}(\mathbf{k}),\tag{A.15}$$

$$\alpha^{(j)}_{q\,\pm\mp}=\pm i\frac{\pi\hbar}{\varepsilon_0}\frac{\omega^2}{c^2}\frac{\beta_j''}{|\beta_j|^2}\left(e^{\mp2i\beta_j'd_j}-1\right)\mathbf{e}^{(j)}_{q\pm}(\mathbf{k})\cdot\mathbf{e}^{(j)*}_{q\mp}(\mathbf{k}).\tag{A.16}$$

Using Eqs. (A.14)–(A.16), we find that the operators

$$\hat{a}_{q\pm}^{(j)}(\mathbf{k},\omega) = \frac{1}{\xi_{q\pm}^{(j)}(\mathbf{k},\omega)} \left[e^{i\beta_j d_j} \hat{E}_{q+}^{(j)}(\mathbf{k},\omega) \pm \hat{E}_{q-}^{(j)}(\mathbf{k},\omega) \right], \tag{A.17}$$

where

$$\xi_{q\pm}^{(j)}(\mathbf{k},\omega) = \frac{2\omega}{c\beta_j} \sqrt{\frac{\pi\hbar}{\varepsilon_0}} e^{-\beta_j'' d_j/2} \times \left[\beta_j' \sinh(\beta_j'' d_j) \mathbf{e}_{q+}^{(j)}(\mathbf{k}) \cdot \mathbf{e}_{q+}^{(j)*}(\mathbf{k}) \pm \beta_j'' \sin(\beta_j' d_j) \mathbf{e}_{q+}^{(j)}(\mathbf{k}) \cdot \mathbf{e}_{q-}^{(j)*}(\mathbf{k}) \right]^{\frac{1}{2}} \tag{A.18}$$

satisfy bosonic commutation relations.

In the limiting case $\varepsilon_{0,n} \to 0$ (when the plate is surrounded by vacuum) one has to distinguish between $\omega/c > k$ (propagating-field components) and $\omega/c \leq k$ (evanescent-field components). In the first case ($\omega/c > k$) we have $\beta_0' = \beta_0 = \beta_n = \beta_n' > 0$, so that Eqs. (A.8)–(A.10) reduce to

$$\left[\hat{E}_{q\,\mathrm{out}}^{(0)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(n)\dagger}(\mathbf{k}',\omega') \right] = 0, \tag{A.19}$$

and Eqs. (A.11)–(A.13) simplify to

$$\left[\hat{E}_{q\,\mathrm{out}}^{(0)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(0)\dagger}(\mathbf{k}',\omega') \right] = \frac{\pi\hbar}{\varepsilon_0} \frac{\omega^2{}^2}{c^2} \frac{1}{\beta_0} \delta_{qq'} \delta(\omega-\omega') \delta(\mathbf{k}-\mathbf{k}'). \tag{A.20}$$

$$\left[\hat{E}_{q\,\mathrm{out}}^{(n)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(n)\dagger}(\mathbf{k}',\omega') \right] = \frac{\pi\hbar}{\varepsilon_0} \frac{\omega^2}{c^2} \frac{1}{\beta_n} \delta_{qq'} \delta(\omega-\omega') \delta(\mathbf{k}-\mathbf{k}'). \tag{A.21}$$

In the second case ($\omega/c \leq k$) we have $\beta_0' = \beta_n' = 0$. Equations (A.8)–(A.10) then lead to

$$\left[\hat{E}_{q\,\mathrm{out}}^{(0)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(n)\dagger}(\mathbf{k}',\omega') \right] = \frac{\pi\hbar}{\varepsilon_0} \frac{\omega^2}{c^2} \frac{2t_{0/n}^{q\,\prime\prime}}{|\beta_0|} \delta_{qq'} \delta(\omega-\omega') \delta(\mathbf{k}-\mathbf{k}'), \tag{A.22}$$

and from Eqs. (A.11)–(A.13) we find

$$\left[\hat{E}_{q\,\mathrm{out}}^{(0)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(0)\dagger}(\mathbf{k}',\omega') \right] = \frac{\pi\hbar}{\varepsilon_0} \frac{\omega^2}{c^2} \frac{2r_{0/n}^{q\,\prime\prime}}{|\beta_0|} \delta_{qq'} \delta(\omega-\omega') \delta(\mathbf{k}-\mathbf{k}'), \tag{A.23}$$

and similarly

$$\left[\hat{E}_{q\,\mathrm{out}}^{(n)}(\mathbf{k},\omega), \hat{E}_{q'\,\mathrm{out}}^{(n)\dagger}(\mathbf{k}',\omega') \right] = \frac{\pi\hbar}{\varepsilon_0} \frac{\omega^2}{c^2} \frac{2r_{n/0}^{q\,\prime\prime}}{|\beta_n|} \delta_{qq'} \delta(\omega-\omega') \delta(\mathbf{k}-\mathbf{k}'). \tag{A.24}$$

A.2 Cavity Mode Operators

To prove the commutation relation for the cavity mode operators (3.159), we recall the definition of the electric field operator in the time domain $\hat{E}_k^{(1)}(z,t)$, namely

$$\hat{E}_k^{(1)}(z,t) = \int_{\Delta_k} \mathrm{d}\omega\, \underline{\hat{E}}^{(1)}(z,\omega,t), \tag{A.25}$$

where $\underline{\hat{E}}^{(1)}(z,\omega,t)$ is defined by Eq. (3.112) for $j=1$. Using Eq. (3.114) and the integral relation of the Green tensor Eq. (3.14) and recalling the basic commutation relations (3.16) and (3.17), after some trivial algebra we find

$$\left[\hat{E}_k^{(1)}(z_1,t), \hat{E}_{k'}^{(1)\dagger}(z_2,t)\right] = \delta_{kk'}\frac{\mu_0\hbar}{\pi\mathcal{A}} \int_{\Delta_k} \mathrm{d}\omega\, \omega^2 \mathrm{Im}\, G^{(11)}(z_1,z_2,\omega). \tag{A.26}$$

To perform the integration we extend the lower (upper) integration limit to $-\infty$ (∞) and write $\mathrm{Im}\, G^{(11)}(z_1,z_2,\omega) = [G^{(11)}(z_1,z_2,\omega) - G^{(11)*}(z_1,z_2,\omega)]/(2i)$. Then, recalling the definition of $G^{(11)}(z_1,z_2,\omega)$ $[G^{(11)*}(z_1,z_2,\omega)]$ from Eq. (B.1) with $j=1$, we evaluate the integral applying the residue theorem for the poles determined by the zeroes of the function $D_1(\omega)$ $[D_1^*(\omega)]$. Thus, for cavities with sufficiently high Q value, $\Gamma_k \ll \Delta_k$ we obtain

$$\left[\hat{E}_k^{(1)}(z_1,t), \hat{E}_{k'}^{(1)\dagger}(z_2,t)\right] = \delta_{kk'}\frac{\hbar\omega_k}{\varepsilon_0|\varepsilon_1| l\mathcal{A}} \sin[\beta_1(\omega_k)z_1]\sin[\beta_1^*(\omega_k)z_2]. \tag{A.27}$$

Comparing Eq. (A.27) with Eq. (3.155) [together with Eq. (3.156)], we then easily see that the equal time commutation relation for the cavity mode operators (3.159) holds.

A.3 Outgoing Field Operators in Time Domain

To obtain the commutating relations for the outgoing field operators in the time domain Eq. (3.180), we first notice, that Eq. (3.177) yields

$$\begin{aligned}&\left[\hat{b}_{k\mathrm{out}}(t), \hat{b}^\dagger_{k'\mathrm{out}}(t')\right]\\ &= \frac{1}{2\pi}\int_{\Delta_k}\mathrm{d}\omega \int_{\Delta_{k'}}\mathrm{d}\omega'\ |\alpha_{\mathrm{out}}|^2\ \frac{\pi\mathcal{A}}{\mu_0 c\hbar\sqrt{\omega\omega'}}\left[\underline{\hat{E}}^{(3)}_{k\mathrm{out}}(0^+,\omega,t), \underline{\hat{E}}^{(3)\dagger}_{k'\mathrm{out}}(0^+,\omega',t')\right],\end{aligned} \tag{A.28}$$

which in the source-quantity representation reads as [cf. Eqs. (3.111) and (3.176)]

$$\big[\hat{b}_{k\mathrm{out}}(t),\hat{b}^{\dagger}_{k'\mathrm{out}}(t')\big]=\frac{1}{2\pi}\int_{\Delta_k}\mathrm{d}\omega\int_{\Delta_{k'}}\mathrm{d}\omega'\;|\alpha_{\mathrm{out}}|^2\,\frac{\pi\mathcal{A}}{\mu_0 c\hbar\sqrt{\omega\omega'}}$$
$$\times\Big\{\big[\underline{\hat{E}}^{(3)}_{k\mathrm{out,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k'\mathrm{out,free}}(0^+,\omega',t')\big]+\big[\underline{\hat{E}}^{(3)}_{k\mathrm{s}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k'\mathrm{s}}(0^+,\omega',t')\big]$$
$$+\big[\underline{\hat{E}}^{(3)}_{k\mathrm{out,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k'\mathrm{s}}(0^+,\omega',t')\big]+\big[\underline{\hat{E}}^{(3)}_{k\mathrm{s}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k'\mathrm{out,free}}(0^+,\omega',t')\big]\Big\}. \tag{A.29}$$

Using Eq. (3.113) it is easy to find

$$\big[\underline{\hat{E}}^{(3)}_{k\mathrm{s}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega_2,t_2)\big]$$
$$=\frac{1}{\pi^2\varepsilon_0^2\mathcal{A}^2}\frac{\omega_1^2\omega_2^2}{c^4}\sum_{AA'}\mathrm{Im}\,G^{(13)}(z_{A'},0^+,\omega_1)\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)$$
$$\times\int\mathrm{d}t'\int\mathrm{d}t''\,\Theta(t_1-t')\Theta(t_2-t'')e^{-i\omega_1(t_1-t')}e^{i\omega_2(t_2-t'')}\big[\hat{d}_{A'}(t'),\hat{d}_A(t'')\big]. \tag{A.30}$$

Further, from Eqs. (3.104)–(3.106) it follows that

$$\big[\hat{f}_{\mathrm{free}}(z',\omega_1,t_1),\hat{d}_A(t')\big]=-\mu_0\omega_1^2\sqrt{\frac{\varepsilon_0}{\hbar\pi\mathcal{A}}\mathrm{Im}\,\varepsilon(z',\omega_1)}$$
$$\times\sum_{A'}\int\mathrm{d}t''\,\Theta(t'-t'')G^*(z_{A'},z',\omega_1)e^{-i\omega_1(t_1-t'')}\big[\hat{d}_{A'}(t''),\hat{d}_A(t')\big]. \tag{A.31}$$

We multiply both sides of Eq. (A.31) by $i\omega_1^2\mu_0[\mathrm{Im}\,\varepsilon_j\,\hbar\varepsilon_0/(\pi\mathcal{A})]^{1/2}\,G^{(3j)}(0^+,z',\omega_1)$ and perform the sum $\sum_{j=1}^3$ and the integrals $\int_{[j]}\mathrm{d}z'$, by making use of Eq. (3.14). Next, we multiply the result by $-i/(\pi\varepsilon_0\mathcal{A})\,(\omega_2^2/c^2)\,\Theta(t_2-t')\,\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)\,e^{i\omega_2(t_2-t')}$, take the sum with respect to A, evaluate the integration with respect to t', and recall the free-field and the source-field definitions (3.131) and (3.132) to obtain

$$\big[\underline{\hat{E}}^{(3)}_{k\mathrm{free}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega_2,t_2)\big]$$
$$=\frac{1}{\pi^2\varepsilon_0^2\mathcal{A}^2}\frac{\omega_1^2\omega_2^2}{c^4}\sum_{AA'}\mathrm{Im}\,G^{(31)}(0^+,z_{A'},\omega_1)\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)$$
$$\times\int\mathrm{d}t'\int\mathrm{d}t''\,\Theta(t_2-t')\Theta(t'-t'')e^{-i\omega_1(t_1-t'')}e^{i\omega_2(t_2-t')}\big[\hat{d}_{A'}(t''),\hat{d}_A(t')\big]. \tag{A.32}$$

On recalling that $\Theta(x)+\Theta(-x)=1$ and using Eqs. (A.30) and (A.32), we then derive,

$$\begin{aligned}&\big[\underline{\hat{E}}^{(3)}_{k\mathrm{s}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega_2,t_2)\big]+\big[\underline{\hat{E}}^{(3)}_{k\mathrm{free}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega_2,t_2)\big]\\&\quad+\big[\underline{\hat{E}}^{(3)}_{k\mathrm{s}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{free}}(0^+,\omega_2,t_2)\big]=-\frac{1}{\pi^2\varepsilon_0^2\mathcal{A}^2}\frac{\omega_1^2\omega_2^2}{c^4}\sum_{AA'}\mathrm{Im}\,G^{(13)}(z_{A'},0^+,\omega_1)\\&\quad\times\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)\int\mathrm{d}t'\int\mathrm{d}t''\,e^{-i\omega_1(t_1-t')}e^{i\omega_2(t_2-t'')}\big[\hat{d}_{A'}(t'),\hat{d}_A(t'')\big]\\&\qquad\times[\Theta(t_1-t')\Theta(t'-t'')\Theta(t''-t_2)+\Theta(t_2-t'')\Theta(t''-t')\Theta(t'-t_1)].\end{aligned}\tag{A.33}$$

To calculate the commutation relation $[\underline{\hat{E}}^{(3)}_{k\mathrm{in,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega',t')]$, where

$$\underline{\hat{E}}^{(3)}_{k\mathrm{in,free}}(z,\omega,t)=e^{-i\beta_3 z}\,\hat{C}^{(3)}_{-}(\omega,t),\tag{A.34}$$

where $\hat{C}^{(3)}_{-}(\omega,t)$ is given by Eq. (3.69), we as the first step multiply the both sides of Eq. (A.31) by $\mu_0\omega_1 c/[2n_3(\omega_1)]\,\sqrt{\hbar\varepsilon_0/(\pi\mathcal{A})}\,\sqrt{\mathrm{Im}\,\varepsilon_3(\omega_1)}\,e^{i\beta_3(\omega_1)z'}$. After performing the integration with respect to z' variable $\int_{[3]}\mathrm{d}z'$, we multiply both sides with $[i/(\pi\varepsilon_0\mathcal{A})]\,(\omega_2^2/c^2)\,\Theta(t_2-t')\,\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)\,e^{i\omega_2(t_2-t')}$, apply integration with respect to t', and sum with respect to A. Some algebra leads to

$$\begin{aligned}&\big[\underline{\hat{E}}^{(3)}_{k\mathrm{in,free}}(0^+,\omega_1,t_1),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega_2,t_2)\big]=-\frac{i}{\pi^2\varepsilon_0^2\mathcal{A}^2}\frac{\omega_1^2\omega_2^2}{c^4}\frac{\omega_1}{2cn_3}\\&\quad\times\int_{[3]}\mathrm{d}z'\,\varepsilon_3''(\omega_1)\sum_{AA'}e^{i\beta_3(\omega_1)z'}G^{(13)*}(z_{A'},z,\omega_1)\mathrm{Im}\,G^{(13)}(z_A,0^+,\omega_2)\\&\quad\times\int\mathrm{d}t'\int\mathrm{d}t''\,\Theta(t_2-t')\Theta(t'-t'')e^{-i\omega_1(t_1-t'')}e^{i\omega_2(t_2-t')}\big[\hat{d}_{A'}(t''),\hat{d}_A(t')\big].\end{aligned}\tag{A.35}$$

Now we calculate the commutation relation $[\underline{\hat{E}}^{(3)}_{k\mathrm{out,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega',t')]$, using the identity

$$\begin{aligned}&\big[\underline{\hat{E}}^{(3)}_{k\mathrm{out,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega',t')\big]\\&\quad=\big[\underline{\hat{E}}^{(3)}_{k\mathrm{free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega',t')\big]-\big[\underline{\hat{E}}^{(3)}_{k\mathrm{in,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)\dagger}_{k\mathrm{s}}(0^+,\omega',t')\big].\end{aligned}\tag{A.36}$$

Combining Eqs. (A.29), (A.32), (A.33), (A.35), and (A.36), we derive

$$\begin{aligned}&\big[\hat{b}_{k\mathrm{out}}(t),\hat{b}^\dagger_{k'\mathrm{out}}(t')\big]=\frac{1}{2\pi}\frac{\pi\mathcal{A}}{\mu_0 c\hbar}\\&\quad\times\int_{\Delta_k}\mathrm{d}\omega\int_{\Delta_{k'}}\mathrm{d}\omega'\,\frac{|\alpha_{\mathrm{out}}|^2}{\sqrt{\omega\omega'}}\big[\underline{\hat{E}}^{(3)}_{k\mathrm{out,free}}(0^+,\omega,t),\underline{\hat{E}}^{(3)}_{k'\mathrm{out,free}}(0^+,\omega',t')\big].\end{aligned}\tag{A.37}$$

It is shown in App. A.1 that the operators $\hat{b}_{\mathrm{out,free}}(\omega,t)$ [and $\hat{b}^{\dagger}_{\mathrm{out,free}}(\omega,t)$] defined according to Eq. (3.177) with $\underline{\hat{E}}^{(3)}_{\mathrm{out,free}}(z,\omega,t)$ in place of $\underline{\hat{E}}^{(3)}_{\mathrm{out}}(z,\omega,t)$ obey the Bose commutation relation [cf. Eq. (A.21)]

$$\left[\hat{b}_{\mathrm{out,free}}(\omega,t),\hat{b}^{\dagger}_{\mathrm{out,free}}(\omega',t')\right] = e^{-i\omega(t-t')}\delta(\omega-\omega'). \tag{A.38}$$

Hence from Eq. (A.37) it follows that

$$\left[\hat{b}_{k\mathrm{out}}(t),\hat{b}^{\dagger}_{k'\mathrm{out}}(t')\right] = \delta_{kk'}\frac{1}{2\pi}\int_{\Delta_k}\mathrm{d}\omega\, e^{-i\omega(t-t')}. \tag{A.39}$$

Extending, within the approximation scheme used, the limits of integration to $-\infty$ and ∞, we conclude that the commutation relation (3.186) is proved.

Appendix B

One-Dimensional Multilayer Planar Structure

The non-local part of the one-dimensional Green function in the frequency domain reads [cf. Eq. (3.53) in Sec. 3.2.1]

$$G^{(jj')}(z,z',\omega) = \tfrac{1}{2}i\big[\mathcal{E}^{(j)>}(z,\omega)\,\Xi^{jj'}\mathcal{E}^{(j')<}(z',\omega)\Theta(j-j') \\ +\mathcal{E}^{(j)<}(z,\omega)\,\Xi^{j'j}\mathcal{E}^{(j')>}(z',\omega)\Theta(j'-j)\big], \tag{B.1}$$

where the functions

$$\mathcal{E}^{(j)>}(z,\omega) = e^{i\beta_j(z-d_j)} + r_{j/3}e^{-i\beta_j(z-d_j)} \tag{B.2}$$

and

$$\mathcal{E}^{(j)<}(z,\omega) \;=\; e^{-i\beta_j z} + r_{j/0}e^{i\beta_j z} \tag{B.3}$$

[cf. Eqs. (3.54), (3.54)], respectively, represent waves of unit strength traveling rightward and leftward in the jth layer and being reflected at the boundary [note that $\Theta(j-j')$ means $\Theta(z-z')$ for $j=j'$]. Further $\Xi^{jj'}$ is defined in terms of the frequency dependent transmission and reflection coefficients by Eq. (3.56).

To calculate the multilayer reflection and transmission coefficients r_{ij} and t_{ij}, respectively, we first note that in the case $|i-j|=1$, i.e., single-interface transmission, they are defined according to

$$r_{ij} \equiv r_{i/j} = \frac{\beta_i-\beta_j}{\beta_i+\beta_j} = -r_{j/i}, \tag{B.4}$$

$$t_{ij} \equiv t_{i/j} = (1+r_{i/j}) = \frac{\beta_i}{\beta_j}\,t_{ji}, \tag{B.5}$$

leading to

$$t_{ij}t_{ji} - r_{ij}r_{ji} = 1. \tag{B.6}$$

In the general case, the relations (see Ref. [60])

$$r_{i/j/k} = \frac{r_{i/j} + (t_{i/j}t_{j/i} - r_{i/j}r_{j/i})r_{j/k}e^{2i\beta_j d_j}}{1 - r_{j/i}r_{j/k}e^{2i\beta_j d_j}}, \tag{B.7}$$

$$t_{i/j/k} = \frac{t_{i/j}t_{j/k}e^{i\beta_j d_j}}{1 - r_{j/i}r_{j/k}e^{2i\beta_j d_j}} \tag{B.8}$$

$[\min(i,k) \leq j \leq \max(i,k)]$ hold. With these formulas at hand, we can calculate recursively all the quantities $r_{i/j}$ and $t_{i/j}$, since

$$r_{ik} \equiv r_{i/k} = r_{i/j/k}, \tag{B.9}$$

$$t_{ik} \equiv t_{i/k} = t_{i/j/k} \tag{B.10}$$

for any j with $\min(i,k) \leq j \leq \max(i,k)$.

In the four-layer system under consideration the condition of perfect reflection from the left-side mirror of the cavity requires $r_{10} = -1$. Then, the following relations can be easily proved:

$$\frac{1}{D_2}\left(1 \pm r_{20}e^{i\beta_2 d}\right) - \frac{1}{D_2'}\left(1 \pm r_{21}e^{i\beta_2 d}\right) = \mp\frac{t_{13}}{D_1 D_2'}\frac{t_{21}}{t_{23}}\left(1 \pm r_{23}e^{i\beta_2 d}\right)e^{2i\beta_1 l}, \tag{B.11}$$

$$r_{30} - r_{31} = -\frac{t_{13}t_{31}}{D_1}e^{2i\beta_1 l}. \tag{B.12}$$

Using Eqs. (3.57), (3.121), and (B.7), the following relations can be easily proved:

$$\frac{1}{D_2}\left(1 \pm r_{20}e^{i\beta_2 d}\right) - \frac{1}{D_2'}\left(1 \pm r_{21}e^{i\beta_2 d}\right) = \mp\frac{t_{13}}{D_1 D_2'}\frac{t_{21}}{t_{23}}\left(1 \pm r_{23}e^{i\beta_2 d}\right)e^{2i\beta_1 l}, \tag{B.13}$$

$$r_{30} - r_{31} = -\frac{t_{13}t_{31}}{D_1}\,e^{2i\beta_1 l}. \tag{B.14}$$

To distinguish the part of the Green function that relates the outgoing field at point z outside the cavity to the sources at z' in the j'th layer, we may write

$$G_{\text{out}}(z,z',\omega) \equiv G_{\text{out}}^{(3j')}(z,z',\omega) = G^{(3j')}(z,z',\omega) - G_{\text{in}}^{(3j')}(z,z',\omega), \tag{B.15}$$

where

$$G_{\text{in}}^{(3j')}(z,z',\omega) = \frac{1}{2\beta_3(\omega)}ie^{-i\beta_3(\omega)z}e^{i\beta_{j'}(\omega)z'}\Theta(z'-z). \tag{B.16}$$

Then, using Eqs. (B.1) and (B.16), we derive by straightforward calculation

$$\frac{\omega^2}{c^2}\int \mathrm{d}z'\,\varepsilon''(z',\omega)G_{\mathrm{in}}^{(3j')}(z,z',\omega)G^{(1j')*}(z_A,z',\omega) = \tfrac{1}{2}iG^{(31)*}(z,z_A,\omega). \tag{B.17}$$

Employing the integral relation (3.14) for the Green tensor and using Eqs. (B.15) and (B.17), we then find that

$$\frac{\omega^2}{c^2}\int \mathrm{d}z'\,\varepsilon''(z',\omega)G_{\mathrm{out}}(z,z',\omega)G^{(1j')*}(z_A,z',\omega) = -\tfrac{1}{2}iG^{(31)}(z,z_A,\omega). \tag{B.18}$$

We may also decompose the Green tensor inside the cavity, $G^{(11)}(z,z',\omega)$, as given by Eq. (B.1), into bulk and scattering parts $G^{(11)0}$ and $G^{(11)S}$, respectively,

$$G^{(11)}(z,z',\omega) = G^{(11)0}(z,z',\omega) + G^{(11)S}(z,z',\omega), \tag{B.19}$$

where

$$G^{(11)0}(z,z',\omega) = \tfrac{1}{2}i\big[e^{i\beta_1(\omega)(z-z')}\Theta(z-z') + e^{-i\beta_1(\omega)(z-z')}\Theta(z'-z)\big], \tag{B.20}$$

and

$$G^{(11)S}(z,z',\omega) = \frac{g(z,z',\omega)}{D_1(\omega)}\,, \tag{B.21}$$

with

$$g(z,z',\omega) = \tfrac{1}{2}\frac{e^{i\beta_1(\omega)l}}{\beta_1(\omega)}i\left[r_{13}(\omega)e^{-i\beta_1(\omega)(z-l)}\mathcal{E}^{(1)<}(z',\omega) - e^{i\beta_1(\omega)z}\mathcal{E}^{(1)>}(z',\omega)\right]. \tag{B.22}$$

Using the resonance properties of the cavity response function, we may transform $\omega^2 G^S(z,z',\omega)$ according to Eqs. (3.126), (3.127) to obtain

$$f(z,z',t) = \int\frac{\mathrm{d}\omega}{2\pi}\,\omega^2 e^{-i\omega t}\,\frac{g(z,z',\omega)}{D_1(\omega)} = \sum_k \frac{c}{2n_1(\Omega_k)l}\,\Theta(t)\Omega_k^2 g(z,z',\Omega_k)e^{-i\Omega_k t}. \tag{B.23}$$

Further, the inverse transformation reads

$$\omega^2 G^{(11)S}(z,z',\omega) = \int \mathrm{d}t\, e^{i\omega t}f(z,z',t) = \sum_k \frac{c}{2n_1(\Omega_k)l}\,\frac{i\Omega_k^2}{\omega-\Omega_k}\,g(z,z',\Omega_k). \tag{B.24}$$

Notice, Eq. (B.24) yields

$$\begin{aligned} G^S(z_A,z_A,\omega) &\equiv G^{(11)S}(z_A,z_A,\omega) \\ &= -\sum_k \frac{c^2}{\Omega_k|n_1(\Omega_k)|^2 l}\frac{1}{\omega-\Omega_k}\sin^2[\omega_k|n_1(\Omega_k)|z_A/c]. \end{aligned} \tag{B.25}$$

The Green tensor of cavity modelled by the four-layer system features the (complex) resonance frequencies of the cavity [cf. Eq. (3.122)]. The solution for the imaginary part Γ_k—the width of a resonancy—can be found by iteration from Eqs. (3.124) and (3.125) in leading order as

$$\Gamma_k = \frac{c}{2n_1 l}(1 - |r_{13}|^2), \tag{B.26}$$

where n_1, r_{13}, and the parameter introduced in the following are taken at the (unperturbed) frequency ω_k. By means of

$$r_{13} = \frac{-r_{21} + r_{23}e^{2i\beta_2 d_2}}{1 - r_{23}r_{21}e^{2i\beta_2 d_2}} \tag{B.27}$$

Eq. (B.26) can be rewritten as

$$\Gamma_k = \frac{1}{|D_2'|^2}\frac{4n_1}{|n_1+n_2|^2}\left[n_2'\left(1-|r_{23}|^2e^{-4\beta_2''d}\right) + in_2''\left(r_{23}^*e^{-2i\beta_2^*d} - r_{23}e^{2i\beta_2 d}\right)\right]. \tag{B.28}$$

Further, from Eqs. (3.137) ($\omega = \omega_k$) and (B.8) we find

$$|T_k|^2 = \frac{16n_1|n_2|^2 n_3'}{|D_2'|^2|n_1+n_2|^2|n_2+n_3|^2}\,e^{-2\beta_2''d}\,. \tag{B.29}$$

Making use of Eqs. (3.135) and (3.136), we derive ($\omega = \omega_k$)

$$\sum_\lambda |A_{k\lambda}|^2 = \frac{4n_1}{|D_2'|^2|n_1+n_2|^2}\,e^{-\beta_2''d}$$
$$\times\left[2n_2'\sinh(\beta_2''d)(1+|r_{23}|^2e^{-2\beta_2''d}) + 2n_2''\sin(\beta_2'd)(r_{23}e^{i\beta_2 d} + r_{23}^*e^{-i\beta_2^*d})\right]. \tag{B.30}$$

Thus, combining Eqs. (B.28)–(B.30), we arrive at

$$\Gamma_k = \frac{c}{2|n_1|l}\left(|T_k|^2 + \sum_\lambda |A_{k\lambda}|^2\right), \tag{B.31}$$

which matches Eq. (3.160) together with Eqs. (3.161) and (3.162).

Appendix C

Derivation of the Input-Output Relation in Frequency Domain

To derive the outgoing field operator in the frequency domain for times $t \geq t_0$) in terms of the cavity mode operator at time t_0, we first show that the integral term on the right-hand side of the Eq. (3.181) can be rewritten as follows:

$$\begin{aligned}\hat{\Xi}_k(t) &\equiv \int \mathrm{d}t'\, \Theta(t-t') e^{-i\Omega_k(t-t')} \left[T_k(\omega)\hat{b}_{k\mathrm{in}}(\omega,t') + \sum_\lambda A_{k\lambda}(\omega)\hat{c}_{k\lambda}(\omega,t') \right] \\ &= \frac{1}{2\pi}\int_{\Delta_k} \mathrm{d}\omega' \int_{t_0}^{t+\Delta t} \mathrm{d}t''\, e^{i(\omega'-\omega)(t-t'')} \int \mathrm{d}t'\, \Theta(t-t') e^{-i\Omega_k(t-t')} \\ &\quad\times \left[T_k(\omega')\hat{b}_{k\mathrm{in}}(\omega',t') + \sum_\lambda A_{k\lambda}(\omega')\hat{c}_{k\lambda}(\omega',t') \right] \end{aligned} \tag{C.1}$$

($\Delta t \gg \Delta_k^{-1}$). To verify, we first perform the t''-integration on the right-hand side of this equation to obtain

$$\begin{aligned}\hat{\Xi}_k(t) &= \frac{1}{2\pi}\int_{\Delta_k} \mathrm{d}\omega' \int \mathrm{d}t'\, \Theta(t-t') e^{-i\Omega_k(t-t')} \frac{e^{i(\omega-\omega')\Delta t} - 1 + 1 - e^{i(\omega-\omega')(t_0-t)}}{i(\omega-\omega')} \\ &\quad\times \left[T_k(\omega')\hat{b}_{k\mathrm{in}}(\omega',t') + \sum_\lambda A_{k\lambda}(\omega')\hat{c}_{k\lambda}(\omega',t') \right]. \end{aligned} \tag{C.2}$$

In the coarse-grained approximation used, i.e., Δt, $t - t_0 \gg \Delta_k^{-1}$, we may let

$$\frac{e^{i(\omega-\omega')\tau} - 1}{i(\omega-\omega')} \mapsto \zeta_\pm(\omega-\omega') = \begin{cases} i\mathrm{P}\dfrac{1}{\omega-\omega'} + \pi\delta(\omega-\omega') & \text{if} \quad \tau > 0, \\ i\mathrm{P}\dfrac{1}{\omega-\omega'} - \pi\delta(\omega-\omega') & \text{if} \quad \tau < 0 \end{cases} \tag{C.3}$$

(P, principal value). Now the ω'-integration can be easily performed to verify that Eq. (C.1) is correct within the approximation scheme used.

Making change of variables on the right-hand side of the Eq. (C.1) according to $t' \to t - t'' + t'$ and performing ω'-integration [$T_k = T_k(\omega_k)$, $A_{k\lambda} = A_{k\lambda}(\omega_k)$], we find

$$\hat{\Xi}_k(t) = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t+\Delta t} \mathrm{d}t'' \int \mathrm{d}t' \Theta(t''-t') e^{-i\Omega_k(t''-t')} e^{-i\omega(t-t'')} \times \left[T_k \hat{b}_{k\mathrm{in}}(t') + \sum_{\lambda} A_{k\lambda} \hat{c}_{k\lambda}(t') \right]. \tag{C.4}$$

Comparing Eq.(C.4) with Eq.(3.157), we see that

$$\hat{\Xi}_k(t) = \left[\frac{c}{2n_1(\omega_k)l} \right]^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t+\Delta t} \mathrm{d}t'' e^{-i\omega(t-t'')} \hat{a}_k(t''). \tag{C.5}$$

Substitution of Eq. (C.5) into Eq. (3.181) eventually yields the input-output relation (3.214).

Zusammenfassung

In dieser Arbeit wurde die makroskopische Quantenelektrodynamik in linearen Medien angewandt, um eine universell gültige Quantentheorie zur Beschreibung der Wechselwirkung des elektromagnetischen Feldes mit atomaren Quellen in high-Q-Cavities zu entwickeln. In dieser Theorie wird eine vollständige Beschreibung der Charakteristika der emittierten Strahlung angegeben. Die Theorie ermöglicht es, die Grenzen der Anwendbarkeit der üblicherweise verwendeten Theorie aufzuzeigen.

Um eine möglichst allgemeingültige Theorie aufzustellen wurde zunächst die Atom-Feld-Wechselwirkung im Rahmen der makroskopischen Quantenelektrodynamik in dispersiven und absorptiven Medien untersucht. Um das elektromagnetische Feld zu beschreiben wurde von den Maxwell-Gleichungen ausgegangen, wobei die Rauschstromdichten, die mit der Absorption des Mediums verknüpft sind, mit einbezogen wurden. Die Lösung dieser Gleichungen drückt die elektromagnetischen Feldvariablen mittels des Greentensors der makroskopischen Maxwellgleichungen durch die Rauschstromdichten aus. Die explizite Quantisierung wird anhand der Rauschstromdichten durchgeführt, wobei ein diagonaler Hamiltonoperator eingeführt wird, der dann die Zeitentwicklung gemäß den Maxwell-Gleichungen und die Erfüllung der fundamentalen gleichzeitigen Vertauschungsregeln der Feldvariablen gewährleistet. Im Falle der Wechselwirkung des mediengestützten Feldes mit Atomen muß der Hamiltonoperator um Atom-Feld-Wechselwirkungsenergien erweitert werden, wobei die kanonischen Kopplungsschemata der minimalen oder multipolaren Kopplung benutzt werden können. Die dielektrischen Eigenschaften der materiellen Körper sowie ihre Gestalt sind im Greentensor der makroskopischen Maxwell-Gleichungen kodiert. Als vorbereitender Schritt wurde zunächst der Greentensor spezialisiert, um dreidimensionale Input-Output-Beziehungen für die elektromagnetischen Feldoperatoren an einer ebenen Vielschichtstruktur herzuleiten. Eine solche allgemeine Beschreibung des elektromagnetischen Feldes erlaubt die Einbeziehung sowohl der Dispersion und Absorption der Medien als auch der möglichen Gegenwart aktiver Quellen innerhalb oder außerhalb der Schichten und kann deshalb beispielsweise zur Beschreibung von Resonator-Cavities benutzt werden.

Die hergeleiteten Beziehungen wurden deshalb zur Untersuchung einer eindimensionalen Resonator-Cavity, die von einem perfekt reflektierenden und einem teilweise transparenten Spiegel begrenzt wird, verwendet. Um den Effekt unerwünschter Verluste zu untersuchen wurde angenommen, dass sowohl das Medium innerhalb der

Cavity als auch der teilweise transparente Spiegel lineare, dispersive und absorptive Dielektrika sind. Es ist wichtig, darauf hinzuweisen, dass der Greentensor der makroskopischen Maxwellgleichungen nicht nur das räumliche, sondern auch das spektrale Verhalten des elektromagnetischen Feldes charakterisiert. Insbesondere legt er die spektrale Response der Resonator-Cavity fest und liefert die Resonanzfrequenzen und zugehörigen Breiten.

Im ersten Teil der Arbeit wurde von der quellenmäßigen Darstellung des elektromagnetischen Feldes im Rahmen der makroskopischen QED Gebrauch gemacht, wobei die Input-Output-Kopplung für eine Resonator-Cavity im Vordergrund stand. Es wurde angenommen, dass sich die aktiven Atome im Innern der Cavity befinden, und die Felder im Innen- und Außenraum wurden berechnet.

Auf einer Zeitskala, die groß gegenüber dem reziproken Abstand zweier benachbarter Resonanzfrequenzen der Cavity ist, wurde gezeigt, dass sich das Feld im Innenraum durch stehende Wellen ausdrücken läßt und sich zugehörige bosonische Cavity-Modenoperatoren einführen lassen. Es wurde bewiesen, dass diese Operatoren Quanten-Langevin-Gleichungen erfüllen, wobei die Strahlungverluste durch die Input-Output-Kopplung und die Absorptionsverluste die Rolle von Langevin-Rauschkräften spielen und dadurch zu den zugeordneten Dämpfungsraten Anlaß geben. Eine unmittelbare Schlußfolgerung besteht darin, dass der intuitive Zugang verwendet werden kann, um Absorptionsverluste im Rahmen der pänomenologischen Markovschen Dämpfungstheorie offener high-Q-Cavities zusätzlich mit einzubeziehen. Die bilineare Wechselwirkungsenergie, die die strahlungsbezogene Input-Output-Kopplung einer offenen Cavity beschreibt, kann nämlich (im Sinne eines heuristischen Ansatzes) durch ähnliche bilineare Terme ergänzt werden, die dann die Wechselwirkung zwischen der Cavity-Mode und geeignet festgelegten Dissipationskanälen beschreiben, welche für die unerwünschten Verluste verantwortlich sind.

Der intuitive Zugang beschreibt allerdings im allgemeinen die Input-Output-Beziehungen nicht richtig. Es wurde gezeigt, dass die mit dem Kopplungsspiegel assoziierten Absorptionsverluste zu weiteren Krafttermen in den Input-Output-Beziehungen Anlaß geben. Diese können nicht aus den Wechselwirkungsenergien zwischen den Cavity-Moden und den Dissipationskanälen erhalten werden. Es stellte sich aber heraus, dass im Rahmen der Beschreibung einer offenen Cavity durch die Markovsche Dämpfungstheorie die mit dem Kopplungsspiegel assoziierten Absorptionsverluste modelliert werden können, indem vor dem Kopplungspiegel (außerhalb der Cavity) ein geeignetes System von Strahlteilern eingeführt wird.

Es ist bemerkenswert, dass die Kommutatoren zwischen dem Cavity-Moden-Operator und den (zur gleichen Zeit genommenen) einlaufenden Feldoperatoren nicht verschwinden. Dies bringt die Tatsache zum Ausdruck, dass sich das Strahlungsfeld, dass man sich zum Beispiel nach einem kontinuierlichen Satz von Moden entwickelt denken kann, über den ganzen Raum erstreckt. Im Gegensatz dazu werden diese Kommutatoren *per definitionem* gleich Null gesetzt, wenn die Beschreibung offener Cavities mittels der Quantum-Noise-Theorie zugrunde gelegt wird.

Die exakten Input-Output-Beziehungen können verwendet werden, um das Problem der Quantenzustandsextraktion zu untersuchen. Es wurde dabei angenommen, dass eine bestimmte Cavity-Mode in einem Quantenzustand so präpariert wurde, dass die Zeitskala der Zustandspräparation hinreichend klein ist gegenüber der Abklingzeit der Cavity-Mode. In diesem Fall können die sogenannte relevante nicht-monochromatische Mode des einlaufenden Feldes und die relevante nicht-monochromatische Mode des auslaufenden Feldes eingeführt werden, d.h. diejenigen Moden, die an die Cavity-Mode koppeln. Die s-parametrisierte Phasenraumfunktion des Quantenzustandes der relevanten Mode des auslaufenden Feldes wurde durch die Phasenraumfunktion des Quantenzustandes der Cavity-Mode, diejenige der relevanten Mode des einlaufenden Feldes und diejenige der dissipativen Freiheitsgrade, welche für die unerwünschten Verluste verantwortlich sind, ausgedrückt. Für Zeiten, die groß gegenüber der Abklingzeit der Cavity-Mode sind, ist die Effizienz der Quantenzustandsextraktion bestimmt durch das Verhältnis der Abklingrate der Cavity-Mode, die mit den unerwünschten Verlusten verknüpft ist, zu derjenigen Rate, die mit den (erwünschten) Strahlungsverlusten verknüpft ist. Dieses Verhältnis muß hinreichend klein sein, wenn eine nahezu perfekte Extraktion erreicht werden soll. Es sollte erwähnt werden, dass es zur Reduktion des Effektes der unerwünschten Verluste nützlich sein kann, die Transmissionsverluste absichtlich zu vergrößern, damit die unerwünschten Verluste im Vergleich dazu klein sind. Weil sich in diesem Falle andererseits der Gütefaktor der Cavity verringert, muß dann der Präparationsprozeß unbedingt in die Rechnung einbezogen werden.

Um eine allgemein anwendbare Theorie für high-Q-Cavities zur Verfügung zu stellen, wurde im zweiten Teil der Arbeit die exakte Beschreibung der Wechselwirkung des Cavity-gestützten elektromagnetischen Feldes mit einem anfänglich angeregten Zweiniveauatom, welches sich im Innern der Cavity befindet, angegeben. Die starke Atom-Feld-Kopplung kann auftreten, wenn eine der Cavity-Resonanzfrequenzen hinreichend nahe der atomaren Übergangsfrequenz liegt. In diesem Falle ist der Quantenzustand der angeregten auslaufenden Mode aufgrund der unvermeidlich auftretenden unerwünschten Verluste immer ein Gemisch eines Einphotonen-Fockzustandes und des Vakuumzustandes. Im Falle einer fortwährenden Atom-Feld-Kopplung ist die Effizienz für die Präparation der angeregten auslaufenden Mode in einem Einphotonen-Fockzustand zeitabhängig und spiegelt die Rabi-Oszillationen der Atom-Feld-Kopplung wider. Dies liegt daran, dass sich das dem emittierten Photon zugeordnete Wellenpaket für Zeiten, die klein im Vergleich zur reziproken Breite der Cavity-Resonanzlinie sind, gleichzeitig über Regionen innerhalb und außerhalb der Cavity erstreckt. Deshalb kann derjenige Teil des Wellenpaketes, der sich noch innerhalb der Cavity befindet, vom Atom erneut absorbiert werden, wodurch sich die zeitabhängige Effizienz verringert. Die Wahrscheinlichkeit, das emittierte Photon mit einem außerhalb der Cavity platzierten Photodetektor zu registrieren, wird nur von dem Teil der angeregten auslaufenden Mode bestimmt, der sich außerhalb der Cavity befindet. Offensichtlich wächst diese Wahrscheinlichkeit monoton mit der Zeit und geht für große Zeiten asymptotisch gegen die Effizienz für die Präparation eines Einphotonen-Fockzustandes.

Die Rechnungen wurden auch für den Fall ausgeführt, dass die Atom-Feld-Wechselwirkung zu einer Zeit τ unterbrochen wird. Die räumlich-zeitliche Form der angeregten auslaufenden Mode zeigt an, dass sich die Mode gleichzeitig über Regionen innerhalb und außerhalb der Cavity erstreckt. Die Effizienz für die Präparation der angeregten auslaufenden Mode in einem Einphotonen-Fockzustand ist für alle Zeiten $t \geq \tau$ konstant. Ihr Wert ist durch den Wert der Effizienz gegeben, der im Falle der fortwährenden Atom-Feld-Kopplung zur Zeit $t = \tau$ beobachtet wird. Die räumlich-zeitliche Form der angeregten auslaufenden Mode hängt empfindlich von der Atom-Feld-Kopplung ab. Wenn speziell die Atom-Feld-Wechselwirkung zu dem Zeitpunkt unterbrochen wird, zu dem das Atom erstmalig im Grundzustand ist, kann das Wellenpaket als Puls mit einem einzigen Peak angesehen werden, dessen Rückflanke sich über Regionen innerhalb und außerhalb der Cavity erstreckt. Wenn die Wechselwirkungszeit hinreichend kurz im Vergleich zur reziproken Breite der Cavity-Resonanzlinie ist, kann der Puls durch seine Rückflanke approximiert werden. Der Teil der Rückflanke, der sich vollständig außerhalb der Cavity befindet, stimmt dann mit der in den Input-Output-Beziehungen auftretenden relevanten Mode überein. Es sollte bemerkt werden, dass der Kommutator zwischen dem Cavity-Moden-Operator und den einlaufenden Feldoperatoren effektiv nur zur Vorderflanke der angeregten auslaufenden Mode beiträgt und deshalb im Falle hinreichend kurzer Atom-Feld-Wechselwirkung vernachlässigt werden kann.

Abschließend kann gesagt werden, dass die ausgearbeitete Theorie der makroskopischen Quantenelektrodynamik von high-Q-Cavities die vollständige Kenntnis der emittierten Strahlung liefert. Sie bildet einen Ausgangspunkt im Hinblick auf viele Zielstellungen der Quanten-Informationstheorie. Insbesondere stellt sie eine ausführliche Beschreibung einer Einphotonenquelle mit vollständiger, kohärenter Kontrolle über die Eigenschaften des emittierten Feldes zur Verfügung. Da die Theorie allgemein für eine beliebige Dauer der Atom-Feld-Wechselwirkung anwendbar ist, kann sie auf einfache Weise erweitert werden, um die Atom-Feld-Wechselwirkung auch für die Konfiguration der „Raman adiabatic passage“ zu beschreiben. In dieser Konfiguration wäre es möglich, die Abhängigkeit der Systemdynamik von der Position des Atoms in der Cavity zu eliminieren. Desweiteren kann in der Theorie auch der Fall angeregter einlaufender Felder betrachtet werden. Hierdurch würde sofort eine Erweiterung auf die Beschreibung der Wechselwirkung des elektromagnetischen Feldes mit zwei oder mehr Atomen, die sich in zugehörigen Cavities befinden, ermöglicht – ein großer Schritt auf dem Weg zur Realisierung von Quantennetzwerken.

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