Weighted function spaces and traces on fractals

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Contents

Zusammenfassung

Das wesentliche Thema in der allgemeinen Theorie der Funktionenräume der vergangenen Jahrzehnte ist die Untersuchung der Zusammenhänge zwischen Funktionenräumen, Fourieranalysis und Spektraltheorie von Differentialoperatoren und in letzter Zeit auch zur Fraktalen Geometie. Wir benutzen in unserer Arbeit sowohl fundamentale Ideen aus der Theorie der Funktionenräume als auch Methoden der Fraktalen Geometrie, um die Zusammenhänge zwischen Fraktalen und gewichteten Funktionenräumen vom Besov- und Triebel-Lizorkin-Typ, bezeichnet mit $B^s_{pq}(\mathbb{R}^n, w)$ bzw. $F^s_{pq}(\mathbb{R}^n, w)$, zu studieren. In der gesamten Arbeit werden wir Gewichtsfunktionen betrachten, welche zu einer Muckenhoupt Klasse $\mathcal{A}_p$ mit $1 < p < \infty$ gehören. Eine positive Funktion $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ heißt $\mathcal{A}_p$-Gewicht, falls eine positive Konstante $A > 0$ existiert, so dass

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)^{1/p} \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\mathrm{d}x\right)^{1/p'} \leq A,$$

wobei B eine beliebige Kugel in $\mathbb{R}^n$ und $|B|$ ihr Lebesgue-Maß ist. Die Klasse $\mathcal{A}_p$ von Gewichten wurde von B. Muckenhoupt in [Muc72a] eingeführt. Er zeigte, dass die $\mathcal{A}_p$-Gewichte w genau diejenigen Gewichte sind, für die der Hardy-Littlewood-Maximaloperator

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n$$

von $L_p(\mathbb{R}^n, w)$ nach $L_p(\mathbb{R}^n, w)$ beschränkt ist. Eine umfassende Darstellung der Muckenhoupt-Gewichte kann man in dem Buch von J. Garcia-Cuerva und J. L. Rubio de Francia [GR85] finden. Ein systematisches Studium der Besov- und Triebel-Lizorkin-Räume mit Muckenhoupt-Gewichten wurde in den Arbeiten von H. Q. Bui begonnen. Um die Beziehung zwischen Fraktaler Geometrie und der Theorie der Funktionenräume besser zu verstehen, führen wir die spezielle Gewichtsfunktion $w_\varkappa^\Gamma$ ein, die ein Maß für den Abstand zwischen einem gegebenen Punkt $x \in \mathbb{R}^n$ und der fraktalen Menge Γ ist. Mögliche Kandidaten für fraktale Mengen Γ sind dabei d-Mengen und ihre Verallgemeinerungen, die (d, Ψ)-Mengen. Ein Ziel unserer Arbeit ist es, die Interaktion zwischen der Struktur der Fraktale und der Glattheit der Grundfunktionen mittels geeigneter Gewichtsfunktion $w_\varkappa^\Gamma$ zu untersuchen. Ein weiteres Ziel ist, eine atomare Zerlegung für gewichtete Funktionenräume mit Muckenhoupt-Gewichten anzugeben, die für den allgemeinen Fall bewiesen werden.

In der Theorie der Funktionenräume sind viele andere Klassen von Gewichtsfunktionen betrachtet worden. Eine der interessantesten Klassen bilden z.B. die sogenannten "zulässigen Gewichte" (admissible weights). Wir verweisen auf [Tri78] und [SchT87] für weitere Informationen.

In Kapitel 2 wiederholen wir grundlegende Definitionen, legen die Notation fest und stellen die Konzepte vor. Insbesondere definieren wir die klassischen Besov- und Triebel-Lizorkin-Räume und stellen einige Ergebnisse bereit. Abschnitt 2.3 dient der Einführung und dem Studium der Muckenhoupt-Gewichte. Außerdem führen wir die Gewichtsfunktion $w_\varkappa^\Gamma(x) = \operatorname{dist}(x, \Gamma)^\varkappa$ in der Umgebung von Γ ein, wobei Γ eine d-Menge mit $0 < d < n$ ist und studieren ihre Eigenschaften. Das Hauptergebnis dieses Kapitels besagt, dass eine Funktion $w_\varkappa^\Gamma$ genau dann zur Muckenhoupt-Klasse $\mathcal{A}_r$ gehört, wenn $-(n-d) < \varkappa < (n-d)(r-1)$ gilt.

Das dritte Kapitel widmet sich dem atomaren Zerlegungstheorem für gewichtete Besov- und Triebel-Lizorkin-Räume. Wir zeigen, dass jede Distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, die Element eines Besov-Raumes $B^s_{pq}(\mathbb{R}^n, w_\varkappa^\Gamma)$ oder eines entsprechenden Triebel-Lizorkin-Raumes $F^s_{pq}(\mathbb{R}^n, w_\varkappa^\Gamma)$ ist, sich als

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \qquad \text{Konvergenz in} \quad \mathcal{S}'(\mathbb{R}^n),$$

darstellen lässt, wobei $a_{\nu m}(x)$ sogenannte Atome und $\lambda_{\nu m}$ Koeffizienten sind. Desweiteren zeigen wir, dass eine Funktion f zu einem Funktionenraum genau dann gehört, wenn die Folge der komplexen Zahlen $(\lambda_{\nu m})$ zum entsprechenden Folgenraum gehört. Die Ergebnisse des zweiten und dritten Kapitels basieren auf einer Zusammenarbeit mit D. D. HAROSKE und sind zur Veröffentlichung angenommen (siehe [HP]).

In Kapitel 4 wenden wir das atomare Zerlegungstheorem an, um die Spur von gewichteten Besov- und Triebel-Lizorkin-Räumen auf d-Mengen zu berechnen. Das Spurproblem für klassische Besov- und Triebel-Lizorkin-Räume ist in der Literatur ausführlich diskutiert worden. Wir verweisen besonderes auf die Arbeiten von H. TRIEBEL [Tri78] und B. JAWERTH [Jaw77]. Das Problem der Spurcharakterisierung auf Fraktalen ist erst in der letzten Zeit interessant geworden. Die bisher wichtigsten Ergebnisse dieser Entwicklung sind in [Tri97, Kapitel 18] zusammengefasst. Unser Hauptergebnis ist das Folgende. Sei tr_Γ der Spuroperator, wie üblich definiert über punktweise Einschränkung glatter Funktionen auf Γ und deren Vervollständigung. Dann gilt für $\varkappa > -(n-d)$

$$\mathrm{tr}_\Gamma\, B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{p\,\min(1,p)}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma), \qquad 0 < p < \infty,$$

wobei wir die Elemente von $L_p(\Gamma)$ als temperierte Distributionen auf $\mathbb{R}^n$ verstehen. Dieses Ergebnis wurde von einem Resultat von H. TRIEBEL für den nicht gewichteten Fall (siehe [Tri97, Kapitel 18]) inspiriert. Wir beweisen folgendes Resultat für F-Räume (siehe Theorem 4.11):

$$\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p}}_{pp}(\mathbb{R}^n) = \mathbb{B}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}_{pp}(\Gamma), \qquad s > \frac{n-d}{p} + \frac{\varkappa}{p}, \quad 0 < p < \infty,$$

wobei der Spur-Raum $\mathbb{B}^s_{pq}(\Gamma)$ in Definition 4.6 gegeben ist. Insbesondere gilt für $0 < p \leq 1$ und $0 < q \leq \infty$

$$\mathrm{tr}_\Gamma\, F^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma).$$

Am Ende des Kapitels betrachten wir gewichtete Sobolev-Räume mit der speziellen Gewichtsfunktion $w_\alpha(x) = |x_n|^\alpha$. Wir charakterisieren die Spuren dieser Räume auf $(n-1)$-dimensionalen Hyperebenen

$$\mathrm{tr}_{\mathbb{R}^{n-1}}\, W^k_p(\mathbb{R}^n, w_\alpha) = \mathbb{B}^{k-\frac{\alpha+1}{p}}_{pp}(\Gamma), \quad k \in \mathbb{N}, \quad k > \frac{\alpha+1}{p}.$$

Die Ergebnisse des vierten Kapitels sind in der Arbeit [Pio] zussamengestellt und zur Veröffentlichung angenomen. In Kapitel 5 verallgemeinern wir die Resultate des vorherigen Kapitels, indem wir die Spuren gewichteter Räume auf (d,Ψ)-Mengen berechnen.

Im letzten Kapitel werden wir mit Hilfe der zuvor bewiesenen Ergebnisse, das Verhalten der Entropiezahlen der kompakten Einbettung

$$B^{s_1}_{p_1q_1}(\mathbb{R}^n, w^\Gamma_\varkappa) \to B^{s_2}_{p_2q_2}(\mathbb{R}^n, w^\Gamma_\varkappa)$$

untersuchen. Hier ist Γ wieder eine d-Menge oder eine (d,Ψ)-Menge. Schließlich geben wir für d-Mengen Γ noch eine Abschätzung der Approximationszahlen des Spuroperators der mit $w^\Gamma_\varkappa$ gewichteten Besov-Räume an, d.h.

$$e_k\left(\mathrm{tr}_\Gamma :\ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma)\right) \sim k^{\frac{1}{d}\left(\frac{n+\varkappa}{p}-s\right)-\frac{1}{p}} \sim a_k(\mathrm{tr}_\Gamma :\ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma)).$$

Chapter 1

Introduction

It is a central topic in the general theory of function spaces during the last decades to investigate the connection between fractal geometry and function spaces, Fourier analysis and spectral theory of differential operators. In this thesis we follow a basic idea to study the interplay between fractal geometry and weighted function spaces of Besov and Triebel-Lizorkin type denoted by $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$, respectively with $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Throughout what follows, we shall only work with weight functions w that belong to some Muckenhoupt class $\mathcal{A}_p$ with $1 < p < \infty$. Recall that a weight w is said to be an $\mathcal{A}_p$ weight, if there exists a positive constant $A > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)^{1/p} \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\mathrm{d}x\right)^{1/p'} \le A,$$

where B is an arbitrary ball in $\mathbb{R}^n$ with Lebesgue measure $|B|$. The class of $\mathcal{A}_p$ weights was introduced by B. Muckenhoupt in [Muc72a], where he showed that the $\mathcal{A}_p$ weights are precisely those weights w for which the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n$$

is bounded from $L_p(\mathbb{R}^n, w)$ to $L_p(\mathbb{R}^n, w)$. A comprehensive treatment of Muckenhoupt weights may be found in the monograph by J. Garcia-Cuerva and J. L. Rubio de Francia [GR85]. A systematic study of Besov and Triebel-Lizorkin spaces with Muckenhoupt weights was initiated in the works of H. Q. Bui et al. [Bui82, Bui84, BPT96, BPT97]. To investigate the interplay between fractal geometry and function spaces with Muckenhoupt weights we introduce the special weight function $w_\varkappa^\Gamma$ that measures the distance of a given point $x \in \mathbb{R}^n$ to a certain fractal set Γ. Possible candidates for fractal sets to consider are d-sets and their generalizations (d, Ψ)-sets. Our main purpose in this thesis is to study the interaction between the structure of fractals and the smoothness of the underlying functions by means of the corresponding weight function $w_\varkappa^\Gamma$. Another aim of this work is to develop atomic decomposition techniques for function spaces with Muckenhoupt weights, which are proved in the greatest generality.

In the theory of function spaces several other classes of weight functions are considered. As a class of particular interest we mention the so-called admissible weights. The interested reader is referred to [Tri78] and [SchT87] for further details.

Let us now present the contents of this thesis in some detail. Chapter 2 collects fundamental notation and concepts. In particular we define the classical Besov and Triebel-Lizorkin spaces and present a few aspects of their theory. Section 2.3 is devoted to a general study of Muckenhoupt weights. Moreover we introduce the weight function $w_\varkappa^\Gamma(x) = \operatorname{dist}(x, \Gamma)^\varkappa$ in a neighbourhood of Γ, where Γ is some d-set, $0 < d < n$ and study its important properties. The main result in this chapter states that the function $w_\varkappa^\Gamma$ belongs to the Muckenhoupt class $\mathcal{A}_r$ if, and only if, $-(n-d) < \varkappa < (n-d)(r-1)$.

In the third Chapter we will be concerned with an atomic decomposition theorem for weighted Besov and Triebel–Lizorkin spaces. It is shown that the element $f \in \mathcal{S}'(\mathbb{R}^n)$ in the Besov space $B^s_{pq}(\mathbb{R}^n, w)$, or in the corresponding Triebel-Lizorkin space $F^s_{pq}(\mathbb{R}^n, w)$ can be represented as

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \qquad \text{convergence in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{1.1}$$

where $a_{\nu m}(x)$'s are the so-called atoms and the sequence of complex numbers $(\lambda_{\nu m})$ belongs to an appropriate sequence space. Moreover, based on these sequence spaces equivalent quasi–norms for corresponding function spaces are derived. The results obtained in the second and third chapter are accepted for publication in the joint paper with D. D. HAROSKE [HP].

In Chapter 4 we apply the atomic decomposition theorem to compute the trace on the d-set Γ of weighted Besov and Triebel-Lizorkin spaces. There is quite an extensive literature concerning trace problems for classical Besov and Triebel-Lizorkin spaces, beginning with the work of H. TRIEBEL [Tri78] as well as of B. JAWERTH [Jaw77]. The problem of characterizing traces on fractals attracted great attention rather recently, and important progress had been made in [Tri97, Chapter 18]. Our main result here is the following. Let tr_Γ be the trace operator understood as the usual extension of the pointwise restriction operator on Γ. Then for $\varkappa > -(n-d)$ we have

$$\mathrm{tr}_\Gamma\, B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{p\,\min(1,p)}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma), \qquad 0 < p < \infty,$$

where we interpret elements of $L_p(\Gamma)$ in the usual way as a tempered distribution on $\mathbb{R}^n$. This result has been inspired by the unweighted results due to H. TRIEBEL [Tri97, Section 18]. We obtain also the following result for F-spaces (Theorem 4.11)

$$\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p}}_{pp}(\mathbb{R}^n) = \mathbb{B}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}_{pp}(\Gamma), \qquad s > \frac{n-d}{p} + \frac{\varkappa}{p}, \quad 0 < p < \infty,$$

where $\mathbb{B}^s_{pq}(\Gamma)$ is a trace space according to Definition 4.6. In particular, for $0 < p \le 1$ and $0 < q \le \infty$ we have

$$\mathrm{tr}_\Gamma\, F^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = L_p(\Gamma).$$

We conclude this chapter by characterizing traces on $n-1$ dimensional hyperplanes of Sobolev spaces with the special weight function given by $w_\alpha(x) = |x_n|^\alpha$

$$\mathrm{tr}_{\mathbb{R}^{n-1}}\, W^k_p(\mathbb{R}^n, w_\alpha) = \mathbb{B}^{k-\frac{\alpha+1}{p}}_{pp}(\Gamma), \quad k \in \mathbb{N}, \quad k > \frac{\alpha+1}{p}.$$

The results obtained in this chapter are contained in [Pio].
In Chapter 5 we generalize the results from the previous chapter, computing traces of weighted spaces on (d,Ψ)-sets instead of d-sets.
In the final chapter, based on the result obtained so far, we investigate the asymptotic behavior of the entropy numbers of the compact embedding

$$B^{s_1}_{p_1q_1}(\mathbb{R}^n, w^\Gamma_\varkappa) \to B^{s_2}_{p_2q_2}(\mathbb{R}^n, w^\Gamma_\varkappa).$$

Here Γ denotes a d-set or (d,Ψ)-set. Finally, we give estimates on approximation numbers of a trace operator of weighted Besov spaces, e.g.

$$e_k\left(\mathrm{tr}_\Gamma :\ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma)\right) \sim k^{\frac{1}{d}\left(\frac{n+\varkappa}{p}-s\right)-\frac{1}{p}} \sim a_k(\mathrm{tr}_\Gamma :\ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma)).$$

Chapter 2

Preliminaries

2.1 Notation and conventions

In this section we collect some needed notation, which remain fixed throughout this work. Moreover, we briefly recall the classical notions and definitions that will be needed in subsequent chapters.

In the sequel, the symbol $\mathbb{K}$ stands as a synonym for the scalar field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Furthermore, we put $\mathbb{N}_0$ for the non-negative integers. For a real number t let $[t]$ represent the greatest integer less than or equal to t, i.e $[t] = \max\{a \in \mathbb{Z} : \ a \leq t\}$. The positive part of a real function f is given by $f_+(x) = \max(f(x), 0)$. For two positive real sequences $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ we mean by $a_k \sim b_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1 a_k \leq b_k \leq c_2 a_k$ for all $k \in \mathbb{Z}$. For two positive functions on general domains or two positive Borel measures, the notation is defined analogously.

We will denote by $\mathbb{R}^n$ the real n-dimensional Euclidean space. The Euclidean scalar product of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is given by $x \cdot y = x_1 y_1 + \ldots + x_n y_n$. We denote by $|\Omega|$ the n-dimensional Lebesgue measure of $\Omega \subset \mathbb{R}^n$. The characteristic function of a measurable set Ω is denoted by χ_Ω. For any measurable subset $\Omega \subset \mathbb{R}^n$ the *Lebesgue space* $L_p(\Omega)$, $0 < p \leq \infty$ consists of all measurable functions for which

$$\|f \ |L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p} \tag{2.1}$$

is finite. In the limiting case $p = \infty$ the usual modification with the essential supremum is required. Taking $\Omega = \mathbb{N}, \mathbb{Z}$ or $\Omega = \{1, \ldots, n\}$ and replacing the Lebesgue measure by the counting measure produces *Lebesgue sequence spaces* denoted as usual by ℓ_p and ℓ_p^n, respectively. It is known that for $1 \leq p < \infty$ the space $L_{p'}(\Omega)$ is isometrically isomorphic to the dual space $L_p(\Omega)'$. Here p' denotes the *conjugate exponent* of p given by $1/p + 1/p' = 1$.

Let us now discuss some basic facts from the theory of distributions. Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. For $m \in \mathbb{N}$, $C^m(\mathbb{R}^n)$ is the collection of all $\mathbb{K}$-valued functions f having bounded continuous derivatives $\mathrm{D}^\alpha f$ with $|\alpha| \leq m$ on $\mathbb{R}^n$, i.e $C^m(\mathbb{R}^n) = \{f : \mathrm{D}^\alpha f \in C(\mathbb{R}^n) \quad \text{for all} \quad |\alpha| \leq m\}$. Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ stands for some multi-index, whose length is denoted by $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and

$$\mathrm{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \alpha \in \mathbb{N}_0^n;$$

$C^m(\mathbb{R}^n)$ is endowed with the norm $\|f|C^m(\mathbb{R}^n)\| = \sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha f|L_\infty(\mathbb{R}^n)\|$. In addition, we denote by $C^\infty(\mathbb{R}^n)$ the class of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{K}$.

Furthermore $\mathcal{D}(\Omega)$ stands for the subset of functions from $C^\infty(\mathbb{R}^n)$ with compact support in Ω. The *Schwartz space* of all complex-valued, rapidly decreasing C^∞-functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$. The space of continuous linear functionals on $\mathcal{D}$ and $\mathcal{S}$ will be denoted by $\mathcal{D}'$ and $\mathcal{S}'$, respectively.

We define the *Fourier transform* of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} \mathrm{d}x.$$

Here $\mathrm{d}x$ denotes n-dimensional Lebesgue measure. The Fourier transform is a one to one mapping from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Moreover,

$$\mathcal{F}^{-1}(\mathcal{F}f) = f, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\mathcal{F}^{-1}f(\xi) = f^\vee(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{ix\xi} \mathrm{d}x.$$

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'$ in the standard way.

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

We are now in a position to introduce an important tool that will be often used in this work. For any locally integrable function f on $\mathbb{R}^n$, we define the *Hardy-Littlewood maximal operator* M to be

$$Mf(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| \,\mathrm{d}y, \tag{2.2}$$

where the supremum is taken over all balls with the center at the point $x \in B$;

$$B(x,r) = \Big\{ y \in \mathbb{R}^n : \ |y-x| < r \Big\}, \quad x \in \mathbb{R}^n, \ \ r > 0.$$

Moreover, we denote by $Q_{\nu m}$ a cube in $\mathbb{R}^n$ with sides parallel to the axes, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$, where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$.

2.2 Classical function spaces

This section gives an introduction to the main topic of this work: Function spaces of Besov and Triebel-Lizorkin type. They may be defined in a variety of ways, e.g. by derivatives, differences of functions, interpolation methods, Fourier-analytical representations, local means, atomic decomposition, etc.. We restrict ourselves to those one, which play a pivotal rôle in our later consideration. Let us begin with the most common Fourier analytic approach. We first need the concept of a smooth dyadic resolution of unity.

DEFINITION 2.1. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R}^n \ : |y| < 2\} \qquad \text{and} \qquad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1.$$

Furthermore, we let $\varphi_0 = \varphi$ and for each $j \in \mathbb{N}$ we put

$$\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x). \tag{2.3}$$

Then

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

The system of functions $\{\varphi_j\}_{j=0}^{\infty}$ is called a *smooth dyadic resolution of unity.*

By virtue of the Paley-Wiener-Schwartz theorem, $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$. In particular $(\varphi_j \widehat{f})^\vee$ makes sense pointwise. Moreover

$$f = \sum_{j=0}^{\infty} (\varphi_j \widehat{f})^\vee,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. The classical Besov and Triebel-Lizorkin spaces are defined in the following way.

DEFINITION 2.2. (i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth dyadic resolution of unity. We define the *Besov spaces* $B^s_{pq}(\mathbb{R}^n)$ to be the collection of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \, | B^s_{pq}(\mathbb{R}^n) \big\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \big\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(\mathbb{R}^n) \big\|^q \right)^{1/q}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth dyadic resolution of unity. We define the *Triebel-Lizorkin spaces* $F^s_{pq}(\mathbb{R}^n)$ to be the collection of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \, | F^s_{pq}(\mathbb{R}^n) \big\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} | \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)|^q \right)^{1/q} | L_p(\mathbb{R}^n) \right\|$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Let us continue by giving some important comments and remarks.

Remark 2.3. The spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ appearing in their definitions. The proof of this fact may be found in [Tri92]. In particular, both $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces and if $p \ge 1$ and $q \ge 1$, then both are Banach spaces. The Fourier-analytic definition of Besov spaces given here is inspired by the monograph of J. PEETRE [Pee76]. The full treatment of both scales of spaces can be found in monographs [Tri83], [Tri92] and [Tri06].

Next, we present the characterization of Besov and Triebel-Lizorkin spaces in terms of local means. Let $k_0, k^0 \in C^\infty(\mathbb{R}^n)$ with compact support in the unit ball $\{y \in \mathbb{R}^n, |y| < 1\}$, such that $\widehat{k_0}(0) \neq 0$ and $\widehat{k^0}(0) \neq 0$. We put

$$k_N(y) = \Delta^N k^0(y) = \Big(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\Big)^N k^0(y) \quad \text{for} \quad N \in \mathbb{N}.$$

Let $t > 0$, $M \in \mathbb{N}_0$, and $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the corresponding *local means* by

$$k_M(t, f)(x) = \int_{\mathbb{R}^n} k_M(y) f(x + ty) \mathrm{d}y, \qquad x \in \mathbb{R}^n. \tag{2.4}$$

We introduce the abbreviations

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ \quad \text{and} \quad \sigma_{pq} = n \left(\frac{1}{\min(p, q)} - 1 \right)_+ . \tag{2.5}$$

THEOREM 2.4. (i) *Let $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Let $N \in \mathbb{N}$ with $2N > \max(s, \sigma_p)$ then*

$$\|k_0(1, f)|L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k_N(2^{-j}, f)|L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{2.6}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$. Let $N \in \mathbb{N}$ with $2N > \max(s, \sigma_p)$ then*

$$\|k_0(1, f)|L_p(\mathbb{R}^n)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} |k_N(2^{-j}, f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n) \right\| \tag{2.7}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $F^s_{pq}(\mathbb{R}^n)$.

For proof and more details on local means we refer the reader to the monograph [Tri92, 2.4.6, 2.5.3] and references given there. Note that if we look at [Tri06, (1.42), Theorem 1.10] we may sufficiently assume above that $2N > s$.

2.3 Muckenhoupt weights

The purpose of this section is to review some known facts and definitions on $\mathcal{A}_p$ Muckenhoupt classes. Recall that this notion is closely related to the characterization of those non-negative measures $\mathrm{d}\mu$ on $\mathbb{R}^n$ that satisfy maximal inequalities of the form

$$\int_{\mathbb{R}^n} (Mf(x))^p \, \mathrm{d}\mu(x) \;\le\; A \int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}\mu(x), \tag{2.8}$$

for $1 < p < \infty$, and all $f \in L_p(\mathbb{R}^n, \mu)$. Here M is the Hardy-Littlewood maximal operator given by (2.2). It turned out that (2.8) holds exactly in the case when $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$ and w belongs to the so-called Muckenhoupt $\mathcal{A}_p$-class.
In the sequel, let w denote a positive, locally integrable function, i.e. $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

DEFINITION 2.5. We say that w belongs to the *Muckenhoupt class* $\mathcal{A}_p$ with $1 < p < \infty$ if there exists a constant $0 < A < \infty$ such that for all balls B the following inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)\, \mathrm{d}x\right)^{1/p} \cdot \left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\, \mathrm{d}x\right)^{1/p'} \le A, \tag{2.9}$$

where p' is the dual exponent to p given by $1/p' + 1/p = 1$ and $|B|$ stands for the Lebesgue measure of the ball B.

For $p = 1$ we modify the above stated definition in the following way.

DEFINITION 2.6. A weight w belongs to the *Muckenhoupt class* $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality

$$Mw(x) \le Aw(x)$$

holds for almost all $x \in \mathbb{R}^n$. We also consider the Muckenhoupt class $\mathcal{A}_\infty$ defined by

$$\mathcal{A}_\infty = \bigcup_{p \ge 1} \mathcal{A}_p. \tag{2.10}$$

Since the pioneering work of B. MUCKENHOUPT [Muc72b], [Muc72a], [Muc73], these classes of weight functions have been studied in great detail, we refer, in particular, to the monographs [GR85], [ST89], and [Ste93, Chapter V] for a complete account on the theory of Muckenhoupt weights.

For convenience, we recall a few basic properties only; in particular, the class $\mathcal{A}_p$ is stable with respect to translation, dilation and multiplication by a positive scalar. We use the abbreviation

$$w(\Omega) = \int_\Omega w(x)\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set.

LEMMA 2.7. *Let* $1 < p < \infty$.

(i) *If* $w \in \mathcal{A}_p$, *then we have* $w^{-p'/p} \in \mathcal{A}_{p'}$, *where* $1/p + 1/p' = 1$.

(ii) $w(\cdot) \in \mathcal{A}_p$ *if, and only if,* $w(a\ \cdot) \in \mathcal{A}_p$ *for* $a > 0$.

(iii) $w(\cdot) \in \mathcal{A}_p$ *if, and only if,* $w(\cdot - h) \in \mathcal{A}_p$ *for* $h \in \mathbb{R}^n$.

(iv) $w \in \mathcal{A}_p$ *possesses the* doubling property, *i.e. there exists a constant* $c > 0$ *such that*

$$w(B_2) \leq c w(B_1) \tag{2.11}$$

holds for arbitrary balls $B_1 = B(x, r)$ *and* $B_2 = B(x, 2r)$ *with* $x \in \mathbb{R}^n$, $r > 0$.

(v) *Let* $1 \leq p_1 < p_2 \leq \infty$. *Then we have* $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$.

(vi) *If* $w \in \mathcal{A}_p$, *then there exists some number* $r < p$ *such that* $w \in \mathcal{A}_r$.

The proof of (i)-(v) is straightforward, cf. [Ste93, Chapter V]. The extension of (i)-(iv) to $p = \infty$ is clear by (2.10), but there are also counterparts for $p = 1$. However, as we are mainly interested in the case $p = \infty$ later on, we shall not discuss it here. We only want to point out that the somehow surprising property (vi) is closely connected with the so-called 'reverse Hölder inequality', a fundamental feature of $\mathcal{A}_p$ weights, see [Ste93, Chapter V, Proposition 3]. In our case this fact will re-emerge in the number

$$r_0 := \inf\{r\ : w \in \mathcal{A}_r\} < \infty, \quad w \in \mathcal{A}_\infty, \tag{2.12}$$

that plays an essential rôle later on.

Remark 2.8. Obviously, one of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_p$, $1 < p < \infty$, is given by $w(x) = |x|^\varrho$ with $-n < \varrho < n(p-1)$. We are, however, more interested in other examples which will be collected below in 2.11.

Let us recall an important concept from fractal geometry.

DEFINITION 2.9. A set $\Gamma \subset \mathbb{R}^n$ is called *d-set*, $0 < d < n$, if there exists a Borel measure μ in $\mathbb{R}^n$ such that supp $\mu = \Gamma$ and there are constants $c_1, c_2 > 0$ such that for arbitrary $\gamma \in \Gamma$ and all $0 < r < 1$ holds

$$c_1 r^d \leq \mu(B(\gamma, r) \cap \Gamma) \leq c_2 r^d.$$

Note that some self-similar fractals are outstanding examples of d-sets. For instance, the usual (middle-third) Cantor set in $\mathbb{R}^1$ is a d-set for $d = \ln 2/\ln 3$, and the Koch curve in $\mathbb{R}^2$ is a d-set for $d = \ln 4/\ln 3$. It is well-known that $\mu \sim \mathcal{H}^d$, the d-dimensional Hausdorff measure, see [Tri97, Chapter 1].

Remark 2.10. The notion of a d-set appears in fractal geometry as well as in the theory of function spaces. We rely here on the version introduced in [Tri97, Definition 3.1] and [JW84], which is different from [Fal85], see also [Mat95]. Furthermore, this concept was extended and generalized to (d, Ψ)-sets in [ET98], [ET99], [Mou01], h-sets in [Bri04], anisotropic d-sets in [FT99], [Tri97].

Example 2.11. As promised above, we discuss some examples now, starting from the trivial (unweighted) case, up to some more interesting ones related to fractal geometry. We shall allude to these functions (keeping also their special labelling) in connection with our results below.

(a) $w_0 \equiv 1$,

(b) Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$ We consider the weight

$$w_\alpha(x) = \begin{cases} |x_n|^\alpha & |x_n| \leq 1 \\ 1 & \text{otherwise}, \end{cases} \tag{2.13}$$

(c) Let Γ denote a d-set with $0 < d < n$ introduced in Definition 2.9 and let $\varkappa \in \mathbb{R}$. We consider the weight

$$w_\varkappa^\Gamma(x) = \begin{cases} \operatorname{dist}(x, \Gamma)^\varkappa & \operatorname{dist}(x, \Gamma) \leq 1 \\ 1 & \text{otherwise}. \end{cases} \tag{2.14}$$

Plainly, $w_0 \equiv 1$ belongs to all $\mathcal{A}_p$ classes. Moreover, our intention is to use suitably weighted spaces and their atomic representations in order to study trace problems afterwards. This should contain, in particular, weight functions of type (b) and (c). Hence, it is reasonable and for our purpose sufficient to consider only weights that are "locally" (near hyper-planes, fractal d-sets etc.) of a certain $\mathcal{A}_p$-type. We now study the criteria for w_α and $w^\Gamma_\varkappa$ to belong to $\mathcal{A}_p$.

PROPOSITION 2.12. *Let* $1 < p < \infty$.

(i) *Then* $w_\alpha \in \mathcal{A}_p$ *if, and only if,* $-1 < \alpha < p-1$.

(ii) *Let* Γ *be a d-set in* $\mathbb{R}^n$, $0 < d < n$. *Then* $w^\Gamma_\varkappa \in \mathcal{A}_p$ *if, and only if,*

$$-(n-d) < \varkappa < (n-d)(p-1). \tag{2.15}$$

Proof. Observe that the first part of our proposition is a direct consequence of part (ii). It follows easily from the second statement by putting $d = n-1$ and $\Gamma \sim \{x \in \mathbb{R}^n \ : x_n = 0\}$.

To prove part (ii), we first remark that by the definition of $w^\Gamma_\varkappa$ it is sufficient to verify the $\mathcal{A}_p$-condition for balls in a neighbourhood of Γ. Furthermore, recall that weights $w \in \mathcal{A}_p$ possess the doubling property, see (2.11). Hence, instead of dealing with arbitrary such balls $B = B(y,r)$ in the $\mathcal{A}_p$-condition we may restrict ourselves to cubes $Q_{\nu m}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ only. In order to check the $\mathcal{A}_p$-condition in this case we estimate the following integral

$$\frac{1}{|Q_{\nu m}|} \int_{Q_{\nu m}} w^\Gamma_\varkappa(x)\mathrm{d}x.$$

For $k \in \mathbb{N}$ we define sets

$$\begin{aligned} S_k &= \Big\{ x \in \mathbb{R}^n : \ \operatorname{dist}(x,\Gamma) \sim 2^{-k} \Big\} \cap \ Q_{\nu m} \\ &= \Big\{ x \in \mathbb{R}^n : \ 2^{-k-1} < \operatorname{dist}(x,\Gamma) \le 2^{-k} \Big\} \cap \ Q_{\nu m}. \end{aligned}$$

Moreover, for $l = 1, \ldots, N_{k,\nu}$ let K_l denote balls with radius approximately 2^{-k} that cover the set S_k; this is indicated in Figure 2.1.

It turns out that $Q_{\nu m}$ can be covered by $\bigcup_{k=\nu}^\infty S_k$. Then we obtain

$$\begin{aligned} \frac{1}{|Q_{\nu m}|} \int_{Q_{\nu m}} w^\Gamma_\varkappa(x)\mathrm{d}x &= 2^{\nu n} \int_{Q_{\nu m}} w^\Gamma_\varkappa(x)\mathrm{d}x \quad \sim \ 2^{\nu n} \sum_{k=\nu}^\infty \int_{S_k} w^\Gamma_\varkappa(x)\mathrm{d}x \\ &\sim \ 2^{\nu n} \sum_{k=\nu}^\infty 2^{-k\varkappa} \int_{S_k} \mathrm{d}x \ \sim \ 2^{\nu n} \sum_{k=\nu}^\infty 2^{-k\varkappa} \sum_{l=1}^{N_{k,\nu}} \int_{K_l} \mathrm{d}x. \end{aligned}$$

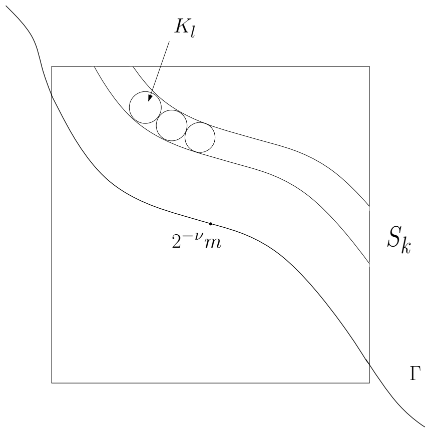

Figure 2.1: Definition of S_k

Using the fact that $|K_l| \sim 2^{-kn}$, $l = 1, \ldots, N_{k,\nu}$, we get

$$\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_{\varkappa}^{\Gamma}(x)\mathrm{d}x \;\sim\; 2^{\nu n} \sum_{k=\nu}^{\infty} 2^{-k(\varkappa+n)} N_{k,\nu}.$$

Taking into account that Γ is a d-set, we conclude that $N_{k,\nu} \sim 2^{(k-\nu)d}$, and this yields

$$\begin{aligned}\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_{\varkappa}^{\Gamma}(x)\mathrm{d}x &\sim 2^{\nu n} \sum_{k=\nu}^{\infty} 2^{-k(\varkappa+n)} 2^{(k-\nu)d} \\ &= 2^{\nu n-\nu(\varkappa+n)} \sum_{k=\nu}^{\infty} 2^{-(k-\nu)(\varkappa+n-d)} = 2^{-\nu\varkappa} \sum_{l=0}^{\infty} 2^{-l(\varkappa+n-d)}.\end{aligned}$$

Obviously, the last series converges if, and only if, $\varkappa > -(n-d)$. Consequently, we obtain

$$\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_{\varkappa}^{\Gamma}(x)\mathrm{d}x \;\sim\; 2^{-\nu\varkappa} \tag{2.16}$$

with equivalence constants independent of $\nu \in \mathbb{N}_0$. On the other hand, with $\gamma := -\varkappa p'/p = -\varkappa(p'-1)$, (2.16) also implies

$$\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} \left(w_{\varkappa}^{\Gamma}(x)\right)^{-p'/p} \mathrm{d}x \;=\; \frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_{\gamma}^{\Gamma}(x)\,\mathrm{d}x \;\sim\; 2^{-\nu\gamma} \tag{2.17}$$

if, and only if, $\gamma > -(n-d)$, which is equivalent to $\varkappa < \frac{p}{p'}(n-d) = (p-1)(n-d)$. Consequently, by (2.16) and (2.17),

$$\left(\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_\varkappa^\Gamma(x) \mathrm{d}x\right) \left(\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} \left(w_\varkappa^\Gamma(x)\right)^{-p'/p} \mathrm{d}x\right)^{p/p'} \sim 2^{-\nu(\varkappa+\gamma\frac{p}{p'})} \sim 1.$$

This shows that $w_\varkappa \in \mathcal{A}_p$, $1 < p < \infty$, if, and only if, $-(n-d) < \varkappa < (p-1)(n-d)$, which finishes the proof.

$\square$

Remark 2.13. In view of Remark 2.8, i.e. $|x|^\varrho \in \mathcal{A}_p$ if, and only if, $-n < \varrho < n(p-1)$, part (i) coincides with this result for $n = 1$, and it would also amount to the 'limiting' case $\Gamma = \{0\}$ (and hence, $d = 0$) of (ii). However, this case is not admitted for the d-set Γ. Moreover, as already mentioned above, we have $w_0 \equiv 1 \in \mathcal{A}_p$, $1 \leq p \leq \infty$, such that we conclude for the corresponding numbers r_0 given by (2.12),

(a) $r_0\left(w_0\right) = 1, \quad w_0 \equiv 1$

(b) $r_0\left(w_\alpha\right) = \max(\alpha+1, 1) = \left\{\begin{array}{ccc} 1 & , & -1 < \alpha \leq 0 \\ \alpha + 1 & , & \alpha > 0 \end{array}\right\}, \quad w_\alpha(x) = |x_n|^\alpha$ locally

(c) $r_0\left(w_\varkappa^\Gamma\right) = \max\left(\frac{\varkappa}{n-d} + 1, 1\right) = \left\{\begin{array}{ccc} 1 & , & -(n-d) < \varkappa \leq 0 \\ \frac{\varkappa}{n-d} + 1 & , & \varkappa > 0 \end{array}\right\},$

$w_\varkappa^\Gamma(x) = \operatorname{dist}(x, \Gamma)^\varkappa$ near a d-set Γ, $0 < d < n$

By (2.10) and Proposition 2.12 we immediately obtain the following result.

COROLLARY 2.14.

(i) $w_\alpha \in \mathcal{A}_\infty$ *if, and only if,* $\alpha > -1$.

(ii) *Let* Γ *be a* d*-set in* $\mathbb{R}^n$ *with* $0 < d < n$. *Then* $w_\varkappa^\Gamma \in \mathcal{A}_\infty$ *if, and only if,* $\varkappa > -(n-d)$.

Chapter 3

Weighted function spaces

In this chapter we deal with weighted function spaces of type $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$, where w is a weight function from the Muckenhoupt class $\mathcal{A}_\infty$. Our goal here is to study atomic decompositions of spaces under consideration.

3.1 Introduction

In this section we define the weighted function spaces and recall their basic properties. In the following, let the weight $w \geq 0$ belong to the class $\mathcal{A}_\infty$ according to (2.10) and let $\{\varphi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity according to Definition 2.1. We define *the weighted Lebesgue spaces* $L_p(\mathbb{R}^n, w)$ with $0 < p < \infty$ as the collection of all measurable functions such that

$$\|f \,|L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \mathrm{d}x \right)^{1/p} \tag{3.1}$$

is finite. Note that for $p = \infty$ one obtains the classical (unweighted) Lebesgue space $L_\infty(\mathbb{R}^n)$; we thus restrict ourselves to $p < \infty$ in what follows.

Our later argument essentially relies on the weighted vector-valued Fefferman-Stein inequality for the Hardy-Littlewood maximal operator, recall its definition (2.2).

THEOREM 3.1. *Suppose that* $1 < p < \infty, 1 < q \leq \infty$ *and* $w \in \mathcal{A}_p$. *Then there is* *a constant* $C > 0$ *such that*

$$\left\| \left(\sum_{k=1}^{\infty} |Mf_k|^q \right)^{1/q} \Big| L_p(\mathbb{R}^n, w) \right\| \leq C \left\| \left(\sum_{k=1}^{\infty} |f_k|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\|$$

holds for any $(f_k) \subset L_p(\mathbb{R}^n, w)$.

A proof of this crucial result may be found in [Kok78, Theorem 1], [AJ80, Theorem 3.1], see also [Bui82, Lemma 1.1], [GR85].
We are now in a position to state the definitions of weighted Besov and Triebel - Lizorkin spaces.

DEFINITION 3.2. Let $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $\{\varphi_j\}$ be a smooth dyadic resolution of unity according to Definition 2.1. Assume $w \in \mathcal{A}_\infty$.

(i) For $0 < p \leq \infty$ we define *weighted Besov spaces* $B^s_{pq}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \,|B^s_{pq}(\mathbb{R}^n, w) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p(\mathbb{R}^n, w) \right\|^q \right)^{1/q} \tag{3.2}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

(ii) For $0 < p < \infty$ we define *weighted Triebel - Lizorkin spaces* $F^s_{pq}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \,|F^s_{pq}(\mathbb{R}^n, w) \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\| \tag{3.3}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 3.3. The spaces $B^s_{pq}(\mathbb{R}^n, w)$ and $F^s_{pq}(\mathbb{R}^n, w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \geq 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, similarly for the F-case, where the first embedding is dense if $q < \infty$; cf. [Bui82]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_\infty$ we re-obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces; we refer, in particular, to the series of monographs by H. TRIEBEL, [Tri78], [Tri83], [Tri92], [Tri97] and [Tri01] for a comprehensive treatment of the unweighted spaces.

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been first studied systematically by H. Q. BUI in [Bui82], [Bui84], with subsequent papers [BPT96], [BPT97]. It turned out that many of the results from the unweighted situation have weighted counterparts : e.g. we have $F^0_{p,2}(\mathbb{R}^n, w) = h_p(\mathbb{R}^n, w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [Bui82, Theorem 1.4], and, in particular, $h_p(\mathbb{R}^n, w) = L_p(\mathbb{R}^n, w) = F^0_{p,2}(\mathbb{R}^n, w)$, $1 < p < \infty$, $w \in \mathcal{A}_p$, see [ST89, Chapter VI, Theorem 1]. Concerning (classical) Sobolev spaces $W^k_p(\mathbb{R}^n, w)$ (built upon $L_p(\mathbb{R}^n, w)$ in the usual way) it holds $W^k_p(\mathbb{R}^n, w) = F^k_{p,2}(\mathbb{R}^n, w)$, $k \in \mathbb{N}_0$, $1 < p < \infty$, $w \in \mathcal{A}_p$, cf. [Bui82, Theorem 2.8]. Further results, concerning, for instance, embeddings, (real) interpolation, extrapolation, lift operators, duality assertions can be found in [Bui82], [Bui84], [GR85], [Rou04b].
Later this topic was revived and extended by V. S. RYCHKOV in [Ryc01], including also approaches for locally regular weights. The latter underwent some renaissance recently in connection with compact embeddings which will be discuss in Chapter 6. In particular, starting from the series of papers [HT94a], [HT94b], [Har95], closely connected with the proto-type $w(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha \in \mathbb{R}$, new contributions were achieved in [HT05], [KLSS06a], [KLSS06b], [Skr], [KLSS], all related to such locally regular weights. Moreover, T. SCHOTT obtained some results for exponential weights [Sch98a], [Sch98b]. In some sense V. S. RYCHKOV introduced a new weight class $\mathcal{A}^{\text{loc}}_p$ that contains both Muckenhoupt as well as such locally regular weights. Recent works, devoted to matrix $\mathcal{A}_p$ weights in Besov spaces are due to S. ROUDENKO [Rou04a], [Rou04b], [FR04], see also [NT] and [Vol97] (for some F-cases). We shall return to this approach in connection with atomic decompositions below.

We recall the definition of Peetre's maximal function as it plays an essential rôle in the proof of the atomic decomposition.

DEFINITION 3.4. Let $r > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. For a sequence $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$ given by (2.3) we define *Peetre's maximal function* by

$$(\varphi^*_j f)_r(x) = \sup_{z \in \mathbb{R}^n} \frac{\left|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(x-z)\right|}{(1 + 2^j |z|)^r} \quad \text{for} \quad x \in \mathbb{R}^n.$$

We now present a fundamental characterization of weighted spaces under consideration. The following result is due to H. Q. BUI [Bui82].

THEOREM 3.5. *Let* $\{\varphi_j\}$ *be a smooth dyadic resolution of unity and let* $0 < p < \infty, 0 < q \le \infty$, $s \in \mathbb{R}$ *and* $w \in \mathcal{A}_\infty$ *with* r_0 *given by* (2.12). *Let* $(\mathcal{F}\varphi)(0) \neq 0$.

(i) *If, in addition,* $r > \frac{nr_0}{p}$, *then*

$$\big\|f \;|B^s_{pq}(\mathbb{R}^n, w)\big\|^* = \left(\sum_{j=0}^{\infty} 2^{jsq} \big\|(\varphi_j^* f)_r|L_p(\mathbb{R}^n, w)\big\|^q\right)^{1/q}$$

(*with usual modification for* $q = \infty$) *is an equivalent quasi-norm in* $B^s_{pq}(\mathbb{R}^n, w)$.

(ii) *If, in addition,* $r > \max\left(\frac{nr_0}{p}, \frac{n}{q}\right)$, *then*

$$\big\|f \;|F^s_{pq}(\mathbb{R}^n, w)\big\|^* = \left\|\left(\sum_{j=0}^{\infty} 2^{jsq} \big|(\varphi_j^* f)_r(\cdot)\big|^q\right)^{1/q} \Big|L_p(\mathbb{R}^n, w)\right\|$$

(*with usual modification for* $q = \infty$) *is an equivalent quasi-norm in* $F^s_{pq}(\mathbb{R}^n, w)$.

For a proof see [Bui82, Theorem 2.4] and also the discussion in [Ryc01, Proposition 2.1]. Note that by our remarks in Example 2.11 we have $r_0 = 1$ in the unweighted case $w_0 \equiv 1$, such that the above setting coincides with the results of J. PEETRE [Pee75] and H. TRIEBEL [Tri92, Theorem 2.3.2].

We recall some characterization of the above spaces in terms of local means. Let $k_N(2^{-j}, f)$ be given by (2.4) with $t = 2^{-j}$. Then H. Q. BUI, M. PALUSZYŃSKI and M. H. TAIBLESON proved in [BPT96], [BPT97] the following result.

THEOREM 3.6. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_\infty$, *and* $N \in \mathbb{N}$ *sufficiently large. Then*

$$\big\|f \;|B^s_{pq}(\mathbb{R}^n, w)\big\|_* = \tag{3.4}$$
$$\|k_0(1, f) \;|L_p(\mathbb{R}^n, w)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \big\|k_N(2^{-j}, f) \;|L_p(\mathbb{R}^n, w)\big\|^q\right)^{1/q}$$

(*with the usual modification for* $q = \infty$) *is an equivalent quasi-norm in* $B^s_{pq}(\mathbb{R}^n, w)$, *and*

$$\left\| f \,|F^s_{pq}(\mathbb{R}^n, w)\right\|_* = \qquad (3.5)$$
$$\|k_0(1, f)\,|L_p(\mathbb{R}^n, w)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left|k_N(2^{-j}, f)(\cdot)\right|^q \right)^{1/q} \Big| L_p(\mathbb{R}^n, w) \right\|$$

(*with the usual modification for* $q = \infty$) *is an equivalent quasi-norm in* $F^s_{pq}(\mathbb{R}^n, w)$.

Remark 3.7. Note that we have actually stated a reformulation of the original result of H. Q. BUI ET AL. in terms of local means. The same argument as in [Tri92, Theorems 2.4.6, 2.5.1] may be applied to our case for clarifying the use of local means instead of convolution. Similar to the unweighted case [Tri92] the number N has to be chosen sufficiently large depending on s, p, q and - in our case - r_0, see also Theorem 3.5 or Theorem 3.11 below.
Note that for F-spaces one also has the so-called 'localization principle' : let $w \in \mathcal{A}_\infty$, $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $\gamma \in \mathcal{S}(\mathbb{R}^n)$, compactly supported, with $\sum_{k\in\mathbb{Z}^n} \gamma(x-k) = 1,\ x \in \mathbb{R}^n$. Then

$$\left\|f|F^s_{pq}(\mathbb{R}^n, w)\right\| \sim \left(\sum_{k\in\mathbb{Z}^n} \left\|\gamma(\cdot - k)f|F^s_{pq}(\mathbb{R}^n, w)\right\|^p \right)^{1/p},$$

see [Ryc01, Theorem 2.21].

Recall that for any $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, let $Q_{\nu m}$ denote an n-dimensional cube with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$ and with side length $2^{-\nu}$. The main goal of this section is to prove an atomic decomposition result for spaces of type $B^s_{pq}(\mathbb{R}^n, w)$, $F^s_{pq}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$. For that reason we introduce these special building blocks, i.e. atoms.

DEFINITION 3.8.

(a) Suppose that $K \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K-*atom* (or simply an 1-*atom*) if the following assumptions are satisfied

(i) $\operatorname{supp} a \subset dQ_{0m}$ for some $m \in \mathbb{Z}^n$,

(ii) $|\mathrm{D}^\alpha a(x)| \le 1$ for $|\alpha| \le K, \quad x \in \mathbb{R}^n$.

(b) Suppose that $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s,p)_{K,L}$*-atom* (or simply an (s,p) -atom) if for some $\nu \in \mathbb{N}_0$ the following assumptions are satisfied

(i) $\operatorname{supp} a \subset dQ_{\nu m}$ for some $m \in \mathbb{Z}^n$,

(ii) $|\mathrm{D}^\alpha a(x)| \le 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}$ for $|\alpha| \le K$, $x \in \mathbb{R}^n$,

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$ for $|\beta| \le L$.

When $L = -1$, we shall mean in (b) that there is no moment condition (iii). In the sequel, we will write $a_{\nu m}$ instead of a, to indicate the localization and size of an $(s,p)_{K,L}$-atom a. In order to obtain an atomic decomposition for the weighted function spaces we still need appropriately weighted sequence spaces $b_{pq}(w)$ and $f_{pq}(w)$. For this purpose we adapt the (matrix-weighted) Besov sequence spaces used by S. Roudenko in [Rou04a] (there are also f-versions in [NT], [Vol97] for instance). These are weighted counterparts of the original ones by M. Frazier and B. Jawerth [FJ85], [FJ90], [FJW91]. Note that S. Roudenko deals with so-called molecules to obtain a corresponding decomposition of matrix-weighted Besov spaces, see [Rou04a, Theorems 11.3, 11.4], whereas we concentrate on very special molecules, that is, atoms. Moreover, we prefer a slightly different normalization as already given in Definition 3.8, part (ii) : Following the notation from [Rou04a] let us write $m_{Q_{\nu m}} = m_{\nu m}$ (with compactly supported molecules in obvious notation). According to condition [Rou04a, (M3), p. 282], we arrive at

$$|\mathrm{D}^\alpha m_{\nu m}(x)| \le 2^{\nu\frac{n}{2}+\nu|\alpha|} \tag{3.6}$$

compared with (ii) in Definition 3.8(b). Thus we put $a_{\nu m} = 2^{-\nu(s-\frac{n}{p})-\nu\frac{n}{2}} m_{\nu m}$ and have to compensate this in the coefficients by $\lambda_{\nu m} = 2^{\nu(s-\frac{n}{p})+\nu\frac{n}{2}}\widetilde{\lambda}_{\nu m}$, where $\widetilde{\lambda}_{\nu m}$ are the coefficients in the corresponding molecular decomposition in [Rou04a]. For a function $f \in B^s_{pq}(\mathbb{R}^n, w)$ we thus get

$$f = \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \widetilde{\lambda}_{\nu m} m_{\nu m}(x) = \sum_{\nu=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x),$$

assuming the first decomposition to hold. In other words, due to this re-normalization we have to modify the sequence $\lambda = \{\lambda_{\nu m}\}_{\nu,m}$ compared with $\widetilde{\lambda}$ from [Rou04a], and are thus led to an adapted sequence space version for $b_{pq}(w)$ accordingly.

For $0 < p < \infty$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we denote by $\chi_{\nu m}^{(p)}$ the p-normalized characteristic function of the cube $Q_{\nu m}$ defined by

$$\chi_{\nu m}^{(p)}(x) = 2^{\frac{\nu n}{p}} \chi_{\nu m}(x) = \begin{cases} 2^{\frac{\nu n}{p}} & \text{for} \quad x \in Q_{\nu m} \\ 0 & \text{for} \quad x \notin Q_{\nu m}. \end{cases} \tag{3.7}$$

Simple computation shows that $\|\chi_{\nu m}^{(p)}|L_p(\mathbb{R}^n)\| = 1$.

DEFINITION 3.9. Let $0 < p < \infty$, $0 < q \le \infty$, $w \in \mathcal{A}_\infty$, and put $\lambda = \{\lambda_{\nu m} \in \mathbb{C} \,:\, \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$. We define

$$b_{pq}(w) = \tag{3.8}$$
$$\left\{ \lambda = \{\lambda_{\nu m}\} \,:\, \big\|\lambda \,|b_{pq}(w)\big\| = \left(\sum_{\nu=0}^{\infty} \Big\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi_{\nu m}^{(p)} \,|L_p(\mathbb{R}^n, w)\Big\|^q \right)^{1/q} < \infty \right\}$$

and

$$f_{pq}(w) = \tag{3.9}$$
$$\left\{ \lambda = \{\lambda_{\nu m}\} \,:\, \big\|\lambda \,|f_{pq}(w)\big\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \Big|\lambda_{\nu m}\chi_{\nu m}^{(p)}(\cdot)\Big|^q \right)^{1/q} \Big| L_p(\mathbb{R}^n, w) \right\| < \infty \right\}$$

(usual modification for $q = \infty$).

We now discuss our Examples 2.11 from Chapter 2.

Example 3.10. (1) Let us first consider the weight $w_0(x) = 1$. Then we obtain

$$\begin{aligned} \big\|\lambda \,|b_{pq}(w_0)\big\| &= \left(\sum_{\nu=0}^{\infty} \Big\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi_{\nu m}^{(p)} \,|L_p(\mathbb{R}^n, w_0)\Big\|^q \right)^{1/q} \\ &\sim \left(\sum_{\nu=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} = \big\|\lambda \,|b_{pq}\big\|. \end{aligned}$$

Consequently, these spaces coincide with the unweighted spaces b_{pq} introduced in [Tri97, (13.27)]. A similar argument works for $f_{pq}(w_0) = f_{pq}$.

(2) We consider $w_{\varkappa}^{\Gamma}$ introduced in Example 2.11(c),

$$w_{\varkappa}^{\Gamma}(x) = \begin{cases} \operatorname{dist}(x,\Gamma)^{\varkappa} & \operatorname{dist}(x,\Gamma) \le 1 \\ 1 & \text{otherwise} \end{cases},$$

where Γ is a d-set, $0 < d < n$. We restrict ourselves to the b-case only.

According to Corollary 2.14 (ii), $w_{\varkappa}^{\Gamma}$ belongs to $\mathcal{A}_\infty$ if, and only if, $\varkappa > -(n-d)$. Next we compute the norm $\left\| \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\|$. From what has been proved in Proposition 2.12(ii), in particular in (2.16), we conclude that

$$\begin{aligned} \left\| \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\| &= \left(2^{\nu n} \int\limits_{Q_{\nu m}} w_{\varkappa}^{\Gamma}(x)\, \mathrm{d}x \right)^{1/p} \\ &= \left(\frac{1}{|Q_{\nu m}|} \int\limits_{Q_{\nu m}} w_{\varkappa}^{\Gamma}(x)\, \mathrm{d}x \right)^{1/p} \sim 2^{-\frac{\nu\varkappa}{p}}. \end{aligned}$$

We thus can summarize the above considerations as follows:

- For $Q_{\nu m} \cap \Gamma = \emptyset$ we obtain that

$$\left\| \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\|^p \sim \left\| \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_0) \right\|^p \sim 1.$$

- For $Q_{\nu m} \cap \Gamma \neq \emptyset$ choose $x^{\nu,m}$ in a neighbourhood of Γ almost at the centers $2^{-\nu}m$ of a cube $Q_{\nu m}$. It follows that $\operatorname{dist}(x^{\nu,m},\Gamma) \sim 2^{-\nu}$ and

$$\left\| \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\|^p \sim 2^{-\nu\varkappa} \sim w_{\varkappa}^{\Gamma}(x^{\nu,m})$$

for all $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

The controlled overlapping of $Q_{\nu m}$ and Definition 3.9 thus lead to

$$\begin{aligned} \|\lambda \,| b_{pq}(w_{\varkappa}^{\Gamma})\| &= \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} \chi_{\nu m}^{(p)} \,| L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\|^q \right)^{1/q} \\ &\sim \left(\sum_{\nu=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_{\varkappa}^{\Gamma}(x^{\nu,m}) \right)^{q/p} \right)^{1/q}. \end{aligned}$$

(3) Replacing $w_{\varkappa}^{\Gamma}$ in (2) by w_α from Example 2.11(b), $\alpha > -1$, we obtain

$$\left\|\chi_{\nu m}^{(p)} \,|L_p(\mathbb{R}^n, w_\alpha)\right\| = \left(2^{\nu n} \int\limits_{Q_{\nu m}} w_\alpha(x)\,\mathrm{d}x\right)^{1/p} \sim 2^{-\frac{\nu\alpha}{p}} \sim w_\alpha\left(x^{\nu,m}\right) \quad (3.10)$$

with $x^{\nu,m} \sim 2^{-\nu}m$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, with $|m_n| < \delta$, and $\delta > 0$ small. Likewise we arrive at

$$\|\lambda\,|b_{pq}(w_\alpha)\| \sim \left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p w_\alpha(x^{\nu,m})\right)^{q/p}\right)^{1/q}.$$

3.2 An atomic decomposition

In recent years it turned out that atomic and sub-atomic (quarkonial), as well as wavelet decompositions of such spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces of the above type, where arguments can be equivalently transferred to the sequence space setting, which is often more convenient to handle. But this applies equally to questions of mapping properties of pseudo-differential operators, to trace problems, and – last but not least – gives a powerful method when dealing with spaces defined on fractals. The idea of atomic decompositions in the above sense leads back to M. FRAZIER and B. JAWERTH in their series of papers [FJ85], [FJ90], [FJW91], see also [Tri97, Section 13].

Recall our notation

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+, \qquad \sigma_{p,q} = n\left(\frac{1}{\min(p,q)} - 1\right)_+, \quad (3.11)$$

where $0 < p < \infty$, $0 < q \le \infty$. Our main result is the following.

THEOREM 3.11. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and* $w \in \mathcal{A}_\infty$ *be a weight with* r_0 *given by* (2.12).

(i) *Let* $K, L+1 \in \mathbb{N}_0$ *with*

$$K \ge (1+[s])_+ \quad \textit{and} \quad L \ge \max\left(-1, [\sigma_{p/r_0} - s]\right). \quad (3.12)$$

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be written as a series*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \qquad \textit{converging in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{3.13}$$

where $a_{\nu m}(x)$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *and* $\lambda \in b_{pq}(w)$*. Furthermore*

$$\|f\|_{\bullet} = \inf \|\lambda | b_{pq}(w)\| \tag{3.14}$$

is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w)$*, where the infimum ranges over all admissible representations* (3.13)*.*

(ii) *Let* $K, L+1 \in \mathbb{N}_0$ *with*

$$K \geq (1+[s])_+ \qquad \textit{and} \qquad L \geq \max\left(-1, [\sigma_{p/r_0,q} - s]\right). \tag{3.15}$$

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $F^s_{pq}(\mathbb{R}^n, w)$ *if, and only if, it can be written as a series* (3.13)*, where* $a_{\nu m}(x)$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *and* $\lambda \in f_{pq}(w)$*. Furthermore*

$$\|f\|_{\bullet} = \inf \|\lambda | f_{pq}(w)\| \tag{3.16}$$

is an equivalent quasi-norm in $F^s_{pq}(\mathbb{R}^n, w)$*, where the infimum ranges over all admissible representations* (3.13)*.*

As preparation for the proof of our atomic decomposition theorem we need the following result by M. FRAZIER and B. JAWERTH [FJ85].

LEMMA 3.12. *There exist functions* $\theta_0, \theta, \varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ *with the following properties*

$$|\widehat{\theta}_0(\xi)| > 0 \quad \textit{for} \quad |\xi| \leq 2, \tag{3.17}$$

$$|\widehat{\theta}(\xi)| > 0 \quad \textit{for} \quad 1/2 \leq |\xi| \leq 2, \tag{3.18}$$

$$\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \ \textit{and} \ |\varphi_0(\xi)| > 0 \ \textit{for} \ |\xi| \leq \delta, \tag{3.19}$$

$$\operatorname{supp} \varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \ \textit{and} \ |\varphi(\xi)| > 0 \ \textit{for} \ 3/5 \leq |\xi| \leq 2 \tag{3.20}$$

and

$$\widehat{\theta}_0(\xi)\varphi_0(\xi) + \sum_{\nu=1}^{\infty} \widehat{\theta}(2^{-\nu}\xi)\varphi(2^{-\nu}\xi) = 1 \ \textit{for all} \ \xi \in \mathbb{R}^n. \tag{3.21}$$

A proof is given in [FJW91, Lemma 5.12]. Moreover, according to [FJ85, p. 783], it is even possible to assume, in addition, that

$$\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x| \le 1\}, \tag{3.22}$$

$$\int_{\mathbb{R}^n} x^\beta \theta(x)\,\mathrm{d}x = 0, \quad |\beta| \le L \tag{3.23}$$

for a given number $L \in \mathbb{N}_0$.

Proof of Theorem 3.11. We divide the proof into three steps. The first two steps closely follow the argument in [Far00, Section 5.1] (related to anisotropic, unweighted spaces), the third step is very similar to [Tri97, Theorem 13.8] (in the unweighted setting).

Step 1. Assume that $f \in B^s_{p,q}(\mathbb{R}^n, w)$ or $f \in F^s_{p,q}(\mathbb{R}^n, w)$, respectively, and let $\theta_0, \theta, \varphi_0$ and φ be the functions introduced in Lemma 3.12. Short computation together with (3.21) give

$$\begin{aligned} f(x) &= \left(\theta_0 * \mathcal{F}^{-1}(\varphi_0 \mathcal{F}f)\right)(x) + \sum_{\nu=1}^{\infty} 2^{\nu n} \left(\theta(2^\nu \cdot) * \mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)\right)(x) \\ &= \sum_{m\in\mathbb{Z}^n} \int_{Q_{\nu m}} \theta_0(x-y)\mathcal{F}^{-1}(\varphi_0\mathcal{F}f)(y)\mathrm{d}y \\ &\quad + \sum_{\nu=1}^{\infty} 2^{\nu n} \sum_{m\in\mathbb{Z}^n} \int_{Q_{\nu m}} \theta(2^\nu(x-y))\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\,\mathrm{d}y \end{aligned} \tag{3.24}$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define the coefficients and atoms in formula (3.13) as follows: for each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$ put

$$\lambda_{\nu m} = 2^{\nu(s-\frac{n}{p})} C \sup_{y\in Q_{\nu m}} \left|\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\right| \tag{3.25}$$

with

$$C = \max_{|\alpha|\le K} \sup_{|x|\le 1} |\mathrm{D}^\alpha \theta(x)|$$

and

$$a_{\nu m}(x) = \lambda_{\nu m}^{-1}\; 2^{\nu n} \int_{Q_{\nu m}} \theta(2^\nu(x-y))\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\,\mathrm{d}y. \tag{3.26}$$

Hence by (3.24) the decomposition (3.13) is satisfied. Let us now check that such $a_{\nu m}$ are atoms in the sense of Definition 3.8 (b). Note that the support and moment conditions are clear by (3.22) and (3.23), respectively. It thus remains to check (ii) in Definition 3.8 (b):

$$\begin{aligned}
|\mathrm{D}^\alpha a_{\nu m}(x)| &= \left| \mathrm{D}^\alpha \left(\lambda_{\nu m}^{-1}\ 2^{\nu n} \int\limits_{Q_{\nu m}} \theta(2^\nu(x-y))\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\,\mathrm{d}y \right) \right| \\
&\le \lambda_{\nu m}^{-1}\ 2^{\nu n} \int\limits_{Q_{\nu m}} \left|(\mathrm{D}^\alpha \theta(2^\nu(x-y)))\,\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\right|\,\mathrm{d}y \\
&\le 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|} 2^{\nu n} \left(\sup_{y\in Q_{\nu m}} |\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)| \right)^{-1} \cdot \\
&\quad \cdot \int\limits_{Q_{\nu m}} |\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)|\,\mathrm{d}y \\
&\le 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|},
\end{aligned}$$

as desired. The modifications for the terms with $\nu = 0$ are obvious.

Step 2. Next we show that there is a constant $c > 0$ such that

$$\|\lambda|b_{pq}(w)\| \le c\|f\ |B^s_{pq}(\mathbb{R}^n, w)\|,$$

similarly for the F-case. For that reason we exploit the equivalent quasi-norms given in Theorem 3.5 involving Peetre's maximal function, see Definition 3.4. Let us fix $\nu \in \mathbb{N}$. Taking into account that $\sum_{m\in\mathbb{Z}^n} \chi_{\nu m}(x) = 1$ and $|x-y| \le c2^{-\nu}$ for $x, y \in Q_{\nu m}$ we obtain

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\chi^{(p)}_{\nu m}(x) &= 2^{\nu(s-\frac{n}{p})} C \sum_{m\in\mathbb{Z}^n} \sup_{y\in Q_{\nu m}} \left|\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(y)\right| 2^{\nu\frac{n}{p}} \chi_{\nu m}(x) \\
&\le C_1\, 2^{\nu s} \left(\sup_{|z|\le c2^{-\nu}} \frac{\left|\mathcal{F}^{-1}(\varphi(2^{-\nu}\cdot)\mathcal{F}f)(x-z)\right|}{(1+2^\nu|z|)^r} (1+2^\nu|z|)^r \right) \\
&\le C_2\, 2^{\nu s} (\varphi^*_\nu f)_r(x) \qquad (3.27)
\end{aligned}$$

for arbitrary $r > 0$, with $\varphi_\nu = \varphi(2^{-\nu}\cdot)$. Hence, combining the monotonicity of the quasi-norm with the summation over $\nu \in \mathbb{N}$ yields

$$\sum_{\nu=1}^{\infty}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{\nu m}^{(p)}|L_p(\mathbb{R}^n,w)\right\|^q\leq c\sum_{\nu=1}^{\infty}2^{\nu sq}\left\|(\varphi_\nu^* f)_r|L_p(\mathbb{R}^n,w)\right\|^q.$$

Finally, choosing $r>\frac{nr_0}{p}$ sufficiently large, the last inequality (which is by the same argument also true for $\nu=0$) jointly with Theorem 3.5 (i) gives

$$\|\lambda|b_{pq}(w)\|\leq c\left(\sum_{\nu=1}^{\infty}2^{\nu sq}\left\|(\varphi_\nu^* f)_r|L_p(\mathbb{R}^n,w)\right\|^q\right)^{1/q}\leq c'\|f\ |B_{pq}^s(\mathbb{R}^n,w)\|.$$

This finishes the proof of the inequality in the B-case. Concerning the F-case, (3.27) and the monotonicity of the quasi-norm imply

$$\left\|\left(\sum_{\nu=1}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\chi_{\nu m}^{(p)}(\cdot)|^q\right)^{1/q}|L_p(\mathbb{R}^n,w)\right\|\leq c\left\|\left(\sum_{\nu=1}^{\infty}2^{\nu sq}(\varphi_\nu^* f)_r(\cdot)^q\right)^{1/q}|L_p(\mathbb{R}^n,w)\right\|.$$

Consequently, the desired inequality follows from Theorem 3.5 (ii) applied in the same way as before, now with $r>\max\left(\frac{nr_0}{p},\frac{n}{q}\right)$. Indeed, we have

$$\|\lambda|f_{pq}(w)\|\leq c\left\|\left(\sum_{\nu=1}^{\infty}2^{\nu sq}(\varphi_\nu^* f)_r(\cdot)^q\right)^{1/q}|L_p(\mathbb{R}^n,w)\right\|\leq c'\|f\ |F_{pq}^s(\mathbb{R}^n,w)\|,$$

which establishes the inequality in the F-case.

Step 3. To prove the converse we assume that $f\in\mathcal{S}'(\mathbb{R}^n)$ possesses the representation

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}(x)$$

with K and L satisfying conditions (3.12), where $a_{\nu m}(x)$ are corresponding atoms according to Definition 3.8. We first consider the F-case and show that $\|f\ |F_{pq}^s(\mathbb{R}^n,w)\|\leq c\|\lambda|f_{pq}(w)\|$, i.e. the 'if'-part in the F-case. Our argument essentially relies on the characterization of function spaces by local means in Theorem 3.6. In view of (3.4), (3.5) we thus have to deal with terms of type $k_N\left(2^{-j},a_{\nu m}\right)$. Let us fix $\nu,j\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$ and $x\in\mathbb{R}^n$. We shall first assume that $j\geq\nu$. According to the definition of local means (2.4), we obtain

$$\begin{aligned} 2^{js}k_N(2^{-j},a_{\nu m})(x) &= 2^{js}\int_{\mathbb{R}^n} k_N(y)\,a_{\nu m}(x+2^{-j}y)\,\mathrm{d}y \\ &= 2^{js}\int_{\mathbb{R}^n} \Delta^N k^0(y)\;a_{\nu m}(x+2^{-j}y)\,\mathrm{d}y. \end{aligned} \tag{3.28}$$

Let us temporarily write the atoms $a_{\nu m}(x)$ as

$$a_{\nu m}(x) = 2^{-\nu(s-\frac{n}{p})}a^{\nu m}(2^\nu x - m) \tag{3.29}$$

such that $a^{\nu m}(x)$ are 1_K-atoms with respect to the unit cube centered at the origin. To simplify our proof we consider only the case K even, say $K = 2M$, and leave it to the reader to find the necessary modifications otherwise. Let us choose $N > M$. Combining (3.28) with (3.29) and integrating by parts leads to

$$\begin{aligned} 2^{js}k_N(2^{-j},a_{\nu m})(x) &= 2^{js-\nu s+\frac{\nu n}{p}}\int_{\mathbb{R}^n} \Delta^N k^0(y)a^{\nu m}(2^\nu x+2^{\nu-j}y-m)\,\mathrm{d}y \\ &= 2^{-(K-s)(j-\nu)+\frac{\nu n}{p}}\int_{\mathbb{R}^n} \Delta^{N-M}k^0(y)\left(\Delta^M a^{\nu m}\right)(2^\nu x+2^{\nu-j}y-m)\,\mathrm{d}y. \end{aligned}$$

The support properties of k^0 and $\Delta^M a^{\nu m}$ imply that

$$2^{js}|k_N(2^{-j},a_{\nu m})(x)| \le c\,2^{-(K-s)(j-\nu)}\widetilde{\chi}^{(p)}_{\nu m}(x) \quad \text{for} \quad j \ge \nu. \tag{3.30}$$

Here $\widetilde{\chi}^{(p)}_{\nu m}(x)$ stands for the p-normalized characteristic function given by (3.7) with $cQ_{\nu m}$ instead of $Q_{\nu m}$.

We now consider the case $j < \nu$. Observe that in this case we may restrict the integration in (3.28),

$$\begin{aligned} 2^{js}k_N(2^{-j},a_{\nu m})(x) &= 2^{js}\int_{\mathbb{R}^n} k_N(y)\,a_{\nu m}(x+2^{-j}y)\,\mathrm{d}y \\ &= 2^{j(s+n)}\int_{|y|\le c2^{-j}} k_N\left(2^j y\right)\;a_{\nu m}(x+y)\,\mathrm{d}y \end{aligned} \tag{3.31}$$

to the set $\{y : |y| \le c2^{-j}\}$ for appropriate $c > 0$. We write the Taylor expansion of $k_N(2^j\cdot)$ at $2^{-\nu}m - x$ up to order L as

$$k_N(2^j y) = \sum_{|\beta| \le L} c_\beta(x)(y - 2^{-\nu}m + x)^\beta + 2^{j(L+1)}\mathcal{O}(|x + y - 2^{-\nu}m|^{L+1}). \tag{3.32}$$

We insert (3.32) into (3.31) and observe that by the assumed moment conditions in Definition 3.8 (iii) the terms with $|\beta| \le L$ vanish. On the other hand, Definition 3.8 (i), (ii) yields that $|a_{\nu m}(x + y)| \le 2^{-\nu(s-\frac{n}{p})}\widetilde{\chi}_{\nu m}(x + y)$, where $\widetilde{\chi}_{\nu m}(x)$ is the characteristic function of the cube $dQ_{\nu m}$, such that

$$\begin{aligned} &2^{js}\left|k_N(2^{-j}, a_{\nu m})(x)\right| \\ &\le 2^{j(s+n)} \int\limits_{|y| \le c2^{-j}} 2^{j(L+1)}\mathcal{O}(|x + y - 2^{-\nu}m|^{L+1})\,|a_{\nu m}(x + y)| \,\mathrm{d}y \\ &\le c2^{j(s+n)-\nu(s-\frac{n}{p})}2^{(j-\nu)(L+1)} \int\limits_{|y| \le c2^{-j}} \widetilde{\chi}_{\nu m}(x + y)\,\mathrm{d}y. \end{aligned} \tag{3.33}$$

Additionally, we have

$$\int\limits_{|y| \le c2^{-j}} \widetilde{\chi}_{\nu m}(x + y)\,\mathrm{d}y \le c\,2^{-\nu n}\chi(c2^{\nu-j}Q_{\nu m})(x), \tag{3.34}$$

where $\chi(c2^{\nu-j}Q_{\nu m})(x)$ denotes the characteristic function of the cube $c2^{\nu-j}Q_{\nu m} =: Q_0$. The last part of the proof is based on estimates for the Hardy-Littlewood maximal function given by (2.2), essentially using Theorem 3.1. We obtain for any $x \in Q_0$,

$$(M\chi_{\nu m})(x) \sim \sup |Q|^{-1} \int\limits_Q |\chi_{\nu m}(y)|\,\mathrm{d}y \ge |Q_0|^{-1}2^{-\nu n} \ge c2^{-(\nu-j)n}, \tag{3.35}$$

where the supremum is taken over all cubes Q with $x \in Q$. Consequently, with $0 < \varrho < \min\left(1, \frac{p}{r_0}, q\right)$ we obtain from (3.34) and (3.35) that

$$\int\limits_{|y| \le c2^{-j}} \widetilde{\chi}_{\nu m}(x + y)\,\mathrm{d}y \le c\,2^{-\nu n}2^{(\nu-j)\frac{n}{\varrho}}\left(M\chi_{\nu m}^{\varrho}\right)^{1/\varrho}(x). \tag{3.36}$$

We now insert the last estimate into the formula (3.33) and replace $\chi_{\nu m}$ by $\chi_{\nu m}^{(p)}$ to conclude

$$2^{js}|k_N(2^{-j}, a_{\nu m})(x)| \le c\,2^{-(\nu-j)(s+L+1+n-\frac{n}{\varrho})}(M\chi_{\nu m}^{(p)\varrho})^{\frac{1}{\varrho}}(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

We observe that by (3.15) and (3.11) the number ϱ may be chosen in such a way that $s+L+1+n-\frac{n}{\varrho}$ is positive, as $L \geq \max\left(-1,[\sigma_{p/r_0,q}-s]\right)$ implies that

$$s+L+1+n > n\max\left(1,\frac{1}{q},\frac{r_0}{p}\right) > \frac{n}{\varrho}.$$

Thus for $\tau>0$ we obtain

$$2^{js}|k_N(2^{-j},a_{\nu m})(x)| \leq c\, 2^{-(\nu-j)\tau}(M\chi_{\nu m}^{(p)\varrho})^{\frac{1}{\varrho}}(x) \quad \text{for} \quad j<\nu, \quad x\in\mathbb{R}^n. \qquad (3.37)$$

Let us first finish the proof for the F-case. Putting together the estimates (3.30) and (3.37) yields for $q\leq 1$,

$$\begin{aligned} &2^{jsq}\left|k_N\left(2^{-j},\sum_{\nu,m}\lambda_{\nu m}a_{\nu m}\right)(x)\right|^q \\ &\leq c\sum_{\nu\leq j}\sum_m|\lambda_{\nu m}|^q 2^{-\rho(j-\nu)q}\widetilde{\chi}_{\nu m}^{(p)q}(x) + \ c\sum_{\nu>j}\sum_m|\lambda_{\nu m}|^q 2^{-\tau(\nu-j)q}(M\chi_{\nu m}^{(p)\varrho})^{\frac{q}{\varrho}}(x)\end{aligned}$$

for some appropriate $\rho,\tau>0$. Observe that the same conclusion can also be drawn for $1<q\leq\infty$. Summing over j, taking the $1/q$-power and afterwards the $L_p(\mathbb{R}^n,w)$-quasi-norm we obtain

$$\begin{aligned} &\left\|\left(\sum_{j=1}^{\infty}2^{jsq}\left|k_N\left(2^{-j},\sum_{\nu,m}\lambda_{\nu m}a_{\nu m}\right)(\cdot)\right|^q\right)^{1/q}\Big|L_p(\mathbb{R}^n,w)\right\| \\ &\leq c\left\|\left(\sum_{\nu,m}|\lambda_{\nu m}|^q\widetilde{\chi}_{\nu m}^{(p)q}(\cdot)\right)^{1/q}\Big|L_p(\mathbb{R}^n,w)\right\| \\ &\quad + \ c\left\|\left(\sum_{\nu,m}|\lambda_{\nu m}|^q(M\chi_{\nu m}^{(p)\varrho})^{\frac{q}{\varrho}}(\cdot)\right)^{1/q}\Big|L_p(\mathbb{R}^n,w)\right\|.\end{aligned}$$

Since we may replace $\widetilde{\chi}_{\nu m}^{(p)}$ by $\chi_{\nu m}^{(p)}$, the first summand on the right-hand side can be estimated from above by $\|\lambda|f_{pq}(w)\|$. To deal with the second summand on the right-hand side, we observe that it may be written as

$$\left\|\left(\sum_{\nu,m}(Mg_{\nu m}^{\varrho})(\cdot)^{\frac{q}{\varrho}}\right)^{\frac{\varrho}{q}}\Big|L_{\frac{p}{\varrho}}(\mathbb{R}^n,w)\right\|^{\frac{1}{\varrho}},$$

with $g_{\nu m}(x) = \lambda_{\nu m}\chi_{\nu m}^{(p)}$. Finally, we apply Theorem 3.1 with $p' = \frac{p}{\varrho} > r_0 \geq 1$, $q' = \frac{q}{\varrho} > 1$ and $w \in \mathcal{A}_{p'}$ (as $p' > r_0$, recall (2.12) and Lemma 2.7 (v)), which establishes the desired inequality.

Concerning the B-case, the above argument can be immediately transferred, now assuming $0 < \varrho < \min\left(1, \frac{p}{r_0}\right)$. Again, we combine estimates (3.30) and (3.37) and use the monotonicity of the $L_p(\mathbb{R}^n, w)$-quasi-norm and triangle inequality to obtain for arbitrary $0 < q \leq \infty$,

$$\begin{aligned} & 2^{jsq}\left\| k_N\left(2^{-j}, \sum_{\nu,m}\lambda_{\nu m}a_{\nu m}\right) | L_p(\mathbb{R}^n, w)\right\|^q \\ & \leq c \sum_{\nu\leq j}\left\|\sum_m \lambda_{\nu m}2^{-\rho(j-\nu)}\widetilde{\chi}_{\nu m}^{(p)} \left| L_p(\mathbb{R}^n, w)\right.\right\|^q \\ & \quad + c \sum_{\nu>j}\left\|\sum_m \lambda_{\nu m}2^{-\tau(\nu-j)}(M\chi_{\nu m}^{(p)\varrho})^{\frac{1}{\varrho}} \left| L_p(\mathbb{R}^n, w)\right.\right\|^q \end{aligned}$$

for some ρ and τ positive. Summing over $j \in \mathbb{N}$ and applying the (scalar) Hardy-Littlewood maximal theorem to the second summand on the right hand side results in the desired inequality. The same conclusion can be drawn for $\nu = 0$ and/or $j = 0$. □

Remark 3.13. The unweighted version of the above decomposition result may be found in [Tri97, Theorem 13.8]. As already mentioned, first results on atomic decompositions of that type go back to M. Frazier and B. Jawerth [FJ85], [FJ90], [FJW91]. In [FJ90, Proposition 10.14] there is even an atomic decomposition result for homogeneous weighted F-spaces, whereas in [Rou04a, Section 11] we find results for (matrix-valued) weighted Besov spaces, $1 \leq p < \infty$, as already mentioned. In the same context an extension to $0 < p \leq 1$, $w \in \mathcal{A}_1$, can be found in [FR04]. For recent results in this direction we also refer to [Bow05] and [BH06].
Finally, related wavelet results are given (in different situations) in [NT, Sections 9, 11], [Vol97] (concerning Haar and bi-orthogonal wavelets), and [Lem94] for $L_p(\mathbb{R}, w)$, $w \in \mathcal{A}_p$, $1 < p < \infty$, with compactly supported wavelets.

For later use we formulate a special case of Theorem 3.11 where $w = w_{\varkappa}^{\Gamma}$ from Example 2.11 (c); recall Remark 2.13 (c).

COROLLARY 3.14. *Let* $0 < d < n$ *and let* Γ *be a* d*-set in* $\mathbb{R}^n$ *in the sense of Definition 2.9. Moreover let* $w_\varkappa^\Gamma$ *be the weight introduced in Example 2.11 (c) with* $\varkappa > -(n-d)$*, and* $r_0 = \max\left(\frac{\varkappa}{n-d} + 1, 1\right)$*. Assume* $0 < p < \infty$*,* $0 < q \leq \infty$*,* $s \in \mathbb{R}$*. Let* $K, L+1 \in \mathbb{N}_0$ *with*

$$K \geq (1 + [s])_+, \quad \textit{and} \quad L \geq \max\left(-1, [\sigma_{p/r_0} - s]\right). \tag{3.38}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^s_{pq}(\mathbb{R}^n, w_\varkappa^\Gamma)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \qquad \textit{converging in} \quad \mathcal{S}'(\mathbb{R}^n), \tag{3.39}$$

where $a_{\nu m}(x)$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *and* $\lambda \in b_{pq}(w_\varkappa^\Gamma)$*. Furthermore, taking the infimum over all admissible representations* (3.39) *of*

$$\|\lambda | b_{pq}(w_\varkappa^\Gamma)\| \sim \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p w_\varkappa^\Gamma(x^{\nu,m}) \right)^{q/p} \right)^{1/q}, \tag{3.40}$$

we obtain an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w_\varkappa^\Gamma)$*, where* $x^{\nu,m} \sim 2^{-\nu} m$*,* $\nu \in \mathbb{N}_0$*,* $m \in \mathbb{Z}^n$*. In particular, for* $-(n-d) < \varkappa \leq 0$ *we can replace* (3.38) *by its unweighted counterparts,*

$$K \geq (1 + [s])_+, \quad \textit{and} \quad L \geq \max\left(-1, [\sigma_p - s]\right), \tag{3.41}$$

such that for $s > \sigma_p$ *no moment conditions are necessary for the corresponding atoms in* (3.39)*.*

Remark 3.15. Plainly, when dealing with $w = w_\alpha$ from Example 2.11 (b), we have a similar result, now with $r_0 = \max(\alpha + 1, 1)$, $\alpha > -1$; cf. Remark 2.13 (b). Likewise, we regain the unweighted conditions (3.41) for $-1 < \alpha \leq 0$, and need consequently no moment conditions if, in addition, $s > \sigma_p$.

Chapter 4

Traces on fractals

The main purpose of this chapter is to present a solution of the trace problem for the weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and weighted Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, where the underlying weight $w^\Gamma_\varkappa$ is a function given by (2.14). The treatment of the fractal trace problem for weighted function spaces has been inspired by the unweighted results due to H. Triebel [Tri97, Chapter 18]. The corresponding trace operator tr_Γ shall map weighted function spaces of type $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ into suitable function spaces on Γ. The basic idea is to investigate the interaction between the structure of fractals and the smoothness of the underlying functions by means of the corresponding weight function. The essential tool in proving our results will be atomic decomposition of function spaces with Muckenhoupt weights, see Chapter 3. The results obtain in this chapter are contained in [Pio].

4.1 Traces

There is a variety of literature on traces on $\mathbb{R}^n$ both for Besov and Triebel-Lizorkin spaces, but the systematic study of trace problems in the framework of fractal sets started rather recently in [Tri97] only. This section contains results on traces of classical Besov spaces on fractals. Let us start by summarizing unweighted results in this direction. Recall that for $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ the mapping

$$\mathrm{tr}_{\mathbb{R}^{n-1}} : f(x) \mapsto f(x', 0) \tag{4.1}$$

is called the trace of f on $\mathbb{R}^{n-1}$. In other words, $\operatorname{tr}_{\mathbb{R}^{n-1}}$ restricts functions on $\mathbb{R}^n$ to the hyperplane $H = \{x \in \mathbb{R}^n : \ x_n = 0\}$. Given a function space $X \subset \mathcal{D}'(\mathbb{R}^n)$, the trace problem consists in finding a space $Y \subset \mathcal{D}'(\mathbb{R}^{n-1})$ such that $\operatorname{tr}_{\mathbb{R}^{n-1}}$ is a bounded linear surjection from X to Y. There is quite an extensive literature concerning trace problems for classical Besov and Triebel-Lizorkin spaces, beginning with the work of H. Triebel [Tri78] as well as of B. Jawerth [Jaw77]. The interested reader is referred to [Tri92, Chapter 4.4] for a new approach to this topic using atomic decompositions and local means techniques. The following theorem gives the complete answers to the trace problem in the case of a hyperplane $\mathbb{R}^{n-1}$.

THEOREM 4.1.

(i) *Let* $0 < p, q \le \infty$ *and* $s - \frac{1}{p} > (n-1)(\frac{1}{p} - 1)_+$. *Then we get*

$$\operatorname{tr}_{\mathbb{R}^{n-1}} B^s_{pq}(\mathbb{R}^n) = B^{s-\frac{1}{p}}_{pq}(\mathbb{R}^{n-1}). \tag{4.2}$$

(ii) *Let* $n \ge 2$, $0 < p < \infty$ *and* $0 < q \le \min(1, p)$. *Then we get*

$$\operatorname{tr}_{\mathbb{R}^{n-1}} B^{\frac{1}{p}}_{pq}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1}). \tag{4.3}$$

Classical references for trace problems in that case are [Tri92, 4.4.1 and 4.4.2]. We shall now extend assertions of type (4.3) to the case of suitable compact d-sets instead of hyperplanes in $\mathbb{R}^{n-1}$. In the sequel any function $f^\Gamma \in L_p(\Gamma)$, $1 \le p \le \infty$, will be interpreted as a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)(\varphi|\Gamma)(\gamma)\mu(\mathrm{d}\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where the restriction $\varphi|\Gamma$ of φ is understood pointwise and μ is a Radon measure on Γ. We explain the fractal counterpart of (4.1) now.

Let us temporarily consider a closed set $\Gamma \subset \mathbb{R}^n$ with $|\Gamma| = 0$ and assume that there exists a Radon measure μ on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) = \Gamma$. Therefore the restriction $\operatorname{tr}_\Gamma \varphi = \varphi|\Gamma$ understood pointwise is well-defined for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Moreover let us suppose that for $s > 0$ and $0 < p, q < \infty$ there is a constant $c > 0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\| \operatorname{tr}_\Gamma \varphi | L_p(\Gamma) \| \le c \| \varphi | B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \|. \tag{4.4}$$

Since the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, the inequality (4.4) may be extended by completion to all $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. The resulting limit of $\operatorname{tr}_\Gamma \varphi$ will be denoted by $\operatorname{tr}_\Gamma f$. Note that it is independent of the approximation of $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ by $\mathcal{S}(\mathbb{R}^n)$-functions due to (4.4).
We first recall what is known on traces of unweighted Besov spaces on a d-set Γ.

THEOREM 4.2. *Let Γ be a d-set with $0 < d < n$. Moreover let $0 < p < \infty$ and $0 < q \le \min(1, p)$. Then*

$$\operatorname{tr}_\Gamma B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma). \tag{4.5}$$

The interpretation of the equality (4.5) is that $\operatorname{tr}_\Gamma f \in L_p(\Gamma)$ for any $f \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$, and that any $f^\Gamma \in L_p(\Gamma)$ is a trace of a suitable $g \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ on Γ in the above described sense with

$$\|f^\Gamma | L_p(\Gamma)\| \sim \inf \|g \,| B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ such that $\operatorname{tr}_\Gamma g = f^\Gamma$.

For a complete discussion and proof we refer to [Tri97, Theorem 18.6, Corollary 18.12] in connection with [Tri01, Remark 9.19]. The interested reader will find there also further references.

4.2 Traces of Besov spaces on fractals: a heuristic approach

From now on let $0 < p < \infty$, $0 < q \le \infty$, $\sigma \in \mathbb{R}$. We will work in the framework of a d-set Γ as introduced in Definition 2.9 with $0 < d < n$. Moreover let $w^\Gamma_\varkappa$ be the weight according to Example 2.11(c) and $\varkappa > -(n-d)$. Recall that by Theorem 3.14 the question whether a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the weighted Besov space $B^\sigma_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ can be equivalently expressed in terms of sequence spaces, $\lambda \in b_{pq}(w^\Gamma_\varkappa)$, where we use the appropriate atomic decomposition in the form

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x) \tag{4.6}$$

with suitable coefficients $\lambda_{\nu m}$ and (σ, p)-atoms $a_{\nu m}$. In the sequel we shall divide the summation over $m \in \mathbb{Z}^n$ in (4.6) with respect to the following "remainder" set

$$\mathrm{I}_{\Gamma,\nu} = \left\{ m \in \mathbb{Z}^n : \operatorname{dist}(\Gamma, \operatorname{supp} a_{\nu m}) > b 2^{-\nu} \right\}, \quad \nu \in \mathbb{N}_0, \tag{4.7}$$

i.e. for $m \in \mathrm{I}_{\Gamma,\nu}$ the supports of the corresponding atoms have an empty intersection with Γ. To shorten the notation we utilize the following abbreviations for respective sums,

$$\sum_{m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \quad \text{and} \quad \sum_{m \in \mathrm{I}_{\Gamma,\nu}} = \sum_{\substack{\Gamma,\nu \\ m \in \mathbb{Z}^n}}, \tag{4.8}$$

such that $\sum^{\Gamma,\nu}$ collects all atoms with a support near to Γ, and $\sum_{\Gamma,\nu}$ the remaining ones, that are less important for trace problems on Γ. This notation allows us to write (4.6) as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) + \sum_{\nu=0}^{\infty} \sum_{\substack{\Gamma,\nu \\ m \in \mathbb{Z}^n}} \lambda_{\nu m} a_{\nu m}(x). \tag{4.9}$$

Subsequently, we simplify the writing by denoting by f^{Γ} and f_{Γ} the first and second sum, respectively, i.e.

$$f^{\Gamma} = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m} \quad \text{and} \quad f_{\Gamma} = \sum_{\nu=0}^{\infty} \sum_{\substack{\Gamma,\nu \\ m \in \mathbb{Z}^n}} \lambda_{\nu m} a_{\nu m}. \tag{4.10}$$

A careful look at (4.7) shows that f_{Γ} has no influence on the trace problem on Γ. It implies that $\operatorname{tr}_{\Gamma} a_{\nu m}(x) = 0$ for $m \in \mathrm{I}_{\Gamma,\nu}$. Consequently, f and f^{Γ} possess the same trace on Γ,

$$\operatorname{tr}_{\Gamma} f = \operatorname{tr}_{\Gamma} f^{\Gamma}.$$

Assume for the moment that $f \in \mathcal{S}(\mathbb{R}^n)$ and the trace is taken pointwise; recall that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^{\Gamma}_{\varkappa})$ for $q < \infty$. Let us now consider the following reformulation of f^{Γ},

$$f^{\Gamma} = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \left(\lambda_{\nu m} 2^{-\nu \frac{\varkappa}{p}} \right) \left(2^{\nu \frac{\varkappa}{p}} a_{\nu m}(x) \right) = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \widetilde{\lambda}_{\nu m} \widetilde{a}_{\nu m}(x), \tag{4.11}$$

where $\widetilde{\lambda}_{\nu m} = \lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}}$ are new coefficients and $\widetilde{a}_{\nu m} = 2^{\nu\frac{\varkappa}{p}} a_{\nu m}$ are $(\sigma - \frac{\varkappa}{p}, p)_{K,L}$-atoms, accordingly. Let $\widetilde{f}^{\Gamma}$ be given by (4.11) with $\widetilde{\lambda}_{\nu m} = 0$, $m \in \mathrm{I}_{\Gamma,\nu}$, $\nu \in \mathbb{N}_0$.
Note that for $m \in \mathrm{I}_{\Gamma,\nu}$ we have $w_{\varkappa}^{\Gamma}(x^{\nu,m}) \sim 1$ and for $m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}$ we obtain that $w_{\varkappa}^{\Gamma}(x^{\nu,m}) \sim 2^{-\nu\varkappa}$. Applying Theorem 3.14 jointly with its unweighted counterpart for $w \equiv 1$, see also, [Tri97, Theorem 3.8 p.75], to (4.11) yields

$$\left\| \widetilde{f}^{\Gamma} | \, B_{pq}^{\sigma - \frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \leq \left\| \tilde{\lambda} | \, b_{pq} \right\| = c \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p 2^{-\nu\varkappa} \right)^{q/p} \right)^{1/q}$$

$$\leq c' \left\| \lambda | \, b_{pq}(w_{\varkappa}^{\Gamma}) \right\| \leq c'' \left\| f | \, B_{pq}^{\sigma}(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\|, \tag{4.12}$$

for suitably chosen $\{\lambda_{\nu m}\}$, i.e. $f \in B_{pq}^{\sigma}(\mathbb{R}^n, w_{\varkappa}^{\Gamma})$ implies $\widetilde{f}^{\Gamma} \in B_{pq}^{\sigma - \frac{\varkappa}{p}}(\mathbb{R}^n)$.
Assume for the moment that $\sigma - \frac{\varkappa}{p} = \frac{n-d}{p}$, i.e. $\sigma = \frac{\varkappa + n - d}{p} > 0$, and $\mathrm{tr}_{\Gamma} f = f^{\Gamma} = \mathrm{tr}_{\Gamma} \widetilde{f}$. Then $f \in B_{pq}^{\sigma}(\mathbb{R}^n, w_{\varkappa}^{\Gamma})$ leads to $\mathrm{tr}_{\Gamma} f \in L_p(\Gamma)$, that is $\mathrm{tr}_{\Gamma} B_{pq}^{\frac{\varkappa+n-d}{p}}(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \subset L_p(\Gamma)$, see Theorem 4.4 below.

4.3 Traces on fractals of weighted Besov spaces

Before we get on to the main point of this section we recall a definition which plays an important rôle in our later consideration.

DEFINITION 4.3. Let Γ be a non-empty Borel set in $\mathbb{R}^n$ with $|\Gamma| = 0$. We say that Γ satisfies the *ball condition* if there is a number $0 < \eta < 1$ such that for any ball $B(x, r)$ centered at $x \in \Gamma$ and of radius $0 < r < 1$ there is a ball $B(y, \eta r)$ centered at some $y \in \mathbb{R}^n$, depending on x, and of radius ηr with

$$B(y, \eta r) \subset B(x, r) \qquad \text{and} \qquad B(y, \eta r) \cap \overline{\Gamma} = \emptyset. \tag{4.13}$$

Note that any d-set possesses this feature, see [Tri01, Proposition 9.18].
We can formulate the first main result of this chapter, which extends Theorem 4.2 to the weighted case.

THEOREM 4.4. *Let $0 < d < n$, $\varkappa > -(n-d)$, $0 < p < \infty$, $0 < q \leq \min(1, p)$ and let Γ be a d-set. Then we have*

$$\mathrm{tr}_{\Gamma} B_{pq}^{\frac{\varkappa}{p} + \frac{n-d}{p}}(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) = L_p(\Gamma), \tag{4.14}$$

in the sense, that $\mathrm{tr}_\Gamma f \in L_p(\Gamma)$ *for any* $f \in B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$ *and any* $f^\Gamma \in L_p(\Gamma)$ *is a trace of a suitable* $g \in B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$ *on* Γ *and*

$$\|f^\Gamma\ |L_p(\Gamma)\| \sim \inf \left\| g\ |B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) \right\|,$$

where the infimum is taken over all $g \in B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$ *such that* $\mathrm{tr}_\Gamma\, g = f^\Gamma$.

Remark 4.5. The discussion on weighted Triebel-Lizorkin spaces $F_{pq}^s(\mathbb{R}^n, w)$ with $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$ will be postponed to the end of this chapter.

Proof. Our proof is based upon ideas found in [Tri97, Theorem 18.6]. We essentially make use of the atomic decomposition techniques from Chapter 3.
Step 1. Let us assume that $0 < p < \infty$, $0 < d < n$ and $0 < q \le \min(1, p)$.
We first prove that

$$\mathrm{tr}_\Gamma\, B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) \subset L_p(\Gamma). \tag{4.15}$$

We start with $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This causes no loss of generality, since the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$, see [Bui82]. We recall that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the restriction operator $\mathrm{tr}_\Gamma\, \varphi = \varphi|\Gamma$ is meant pointwise. We consider an optimal atomic decomposition according to Theorem 3.14 of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ in $B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)$,

$$\varphi = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \tag{4.16}$$

such that

$$\left\| \varphi |\ B_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| \sim \left\| \lambda |\ b_{pq}(w_\varkappa^\Gamma) \right\|. \tag{4.17}$$

Here the coefficients $\lambda_{\nu m}$ and the $\left(\frac{n-d+\varkappa}{p}, p\right)$-atoms $a_{\nu m}$ have the same meaning as explained in Definitions 3.8 and (3.8). In particular, according to Definition 3.8 we have that supp $a_{\nu m} \subset bQ_{\nu m}$ and

$$|a_{\nu m}(x)| \le 2^{-\nu\left(\frac{n-d+\varkappa}{p}-\frac{n}{p}\right)} = 2^{\frac{\nu(d-\varkappa)}{p}}, \quad m \in \mathbb{Z}^n, \nu \in \mathbb{N}_0. \tag{4.18}$$

Proceeding exactly as in Section 4.2 let us consider a decomposition $\varphi = \varphi^\Gamma + \varphi_\Gamma$, such that φ^Γ collects all atoms with a non-empty intersection of their support with Γ, and φ_Γ being the rest.

Assume first that $0 < p \le 1$. In view of (4.17), to prove (4.15) we have to find an estimate from above of the quasi-norm

$$\|\mathrm{tr}_\Gamma\, \varphi |\; L_p(\Gamma)\|^p = \int_\Gamma |\varphi^\Gamma(\gamma)|^p \mu(\mathrm{d}\gamma) + \int_\Gamma |\varphi_\Gamma(\gamma)|^p \mu(\mathrm{d}\gamma) \tag{4.19}$$

by the quasi-norm $\left\|\lambda |\; b_{pq}(w^\Gamma_\varkappa)\right\|$. Taking into account that $a_{\nu m} \cap \Gamma = \emptyset$ for all atoms belonging to the representation of φ_Γ, we immediately get that the last integral in (4.19) does vanish, since then $\int_\Gamma |\varphi_\Gamma(\gamma)|^p \mu(\mathrm{d}\gamma) = 0$. Hence we have

$$\begin{aligned} \|\mathrm{tr}_\Gamma\, \varphi |\; L_p(\Gamma)\|^p &\le \sum_{\nu=0}^{\infty} \int_\Gamma \left| \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(\gamma)\right|^p \mu(\mathrm{d}\gamma) \\ &\le c \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p \int_\Gamma |a_{\nu m}(\gamma)|^p \mu(\mathrm{d}\gamma). \end{aligned} \tag{4.20}$$

Recall that

$$\sum_{m\in\mathbb{Z}^n\setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu},$$

i.e. we consider only atoms with a support near Γ. The rest of the atoms play no rôle for a trace problem on Γ.

Let us turn our attention to the last integral in (4.20). Since $\mu(\Gamma \cap Q_{\nu m}) \sim 2^{-\nu d}$ by Definition 2.9 and (4.18) we obtain

$$\int_\Gamma |a_{\nu m}(\gamma)|^p \mu(\mathrm{d}\gamma) \le c 2^{\nu(d-\varkappa)} \mu(\Gamma \cap Q_{\nu m}) \sim c 2^{-\nu\varkappa}.$$

Plugging the above estimate into the last term in (4.20) yields

$$\|\mathrm{tr}_\Gamma\, \varphi |\; L_p(\Gamma)\|^p \le c' \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} |\lambda_{\nu m}|^p 2^{-\nu\varkappa} \le c'' \left\|\lambda |\; b_{pq}(w^\Gamma_\varkappa)\right\|^p, \tag{4.21}$$

where the last inequality holds by virtue of $q \leq p$ and (3.40). Consequently, by (4.17) for $0 < p \leq 1$ and $q \leq p$ we have

$$\|\mathrm{tr}_\Gamma\,\varphi\;|L_p(\Gamma)\| \leq c'\left\|\lambda \mid b_{pq}(w^\Gamma_\varkappa)\right\| \leq c''\left\|\varphi\;|B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\|. \tag{4.22}$$

For $p > 1$ we use the triangle inequality to get

$$\|\mathrm{tr}_\Gamma\,\varphi|\; L_p(\Gamma)\| \leq c'\left(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu}|\lambda_{\nu m}|^p 2^{-\nu\varkappa}\right)^{1/p} \leq c'\left\|\lambda\;|b_{p1}(w^\Gamma_\varkappa)\right\| \leq c''\|\lambda\;|b_{pq}(w^\Gamma_\varkappa)\|. \tag{4.23}$$

Again, the last inequality holds by virtue of $q \leq 1$. Finally, we arrive at

$$\|\mathrm{tr}_\Gamma\,\varphi|\; L_p(\Gamma)\| \leq c\left\|\varphi|\;\; B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \tag{4.24}$$

with $0 < p < \infty,\; 0 < q \leq \min(p, 1)$, which proves the inclusion (4.15).

Step 2. Let $0 < q \leq \min(p, 1)$ and $\max\left(\frac{d-\varkappa}{n}, 0\right) = \left(\frac{d-\varkappa}{n}\right)_+ < p < \infty$. We give a proof of the reverse inclusion

$$L_p(\Gamma) \subset \mathrm{tr}_\Gamma\, B^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa). \tag{4.25}$$

We shall adapt the arguments used in Step 2 of the proof of Theorem 18.2 of [Tri97]. It is known that $\mathcal{D}|_\Gamma$ is dense in $L_p(\Gamma)$. Thus, we may work without loss of generality with $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Moreover assume that $\varphi|_\Gamma \neq 0$ and consider the neighborhood of Γ given by

$$\Gamma_k = \left\{x \in \mathbb{R}^n :\; \mathrm{dist}(x, \Gamma) < 2^{-k}\right\}.$$

By compactness of Γ together with properties of the Hausdorff measure, there are open balls $B(x_j, r)$ with $j = 1, \ldots, N$ centered at Γ with the same radius $r > 0$ depending on the covering that cover Γ. Note that $\overline{\Gamma}_k \subset \bigcup_{j=1}^N B(x_j, r)$, where k depends on the given covering.

Now, let $\{\varphi_j\}_{j=1}^N$ be a smooth resolution of unity in a neighborhood Γ_k of $\Gamma \cap \operatorname{supp} \varphi$ adapted to $(B(x_j,r))_{j=1}^N$. In particular, we have $\sum_{j=1}^N \varphi_j(x) = 1$ for $x \in \operatorname{supp} \varphi$ and $\operatorname{supp} \varphi_j \subset B(x_j,r)$. Let us now put $\lambda_j = \max_{x \in B(x_j,r)} |\varphi(x)|$. Then, by the properties of the above defined resolution of unity we get

$$\varphi(x) = \sum_{j=1}^N \varphi(x)\varphi_j(x) = \sum_{j=1}^N \lambda_j r^{\frac{d-\varkappa}{p}} \left[r^{-\frac{d-\varkappa}{p}} \lambda_j^{-1} \varphi(x)\varphi_j(x) \right], \tag{4.26}$$

where terms with $\lambda_j = 0$ are omitted. Let us define

$$\widetilde{\lambda}_j = \lambda_j r^{\frac{d-\varkappa}{p}} \quad \text{and} \quad a_j(x) = r^{-\frac{d-\varkappa}{p}} \lambda_j^{-1} \varphi(x)\varphi_j(x).$$

We obtain that $\operatorname{supp} a_j \subset B(x_j,r)$. Furthermore, choosing $r > 0$ small enough, we get

$$|a_j(x)| = \frac{|\varphi(x)|}{\lambda_j} r^{-\frac{d-\varkappa}{p}} |\varphi_j(x)| \le c' r^{\frac{n-d}{p} + \frac{\varkappa}{p} - \frac{n}{p}}$$

and analogous estimates for all $\mathrm{D}^\alpha a_j$. We thus can consider a_j as $\left(\frac{n-d+\varkappa}{p}, p\right)_{K,L}$-atoms according to Definition 3.8. It follows from the assumption $p > \left(\frac{d-\varkappa}{n}\right)_+$ that $\frac{n-d+\varkappa}{p} > n\left(\frac{1}{p} - 1\right)_+$. Therefore, moment conditions as needed in (3.41) may be omitted. Once again, using the atomic decomposition method together with properties of the weight $w_\varkappa^\Gamma$ we may estimate the quasi-norm of (4.26) as follows,

$$\begin{aligned} \|\varphi \,|B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| &\le \|\tilde{\lambda} \,|b_{pq}(w_\varkappa^\Gamma)\| \\ &\le c \left(\sum_{j=1}^N |\lambda_j|^p r^{d-\varkappa} \left\| \chi_{B(x_j,r)}^{(p)} \,|L_p(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| \right)^{1/p}. \end{aligned}$$

Let us again choose $r > 0$ arbitrarily small. A straightforward computation shows that $\|\chi_{B(x_j,r)}^{(p)} \,|L_p(\mathbb{R}^n, w_\varkappa^\Gamma)\| \sim r^\varkappa$. Moreover we have $\mu(B(x_j,r)) \sim r^d$ by Definition 2.9. Proceeding further as in the Riemann integral construction we arrive at

$$\left(\sum_{j=1}^N |\lambda_j|^p r^{d-\varkappa} \left\| \chi_{B(x_j,r)}^{(p)} \,|L_p(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| \right)^{1/p} \le c \,\|\operatorname{tr}_\Gamma \varphi \,|L_p(\Gamma)\|. \tag{4.27}$$

Hence, we have proved that

$$\|\varphi \,|B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \le c\| \operatorname{tr}_\Gamma \varphi \,|L_p(\Gamma)\|, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{4.28}$$

The rest of the proof goes through as for [Tri97, Theorem 18.6], with hardly any changes: for convenience, we include the argument here. It follows from density of $\mathcal{D}|_\Gamma$ in $L_p(\Gamma)$ that any $f \in L_p(\Gamma)$ can be represented in the form

$$f(\gamma) = \sum_{j=1}^{\infty} f_j(\gamma), \quad \gamma \in \Gamma, \quad f_j \in \mathcal{D}(\mathbb{R}^n) \tag{4.29}$$

with

$$0 < \| \operatorname{tr}_\Gamma f_j \,|L_p(\Gamma)\| \le c\, 2^{-j} \|f|\, L_p(\Gamma)\|, \quad j \in \mathbb{N}. \tag{4.30}$$

Thus by (4.28) we have

$$\|f_j \,|B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \le c' \| \operatorname{tr}_\Gamma f_j \,|L_p(\Gamma)\|. \tag{4.31}$$

Now we may define an extension operator in the following way,

$$\operatorname{ext} f = \sum_{j=1}^{\infty} f_j \in B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma), \quad \operatorname{tr}_\Gamma \operatorname{ext} f = f. \tag{4.32}$$

By virtue of (4.29) and (4.30) we obtain

$$\|\operatorname{ext} f \,|B_{pq}^{\frac{n-d+\varkappa}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma)\| \le c' \|f \,|L_p(\Gamma)\|. \tag{4.33}$$

This finishes the proof of (4.25).

Step 3. To complete our proof we have to extend the result of Step 2 to $p > 0$, i.e. for $\varkappa < d$. Let us assume now that $0 < q < p \le \frac{d-\varkappa}{n}$. Analysis similar to that in the proof of [Tri97, Corollary, 18.12] shows that for $\left(\frac{n-d+\varkappa}{p}, p\right)_{K,L}$- atoms we do not have moment conditions for $\varphi\varphi_j$ in (4.26) by property (3.38). Let $B(y_j, \eta r)$ be a ball with the condition (4.13) which can be written, after easy reformulation, in the following form

$$\operatorname{dist}(B(y_j, \eta r), \overline{\Gamma}) \ge \eta r. \tag{4.34}$$

We follow the argument in Step 2 replacing $\varphi\varphi_j$ by the function

$$\psi_j(x) = (\varphi\varphi_j)(x) + \chi_j(x),$$

where supp $\chi_j \subset B(y_j, \eta r)$ and ψ_j is an $\left(\frac{n-d+\varkappa}{p}, p\right)$-atom with moment conditions according to Definition 3.8 with $L \geq \max(-1, \sigma_{p/r_0} - s)$. This is a somewhat tricky construction and can be found in the proof of [TW96, Theorem 3.6]. The atoms $\varphi\varphi_j$ and ψ_j coincide in a neighbourhood of Γ due to (4.34). Now we can use the argument of Step 2 again. The proof of Theorem 4.4 is thus complete.

□

In the concluding part of this section we shall work with Besov spaces introduced in terms of traces on fractals, and recall their definition first.

DEFINITION 4.6. Let Γ be a d-set in $\mathbb{R}^n$ according to Definition 2.9 with $0 < d < n$. Let $s > 0$, $0 < p \leq \infty$, and $0 < q \leq \infty$. Let us define

$$\mathbb{B}^s_{pq}(\Gamma) = \mathrm{tr}_\Gamma\, B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n). \tag{4.35}$$

We equip this space with the quasi-norm

$$\left\| f\ |\mathbb{B}^s_{pq}(\Gamma)\right\| = \inf \left\| g|\ B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\right\|, \tag{4.36}$$

where the infimum ranges over all $g \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma\, g = f$.

In a natural way we extend this notation to weighted spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$: by $\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ we mean the collection of all $f \in L_p(\Gamma)$ such that there exists some $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ with $\mathrm{tr}_\Gamma\, g = f$, and $\|f\ |\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\| = \inf\|g\ |B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\|$, where the infimum is taken over all $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma\, g = f$. Concerning the fractal trace problem we get the following statement.

THEOREM 4.7. *Let $0 < d < n$, $s > 0$, $0 < p < \infty$, $0 < q \leq \infty$ and $-(n-d) < \varkappa < sp - (n-d)$. Then*

$$\mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}_{pq}(\Gamma).$$

Proof. The idea of the proof is to use Definition 4.6 together with the observation that

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n), \tag{4.37}$$

with the parameters given above. Afterwards we apply (4.35) to $s' = s - \frac{\varkappa}{p} - \frac{n-d}{p} > 0$, i.e. such that $s' + \frac{n-d}{p} = s - \frac{\varkappa}{p}$. This leads to

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^{s'}_{pq}(\Gamma),$$

that is, the desired result. Moreover, as will be clear from the argument below, it is sufficient to deal with the inclusion

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \hookrightarrow \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p}}_{pq}(\mathbb{R}^n) \tag{4.38}$$

only, the converse assertion follows by parallel observations.
We consider some $f \in \mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. Let $\varepsilon > 0$. By the definition of this space there is some $g \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f$ and

$$\big\|g|\ B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\big\| \le \big\|f|\ \mathrm{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\big\| + \frac{\varepsilon}{2}. \tag{4.39}$$

We take the atomic decomposition of g in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$,

$$g = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x), \tag{4.40}$$

where $\lambda_{\nu m} \in \mathbb{C}$ are coefficients and $a_{\nu m}(x)$ are $(s,p)_{K,L}$-atoms in the sense of Definition 3.8. In view of Theorem 3.14 we have to choose $K > s$, $L \ge \max(-1, [\sigma_{p/r_0} - s])$ with $r_0 = \max(\frac{\varkappa}{n-d} + 1, 1)$; so let us assume

$$K > \max(s, s - \frac{\varkappa}{p})$$
$$\text{and} \tag{4.41}$$
$$L > \max\left(-1, \left[\sigma_{p/r_0} - s\right], \left[\sigma_p - s + \frac{\varkappa}{p}\right]\right).$$

Thus Theorem 3.14 implies that we find a corresponding atomic decomposition (4.40) with (4.41) and

$$\big\|\lambda|\ b_{pq}(w^\Gamma_\varkappa)\big\| \le c\,\big\|g|\ B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)\big\| + \frac{\varepsilon}{2}. \tag{4.42}$$

We now proceed similar to Section 4.2. Recall our notation

$$\mathrm{I}_{\Gamma,\nu} = \left\{ m \in \mathbb{Z}^n : \ \mathrm{dist}(\Gamma, \mathrm{supp} a_{\nu m}) > b 2^{-\nu} \right\}, \quad \nu \in \mathbb{N}_0,$$

and

$$\sum_{m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}} = \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu}, \qquad \sum_{m \in \mathrm{I}_{\Gamma,\nu}} = \sum_{m \in \mathbb{Z}^n}{}_{\Gamma,\nu}.$$

We decompose

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) + \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}{}_{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) := g^{\Gamma} + g_{\Gamma}$$

with $\mathrm{tr}_{\Gamma}\, g = \mathrm{tr}_{\Gamma}\, g^{\Gamma}$, $\mathrm{tr}_{\Gamma}\, g_{\Gamma} = 0$. We extend g^{Γ} by 0 outside,

$$\begin{aligned} \widetilde{g} &= \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}^{\Gamma,\nu} \left(\lambda_{\nu m} 2^{-\nu \frac{\varkappa}{p}} \right) \left(2^{\nu \frac{\varkappa}{p}} a_{\nu m}(x) \right) + \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n}{}_{\Gamma,\nu} 0 \left(2^{\nu \frac{\varkappa}{p}} a_{\nu m}(x) \right) \\ &= \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \widetilde{\lambda}_{\nu m} \widetilde{a}_{\nu m}(x), \end{aligned} \tag{4.43}$$

obtaining an atomic decomposition of $\widetilde{g}$ with

$$\widetilde{\lambda}_{\nu m} = \begin{cases} \lambda_{\nu m} 2^{-\nu \frac{\varkappa}{p}} & \text{for } m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma,\nu}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.44}$$

Moreover $\widetilde{a}_{\nu m} = 2^{\nu \frac{\varkappa}{p}} a_{\nu m}$ are $\left(s - \frac{\varkappa}{p}, p \right)_{K,L}$-atoms. We benefit from our assumption (4.41) and can apply the unweighted version of Theorem 3.14 ($\varkappa = 0, r_0 = 1$), see [Tri92, Theorem 3.10], to obtain

$$\left\| \widetilde{g} \, | B_{pq}^{s - \frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \le c \left\| \widetilde{\lambda}_{\nu m} \, | b_{pq} \right\|. \tag{4.45}$$

On the other hand, $\mathrm{tr}_{\Gamma}\, \widetilde{g} = \mathrm{tr}_{\Gamma}\, g^{\Gamma} = \mathrm{tr}_{\Gamma}\, g = f$, and

$$\left\| \widetilde{\lambda}_{\nu m} \, | b_{pq} \right\| \le c \left\| \lambda_{\nu m} \, | b_{pq}(w_{\varkappa}^{\Gamma}) \right\| \tag{4.46}$$

by (4.44) and (3.8), recall $\left\| \chi_{\nu m}^{(p)} \, | L_p(\mathbb{R}^n, w_{\varkappa}^{\Gamma}) \right\| \sim 2^{\nu \frac{\varkappa}{p}}$, $m \in \mathbb{Z}^n \setminus \mathrm{I}_{\Gamma\nu}$, $\nu \in \mathbb{N}_0$. Combining (4.39), (4.42), (4.45) and (4.46) we obtain

$$\left\| \widetilde{g} \,|B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \le c \left\| f \,|\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| + \varepsilon,$$

that is, we have found some $\widetilde{g} \in B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma \widetilde{g} = f$ and the above norm estimate. Hence, $f \in \mathrm{tr}_\Gamma B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$, and for $\varepsilon \searrow 0$,

$$\left\| f \mid \mathrm{tr}_\Gamma B_{pq}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \le c \left\| f \,|\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| .$$

This proves (4.38). □

In view of (2.9) it is clear that the theory of Besov spaces with Muckenhoupt weights covers only weights $w_\varkappa^\Gamma$ from (2.11)(c) with $\varkappa > -(n-d)$. Theorem 4.4 above concerns weights $w_\varkappa^\Gamma$ with $\varkappa < sp-(n-d)$, $s > 0$, $0 < p < \infty$, where $f \in B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ possesses a trace $\mathrm{tr}_\Gamma f \in \mathbb{B}_{pq}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}(\Gamma)$.
Similarly for $\varkappa = sp-(n-d)$, $0 < q \le \min(1,p)$, see Theorem 4.4. A natural question to ask is what happens for stronger weights, that is, $\varkappa > sp-(n-d)$ or $\varkappa = sp-(n-d)$ with $q > \min(1,p)$, respectively? The final answer to this question in the unweighted case is due to H. TRIEBEL [Tri06, Theorem 1.174], see also [Tri06, Corollary 7.21]. Roughly speaking, the result given there states that for $s < \frac{n-d}{p}$, $0 < p, q \le \infty$, or $s = \frac{n-d}{p}$, $q > \min(p,1)$, the trace space $\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n)$ does not exist. Below we show how to transfer this observation to our situation.

COROLLARY 4.8. *Let $0 < d < n$, $s > 0$, $1 < p < \infty$, $0 < q \le \infty$ and $\varkappa > -(n-d)$. Then $\mathrm{tr}_\Gamma B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ exists if, and only if,*

$$\varkappa < sp-(n-d)$$

or

$$\varkappa = sp-(n-d) \quad \text{and} \quad 0 < q \le 1.$$

Moreover, if $\varkappa > sp-(n-d)$, then $\mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ is dense in $B_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$.

Proof. The sufficiency follows from Theorems 4.4 and 4.7, concerning the necessity we refer to [Tri06, Corollary 7.21] for the unweighted case and (4.37). Note that the additional assumption $\varkappa < sp-(n-d)$ or $0 < q \le \min(p,1)$ when $\varkappa = sp-(n-d)$ are needed only later on to determine the trace space explicitly.

It remains to show the density of $\mathcal{D}(\mathbb{R}^n\backslash\Gamma)$ in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ when $\varkappa > sp-(n-d)$. Clearly, by the embeddings

$$B^{s+\varepsilon}_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \hookrightarrow B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) \hookrightarrow B^{s-\varepsilon}_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$$

for all $0 < q \leq \infty$, and $\varepsilon > 0$ small, it is enough to deal with spaces $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ only, where $\varkappa > sp-(n-d)$ and $1 < p < \infty$. Then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ and we can restrict ourselves to show that for all $\varepsilon > 0$ and all $\psi \in \mathcal{S}(\mathbb{R}^n)$ there is some $\varphi \in \mathcal{D}(\mathbb{R}^n\backslash\Gamma)$, i.e. $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi) \subset \mathbb{R}^n\backslash\Gamma$, such that

$$\left\|\psi - \varphi \mid B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| < \varepsilon. \tag{4.47}$$

We continue by assuming that $\operatorname{supp}\psi \cap \Gamma \neq \emptyset$. Otherwise, $\operatorname{dist}(\Gamma, \operatorname{supp}\psi) = \delta > 0$ and we can take $\varphi = \psi$, appropriately modified, if $\operatorname{supp}\psi$ is not compact. Let Γ_k be some neighbourhood of $\Gamma \cap \operatorname{supp}\psi$. For $j \in \mathbb{N}$, consider a covering of Γ_k with balls centered at Γ and with radius 2^{-j}. Since Γ is a compact d-set one needs $M_j \sim 2^{jd}$ balls to cover it. Let $\{\varphi_r\}_{r=1}^{M_j}$ be an associated smooth partition of unity such that $\varphi_r \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi_r \subset B_{r,j} = B(\gamma_r, 2^{-j})$, $\gamma_r \in \Gamma$ and $\sum_{r=1}^{M_j}\varphi_r(x) = 1$ with $x \in \Gamma_k$. Recall that $\left\|\chi^{(p)}_{B_{r,j}} \,|L_p(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \sim 2^{-j\frac{\varkappa}{p}}$. Let $\gamma \in \mathcal{D}(\mathbb{R}^n)$ with $\gamma = 1$ on $\Gamma_{k/2}$ and $\operatorname{supp}\gamma \subset \Gamma_k$. Taking into account Definition 3.8 and Theorem 3.14 we obtain

$$\gamma = \sum_{r=1}^{M_j}(\varphi_r\gamma)(x) = \sum_{r=1}^{M_j} 2^{j(s-\frac{n}{p})}2^{-j(s-\frac{n}{p})}(\varphi_r\gamma)(x), \quad x \in \Gamma_k. \tag{4.48}$$

The sum on the right-hand side of (4.48) may be viewed as an atomic decomposition of γ in $B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)$ with atoms given by $2^{-j(s-\frac{n}{p})}(\varphi_r\gamma)(x)$ and coefficients $\lambda_r = 2^{j(s-\frac{n}{p})}$. For convenience let us assume once more that we do not need moment conditions, otherwise (4.48) has to be modified. Then Theorem 3.14 and (3.8) imply

$$\begin{aligned}\left\|\gamma \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| &\leq \left(\sum_{r=1}^{M_j} 2^{j(s-\frac{n}{p})p}\left\|\chi^{(p)}_{B_{r,j}}|\, L_p(\mathbb{R}^n, w^\Gamma_\varkappa)\right\|^p\right)^{1/p} \\ &\leq 2^{j(s-\frac{n}{p})-j\frac{\varkappa}{p}}\left(\sum_{r=1}^{M_j}1\right)^{1/p} = c2^{j(s-\frac{n-d}{p}-\frac{\varkappa}{p})}.\end{aligned}$$

It follows from the assumption $s < \frac{\varkappa+n-d}{p}$ that

$$\left\|\gamma \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| < \varepsilon,$$

choosing in our construction j sufficiently large. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we thus arrive at

$$\begin{aligned} \left\|\psi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| &= \left\|\psi\gamma + (1-\gamma)\psi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \\ &\leq \left\|\psi\gamma \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| + \left\|(1-\gamma)\psi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \\ &\leq \left\|\psi \,|C^k(\mathbb{R}^n)\right\| \left\|\gamma \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| + \left\|(1-\gamma)\psi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| \\ &< \varepsilon' + \left\|(1-\gamma)\psi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\|, \end{aligned}$$

where $k \in \mathbb{N}$ is chosen large enough. On the other hand, we obtain $(1-\gamma)\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{dist}(\operatorname{supp}((1-\gamma)\psi), \Gamma) > 0$. Hence, there exists some $\varphi \in \mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ with

$$\left\|(1-\gamma)\psi - \varphi \,|B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa)\right\| < \varepsilon.$$

This concludes the proof of (4.47). □

Remark 4.9. Corollary 4.8 explains, at least in some cases, the impossibility to have a trace of $f \in B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, $\varkappa > sp - (n-d)$ in the sense of $L_p(\Gamma)$. We only get the trivial counterpart of (4.4), i.e. for the dense subset $\mathcal{D}(\mathbb{R}^n\backslash\Gamma)$ in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ the left-hand side in (4.4) always vanishes unlike the right-hand side. But then it is not possible to explain $\operatorname{tr}_\Gamma f$ in a reasonable (standard) way, as the independence of the approximating sequence fails. One would like to have a real alternative in the sense that either $\operatorname{tr}_\Gamma B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ exists or $\mathcal{D}(\mathbb{R}^n\backslash\Gamma)$ is dense in $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$. But this remains open so far - as in the unweighted case.

4.4 Traces on fractals of weighted Triebel-Lizorkin spaces and applications

In this section we discuss traces on fractals of weighted Triebel-Lizorkin spaces. Our main aim here is to extend known results on traces of unweighted Triebel-Lizorkin spaces to the weighted case. The last part of this section is devoted to give an application of our results for F-spaces to traces of weighted Sobolev spaces on $(n-1)$-dimensional hyperplanes. Let us start by recalling needed definitions. The best references here are [Bui82] and [HP].

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty$. Moreover let $\{\varphi_j\}_{j=0}^\infty$ be a smooth partition of unity as introduced in Section 2.2. Recall that the weighted Triebel -

Lizorkin space $F^s_{pq}(\mathbb{R}^n, w)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \,|F^s_{pq}(\mathbb{R}^n, w)\right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)|^q \right)^{1/q} |L_p(\mathbb{R}^n, w) \right\|$$

is finite, see (3.3). In the limiting case $q = \infty$ the usual modification is required. Taking in (3.3) $1 < p < \infty$, $s \in \mathbb{N}_0$, $q = 2$ and $w \equiv 1$ we obtain classical Sobolev spaces, i.e.

$$F^s_{p2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n)$$

see [Tri83, Section 2], [Tri92, Section 1.2.5] and [Tri97, Section 10.5].
The unweighted trace result due to H. Triebel [Tri97, Corollary 18.12] reads as follows.

THEOREM 4.10. *Let Γ be a d-set, $0 < d < n$. Let $0 < p \leq 1$ and $0 < q \leq \infty$. Then we get*

$$\operatorname{tr}_\Gamma F^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma)$$

with the usual interpretation.

We also recall the definition of the corresponding Triebel-Lizorkin sequence spaces. Let $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}_\infty$. Furthermore let $\chi^{(p)}_{\nu m}$ denote the p-normalized characteristic function of the cube $Q_{\nu m}$ defined by (3.7). Then $f_{pq}(w)$ is the collection of all sequences $\lambda = \{\lambda_{\nu m}\} \in \mathbb{C}$ such that

$$\left\| \lambda \,|f_{pq}(w)\right\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \left| \lambda_{\nu m} \chi^{(p)}_{\nu m}(\cdot) \right|^q \right)^{1/q} \left| L_p(\mathbb{R}^n, w) \right. \right\| \tag{4.49}$$

is finite (usual modification for $q = \infty$).
In the sequel, we again consider the weight $w^\Gamma_\varkappa$ as introduced in Example 2.11(c). We now present a generalization of Theorem 4.10 to the weighted case.

THEOREM 4.11. *Let Γ be a d-set, $0 < d < n$. Let $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$, $-(n-d) < \varkappa < sp - (n-d)$, or $\varkappa = sp - (n-d)$ if $0 < p \leq 1$. Then*

$$\operatorname{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa) = \operatorname{tr}_\Gamma B^{s-\frac{\varkappa}{p}}_{pp}(\mathbb{R}^n). \tag{4.50}$$

In particular,

$$\operatorname{tr}_\Gamma F_{pq}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n, w_\varkappa^\Gamma) = L_p(\Gamma) \tag{4.51}$$

for $0 < p \leq 1$, $0 < q \leq \infty$, *and*

$$\operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) = \mathbb{B}_{pp}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}(\Gamma), \tag{4.52}$$

provided that $\varkappa < sp - (n-d)$, $0 < p < \infty$ *and* $0 < q \leq \infty$.

Proof. The proof is based on the argument given in the proof of Theorem 4.7 combined with [Tri06, Proposition 9.22]. We only outline the main ideas of the proof for

$$\operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \subset \operatorname{tr}_\Gamma B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n). \tag{4.53}$$

The proof of the converse inclusion is done analogously. Let $f \in \operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$. Following the same consideration as in Step 1 of the proof of Theorem 4.7 we arrive at the atomic decomposition of g in $F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$ and its reformulation for $\widetilde{g}$ as in (4.43). We conclude that $\widetilde{g} \in B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n)$, since

$$\left\| \widetilde{g} | \, B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \leq c_1 \left\| \widetilde{\lambda} | \, b_{pp} \right\| \leq c_2 \left\| \widetilde{\lambda} | \, f_{pq} \right\| \leq c_3 \left\| \lambda | \, f_{pq}(w_\varkappa^\Gamma) \right\|, \tag{4.54}$$

where the equation $\left\| \widetilde{\lambda} | \, b_{pp} \right\| \sim \left\| \widetilde{\lambda} | \, f_{pq} \right\|$ follows from [Tri06, Proposition 9.22 (ii)], since d-sets satisfy the ball condition what means that they are porous in the notation used in [Tri06]. Consequently, we have for $\widetilde{g}$ with $\operatorname{tr}_\Gamma \widetilde{g} = f$,

$$\left\| \widetilde{g} | \, B_{pp}^{s-\frac{\varkappa}{p}}(\mathbb{R}^n) \right\| \leq c \left\| g | \, F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| + \frac{\varepsilon}{2} \leq c \left\| f | \operatorname{tr}_\Gamma F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma) \right\| + \varepsilon, \tag{4.55}$$

which completes the proof. ☐

Remark 4.12. It turns out that the index q plays no rôle in the consideration of traces on d-sets of $F_{pq}^s(\mathbb{R}^n, w_\varkappa^\Gamma)$. More precisely, for $0 < q_0 < q_1 < \infty$ we get

$$\operatorname{tr}_\Gamma F_{pq_0}^s(\mathbb{R}^n, w_\varkappa^\Gamma) = \operatorname{tr}_\Gamma F_{pq_1}^s(\mathbb{R}^n, w_\varkappa^\Gamma),$$

as in the unweighted case, see [Tri01, Theorem 9.21].

We have the following counterpart of Corollary 4.8.

COROLLARY 4.13. *Let* $0 < d < n$, $s > 0$, $1 < p < \infty$, $0 < q \leq \infty$ *and* $\varkappa > -(n-d)$. *Then* $\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ *exists if, and only if,* $\varkappa < sp-(n-d)$. *Moreover, if* $\varkappa > sp-(n-d)$, *and* $1 < p,q < \infty$, *then* $\mathcal{D}(\mathbb{R}^n \backslash \Gamma)$ *is dense in* $F^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$.

We conclude this section with a well-known example of Sobolev spaces and a d-set Γ with $d = n-1$, i.e. $\Gamma \sim \{x \in \mathbb{R}^n\ x_n = 0\}$. We characterize traces on $(n-1)$-dimensional hyperplanes of Sobolev spaces. We first discuss a special case of the weight function $w^\Gamma_\varkappa$ for $d = n-1$.

Example 4.14. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Note that for $d = n-1$ and taking $\varkappa = \alpha$ the weight $w^\Gamma_\varkappa$ transforms into

$$w_\alpha(x) = \begin{cases} |x_n|^\alpha & |x_n| < 1 \\ 1 & \text{otherwise .} \end{cases} \tag{4.56}$$

As shown in Proposition 2.12(i), $w_\alpha(x)$ belongs to the Muckenhoupt class $\mathcal{A}_r$ if, and only if, $-1 < \alpha < r-1$.

We recall briefly the definition of Sobolev spaces.

DEFINITION 4.15. Let $k \in \mathbb{N}$, $1 \leq p < \infty$ and $w \in \mathcal{A}_\infty$. The *Sobolev space* $W^k_p(\mathbb{R}^n, w)$ is the collection of all $f \in L_p(\mathbb{R}^n, w)$ such that the norm

$$\left\| f |\ W^k_p(\mathbb{R}^n, w) \right\| = \left(\sum_{|\beta| \leq k} \left\| \mathrm{D}^\beta f |\ L_p(\mathbb{R}^n, w) \right\|^p \right)^{1/p}$$

is finite.

It is well-known that for $k \in \mathbb{N}_0$, $1 < p < \infty$, and $w_\alpha \in \mathcal{A}_p$, i.e. $-1 < \alpha < p-1$, we have

$$F^k_{p,2}(\mathbb{R}^n, w_\alpha) = W^k_p(\mathbb{R}^n, w_\alpha). \tag{4.57}$$

This can be found, for instance in [Ryc01, Proposition 1.9]. We are now in a position to state the last result of this section.

PROPOSITION 4.16. *Let $1 < p < \infty$ and $-1 < \alpha < p-1$. Then for any $k \in \mathbb{N}$*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, w_\alpha) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma).$$

Proof. Using (4.57) and Remark 4.12 combined with Theorems 4.7 and 4.11, we obtain

$$\begin{aligned}\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, w_\alpha) &= \mathrm{tr}_{\mathbb{R}^{n-1}} F_{p,2}^k(\mathbb{R}^n, w_\alpha) = \mathrm{tr}_{\mathbb{R}^{n-1}} B_{pp}^{k-\frac{\alpha}{p}}(\mathbb{R}^n)\\ &= \mathbb{B}_{pp}^{k-\frac{\alpha}{p}-\frac{1}{p}}(\Gamma) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma).\end{aligned}$$

Note that our assumption for α to imply $w_\alpha \in \mathcal{A}_p$, i.e. $\alpha < p-1$, already ensures $\alpha < kp-1$, $k \in \mathbb{N}$, needed in Theorem 4.11. ☐

Remark 4.17. This result was first proved in [Tri78, Section 3.6] using tricky interpolation techniques.

Chapter 5

Weighted function spaces of generalized smoothness and traces on related (d, Ψ)-sets

In this chapter we present a generalization of the setting described in Chapter 4. Our main purpose is to prove results concerning traces on fractals replacing classical d-sets by so-called (d, Ψ)-sets.

5.1 Function spaces $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ and (d, Ψ)-sets

This section covers definitions and results on spaces of generalized smoothness and related (d, Ψ)-sets that will be of importance in the subsequent sections. We start by recalling needed definitions.

DEFINITION 5.1. A positive monotone function Ψ on the interval $(0, 1]$ is called *admissible* if

$$\Psi(2^{-k}) \sim \Psi(2^{-2k}), \qquad k \in \mathbb{N}_0. \tag{5.1}$$

Example 5.2. We check at once that for $b \in \mathbb{R}$

$$\Psi_b(x) = (1 + |\log x|)^b, \quad x \in (0, 1],$$

where log is taken with respect to base 2, is an admissible function according to the above definition.

Below we list some simple but useful properties of an admissible function Ψ.

PROPOSITION 5.3. *Let Ψ be an admissible function on the interval* $(0, 1]$.

(i) *Let $\delta \in \mathbb{R}$. Then Ψ^δ is also admissible.*

(ii) *Let $a \in \mathbb{R}^+$. Then*

$$\lim_{x \to 0^+} x^a \Psi(x) = 0.$$

(iii) *There are positive numbers c_1, c_2, b and c, with $c \in (0, 1)$ such that*

$$c_1 |\log(cx)|^{-b} \leq \Psi(x) \leq c_2 |\log(cx)|^b, \quad x \in (0, 1].$$

(iv) *There is a positive constant c such that*

$$\Psi(2x) \leq c\Psi(x), \quad x \in (0, 1/2].$$

(v) *If $a \in \mathbb{R}^+$, then there exists $j_0 \in \mathbb{N}_0$ such that for any $j \in \mathbb{N}_0$ with $j \geq j_0$*

$$\Psi(a2^{-j}) \sim \Psi(2^{-j}) \quad \textit{and} \quad \Psi(2^{-aj}) \sim \Psi(2^{-j}).$$

For a proof and more details we refer the reader to [Mou01] and [ET99].
Let $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth resolution of unity introduced in Definition 2.1.

DEFINITION 5.4. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Moreover let Ψ be an admissible function according to the Definition 5.1. Then $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\left\| f \,|B^{s,\Psi}_{pq}(\mathbb{R}^n) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \Psi(2^{-j})^q \left\| (\varphi_j \widehat{f})^\vee \,|L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} \tag{5.2}$$

(with the usual modification for $q = \infty$) is finite.

Remark 5.5. The spaces $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ were introduced by D. E. EDMUNDS and H. TRIEBEL in [ET98]. For a complete treatment of these spaces we refer the reader the work of S. D. MOURA, [Mou01], see also [ET96], [Tri97] and [Tri01] for more details. One may also consider the Triebel-Lizorkin spaces of generalized smoothness $F^{s,\Psi}_{pq}(\mathbb{R}^n)$, $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ by interchanging the order of ℓ_q- and L_p- quasi-norms in (5.2). We shift this case to the end of the present chapter. The spaces $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$). It is known that the space $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ does not depend on the chosen smooth resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ (in the sense of equivalent quasi-norms). In particular, if $\Psi = 1$ we obtain classical Besov spaces $B^s_{pq}(\mathbb{R}^n)$, studied in detail in [Tri83] and [Tri92], see also Section 2.2.

We need the following counterpart of Definition 3.8.

DEFINITION 5.6.

(a) Let $K \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K-*atom* (or simply an 1-*atom*) if the following assumptions are satisfied

(i) supp $a \subset dQ_{0m}$ for some $m \in \mathbb{Z}^n$,

(ii) $|\mathrm{D}^\alpha a(x)| \leq 1$ for $|\alpha| \leq K, \quad x \in \mathbb{R}^n$.

(b) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s, p, \Psi)_{K,L}$-*atom* if for some $\nu \in \mathbb{N}_0$ the following assumptions are satisfied

(i) supp $a \subset dQ_{\nu m}$ for some $m \in \mathbb{Z}^n$,

(ii) $|\mathrm{D}^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}\Psi(2^{-\nu})^{-1}$ for $|\alpha| \leq K, \ x \in \mathbb{R}^n$,

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$ for $|\beta| \leq L$.

Analogously to the case of $(s, p)_{K,L}$-atoms from Definition 3.8, we will write $a_{\nu m}$ instead of a, to indicate the localization and size of an $(s, p, \Psi)_{K,L}$-atom a. Below we state the atomic decomposition of weighted Besov spaces of generalized smoothness $B^{s,\Psi}_{pq}(\mathbb{R}^n)$. This was proved by S. D. MOURA [Mou01, Theorem 1.3.5].

THEOREM 5.7. *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ *and* Ψ *an admissible function. Let* $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K \geq (1 + [s])_+ \qquad \textit{and} \qquad L \geq \max(-1, [\sigma_p - s])$$

be fixed. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^{s,\Psi}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \qquad \textit{convergence being in } \mathcal{S}'(\mathbb{R}^n), \tag{5.3}$$

where $a_{\nu m}$ *are* 1_K*-atoms* $(\nu = 0)$ *or* $(s, p, \Psi)_{K,L}$*-atoms* $(\nu \in \mathbb{N})$ *and* $\lambda \in b_{pq}$. *Furthermore*

$$\inf \|\lambda \,| b_{pq}, \| \tag{5.4}$$

where the infimum is taken over all admissible representations (5.3), *is an equivalent quasi-norm in* $B^{s,\Psi}_{pq}(\mathbb{R}^n)$.

DEFINITION 5.8. Let Γ be a non-empty closed subset of $\mathbb{R}^n$.

(i) Let $0 < d < n$ and let Ψ be an admissible function according to Definition 5.1. Then Γ is called a (d, Ψ)*-set* , if there exist a Radon measure μ on $\mathbb{R}^n$ with supp $\mu = \Gamma$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \Psi(r) \le \mu(B(\gamma, r)) \le c_2 r^d \Psi(r) \tag{5.5}$$

for any ball $B(\gamma, r)$ in $\mathbb{R}^n$ centered at $\gamma \in \Gamma$ and of radius $r \in (0, 1)$.

(ii) Let Ψ be a decreasing admissible function according to Definition 5.1 with $\Psi(x) \to \infty$, if $x \to 0$. Then Γ is called a (n, Ψ)*-set* , if there is a Radon measure μ in $\mathbb{R}^n$ with the above properties and $d = n$ in (5.5).

Remark 5.9. Note that for $\Psi \equiv 1$ we obtain d-sets with $0 < d < n$ as introduced in Definition 2.9. Let $0 < d < n$ and let Ψ an admissible function, then for any couple (d, Ψ) there exists a (d, Ψ)-set in $\mathbb{R}^n$, see [ET99, Proposition 2.8]. Furthermore any (d, Ψ)-set in $\mathbb{R}^n$ with $d < n$ satisfies the ball condition, see (4.13) and [Tri01, Proposition 22.6(iv)].

Example 5.10. Let Ψ_b be as in Example 5.2, then Γ_b is a (d, Ψ_b)-set, for $0 < d < n$, if there exist two positive constants c_1 and c_2 such that

$$c_1 r^d (1 + |\log r|)^b \le \mu(B(\gamma, r)) \le c_2 r^d (1 + |\log r|)^b,$$

where μ is a Radon measure in $\mathbb{R}^n$ with supp $\mu = \Gamma_b$ and $B(\gamma, r)$ is a ball centered at $\gamma \in \Gamma$ and of radius r, $0 < r < 1$.

We consider the following example of a weight function which is a generalization of Example 2.11 to (d, Ψ)-sets.

Example 5.11. Let Γ be a (d,Ψ)-set, $0<d<n$, Ψ an admissible function, $\varkappa\in\mathbb{R}$

$$v_{\varkappa}^{\Gamma}(x) := \begin{cases} (\mathrm{dist}(x,\Gamma))^{\varkappa}\,\Psi(\mathrm{dist}(x,\Gamma)), & \text{for} \quad \mathrm{dist}(x,\Gamma)\le 1 \\ \Psi(1), & \text{otherwise.} \end{cases} \tag{5.6}$$

In particular, for $\Psi=\Psi_b$ given by Example 5.2, and Γ_b defined in Example 5.10, we obtain

$$\begin{aligned} v_{\varkappa}^{\Gamma_b}(x) : &= \begin{cases} (\mathrm{dist}(x,\Gamma_b))^{\varkappa}\,\Psi_b(\mathrm{dist}(x,\Gamma_b)), & \text{for} \quad \mathrm{dist}(x,\Gamma_b)\le 1 \\ \Psi(1), & \text{otherwise.} \end{cases} \\ &= \begin{cases} (\mathrm{dist}(x,\Gamma_b))^{\varkappa}\,(1+|\log(\mathrm{dist}(x,\Gamma_b))|)^{b}\,, & \text{for} \quad \mathrm{dist}(x,\Gamma_b)\le 1 \\ \Psi(1), & \text{otherwise.} \end{cases} \end{aligned} \tag{5.7}$$

PROPOSITION 5.12. *Let $1<p<\infty$ and let Γ be a (d,Ψ)-set with $0<d<n$ and Ψ an admissible function. Then $v_{\varkappa}^{\Gamma}\in\mathcal{A}_p$ if, and only if,*

$$-(n-d)<\varkappa<(n-d)(p-1). \tag{5.8}$$

Proof. The proof is similar to that given for Proposition 2.12. Again we restrict ourselves to cubes $Q_{\nu m}$, $\nu\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$ only . To verify condition (2.9) we estimate the first integral

$$\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x.$$

We consider a covering of the cube $Q_{\nu m}$ with sets

$$S_k=\left\{x\in\mathbb{R}^n:\;\; 2^{-k-1}<\mathrm{dist}(x,\Gamma)\le 2^{-k}\right\}\cap\, Q_{\nu m}$$

i.e. $Q_{\nu m}\subset\bigcup_{k=\nu}^{\infty}S_k$, see Figure 2.1. Furthermore, let K_l, $l=1,\ldots,N_{k,\nu}$ denote balls with radius $r\sim 2^{-k}$ that cover the set S_k. Hence we obtain

$$\begin{aligned} \frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x &= 2^{\nu n}\int\limits_{Q_{\nu m}} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x \sim 2^{\nu n}\sum_{k=\nu}^{\infty}\int\limits_{S_k} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x \\ &\sim 2^{\nu n}\sum_{k=\nu}^{\infty}2^{-k\varkappa}\Psi(2^{-k})\int\limits_{S_k}\mathrm{d}x \sim 2^{\nu n}\sum_{k=\nu}^{\infty}2^{-k\varkappa}\Psi(2^{-k})\sum_{l=1}^{N_{k,\nu}}\int\limits_{K_l}\mathrm{d}x. \end{aligned}$$

The Lebesgue measure of a ball K_l in $\mathbb{R}^n$ is approximately equal to 2^{-kn}. Moreover carefully looking at the condition (5.5) we infer that

$$N_{k,\nu} \sim \frac{2^{-\nu d}\Psi(2^{-\nu})}{2^{-kd}\Psi(2^{-k})}.$$

This provides that

$$\begin{aligned}\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x &\sim 2^{\nu n}\sum_{k=\nu}^{\infty} 2^{-k(\varkappa+n)}\Psi(2^{-k})N_{k,\nu}\\ &\sim 2^{\nu n}\sum_{k=\nu}^{\infty} 2^{-k(\varkappa+n)}\Psi(2^{-k})\frac{2^{-\nu d}\Psi(2^{-\nu})}{2^{-kd}\Psi(2^{-k})}\\ &\sim 2^{\nu n}\sum_{k=\nu}^{\infty} 2^{-k(\varkappa+n)}2^{d(k-\nu)}\Psi(2^{-\nu})\\ &\sim 2^{-\nu\varkappa}\Psi(2^{-\nu})\sum_{k=\nu}^{\infty} 2^{-(k-\nu)(\varkappa+n-d)}.\end{aligned}$$

Certainly, the last series converges if, and only if, $\varkappa > -(n-d)$, and thus

$$\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} v_{\varkappa}^{\Gamma}(x)\,\mathrm{d}x \sim 2^{-\nu\varkappa}\Psi(2^{-\nu}). \tag{5.9}$$

Furthermore, looking at the second integral in (2.9) with (5.9) we have that

$$\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} \left(v_{\varkappa}^{\Gamma}(x)\right)^{-p'/p}\,\mathrm{d}x = \frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} \widetilde{v}_{\vartheta}^{\Gamma}(x)\,\mathrm{d}x \tag{5.10}$$

where $\vartheta = -\varkappa p'/p = -\varkappa(p-1)$ and

$$\widetilde{v}_{\vartheta}^{\Gamma}(x) = [\mathrm{dist}(x,\Gamma)]^{-\varkappa p'/p}\,\Psi(\mathrm{dist}(x,\Gamma))^{-p'/p}.$$

We put $\widetilde{\Psi}(\mathrm{dist}(x,\Gamma)) = \Psi(\mathrm{dist}(x,\Gamma))^{-p'/p}$. As $\widetilde{\Psi}$ is admissible according to Proposition 5.3(i) we obtain in the same way that $\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} \left(v_{\varkappa}^{\Gamma}(x)\right)^{-p'/p}\,\mathrm{d}x$ is finite if, and only if, $\vartheta > -(n-d)$, i.e. $\varkappa < \frac{p}{p'}(n-d) = (n-d)(p-1)$. Consequently, we get that

$$\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} \widetilde{v}_{\vartheta}^{\Gamma}(x)\,\mathrm{d}x \sim 2^{-\nu\vartheta}\widetilde{\Psi}(2^{-\nu}) = 2^{-\nu\vartheta}\left(\Psi(2^{-\nu})\right)^{-p'/p} = \left[2^{-\nu\varkappa}\Psi(2^{-\nu})\right]^{-p'/p}$$

and by (5.9) that $v_{\varkappa}^{\Gamma}$ satisfies the $\mathcal{A}_p$- condition (2.9) if, and only if, $-(n-d) < \varkappa < (p-1)(n-d)$ for all admissible Ψ. □

5.2 Traces on (d,Ψ)-sets of weighted Besov spaces

We consider weighted Besov spaces with weights $v_\varkappa^\Gamma$ introduced in Example 5.11. Let $\chi_{\nu m}^{(p)}$ be the p-normalized characteristic function on the cube $Q_{\nu m}$ according to Definition 3.7. It follows from (5.9) that

$$\left\|\chi_{\nu m}^{(p)}\,|L_p(\mathbb{R}^n,v_\varkappa^\Gamma)\right\| = \left(2^{\nu n}\int\limits_{Q_{\nu m}} v_\varkappa^\Gamma(x)\,\mathrm{d}x\right)^{1/p} = \left(\frac{1}{|Q_{\nu m}|}\int\limits_{Q_{\nu m}} v_\varkappa^\Gamma(x)\,\mathrm{d}x\right)^{1/p} \sim 2^{\frac{-\nu\varkappa}{p}}\Psi(2^{-\nu})^{\frac{1}{p}}. \tag{5.11}$$

A straightforward calculation shows that

$$\begin{aligned}\left\|\lambda_{\nu m}\,|b_{pq}(v_\varkappa^\Gamma)\right\| &\sim \left(\sum_{\nu=0}^{\infty}\left\|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\chi_{\nu m}^{(p)}\,|L_p(\mathbb{R}^n,v_\varkappa^\Gamma)\right\|^q\right)^{1/q}\\ &\sim \left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p\left\|\chi_{\nu m}^{(p)}\,|L_p(\mathbb{R}^n,v_\varkappa^\Gamma)\right\|^p\right)^{q/p}\right)^{1/q}\\ &\sim \left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^p v_{\nu m}\right)^{q/p}\right)^{1/q},\end{aligned} \tag{5.12}$$

where

$$v_{\nu m}\sim\begin{cases}2^{-\nu\varkappa}\Psi(2^{-\nu}), & \text{if}\quad \operatorname{dist}(2^{-\nu}m,\Gamma)\le 1\\ \Psi(1), & \text{otherwise}\ .\end{cases}$$

In the sequel we will consider the following extension of the notion of trace spaces on d-sets, as introduced in Definition 4.6, to (d,Ψ)-sets. We consider Definition 2.2.7 in [Mou01] with $a=0$.

DEFINITION 5.13. Let $0<p,q\le\infty$, $s>0$, Ψ be an admissible function and let Γ be a (d,Ψ)-set in $\mathbb{R}^n$ with $0<d<n$. We define

$$\mathbb{B}_{pq}^s(\Gamma) := \operatorname{tr}_\Gamma B_{pq}^{s+\frac{n-d}{p},\Psi^{1/p}}(\mathbb{R}^n). \tag{5.13}$$

We equip this space with the quasi-norm

$$\|f\,|\mathbb{B}^s_{pq}(\Gamma)\| = \inf \left\| g\,|B^{s+\frac{n-d}{p},\Psi^{1/p}}_{pq}(\mathbb{R}^n) \right\|, \tag{5.14}$$

where the infimum is taken over all $g \in B^{s+\frac{n-d}{p},\Psi^{1/p}}_{pq}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma\, g = f$.

Note that for $\Psi \equiv 1$, Γ is a d-set according to Definition 2.9, and then the above definition covers the Definition 4.6.
Let $v^\Gamma_\varkappa$ be the Muckenhoupt weight introduced in Example 5.11. We have the following generalization of Theorem 4.7.

THEOREM 5.14. *Let $0 < d < n$, $s \in \mathbb{R}$, $-(n-d) < \varkappa$, $0 < p < \infty$, $0 < q \le \infty$, Ψ be an admissible function and let Γ be a (d,Ψ)-set according to Definition 5.8(i). Then we have*

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) = \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pq}(\mathbb{R}^n), \tag{5.15}$$

whenever these spaces exist.

We compare this theorem with Definition 5.13 and we have

COROLLARY 5.15. *Let $0 < d < n$, $0 < p < \infty$, $0 < q \le \infty$, Ψ be an admissible function and let Γ be a (d,Ψ)-set according to the Definition 5.8(i). Let $-(n-d) < \varkappa < sp - (n-d)$ then*

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) = \mathbb{B}^{s-\frac{\varkappa}{p}-\frac{n-d}{p}}_{pq}(\Gamma). \tag{5.16}$$

of Theorem 5.14. The proof follows analogous ideas to that of Theorem 4.7. We present only one inclusion

$$\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) \hookrightarrow \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pq}(\mathbb{R}^n). \tag{5.17}$$

The second one is proved in a similar way. Let us start with a function $f \in \mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$. Let $\varepsilon > 0$, then there exists some $g \in B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma\, g = f$ and

$$\left\| g\,|B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) \right\| \le \left\| f\,|\,\mathrm{tr}_\Gamma\, B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) \right\| + \frac{\varepsilon}{2}. \tag{5.18}$$

We consider an atomic decomposition of g in $B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ as in (3.13)

$$g = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}(x),$$

where $a_{\nu m}(x)$ are $(s,p)_{K,L}$-atoms according to Definition 3.8 and $\lambda_{\nu m} \in \mathbb{C}$ are coefficients. Furthermore, carefully looking at Theorem 3.11 with K and L according to (4.41) yields

$$\left\|\lambda \,|b_{pq}\left(v^\Gamma_\varkappa\right)\right\| \le c \left\|g\,|B^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\right\| + \frac{\varepsilon}{2}, \tag{5.19}$$

for a suitably chosen atomic decomposition. Let

$$\mathrm{I}_{\Gamma,\nu} = \left\{m \in \mathbb{Z}^n : \operatorname{dist}(\Gamma, \operatorname{supp} a_{\nu m}) > b2^{-\nu}\right\}, \quad \nu \in \mathbb{N}_0, \quad b > 0.$$

Once again as in the proof of Theorem 4.7, we arrive at the following decomposition

$$g = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) + \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}{}_{\Gamma,\nu} \lambda_{\nu m} a_{\nu m}(x) := g^\Gamma + g_\Gamma,$$

where the first term collects all atoms with a support near to Γ, and the second one the rest, see (4.8) and (4.10). Moreover we have that $\operatorname{tr}_\Gamma g = \operatorname{tr}_\Gamma g^\Gamma$ and $\operatorname{tr}_\Gamma g_\Gamma = 0$. We extend now g^Γ by 0 outside

$$\begin{aligned} \widetilde{g} =& \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}^{\Gamma,\nu} \left(\lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}} \Psi(2^{-\nu})^{1/p}\right) \left(2^{\nu\frac{\varkappa}{p}} \Psi(2^{-\nu})^{-1/p} a_{\nu m}(x)\right) \\ &+ \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n}{}_{\Gamma,\nu} 0 \left(2^{\nu\frac{\varkappa}{p}} \Psi(2^{-\nu})^{-1/p} a_{\nu m}(x)\right) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \widetilde{\lambda}_{\nu m} \widetilde{a}_{\nu m}(x), \end{aligned} \tag{5.20}$$

where $\widetilde{a}_{\nu m} = 2^{\nu\frac{\varkappa}{p}} \Psi(2^{-\nu})^{-1/p} a_{\nu m}$ are $\left(s - \frac{\varkappa}{p}, p, \Psi^{1/p}\right)_{K,L}$-atoms according to Definition 5.6 with K, L sufficiently large as in (4.41) and the coefficients are given by

$$\widetilde{\lambda}_{\nu m} = \begin{cases} \lambda_{\nu m} 2^{-\nu\frac{\varkappa}{p}} \Psi(2^{-\nu})^{1/p} & \text{for } m \in \mathbb{Z}^n \backslash \mathrm{I}_{\Gamma,\nu}, \\ 0 & \text{otherwise}. \end{cases} \tag{5.21}$$

From the unweighted atomic decomposition Theorem 5.7 for (d, Ψ)-sets , (see [Mou01, Theorem 1.3.5(ii)]) we obtain

$$\left\|\widetilde{g}\,|B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pq}(\mathbb{R}^n)\right\| \le c \left\|\widetilde{\lambda}_{\nu m}\,|b_{pq}\right\|. \tag{5.22}$$

It is easy to see that (5.12) and the definition of the b_{pq}-norm yield

$$\left\|\widetilde{\lambda}_{\nu m}\,|b_{pq}\right\| \le c' \left\|\lambda_{\nu m}\,|b_{pq}(v_\varkappa^\Gamma)\right\|. \tag{5.23}$$

Furthermore, $\mathrm{tr}_\Gamma\,\widetilde{g} = \mathrm{tr}_\Gamma\,g^\Gamma = \mathrm{tr}_\Gamma\,g = f$. Combining the inequality (5.23) with (5.18), (5.19) and (5.22) gives

$$\left\|\widetilde{g}\,|B_{pq}^{s-\frac{\varkappa}{p},\Psi^{1/p}}(\mathbb{R}^n)\right\| \le c\left\|f\,|\,\mathrm{tr}_\Gamma\,B_{pq}^s(\mathbb{R}^n,v_\varkappa^\Gamma)\right\| + \varepsilon,$$

i.e. we have found $\widetilde{g} \in B_{pq}^{s-\frac{\varkappa}{p},\Psi^{1/p}}(\mathbb{R}^n)$ with $\mathrm{tr}_\Gamma\,\widetilde{g} = f$. Consequently, for $\varepsilon \searrow 0$,

$$\left\|f\,|\,\mathrm{tr}_\Gamma\,B_{pq}^{s-\frac{\varkappa}{p},\Psi^{1/p}}(\mathbb{R}^n)\right\| \le c\left\|f\,|\,\mathrm{tr}_\Gamma\,B_{pq}^s(\mathbb{R}^n,v_\varkappa^\Gamma)\right\|,$$

which proves our claim. □

The next example ties together the concepts of Theorem 5.14 and Corollary 5.15 with the example of the special admissible function $\Psi_b(x) = (1+|\log(x)|)^b$ with $x \in (0,1]$.

Example 5.16. Let $b \in \mathbb{R}$. Let Ψ_b be an admissible function introduced in Example 5.2, and let Γ_b be a (d,Ψ_b)-set according to Example 5.10, $0<d<n$. For the weight function $v_\varkappa^{\Gamma_b}$ given by (5.7) we have

$$\mathrm{tr}_{\Gamma_b}B_{pq}^s(\mathbb{R}^n,v_\varkappa^{\Gamma_b}) = \mathrm{tr}_{\Gamma_b}B_{pq}^{s-\frac{\varkappa}{p},\Psi^{1/p}}(\mathbb{R}^n) = \mathrm{tr}_{\Gamma_b}B_{pq}^{s-\frac{\varkappa}{p},b/p}(\mathbb{R}^n) = \mathbb{B}_{pq}^{s-\frac{\varkappa}{p}-\frac{n-d}{p}}(\Gamma_b),$$

where $B_{pq}^{s-\frac{\varkappa}{p},b/p}(\mathbb{R}^n)$ is the space used by H. G. LEOPOLD in [Leo00].

COROLLARY 5.17. (i) *Let $0<p<\infty$, $-(n-d)<\varkappa$ and let Γ be a (d,Ψ)-set with $0<d<n$. Then for $0<q\le\min(p,1)$ we have*

$$\mathrm{tr}_\Gamma\,B_{p,q}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n,v_\varkappa^\Gamma) = L_p(\Gamma), \tag{5.24}$$

in the sense, that $\mathrm{tr}_\Gamma\,f \in L_p(\Gamma)$ for any $f \in B_{p,q}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n,v_\varkappa^\Gamma)$ and any $f^\Gamma \in L_p(\Gamma)$ is a trace of a suitable $g \in B_{p,q}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n,v_\varkappa^\Gamma)$ on Γ and

$$\left\|f^\Gamma\,|L_p(\Gamma)\right\| \sim \inf\left\|g\,|B_{p,q}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n,v_\varkappa^\Gamma)\right\|,$$

where the infimum is taken over all $g \in B_{p,q}^{\frac{\varkappa}{p}+\frac{n-d}{p}}(\mathbb{R}^n,v_\varkappa^\Gamma)$ such that $\mathrm{tr}_\Gamma\,g = f^\Gamma$.

(ii) *Let $1 < p < \infty$ and let Γ be an (n,Ψ)-set according to Definition 5.8(ii). Let*

$$\sum_{j=0}^{\infty} \Psi^{-1/p}(2^{-j}) < \infty.$$

Then

$$\mathrm{tr}_\Gamma B^{\frac{\varkappa}{p}}_{p1}(\mathbb{R}^n, v^\Gamma_\varkappa) = L_p(\Gamma). \tag{5.25}$$

Proof. The proof of both parts is an easy consequence of Theorem 5.14 above and [Tri01, Theorem 22.18]. See also [Mou01, Proposition 2.2.4, Remark 2.2.5]. ☐

5.3 Traces on (d,Ψ)-sets of weighted Triebel-Lizorkin spaces

This section treats the trace problem on the perturbed d-sets for the weighted Triebel-Lizorkin spaces. In what follows, let $v^\Gamma_\varkappa$ be the weight function introduced in Example 5.11. Recall that $F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the quasi-norm $\|f\ |F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\|$ given by (3.3), with $w = v^\Gamma_\varkappa$, is finite. Furthermore, $\mathbb{B}^s_{pq}(\Gamma)$ is the trace space according to Definition 5.13. The following theorem gives the answer for the question about the trace problem on (d,Ψ)-sets for the weighted F-spaces.

THEOREM 5.18. *Let $0 < d < n$, $0 < p < \infty$, $0 < q \le \infty$, $-(n-d) < \varkappa < sp - (n-d)$, Ψ be an admissible function and let Γ be a (d,Ψ)-set according to Definition 5.8(i). Then*

$$\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) = \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pp}(\mathbb{R}^n). \tag{5.26}$$

In particular, for $0 < p \le 1$, $0 < q \le \infty$, $-(n-d) < \varkappa$

$$\mathrm{tr}_\Gamma\, F^{\frac{\varkappa}{p}+\frac{n-d}{p}}_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) = L_p(\Gamma), \tag{5.27}$$

and

$$\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) = \mathbb{B}^{s-\frac{n-d}{p}-\frac{\varkappa}{p}}_{pp}(\Gamma), \tag{5.28}$$

provided that $\varkappa < sp - (n-d)$, $0 < p < \infty$ and $0 < q \le \infty$.

Proof. The proof is similar to that given in Chapter 4 for the case of *d*-sets, see Theorem 4.11 and the proof of Theorem 5.14. Thus we show only crucial modifications. We start by showing the first inclusion

$$\mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa) \subset \mathrm{tr}_\Gamma\, B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pp}(\mathbb{R}^n). \tag{5.29}$$

Let $f \in \mathrm{tr}_\Gamma\, F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ and $\varepsilon > 0$. Then there exists some $g \in F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ such that $\mathrm{tr}_\Gamma g = f$ and

$$\|g\ |F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\| \le \|f\ |\mathrm{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\| + \frac{\varepsilon}{2}.$$

Further, it follows as in the proof of Theorem 5.14, that

$$\left\|\lambda\ |f_{pq}(v^\Gamma_\varkappa)\right\| \le c \left\|g\ |F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\right\| + \frac{\varepsilon}{2}, \tag{5.30}$$

if we choose the atomic decomposition of g in $F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)$ appropriately, see (3.13). Moreover we obtain $\widetilde{g}$ as in (5.20). At this stage we appeal to [Tri06, Proposition 9.22(i)] to deduce that $\widetilde{g} \in B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pp}(\mathbb{R}^n)$, see (4.54) and arguments given there. Furthermore we have that $\mathrm{tr}_\Gamma \widetilde{g} = f$. As a conclusion we have

$$\left\|f\ |\mathrm{tr}_\Gamma B^{s-\frac{\varkappa}{p},\Psi^{1/p}}_{pp}(\mathbb{R}^n)\right\| \le c \left\|f\ |\mathrm{tr}_\Gamma F^s_{pq}(\mathbb{R}^n, v^\Gamma_\varkappa)\right\|,$$

which proves (5.29). The proof of the converse direction follows similarly. □

We end this section with an example for Sobolev spaces and a (d,Ψ)-set Γ with $d = n-1$, parallel to Example 4.14. We treat the special case of the weight function $v^\Gamma_\varkappa$ introduced in Example 5.11 with $\varkappa = \alpha$.

Example 5.19. Let Γ be an $(n-1,\Psi)$-set, Ψ an admissible function. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$. We put $\varkappa = \alpha$. Then, we consider

$$v_\alpha(x) := \begin{cases} |x_n|^\alpha \Psi(|x_n|), & \text{for } |x_n| \le 1 \\ \Psi(1), & \text{otherwise.} \end{cases} \tag{5.31}$$

An obvious consequence of Proposition 5.12 is the following.

PROPOSITION 5.20. *Let $1 < p < \infty$ and let Γ be a (d,Ψ)-set with $d = n-1$ and Ψ an admissible function. Then v_α according to (5.31) belongs to the Muckenhoupt class $\mathcal{A}_p$ if, and only if,*

$$-1 < \alpha < p-1. \tag{5.32}$$

Let v_α be the weight function given by (5.31). Recall that the weighted Sobolev space $W_p^k(\mathbb{R}^n, v_\alpha)$ is the collection of all $f \in L_p(\mathbb{R}^n, v_\alpha)$ such that $\mathrm{D}^\beta f \in L_p(\mathbb{R}^n, v_\alpha)$ for all multi-indices $|\beta| \le k$. Moreover, $\mathbb{B}^s_{pq}(\Gamma)$ is the trace space introduced in Definition 5.13. We have, for the weight v_α

$$W_p^k(\mathbb{R}^n, v_\alpha) = F^k_{p2}(\mathbb{R}^n, v_\alpha), \quad 1 < p < \infty, \quad -1 < \alpha < p-1, \tag{5.33}$$

see [Ryc01, Proposition 1.19] and also [Bui82, Theorem 1.4]. We can now formulate the last result of this section.

PROPOSITION 5.21. *Let $1 < p < \infty$, $-1 < \alpha < p-1$ and let Γ be a (d,Ψ)-set according to Definition 5.8(i) with $d = n-1$. Then for any $k \in \mathbb{N}$*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, v_\alpha) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma).$$

Proof. We first apply (5.33), then we take Theorem 5.18 with Definition 5.13 and we obtain

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W_p^k(\mathbb{R}^n, v_\alpha) = \mathrm{tr}_{\mathbb{R}^{n-1}} F^k_{p2}(\mathbb{R}^n, v_\alpha) = \mathrm{tr}_{\mathbb{R}^{n-1}} B_{pp}^{k-\frac{\alpha}{p},\Psi^{1/p}}(\mathbb{R}^n) = \mathbb{B}_{pp}^{k-\frac{\alpha+1}{p}}(\Gamma),$$

what is just what we want. Note that $-1 < \alpha < p-1$ implies $-1 < \alpha < kp-1$, as required in Theorem 5.18. □

Chapter 6

Entropy and approximation numbers of embeddings between weighted Besov spaces

The aim of this chapter is to study the compactness of the trace operator. More precisely, we use known results on entropy numbers to investigate the behavior of those numbers of compact embeddings between weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$, where $w^\Gamma_\varkappa$ is given by (2.14). In particular, we consider the trace operator from spaces $B^s_{pq}(\mathbb{R}^n, w^\Gamma_\varkappa)$ into Lebesgue spaces $L_p(\Gamma)$, where Γ is a d-set according to Definition 2.9. Furthermore, we generalize the result to the (d, Ψ)-set according to Definition 5.8. Moreover we compute approximation numbers of the embeddings between function spaces of the above type.

6.1 Entropy numbers

Let X and Y be quasi-Banach spaces and let $T\ : X \to Y$ be a bounded linear operator. Let

$$U_X := \{x \in X : \ \|x\,|X\| \leq 1\}$$

be the unit ball in the quasi-Banach space X. An operator T is called compact if for any given $\varepsilon > 0$ we can cover the image of the unit ball U_X with finitely many balls in Y of radius ε.

DEFINITION 6.1. Let X, Y be quasi-Banach spaces and let $T \in L(X, Y)$. Then for all $k \in \mathbb{N}$, the *kth dyadic entropy number $e_k(T)$ of T* is defined by

$$e_k(T) = \inf \left\{ \varepsilon > 0 : \; T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} (y_j + \varepsilon U_Y) \;\text{ for some }\; y_1, \ldots, y_{2^{k-1}} \in Y \right\},$$

where U_X und U_Y denote the unit balls in X and Y, respectively.

These numbers have various elementary properties summarized in the following lemma.

LEMMA 6.2. *Let X, Y and Z be quasi-Banach spaces, let $S, T \in L(X, Y)$ and $R \in L(Y, Z)$.*

(i) **(Monotonicity):** $\|T\| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0$. *Moreover* $\|T\| = e_1(T)$, *provided that Y is a Banach space.*

(ii) **(Additivity):** *If Y is a p-Banach space* $(0 < p \leq 1)$, *then for all* $j, k \in \mathbb{N}$

$$e^p_{j+k-1}(S + T) \leq e^p_j(S) + e^p_k(T).$$

(iii) **(Multiplicativity):** *For all* $j, k \in \mathbb{N}$

$$e_{j+k-1}(RT) \leq e_j(R) e_k(T).$$

(iv) **(Compactness):** *T is compact if, and only if,* $\lim_{k\to\infty} e_k(T) = 0$.

Proofs of the above properties may be found for instance in [ET96, Lemma 1.3.1/1]. For more information, we recommend the monographs [ET96] and [CS90].

Remark 6.3. Let us briefly discuss the connection between eigenvalues of a compact linear map and its entropy numbers, though applications of that kind are out of the scope of this thesis. Let $T : X \to X$ be a compact linear operator in a quasi-Banach space X and let $(\lambda_n(T))$ be the sequence of all nonzero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq 0$. Then the *Carl's inequality* states

$$|\lambda_n(T)| \leq \sqrt{2} e_n(T).$$

A general reference here is again [ET96] and [CS90]. Based on this inequality, and having in mind application to spectral theory of certain pseudo-differential operators, D. D. HAROSKE, D. E. EDMUNDS, and H. TRIEBEL initiated a program to investigate the behavior of the entropy numbers in the context of weighted function spaces of Besov and Triebel-Lizorkin type, see [ET96] and [HT94a, HT94b]. For a recent account we refer to the series of papers by T. KÜHN ET AL. [KLSS06a, KLSS06b, KLSS].

Let us recall a result for entropy numbers which is due to H. TRIEBEL , see [Tri97, Theorem 20.6].

THEOREM 6.4. *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ according to Definition 2.9. Let $\mathbb{B}^s_{pq}(\Gamma)$ be the spaces introduced in Definition 4.6, notationally complemented by $\mathbb{B}^0_{pq}(\Gamma) = L_p(\Gamma)$ for any $0 < p \leq \infty$ and $0 < q \leq \infty$. Let*

$$0 \leq s_2 < s_1 < \infty, \;\; 0 < p_1, p_2 \leq \infty, \;\; 0 < q_1, q_2 \leq \infty,$$

and

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0.$$

Then the embedding

$$\mathrm{id}: \; \mathbb{B}^{s_1}_{p_1 q_1}(\Gamma) \to \mathbb{B}^{s_2}_{p_2 q_2}(\Gamma)$$

is compact and for the related entropy numbers holds

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1 - s_2}{d}}, \quad k \in \mathbb{N}. \tag{6.1}$$

Remark 6.5. Recall that equivalence $\sim$ in (6.1) means that there exist two positive numbers c_1 and c_2 such that for all $k \in \mathbb{N}$,

$$c_1 k^{-\frac{s_1 - s_2}{d}} \leq e_k(\mathrm{id}) \leq c_2 k^{-\frac{s_1 - s_2}{d}}.$$

Assume that $0 < q \leq \infty$ and

$$s - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0.$$

Then (6.1) with $s_2 = 0$ can be rewritten in the form

$$e_k\left(\mathrm{tr}_\Gamma :\ B^{s+\frac{n-d}{p_1}}_{p_1 q}(\mathbb{R}^n) \to L_{p_2}(\Gamma)\right) \sim k^{-\frac{s}{d}}, \quad k \in \mathbb{N}. \tag{6.2}$$

For more details, see [Tri97, Chapter IV, p.172].

We are now in a position to present results on entropy numbers for weighted Besov spaces.

THEOREM 6.6. *Let Γ be a d-set in $\mathbb{R}^n$ with $0 < d < n$ according to Definition 2.9. Let $\frac{n-d+\varkappa}{p_2} \le s_2 < s_1 < \infty$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$, and*

$$-(n-d) < \varkappa < \min(s_1 p_1, s_2 p_2) - (n-d). \tag{6.3}$$

Let

$$s_1 - s_2 > (\varkappa + n - d)\left(\frac{1}{p_1} - \frac{1}{p_2}\right) + d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+. \tag{6.4}$$

Then for the weight $w^\Gamma_\varkappa$ introduced in Example 2.11(c) the embedding

$$\mathrm{id} :\ \mathrm{tr}_\Gamma B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w^\Gamma_\varkappa) \longrightarrow \mathrm{tr}_\Gamma B^{s_2}_{p_2 q_2}(\mathbb{R}^n, w^\Gamma_\varkappa) \tag{6.5}$$

is compact and for the related entropy numbers holds

$$e_k(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{d} + \left(\frac{\varkappa+n-d}{d}\right)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}, \quad k \in \mathbb{N}. \tag{6.6}$$

Proof. The proof is a simple consequence of Theorem 4.7 and Theorem 6.4. We have the following

$$\begin{aligned} & e_k\left(\mathrm{id} :\ \mathrm{tr}_\Gamma B^{s_1}_{p_1 q_1}(\mathbb{R}^n, w^\Gamma_\varkappa) \longrightarrow \mathrm{tr}_\Gamma B^{s_2}_{p_2 q_2}(\mathbb{R}^n, w^\Gamma_\varkappa)\right) = \\ & e_k\left(\mathrm{id} :\ \mathbb{B}^{s_1 - \frac{\varkappa}{p_1} - \frac{n-d}{p_1}}_{p_1 q_1}(\Gamma) \longrightarrow \mathbb{B}^{s_2 - \frac{\varkappa}{p_2} - \frac{n-d}{p_2}}_{p_2 q_2}(\Gamma)\right). \end{aligned} \tag{6.7}$$

By virtue of (6.1) with $\bar{s}_1 - \bar{s}_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0$ we obtain

$$e_k\left(\mathrm{id} :\ \mathbb{B}^{s_1 - \frac{\varkappa}{p_1} - \frac{n-d}{p_1}}_{p_1 q_1}(\Gamma) \longrightarrow \mathbb{B}^{s_2 - \frac{\varkappa}{p_2} - \frac{n-d}{p_2}}_{p_2 q_2}(\Gamma)\right) \sim k^{-\frac{\bar{s}_1 - \bar{s}_2}{d}}, \quad k \in \mathbb{N},$$

where

$$\bar{s}_i = s_i - \frac{\varkappa}{p_i} - \frac{n-d}{p_i} \quad \text{for} \quad i = 1, 2.$$

One immediately checks the compatibility (6.3) and (6.4). This finishes the proof.

□

Remark 6.7. Let $s = \delta + \frac{n-d}{p_1} + \frac{\varkappa}{p_1}$. From Theorem 4.7 we conclude that

$$\mathrm{tr}_\Gamma B^s_{p_1 q}(\mathbb{R}^n, w^\Gamma_\varkappa) = \mathbb{B}^\delta_{p_1 q}(\Gamma), \quad \delta > 0,$$

Furthermore, by Definition 4.6 we get

$$\mathbb{B}^\delta_{p_1 q}(\Gamma) = \mathrm{tr}_\Gamma B^{\delta + \frac{n-d}{p_1}}_{p_1 q}(\mathbb{R}^n).$$

Comparing this with (6.2) and the above theorem we obtain the following result.

PROPOSITION 6.8. *Let Γ be a d-set in $\mathbb{R}^n$ with $0 < d < n$ according to Definition 2.9. Let $0 < p_1, p_2 < \infty$, $0 < q \le \infty$, $-(n-d) < \varkappa < sp_1 - (n-d)$, and let $w^\Gamma_\varkappa$ be a weight function given by* (2.14)*. Moreover let*

$$s - \frac{1}{p_1}(\varkappa + n - d) - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0.$$

The trace operator tr_Γ of $B^s_{p_1 q}(\mathbb{R}^n, w^\Gamma_\varkappa)$ into $L_{p_2}(\Gamma)$ is compact and the related entropy numbers satisfy

$$e_k\left(\mathrm{tr}_\Gamma : \ B^s_{p_1 q}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_{p_2}(\Gamma)\right) \sim k^{\frac{1}{d}\left(\frac{n+\varkappa}{p_1} - s\right) - \frac{1}{p_1}}. \tag{6.8}$$

One can extend this result to the (d, Ψ)-sets, where Ψ is an admissible function according to Definition 5.8. In [ET99, Theorem 2.24] we get desired generalization but only for $1 < p_1, p_2 \le \infty$ and with the target space L_p. The case $0 < p \le 1$ has been considered by S. D. Moura in [Mou01, Theorem 3.3.2]. She deals with target spaces of Besov type. Let Ψ be an admissible function according to Definition 5.1 and let Γ be a (d, Ψ)-set according to Definition 5.8. Let now $\mathbb{B}^s_{pq}(\Gamma)$ be the trace spaces introduced in Definition 5.13. By assumption we have $0 < p_1, p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, and $s_1, s_2 \ge 0$ such that

$$s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0,$$

that the embedding

$$\mathrm{id}:\ \mathbb{B}^{s_1}_{p_1q_1}(\Gamma) \to \mathbb{B}^{s_2}_{p_2q_2}(\Gamma)$$

is compact. Furthermore, for the related entropy numbers holds

$$e_k(\mathrm{id}) \sim \left[k\Psi(k^{-1})\right]^{-\frac{s_1-s_2}{d}}, \quad k \in \mathbb{N}. \tag{6.9}$$

Recall, that we take $a_1 = a_2 = 0$ in the original version of Theorem 3.3.2 in [Mou01], such that Definition 2.2.7 in [Mou01] of $\mathbb{B}$-spaces covers Definition 5.13. The best general reference here is [Mou01, Chapter 3] and also [ET98] and [ET99].
We can now give an extension of Theorem 6.6 to the (d,Ψ)-set.

PROPOSITION 6.9. *Let Ψ be an admissible function, and let Γ be a (d,Ψ)-set according to Definition 5.8. Let $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \le \infty$, and*

$$-(n-d) < \varkappa < \min(s_1p_1, s_2p_2) - (n-d).$$

Moreover let s_1, s_2 be as in (6.4) *with $s_2 > \frac{n-d+\varkappa}{p_2}$. Then for the weight $v^\Gamma_\varkappa$ introduced in* (5.11) *the embedding*

$$\mathrm{id}:\ \mathrm{tr}_\Gamma B^{s_1}_{p_1q_1}(\mathbb{R}^n, v^\Gamma_\varkappa) \longrightarrow \mathrm{tr}_\Gamma B^{s_2}_{p_2q_2}(\mathbb{R}^n, v^\Gamma_\varkappa) \tag{6.10}$$

is compact and for the related entropy numbers holds

$$e_k(\mathrm{id}) \sim \left[k\Psi(k^{-1})\right]^{\frac{s_1-s_2}{d}-\left(\frac{\varkappa+n-d}{d}\right)\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}, \qquad k \in \mathbb{N}. \tag{6.11}$$

Proof. We follow the proof of Theorem 6.6. We consider Theorem 5.14 and Definition 5.13 and arrive at

$$e_k\left(\mathrm{id}:\ \mathrm{tr}_\Gamma B^{s_1}_{p_1q_1}(\mathbb{R}^n, v^\Gamma_\varkappa) \longrightarrow \mathrm{tr}_\Gamma B^{s_2}_{p_2q_2}(\mathbb{R}^n, v^\Gamma_\varkappa)\right) =$$
$$e_k\left(\mathrm{id}:\ \mathbb{B}^{s_1-\frac{\varkappa}{p_1}-\frac{n-d}{p_1}}_{p_1q_1}(\Gamma) \longrightarrow \mathbb{B}^{s_2-\frac{\varkappa}{p_2}-\frac{n-d}{p_2}}_{p_2q_2}(\Gamma)\right).$$

Combining this with (6.9) completes the proof. □

6.2 Approximation numbers

In this section we recall the basic definitions and properties concerning approximation numbers and apply it in weighted Besov spaces.

DEFINITION 6.10. Let $T \in L(X,Y)$, $k \in \mathbb{N}$. The *kth approximation number* a_k *of* T is defined by

$$a_k(T) = \inf \{\|T - L\| \ : \ L \in L(X,Y), \ \ \mathrm{rank} L < k\}, \tag{6.12}$$

where rank$L$ is the dimension of the range of L.

We have also for approximation numbers analogous properties as for entropy numbers. We present them in the following lemma.

LEMMA 6.11. *Let X, Y and Z be quasi-Banach spaces, let $S, T \in L(X,Y)$ and $R \in L(Y,Z)$.*

(i) **(Monotonicity):** $\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq 0.$

(ii) **(Additivity):** *If Y is a p–Banach space $(0 < p \leq 1)$, then for all $j, k \in \mathbb{N}$*

$$a^p_{j+k-1}(S+T) \leq a^p_j(S) + a^p_k(T).$$

(iii) **(Multiplicativity):** *For all $j, k \in \mathbb{N}$*

$$a_{j+k-1}(RT) \leq a_j(R) a_k(T).$$

(iv) **(Rank property):**

$$a_n(T) = 0 \ \textit{if, and only if,} \ \mathrm{rank} T < n.$$

The best general references here are [CS90] and [ET96]. In the sequel, we restrict ourselves to d-sets and formulate our result. Recall that the function $w^\Gamma_\varkappa$ is a weight given by (2.14). We now state the main result for approximation numbers.

THEOREM 6.12. *Let $0 < d < n$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{n-d+\varkappa}{p} < s \leq \frac{n+\varkappa}{p}$, and $-(n-d) < \varkappa$. Let Γ be a d-set according to Definition 2.9. Then the trace operator*

$$\mathrm{tr}_\Gamma : \ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma) \tag{6.13}$$

is compact and for the related approximation numbers a_k holds

$$a_k(\mathrm{tr}_\Gamma : \ B^s_{pp}(\mathbb{R}^n, w^\Gamma_\varkappa) \to L_p(\Gamma)) \sim k^{\frac{1}{d}(\frac{n+\varkappa}{p} - s) - \frac{1}{p}}, \quad k \in \mathbb{N}. \tag{6.14}$$

Proof. As a consequence of Theorem 4.7 and the Definition 4.6, from embedding (6.13) we get

$$\mathrm{tr}_\Gamma : \; B^s_{pp}(\mathbb{R}^n) \to L_p(\Gamma).$$

Combining this with Theorem 2 and Remark 9 (Example) in [Tri04] we obtain the desired estimate (6.14). The compactness is covered by Proposition 6.8 with $p_1 = p_2 = q$. □

Remark 6.13. Note that (6.8) coincides with (6.14) for $p_1 = p_2 = q = p$. One should expect a different behaviour of $e_k(\mathrm{tr}_\Gamma)$ and $a_k(\mathrm{tr}_\Gamma)$ for $p_1 \neq p_2$. This study is postponed to later occasion, as well as the counterpart of Proposition 6.9 for approximation numbers.

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