Instanton-Induced Defects in Gauge Theories

Dissertation
zu Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der Physikalisch-Astronomischen Fakultät
der Friedrich-Schiller-Universität Jena

von Dipl.-Phys. Falk Bruckmann
geboren am 9. Januar 1973 in Halle/Saale

Gutachter

1. Prof. Dr. Andreas Wipf, Jena

2. Prof. Dr. Pierre van Baal, Leiden, NL

3. Prof. Dr. Michael Müller-Preußker, Berlin

Tag der letzten Rigorosumsprüfung: 21.09.2001

Tag der öffentlichen Verteidigung: 30.10.2001

Contents

‘... the forces might become large enough to **confine** the quarks. That is the foremost problem of QCD.’ [1]

‘The outstanding problem in QCD is to explain long distance phenomena, in particular why we do not see quarks and gluons as physical objects - the so called problem of ”quark **confinement**”’. [2]

‘It is unlikely that one will ever prove from first principles that permanent **confinement** takes place...’ [3]

‘Quark **confinement** is not yet completely solved.’ [4]

‘The most essential property of QCD is **confinement**.’ [5]

‘Die spektakulärste Aussage der QCD ist zweifellos das Verbot freier, nicht in Hadronen gebundener Quarks.’ [6]

‘A long standing and yet unsolved problem is to explain color **confinement** in QCD.’ [7]

‘Over the last two decades various attempts have been aiming at a qualitative understanding and modelling of two basic properties of QCD: quark **confinement** and chiral symmetry breaking.’ [8]

‘Therefore, *understanding* **confinement**, in my opinion, is one of the most exciting challenges of modern physics.’ [9]

‘**Confinement** is something of a mystery. It is certainly the most striking qualitative phenomenon in QCD. Still we do not even have a satisfactory definition of what exactly is meant by this word.’ [10]

1. Introduction

1.1. The Gauge Theory of Strong Interactions

At present-day energies the *Standard Model* is the fundamental theory of elementary particles and their interactions (for a recent review see [11]). Its matter content consists of fermionic fields carrying a representation of the gauge group $U(1) \times SU(2) \times SU(3)$. The interactions are provided by gauge fields, i.e. vector fields in the Lie algebra of this gauge symmetry[1]. The part containing the strong interactions is named *Quantum Chromodynamics* (QCD); the Lagrangian density reads,

$$L = \sum_{k=1}^{N_f} \bar{\psi}^k (i\gamma^\mu \mathrm{D}_\mu - m^k)\psi^k - \frac{1}{2}\,\mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}\,, \tag{1.1}$$

where $\{\psi^k\}_{k=1,\dots,N_f}$ stands for the fields of quark flavours ($N_f = 6$: up, down, strange, charm, bottom, top), m^k for their masses, $A_\mu = \sum_{a=1}^{8} A_\mu^a T_a$ for the gluon fields, $\mathrm{D}_\mu = \partial_\mu - iA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ for the covariant derivative and the field strength, respectively. $\{T_a\}_{a=1,\dots,8}$ are the Gell-Mann matrices, i.e. the generators of (the fundamental representation of) the Lie algebra $su(3)$ and $\{\gamma^\mu\}_{\mu=0,\dots,3}$ are the Dirac matrices in Minkowski space. Furthermore, the coupling constant g is set to unity, so are $\hbar$ and c; we suppress spinor indices and use the Einstein summation convention.

This Lagrangian is invariant under local gauge transformations with $g(x) \in SU(3)$,

$$\psi \to g\psi\,, \quad \bar{\psi} \to \bar{\psi} g^\dagger\,, \quad A_\mu \to g A_\mu g^\dagger + ig\partial_\mu g^\dagger\,, \quad (\mathrm{D}_\mu \to g\mathrm{D}_\mu g^\dagger\,, F^{\mu\nu} \to gF_{\mu\nu} g^\dagger)\,, \tag{1.2}$$

which may be thought of as rotations in a colour space spanned by 'red', 'green' and 'blue'. Due to the character of the matrix group $SU(3)$, such a gauge theory is called non-Abelian. Its quantum version exhibits very interesting features already at the perturbative level: the gauge fields interact among themselves via 3- and 4-vertices, they carry colour themselves. As a consequence the perturbative β-function shows[2] that the running coupling constant is small at high energies/short distances and large at low energies/long distances, respectively.

[1] Although the gravitational interaction can be described as a gauge theory as well, a quantum version of it is not well-defined yet; in our considerations gravitational effects are negligible.

[2] if there are not too many flavours as is realised in nature

The first fact, the so-called *asymptotic freedom* in the ultraviolet region, is the basis for many confirmations of QCD in deep inelastic scattering: probing strongly interacting particles at high energies one finds the quark constituents ('partons') to be essentially free. The second fact signals the occurence of *non-perturbative effects* in the infrared region of the theory. Indeed, *the fundamental quarks in the Lagrangian do not appear as asymptotically free states in nature.* Instead they are bound together to mesons, $\psi\bar{\psi}$-states, and baryons, $\psi\psi\psi$-states. These hadrons[3] are all singlets under the colour group $SU(3)$. In other words, *free coloured states have never been observed.* This effect is called colour *confinement.* It is generally believed to be a consequence of the non-Abelian nature of the gauge group, i.e. it should occur for all $SU(N)$, $N \geq 2$. However, its derivation from the Lagrangian (1.1) remains an *open problem of the Standard Model.*

One expects a similar effect to happen in the pure glue sector of QCD: at low energies *glueballs*, bound states of gluons, should appear. These objects have not been observed in experiments yet. Nevertheless, QCD sum rules and lattice simulations predict their masses to be around 1.5 GeV (see [12] for a review). Such a *mass gap* would force any correlation function in this theory to decay exponentially thus explaining the absence of long-ranged fields in QCD. Being one of the 'Millenium Prize Problems' [13] it is also interesting from a purely mathematical point of view.

In the chiral limit where the quark masses are neglected, QCD shows another non-perturbative property, the *chiral symmetry breaking.* Left handed and right handed quarks decouple in the Lagrangian (1.1) when $m = 0$. This amounts to two commuting flavour symmetry groups which can be rewritten as a product of a vector and an axial symmetry[4]. The latter would predict all hadrons to come in pairs of opposite parity, which is not the case. Chiral symmetry is broken by the *chiral condensate* $\langle\bar{\psi}\psi\rangle$ which couples left handed to right handed quarks like a mass term. The would-be Goldstone bosons for $SU(N_{\mathrm{f}} = 2)$ are the pions.

Presumably, QCD undergoes a phase transition at sufficiently high temperatures and/or densities: hadrons start to overlap and quarks and gluons are free to travel. Beyond this *deconfinement phase transition* a new state of matter occurs, the *quark gluon plasma.* It is assumed to be realised in the early universe and in neutron stars. Lattice simulations [14] predict the critical temperature[5] to be 170 MeV, but the observation of

[3] To be precise: hadrons have the same quantum numbers as if they consist of the given *valence* quarks; in fact, they also contain *sea* quarks and gluons induced by quantum fluctuations.

[4] In fact, the vector symmetry is a subgroup of the flavour symmetry group, while the axial symmetry is only a coset.

[5] for two flavours in the chiral limit

the quark gluon plasma in heavy ion collisions has not been achieved yet.

All these non-perturbative phenomena – as well as others we have not mentioned like the $U_A(1)$ problem – should follow from QCD as the fundamental theory or a proper effective theory thereof. In many cases the effects are mainly due to the pure glue part and it is easier to look at their remnants in *pure Yang-Mills* (YM) *theories* which are defined by neglecting the quark term in the Lagrangian (1.1). This is tantamount to treating the quarks as very heavy non-dynamical objects. Therefore, this approximation is also called *quenched QCD*. We will mainly adopt this point of view in due course.

To make progress in a better understanding of colour confinement is the main motivation of this work. Therefore, this phenomenon is described in detail in the next section, followed by a discussion of lattice gauge theory and two effective theories modelling confinement and glueball formation.

1.2. Confinement

The intuitive picture of confinement is the following (Figure 1.1): In order to separate a quark q and an anti-quark $\bar{q}$ (or three quarks) one has to bring more and more energy into the system. This energy is used to create a new quark anti-quark pair $q'\bar{q}'$ from the vacuum and one ends up with two hadrons instead of free quarks. In a field theoretic description this means that the lines of the gluon field are concentrated in narrow tubes between the quarks. The latter are the sources of the *chromoelectric field*. In contrast to that, the lines of the electric field between two electric sources are spread, leading to the well-known Coulomb potential which does not confine[6]. The pair production described above is also called *string breaking*, because it breaks the flux tube into two pieces.

One can describe this phenomenon in *pure* YM theory by introducing a *heavy quark potential* $V_{q\bar{q}}(R)$ (Figure 1.1). For large separations R it rises linearly with R [16],

$$V_{q\bar{q}}(R) \to \sigma R \qquad \text{for large } R\,. \tag{1.3}$$

The quarks experience a *constant force*, as is very intuitive, since the densitiy of field lines is independent of R. We will refer to such a potential as confinement; the quarks are confined because an arbitrarily large amount of energy is needed to separate them.

The factor σ is called *string tension*. It can be estimated from the spectrum of charmonium J/Ψ (see e.g. [17]) since the charm quarks forming these hadrons are rather heavy. In the modern literature, the value of the string tension is $\sigma \simeq 1$ GeV/fm.

[6] One can show that in electrodynamics a tube-like configuration between electric sources is unstable and evolves in time to the Coulombic configuration [15].

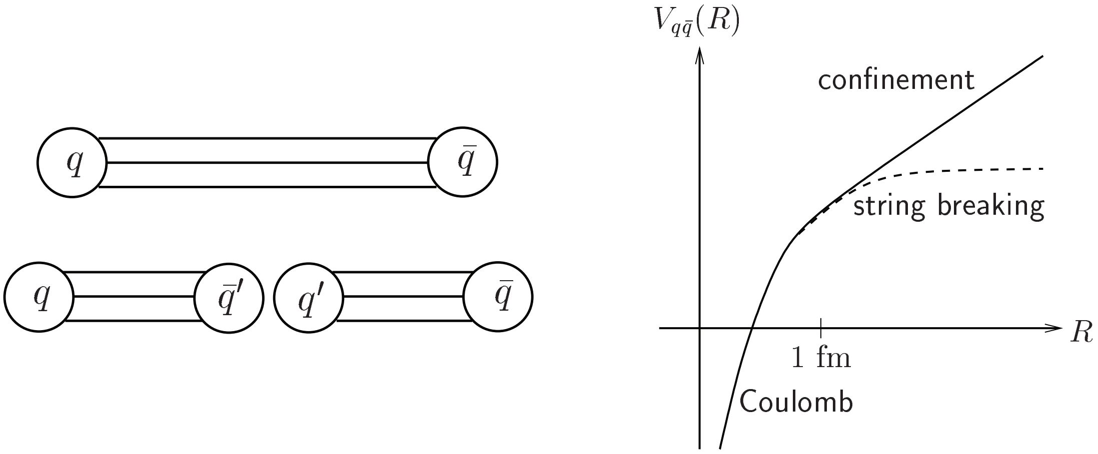

Fig. 1.1.: The picture of confinement as a tube of glue flux lines between a quark-antiquark pair $q\bar{q}$. The quarks are invisible either because their potential energy becomes arbitrarily large (without dynamical quarks, cf. (1.3)) or because a virtual quark pair $q'\bar{q}'$ becomes real (string breaking).

Some remarks are in order. At small distances the potential becomes Coulomb-like due to one gluon-exchange. Compared to the $U(1)$-theory a multiplicative factor enters. It is the second-order Casimir of the representation of the gauge group. The crossover between these regimes takes place at a scale of about 1 fm, the typical size of a proton. In full QCD the string breaking will lead to a flattening of the potential for large R and to screening of charges. Then the linear behaviour is valid only in an *intermediate range* between 0.3 fm and 1.5 fm [18].

The string also breaks if the quarks come in the adjoint (or an even higher) representation of the gauge group. The string tension in the intermediate range still scales with the Casimir [1]. Furthermore, there are always contributions to the interquark potential coming from self-energies and the universal Lüscher term, which we both neglected above.

How to relate the behaviour of the quark anti-quark potential (1.3) to the fundamental objects of pure YM theories? The answer is the well-known Wilson criterion [19]: *the expectation values of large Wilson loops decay exponentially with an area law.* First of all, for the influence of gauge fields on test charges the *loop space picture* of gauge theories [20, 21] is appropriate. It states that the wavefunction of a particle moving on a worldline γ is multiplied by the path ordered exponential of the gauge field along this contour,

$$\psi \longrightarrow \mathcal{P}[A;\gamma]\psi\,, \qquad \mathcal{P}[A;\gamma] \equiv \mathcal{P}\exp(i\int_\gamma A_\mu \mathrm{d}x^\mu)\,. \tag{1.4}$$

In other words, the effect of the gauge field is such that ψ is parallel transported along γ with the *holonomy* $\mathcal{P}[A;\gamma]$.

Now we consider the following process: the creation of a quark anti-quark pair in the past $-T/2$, its separation to a distance R, its propagation over a long time period and finally its annihilation in the future $T/2$, respectively. Such a worldline is approximated by a rectangle C with spatial extension R and time extension T. Following the loop space formulation, the quark anti-quark wave function is multiplied by the Wilson loop [22, 19], $W[T \times R] = \mathcal{P}[A; T \times R]$. Its trace is gauge invariant. In Euclidean space and in the limit of small temperatures (large $\beta = T$), the expectation value of the Wilson loop gives the ground state energy in the presence of the quark-antiquark pair, hence the heavy quark potential. Thus, if the latter is of the confining form (1.3), one has,

$$\langle \mathrm{tr}\, W[\beta \times R] \rangle \longrightarrow \exp(-\beta V_{q\bar{q}}(R)) = \exp(-\sigma \beta R)\,. \qquad (1.5)$$

Because of rotational invariance this behaviour is valid then for all large Wilson loops with βR replaced by the *area* enclosed by the loop. In contrary a non-confining potential would give a *perimeter law*.

The holonomy is a basic ingredient for lattice gauge theory we (briefly) sketch now.

1.3. Lattice Gauge Theory

When quantising non-Abelian gauge theories one encounters unavoidable singularities which call for regularisation and renormalisation. One of the most useful UV-regulators for quantum field theories is the space-time *lattice*. As in the derivation of the path integral the space-time continuum is *discretised*, i.e. replaced by a finite lattice of spacing a. The fields of the theory live on lattice points or bonds; by definition their momenta are cut off at a scale $1/a$ in momentum space. Usually periodic boundary conditions are imposed, that is, a torus is used as an IR-regulator. This leads to a finite number of degrees of freedom and allows to define the system on a computer.

Lattice gauge theories were established by Wilson in his pioneering work [19]. The fundamental degrees of freedom are *group valued* fields $U_\mu(x)$ (see the texts [23, 24]). They are defined on oriented bonds connecting lattice points and are therefore labelled by two discrete indices, x for the position of the starting point and μ for the direction. The relation to the gauge fields is provided by the holonomy from the point x to the point $x + \hat{\mu}a$, being its neighbour in the μ-direction,

$$U_\mu(x) = \mathcal{P}\exp(ig_0 \int_x^{x+\hat{\mu}a} A_\nu(y) \mathrm{d}y^\nu) \longrightarrow \exp(ig_0 A_\mu(x) a) \qquad \text{for small } a\,. \qquad (1.6)$$

g_0 is the bare coupling of the theory. The inverse of U is attached to the bond with inverse orientation. Since the field strength in the continuum is related to the parallel transport along a small closed loop, it is translated into the group element obtained by multiplying the group elements of bonds enclosing a *plaquette*, an elementary square of the lattice,

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu}a)U_\mu^{-1}(x+\hat{\nu}a)U_\nu^{-1}(x) \longrightarrow \exp(ig_0 F_{\mu\nu}(x)a^2) \quad \text{for small } a\,. \tag{1.7}$$

The Wilson action [19],

$$S = \frac{1}{\beta} \sum_{\text{plaquettes}} \left(1 - \frac{1}{N}\,\mathrm{tr}\,\mathrm{Re}\,U_{\mu\nu}\right), \tag{1.8}$$

is built out of these plaquette variables and turns into the YM action in the continuum limit if $\beta = 2N/g_0^2$. When considered in Euclidean space (and equipped with the Haar measure for the gauge group) the quantum gauge theory can be identified in the usual way with a statistical system at temperature $1/\beta$. From the renormalisation group equation it is clear that the continuum limit is $a \to 0$, $g_0 \to 0$, $\beta \to \infty$. The correlation length becomes infinite cancelling all discretisation effects. In the language of statistical physics the system is at a critical point. Physical quantities must fulfil an *asymptotic scaling* in this limit.

The feature that makes lattice gauge theory so attractive in our context is that it *does not refer to perturbation theory*, i.e. it is capable to describe non-perturbative effects. The most prominent example of an analytic result on the lattice is confinement in the *strong coupling* regime of the theory (by virtue of diagrams in a high temperature expansion [19]). Since this is just the opposite corner to the continuum limit, it has to be checked that there is no phase transition inbetween. In fact, *compact quantum electrodynamics* shows confinement in the strong coupling regime, too, but there is a phase transition to a Coulomb phase at weak coupling [25, 26].

In the last decades numerical simulations in lattice gauge theory have become a branch of research in its own right. Modern computer simulations are based on the Monte Carlo method. Both confinement and asymptotic freedom have been shown beyond doubt [27]. The heavy quark potential including string breaking excellently coincides with the intuitive picture in Figure 1.1 (see e.g. [9] and references therein). We note that, due to the finite number of degrees of freedom, lattice gauge theories need not be gauge fixed[7]. Many ideas especially concerning the explanation of confinement by Abelian projections have been tested on the lattice, as we will see later.

[7]as long as gauge invariant operators are studied

However, since numerical simulations are not as transparent as physical models, we now come to the dual superconductor picture of confinement.

1.4. The Dual Superconductor and the Dual Abelian Higgs Model

What kind of physical system admits flux tubes? It is the *superconductor of type II*. The Meissner effect states that the superconductor repels weak magnetic fields, while strong magnetic fields penetrate the superconductor in flux tubes. These normalconducting tubes are shielded by supercurrents. In solid states physics these tubes or strings go under the name of Abrikosov. An Aharonov-Bohm gedanken experiment shows that the magnetic flux is quantised in $1/q$ where $q = 2e$ is the electric charge of the Cooper pairs.

If there were magnetic monopoles inside the superconductor, they would certainly be connected by such flux tubes, too. Now for the *dual superconductor picture of QCD* [28, 29, 30, 31] one simply replaces 'electric' by 'chromomagnetic' and 'magnetic' by 'chromoelectric', respectively. Then chromomagnetic monopoles (as the dual of the Cooper pairs) condense and force the flux between chromoelectric sources into flux tubes thereby confining the quarks. But there are no magnetic monopoles in a pure unbroken Yang-Mills theory! The reason is purely topological (see e.g. [32, FB3]): the magnetic charge characterises the second homotopy group of the moduli space (the coset), when the gauge group is spontaneously *broken* in a particular manner. 't Hooft proposed the *Abelian projections* [33], the generic defects of which are magnetic monopoles. We will discuss this topic in the body of this work.

To be more precise, the relativistic generalisation of the Ginzburg-Landau theory [34],

$$L = (\mathrm{D}^\mu\phi)^*(\mathrm{D}_\mu\phi) - V(\phi) - \frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}\,, \quad D_\mu = \partial_\mu - iA_\mu\,,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.9}$$

describes a superconductor near its critical temperature. ϕ is the complex many-particle wave function[8] for the Bose-Einstein condensed Cooper pairs coupled to an Abelian gauge field A_μ. ϕ serves as an order parameter which develops a vacuum expectation value in the superconducting phase below the critical temperature. Therefore the potential $V(\phi)$ is of the well-known Mexican-hat shape in this phase. The London equation, i.e. superconductivity, follows easily from the conserved electromagnetic current. The field equations of this theory reflect the penetration depth of the magnetic field and the coherence length of the order parameter. They describe the decrease of the magnetic

[8] The underlying microscopic theory was given later by Bardeen, Cooper and Schrieffer. It explains the attractive force between electrons by means of interactions with phonons and also the energy gap in the quasi-particle spectrum near the Fermi energy.

field and the increase of the order parameter away from the surface or the position of the Nielsen-Olesen vortex [35]. Their ratio indicates repulsion or attraction of the vortices and thus distinguishes type I and type II supercondutors. In the language of spontaneous symmetry breaking these lengths are nothing but the inverse masses of the photon and the Higgs particle, respectively.

Accepting the dual superconductor picture the dual Abelian Higgs model (DAHM) is supposed to be an effective theory for YM [4, 36]. Now A_μ is the *dual photon* and ϕ describes *magnetic objects* which have to condense. As already mentioned the latter are obtained from the original theory by the method of Abelian projection. The Nielsen-Olesen vortex is now an electric object connecting external quarks. It has finite energy per unit length, i.e. finite string tension. First evidence for such electric flux tubes on the lattice was reported in [37] followed by a high precision study in [38], where the authors found the energy density and action density to be well concentrated on a string between the external charges. The masses have been fitted to be about 1 GeV (depending on the gauge group), which suggests the superconductor to be at the border between the two types (Bogomol'nyi limit) [9, 39, 40, 41, 42, 43]. We will come back to the monopole interpretation after a proper definition of monopoles.

A weak point of this model is that it is not clear whether these mass *parameters* are attached to physical quantities like masses or decay constants. If so, one would expect the glueballs to arise here [10]. But the dual gluon has $J^{PC} = 1^{+-}$ while the lightest glueballs are 0^{++} and 2^{++} states. Recent studies suggest that glueballs could emerge in this model as *flux tube rings* [41]. Another criticism concerns fields which are not charged w.r.t. the Abelian theory: they are not confined within the DAHM.

From this Higgs model one can further derive effective string theories for QCD (for a review see e.g. [44]). We will follow a different path and present an alternative effective model which is more adapted to glueballs.

1.5. The Faddeev-Niemi Action

In the 70's Faddeev proposed a model for a unit vector in three dimensions [45, 46]. Its Hamiltonian is derived from the action,

$$S = \int \mathrm{d}^4x \left(m^2 (\partial_\mu n, \partial_\mu n) + \lambda H_{\mu\nu}^2 \right) , \qquad H_{\mu\nu} = (n, \partial_\mu n \times \partial_\nu n) \,. \tag{1.10}$$

The field n fulfils $(n, n) = 1$, where we have used the scalar product and vector product in internal space. This action is a generalisation of the Heisenberg model of three-dimensional spins of unit length, called non-linear σ model. It can also be obtained by $U(1)$-gauging the Skyrme model [47]. The coupling m^2 of the σ model term has dimension mass squared, while the coupling λ of the Skyrme term is dimensionless.

This model is claimed to have numerous physical applications, since it possesses stable solitons, i.e. static solutions minimising the Hamiltonian. Their existence is supported by the Hobart-Derrick scaling argument [48, 49]. The stationary point has kinetic energy equal to potential energy and obeys the virial theorem [50]. In other words, the Skyrme term is necessary to support solitons in 3+1 dimensions. These solitons where proposed to be closed vortices stabilised by twist against shrinkage.

Finite energy solutions must have a definite limit at spatial infinity; they effectively live on the three-sphere $S^3 \cong \dot{\mathbb{R}}^3$. Accordingly, the field n is a mapping from S^3 to S^2. Such mappings are characterised by the Hopf invariant H (cf. Section 4.2.1). The lower bound estimate is quite exotic,

$$E \geq c\,\sqrt{\lambda}\,|H|^{3/4}\,, \tag{1.11}$$

with a numerical constant c as derived in [51, 52, 53] for $SU(2)$[9].

The model regained interest recently due to advances in computer performance. Notice that, compared to the Skyrme model, exact solutions are harder to find since solitons with non-trivial Hopf invariant can at most be axially symmetric[10]. Faddeev and Niemi could prove the existence of closed solitons numerically [54, 50]. More elaborated computations revealed a rich variety of such solitons among them *linked* ($H \geq 6$) and *knotted* ($H \geq 7$) ones [55, 56, 57, 58, 59].

Moreover and most interesting in our context, the model serves as a possible low energy effective action for pure YM theory [60]. The main goal in this program is the *identification of the solitons with glueballs.* To make contact with YM, the following parametrisation was proposed to be valid on-shell,

$$A_\mu = C_\mu\, n + n \times \partial_\mu n + \rho\,\partial_\mu n + \sigma n \times \partial_\mu n\,, \tag{1.12}$$

The full $SU(2)$ gauge field A is decomposed into an Abelian gauge field C, a unit vector n in isospace and a complex scalar field $\Phi = \rho + i\sigma$. The counting of degrees of freedom is correct on-shell since $6 = 2+2+2$. This parametrisation is verified in the sense that one obtains the same equations of motion either by first plugging in the parametrisation and varying the action w.r.t. those fields or by first varying w.r.t. the full field A and then using the parametrisation [60]. We will see that a normalised field n is a natural object

[9] and later generalised to $SU(N)$ by Shabanov [15].

[10] The axial symmetry of the standard mapping of this type is shown in Section 4.3.1.

in Abelian projections; therefore we will discuss possible off-shell parametrisations later (Section 3.5).

How does the Faddeev-Niemi action arise from pure Yang Mills theory by virtue of the above parametrisation? The Skyrme term is just the square of the (full non-Abelian) field strength of the pure n term,

$$F_{\mu\nu}(n \times \partial_\mu n) = \partial_\mu n \times \partial_\nu n \,, \qquad \mathrm{tr}\, F_{\mu\nu}^2 = H_{\mu\nu}^2 \,. \tag{1.13}$$

Although the σ model term is the most natural one in a gradient expansion, its derivation from pure YM is highly non-trivial. The gauge coupling in four dimensions is dimensionless, thus any mass parameter m can only emerge via regularising and renormalising the quantum theory. The effect that a mass enters a massless theory via radiative corrections is called *dimensional transmutation* [61]. In the case at hand, m is expected to arise upon *integrating out all degrees of freedom but* n [60]. This has been done using various approximation schemes [62, 63, 64, 65, 66]. All these approaches support the existence of a positive parameter m^2, but question the uniqueness of the action (1.10). This action is 'unique in the sense that it contains all such infrared relevant and marginal, local Lorentz invariant operators of n which are at most quadratic in time derivatives, as is necessary for a Hamiltonian interpretation' [60]. The above considerations render the last property artificial and one is left with three independent terms of fourth order in derivatives,

$$(\partial_\mu n)^4 \,, \qquad (\partial_\mu n, \partial_\nu n)^2 \,, \qquad (\Box n)^2 \,. \tag{1.14}$$

The first two of them enter $H_{\mu\nu}^2$ in a special combination.

The outline of this work is the following: Chapter 2 summarises the main facts about the mathematical setting and solitons of gauge theories. Both will play an important role in Abelian projections. The latter are the main topic of this work. The definitions of Abelian gauges as well as their basic properties and problems are given in Chapter 3, while Chapter 4 is devoted to the discussion of defects. Our focus lies on topological properties of defects enforced by instantons. The authors main results are contained in Sections 3.2.1, 3.2.2, 3.3.1, 3.4.1 and 4.3, outlooks are given at the end of these chapters. We conclude with a summary.

2. Preliminaries

2.1. Fibre Bundles

In this section we sketch the mathematical formalism for gauge theories, namely the theory of fibre bundles (see e.g. [67, 68, 69, 70, 71, 72]). It is the natural setting when dealing with global properties of non-trivial configurations, and in addition it is also very elegant. For the same reason we pass over to the language of differential forms now.

Fibre bundles may be seen as to implement the concept of gauge invariance. Gauge *variant* objects cannot be measured, they may even *not be defined globally, but vary invisibly* over the space-time. To be precise, the gauge fields A_1 and A_2 might be defined only on some open sets U_1 and U_2, but must give the same gauge *invariant* objects on the overlap $U_1 \cap U_2$, i.e. are related by a gauge transformation, $A_2 = {}^tA_1$.

The definition of a *fibre bundle* consists of a *base* (space-time) manifold M and a *fibre* manifold F, in which a field takes its values, subject to the action of a Lie group G called *structure group*. The *total space* manifold E is locally $M \times F$, but not globally in the sense that there exists

→ a *projection* from E to M the inverse image of which is the fibre,

→ a set of *local trivialisations* ϕ_i on open sets/charts U_i providing an identification of parts of E with the direct product $U_i \times F$ and

→ a set of *transition functions* t_{ij} from the overlaps $U_i \cap U_j$ to G gluing together the local trivialisations. *Sections* s_i from M to E invert the projection, $\pi s = \mathrm{id}_M$, but may only be defined locally.

Given M, F and G the bundle can fully be reconstructed from the transition functions. The latter measure the deviation from the trivial bundle $M \times F$, where all transition functions can be chosen to be the identity. If the base space is contractible to a point, any fibre bundle is trivial. In gauge theories non-trivial base manifolds and non-trivial bundles occur either by boundary conditions (e.g. the torus as infrared cut-off) or by demanding finite action/finite energy (then fields typically approach a constant value at infinity, like for instantons/solitons) or by excluding points from the space-time (where singularities are located, like for the Dirac monopole).

2.1.1. Principal Bundles and Associated Bundles

To arrive at the gauge field one has to consider *principal* bundles. Within these bundles, the fibre F is identical with the structure group G – acting on itself by left multiplication – and the total space is usually denoted by P. The need for local sections is expressed by the theorem that a principal bundle is trivial if and only if a global section exists. The *connection one-form* $\bar{A}$ is a one-form on the total space with values in the Lie algebra of the structure group. It provides a separation of the tangent space into a *vertical* part (along the fibre) and a *horizontal* part, respectively. The *curvature two-form* $\bar{F}$ is defined as the *covariant derivative* D of the connection one-form $\bar{A}$.

Both quantities are defined globally on the total space. With the help of local sections one can pull them back onto the base space and define the gauge field $A_i \equiv s_i^*(\bar{A})$ and the field strength $F_i \equiv s_i^*(\bar{F})$ as local Lie algebra valued forms. Cartan's structure equation translates into the well-known relation (cf. B.2),

$$F = \mathrm{D}A = \mathrm{d}A - iA \wedge A\,, \tag{2.1}$$

from which the homogeneous YM equation $\mathrm{D}F = 0$ follows as a Bianchi identity.

Apart from the transition functions there are two notions of *gauge transformations* in the bundle approach. The active one is related to a change of the connection by virtue of vertical automorphisms of the principle bundle, compatible with the action of the structure group. In the passive sense this amounts to a change of local sections,

$$s_2(x) = g(x)s_1(x)\,, \qquad A_2(x) = g(x)A_1(x)g^\dagger(x) + ig(x)\mathrm{d}g^\dagger(x)\,. \tag{2.2}$$

Matter fields in a gauge theory are described by *associated* bundles. The structure group G is assumed to have an action ρ on a manifold F. Then the associated bundle to a principal bundle $P(M, G)$ has the total space $E = P \times F/G$ and the same base space M, respectively. It inherits a set of local trivialisations from the principal bundle, which identifies it locally with $U_i \times F$. In other words, the fibre of E is F and the structure group is G, as expected. Another important point is that it has the same transition functions up to representation, $t_{ij}^E = \rho(t_{ij}^P)$. With the help of a connection, a covariant derivative can be defined on the associated bundle as well. It acts on differential forms[1] with values in sections in E and is the ingredient in building gauge invariant objects. Locally it reads,

$$\mathrm{D}\phi_i = \mathrm{d}\phi_i + \rho(A_i)\phi_i\,, \tag{2.3}$$

with the corresponding representation ρ of the Lie algebra.

[1]or functions, in the simplest case

2.1.2. Characteristic Classes

Characteristic classes serve as a tool to classify bundles. They all rest upon invariant polynomials. We will only consider the *Chern characters* ch_n which are (wedge) powers of the field strength,

$$\mathrm{ch}_n(F) \equiv \frac{1}{(2\pi)^n n!}\,\mathrm{tr}\, F^n\,, \qquad \mathrm{ch}_1(F) = \frac{1}{2\pi}\,\mathrm{tr}\, F\,, \qquad \mathrm{ch}_2(F) = \frac{1}{8\pi^2}\,\mathrm{tr}\, F\wedge F\,. \tag{2.4}$$

They play a very important role in index theorems and anomalies, too. With the help of the Chern-Weil theorem, the Chern characters are related to elements of the de Rham cohomology group of the base manifold. In this way they describe topological properties – in particular the nontriviality – of the bundle.

The Chern characters belong to *integer* cohomology classes. That is, combinations of Chern characters integrated over M yield integers, the *Chern numbers*. In practice, only the lowest Chern numbers occur; they vanish for $2n > \dim M$ anyhow. As demonstrated in the next chapter, the Chern numbers for $n = 1$ and $n = 2$ are the magnetic charge and the instanton number, respectively.

Since $\mathrm{d}\,\mathrm{ch}_n = 0$, the *Chern-Simons forms* of the Chern characters can be defined by,

$$\mathrm{ch}_n(F) \equiv \mathrm{d}\,\mathrm{cs}_{2n-1}(A)\,, \quad \mathrm{cs}_1(A) = \frac{1}{2\pi}\,\mathrm{tr}\, A\,, \quad \mathrm{cs}_3(A) = \frac{1}{8\pi^2}\,\mathrm{tr}\,(A\wedge \mathrm{d}A - \frac{4i}{3}A^3)\,, \tag{2.5}$$

at least locally. These are not gauge invariant.

2.1.3. Reducibility and Bundle Reduction

We report on this particular subject, since it will become important at several points below.

The *holonomy group* H_p of a point p in a principal bundle is obtained by horizontal lifts[2] of all closed curves γ starting and ending in the corresponding point $x = \pi(p)$ of the base space[3]. H is a Lie subgroup of the structure group G. Moreover, it is independent of p in the sense that the holonomy groups at different points are G-conjugate to each other, hence isomorphic. The elements h of H are solutions of an ordinary differential equation and can locally be expressed by path ordered exponentials of the gauge field, $h_A[\gamma] = \mathcal{P}[A;\gamma]$.

The idea of bundle reductions is to relate a principal bundle to a subbundle with the same base M, but a subgroup H of G as structure group. The reduction theorem guarantees that P[4] can be reduced to a bundle with the holonomy group H as structure

[2] given a connection one-form

[3] which we assume here to be connected and paracompact

[4] better that part of P reachable from p by horizontal lifts

group. Furthermore the connection one-form is also valid on that reduced bundle. The possibility of such a reduction is in one-to-one correspondence with the existence of transition functions taking their values in H and with a global section in the associated bundle with fibre G/H.

The subgroup H can be characterised by its centraliser C, the subgroup of elements of G 'commuting' with all elements of H,

$$C = \{k \in G | k^{-1} h k = h \ \forall h \in H\} . \tag{2.6}$$

The bigger the holonomy group the smaller its centraliser. The *reducibility* of a gauge field A is defined accordingly. Gauge fields with no restriction on the holonomy group, $H = G$, are called *irreducible*. The centraliser is as small as possible, it consists of the center only, $C = Z(G)$. The opposite extreme are gauge fields with trivial holonomy $H = \{e\}$, $C = G$, which we call *extremely reducible*. The vacuum $A = 0$ belongs to this type.

Now we further assume M to be simply connected (like the sphere). Then the holonomy group is connected. In other cases (like for the torus) the appropriate tool is the restricted holonomy group, the identity component of H. For $G = SU(2)$ and $M = S^4$ there is only one intermediate type [73], *Abelian* gauge fields with $H = U(1)$ (for a detailed analysis of other groups and other base spaces see [74]). Since this type is important for Abelian projections of $SU(2)$ we elaborate on it in detail. We parametrise the holonomy group H by one (normalised) Lie algebra generator n,

$$H \ni h(x) = \exp(i\lambda n(x)) , \qquad (n(x), n(x)) = 1 \ \forall x . \tag{2.7}$$

The centraliser has to commute with all h and – upon expanding in λ – with n, too. Therefore its elements are of the same form,

$$C \ni k(x) = \exp(i\lambda n(x)) , \tag{2.8}$$

containing the center $Z(SU(2)) = \pm \mathbb{1}_2$ for $\lambda \in \{0, \pi\}$. The centraliser is again a $U(1)$ subgroup[5], which is embedded in $SU(2)$ by $n(x)$ in the same way as H.

There are three equivalent characterisations of reducibility; we shall explain them for the Abelian gauge fields. First, the Ambrose-Singer theorem states that the Lie algebra of the holonomy group H is spanned by the field strength F [77]. This follows easily from the picture that the field strength measures the non-closure of infinitesimal parallel

[5]For higher groups, this coincidence holds only for the maximal Abelian subgroup! [75, 73, 76]

transports. Now it is obvious that all flat connections are extremely reducible. For the Abelian fields the field strength must be parallel to $n(x)$ in colour space.

The second property is that elements k of the centraliser build up the *stabiliser* of A,

$$ {}^{k}A = A\,. \tag{2.9} $$

The reason behind this correspondence is that the holonomy transforms covariantly, $\mathcal{P}[{}^{k}A] = k^{-1}\mathcal{P}[A]k$. Again, the more reducible A, the bigger its stabiliser. For the vacuum, for instance, it consists of all *constant* $SU(2)$ gauge transformations[6].

A third completely equivalent characterisation is by demanding $n(x)$ to be covariantly constant (in the adjoint representation),

$$ 0 = \mathrm{D}_A n \equiv \mathrm{d}n - i[A, n]\,. \tag{2.10} $$

The scalar product of n with this equation, $0 = \mathrm{d}(n, n)/2$, shows again that n is normalised. The solution of this set of differential equations is

$$ n(x) = \mathcal{P}[A, \gamma]n(x_0)\mathcal{P}[A, \gamma]^{-1}\,. \tag{2.11} $$

Here γ is a path connecting the point x with some reference point x_0. The independence of n of the choice of this path translates exactly into the centraliser description above. The equivalence to the stabiliser picture is seen when expanding equation (2.9) in powers of λ. Now the field strength picture emerges as an integrability condition using the commutator of two covariant derivatives, $0 = [F, n]$, which is equivalent to $F \parallel n$.

Equation (2.10) (via its commutator with n) fixes the part of A which is perpendicular to n. Thus, the most general ansatz for a $U(1)$-reducible gauge field is the one written down by Cho [78],

$$ \hat{A} = Cn + i[n, \mathrm{d}n]\,. \tag{2.12} $$

2.2. Configuration Space of Yang-Mills Theories

Gauge fixing is necessary in YM theories both in the Hamiltonian approach – in order to quantise systems with constraints like Gauss' law (cf. [79]) – and in the path integral approach – where one a priori expects any observable to come with an infinite volume factor stemming from integration over the gauge group – respectively. In this section, we will define the physical configuration space of YM theories as the quotient of the space of connections and the gauge group. The main aspects can be studied by means of the Christ-Lee model [80] which is our starting point.

[6] Notice that by the fixed space dependence of $k(x)$ coming from $n(x)$ in (2.8), the stabiliser group becomes isomorphic to a subgroup of the structure group, not of the gauge group we define later.

2.2.1. Christ-Lee Model

The path integral becomes an ordinary one in the degenerate case of a 'zero-dimensional field theory'. That is, the 'field' ϕ 'depends on nothing', and $I = \int \mathrm{d}\phi \exp(-S(\phi))$. For simplicity we consider a two-dimensional internal space, $\phi = (x, y)^\top$. By assumption, the action $S = S(r)$ is invariant under rotations $\phi \to R(\lambda)\phi$, i.e. under the action of the gauge group $SO(2)$. The orbits of this action are concentric circles plus the origin of $\mathbb{R}^2$. They have different dimensions since the stabiliser of the origin is the whole group $SO(2)$. In analogy to the last section, the origin is (extremely) reducible.

Polar coordinates obviously provide a splitting of the field variable ϕ into its gauge variant and invariant part, respectively. The polar angle φ is the coordinate along each orbit (but the origin), and $r \in [0, \infty)$ labels the set of orbits. The latter is the physical configuration space, $\mathfrak{M} = \mathbb{R}^2/SO(2) \cong \mathbb{R}_0^+$. Accordingly, the φ-integration factors out from the integral,

$$I = \int_0^\infty \mathrm{d}r\, r \exp(-S(r)) \int_0^{2\pi} \mathrm{d}\varphi\,, \tag{2.13}$$

giving the volume of the gauge group, $\mathrm{vol}(SO(2)) = \int_0^{2\pi} \mathrm{d}\varphi = 2\pi$. (In zero dimensions the volume of the gauge group is finite. Hence, the overcounting is also finite, and a gauge fixing is not necessary.)

The Faddeev-Popov trick is an alternative procedure to split off the volume of the gauge group. It is based on the well-known formula for the δ-distribution of a function,

$$1 = \int \mathrm{d}\lambda\, \delta(\chi(\lambda))\, \chi'(\lambda_0)\,, \qquad \chi(\lambda_0) = 0\,, \tag{2.14}$$

which we insert into the integral,

$$I = \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\lambda \exp(-S(x, y))\delta(\chi(\lambda))\, \chi'(\lambda_0)\,. \tag{2.15}$$

Since we can reach any point on a fixed orbit by a rotation, we can write

$$\chi(\lambda) = y = r \sin \lambda\,, \qquad \lambda_0 \in \{0, \pi\}\,, \qquad \chi'(\lambda_0) = x\,. \tag{2.16}$$

Then the integral becomes

$$I = \frac{1}{2} \int_{-\infty}^\infty \mathrm{d}x\, x \exp(-S(x)) \int_0^{2\pi} \mathrm{d}\lambda\,. \tag{2.17}$$

which provides a splitting of variables just like (2.13). Notice that the non-triviality of the measure $x\, \mathrm{d}x$ comes from the derivative of the condition χ in (2.16). We have

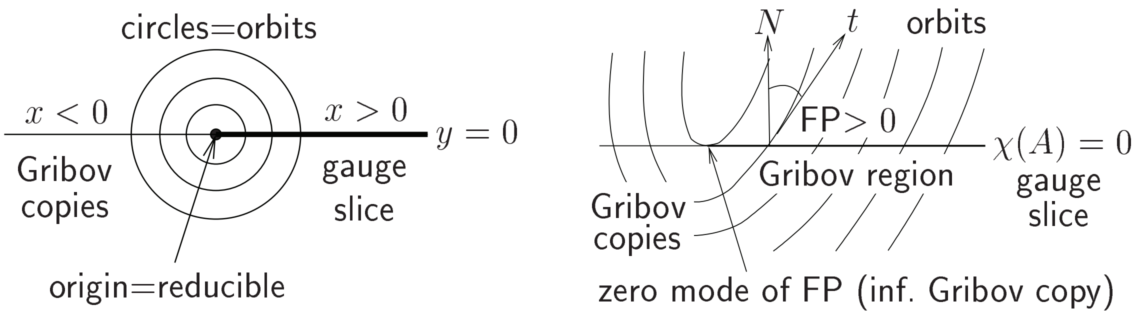

Fig. 2.1.: Left: The Christ-Lee model as a toy model for gauge fixing. Right: A typical fraction of the configuration space with a zero mode of the FP operator, where the normal vector N to the gauge slice and the tangent vector t (the velocity) along the orbit become orthogonal.

corrected a factor 2 from the sum over λ_0 in (2.14) we suppressed so far. In fact, every orbit (but the origin) intersects the line $\chi(\lambda) = y = 0$ twice (see Figure 2.1). To avoid this, one can supplement this equation by the inequality $x > 0$. Then the analogy to the variable r is perfect. We stress that both conditions, $y = 0$ and $x > 0$, can be summed up in demanding the function $F(\lambda) = -x = -r \cos \lambda$ to be *minimal.*

2.2.2. Gauge Bundle, Space of Connections and Gauge Group

Given a principal bundle $P(M, G)$ the space of all smooth connections is called *pre-configuration space* $\mathfrak{A}$. It has the nice properties of being affine and contractible and can be interpreted as a real infinite-dimensional vector space with values in the Lie algebra of the structure group G. However, it is a 'bigger-than-real-life' space [81] in the sense that different connections are related by vertical automorphisms (cf. Subsection 2.1.1). The group of vertical automorphisms is called *gauge group* $\mathfrak{G}$. A more intuitive picture is provided by virtue of the *gauge bundle* $P \times G/G$. This is an associated bundle to $P(M, G)$ where the structure group G acts on itself by conjugation. Now the elements of the gauge group can be interpreted locally as sections in the gauge bundle, i.e. mappings from the base space into the group, therefore the notion $g(x)$ for elements of $\mathfrak{G}$ is mostly used in physics.

Any physical observable is by definition invariant under gauge transformations. Accordingly, the *physical configuration space* $\mathfrak{M}$ is the *orbit space* obtained from the pre-configuration space after dividing out the gauge group, $\mathfrak{M} = \mathfrak{A}/\mathfrak{G}$. (The total configuration space is obtained by the union of all $\mathfrak{M}$'s with different instanton number.) It is this quotient that makes the physical configuration space highly non-trivial. Its topological features are of interest when dealing with non-perturbative effects: the non-perturbative wave functional spreads out (it is not localised around the vacuum $A = 0$) and becomes sensitive to the non-trivial geometry of the configuration space [82, 83].

$\mathfrak{M}$ is not a manifold, it rather contains singularities. The reason is that the action of $\mathfrak{G}$ on $\mathfrak{A}$ is not free.The properties of $\mathfrak{M}$ are condensed in the term *stratified variety* [84]. Recall that the set of gauge fields can be classified according to their reducibility (cf. Section 2.1.3). Therefore, $\mathfrak{M}$ is the disjoint union of countably many *strata*, which are the orbits with identical stabilisers[7]. Each stratum itself is a manifold[8]. The stratum of orbits of irreducible connections is called the *main stratum* $\mathfrak{B}$. The structure of $\mathfrak{M}$ is such that every stratum is open and dense in the set of all strata having the same or a bigger stabiliser. That is, the main stratum is dense in $\mathfrak{M}$. One might think of 'cones over cones': the generic stratum forms a cone, the tip of which are the Abelian connections, and the latter themselves form another cone, the tip of which are the flat connections [73]. This structure is reflected in lower-dimensional toy models [85], among them $\mathfrak{M} = \mathbb{R}_0^+$ of the Christ-Lee model.

2.2.3. Gauge Condition, Faddeev-Popov Method and Gribov Problem

In practice, the physical configuration space $\mathfrak{M}$ is to be modelled as a subspace of the pre-configuration space $\mathfrak{A}$ modulo boundary identifications, called the *fundamental modular domain* Λ [73]. Put differently, this space should intersect every orbit once and only once. The procedure of finding Λ is called *gauge fixing*. Usually, the first step is to write down a *gauge condition*,

$$\chi(A) = 0\,. \tag{2.18}$$

We will use a condensed notation, χ, for a set of equations, their number being the dimension of the structure group. The subspace spanned by the solution of this equation is the *gauge slice* Γ. χ must be a gauge variant function of A. For instance, the conditions for the axial and Lorenz gauge[9] read $\chi(A) \equiv A_0$ and $\chi(A) \equiv \partial_\mu A_\mu = *\,\mathrm{d}{*A}$, respectively. For the latter it was found that the gauge condition (2.18) is not a complete gauge fixing [86]: there exist *Gribov copies*, for example of $A = 0$, a problem which is referred to as the *Gribov problem*. Gribov copies are configurations which at the same time fulfil the gauge condition and belong to the same orbit. It was proven by Singer that it is impossible to fix the gauge by any continuous gauge condition [87]. The reason is that

[7] up to conjugation

[8] and the main stratum related to a subbundle of P

[9] also called Landau gauge or covariant gauge

the principal bundle consisting of the main stratum $\mathfrak{P}$ as total space, $\tilde{\mathfrak{G}} = \mathfrak{G}/Z(G)$ as gauge group and a base manifold $\mathfrak{N}$ which is open and dense in the physical configuration space $\mathfrak{M}$ *is not a trivial bundle*, hence admits no global section. In fact, the proof was given for space-times S^3 and S^4, but is believed to persist for any compact space-time [88].

The gauge condition (2.18) is implemented in the path integral by the Faddeev-Popov trick [89] (like in the Christ-Lee model, cf. (2.15)):

$$I = \int \mathrm{D}A \exp(-S_{\mathrm{YM}}(A)\, \delta(\chi(A)) \det \mathrm{FP}(A)\,, \tag{2.19}$$

where the Faddeev-Popov (FP) operator

$$\mathrm{FP}(A)\, \delta\lambda \equiv \frac{\delta \chi}{\delta A} \mathrm{D}_A\, \delta\lambda\,, \tag{2.20}$$

acts on gauge parameters λ (in the adjoint representation) and has to be evaluated at the gauge slice $\chi(A) = 0$. It can be visualised as the scalar product of the normal vector $N \equiv \delta\chi/\delta A$ at the gauge slice Γ and the velocity $\mathrm{D}_A\lambda$ of a fictituous motion along the orbit (see Figure 2.1).

Zero modes of the FP operator detect *infinitesimal Gribov copies*: The gauge fields A and $A + \delta A$, $\delta A = \mathrm{D}_A\, \delta\lambda$ are both on the gauge slice Γ if and only if

$$\chi(A + \delta A) = \mathrm{FP}(A)\, \delta\lambda = 0\,. \tag{2.21}$$

From (2.20) one infers that there are two generic reasons for this to happen. First $\delta\lambda$ can be a zero mode already of D_A; the velocity vanishes and the action of the gauge group has a fixed point. This is just one notion of reducibility (cf. (2.10)). The second possibility for det FP to vanish is when the normal vector and the velocity are orthogonal. From Figure 2.1 one expects finite Gribov copies on neighbouring orbits. This is what usually happens for background type gauges like the Lorenz gauge.

We go on to describe the identification of the fundamental domain Λ in the Lorenz gauge [73, 83], since this gauge is similar to the Maximally Abelian gauge. Like for the Christ-Lee model the second gauge fixing step uses a *Morse functional* $F[A]$ along the orbit. It must be gauge variant such that its extremum condition ('equation of motion') gives the gauge condition $\chi(A)$. For the Lorenz gauge we have,

$$F_{\mathrm{Lorenz}}[A] = \int \mathrm{tr}\, A \wedge *A = \int (A_\mu^a)^2\, \mathrm{d}V\,. \tag{2.22}$$

The subspace spanned by the *minima* of F is the *Gribov region* Ω. It can be described by a positive FP operator[10], since the latter is the Hessian ('fluctuation operator') of F. The Gribov region Ω is convex and bounded, and every orbit intersects it at least once. The boundary of the Gribov region is the *Gribov horizon* $\delta\Omega$, where the lowest eigenvalue of the FP operator vanishes. We have described this situation above. The intuition behind the Gribov horizon is a bifurcation of minima [83, 90].

In a third step, one further restricts to a subspace of Ω, the set Λ' of *absolute minima* of the functional F. This space Λ' is again convex and bounded. Its interior is void of Gribov copies. On its boundary, configurations which are associated with degeneracies of absolute minima of F are to be identified. Eventually, one arrives at the fundamental modular domain Λ after dividing out the structure group[11] G, $\Lambda = \Lambda'/G$. It is the boundary identifications in Λ' that restore the non-trivial topology of $\Lambda \cong \mathfrak{A}/\mathfrak{G}$.

On the lattice, the detection of Gribov copies has been reported for the first time in [91]. It turns out that some of these copies are lattice artifacts while others survive in the continuum limit [92]. In a sense, therefore, the Gribov problem becomes even more pronounced upon gauge fixing on the lattice (for a recent review see [93]). In order to extract physical results one clearly has to control the influence of Gribov copies. This is of particular relevance for the lattice studies of confinement.

2.3. Solitonic Objects in Gauge Theories

In this chapter we will consider some of the solitonic configurations playing a role in gauge theories, namely monopoles and instantons. We will mainly concentrate on kinematical aspects like topology and symmetry.

2.3.1. Dirac Monopole

The Maxwell equations with sources, $\mathrm{d}f = 0$, $\delta f = *j$, are not invariant under exchanging 'electric' and 'magnetic' (we use small letters for Abelian fields). In order to establish such a *duality*, Dirac has considered the *magnetic monopole* [94, 95]. The Bianchi identity becomes $\mathrm{d}f = *k$, and the introduction of a vector potential, $f = \mathrm{d}a$, is only possible outside the monopole. The original Dirac monopole is static. Excluding

[10] To be precise: this statement is fully correct for background gauges with irreducible background. The Lorenz gauge is a background gauge with vanishing, i.e. extremely reducible background, and therefore FP has always zero modes.

[11] For other backgrounds than the trivial one, the stabiliser has to be divided out.

the monopole position one arrives at the space $\mathbb{R}^3\backslash\{0\}$, which is of the same homotopy type as the two-sphere S^2.

Not surprisingly, the Dirac monopole can be described by a non-trivial principal bundle [67, 96, 97, 98] over S^2. The fibre and the total space are $U(1) \cong S^1$ and S^3, respectively. The latter differs from $S^2 \times S^1$ globally[12]. The projection $\pi : S^3 \to S^2$ is chosen to be the standard Hopf map $n_{\rm H}$ discussed in Section 4.1.1. Using polar angles $(\theta, \varphi_{12}, \varphi_{34})$ on the total space S^3 and (ϑ, φ) on the base manifold S^2 (see A.1), it reads,

$$\pi = n_{\rm H} : S^3 \to S^2\,, \qquad \vartheta = 2\theta\,,\;\; \varphi = \varphi_{12} - \varphi_{34}\,. \tag{2.23}$$

Local sections around the north and south pole are given by

$$s_{N,S} : S^2 \to S^3, \qquad \theta = \vartheta/2\,, \quad \varphi_{12} - \varphi_{34} = \varphi, \quad \varphi_{12} + \varphi_{34} = \mp\varphi - \pi\,, \tag{2.24}$$

respectively, chosen such that they are well-defined in their domains.

A particular choice for the connection one-form and curvature two-form on S^3 is,

$$\bar{a} = g(-\mathrm{d}(\varphi_{12} + \varphi_{34}) - \cos(2\theta)\mathrm{d}(\varphi_{12} - \varphi_{34}))\,,\; \bar{f} = -2g\sin(2\theta)\mathrm{d}\theta \wedge \mathrm{d}(\varphi_{12} - \varphi_{34})\,, \tag{2.25}$$

where the constant g denotes the strength/charge of the monopole. Pulling $\bar{a}$ down onto the base space one arrives at the Wu-Yang vector potentials [99],

$$a_N = g(1 - \cos\vartheta)\mathrm{d}\varphi\,, \qquad a_S = g(-1 - \cos\vartheta)\mathrm{d}\varphi \tag{2.26}$$

They differ by a $U(1)$ gauge transformation h

$$a_N = a_S + 2g\,\mathrm{d}\varphi\,, \qquad h = \exp(2ig\varphi)\,, \tag{2.27}$$

which is well-defined only if the Dirac quantisation condition,

$$g = n/2\,, \tag{2.28}$$

holds[13]. Of course, both gauge fields give the same field strength, a magnetic field of Coulomb type,

$$f = g\sin\vartheta\,\mathrm{d}\vartheta \wedge \mathrm{d}\varphi = -\frac{g}{2}\,\epsilon_{ijk}\hat{x}_i\,\mathrm{d}\hat{x}_j \wedge \mathrm{d}\hat{x}_k\,, \qquad \vec{B} = g\frac{\vec{r}}{r^3}\,. \tag{2.29}$$

An electric field is not present a priori, but can easily be added by introducing a non-vanishing A_0 resulting in a *dyon* [100].

[12] as is obvious from the differing homotopy groups π_1 and π_2

[13] Reintroducing the electric charge e the Dirac condition reads $eg = n/2$; all electric charges would be quantised by just one magnetic monopole in the universe.

The monopole bundle is characterised by the transition function h from (2.27). It is a mapping from the vicinity of the equator, which is homotopic to S^1, to the gauge group $U(1) \cong S^1$, and $2g = n$ is just its *winding number* (degree),

$$\deg(h) = \frac{1}{2\pi} \int_{S^1} h^\dagger \mathrm{d}h = 2g \in \pi_1(S^1) \cong \mathbb{Z}\,. \tag{2.30}$$

The first Chern number is fully equivalent,

$$n = \int_{S^2} \mathrm{ch}_1(f) = \frac{g}{2\pi} \int_{S^2} \mathrm{d}V(S^2) = 2g\,. \tag{2.31}$$

From the discussion on the triviality of principal bundles (Section 2.1.1) it is clear that a single section s is not sufficient. The local sections (2.24) become singular when irregularly extended to the opposite pole. The same is true for the transition function (2.27) and the gauge fields (2.26). The latter develop the well-known *Dirac strings* along the z-axis,

$$\mathrm{d}a_{N,S} = f + f^{\mathrm{sg}}_{N,S}\,, \qquad f^{\mathrm{sg}}_{N,S} = 4\pi g\, \theta(\mp z)\delta^{(2)}(x,y)\, \mathrm{d}x \wedge \mathrm{d}y\,, \tag{2.32}$$

due to the important relation (cf. [101]),

$$\mathrm{d}^2\varphi = 2\pi\delta^{(2)}(x,y)\, \mathrm{d}x \wedge \mathrm{d}y\,, \tag{2.33}$$

which can be understood via Stokes' theorem or the Green's function of the Laplacian in two dimensions. In physical terms, the Dirac strings provide the flux spread out by the Coulombic field,

$$\Phi_{\mathrm{in}} = \int_{S^2} f^{\mathrm{sg}}_{N,S} = -4\pi g\,, \qquad \Phi_{\mathrm{out}} = \int_{S^2} f = 4\pi g\,. \tag{2.34}$$

The latter equality also holds in its local form,

$$\mathrm{div}\vec{B} = \mathrm{d}f = 4\pi g\, \delta^{(3)}(x,y,z)\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = *k = -\mathrm{d}f^{\mathrm{sg}}_{N,S}\,. \tag{2.35}$$

The picture of magnetic monopoles as endpoints of Dirac strings is useful for practical purposes[14].

[14]e.g. the Dirac quantisation can be derived by an Aharonov-Bohm gedanken experiment around the Dirac string (see e.g. [FB3])

2.3.2. 't Hooft-Polyakov Monopole

Another kind of monopole occurs in non-Abelian gauge theories with Higgs fields[15], for instance in the Georgi-Glashow model [102],

$$L = \mathrm{tr}\, \mathrm{D}\phi \wedge *\mathrm{D}\phi - V(|\phi|) + L_{\mathrm{YM}}(A)\,, \qquad L_{\mathrm{YM}}(A) = -\,\mathrm{tr}\, F \wedge *F\,. \tag{2.36}$$

It contains a triplet of scalar fields ϕ coupled to an $SU(2)$ gauge field A, i.e. the Higgs field belongs to the adjoint representation. The YM part is the one from (1.1). The potential V is of Mexican-hat shape with a minimum of value zero at say $|\phi| = v$.

The 't Hooft-Polyakov monopole is a *static solution with finite energy and no electric field*. Via the Bogomol'nyi trick [103] one can derive a lower bound for the energy, namely,

$$E = \int_{\mathbb{R}^3} \Big(\mathrm{tr}\, (\vec{\mathrm{D}}\phi)^2 + \mathrm{tr}\, \vec{B}^2 + V(\phi) \Big) = \int_{\mathbb{R}^3} \Big(\mathrm{tr}\, (\vec{\mathrm{D}}\phi \pm \vec{B})^2 \mp 2\, \mathrm{tr}\, \vec{\mathrm{D}}\phi\, \vec{B} + V(\phi) \Big)\,, \tag{2.37}$$

where we used the three-dimensional notions $\vec{\mathrm{D}}\phi \equiv \vec{\partial}\phi - i[\vec{A}, \phi]$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The energy splits into a positive bulk term E_{bulk}, a sum of squares, and a boundary term E_{boundary}, a total derivative,

$$E_{\mathrm{boundary}} = \mp 2 \int_{\mathbb{R}^3} \mathrm{tr}\, \vec{\mathrm{D}}\phi\, \vec{B} = \mp \int_{S^2_\infty} (\phi, F)\,, \tag{2.38}$$

respectively. We have introduced the Killing form, a scalar product in isospace (see B.1), $(X, Y) \equiv 2\, \mathrm{tr}\, XY = X_a Y_a$. The result is $E \geq |E_{\mathrm{boundary}}|$, where the bound can be saturated for vanishing potential.

In addition, E_{boundary} has a topological meaning. Notice that for finite energy the asymptotic value of $|\phi|$ must be v. Therefore, we identify it with a normalised Higgs field n,

$$|\vec{x}| \to \infty : \quad |\phi| = v, \quad \phi \equiv vn\,. \tag{2.39}$$

n is a mapping from S^2_∞ to the coset $S^2 \cong SU(2)/U(1)$, where the $U(1)$ is the group of rotations around n not affected by symmetry breaking. The integrand of the boundary term is part of 't Hooft's field strength tensor [104],

$$G \equiv (F, n) + (i\mathrm{D}n \wedge \mathrm{D}n, n)\,. \tag{2.40}$$

[15] e.g. for spontaneously broken gauge theories

Each term in this expresssion is gauge invariant in itself. There is also a formula for G as a sum of gauge variant terms,

$$G = \mathrm{d}(A,n) + (i\mathrm{d}n \wedge \mathrm{d}n, n)\,. \tag{2.41}$$

Now it is obvious that the flux of G is a topological quantity,

$$q = \int_{S^2_\infty} G = \int_{S^2_\infty} (i\mathrm{d}n \wedge \mathrm{d}n, n) = 4\pi \deg(n)\,, \qquad \deg(n) \in \pi_2(S^2) \cong \mathbb{Z} \tag{2.42}$$

namely the winding number of $n: S^2 \to S^2$, also called *Brouwer degree* [105]. It counts how many times the image sphere is covered by the preimage sphere. Being an integer it is conserved during time evolution. But it does not generate a Noether symmetry (the equations of motion are not involved in its derivation).

The explicit expressions for the 't Hooft-Polyakov monopole in the so-called *radial gauge* read [104, 106],

$$A = A(|\vec{x}|)\epsilon_{ija} x_i \mathrm{d}x_j \tau_a\,, \qquad \phi = \phi(|\vec{x}|)\delta_{ia} x_i \tau_a\,, \tag{2.43}$$

with the following asymptotic behaviour of the profile functions,

$$|\vec{x}| \to \infty: \quad A(|\vec{x}|) \to 1/|\vec{x}|^2\,, \quad \phi(|\vec{x}|) \to v/|\vec{x}|\,. \tag{2.44}$$

$n = x_a\tau_a/|\vec{x}|$ is simply the identity $S^2 \to S^2$ – the hedgehog – with $\deg(n) = 1$, $q = 4\pi$. The mixing between coordinate space and colour space is expressed by the mixed indices of ϵ and δ in (2.43). We stress that for a winding in n, ϕ is required to vanish somewhere and n becomes singular there. Within the ansatz (2.43), ϕ vanishes at the spatial origin; in the BPS limit of vanishing potential it grows linearly with $|\vec{x}|$.

The 't Hooft-Polyakov monopole turns into the Dirac monopole upon diagonalising ϕ with a gauge transformation g such that ${}^g n = \tau_3$. This so-called *unitary gauge* is used to extract the physical degrees of freedom after spontaneous symmetry breaking[16]. But the diagonalisation of ϕ is only possible with a singular gauge transformation. This point can easily be understood by visualising g as the rotation from n onto the positive three-axis in colour space: it is not continuous when n is on the negative third axis. Therefore g transforms A into the gauge field A_N of the Dirac monopole (with the Dirac string along the negative z-axis, cf. (2.26) with $g = 1$). Interestingly, the contribution to G in (2.41) has changed from the second term to the first term, and q is now the flux of an Abelian magnetic field of Coulomb type ($\Phi_{\text{out}} = 4\pi$ from (2.34)).

[16] In fact, q is proportional to the mass of the W boson.

2.3.3. Instantons

By virtue of the Bianchi identity $\mathrm{D}F = 0$, the classical YM equations $D * F = 0$ are fulfilled for (anti-)selfdual fields, $*F = \pm F$ ($\vec{E} = \pm\vec{B}$). These first order differential equations are easier to solve than the original second order ones. Their non-trivial solutions are called *instantons* ([107, 108], for a review see [109]). They are minima of the action since the inequality $\mathrm{tr}\,(F \pm *F)^2 \geq 0$ yields,

$$S_{\mathrm{YM}}(A) \geq 8\pi^2 |\nu(A)| \tag{2.45}$$

where the *instanton number* ν is nothing but the second Chern number

$$\nu(A) \equiv \frac{1}{8\pi^2} \int \mathrm{tr}\, F \wedge F = \int \mathrm{ch}_2(F) \in \mathbb{Z}\,. \tag{2.46}$$

The equality in (2.45) holds exactly for (anti-)selfdual fields.

Most of the known results about instantons refer to the four-sphere S^4. It enters the game via one-point compactification: demanding *finite action* the gauge field has to approach a *pure gauge* at infinity,

$$A \to i g^\dagger \mathrm{d}g \quad \text{for } r \to \infty\,. \tag{2.47}$$

This is exactly the situation allowing for the application of the compactifiability theorem by Uhlenbeck [110]: such a gauge field can be extended to $\mathbb{R}^4 \cup \{\infty\} \cong S^4$. Of course the above gauge field is smooth only on one chart[17] of S^4 and has to be supplied by another local gauge field.

By virtue of the associated Chern-Simons form (2.5), the instanton number can be related to the *winding number of the gauge transformation* g,

$$\nu(A) = \int_{\mathbb{R}^4} \mathrm{ch}_2(F) = \int_{S^3_\infty} \mathrm{cs}_3(A) = \frac{1}{24\pi^2} \int_{S^3_\infty} \mathrm{tr}\,(g\mathrm{d}g^\dagger)^3 = \deg(g) \in \pi_3(S^3) \cong \mathbb{Z}\,, \tag{2.48}$$

where g is a mapping from the boundary of space-time S^3_∞ to the gauge group $SU(2) \cong S^3$. Gauge transformations with non-vanishing degree (which cannot be obtained by a deformation from the identity) are called 'large gauge transformations'. In the language of principal bundles one needs two charts for the sphere S^4 and g is just the transition function living on the overlap which is homotopic to S^3.

The number of parameters/the dimension of the moduli space of (anti-)selfdual solutions with instanton number $\nu(A) = k$ is known (cf. [69] and references therein). For

[17] namely on the one around the pole which is mapped to the origin by stereographic projection

$SU(2)$ it is $8k-3$ where $8k = 4k+k+3k$ refers to the position, size and colour orientation of k single instantons, respectively, and a number three from overall global colour rotations has to be subtracted. The general solution is realised by the algebraic ADHM construction [111].

Much less is known about *torus*-type manifolds $\mathbb{T}^n \times \mathbb{R}^{4-n}$, $n = 1, 2, 3, 4$. An ADHM-like formalism has been exploited yielding the so-called *calorons*, instantons over $S^1 \times \mathbb{R}^3$ with instanton number one [112, 113, 114]. Constructions of instantons become very complicated for higher n [115]. On $\mathbb{T}^4$ the existence of instanton with $k \geq 2$ was established by Taubes [116]. Although there are configurations with $k = 1$, instantons as solutions of the equations of motion are ruled out [117, 118] by the Nahm transformation [119]. The latter connects $U(N)$ instantons with charge k to $U(k)$ instantons with charge N on the dual torus.

Abelian Instantons on the Torus

For particular values of the periods L_μ there exist *Abelian* instantons on the torus [120, 121],

$$A = 4\pi(k'\xi_3\, \mathrm{d}\xi_4 + k''\xi_1\, \mathrm{d}\xi_2)\tau_3\,, \qquad \xi_\mu \equiv x_\mu/L_\mu \text{ (no sum)}\,, \tag{2.49}$$

with constant field strength

$$F = 4\pi(k'\mathrm{d}\xi_3 \wedge \mathrm{d}\xi_4 + k''\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2)\tau_3\,, \tag{2.50}$$

and even instanton number $k = 2k'k''$. They have the following Abelian transition functions

$$t_0 = t_2 = \mathbb{1}_2\,, \qquad t_1 = \exp(4\pi i k''\xi_2\tau_3)\,, \qquad t_3 = \exp(4\pi i k'\xi_4\tau_3)\,, \tag{2.51}$$

but are solutions only if $k''/L_1L_2 = k'/L_4L_3$ (then $F = \text{const}\, \eta^3_{\mu\nu}\mathrm{d}x_\mu \wedge \mathrm{d}x_\nu\tau_3$ is selfdual). They represent singular points of the moduli space.

The Single Instanton

The following ansatz for $SU(2)$-instantons is often used [122, 123, 124, 125], although it is general only for $k = 1$,

$$A = -\bar{\eta}^a_{\mu\nu}\, \partial_\nu \ln \Pi\, \mathrm{d}x_\mu\tau_a\,. \tag{2.52}$$

The η-tensor [126] relates Lorentz and colour indices like ϵ and δ for the 't Hooft-Polyakov monopole (cf. (2.43)). Now the self-duality problem has reduced to a Laplace equation

$\Box\Pi/\Pi = 0$ solved by Coulomb-type Green's functions which are squares of the inverse distances,

$$\Pi(x) = 1 + \sum_{i=1}^{k} \frac{\rho_i}{(x - x_i)^2}\,, \qquad \nu(A) = k\,. \tag{2.53}$$

The occurence of only $5k$ parameters (ρ_i, x_i) reflects the fact that all constituent instantons of this ansatz are equally oriented in colour space.

We specialise immediately to the case $k = 1$ ($\rho_1 \equiv \rho$, $x_1 = 0$), which yields the single instanton *in singular gauge*,

$$A^{\rm sg} = \frac{2\bar\eta^a_{\mu\nu} x_\nu \rho^2}{r^2(r^2+\rho^2)}\, {\rm d}x_\mu \tau_a\,. \tag{2.54}$$

The singularity at the origin is a gauge artefact. Indeed, one should better use $A^{\rm sg}$ around infinity and the gauge-related single instanton *in regular gauge* [107],

$$A^{\rm reg} = \frac{2\eta^a_{\mu\nu} x_\nu}{r^2+\rho^2}\, {\rm d}x_\mu \tau_a\,, \tag{2.55}$$

around the origin. For brevity we will refer to these configurations as singular and regular instanton, respectively. In the bundle approach, these are two local gauge fields in the quaternionic Hopf bundle with total space S^7, base space S^4, fibre $S^3 \cong SU(2)$ and another Hopf fibering as the projection $\pi : S^7 \to S^3$, respectively. The transition function is the *identity mapping* from S^3 to $S^3 \cong SU(2)$,

$$\hat g \equiv i\hat x_\mu \sigma^{(+)}_\mu \equiv \hat x_4 \mathbb{1}_2 + i\hat x_k \sigma_k\,, \qquad A^{\rm reg} = {}^{\hat g}\!A^{\rm sg}\,, \tag{2.56}$$

which clearly has $\deg(\hat g) = 1$. Like for the Dirac monopole this gauge transformation must be singular and so it is, namely at the origin and infinity. Certainly it also relates the field strengths, while the instanton density is gauge invariant,

$$\frac{1}{8\pi^2}\,{\rm tr}\, F\wedge F = \frac{6}{\pi^2}\frac{\rho^4}{(r^2+\rho^2)^4}\,{\rm d}V(\mathbb{R}^4)\,. \tag{2.57}$$

It correctly yields unity when integrated over $\mathbb{R}^4$ and approaches the four-dimensional δ-distribution for vanishing size, $\rho \to 0$.

Up to now we have only considered the *topological* four-sphere $\mathbb{R}^4 \cup \{\infty\} \cong S^4$. To really move to the *geometrical* four-sphere S^4 with non-flat metric, we benefit from the fact that classical YM theories are conformally invariant. If we use conformal coordinates[18] x_μ the metric is conformally flat,

$$ {\rm d}s^2 = e^{\alpha_R(r)}\delta_{\mu\nu}\,{\rm d}x_\mu\,{\rm d}x_\nu\,, \qquad e^{\alpha_R(r)} = \frac{4R^4}{(r^2+R^2)^2}\,. \tag{2.58}$$

[18] which are simply the Cartesian coordinates of the point stereographically projected onto $\mathbb{R}^4$

Field configurations minimising the YM action on $\mathbb{R}^4$ are also minimising configurations on the sphere, if the Cartesian coordinates are substituted by conformal coordinates. Thus, we can simply use the expressions (2.54) and (2.55) for the single instanton on the sphere, too.

For actual computations, symmetries of the configurations involved are very important (see [127] and references therein). Since gauge fields are gauge dependent, the symmetry concept is slightly enlarged: Under a symmetry a gauge field A has to come back to itself *up to a gauge transformation*, for instance under rotations R, ${}^gA = RA(R^{-1}x)$.

The single instanton is highly symmetric: On $\mathbb{R}^4$ it is invariant under SO(4) rotations and a combination of translations and special conformal transformations,

$$\delta x_\mu = \omega_{\mu\nu} x^\nu + 2\, c \cdot x\, x_\mu/\rho - c_\mu(x^2+\rho^2)/\rho\,, \tag{2.59}$$

up to a compensating (infinitesimal) gauge transformation, $\delta A = \mathrm{D}\lambda$, with

$$\lambda = (\frac{1}{2}\,\omega_{\,\mu\nu}\eta^a_{\mu\nu} - 2\,c_{\,\mu}\,\eta^a_{\mu\nu}\,x_\nu)\tau_a\,, \qquad \text{(reg. gauge)}\,. \tag{2.60}$$

Altogether, these transformations form (a non-linear representation of) the group SO(5). This symmetry is preserved on S^4 when the radius R of S^4 embedded in $\mathbb{R}^5$ coincides with the instanton size ρ. To illustrate this point, we note that the gauge invariant instanton density,

$$\mathrm{tr}\, F \wedge F \propto \frac{\rho^4}{R^8}\,\frac{(r^2+R^2)^4}{(r^2+\rho^2)^4}\,, \qquad \text{(sg. and reg. gauge)}\,. \tag{2.61}$$

is constant on S^4 (and thus SO(5)-invariant) only if $R = \rho$. For $R \neq \rho$ the explicit appearance of r, which is only SO(4)-invariant, breaks SO(5) down to SO(4). The single instanton is the only gauge field within the ansatz (2.52) and (2.53) with such a high symmetry. The occurence of at least two different positions of single instantons for higher k certainly breaks the full rotational symmetry.

We conclude this section by noting that the gauge transformation $\hat{g}$ is invariant under $SO(4)$ rotations, provided selfdual rotations are compensated by right multiplication and anti-selfdual rotations are compensated by left multiplication, respectively,

$$\delta\hat{g} = \hat{g}\cdot i\lambda\,, \quad \lambda = \frac{1}{2}\,\omega_{\mu\nu}\eta^a_{\mu\nu}\tau_a\,, \qquad \delta\hat{g} = i\lambda\cdot\hat{g}\,, \quad \lambda = \frac{1}{2}\,\omega_{\mu\nu}\bar{\eta}^a_{\mu\nu}\tau_a\,. \tag{2.62}$$

3. Abelian Projections

In the introductory chapters we have discussed both the rich structure of pure YM theories due to the non-Abelian nature of the structure group and the fundamental property of confinement, which, as a non-perturbative phenomenon, is difficult to derive from first principles. The *Abelian projections* invented by 't Hooft [33] were a breakthrough in that they isolate the non-Abelian part of the theory which very probably is responsible for confinement.

In order to trace out this part, one first makes use of an *Abelian gauge*. This is a partial gauge fixing which leaves the maximal Abelian subgroup untouched. For $SU(2)$ the latter is $U(1)$, realised, for instance, in terms of diagonal matrices. In this way one naively expects to obtain a version of electrodynamics. Indeed, under the residual Abelian gauge freedom the diagonal part and the off-diagonal part of the gauge fields transform as photons and matter (in the adjoint representation), respectively.

The off-diagonal gauge field is just another matter field like the quarks; no symmetry prevents it from becoming massive in the quantum theory. Therefore, it is assumed not to induce confinement and is neglected in a first approximation, known as *Abelian projection*.

So far one would arrive at a rather trivial theory – an Abelian gauge theory, which is not confining – if there were no remnants from the coset part of the gauge group. These effects go under the name of *defects*. As we will show, the gauge transformations that transform into the Abelian gauge cannot be smooth for all gauge fields. The associated singularities can be characterised by topological quantities representing obstructions against the diagonalisation of non-trivial mappings.

Furthermore, for generic gauge fields, the defects are localised. Since the Abelian gauge is part of a complete gauge fixing, the defects might be viewed as local realisations of the Gribov problem. Generic defects in four dimensions are worldlines of magnetic monopoles carrying unit magnetic charge. By the duality argument of Section 1.4, the condensation of these defects is supposed to lead to confinement[1]. Strong support of this picture is provided by lattice simulations.

In the forthcoming sections we discuss the mechanism of Abelian projections in more

[1] We will not concern ourselves with perturbative and renormalisability issues [128, 90, 129, 130].

detail. We will restrict ourselves to $SU(2)$ for simplicity and summarise the main modifications due to $SU(N)$ in a separate section. In the main part, we survey the definitions of the three most popular Abelian gauges and analyse their properties including the Gribov problem. Afterwards, the relations of Abelian gauges to the Faddeev-Niemi model and fermionic zero modes are given.

3.1. Definition of Abelian Gauges

This section summarises the technicalities involved in Abelian projections.

3.1.1. Cartan Decomposition

The Cartan subalgebra $\mathcal{H}$ of a Lie algebra $\mathcal{G}$ is the maximal subset which consists of mutually commuting elements/matrices. We choose[2] $\mathcal{H}$ to consist of the diagonal matrices in $\mathcal{G}$. For $SU(2)$ the Cartan subalgebra $\mathcal{H} = u(1)$ is generated just by the third Pauli matrix,

$$\mathcal{H} = \{\lambda\tau_3 | \lambda \in \mathbb{R}\} . \tag{3.1}$$

Any element of the Lie algebra can be decomposed into its diagonal (Cartan, 'parallel') and off-diagonal ('perpendicular') part, respectively,

$$X \in \mathcal{G} : \; X = X^{\|} + X^{\perp} , \qquad X^{\|} \in \mathcal{H} , \qquad X^{\perp} \in \mathcal{H}^{\perp} . \tag{3.2}$$

We define the associated projectors,

$$X^{\|} = \mathbb{P}^{\|}(X) = (X, \tau_3)\tau_3 , \quad X^{\perp} = \mathbb{P}^{\perp}(X) = (X, \tau_1)\tau_1 + (X, \tau_2)\tau_2 = [\tau_3, [\tau_3, X]] . \tag{3.3}$$

The projections have the following properties,

$$(X^{\|}, Y^{\perp}) = 0 , \qquad [X^{\|}, Y^{\|}] = 0 , \quad [X^{\|}, Y^{\perp}], [X^{\perp}, Y^{\|}] \in \mathcal{H}^{\perp} , \quad [X^{\perp}, Y^{\perp}] \in \mathcal{H} . \tag{3.4}$$

We stress that the last property does not hold for higher structure groups. The maximal Abelian subgroup[3] H is the subgroup of G generated by the Cartan subalgebra $\mathcal{H}$,

$$U(1) = H = \exp(i\mathcal{H}) = \{\exp(i\lambda\tau_3) = \cos\frac{\lambda}{2}\mathbb{1}_2 + i\sin\frac{\lambda}{2}\tau_3 | \lambda \in (0, 2\pi)\} , \tag{3.5}$$

which again consists of mutually commuting, i.e. diagonal matrices.

For later use we note that,

$$h \in H : \quad (X^{\|}, hY^{\perp}h^{\dagger}) = (h^{\dagger}X^{\|}h, Y^{\perp}) = (X^{\|}, Y^{\perp}) = 0 . \tag{3.6}$$

[2] The Cartan subalgebra is defined only up to conjugations.
[3] also called *maximal torus*

3.1.2. Gauge Condition, Functional, and Higgs Field

As already mentioned, an *Abelian gauge* is a partial gauge fixing leaving the maximal Abelian subgroup H unfixed. That is, if h is an element of H, together with A also the gauge transformed ${}^{h}A$ lies on the gauge fixing hypersurface[4],

$$\chi(A) = 0 \Rightarrow \chi({}^{h}A) = 0 \,. \tag{3.7}$$

On the Gribov region, we minimise a functional $F[A]$, which under h must be invariant, $F[A] = F[{}^{h}A]$.

It turns out that such a functional can be found by a slight modification of the one corresponding to the Lorenz gauge (2.22). It contains some projector $\mathbb{P}$, to be specified later [FB5],

$$F[A] = \int \operatorname{tr} \mathbb{P}A \wedge * \mathbb{P}A = \int (\mathbb{P}A^a_\mu)^2 \, \mathrm{d}V = \langle \mathbb{P}A, \mathbb{P}A \rangle \,. \tag{3.8}$$

For brevity we have introduced a scalar product,

$$\langle X, Y \rangle \equiv \int \operatorname{tr} X \wedge *Y \,, \tag{3.9}$$

w.r.t. to which $\mathbb{P}$ is assumed self-adjoint.

In order to derive the gauge condition we have to vary $F[A]$ along the orbit, i.e. under the action of a gauge transformation $g = \exp(i\lambda)$,

$$F({}^{g}A) = F(A + \mathrm{D}_A\lambda + O(\lambda^2)) = F(A) + F^{(1)}(A;\lambda) + O(\lambda^2) \,. \tag{3.10}$$

We further rewrite the term of first order in λ

$$F^{(1)}(A;\lambda) = 2\langle \mathbb{P}A, \mathbb{P}\mathrm{D}_A\lambda \rangle = 2\langle \mathbb{P}\mathbb{P}A, \mathrm{D}_A\lambda \rangle = 2\langle \mathbb{P}A, \mathrm{D}_A\lambda \rangle = -2\langle * \mathrm{D}_A * \mathbb{P}A, \lambda \rangle \,. \tag{3.11}$$

This term vanishes for all $\lambda(x)$ if and only if

$$\chi(A) \equiv * \mathrm{D}_A * \mathbb{P}A = 0 = \mathrm{D}_{A_\mu}\mathbb{P}A_\mu \,. \tag{3.12}$$

As in Section 2.2.3 we calculate the FP operator as the variation of χ,

$$\chi({}^{g}A) = \chi(A + \mathrm{D}_A\lambda + O(\lambda^2)) = \chi(A) + * \mathrm{D}_A * \mathbb{P}\mathrm{D}_A\lambda - i * \operatorname{ad}_{\mathrm{D}_A\lambda} * \mathbb{P}A + O(\lambda^2), \tag{3.13}$$

implying,

$$\mathrm{FP} = *(i \operatorname{ad}_{\mathbb{P}A} + \mathrm{D}_A\mathbb{P}) * \mathrm{D}_A = (i \operatorname{ad}_{\mathbb{P}A_\mu} + \mathrm{D}_{A_\mu}\mathbb{P})\mathrm{D}_{A_\mu} \,. \tag{3.14}$$

[4] We slightly abuse the notions defined in Section 2.2.3, although we are not aiming to fix the gauge completely. It can be done by some Abelian gauge fixing.

It agrees with the general form (2.20) discussed in Section 2.2.3. Equivalently, one can vary the functional F up to second order, and $F^{(2)}(A;\lambda) = -\langle \lambda, \mathrm{FP}\lambda \rangle$.

A gauge fixing which is more adapted to Abelian gauges uses an 'auxiliary Higgs field'. One can think of an Abelian gauge (AG) as assigning to each gauge field A a field ϕ transforming in the adjoint representation of the gauge group,

$$\mathrm{AG}: \; A \to \phi \qquad ({}^{g}\!A \to {}^{g}\!\phi = g\phi g^{\dagger}) \,. \tag{3.15}$$

In the spirit of Section 2.1.1, we demand ϕ to be a section in an associated bundle. An obvious and early used choice is a component of the field strength, say $\phi \equiv F_{41}$ [33], having the disadvantage of not being Lorentz invariant.

By definition, *the gauge transformation g that brings A into the Abelian gauge is the one which diagonalises ϕ.* For most applications ϕ is Lie algebra valued, and we can write,

$$\chi(A) = 0 \rightleftharpoons \chi(\phi) = \phi^{\perp} = 0 \rightleftharpoons \phi = \phi^{\parallel} \in \mathcal{H} \,. \tag{3.16}$$

For $SU(2)$ the Higgs field simply points into the third colour direction, $\phi^{\parallel} = |\phi|\tau_3 \simeq (0,0,|\phi|)^{\top}$. The residual gauge freedom clearly consists of diagonal matrices H which commute with ϕ and hence do not change it. By virtue of the property (3.6), one has (in analogy with (3.7)),

$$\phi^{\perp} = 0 \Rightarrow ({}^{h}\phi)^{\perp} = h\phi^{\perp}h^{\dagger} = 0 \,. \tag{3.17}$$

For $SU(2)$ the gauge transformation h acts as a rotation[5] around the third axis in colour space and (3.17) means that it cannot create a component perpendicular to it.

It is obvious that the length of the Higgs field does not matter for the definition of the gauge transformation, i.e. one can use a normalised Higgs field $n = \phi/|\phi|$ as well.

For completeness we note that the residual gauge freedom includes the Weyl group as well. This group permutes the diagonal entries (eigenvalues) of $\phi^{\parallel}$ and can easily fixed by an ordering prescription [33].

In the Higgs field language not only the gauge condition χ but also the Faddeev-Popov operator is purely algebraic,

$$\chi({}^{g}\phi) = \chi(\phi - i[\phi,\lambda] + O(\lambda^2)) = \chi(\phi) - i\,\mathbb{P}^{\perp}\mathrm{ad}_{\phi}\lambda + O(\lambda^2) \,. \tag{3.18}$$

On the gauge slice Γ one has $\phi = \phi^{\parallel}$, so that

$$\mathrm{FP} = -i\,\mathbb{P}^{\perp}\mathrm{ad}_{\phi^{\parallel}} = -i|\phi|\,\mathbb{P}^{\perp}\mathrm{ad}_{\tau_3} \,, \qquad \mathrm{ad}_{\tau_3} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.19}$$

[5] since the adjoint representation of $SU(2)$ is the group $SO(3)$

The zero entries in the third row and column are typical for Abelian gauges; the associated zero modes $\lambda^{\|} = |\lambda|\tau_3 \simeq (0, 0, |\lambda|)^\top$ reflect the residual Abelian gauge freedom $h \in \mathcal{H}$ (cf. (3.7) and (3.17)) on the infinitesimal level. The remaining two by two matrix σ_2 in the perpendicular block of ad_{τ_3} can easily be diagonalised and yields eigenvalues ± 1

Multiplying by $-i|\phi|$ we conclude that FP acts as a *multiplicative operator*. It leads to δ-like eigenfunctions[6]. Therefore, we change to a lattice regularisation of space-time for the moment. Then the spectrum of FP is $\pm i|\phi|(x)$ and the FP determinant becomes

$$\det \mathrm{FP} = \prod_x |\phi|^2(x)\,. \tag{3.20}$$

A multiplication over space-time points being left over in the FP determinant, is known from the Coulomb gauge which is 'ultralocal' (involves no derivatives) in time [131, 132].

It can be exponentiated, but an UV-divergent factor $1/a^4$ (the volume of momentum space) is needed in order to make the exponent dimensionless,

$$\det \mathrm{FP} = \exp\left(\sum_x \ln|\phi|^2(x)\right) \to \exp\left(\frac{1}{a^4}\int \ln|\phi|^2(x)\mathrm{d}V\right)\,. \tag{3.21}$$

This term enters the one-loop effective potential [133, 134].

Unfortunately, the problem of evaluating the FP determinant for Abelian gauges is not really solved by turning to the Higgs field language: For the path integral (2.19), both the gauge condition and the Faddeev-Popov operator must be functions of the gauge fields to integrate over. In principle this is true also in the Higgs field language. The function $\phi(A)$ (the assignment (3.15)), however, can be very implicit and so is any involved Jacobian which has to be calculated when one tries to evaluate the path integral.

3.1.3. Abelian Projections: Space Fixed Frame and Body Fixed Frame

A given gauge field A has to be transformed by a gauge transformation g to yield the associated gauge field in the Abelian gauge. For definiteness we will denote $A_{\mathrm{AG}} \equiv {}^gA$ in this section (it is A_{AG} that fulfils $\chi(A_{\mathrm{AG}}) = 0$). This gauge field is to be decomposed into its diagonal and off-diagonal part,

$$A_{\mathrm{AG}} = A^{\|}_{\mathrm{AG}} + A^{\perp}_{\mathrm{AG}}\,. \tag{3.22}$$

[6] like a pure potential in a Schrödinger equation

Under the residual gauge freedom $h = \exp(i\lambda\tau_3) \in \mathcal{H}$ the diagonal part transforms as a 'photon',

$$A^{||}_{\mathrm{AG}} \to A^{||}_{\mathrm{AG}} + \mathrm{d}\lambda\,\tau_3\,. \tag{3.23}$$

The off-diagonal part transforms as a matter field in the adjoint representation,

$$A^{\perp}_{\mathrm{AG}} \to hA^{\perp}_{\mathrm{AG}}h^{\dagger} \tag{3.24}$$

where we used (3.6) to show that the r.h.s. has no parallel part. This proves the claims about transformation properties made in the beginning of this chapter.

In the *Abelian projection* (AP) the perpendicular part is simply neglected[7],

$$\mathrm{AP}:\; A_{\mathrm{AG}} \to A^{||}_{\mathrm{AG}}\,. \tag{3.25}$$

The remaining gauge field is *Abelian* which is just a special case of being *reducible* ! One can easily check all the properties described in Section 2.1.3: the holonomy group is $U(1)$ (diagonal), the centraliser of which is again $U(1)$ (diagonal); $A^{||}_{\mathrm{AG}}$ is left invariant by a $U(1)$ subgroup (diagonal and constant); the field strength $F(A^{||}_{\mathrm{AG}})$ commutes with τ_3 (is diagonal, too); finally τ_3 is covariantly constant w.r.t. $A^{||}_{\mathrm{AG}}$, since $[A^{||}_{\mathrm{AG}}, \tau_3] = 0 = \mathrm{d}\tau_3$.

Following [135] we refer to (3.22) as the decomposition in the *space fixed frame*. Under certain circumstances, e.g. for the Faddeev-Niemi action of Section 1.5, it is helpful to perform an analogous decomposition without transforming the gauge field A. This is called decomposition in the *body fixed frame*. The normalised Higgs field n is a natural candidate to provide the decomposition of A without transforming it,

$$A = A^{||n} + A^{\perp n}\,. \tag{3.26}$$

However, the naive generalisation of (3.3), $A^{||n} = (A,n)n$, fails, if we *demand* $A^{||n}$ *to transform as a gauge field under the full group* $SU(2)$ (c.f. Figure 3.1), and in particular under the residual group $U(1)$. The correction coming from the inhomogeneous term can be obtained most easily from transforming back $A^{||}_{\mathrm{AG}}$,

$$A^{||n} \equiv {}^{g^\dagger}(A^{||}_{\mathrm{AG}}) \qquad \text{as a gauge field}\,. \tag{3.27}$$

We use the ad-invariance of the Killing form to rewrite,

$$\begin{aligned} A^{||n} &= (gAg^\dagger + ig\mathrm{d}g^\dagger, \tau_3)g^\dagger\tau_3 g + ig^\dagger\mathrm{d}g = (A,n)n - i(g^\dagger\mathrm{d}g, n)n + ig^\dagger\mathrm{d}g \\ &= \mathbb{P}^{||n}(A) + \mathbb{P}^{\perp n}(ig^\dagger\mathrm{d}g)\,. \end{aligned} \tag{3.28}$$

[7] in the observables, not in the path integral measure, cf. Section 4.1.3

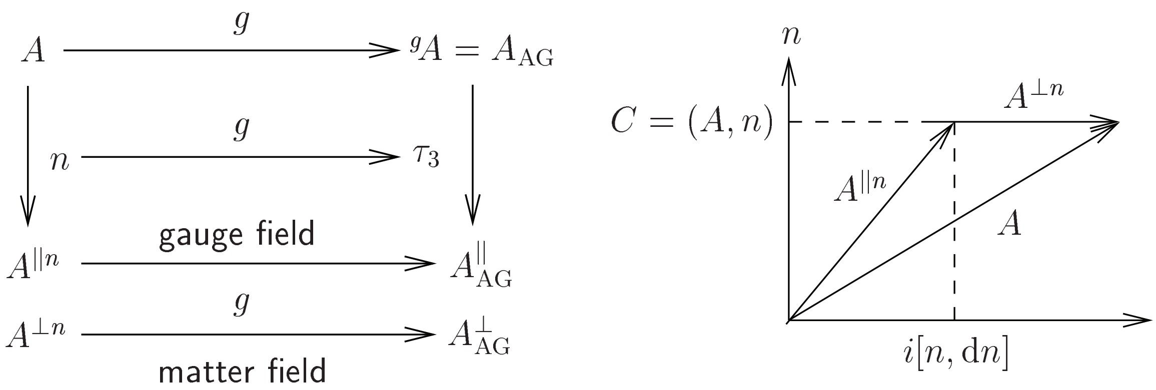

Fig. 3.1.: Left: The gauge transformation g relates the two frames described in the text. Right: In the body fixed frame the gauge field $A^{||n}$ inherits a part perpendicular to n from the inhomogeneous transformation.

where we define the associated projectors as (cf. (3.3)),

$$\mathbb{P}^{||n}(X) = (X, n)n\,, \qquad \mathbb{P}^{\perp n}(X) = X - (X, n)n = [n, [n, X]]\,. \tag{3.29}$$

By virtue of $\mathrm{d}n = -i[n, ig^\dagger \mathrm{d}g]$ (see B.3) we arrive at the final formula

$$A^{||n} = \mathbb{P}^{||n}(A) + i[n, \mathrm{d}n] = Cn + i[n, \mathrm{d}n]\,, \qquad C \equiv (A, n)\,. \tag{3.30}$$

This is the general form of an $U(1)$-reducible connection discussed in Section 2.1.3. The properties of $A^{||n}$ are essentially the same as for $A^{||}$, if τ_3 is replaced by n. In particular, the residual Abelian gauge freedom is,

$$h(x) = \exp(i\lambda(x)n(x)) : \qquad A^{||n} \to A^{||n} + \mathrm{d}\lambda\, n \qquad (n \to n,\ C \to C + \mathrm{d}\lambda)\,. \tag{3.31}$$

The stabiliser subgroup of $A^{||n}$ arises as a special case of this formula, namely for constant gauge parameter λ (cf. (2.8) and (2.9)).

Accordingly, the field $A^{\perp n}$ transforms as a matter field in the adjoint representation,

$$A^{\perp n} \equiv {}^{g^\dagger}(A^{\perp}_{\mathrm{AG}}) \qquad \text{homogeneously}\,. \tag{3.32}$$

Consistently with (3.26) and (3.28) we obtain,

$$A^{\perp n} = \mathbb{P}^{\perp n}(A) - \mathbb{P}^{\perp n}(ig^\dagger \mathrm{d}g) = A - (A, n)n - i[n, \mathrm{d}n] = -i[n, \mathrm{D}_A n]\,. \tag{3.33}$$

Notice that this field is really perpendicular to n. The decomposition in the body fixed frame is visualised in Figure 3.1.

For later use we also calculate the field strength in both frames. In the space fixed frame it is simply,

$$F(A^{||}_{\mathrm{AG}}) = \mathrm{d}A^{||}_{\mathrm{AG}}\,. \tag{3.34}$$

since the commutator term is absent. In the body fixed frame one has to employ the full non-Abelian expression, and a lengthy but straightforward computation gives,

$$F(A^{\|n}) = G\,n, \qquad G = \mathrm{d}C + i(n, \mathrm{d}n \wedge \mathrm{d}n), \qquad C \equiv (A, n) \tag{3.35}$$

The field strength has only a component parallel to n, in accordance with the Ambrose-Singer theorem (cf. Section 2.1.3). The coefficient is the 't Hooft tensor G from Section 2.3.2.

3.1.4. Remarks on Higher Gauge Groups

The idea of Abelian projections undergoes some modifications when the structure group is taken to be $SU(N)$. The Cartan subalgebra becomes $(N-1)$-dimensional and the maximal Abelian subgroup is $U(1)^{N-1}$. After the Abelian projection one expects $N-1$ copies of electrodynamics plus the defects which are again ambiguities in the diagonalising gauge transformation. *Basic defects* occur when two eigenvalues of the Higgs field coincide [33]. In other words, these are just $SU(2)$-defects embedded into $SU(N)$, and the residual symmetry is locally enlarged to the non-Abelian group $SU(2) \times U(1)^{N-2}$. There exist N types of basic defects. Observe that the Higgs field itself need not vanish (a group valued field need not be proportional to the identity), but its scalar product with some simple root does. *Non-basic defects* arise when more than two eigenvalues coincide. The residual symmetry is $SU(3) \times U(1)^{N-3}$ (for three coinciding eigenvalues) or $SU(2) \times SU(2) \times U(1)^{N-3}$ (for two pairs of coinciding eigenvalues) or even larger; and the scalar products with two or more roots vanish. In root space, defects are characterised by the boundary faces of the fundamental domain on which they are located [136].

Generic defects in four dimensions again form closed loops and turn into magnetic monopoles upon diagonalising. There are $N-1$ separately quantised magnetic charges [137] characterising $\pi_2(SU(N)/U(1)^{N-1}) \cong \mathbb{Z}^{N-1}$.

For the projection of the gauge field we use the Weyl basis with $[H_k, E_\alpha] = \alpha_k E_\alpha$ [138]. With proper normalisations the projectors (3.3) in the space fixed frame become,

$$\mathbb{P}^{\|}(X) = \sum_k (X, H_k) H_k\,, \qquad \mathbb{P}^{\perp}(X) = \sum_\alpha (X, E_{\pm\alpha}) E_{\pm\alpha}\,. \tag{3.36}$$

Notice that one Higgs field n defines a whole set $\{n_k\}$ of mutually commuting normalised Higgs fields which generate the rotated Cartan subalgebra [135]. Therefore the projectors in the body fixed frame are,

$$\mathbb{P}^{\|}(X) = \sum_k (X, n_k) n_k\,, \quad n_k = g^\dagger H_k g\,, \qquad \mathbb{P}^{\perp}(X) = X - \mathbb{P}^{\|}(X)\,, \tag{3.37}$$

with H_k being normalised, too.

However, the decomposition $A^{\|n}$ as well as the Cho connection $\hat{A}$ used in the FN decomposition (see (2.12), (3.3) and the previous chapter) rely on the double commutator. The $SU(N)$-analogue is,

$$\sum_k [H_k, [H_k, X^\perp]] = \sum_\alpha (X, E_{\pm\alpha}) \sum_k [H_k, [H_k, E_{\pm\alpha}]] = \sum_\alpha (X, E_{\pm\alpha}) \vec{\alpha}^2 E_{\pm\alpha} \,, \quad (3.38)$$

which is proportional to $X^\perp$ because *all roots* $\vec{\alpha}$ *are of the same length*[8]. The inhomogeneous term in the decomposition becomes $i\sum_k [n_k, \mathrm{d}n_k]$. Actually, it acquires a group theoretical factor N, so in the large N limit this topological term dominates [139].

3.2. Maximally Abelian Gauge

The *Maximally Abelian gauge* (MAG) was introduced by 't Hooft in his seminal work [33]. The gauge condition is,

$$\chi_{\mathrm{MAG}}(A) \equiv *\,\mathrm{D}_A * A^\perp = 0 = \mathrm{D}_{A_\mu} A^\perp_\mu = \mathrm{D}_{A^\|_\mu} A^\perp_\mu = \partial_\mu A^\perp_\mu - i[A^\|_\mu, A^\perp_\mu] \,. \quad (3.39)$$

For $SU(2)$ it can be formulated as,

$$\chi_{\mathrm{MAG}}(A) = (\partial_\mu \pm i A^3_\mu) A^\pm_\mu = 0 \,. \quad (3.40)$$

These expressions look like a background gauge, but the 'background' $A^\|$ is not independent of the gauge field A under consideration. Accordingly, the MAG-condition is not linear but quadratic in the gauge field A. Notice that χ as an isovector has no parallel component leaving a $U(1)$-freedom. The latter can be shown very easily by virtue of (3.23) and (3.24): Like $\mathrm{D}_{A^\|}$ and $A^\perp$, χ is just conjugated by h, $\chi({}^h\!A) = h\chi(A)h^\dagger$, which reflects the residual Abelian freedom, cf. (3.7).

Comparing χ_{MAG} with the general form (3.12) one can immediately read off the MAG-projector, $\mathbb{P}_{\mathrm{MAG}} \equiv \mathbb{P}^\perp$. Indeed, the MAG-functional is [33],

$$F_{\mathrm{MAG}}[A] \equiv \langle A^\perp, A^\perp \rangle = \int (A^{\bar{a}}_\mu)^2 \,\mathrm{d}V \,, \quad (3.41)$$

where $\bar{a}$ runs over the non-Cartan generators ($\bar{a} = 1, 2$ for $SU(2)$). This formula offers some physical intuition behind the *Maximally Abelian* gauge: the off-diagonal part of the gauge field is minimised along the orbit, in order to have a good approximation when finally neglecting this part in the Abelian projection; it is a 'smooth gauge' [140].

[8] This remains true for the D-groups $SO(2N)$ and the exceptional groups E_6, E_7 and E_8.

Performing a gauge transformation $\tau_3 \to n$, the MAG can be formulated in the body fixed frame as well:

$$\chi_{\rm MAG}(A;n) \equiv *\,\mathrm{D}_A * A^{\perp n} = [n, *\,\mathrm{D}_A * \mathrm{D}_A n] = [n, \mathrm{D}^2_{A_\mu} n]\,, \tag{3.42}$$

$$\begin{aligned} F_{\rm MAG}[A;n] &\equiv \langle A^{\perp n}, A^{\perp n}\rangle = \langle i[n,\mathrm{D}_A n], i[n,\mathrm{D}_A n]\rangle \\ &= \langle \mathrm{D}_A n, \mathrm{D}_A n\rangle = \int (\mathrm{D}_{A_\mu} n)^2\, \mathrm{d}V\,, \end{aligned} \tag{3.43}$$

where we used that $\mathrm{D}_A n = -i\,\mathrm{ad}_{A^{\perp n}} n$ is perpendicular to n. Equation (3.42) provides the Higgs field formulation of the MAG: Keeping A fixed one seeks a normalised Higgs field such that $\chi_{\rm MAG}(A;n)$ vanishes,

$$[n, D^2_{A_\mu} n] = 0 \rightleftharpoons D^2_{A_\mu} n \parallel n \rightleftharpoons D^2_{A_\mu} n = E(x) n\,, \tag{3.44}$$

with some proportionality factor $E(x)$. Again, $F_{\rm MAG}[A;n]$ has to be minimal [140], while (3.43) means that n should be 'as reducible as possible'.

The FP operator follows from (3.14),

$$\mathrm{FP}_{\rm MAG} = *(i\,\mathrm{ad}_{A^\perp} + \mathrm{D}_A \mathbb{P}^\perp) * \mathrm{D}_A = (i\,\mathrm{ad}_{A^\perp_\mu} + \mathrm{D}_{A_\mu}\mathbb{P}^\perp)\mathrm{D}_{A_\mu}\,. \tag{3.45}$$

It inherits its non-linearity in A from the gauge condition (3.39). Evaluated on the gauge slice $\chi(A) = 0$, it acts in the perpendicular sector only, leading to the expected trivial zero modes from the residual Abelian freedom (see C.1). For explicit calculations the following formula is better to handle [FB1],

$$\mathrm{FP}_{\rm MAG} = \mathbb{P}^\perp(\mathrm{D}^2_{A^\parallel_\mu} + \mathrm{ad}^2_{A^\perp_\mu} - i\,\mathrm{ad}_{A^\perp_\mu}\mathbb{P}^\perp \mathrm{D}_{A_\mu})\mathbb{P}^\perp\,, \tag{3.46}$$

which simplifies considerably for $SU(2)$, where the third term simply vanishes ([129], see C.1).

Let us also look for explicit configurations in the MAG. As a matter of fact, the single instanton both in regular and singular gauge fulfils the MAG-condition (3.39); due to the particular Lorentz and colour structure of the η-tensor, the terms $\partial_\mu A^\perp_\mu$ and $[A^\parallel_\mu, A^\perp_\mu]$ vanish separately. However, the MAG-functional for these configurations [141],

$$F_{\rm MAG}[A^{\rm sg}] = 2\,\pi^2 \int_0^\infty \frac{\mathrm{d}r\; r^3 \rho^4}{r^2(r^2+\rho^2)^2} = 4\pi^2\rho^2\,, \tag{3.47}$$

$$F_{\rm MAG}[A^{\rm reg}] = 2\,\pi^2 \int_0^\infty \frac{\mathrm{d}r\; r^5}{(r^2+\rho^2)^2} \to \infty\,, \tag{3.48}$$

'prefers'[9] $A^{\rm sg}$. Nevertheless, to prove that $A^{\rm sg}$ is in the fundamental modular domain, one still has to check that there is no other gauge field on the same orbit with an even lower value of the MAG-functional.

[9]meaning that the functional for $A^{\rm sg}$ is lower than the one for $A^{\rm reg}$

Other configurations in the MAG are the Abelian instantons (2.49) and the Yung ansatz for an instanton-antiinstanton pair [142]. The MAG-functional of the latter diverges (it is in regular gauge), but the application of the standard gauge transformation $\hat{g}$ (2.56) maps it to a configuration (in singular gauge) with a finite value of the MAG-functional [143].

The lattice formulation of the MAG goes back to Kronfeld et al. [144, 145]. The MAG-functional is formulated in group valued variables, the fundamental objects on the lattice,

$$F_{\mathrm{MAG}}[U] = 4a^2 \sum_{x,\mu} \mathrm{tr}\, \tau_3 U_\mu(x) \tau_3 U_\mu^\dagger(x) \,. \tag{3.49}$$

By virtue of the correspondence (1.6), $F_{\mathrm{MAG}}[U]$ approaches $-F_{\mathrm{MAG}}[A]$ in the continuum limit. Therefore, it has to be *maximised* on the lattice. In the body fixed frame,

$$F_{\mathrm{MAG}}[U; n] = 4a^2 \sum_{x,\mu} \mathrm{tr}\, n(x) U_\mu(x) n(x + \hat{\mu}a) U_\mu^\dagger(x) \,, \tag{3.50}$$

defines the Higgs field n on the lattice [146].

Unfortunately, investigating the critical points of (3.49) is similar to a *spin glass problem* due to the high number of local maxima: Finding the absolute maximum is a numerical problem of non-polynomial complexity [147]. Iterative minimalisation procedures often get stuck in a local maximum. Gribov copies have been detected numerically, for the first time in [148] and with refined techniques in [149, 150]. One should keep in mind, though, that some (if not all) of these copies can be lattice artifacts which do not survive the continuum limit.

3.2.1. A Toy Model

In order to have an illustration of the somewhat abstract notions of the preceding section, we now analyse an example with a finite number of degrees of freedom (and structure group $SU(2)$) [FB1]. To this end we employ a Hamiltonian formulation in 2 + 1 dimensions and consider only gauge potentials A which are spatially constant. Renaming $A_i^a = x_i^a$, $i = 1, 2$, $a = 1, 2, 3$, the Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}(\mathrm{D}_0^{ab} x_i^b)^2 \equiv \frac{1}{2}(\dot{x}_i^a - \epsilon^{abc} A_0^c x_i^b)^2 \,. \tag{3.51}$$

One way of arriving at this Lagrangian is by gauging a free particle Lagrangian $\mathcal{L}_0 = \dot{x}_i^a \dot{x}_i^a / 2$ via minimal substitution, i.e. by replacing the ordinary time derivative ∂_0 with

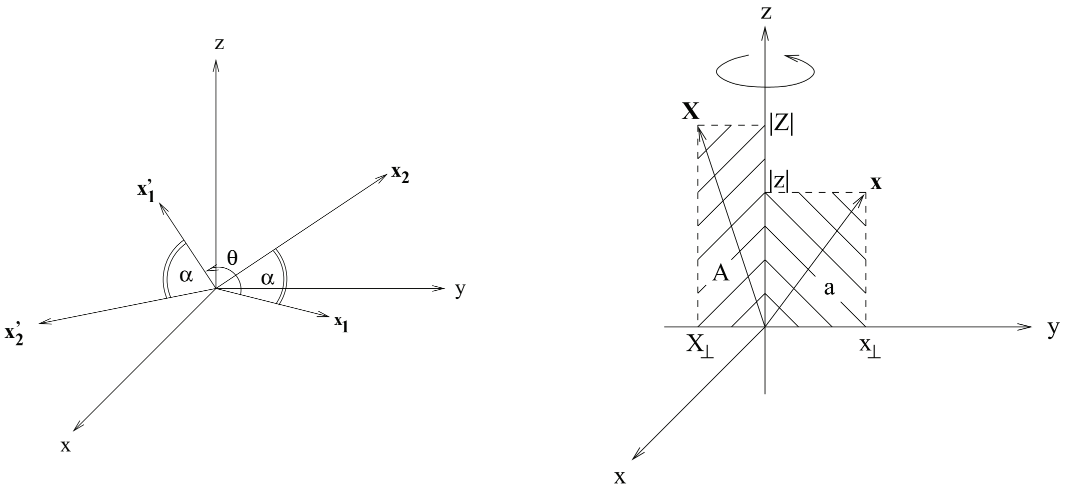

Fig. 3.2.: Left: An isospace (gauge) rotation by an angle θ in the toy model, keeping the lengths of the vectors and the angle α inbetween them are invariant. Right: The MAG condition in the toy model. The areas A and a have to be the same. We have arbitrarily chosen $\boldsymbol{x}$ and $\boldsymbol{X}$ to lie in the yz-plane. The residual $U(1)$ gauge freedom corrresponds to rotations around the z-axis.

the covariant derivative D_0. To keep things as simple as possible, we have not introduced any (YM type) interaction; here we are anyhow only interested in the kinematics of the problem.

Defining the canonical momenta $p_i^a = \mathrm{D}_0^{ab} x_i^b$, the Lagrangian (3.51) can be recast in first order form,

$$\mathcal{L} = p_i^a \dot{x}_i^a - \frac{1}{2} p_i^a p_i^a + A_0^a G^a \, , \tag{3.52}$$

where we have introduced the operator G^a leading to Gauss's law,

$$G^a \equiv \epsilon^{abc} x_i^b p_i^c \equiv \mathrm{D}_i^{ab} p_i^b = 0 \, . \tag{3.53}$$

Obviously, G^a is the total angular momentum of two point particles in $\mathbb{R}^3$ (= colour isospace) with position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Gauge transformations are thus $SO(3)$ rotations of these vectors which do not change their relative orientation (i.e. the angle α inbetween them). This is illustrated in Figure 3.2. To some extend, this model is similar to the Christ-Lee model of Section 2.2.1.

As usual we will work in the Weyl gauge, $A_0 = 0$, so that Gauss's law has to be imposed 'by hand', and, after quantisation, holds upon acting on physical states. Once the Weyl gauge has been chosen, there still is the freedom of performing time independent gauge transformations. This will be (partially) fixed using the MAG,

$$\chi(x) \equiv \mathrm{D}_k x_k^{\perp} = 0 = -i[x_k^{||}, x_k^{\perp}] \, . \tag{3.54}$$

To avoid writing too many indices we denote $\boldsymbol{x}_1 \equiv \boldsymbol{x} = (x, y, z)$, $\boldsymbol{x}_2 \equiv \boldsymbol{X} = (X, Y, Z)$, where z and Z stand for the Cartan parts $x_k^{\|}$. Then the two components of $\chi(x)$ read,

$$\chi^1 = -zy - ZY = 0 \,, \qquad \chi^2 = zx + ZX = 0 \,. \tag{3.55}$$

This can easily be visualised. The projections $\boldsymbol{x}_\perp$ and $\boldsymbol{X}_\perp$ have to be collinear, their magnitudes being related through $|z|\, x_\perp = |Z|\, X_\perp$. The MAG is thus obtained by rotating the configuration $(\boldsymbol{x}, \boldsymbol{X})$ in such a way that both vectors are as close to the z-axis as possible. This is achieved as shown in Figure 3.2. $\boldsymbol{x}$ and $\boldsymbol{X}$ are the diagonals of two rectangles with sides $|z|$, $x_\perp$ and $|Z|$, $X_\perp$, respectively. If the areas a and A of the rectangles coincide, $a = A$, the configuration is in the MAG. Algebraically, the notion of being 'close to the z–axis' is measured by the function,

$$F(\boldsymbol{x}, \boldsymbol{X}) \equiv x_\perp^2 + X_\perp^2 \,. \tag{3.56}$$

One can easily show that the conditions (3.54) or (3.55) minimise F and thus make the 'off-diagonal' components of $\boldsymbol{x}$ and $\boldsymbol{X}$ as small as possible. We mention in passing that the trivial solution of (3.55) given by $z = Z = 0$ corrresponds to a maximum of F so that we can always assume $z \neq 0$ or $Z \neq 0$ (except for the zero-configuration representing the origin).

It is obvious from Figure 3.2 that rotations around the z–axis leave both F and the MAG condition invariant and thus correspond to a residual $U(1)$ gauge freedom. As expected, this situation is reflected in the FP operator,

$$\mathrm{FP}^{ab} = \{\chi^a, G^b\}\big|_{\boldsymbol{\chi}=0} \,, \tag{3.57}$$

which, in matrix notation, can be written as ,

$$\mathrm{FP} = \begin{pmatrix} z^2 + Z^2 - y^2 - Y^2 & xy + XY & 0 \\ xy + XY & z^2 + Z^2 - x^2 - X^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.58}$$

The zero entries in the third row and column correspond to the action of the $\mathbb{P}^\perp$-projection in (3.46). The eigenvalues of FP are found to be

$$E_+ = z^2 + Z^2 \,, \qquad E_- = z^2 + Z^2 - x_\perp^2 - X_\perp^2 \qquad (E_3 = 0) \,. \tag{3.59}$$

Let us concentrate on the eigenvalues $E_\pm$ which are not related to the residual Abelian gauge freedom. Configurations where one of these vanishes are located on the Gribov horizon and reflect some non-trivial residual gauge freedom different from the $U(1)$

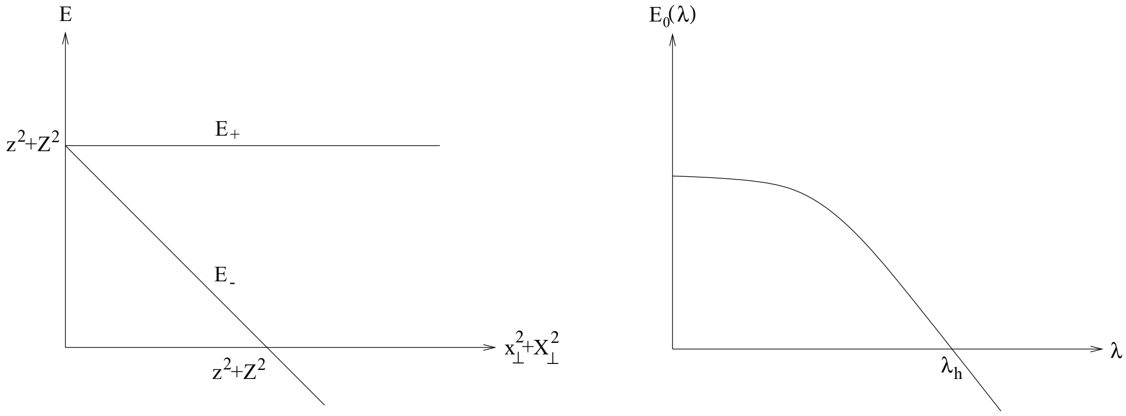

Fig. 3.3.: Behaviour of the eigenvalues of FP as a function of the magnitude $x_\perp^2 + X_\perp^2$ of the 'off-diagonal' components in the toy model (left) and as a function of the 'flow parameter' μ in the full MAG (right). The configurations with vanishing lowest eigenvalue are on the Gribov horizon. Copied from [FB1], where μ is denoted by λ.

above. A particular (in some sense trivial) class of horizon configurations consists in the reducible configurations as discussed in Section 2.2.3. They are fixed points under the action of (a subgroup of) the gauge group. Technically, they show up by inducing zero modes of the Laplacian $\Delta^{ab} = \mathrm{D}_i^{ac}\mathrm{D}_i^{cb}$ [FB1]. Within our example, the reducible configurations are readily identified [85, 151] by simple symmetry considerations. The origin is invariant under the whole action of $SO(3)$, while configurations with $\boldsymbol{x}$ and $\boldsymbol{X}$ collinear are invariant under rotations around their common direction which clearly corresponds to a $U(1)$. This is nicely reflected in the spectrum of FP. At the origin, both $E_\pm$ vanish, while a collinear configuration can always be rotated in the z-axis so that its stabiliser coincides with the standard residual $U(1)$ corresponding to $E_3 = 0$. This $U(1)$ stabiliser is thus 'hidden' in the residual $U(1)$. Fixing the latter by demanding e.g. $x = X = 0$, does, however, not affect configurations collinear along the z-axis so that these will induce zero modes of FP even after residual gauge fixing [85].

There is a remaining possibility for a vanishing eigenvalue. While E_+ is always positive, E_- vanishes if $z^2 + Z^2 = x_\perp^2 + X_\perp^2$. This happens for configurations where $\boldsymbol{x}$ and $\boldsymbol{X}$ are of the same length and orthogonal to each other. Elementary trigonometry implies that in this case the two areas a and A are always the same, irrespective of the location of the configuration relative to the z-axis. Thus, there is an additional residual $U(1)$ gauge freedom for such exceptional configurations. This can be nicely illustrated in terms of a 'spectral flow' as a function of $x_\perp^2 + X_\perp^2$ (see Figure 3.3).

3.2.2. Gribov Problem

The MAG plays a major role in lattice simulations of Abelian projections. However, the Gribov problem on the lattice is rather non-transparent. Therefore, the FP operator of the MAG in the continuum has been analysed in [FB1].

A very general argument shows that generically there have to be Gribov copies within the MAG if the off-diagonal components $A^\perp$ of the gauge fields become sufficiently large: More explicitly, the FP operator (3.46) for $SU(2)$,

$$\mathrm{FP}_{\mathrm{MAG}} = -\mathbb{P}^\perp(\mathrm{D}^2_{A^\parallel_\mu} + \mathrm{ad}^2_{A^\perp})\mathbb{P}^\perp \, , \tag{3.60}$$

is the *difference of two positive semidefinite operators* which we abbreviate for the time being as X and Y,

$$\mathrm{FP} = X - Y \, , \qquad X, Y \geq 0 \, . \tag{3.61}$$

Being the negative of a covariant Laplacian, $-\mathrm{D}^2_{A^\parallel_\mu}$ is obviously nonnegative; the proof that $\mathrm{ad}^2_{A^\perp}$ is nonnegative, too, is given in the Appendix, C.1. The identity (3.61) already suggests that if Y is 'sufficiently large', FP will develop a vanishing eigenvalue. Let us make this statement slightly more rigorous. To this end we modify an argument used in [152, 153] for background type gauges.

First of all we note that together with the configuration $A = A^\parallel + A^\perp$ also the 'scaled' configuration $A' \equiv A^\parallel + \mu\, A^\perp$, with μ some (positive real) parameter, will be in the MAG (cf. (3.39)). The associated FP operator is,

$$\mathrm{FP}(\mu) = X - \mu^2 Y \, . \tag{3.62}$$

Let us denote the lowest eigenvalue and the associated eigenfunction of $\mathrm{FP}(\mu)$ by $E_0(\mu)$ and $\lambda_0(\mu)$, respectively. From (3.61) one must have $E_0(0) \geq 0$. If we turn on μ, a straightforward application of the Hellmann-Feynman theorem leads to,

$$\frac{\partial}{\partial \mu} E_0(\mu) = -2\mu \, \langle \lambda_0(\mu), Y \, \lambda_0(\mu) \rangle \leq 0 \, , \tag{3.63}$$

whence the function $E_0(\mu)$ has negative slope. In addition, it has to be concave [154][10] so that, for μ sufficiently large, there will be a zero-mode at some value, say μ_h. Notice the similarity with the 'spectral flow' of E_- in the toy model (Figure 3.3). In a way we have thus determined a path within the MAG fixing hypersurface that leads us from the

[10] It is exactly for this reason that the second order perturbation theory correction to any groundstate is always negative.

interior of the Gribov region ($\mu = 0$) to its boundary ($\mu = \mu_h$).

The second result concerns the Faddeev-Popov operator in the background of the singular instanton [FB1]: The fact that the MAG-functional for the single instanton in the two gauges is finite and infinite, respectively, (cf. (3.47)), is corroborated by the work of Brower et al. [141] which, when translated into our language, amounts to the following: One numerically constructs a path $\gamma(R)$ in the gauge slice Γ connecting $A^{\rm sg}$ with $A^{\rm reg}$. Along this path[11] beginning at the singular instanton the MAG-functional is monotonically rising. The configurations $A(R)$ along the path are determined by applying a (singular) gauge transformation g which takes the singular instanton to $A(R)$, i.e. $A(R) = {}^gA^{\rm sg}$. Hence, $\gamma(R)$ is a path *both* within Γ and the single instanton orbit. Accordingly, there must be an infinitesimal gauge transformation of the singular instanton that does not leave Γ and thus must be a zero mode of FP in this background. In what follows we will explicitly determine this zero mode.

The first step of this program consists in the calculation of the FP operator in the background of a singular instanton. Plugging (2.54) into (3.60) one obtains the result,

$$\mathrm{FP} = \begin{pmatrix} -\Box & -2i\, a(r)(L_{12} - L_{34}) \\ 2i\, a(r)(L_{12} - L_{34}) & -\Box \end{pmatrix} = -\Box \mathbb{1}_2 + 4a(r) M_3\, \sigma_2\,, \quad (3.64)$$

where we have discarded the vanishing third row and column and introduced the instanton profile function,

$$a(r) = 2\, \frac{\rho^2/r^2}{r^2 + \rho^2} = 2 \left(\frac{1}{r^2} - \frac{1}{r^2 + \rho^2} \right) . \quad (3.65)$$

We also use the generators of $so(4)$, the symmetry group in Euclidean four-space, described in the Appendix, A.2. Using the splitting of the four-dimensional Laplacian (A.20) and the fact that we can diagonalise the operators $\{\vec{M}^2 = \vec{N}^2, M_3, \sigma_2\}$ simultaneously, we arrive at,

$$\mathrm{FP} = -\partial_r^2 - \frac{3}{r}\, \partial_r + \frac{2m(m+1)}{r^2} + 4\, m_3 s\, a(r)\,, \quad (3.66)$$

with $s = \pm 1$ denoting the eigenvalues of σ_2. As described in the Appendix, A.2, m and $m_3 \in \{-m, -m+1, \ldots, m\}$ are half-integers corresponding to the total angular momentum and its third component. By virtue of the angles introduced in the Appendix, A.1, the eigenfunction $\lambda(x)$ can be written as follows (cf. (A.16)),

$$\lambda_{m,m_3,n_3,s}(x) = f_{m,m_3}(r)\, \Theta_{m,m_3,n_3}(\theta)\, e^{i(m_3+n_3)\varphi_{12}} e^{-i(m_3-n_3)\varphi_{34}}\, \chi_s\,. \quad (3.67)$$

[11] In [141] the parameter R is the radius of a monopole loop associated with the configuration $A(R)$ located on γ somewhere inbetween $A^{\rm sg} = A(R=0)$ and $A^{\rm reg} = A(R=\infty)$.

The χ_s are the eigenspinors of σ_2,

$$\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} . \tag{3.68}$$

Introducing the dimensionless variable $\bar{r} = r/\rho$ and defining a function $g(\bar{r})$ via,

$$f(\bar{r}) \equiv g(\bar{r})/\bar{r}^{3/2} , \tag{3.69}$$

(we omit the subscripts of f) the radial equation for the zero mode reads,

$$\left(-\partial_{\bar{r}}^2 + \frac{4m(m+1) + 3/4}{\bar{r}^2} + \frac{8m_3 s}{\bar{r}^2(1+\bar{r}^2)} \right) g(\bar{r}) = 0 . \tag{3.70}$$

We are looking for a normalisable zero mode, or, in other words, a bound state with vanishing energy. For this we need an attractive potential. We thus must have $m_3 s < 0$, and we choose $s = -1$, $m_3 > 0$ in what follows. The bound state equation (3.71) thus finally becomes,

$$\left(-\partial_{\bar{r}}^2 + \frac{4m(m+1) - 8m_3 + 3/4}{\bar{r}^2} + \frac{8m_3}{1+\bar{r}^2} \right) g(\bar{r}) = 0 . \tag{3.71}$$

This equation has already been obtained by Brower et al. [141] in the stability analysis of their monopole solutions. These authors, however, have overlooked the fact that m is half-integer which is crucial for obtaining the correct solution (see below). In addition they approximated the profile function $a(r)$ by $1/r^2$ in the limit of small monopole loops. We will instead solve (3.71) exactly. The latter is an effective one-dimensional Schrödinger equation with a Hamiltonian,

$$H \equiv -\partial_{\bar{r}}^2 + V_1(\bar{r}) + V_2(\bar{r}) . \tag{3.72}$$

The second potential term, V_2, is always positive (for $m_3 > 0$). Only the first term, V_1 has a chance of becoming negative leading to attraction. As m_3 is bounded by m, the Casimir term $m(m+1)$ in (3.71) will always win for large m. We thus should make m as small and m_3 as large as possible, implying $m_3 = m$. From (3.71), there is exactly one solution for m which makes V_1 negative, namely $m = 1/2 = m_3$. We have explicitly checked that for $m > 1/2$ there is no bound state solution[12]. The associated potential $V_1 + V_2$ is plotted in Figure 3.4. For $m = 1/2$, *there is a normalisable solution* of (3.71),

$$g(\bar{r}) = \sqrt{\bar{r}} \left(1 - (1+\bar{r}^2) \ln\left(1 + \frac{1}{\bar{r}^2}\right) \right) . \tag{3.73}$$

[12]The claim in [141], that attraction occurs for $m_3 = 1$ with the ground state having $m = 1$, thus cannot be substantiated.

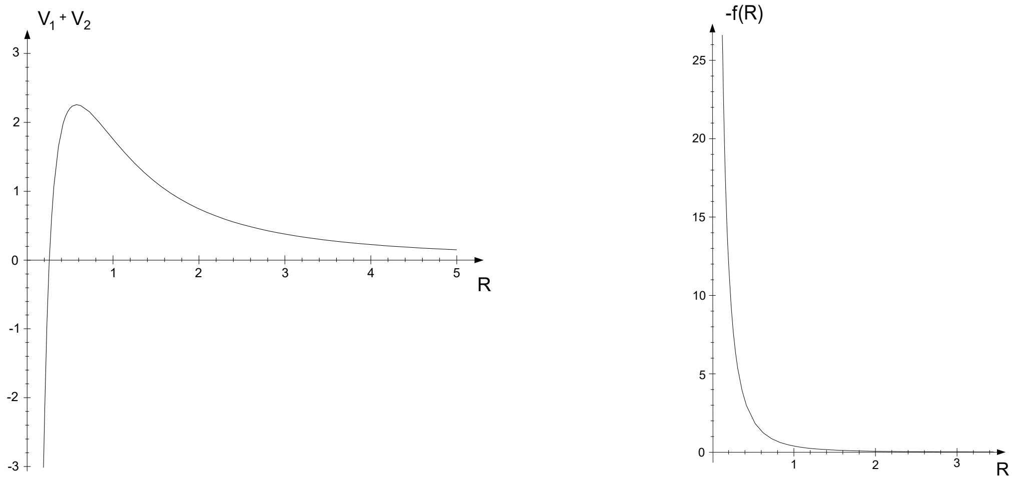

Fig. 3.4.: The bound-state potential $V_1 + V_2$ (left) and the radial wave function $-f$ of the zero mode (right) as a function of $\bar{r} = r/\rho$ for the attractive case (quantum numbers $m = m_3 = 1/2$). Copied from [FB1], where $\bar{r}$ is denoted by R.

Close to the origin, $f(\bar{r}) = g(\bar{r})/\bar{r}^{3/2}$ behaves as

$$f(\bar{r}) = \frac{1}{\bar{r}}(1 + 2\ln\bar{r}) - \bar{r}(1 - 2\ln\bar{r}) + O(\bar{r}^2)\,, \tag{3.74}$$

while asymptotically it drops as $1/\bar{r}^3$ (Figure 3.4). Both types of behaviour are sufficient to make f (and λ) normalisable. f has no nodes and therefore corresponds to the ground state in the sector with $m = 1/2$ (cf. the analogous reasoning in [155]). Accordingly, *the singular instanton is on the Gribov horizon of the Maximally Abelian Gauge.*

The degeneracy of the solution is found as follows. FP does not depend on N_3, therefore n_3 can arbitrarily be chosen as an half-integer from $\{-1/2, 1/2\}$. Furthermore, FP is invariant under $(m_3, s) \to (-m_3, -s)$, so that, altogether, there is a four-fold degeneracy. In terms of abstract states $|m, m_3, n_3, s\rangle$, the zero modes are linear combinations of the four degenerate basis states $|1/2, 1/2, \pm 1/2, -1\rangle$ and $|1/2, -1/2, \pm 1/2, +1\rangle$.

To explicitly determine the zero mode, we have to give the functions $\Theta_{m,m_3,n_3}(\theta)$ in (3.67). In the extremal case $m_3 = \pm m$ at hand they have a simple form (cf. (A.19)),

$$\Theta_{1/2,-1/2,-1/2}(\theta) = \cos\theta = \Theta_{1/2,+1/2,+1/2}(\theta)\,, \tag{3.75}$$

$$\Theta_{1/2,-1/2,+1/2}(\theta) = \sin\theta = \Theta_{1/2,+1/2,-1/2}(\theta)\,. \tag{3.76}$$

Hence the four degenerate zero modes are the following,

$$\lambda_{1/2,-1/2,-1/2,+1}(x) = c f(r) \cos\theta\, e^{-i\varphi_{12}} \chi_+\,, \tag{3.77}$$

$$\lambda_{1/2,-1/2,+1/2,+1}(x) = c f(r) \sin\theta\, e^{i\varphi_{34}} \chi_+\,, \tag{3.78}$$

$$\lambda_{1/2,+1/2,+1/2,-1}(x) = cf(r)\cos\theta\, e^{i\varphi_{12}}\chi_-\,, \tag{3.79}$$
$$\lambda_{1/2,+1/2,-1/2,-1}(x) = cf(r)\sin\theta\, e^{-i\varphi_{34}}\chi_-\,, \tag{3.80}$$

where c denotes a normalisation constant. To find its value we use the measure (A.4) and calculate the integral (λ denoting any of the basic zero modes),

$$\int \mathrm{d}V(\mathbb{R}^4)\ \lambda^*(x)\cdot\lambda(x) = c^2\rho^4\ \frac{\pi^2}{6}\left(1+\frac{\pi^2}{3}\right) \overset{!}{=} 1\,. \tag{3.81}$$

This determines the normalisation c. Any normalisable zero mode λ of FP must be a linear combination of the four basis modes (3.80). It can be shown further that a general linear combination assumes the form,

$$\lambda^{\bar a}(x) \equiv \frac{\sqrt{2}}{c} n_\mu \Psi^{\bar a}_\mu \equiv n_\mu\, \bar\eta^{\bar a}_{\mu\nu}\, x_\nu F(r)\,, \qquad \bar a\in\{1,2\}\,, \tag{3.82}$$

where n_μ is a constant four vector, and $F(r) \equiv f(r)/r$.

At this point it is due time to ask for the physics associated with these zero modes λ. According to the argument on FP zero modes given in the Section 2.1.3, it is clear that with the singular instanton A^{sing} also its infinitesimal neighbour, $A^{\text{sing}} + \mathrm{D}_{A^{\text{sing}}}\lambda$, is in the MAG. However, applying the *finite* gauge transformation $g = \exp(i\lambda^{\bar a}\tau_{\bar a})$ to the singular instanton leads to a configuration *that is no longer in the MAG*. This is at variance with the solution g_R found by Brower et al. [141] which yields a monopole configuration *within* the MAG.

One might speculate that the zero mode (3.82) is induced by some space-time symmetry of the instanton. We have discussed this issue already in Section 2.3.3. It should be noted that a symmetry of a configuration does not necessarily imply a symmetry of the MAG functional as the latter is not gauge invariant. *If*, on the other hand, there is such a symmetry, then, by Goldstone's argument, zero modes of the Hessian must be present. In covariant background gauges, for instance, there indeed appears a whole S^4 of gauge equivalent configurations induced by an $SO(5)$ symmetry transformation of the instanton [156]. Let us, therefore, have a closer look at this possibility in the context of the MAG. A subclass of the $SO(5)$ symmetry is given by the combination of conformal transformations and translations. The associated compensating gauge transformation $\delta A = \mathrm{D}\lambda$ is [127],

$$\lambda = -2n_\mu \bar\eta^a_{\mu\nu} x_\nu \tau_a\,. \tag{3.83}$$

This is nothing but Equation (2.60) for the instanton in singular gauge, setting $\omega = 0$, $c_\mu = n_\mu$. If we extend our zero mode (3.82) by adding the appropriate colour 3-

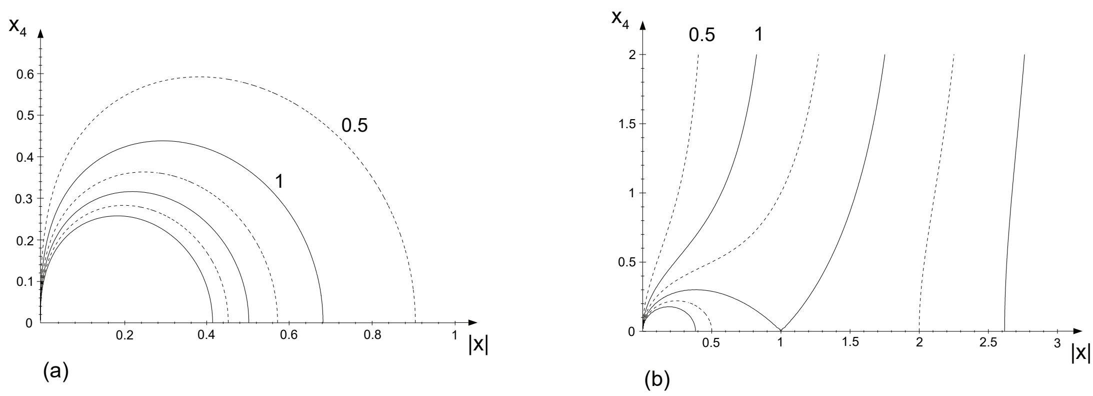

Fig. 3.5.: Lines of constant modulus of the zero modes as a function of $|\vec{x}|$ and x_4. Left: Normalisable zero mode with profile function $F(r)$; right: non-normalisable zero mode with profile function $G(r)$. The modulus increases in half-integer steps from 0.5 to 3. Full lines correspond to integer values. The 'conformal mode' (3.83) would be given by straight vertical lines.

component, $\bar{a} \to a$, (which is always possible as FP is blind against this component), we note that it is a 'deformation' of (3.83): the constant prefactor -2 gets replaced by the profile function $F(r)$. The space-time dependence of the zero mode (3.82) is thus much more complicated than that of the compensating algebra element (3.83). Interestingly, there is also a non-normalisable zero mode of FP that 'interpolates' between the two. It is given by changing the profile function $F(r)$ to,

$$G(r) = \frac{r^2 + \rho^2}{r^2} \,, \tag{3.84}$$

which goes to a constant at large r. Asymptotically, this zero mode thus approaches the 'conformal mode' (3.83) (up to a factor -2). For small r, the two zero modes coincide, as both F and G go like $1/r^2$. The situation is depicted in Figure 3.5.

We thus conclude that our zero mode (3.82) is *not* generated by any of the space-time symmetries of the instanton. It is maybe not too surprising that this result differs from the one obtained in [156]. After all, the MAG represents a non-linear gauge fixing implying a curved gauge fixing hypersurface. Background type gauges, on the other hand, are linear in the gauge potential A. They thus lead to gauge fixing hyper*planes* with constant normals. The residual gauge symmetries in both cases will thus in general be distinct. Nevertheless, a common feature of both gauges is the fact that the singular instanton is located on the associated Gribov horizons. For the MAG, this has been unambiguously shown by the explicit construction of the zero mode (3.82).

3.3. Polyakov Gauge

The *Polyakov gauge* (PG) emerged when it was noticed that the Weyl gauge, $A_0 = 0$, is in general incompatible with finite temperature and periodic boundary conditions in (Euclidean) time [133]. A priori, the time-like component of the gauge field can always be forced to vanish by employing the Polyakov line[13],

$$\mathcal{P}(x_0, \vec{x}) = \mathcal{P} \exp(i \int_0^{x_0} A_0(\tau, \vec{x})\, \mathrm{d}\tau)\,, \tag{3.85}$$

Here, $\mathcal{P}$ is nothing but time ordering. Therefore, the Polyakov line solves the differential equation (we suppress the spatial arguments),

$$\partial_0 \mathcal{P}(x_0) = iA_0(x_0)\mathcal{P}(x_0)\,, \tag{3.86}$$

and its action on the gauge field yields,

$$A_0(x_0) \to {}^{\mathcal{P}^\dagger(x_0)}A_0(x_0) = \mathcal{P}^\dagger(x_0)(A_0(x_0) + i\partial_0)\mathcal{P}(x_0) = 0\,, \tag{3.87}$$

But the transition function in the time-like direction, t_0, will change[14] under the non-periodic gauge transformation $\mathcal{P}(x_0)$ according to [7],

$$t_0(x_0) = \mathbb{1} \to \mathcal{P}^\dagger(x_0)t_0(x_0)\mathcal{P}(x_0 + \beta) = \mathcal{P}(\beta) \neq \mathbb{1}\,. \tag{3.88}$$

All one can achieve is to make A_0 *static and diagonal* [133],

$$\chi_{\mathrm{PG},1}(A) \equiv \partial_0 A_0^{||} = 0\,, \qquad \chi_{\mathrm{PG},2}(A) \equiv A_0^{\perp} = 0\,. \tag{3.89}$$

The Euclidean symmetry is obviously broken by these conditions. Together with A_0 all Polyakov lines $\mathcal{P}(x_0)$ become diagonal, which explains the name of this gauge. The Polyakov gauge is peculiar in that the residual gauge freedom consists of *time-independent* Abelian gauge transformations, since,

$$h = \exp(i\lambda^{||}) \;: \qquad \chi_{\mathrm{PG},1}({}^{h}A) = \chi_{\mathrm{PG},1}(A) + \partial_0^2\lambda^{||} \qquad \big(\chi_{\mathrm{PG},2}({}^{h}A) = \chi_{\mathrm{PG},2}(A)\big)\,. \tag{3.90}$$

Periodicity of $\lambda^{||}$ then requires $\partial_0\lambda^{||}{=}0$, and the residual freedom is

$$h = \exp(i\lambda^{||}(\vec{x}))\,. \tag{3.91}$$

[13] We will use the index 0 and denote the length in this direction by the inverse temperature, $L_0 \equiv \beta$.
[14] visible in the change of A_i in time

The gauge transformation g achieving (3.89) is known explicitly [157]: after applying (3.87) one has to correct for periodicity (reintroduce the Polyakov loop), and, in a third step, diagonalise the Polyakov loop applying a gauge transformation W,

$$g(x_0) = W\mathcal{P}^\dagger(\beta)^{-x_0/\beta}\mathcal{P}^\dagger(x_0)\,, \tag{3.92}$$

where the rightmost transformation is applied first. The gauge fixed configurations are,

$$A_{\mathrm{PG},0} = \frac{2\pi}{\beta}\,\mathrm{diag}(\rho,-\rho)\,, \qquad W\mathcal{P}(\beta)W^{-1} = \mathcal{P}_{\mathrm{PG}}(\beta) = \mathrm{diag}(e^{2\pi i\rho}, e^{-2\pi i\rho})\,. \tag{3.93}$$

The fractional power in (3.92) is defined via diagonalisation. W only depends on the space variables $\vec{x}$ (which we did not make explicit) and contains the residual gauge freedom (3.91).

The FP operator of the Polyakov gauge evaluated on the gauge slice reads [158, 159],

$$\mathrm{FP}_{\mathrm{PG}} = \begin{pmatrix} \mathrm{D}_{A_0^{\|}} & \\ & \partial_0^2 \end{pmatrix}. \tag{3.94}$$

In the parallel sector, ∂_0^2 has the trivial (field independent) zero modes $\lambda^{\|}(\vec{x})$ from (3.91). The operator $\mathrm{D}_{A_0^{\|}} = \partial_0 - i\,\mathrm{ad}_{A_0^{\|}}$ in the perpendicular sector is an ordinary differential operator and consists of two simple operators which due to $\chi_{\mathrm{PG},1}$ (3.89) commute. The determinant can be evaluated by means of the product formula for the sin-function [158]. The result is the *reduced Haar measure* [160] evaluated at the Polyakov loop,

$$\det \mathrm{FP}_{\mathrm{PG}} \sim \prod_{\vec{x}} \mathrm{d}\mu_H(\mathcal{P}_{\mathrm{PG}}(\beta,\vec{x}))\,. \tag{3.95}$$

On the gauge slice the Polyakov loop belongs to the maximal Abelian subgroup according to (3.93); the reduced Haar measure is then a function of the eigenvalues $\pm\rho$,

$$\mathrm{d}\mu_H(\mathcal{P}_{\mathrm{PG}}(\beta)) = \mathrm{d}\rho\,\sin^2(2\pi\rho)\,. \tag{3.96}$$

It has been shown that the reduced Haar measure occurs in the partition function due to expectation values between physical (gauge invariant) states [133, 158]. In two dimensions, the FP determinant cancels exactly against the non-Abelian part in the action ('Abelianisation') [7].

The Higgs field of the Polyakov gauge is the Polyakov loop $\mathcal{P}(\beta,\vec{x})$. While it is group valued, one can locally pass to an algebra valued field by 'taking the logarithm' [136],

$$\mathcal{P}(\beta,\vec{x}) = \pm e^{i\phi(\vec{x})} \qquad \text{around } \mathcal{P} = \pm\mathbb{1}_2\,. \tag{3.97}$$

Then it is easy to see that the FP determinant (3.95) takes the general form (3.20).

Examples for configurations in the Polyakov gauge are the reducible instantons of Section 2.3.3.

3.3.1. The Gauge Fixing Functional

A functional which almost implies the Polyakov gauge is [FB5],

$$F_{\mathrm{PG}}[A] \equiv \langle A_0^{||}, A_0^{||} \rangle = \int (A_0^i)^2 \mathrm{d}V \,, \tag{3.98}$$

where i runs over the Cartan generators ($i = 3$ for $SU(2)$). The associated projector is $\mathbb{P}_{\mathrm{PG}} = \mathbb{P}^{||}\mathbb{P}_0$. From (3.12) the gauge condition is,

$$\tilde{\chi}_{\mathrm{PG}}(A) = \mathrm{D}_{A_0} A_0^{||} = \mathrm{D}_{A_0^\perp} A_0^{||} = 0 \,, \tag{3.99}$$

which, decomposed into Cartan and non-Cartan part, gives,

$$\tilde{\chi}_{\mathrm{PG},1}(A) = \partial_0 A_0^{||} = 0 \,, \qquad \tilde{\chi}_{\mathrm{PG},2}(A) = [A_0^\perp, A_0^{||}] = 0 \,. \tag{3.100}$$

Comparing this with (3.89) one immediately sees that the gauge fixing conditions differ in the second requirement. $\tilde{\chi}_{\mathrm{PG},2}$ is *weaker* than $\chi_{\mathrm{PG},2}$: one can have a non-vanishing perpendicular component $A_0^\perp$ whenever the Killing form of the parallel component $A_0^{||}$ with some root vanishes ($A_0^{||}$ has degenerate eigenvalues/a non-Abelian stabiliser) [FB5], for $SU(2)$ whenever $A_0^3 = 0$. This seems to imply additional defects. However, functions $A_0^\perp$ with such a small support (generically two-dimensional hypersurfaces in three-space) are very peculiar. Accordingly, the gauge slice $\tilde{\Gamma}$ is ('slightly') bigger than Γ.

This is reflected in the residual gauge freedom – on $\tilde{\Gamma}$ it is enlarged to a (non-Abelian) projection from $\tilde{\Gamma}$ to Γ followed by Abelian transformations *within* Γ – and the FP operator,

$$\tilde{\mathrm{FP}}_{\mathrm{PG}} = \begin{pmatrix} -\mathrm{ad}_{A_0^\perp}\mathbb{P}^{||}\mathrm{ad}_{A_0^\perp} + i\mathrm{ad}_{A_0^{||}}\mathrm{D}_{A_0^{||}} & -i\mathrm{ad}_{A_0^\perp}\partial_0 \\ -i\partial_0\mathrm{ad}_{A_0^\perp} & \partial_0^2 \end{pmatrix} \,, \tag{3.101}$$

Its determinant can be rewritten as (cf. [79]),

$$\det \tilde{\mathrm{FP}}_{\mathrm{PG}} = \det(\partial_0^2) \cdot \det(\mathrm{ad}_{A_0^\perp}(-\mathbb{P}^{||} + \partial_0\partial_0^{-2}\partial_0)\mathrm{ad}_{A_0^\perp} + i\mathrm{ad}_{A_0^{||}}\mathrm{D}_{A_0^{||}}) \,, \tag{3.102}$$

which again comes close to the one of the PG (3.94). One expects additional zero modes induced by the additional defects.

Similar things happen in the *Palumbo gauge* [161]. The latter is a complete gauge fixing on the d-dimensional torus, given by

$$\chi_0(A) = A_0^{(0)} = 0 \,, \quad \chi_1(A) = A_1^{(1)} = 0 \,, \quad \ldots \quad \chi_{d-1}(A) = A_{d-1}^{(d-1)} = 0 \,, \tag{3.103}$$

where

$$X^{(\nu)} \equiv \int \mathrm{d}\xi_0 \ldots \mathrm{d}\xi_{\nu-1} X - \int \mathrm{d}\xi_0 \ldots \mathrm{d}\xi_\nu X \,. \tag{3.104}$$

It can almost be derived from the functional [FB5],

$$F[A] = \langle A_0^{(0)}, A_0^{(0)} \rangle + \langle A_1^{(1)}, A_1^{(1)} \rangle + \ldots + \langle A_{d-1}^{(d-1)}, A_{d-1}^{(d-1)} \rangle \,. \tag{3.105}$$

in that the gauge fixing condition obtained from this functional again differs only slightly from the explicit one (3.103), by some 'defects'. For instance,

$$\tilde{\chi}_{d-1}(A) = \mathrm{D}_{a_{d-1}} A_{d-1}^{(d-1)} = 0 \,, \qquad a_{d-1} \equiv \int \mathrm{d}\xi_0 \ldots \mathrm{d}\xi_{d-1} A_{d-1} \,, \tag{3.106}$$

which only in the generic case implies χ_{d-1}.

We finish the discussion of the PG-functional by a no-go conjecture. According to the general results (3.10) – (3.12), any χ obtained from a functional – which might be of a more general form than (3.8) – always contains derivatives and commutators of the gauge field A. Therefore it is evident that neither the Polyakov gauge nor the Palumbo gauge as axial-type gauges can exactly be derived from a gauge fixing functional. It would be interesting to see whether there is some physics behind this discrepancy. Therefore, we propose to study these gauges on the lattice by extremising (in the spirit of the MAG) the functional

$$F_{\mathrm{PG}}[U] = 2a^2 \sum_x \left(\sum_a \operatorname{tr} \tau_a U_0(x) \tau_a U_0^\dagger(x) - 2 \operatorname{tr} \tau_3 U_0(x) \tau_3 U_0^\dagger(x) \right) \,, \tag{3.107}$$

and comparing this gauge to the direct implementation of the PG.

3.4. Laplacian Abelian Gauge

The *Laplacian Abelian gauge* (LAG) was invented by van der Sijs in order to circumvent the spin glass problem of the MAG [146, 140]. It is based on an idea of Vink and Wiese in the context of a complete gauge fixing on the lattice [147, 162]: by modifying the gauge fixing functional, the Landau gauge turns into the eigenvalue problem of the covariant Laplacian in the *fundamental* representation. This gauge is called *Laplacian gauge* (LG). For the Laplacian *Abelian* gauge one simply relaxes the length-one constraint of the Higgs field n of the MAG, such that the gauge fixing functional is maximised by the lowest eigenfunction of the covariant Laplacian in the *adjoint* representation (see below).

In the continuum, this condition is endowed with *normalisability* of the Higgs field ϕ in order to render the eigenvalue problem well-defined. With the help of a Lagrange multiplier, the 'kinetic energy' E, the functional reads,

$$F_{\mathrm{LAG}}[A;\phi] = \langle \mathrm{D}_A\phi, \mathrm{D}_A\phi\rangle + E(\langle\phi,\phi\rangle - 1) = \int (\mathrm{D}_\mu\,\phi^a)^2\,\mathrm{d}V - E(\int(\phi^a)^2\,\mathrm{d}V - 1)\,. \quad (3.108)$$

As already mentioned, ϕ is in the adjoint representation, $\mathrm{D}\phi = \mathrm{d}\phi - i[A,\phi]$, leaving rotations around it as the Abelian gauge freedom. Minimising $F_{\mathrm{LAG}}[A;\phi]$ represents a variational problem of Rayleigh-Ritz type and is solved by the lowest eigenfunction/eigenvalue of (minus) the gauge covariant Laplacian in the background A,

$$\chi_{\mathrm{LAG}}(A;\phi) = -*\mathrm{D}_A * \mathrm{D}_A\,\phi - E_0\,\phi = 0 = -\mathrm{D}^2_{A_\mu}\phi - E_0\,\phi\,. \quad (3.109)$$

In physical terms, F_{LAG} is the quadratic action for a Yang-Mills-Higgs system[15], while χ_{LAG} is its equation of motion. Since the latter represents a four-dimensional (time-independent) Schrödinger problem with a potential essentially given by A^2, we call ϕ and E_0 the ground state wave function and the ground state energy, respectively. Equation (3.109) should be compared with (3.44): In the MAG, the 'Lagrange multiplier' $E(x)$ is a function demanding the Higgs field n to be *normalised pointwise*. For the LAG, the ground state energy E_0 is a constant, so that the condition on the Higgs field is relaxed to only *square integrability over the whole space-time*. Writing $\phi = |\phi|\cdot n$, we make use of our knowledge from the MAG (3.43) to rewrite the functional [146, 140],

$$F_{\mathrm{LAG}}[A;\phi] = \langle \mathrm{d}|\phi|, \mathrm{d}|\phi|\rangle + \langle |\phi| A^{\perp n}, |\phi| A^{\perp n}\rangle + E_0(\langle|\phi|,|\phi|\rangle - 1)\,. \quad (3.110)$$

Variations with respect to the fields $|\phi|$ and n yield three equations [146], the analogues of (3.109),

$$\chi_{\mathrm{LAG},1}(A;\phi) = \mathrm{D}_{A_\mu}(|\phi|^2 A_\mu^{\perp n}) = 0\,, \quad (3.111)$$

$$\chi_{\mathrm{LAG},2}(A;\phi) = -\Box|\phi| + (A_\mu^{\perp n})^2|\phi| - E_0|\phi| = 0\,. \quad (3.112)$$

Not surprisingly, the gauge conditions for the LAG are very similar to the ones for the LG [90]. Moreover, the 'coset equation' $\chi_{\mathrm{LAG},1}$ turns into the MAG-condition (3.42) upon setting $|\phi| = 1$ (i.e. replacing ϕ by n), while $\chi_{\mathrm{LAG},2}$ is an equation for the 'additional degree of freedom' $|\phi|$. We remind the reader that $|\phi|$ is a gauge invariant object ('depends on the orbit as a whole only' [90, 146]) and so is the entire equation $\chi_{\mathrm{LAG},2}$.

[15] Remember that the background A is kept fixed, otherwise the action is gauge invariant and would be useless for our purposes.

Gauge variations of $\chi_{\mathrm{LAG},1}$ give the FP operator of the LAG which, as expected, is very similar to the one of the MAG (3.46),

$$\mathrm{FP}_{\mathrm{LAG}} = |\phi|^2 \mathrm{FP}_{\mathrm{MAG}} + \mathbb{P}^{\perp} 2|\phi|\partial_\mu|\phi|\, \mathrm{D}_{A_\mu^{\parallel}} \mathbb{P}^{\perp} \,. \tag{3.113}$$

Obviously, this operator vanishes on parallel gauge parameters $\lambda^{\parallel}$ as it should.

So far we have been working in the body fixed frame, looking for the Higgs field. The gauge invariant modulus $|\phi|$ will persist to show up in the space fixed frame. We simply translate n into τ_3 (and $\perp n$ into $\perp$),

$$F_{\mathrm{LAG}}[A;|\phi|] = \langle \mathrm{d}|\phi|, \mathrm{d}|\phi|\rangle + \langle |\phi| A^{\perp}, |\phi| A^{\perp}\rangle + E_0(\langle |\phi|, |\phi|\rangle - 1)\,, \tag{3.114}$$

$$\chi_{\mathrm{LAG},1}(A;|\phi|) = \mathrm{D}_{A_\mu}(|\phi|^2 A_\mu^{\perp}) = 0\,, \tag{3.115}$$

$$\chi_{\mathrm{LAG},2}(A;|\phi|) = -\Box|\phi| + (A_\mu^{\perp})^2|\phi| - E_0|\phi| = 0\,. \tag{3.116}$$

This clearly indicates that – unlike for other Abelian gauges – there is no simple condition $\chi_{\mathrm{LAG}}(A)$ without Higgs field. One always has to solve an eigenvalue problem like (3.116) for $|\phi|$. Accordingly, *given a configuration A one cannot immediately decide whether this configuration is in the LAG or not.* One can understand this fact as follows: In the LAG one is directly led to the set Λ' (cf. Section 2.2.3) of absolute minima of the functional (3.108). Thus, solving the LAG of course requires more efforts than just checking a gauge condition (a differential equation), but hopefully not as many as for the MAG.

Another subtlety of the LAG is the definition of a normalisable ground state on a space-time with infinite volume. Since $-D_A^2$ is a non-negative operator we have $E \geq 0$. Moreover, whenever the gauge field A tends to zero at infinity, the eigenfunctions with $E > 0$ are non-normalisable scattering states. A normalisable *zero* mode, $\mathrm{D}_A\phi = 0$, only exists for reducible backgrounds (cf. (2.10)). Thus, for a generic background, the covariant Laplacian $-D_A^2$ does not have a normalisable ground state, and the LAG is not straightforwardly defined. The situation is quite analogous to the quantum mechanics of the ordinary Laplacian ∂_x^2 on the real line: this operator has a spectrum $E > 0$ consisting of scattering states, but one can avoid this 'problem' by considering the system in a box with periodic boundary conditions, i.e. on a circle, which leads to a purely discrete spectrum. For the LAG we will proceed analogously by conformally transforming the gauge fields onto the four-sphere S^4 (cf. Section 2.3.3), which yields a discrete spectrum of the associated gauge covariant Laplacian.

Apart from these more technical problems, the LAG has two sources of ambiguities (like the LG [162]). First, if the ground state is degenerate. The diagonalisation of any of the ground states by definition gives a configuration in the LAG. Since these configurations are on the same orbit, they are Gribov copies of each other. Due to the matrix structure of the Laplacian this is already true for the vacuum $A = 0$: $-\mathrm{D}_A^2$ reduces to $-\Box$ and the eigenfunctions are constants (hence normalisable on the sphere) with a threefold degeneracy given by the canonical dreibein $\hat{e}_a$ in isospace. Second, the (expected) defects occur, if the ground state ϕ has zeros[16]. Having a well-known Schrödinger problem at hand, one can potentially use node and uniqueness theorems to analyse these issues [163].

On the lattice, the Laplacian Abelian gauge shares many features with the Laplacian gauge. Both of them can be achieved by the Lanczos algorithm [147, 140], which is computationally cheaper than the minimisation algorithms of the Maximally Abelian gauge/Landau gauge. The first type of ambiguity occurs when the two lowest-lying eigenstates of the lattice Laplacian agree within numerical precision. Such configurations are exceptional ('of measure zero') and do not appear in practice, as explicitly shown in [162]. Therefore, LAG and LG are unambiguous on the lattice, i.e. free of Gribov copies. The second type of ambiguity, the defects, will be discussed later.

3.4.1. Solitonic Backgrounds

Until recently the only known property of the LAG (for continuum configurations) was a qualitative argument concerning the dyon (and the 't Hooft-Polyakov monopole in the BPS limit) [146, 140]: At the dyon position, the electric field (i.e. the Higgs field) vanishes linearly with the spatial distance $|\phi| \sim |\vec{x}|$ (cf. Section 2.3.2), while the gauge field $A^\perp$ diverges at the origin (in unitary gauge). The latter contributes 'in the wrong way' to the functionals of the LAG (3.114) and MAG (3.41), since these have to be minimised. The Higgs field is the only difference between them, hence in the LAG the vanishing modulus of the Higgs field can compensate the large contribution from the gauge field. Therefore, one concludes that the dyon *is not suppressed in an unnaturally strong fashion in the LAG* [140]. One should keep in mind, however, that the values of $F_{\mathrm{AG}}[A]$ have to be compared *along one orbit within one Abelian gauge*.

The LAG-Higgs field in the background of the single instanton was studied in depth in [FB2]. An analytical treatment is possible because of the high symmetry of this background (see Section 2.3.3). To be on safe grounds, the system is studied on a geometrical four-sphere of radius R embedded in five-space. The metric in conformal

[16]Nodes are a consequence of degeneracy only for scalar wave functions.

coordinates is (cf. (2.58)),

$$g_{\mu\nu}(x) = e^{\alpha_R(r)}\delta_{\mu\nu}\,, \qquad e^{\alpha_R(r)} = \frac{4R^4}{(r^2+R^2)^2}\,, \qquad \sqrt{g} = e^{2\alpha_R(r)}\,. \tag{3.117}$$

Accordingly, the LAG-functional becomes,

$$F_{\rm LAG}[A;\phi] = \int_{S^4} \left(\mathrm{D}_\mu\phi^a \mathrm{D}_\nu\phi^a g^{\mu\nu} - E_0\phi^a\phi^a\right)\sqrt{g}\,\mathrm{d}^4x\,. \tag{3.118}$$

The LAG-condition is now given in terms of the gauge covariant Laplace-Beltrami operator,

$$\chi_{\rm LAG}(A;\phi) = -\frac{1}{\sqrt{g}}\,\mathrm{D}_\mu\sqrt{g}g^{\mu\nu}\mathrm{D}_\nu\phi - E_0\phi = 0\,. \tag{3.119}$$

Although this equation is understood in the adjoint representation of $SU(2)$, we keep the isospin $\vec{T}$ arbitrary and write,

$$\vec{T}^2 \to t(t+1)\,, \qquad t = 1\,. \tag{3.120}$$

Furthermore we treat singular gauge and regular gauge separately and come back to the bundle picture afterwards. We know right from the beginning that, since the backgrounds are related by the gauge transformation $\hat{g}$ (2.56), so are the solutions,

$$\phi^{\rm reg} = {}^{\hat{g}}\phi^{\rm sg} = \hat{g}\phi^{\rm sg}\hat{g}^\dagger\,. \tag{3.121}$$

Similar to what happens in Section 3.2.2, Equation (3.119) becomes a radial one,

$$e^{-\alpha_R(r)}\Big(-\partial_r^2 - \frac{3}{r}\,\partial_r + \frac{4\vec{M}^2}{r^2} + \frac{4\rho^2(\vec{J}^2-\vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} + \frac{4r\partial_r}{r^2+R^2}\Big)\phi^{\rm sg} = E_0\phi^{\rm sg}, \tag{3.122}$$

$$e^{-\alpha_R(r)}\Big(-\partial_r^2 - \frac{3}{r}\,\partial_r + \frac{4\vec{N}^2}{r^2} + \frac{4(\vec{J}^2-\vec{N}^2)}{(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} + \frac{4r\partial_r}{r^2+R^2}\Big)\phi^{\rm reg} = E_0\phi^{\rm reg}. \tag{3.123}$$

These equations differ from those in Euclidean space by a metric factor, $e^{-\alpha_R(r)}$, and a dilatation term, $r\partial_r$. The operator $A_\mu\partial_\mu$ is proportional to $\vec{T}\cdot\vec{M}$ and $\vec{T}\cdot\vec{N}$ for the two gauges, respectively (for the generators $\vec{M}$ and $\vec{N}$ of the Lie algebra $so(4)$, see A.2). Therefore we have introduced the total angular momentum $\vec{J}$ ('spin from isospin', [164, 165, 166]),

$$\vec{J} \equiv \vec{L} + \vec{T}\,, \qquad \vec{J}^2 \to j(j+1)\,, \qquad j \in \{l-1, l, l+1\}\,, \tag{3.124}$$

where $\vec{L}$ denotes $\vec{M}$ or $\vec{N}$, respectively. The eigenvalues j are integer or half-integer, like l. Replacing angular momenta by their eigenvalues and exchanging $j \to n$, $m \to j$, equation (3.122) turns into (3.123). This amounts exactly to the action of the gauge transformations $\hat{g}$ from (3.121).

The symmetry considerations above suggest the following form of the ground state,

$$\phi(x) = Y_{(j,l)}(\hat{x})\varphi(r)\,, \tag{3.125}$$

where the Y's denote the isovector spherical harmonics on S^3 [167]. Note that there are two competing angular momentum terms in (3.122) and (3.123), so that it is not obvious in which angular momentum sector the groundstate will be. By simply looking at the radial potentials in the different sectors, we can only state the following bound on the energy in an arbitrary sector,

$$E_{(j,l)} \geq \min\{E_{(0,1)},\, E_{(1/2,1/2)},\, E_{(1,0)}\}\,. \tag{3.126}$$

The quantum numbers of the ground state candidates on the r.h.s. correspond to the representations (0,1), (1/2, 1/2) and (1,0) of $su(2)_j \oplus su(2)_l$ and thus have degeneracies 3, 4 and 3, respectively. Note that the singlet $(0,0)$ is excluded by the selection rules (3.124), whenever $t \neq 0$. Accordingly, for any of the possible choices in (3.126), the groundstate will be degenerate. The spherical harmonics for the three cases are listed in the Appendix, A.3.

At this point two further remarks are in order: First, the radial part φ shows power law behaviour in r, both for small and large r, independently of R and ρ,

$$\varphi^{\text{sg}}(r \to 0) \to r^{2j}\,, \qquad \varphi^{\text{sg}}(r \to \infty) \to r^{-2m}\,, \tag{3.127}$$

$$\varphi^{\text{reg}}(r \to 0) \to r^{2n}\,, \qquad \varphi^{\text{reg}}(r \to \infty) \to r^{-2j}\,. \tag{3.128}$$

Second, upon substituting $\varphi \equiv (r^2 + R^2) \cdot \chi$ and $\lambda \equiv ER^2 + 2$, one can absorb the dilatation term,

$$(-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{M}^2}{r^2} + \frac{4\rho^2(\vec{J}^2 - \vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} - \frac{4\lambda R^2}{(r^2+R^2)^2})\chi^{\text{sg}} = 0\,, \tag{3.129}$$

$$(-\partial_r^2 - \frac{3}{r}\partial_r + \frac{4\vec{N}^2}{r^2} + \frac{4(\vec{J}^2 - \vec{N}^2)}{(r^2+\rho^2)} - \frac{4\vec{T}^2\rho^2}{(r^2+\rho^2)^2} - \frac{4\lambda R^2}{(r^2+R^2)^2})\chi^{\text{reg}} = 0\,. \tag{3.130}$$

Setting $R = \rho$, the differential equation (3.129) coincides with the one considered by 't Hooft in his analysis of the quantum fluctuations around an instanton [126]. The eigenvalues are $\lambda_k = (k+j+l+1-t)(k+j+l+t+2)$. The lowest energy corresponds to $k = 0$ and $j+l = 1$, consistent with the three possible groundstates of (3.126). Together

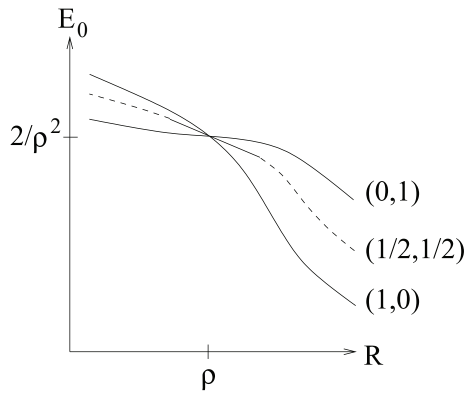

Fig. 3.6.: Energy of the lowest-lying states in the relevant angular momentum sectors as functions of the compactification radius R (singular gauge). At the point $R = \rho$ the two triplets and the quadruplet meet, while for $R \to \infty$ the triplet $(1, 0)$ has lowest energy. For symmetry reasons we expect the dashed line to stay inbetween the other two for $R \neq \rho$.

they form the 10-dimensional adjoint representation[17] of SO(5) [138]. The value of the ground state energy is $E = 2/R^2$. The radial eigenfunctions are rational,

$$\varphi^{\text{sg}}(r) = R\,\frac{(r/R)^{2j}}{r^2 + R^2}\,, \qquad \varphi^{\text{reg}}(r) = R\,\frac{(r/R)^{2-2j}}{r^2 + R^2}\,, \tag{3.131}$$

and obey the asymptotics (3.127) and (3.128), respectively.

For the cases $R > \rho$ and $R < \rho$ we cannot solve the radial equation analytically. However, we are able to prove the following statements: For the singular gauge and $R > \rho$, the triplet $(1, 0)$ has lower energy than the triplet $(0, 1)$. For $R < \rho$, the situation is reversed and the triplet $(0, 1)$ has lower energy (see Fig. 3.6). Analogous statements hold for the regular gauge. These results are a straightforward consequence of the Feynman-Hellmann theorem (see C.2). For the quadruplet (1/2, 1/2), the situation is somewhat more complicated. Using perturbation theory in $\delta = \rho^2 - R^2$ (see C.2), one finds that these states have energy inbetween the two disjoint triplet states. For symmetry reasons we do not expect the spectral flow $E_{(1/2,1/2)}(R)$ to intersect the others for some $R \neq \rho$ (see Figure 3.6).

In the above one encounters the following subtlety: Near the origin, the (0,1) wave functions in the singular gauge are bilinear in $\hat{x}_\mu$ and thus *discontinuous* there. They inherit this singularity from the instanton field, which results in the asymptotics $\varphi(r) \sim$

[17] Using the conventions of [138], this representation is labelled by the integers $\{n_1, n_2\} = \{0, 2\}$ which are the coefficients of the highest weight when expanded in terms of the fundamental weights.

r^0, see (3.127). Nevertheless, the wave functions are square integrable on S^4 due to the measure factor r^3. The same, of course, is true for the regular gauge states near infinity. At this point it is appropriate to use the bundle picture. The Higgs field is a section in an associated fibre bundle: on each of the two charts there is a Higgs field (and a gauge field). The regular gauge is valid around the origin (southern hemisphere), while the singular gauge is valid around infinity (northern hemisphere). Extending them over the whole sphere will lead to the above singularities. The results obtained so far can immediately be carried over to the bundle picture since, for every solution in one gauge, there is a corresponding 'mirror' solution in the other gauge with the same energy, obtained by the action of $\hat{g}$. Moreover, the latter cannot change the zeros of the wave function (the modulus of ϕ is gauge invariant), and the angular momenta are interchanged, $j \to n$, $m \to j$, in such a way that the *radial* wave functions $\varphi(r)$ (e.g. (3.131)) are smoothly defined on the whole of S^4. Hence the complete eigenfunctions ϕ are continuous in their respective charts but 'jump' in their isospin direction in the transition region.

Coming back to the physical region, $R > \rho$, which includes the infinite-volume limit, $R \to \infty$, we know that around the origin (regular gauge) and around infinity (singular gauge) we have to take $(j, n) = (0, 1)$ and $(j, m) = (1, 0)$, respectively, since these multiplets contain the lowest-lying states. Hence the LAG-Higgs field is of the following form,

$$\phi(x) = \begin{cases} Y^{\text{reg}}_{(0,1)}(\hat{x})\, \varphi(r) & \text{around the origin,} \\ Y^{\text{sg}}_{(1,0)}\, \varphi(r) & \text{around infinity,} \end{cases} \tag{3.132}$$

where according to (3.128) ϕ *vanishes quadratically at the origin*. The remaining task is to diagonalise this ground state Higgs field. Around infinity the spherical harmonics $Y^{\text{sg}}_{(1,0)}$ are constants (see (A.21)). If we choose the third of them, ϕ is already diagonal (points in the third colour direction). On this chart there is nothing left to be done. No gauge transformation is needed and the gauge field remains in the singular gauge. On the other hand, around the origin, the spherical harmonics $Y^{\text{reg}}_{(0,1)}(\hat{x})$ are Hopf maps (see (A.25)). We still have to diagonalise them. For the third of these maps, this is achieved by the gauge transformation $\hat{g}$, which transforms the gauge field A from regular to singular gauge. No matter where we choose the transition region, *the LAG-fixed configuration on the orbit of the single instanton is in singular gauge* (for $R > \rho$). Observe that the MAG on the sphere also 'prefers' the instanton in singular gauge for $R > \rho$ since the ratio of the MAG-functional is $F_{\text{MAG}}[A^{\text{reg}}]/F_{\text{MAG}}[A^{\text{sg}}] = R^2/\rho^2$ [FB2] (giving back (3.47) for $R \to \infty$).

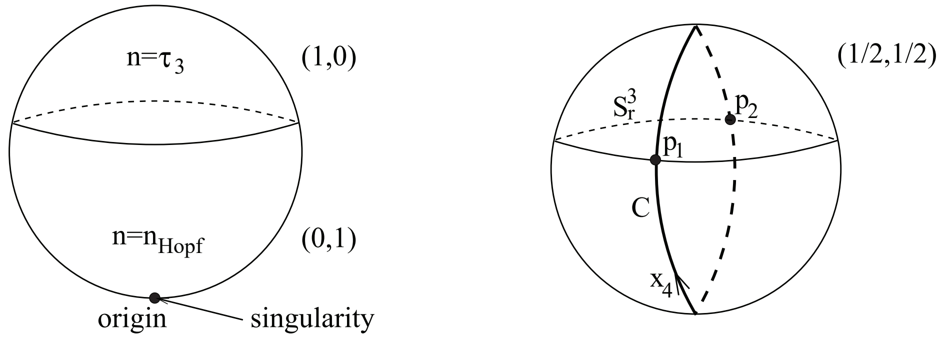

Fig. 3.7.: Submanifolds of S^4 on which the ground state wave function vanishes so that the normalised Higgs field n becomes singular. The generic sectors (0,1) and (1,0) lead to pointlike singularities with Hopf index as topological invariant (left), while the sector (1/2,1/2) gives rise to monopole loops C (right).

Moreover, we have found *a whole S^3 of gauge equivalent configurations* obtained by global $SU(2)$ rotations of A^{sg} (similar to what has been observed in [156]) located on the gauge fixing hypersurface. These are Gribov copies of each other, generated by both finite and infinitesimal gauge transformations. The latter give rise to three flat directions in the configuration space along which the gauge fixing functional does not change. Only one of the flat directions is covered by the residual U(1) freedom. The other two are related to zero modes of the (coset part of the) Faddeev-Popov operator. Accordingly, *the single instanton lies on the Gribov horizon of the Laplacian Abelian gauge.*

Finally, we summarise the most relevant features of the Higgs field for the investigation of defects: The node theorem for the one-dimensional radial equations (3.129) and (3.130) guarantees that $\varphi(r)$ has no zeros apart from $r = 0$ and $r = \infty$; in accordance with the asymptotics (3.127) and (3.128). These properties can be checked explicitly for the known solution (3.131) at $R = \rho$. Moreover, in the Appendix, A.3, we show that the spherical harmonics for the triplets have no zeros, while the ones for the quadruplet vanish at two points on the three-sphere. We conclude that for $R \neq \rho$ the Higgs field ϕ may vanish at the origin – the south pole of the sphere – or at infinity – the north pole. For $R = \rho$ it generically vanishes on loops: the zeros of $Y_{(1/2,1/2)}$ extend via r to great circles over the whole sphere. When adding a triplet state to it (in the sense of a linear combination of degenerate eigenstates) the loop becomes *tilted*. As an example, take a combination of the sectors (1,0) and (1/2, 1/2),

$$\phi^{\text{reg}} = \frac{1}{\sqrt{2}}\left(\phi^{\text{reg}}_{(1,0)} - \phi^{\text{reg}}_{(1/2,1/2)}\right) = \frac{1}{\sqrt{2}(r^2 + R^2)}\,(x_1, x_2, R - x_3)^\top . \tag{3.133}$$

This Higgs field vanishes for $x_\mu = (0, 0, R, x_4)$, a set of zeros which is still a great circle but does no longer include the poles.

The formalism is easily adapted to the Laplacian gauge [FB2], simply by choosing the fundamental representation, $t = 1/2$, in terms of the Pauli matrices $T_a = \sigma_a/2$. For $R = \rho$ one again has to minimise $j + l$, whence $(j, l) = (0, 1/2)$ or $(j, l) = (1/2, 0)$. As before, these states form an irreducible representation[18] of SO(5). For $R > \rho$ and the singular gauge, the state $(1/2, 0)$ has lowest energy (by the same Feynman-Hellmann argument) so that the singular gauge instanton again satisfies the gauge condition. The relevant spherical harmonics are nonzero throughout S^4 (see (A.27) and (A.28)). Due to the asymptotic behaviour (3.128) of the radial part, the Higgs field *vanishes linearly at the origin*. Thus, in the LAG as well as in the LG the modulus of the Higgs field vanishes at the origin where the topological charge of the instanton is concentrated. This perfectly agrees with lattice results: The correlation between the modulus of the Higgs field and the instanton density was clearly demonstrated in Ref. [168][19] including the linear and quadratic dependence on the four-dimensional distance r for the LG and the LAG [169].

In the same manner, the Higgs field of the *meron* was studied in [170]. A meron is a topological object in YM theories that has a tunnelling intepretation like an instanton, but half-integer instanton number [171]. The single meron can be obtained from a variant of the ansatz (2.52),

$$A = -\eta^a_{\mu\nu}\, \partial_\nu \ln \Pi \, \mathrm{d}x_\mu \tau_a \,, \qquad \Pi(x) = \frac{1}{r} \,. \tag{3.134}$$

It can be viewed as half an instanton of vanishing radius (in regular gauge),

$$A_{\text{meron}} = \frac{1}{2}\, A^{\text{reg}}_{\text{inst}}|_{\rho=0} \,, \qquad \nu(A) = \int \frac{1}{2}\, \delta^{(4)}(x)\mathrm{d}V = \frac{1}{2} \,, \tag{3.135}$$

but its action is infinite. Nevertheless, merons possess fermionic zero modes [172] and there are speculations that merons may describe confinement [173, 174].

From (3.135) and (3.129) it is not difficult to infer that the ground states in the meron background form a quadruplet, $j = n = 1/2$ [170]. Accordingly, also the meron lies on the Gribov horizon and has defects on loops.

All configurations considered so far are highly symmetric. In technical terms, the partial differential equation of the LAG (3.109) thus turns into an ordinary differential

[18] the four-dimensional spinor representation labelled by $\{n_1, n_2\} = \{0, 1\}$ in [138].

[19] Those authors even proposed to use this fact for studying instanton excitations without cooling.

equation in the radius. For more generic configurations, the symmetry will be reduced. Hence one cannot expect to solve the eigenvalue problem of the covariant Laplacian in full analytic glory.

One method to handle *configurations near the single instanton* is perturbation theory à la Schrödinger[20] [FB4]. We start with the regular instanton with its Higgs field vanishing at the instanton core and its colour structure being the Hopf map $n_{\rm H}$ (for $R > \rho$, see (3.132)). The new configuration is a field on a different gauge orbit[21], obtained by $A = A^{\rm reg}_{\rm inst} + \lambda\delta A$, with perturbation parameter λ. The usual Schrödinger perturbation theory for the change of the groundstate, $\phi = \phi_{\rm inst} + \lambda\delta\phi$, requires access to all eigenvalues and eigenfunctions of $-{\rm D}^2_{A_{\rm inst}}$. Unfortunately, neither the full spectrum nor the radial dependence of the wave functions are known analytically (for $R > \rho$, cf. the discussion after (3.131)). Nevertheless, if perturbation theory is valid, the size of the defect manifold is small. Therefore, we can restrict ourselves to the vicinity of the origin. There we can Taylor expand $\delta\phi$; for our purposes even the lowest order approximation is sufficient. Thus, the Higgs field of a generic configuration A *close to the single instanton* (in orbit space) and *near the origin* (in coordinate space) is,

$$\phi = \phi_{\rm inst} + \lambda\delta\phi = r^2\, n_{\rm H} + R^2\,{\rm const}\,, \tag{3.136}$$

where we have introduced a new length parameter[22] R, since the Higgs field in our convention is of dimension length squared. Without loss of generality we specialise to a perturbation pointing in the third colour direction,

$$\phi = r^2\, n_{\rm H} - R^2(0,0,1)^\top\,, \tag{3.137}$$

all other cases can be obtained by rotations. A straightforward calculation shows that the zeros of ϕ are then on the *circle* $C:\ x_1^2 + x_2^2 = R^2,\ x_3 = x_4 = 0$. Its size scales with the perturbation parameter $R = \sqrt{\lambda}$. The perturbation has *enlarged the defect manifold from a point to a loop*, thereby breaking the spherical symmetry. The associated defect, a magnetic monopole, will be analysed in detail in Sections 4.1.2 and 4.3.

How could a background inducing a monopole loop look like? In the next chapter we will elaborate more on the correlation between the position of the monopole and the maximum of the topological density. In view of that, the answer should be a non-trivial configuration with *instanton density localised on a circle*. This is the case for the $k = 2$ instanton when the parameters are tuned appropriately [175]. Another approach

[20] not to be confused with perturbation theory in the YM coupling

[21] and not a solution of the YM equations anymore

[22] not to be confused with the radius of the four-sphere when embedded into five-space

is based on an observation by Rossi [176]: The static 't Hooft-Polyakov monopole (as well the Harrington-Shepard instanton) can be obtained by placing an infinite array of instantons (in singular gauge) along the time axis. One can do the same on a finite circle in four dimensions. As the number N of instantons becomes infinite, their size ρ should tend to zero keeping $N\rho^2$ fixed. Indeed, the instanton density is localised on the circle. However, in order to exhibit the monopole content it might be necessary to allow for varying orientations of the constituent instantons. Accordingly, one has to use the ADHM construction [177].

3.5. Faddeev-Niemi Decomposition

In this section we demonstrate how the Faddeev-Niemi on-shell parametrisation (1.12),

$$A = Cn + i[n, \mathrm{d}n] + \rho \mathrm{d}n + i\sigma[n, \mathrm{d}n]\,, \tag{3.138}$$

can be extended off-shell *with the help of Abelian gauges.* The general ansatz,

$$A = Cn + i[n, \mathrm{d}n] + W\,, \tag{3.139}$$

goes under the name 'Cho-Faddeev-Niemi-Shabanov decomposition'. It expresses a general connection as the sum of the Cho connection $\hat{A} = Cn + i[n, \mathrm{d}n]$ – which was shown to be the general form of a $U(1)$ reducible connection (cf. Section 2.1.3) – and a rest W. Counting of degrees of freedom shows a severe mismatch since $12 < 4 + 2 + 12$. Therefore, one generally proceeds by assuming that W is perpendicular to n [139, 178, 179, 64, 66, 180] which in turn means that C is the n-projection of A,

$$(W, n) = 0 \rightleftharpoons C = (A, n)\,. \tag{3.140}$$

This is just *the decomposition in the body fixed frame* (3.26) upon identifying,

$$A = A^{\|n} + A^{\perp n}\,, \qquad A^{\|n} = \hat{A}\,, \qquad A^{\perp n} = W\,. \tag{3.141}$$

The condition (3.140) has reduced the number of degrees of freedom in W from 12 to 8. Still, we are overcounting by two: the dependent fields $C = (A, n)$ and $W = -i[n, \mathrm{D}_A n]$ can be obtained from A *and* n. Therefore, *fixing the redundant variables is in one-to-one correspondence with an Abelian gauge.* The latter makes n a dependent variable (see (3.15)), but can also be read as providing two further restrictions on W. Since the restrictions fix the coset, they are perpendicular to n,

$$\chi(C, n; W) = 0\,, \qquad (\chi, n) = 0\,. \tag{3.142}$$

Indeed, the MAG-condition $\chi_{\text{MAG}}(C,n;W) = \mathrm{D}_{\hat{A}_\mu(C,n)} W_\mu = 0$ (cf. (3.42)) was originally proposed by Shabanov [139]. Faddeev and Niemi, on the other hand, proposed an explicit parametrisation of W in terms of six variables [178], inspired by the use of Darboux variables in an Abelian gauge theory [181]. It can be written as follows [66],

$$W = \rho_1 \mathrm{d}n + i\sigma_1[n, \mathrm{d}n] + g'^{-1}(\rho_2 \mathrm{d}n' - i\sigma_2[n', \mathrm{d}n'])g' \,, \tag{3.143}$$

where $\Phi_1 = \rho_1 + i\sigma_1$ is the complex scalar from the on-shell parametrisation, while $\Phi_2 = \rho_2 + i\sigma_2$ and $n' = g'^{-1} n g'$ denote a second scalar and Higgs field, respectively. In the space fixed frame this parametrisation fulfils a *cubic* gauge condition introduced by Kashaev (as quoted in [178]),

$$\chi(A) = *\,(\mathrm{d}A^\perp \wedge A^\perp \wedge A^\perp)\,. \tag{3.144}$$

The properties $A^\perp \wedge A^\perp \in \mathcal{H}$ and $(A^\perp)^3 = 0$ imply that this is another Abelian gauge.

3.6. Outlook 1: Fermionic Zero Modes

Fermionic zero modes in gauge theories obey the Atiyah-Singer index theorem [182]: the difference of left-handed and right-handed zero modes is the instanton number, the second Chern number. One can try to use this knowledge to define another Abelian gauge. In the spirit of the LAG[23], one looks for the ground state ψ of the Dirac operator in some background A, $(i\not{D}_A)^2\psi = E_0\psi$. Since $(i\not{D}_A)^2$ is a non-negative operator, zero modes, if present, are ground states. (However, one cannot expect zero modes for generic backgrounds in the perturbative sector having $\nu(A) = 0$.) The Higgs field of this *Dirac Abelian gauge* should be a Lorentz scalar and isovector built out of ψ. Explicit expressions for ψ are known in instanton backgrounds fulfilling the ansatz (2.52) [183] (and for the caloron [184]). Typically, the zero modes are localised at the instanton core (the constituent monopole), a fact which is of great interest when studying the defects. This Abelian gauge has not yet been investigated, partly because there are difficulties in defining chiral fermions on the lattice (see e.g. [185] and references therein).

On the other hand, the Banks-Casher relation states that the density of quasi zero modes is proportional to the chiral condensate [186], a clue towards explaining chiral symmetry breaking (see e.g. [109] and references therein). These quasi zero modes are expected to emerge from exact zero modes of instantons, when the latter form an

[23] In fact, for (anti-)selfdual fields $(i\not{D}_A)^2$ reduces to $-\mathrm{D}_A^2$ on the left (right) handed subspace.

ensemble. Furthermore, confinement and chiral symmetry breaking are claimed to have the same critical temperature [187]. Therefore, it is physically reasonable to search for *fermionic zero modes in Abelian projections.* The best candidates are of course Abelian projected instantons. For symmetry reasons, we investigate the single instanton in regular gauge. The projected field reads,

$$a \equiv (A^{\rm reg})_3 = \frac{2\eta^3_{\mu\nu}x_\nu}{r^2+\rho^2}\,{\rm d}x_\mu\tau_3\,. \tag{3.145}$$

It has a different topological density then the full instanton field $A^{\rm reg}$ (cf. (2.57)),

$$\frac{1}{8\pi^2}\,{\rm tr}\,f\wedge f = \frac{2}{\pi^2}\frac{\rho^2}{(r^2+\rho^2)^3}\,{\rm d}V(\mathbb{R}^4)\,, \tag{3.146}$$

but still instanton number one. This fact is not a coincidence and will become clearer upon discussing the Chern-Simons form in Section 4.3.3. On the other hand, self-duality is lost and the action becomes infinite[24]. This is to be expected, else the Uhlenbeck argument would lift the configuration a to a reducible instanton on S^4, which does not exist (see Section 4.2). The situation is the one of the LAG with $R < \rho$ ($n = \tau_3$, cf. Section (3.4.1)) with a defect at infinity.

Still the symmetry is sufficiently high to solve the zero mode equation exactly, the expression for $(i\not{D}_a)^2$ has a structure similar to (3.130). As a preliminary result we quote, that there are no zero modes in the fundamental, but in higher representations. The latter come in the same chiral sector as the ones from the original instanton background.

One might further ask whether the index theorem survives the Abelian projection. In order to answer this question we have to consider the index theorem for manifolds with boundaries [188]. In this case, a correction from the boundary, the η-invariant, has to be added to the instanton number to yield the index. The η-invariant has been computed for a very similar gauge field in [189] and its contribution points into the same direction as the explicit solutions. This is work in progress with A. Kirchberg and A. Wipf.

[24] Put differently, the instanton number of this configuration exhibits Abelian dominance, while the self-duality relation and the action do not.

4. Defects

In the last chapter we have investigated various Abelian gauges in detail, but only sketched the appearance of defects. Recall that *defects* are positions $\bar{x}$ in space-time, where the gauge transformation achieving the Abelian gauge becomes *ill-defined*. Put differently, at the defect manifold the residual gauge freedom is *locally enhanced*.

In this chapter we will perform a general analysis of the topological properties of defects by means of Higgs fields. We have already made use of two kinds of Higgs fields, an unconstrained one called ϕ (in the LAG) and a normalised one called n (in the MAG). Hence, the defects appear as zeros of ϕ and singularities of n, respectively. From the FP determinant (3.20) it follows immediately that *defects are located at the Gribov horizon* [15], with the qualifications discussed in that section.

The outline of this chapter is the following: We first quote the well-known arguments, why generic defects are magnetic monopoles characterised by their magnetic charge. Being measured near the defect manifold, we refer to this property as 'local'. Globally, the defects should account for a Hopf invariant which is related to the instanton number of the background gauge field. When relating local and global properties, we will find that non-trivial configurations enforce the existence of defects. It turns out that additional topological quantities, like twists, have to be present. We shall demonstrate this for an example. An outlook concerning center vortices is given at the end.

4.1. Local Properties

4.1.1. Generic Defects as Magnetic Monopoles

Generic defects in four dimensions are closed wordlines of magnetic monopoles carrying unit magnetic charge [33]. There are several reasons supporting this statement: First, there are three equations $\phi^a(\bar{x}_\mu)=0$ defining the defect locations on a four-dimensional manifold. Therefore, the defect manifold is generically a set of lines (it has co-dimension three). Second, we can expand the Higgs field around its zeros,

$$\phi(x) = M^{a\mu}(\bar{x})(x-\bar{x})_\mu \tau_a + O((x-\bar{x})^2)\,, \qquad M^{a\mu}(\bar{x}) = \frac{\partial \phi^a}{\partial x_\mu}(\bar{x})\,. \tag{4.1}$$

The gradient M is a three-by-four matrix, which generically (i.e. if there are no further restrictions) has rank three. Hence, by a coordinate transformation one can bring M to a form where all entries of one column vanish. The Higgs field is independent of that direction, it stays zero along the above mentioned line.

At this stage we simplify the analysis by assuming that the Higgs field is static and that the defect lies at the spatial origin,

$$\bar{x} = (\vec{0}, x_4) \Rightarrow \phi(x) = M^{ab} x_b \tau_a \,. \tag{4.2}$$

Consider the plane $x_4 = 0$ such that M is just a constant matrix. In that plane, we define the normalised Higgs field n on a two-sphere surrounding the origin,

$$n(x) : \; S^2_{x_4=0,\,|\vec{x}|=\text{const.}} \to S^2 \,. \tag{4.3}$$

The generalisation to an arbitrary defect line is straightforward: one has to go to *the two-sphere in the three-dimensional hyperplane perpendicular to the defect line.* At this stage it is obvious why magnetic monopoles are not genuine parts of pure YM theories: the mappings $S^2 \to SU(N)$ are trivial and cannot give rise to topological objects.

Third, if M were the identity matrix, the Higgs field n would be just the hedgehog field discussed in Section 2.3.2. One can show that M is plus or minus the identity up to topological deformations ([33] and references therein). That is, n covers the image sphere in colour space exactly once and *has winding number* ± 1. The conservation of this integer forces the defect lines to be either *closed loops* or to extend to infinity.

Though we did not specify the gauge field A, the situation is exactly that of the 't Hooft-Polyakov monopole (in the BPS limit). The only difference is that in that model the Higgs field is part of the field content, while in Abelian projections it is artificially introduced via gauge fixing. In terms of the Abelian projection, radial and unitary gauge amount to the decomposition in the body and space fixed frame, respectively. The (n-projected) flux in the body fixed frame is the winding number of n (cf. (3.35)),

$$\int_{S^2} (F(A^{\|n}), n) = \int_{S^2} G = \int_{S^2} (n, i\mathrm{d}n \wedge \mathrm{d}n) = \deg(n) = \pm 1 \,. \tag{4.4}$$

n cannot be diagonalised smoothly due to a topological obstruction: Its winding number is conserved under smooth gauge transformations. Hence, the gauge transformation to the trivial mapping $n = \tau_3$ cannot be smooth. From the discussion of the 't Hooft-Polyakov monopole we know what happens in the space fixed frame: *after diagonalising n, the defect becomes a Dirac monopole with Dirac string.* The set of all Dirac strings forms a two-dimensional *Dirac sheet* bounded by the worldline of the monopole. Since the sheet can be moved around by singular Abelian gauge transformations, its position is ambiguous. But its presence is unavoidable, unless one uses the bundle approach. Then one has to excise the monopole worldlines from the base manifold (space-time) for each configuration.

To illustrate these points, we analyse an example obtained from the LAG in a non-trivial background (Section 3.4.1) in more detail. It is helpful to introduce polar angles in colour space,

$$n = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix}, \qquad \alpha \in (0, 2\pi)\,, \;\; \beta \in (0, \pi)\,. \tag{4.5}$$

The topological density is given by $(n, i\mathrm{d}n \wedge \mathrm{d}n) = \sin\beta\mathrm{d}\beta \wedge \mathrm{d}\alpha$, and the diagonalising gauge transformation becomes,

$$g = e^{i\gamma\tau_3} e^{i\beta\tau_2} e^{i\alpha\tau_3}\,, \tag{4.6}$$

with the residual Abelian gauge freedom inherent in γ.

4.1.2. Example: A Small Monopole Loop

The defect manifold for a perturbed instanton in the LAG is the circle $C: r_{12} = R$, $r_{34} = 0$. From the perturbation (3.137) of the Higgs field ϕ we read off the angles of n,

$$\alpha = \varphi_{12} - \varphi_{34}\,, \qquad \beta = \arctan\frac{r^2\sin(2\theta)}{r^2\cos(2\theta) - R^2} = \arctan\frac{2r_{12}r_{34}}{r_{12}^2 - r_{34}^2 - R^2}\,. \tag{4.7}$$

It turns out that this Higgs field coincides with the one considered in [141, 190],

$$\beta = \theta_+ + \theta_-\,, \qquad \tan\theta_\pm = r_{34}/(r_{12} \pm R)\,. \tag{4.8}$$

Lines of constant β are depicted in Figure 4.1. From this figure and (4.7) one infers that, right at the loop, both angles α and β are singular, while in a vicinity perpendicular to the loop (at fixed worldline coordinate φ_{12}) they *take on all possible values exactly once.* Hence, n has unit winding number. For an explicit check the form (4.8) is best suited: Close to the worldline we can set θ_- to zero, while the two-sphere perpendicular to the circle is parametrised by $\theta_+ \in (0, \pi)$ and $\varphi_{34} \in (0, 2\pi)$. Then,

$$\deg(n) = \frac{1}{4\pi}\int_{S^2} \sin\beta\mathrm{d}\beta \wedge \mathrm{d}\alpha = \frac{1}{4\pi}\int_0^\pi \sin\theta_+\mathrm{d}\theta_+ \int_0^{2\pi} \mathrm{d}\varphi_{34} = 1\,. \tag{4.9}$$

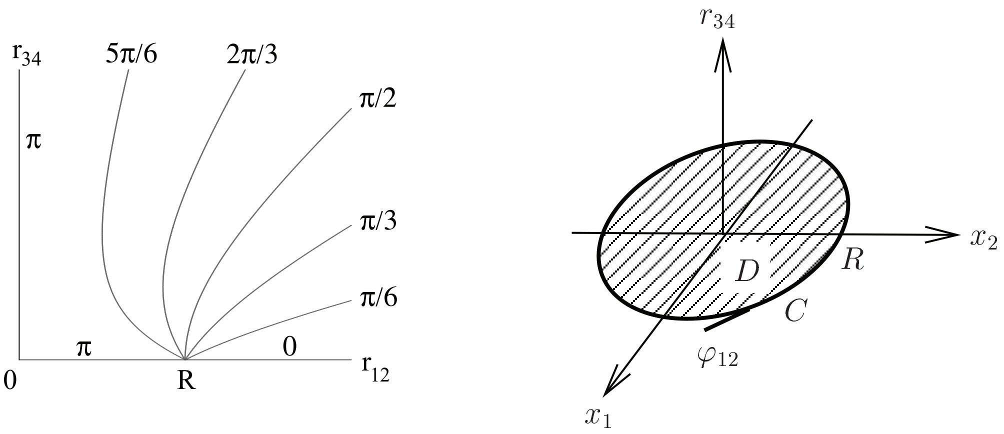

Fig. 4.1.: Left: Lines of constant β for a monopole loop at $r_{12} = R$, $r_{34} = 0$ (cf. (4.7), same as in [141, 190]). Right: With our choice (4.10), the Dirac sheet is the disc D in the x_1x_2-plane.

This is the generic situation; after diagonalisation there is a unit charge magnetic monopole, its worldline being the circle C. With the choice,

$$\gamma = \varphi_{12} + \varphi_{34}\,, \tag{4.10}$$

the Dirac sheet is the disc $D:\ r_{12} \leq R,\ r_{34} = 0$ (see Figure 4.1).

At this point we are ready to discuss a sample of lattice results that support the mechanism of Abelian projections.

4.1.3. Monopoles on the Lattice

The idea of Abelian projections has been successfully tested on the lattice. The main achievements go under the names of *Abelian dominance* [191] and *monopole dominance* [192]. In analogy with the Cartan decomposition in the continuum, the group valued link variables U are written as products of an off-diagonal coset part and a diagonal Abelian part $u = \exp(i\theta\tau_3)$. Abelian dominance means that about 92% of the string tension[1] in the heavy quark potential (1.3) is reproduced upon replacing the full field by the Abelian field. It confirms the validity of Abelian projections a posteriori.

Furthermore, the Abelian field strength[2] $f = \mathrm{d}\theta$ on each plaquette can be decomposed into a fractional part $\bar{f}$ and an integer part m, respectively,

$$f = \bar{f} + 2\pi m\,, \qquad \bar{f} \in (-\pi, \pi)\,,\ \ m \in \{-2, -1, 0, 1, 2\}\,. \tag{4.11}$$

[1] Abelian dominance does not hold for the self-energy and the Coulomb part [149].
[2] For the definition of forms on the lattice see e.g. [193].

The latter measures the magnetic flux through the plaquette, i.e. the presence of Dirac strings. Like in the continuum, endpoints of Dirac strings are magnetic monopoles, and the monopole current on the lattice is $k = *\,\mathrm{d}m$. The magnetic charge is conserved (and bounded), thus, monopole worldlines are closed (on the dual lattice). This operational definition of magnetic monopoles, called the *elementary cube procedure*, is due to DeGrand and Toussaint [26]. Monopole dominance means that 95% of the Abelian string tension is reproduced by keeping only the monopoles, i.e. the singular fields. It has also been checked that the photons, i.e. the regular fields, do not confine.

Monopole condensation in the YM vacuum has been shown considering percolation properties of monopole currents [194, 195, 196, 197], and the thermal partition function in a monopole background[3] [198]. In the language of statistical physics, condensation is intimately connected to an *order parameter*, the vacuum expectation value of some operator. A monopole creation operator for compact electrodynamics has been given by Fröhlich and Marchetti [199]. For YM theories, several versions of such an operator have been suggested [200, 201, 202, 203]. Numerical simulations show that their vacuum expectation values indeed serve as order parameters for monopole condensation and hence dual superconductivity: They have a non-vanishing value in the confining (broken) phase and vanish above the deconfinement phase transition. A constraint effective potential behaves accordingly [204]. The dual superconductor is claimed to be of second kind [205]. Effective actions for monopole currents have been derived on the lattice as well [197].

However, the weakness of these findings is at least four-fold: First, the underlying ensemble has been generated with the Monte Carlo method using the full $SU(2)$ Haar measure. In a path integral language this means that the replacement of the full fields by the Abelian/monopole fields takes place only in the observables, but not in the measure. Second, the Gribov problem affects the value of the string tension by about 10%. The better the MAG is fixed, i.e. the bigger the MAG-functional, the less monopoles and the smaller the string tension [149, 150, 148]. We will come back to this point (in the continuum) in the next section. Third, the mechanism of Abelian projections has conceptual problems with sources in the adjoint representation [206, 149, 207, 208]. Fourth, the monopoles from the elementary cube procedure are not necessarily the defects of the MAG. In fact, one cannot really find the latter, since the notion of singularities in n (or g) makes no sense on the discretised lattice. A way out is provided by the LAG: the position of the monopoles in MAG and LAG are correlated [8]. For the LAG one

[3] If monopoles are condensed it will cost no free energy to add another one.

can define defects as regions of low modulus $|\phi|$, which are indeed correlated with the elementary cube monopoles [140, 209]. Still, very small defects will fall through the lattice.

Since numerical results cannot replace physical insights, we will continue by analysing the role of defects in the continuum.

4.1.4. Defects in Different Abelian Gauges: A Synopsis

In the continuum, the defects of the Polyakov gauge are best understood. They occur whenever the group-valued Polyakov loop $\mathcal{P}(\beta,\vec{x})$ becomes $\pm\mathbb{1}_2$ [133]. Notice that this is a gauge invariant statement, hence one can predict the position of the defects without actually transforming to the PG. Equivalently, the algebra-valued Higgs field ϕ from (3.97) vanishes. Since these fields do not depend on x_4, *the defects of the Polyakov gauge are static*. This is of course a remnant of the particular gauge condition (3.89). Again, generic defects are unit charge magnetic monopoles[4]. The hedgehog behaviour can be easily checked [210] for the Harrington-Shepard instanton [211] and has been elaborated in detail for the constituent monopoles of the caloron [212]. On a space-time of finite volume like the four-torus, the magnetic charges obey an overall charge neutrality, and the Dirac strings form a network between the monopoles [136]. The monopoles are located on the Gribov horizon, because the FP determinant (3.95), being the reduced Haar measure, vanishes exactly where the Polyakov loop has degenerate eigenvalues.

The few analytical results about defects in the LAG are collected in Section 3.4.1. For the MAG the situation is even worse. The small monopole loop from above as well as a static monopole were put in by hand into the singular instanton background [141, 213]. Against the hope of the authors, however, *the MAG suppresses these monopole loops*; the MAG-functional is larger than for the singular instanton[5], or it even diverges. It seems to *prefer singularites with large supports*: After all, the small monopole loop provides a transition from the singular gauge ($n \sim \tau_3$) at large distances to the regular gauge ($n \sim n_{\rm H}$) inside the loop [214]. As we allow for singularities anyhow, let us define the following (naively patched, or 'bundle') configuration,

$$A = \begin{cases} A^{\rm reg} & (n = \tau_3) \quad \text{for } r < \rho\,, \\ A^{\rm sg} & (n = n_{\rm H}) \quad \text{for } r > \rho\,. \end{cases} \tag{4.12}$$

[4] The Abelian instantons (2.49) are non-generic as the Polyakov loop equals $\mathbb{1}_2$ on higher-dimensional defect manifolds (two-dimensional walls, $x_3 =$, const. or even everywhere in space).

[5] One might speculate whether a perturbation of the single instanton could stabilise the monopole loop.

which jumps by a gauge transformation on a whole three-sphere $S^3_{r=\rho}$. Nevertheless, it leads to a value of the MAG-functional which is even lower than the one for the singular instanton[6]. The reason is that the density of the MAG-functional in the two gauges differs by a factor r^2/ρ^2 and is lower for $A^{\rm reg}$ ($A^{\rm sg}$) in the region $r < \rho$ ($r > \rho$) [141]. Squeezing the singularity from the three-sphere into the monopole loop, the MAG-functional blows up again, exceeding the value for the singular instanton.

Nonetheless, a monopole loop clearly shows up in the LAG (on an orbit close to the single instanton orbit) which supports the remark made in the beginning of Section 3.4.1.

We have seen that *defects depend on the chosen Abelian gauge*. This is also a problem on the lattice. As expected, there are always slightly more monopoles in the LAG than in the MAG [146, 8]. Nevertheless, these two gauges are very similar w.r.t. properties like Abelian dominance, total monopole length and space-time asymmetry [215]. For the Polyakov gauge, Abelian dominance is trivial when extracted from Polyakov line correlators, but absent for Wilson loops [216]; monopole dominance holds [217]. The total monopole loop length and space-time asymmetry in this gauge do not change at the deconfinement phase transition, so that the monopoles in this gauge 'apparently lack a dynamical relevance for confinement' [8]. There are also negative results about the Abelian gauge diagonalising a field strength component [200, 201] and an anisotropic version of the MAG [207], confronted with encouraging results about an interpolation between the MAG and no gauge fixing [218]. Altogether, the gauge dependence of the Abelian projections remains controversial in the lattice community, and it seems that every author is drawing his or her own conclusions.

An argument in favour of Abelian projected monopoles as physical objects is the correlation with gauge invariant quantitites. Monopoles contribute significantly to the instanton number [219]. The probability of finding monopoles increases *locally* with the value of the topological charge and action density [220, 221, 222, 223]. Put differently, monopoles come with an excess of topological charge and action, which decreases in the deconfinement phase [209]. Together with the observation that strong gauge fields are nearly (anti)-selfdual [215], monopoles are rather dyons [224]. In order to better understand the correlation between instantons (lumps of topological charge) and monopoles, equilibrium configurations were cooled down to semi-classical ones. The surviving monopole loops are clearly correlated to the positions of the instantons, both in the MAG [225, 224, 226] and in the LAG [227][7]. The formation of monopole loops has also been investigated from the other extreme: Single instantons (and merons [228])

[6] To obtain the lowest value of the functional, even the transition region is determined to be at $r = \rho$.

[7] on top of center vortices

contain single small planar monopole loops (cf. the perturbed instanton in the LAG). Monopole loops of (anti-)instanton pairs are liberated to form one large loop enclosing the instanton centers. This process was found to go on for three and more instantons with a specific group orientation [225].

A similar effect has been observed for an instanton-antiinstanton pair in the continuum. This configuration has a *common monopole loop surrounding the two cores* as shown numerically for the MAG [141] and the LAG [170].

The implication of this correlation for the dynamics of the theory could be that *instantons induce confinement indirectly via the creation of large monopole loops*[8]. However, most of the lattice simulations within the dilute instanton gas [229] and the instanton liquid [230] show, that instantons are not the most relevant configurations for confinement ([231, 232, 233, 234, 235, 236, 237, 238], see e.g. [239, 240]). One should keep in mind, though, that instantons, i.e. lumps of action/topological charge, become indistinguishable from perturbative fluctuations, if coming close to each other [241, 242, 243].

We will come back to the intimate relation between monopoles and instantons in Section 4.3; before that, however, we have to discuss the analogue of the instanton number (i.e. the global properties) for the Higgs field.

4.2. Global Properties

In order to discuss global properties it is appropriate to use the language of fibre bundles. Both kinds of Higgs fields describing Abelian gauges live on bundles associated with the principal bundle. The demand for the same transition functions ensures that on the overlap of two charts, the gauge fields A_i and A_j will lead to the same gauge fixed configuration $A_{\mathrm{AG}} = {}^{g_i}A_i = {}^{g_j}A_j$, up to the residual Abelian gauge freedom $g \to hg$.

The fibre of the ϕ-bundle is the full Lie algebra $\mathcal{G} = su(2) \cong \mathbb{R}^3$. As always, such a vector bundle is characterised by its transition functions, but not by the existence of sections: any vector bundle admits a global section, the *null section.* Although the use of ϕ is advantageous in practical computations (smoothness, LAG vs. MAG), for the topological description one better passes to the normalised field $n = \phi/|\phi|$. The fibre of the n-bundle is the coset $G/H = SU(2)/U(1)$, represented by normalised elements of the Lie algebra $S^2 \subset su(2)$; a null section does not exist.

We start the investigations by stating a related fact, namely that on the four-sphere *the instanton number of every reducible connection vanishes* [87, 73]. In other words,

[8] which is equivalent to monopole condensation in the QCD vacuum.

there are no reducible instantons. In order to verify this statement, one has to check whether non-trivial transition functions can take their values only in $U(1)$ (or even Z_2 for extremely reducible connections). On the four-sphere, there is just one transition function t, being a mapping from S^3 into the group $SU(2) \cong S^3$ (cf. Section 2.3.3). Transition functions whose images are reduced to a subgroup of $SU(2)$ cannot cover the whole S^3 anymore. Thus, they cannot have a winding number $\pi_3(S^3) = \mathbb{Z}$, so that they have vanishing instanton number. To be explicit, the topological density $\mathrm{tr}\,(t\mathrm{d}t^\dagger)^3$ in (2.48) vanishes for an Abelian function, say $t = \exp(i\lambda\tau_3)$, due to both the colour structure, $\mathrm{tr}\,(\tau_3)^3 = 0$, and the wedge product, $\mathrm{d}\lambda \wedge \mathrm{d}\lambda = 0$. In other words, the trivial mappings $S^3 \to U(1)$ induce trivial reduced bundles.

Accordingly, there is no smooth section n (cf. Section 2.1.3). This is the first *correlation between instantons and defects* we have encountered: *For non-trivial backgrounds on the four-sphere, there have to be defects in any Abelian gauge.* Following the considerations above we can even conclude that *there have to be defects for every irreducible background.*

We stress that similarly strong statements cannot hold for the torus. Indeed, the Abelian instantons of Section 2.3.3 have a reduced holonomy group – the maximal Abelian subgroup – and obey Abelian transition functions. Accordingly, the simple choice $n = \tau_3$ is a proper[9] Higgs field *without any defects.* In fact, τ_3 serves both as the MAG-Higgs field (since the configuration is already in the MAG) and the LAG-Higgs field (since, due to the reducibility relation, $\mathrm{D}_A\tau_3 = 0$, τ_3 is the ground state). The topology of the transition functions can be understood as follows: Without loss of generality, two of the four transition functions of the torus can be chosen as the identity. The instanton number receives contributions from integrals of the remaining transition functions over three- and two-subtori, respectively [244, 136]. Two Abelian transition functions, say $t_1 = \exp(i\lambda_1\tau_3)$ and $t_2 = \exp(i\lambda_2\tau_3)$, can contribute to the instanton number[10] since $\mathrm{tr}\,(\tau_3)^2 \neq 0$, and there is a non-vanishing two-form $\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \neq 0$ to be integrated over. This fact is quite plausible since *there are* non-trivial mappings $\mathbb{T}^n \to U(1) \cong S^1$ which can prevent the reduced bundle from being trivial.

Back on the sphere, the Higgs fields on the overlap are mappings from S^3 to S^2 which are characterised by the integer *Hopf invariant.* This invariant provides some insight about what happens in the transition region as will be explained in the following section. Since this invariant may not be of common knowledge, we begin with a brief introduction.

[9] with the same transition functions

[10] Note, however, that Abelian transition functions automatically lead to *even* instanton numbers [136].

4.2.1. The Hopf Invariant

The Hopf invariant $H(n) \in \pi_3(S^2) \cong \mathbb{Z}$ characterises mappings n from the three-sphere to the two-sphere. Its definition via cohomology is fairly abstract. It starts with the pullback[11] of the volume form on S^2,

$$\eta_{(2)} \equiv n_*(\mathrm{d}V(S^2)) = (n, i\,\mathrm{d}n \wedge \mathrm{d}n)\,. \tag{4.13}$$

Integrating this two-form over M gives the degree of n, if n is a mapping from a *two*-dimensional manifold M into the two-sphere (cf. the Brouwer degree in the monopole case, Section 2.3.2). For the Hopf invariant, however, one needs a three-from to be integrated over. A one-form $\eta_{(1)}$ with $\mathrm{d}\eta_{(1)} = \eta_{(2)}$ exists because the volume form on S^2 is obviously closed, and so is its pullback $\eta_{(2)}$; and the second cohomology group of the three-sphere is trivial[12]. Now $\eta_{(1)} \wedge \eta_{(2)}$ is the desired three-form defining the Hopf invariant,

$$H(n) = \frac{1}{16\pi^2}\int_{S^3} \eta_{(1)} \wedge \eta_{(2)}\,, \qquad \mathrm{d}\eta_{(1)} = \eta_{(2)}\,. \tag{4.14}$$

A more geometrical picture is based on *regular points* on the image sphere, the preimages of which are loops [245]. Accordingly, the preimages of two regular points on S^2 under n are two loops on S^3 which do not intersect. The Hopf invariant is *the linking number of these two loops*, irrespective of the chosen points on S^2. The linking number can be understood from Biot-Savart's and Ampere's laws of electrodynamics [101].

An exact sequence $0 \to \pi_3(S^2) \to \pi_3(S^3) \to 0$ relates the Hopf invariant to an ordinary winding number (cf. [245]). Not surprisingly, the latter is *the winding number of the gauge transformation g that diagonalises n*,

$$n = g^\dagger \tau_3 g\,, \qquad H(n) = \deg(g)\,. \tag{4.15}$$

As before, g is a mapping from S^3 to $SU(2)$, defined up to rotations around τ_3.

Notice that the Hopf invariant is not independent of gauge transformations and can be transformed away *smoothly*. This is at variance with the monopole charge, which is gauge invariant and can be transformed away only at the expense of introducing Dirac strings. The difference is plausible: in the monopole case, g cannot carry the topology of n, since the mappings $S^2 \to SU(2)$ are trivial.

In the next section we analyse the implications of the Hopf invariant for the defects.

[11] This is the only place where we explicitly write the pullback, in principle it is also present for all winding numbers.

[12] at variance with the torus, where $H^2(\mathbb{T}^3) \cong \mathbb{R}^3$ due to non-contractible circles.

4.3. Relating Local and Global Properties

With the help of the Hopf invariant we will first show the inevitability of defects in non-trivial backgrounds. From its connection (4.15) to a winding number, the Hopf invariant inherits the following property: If n is smooth inside a topologically trivial four-volume V^4, it has no Hopf invariant at the boundary $\partial V^4 \cong S^3$. (Contracting the latter to a point, one immediately sees that g approaches a constant and hence is trivial.) Furthermore, if the two Higgs fields n_N and n_S are connected by the transition function t, the degree of t is the difference of their Hopf invariants[13],

$$n_H = {}^t n_S\,, \qquad H(n_N) - H(n_S) = \deg(t)\,. \tag{4.16}$$

As a consequence, at least one of the Higgs fields has to have a Hopf invariant, hence a defect in its chart, if $\deg(t) = \nu(A) \neq 0$. (In a similar way, the Higgs field of the 't Hooft-Polyakov monopole has to vanish somewhere in order to develop a magnetic charge.) Thus, *the Hopf invariant proves that defects are unavoidable in non-trivial backgrounds.*

For the same reason, the Hopf invariant has a *residue property*: the total Hopf invariant is the sum of Hopf invariants measured on little three-spheres around each defect [190]. In principle, we can discriminate between a number of necessary defects (to generate the instanton number) and additional defects (the Hopf invariants of which cancel in the sum); the latter are to be expected also in the perturbative sector. This property suggests a 'localisation of the topology', meaning *the instanton number of the background should be determined by the local properties of an ensemble of defects.*

A similar localisation was observed in the Polyakov gauge by a number of authors before [246, 247, 248, 136, 210, 76, 159]. In this gauge, the instanton number is related to the magnetic charges of the static monopoles,

$$\nu(A) = \sum_{\mathcal{P}(\beta,\vec{x})=-\mathbb{1}_2} \deg(n)|_{\vec{x}} = \sum_{\mathcal{P}(\beta,\vec{x})=-\mathbb{1}_2} q_{\text{mag}}|_{\vec{x}}\,. \tag{4.17}$$

However, we cannot expect such a simple relation to hold for other Abelian gauges, and especially not on the four-sphere S^4.

The rest of this section is devoted to the relation between global and local properties of defects – Hopf invariant vs. magnetic charge – for arbitrary Abelian gauges. To gain more intuition we first study pointlike defects.

[13] Thus, in (4.15), we could have chosen any constant instead of τ_3; we continue with diagonal matrices.

4.3.1. Pointlike Defects and the Standard Hopf Map

For pointlike defects the first order Taylor expansion (4.1) is not sufficient[14]. Accordingly, viewing the normalised field n as a mapping from a two-sphere like in (4.3) does not make sense[15]. Instead, n has to be defined on a *three*-sphere in the four-dimensional space-time, $n:\ S^3_{r=\text{const.}} \to S^2$. It follows that pointlike defects are characterised by the Hopf invariant 'already locally'. If there is no second defect present, the Hopf invariant will naturally 'evolve' from the little three-sphere around the origin to the large three-sphere, where the transition takes place.

Pointlike defects, being non-generic, are related to highly symmetric backgrounds. We have explicitly shown the occurence of a pointlike defect in the LAG in the single instanton background[16], see Section 3.4.1. In fact, the field ϕ in (3.132) vanishes quadratically at the origin, and the normalised Higgs field n is just the *standard Hopf map* [249],

$$n_{\mathrm{H}}(\hat{x}_\mu) = \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_4 + \hat{x}_2\hat{x}_3) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix}, \qquad \hat{x}_\mu = x_\mu/r\,. \tag{4.18}$$

With the help of the polar coordinates of Section 4.1.1, this standard mapping is simply given by,

$$\alpha_{\mathrm{H}} = \varphi_{12} - \varphi_{34}\,, \qquad \beta_{\mathrm{H}} = 2\theta = \arctan\frac{2r_{12}r_{34}}{r_{12}^2 - r_{34}^2}\,. \tag{4.19}$$

The lines of constant β are displayed in Figure 4.2. Recalling that the standard Hopf map is diagonalised by the gauge transformation $\hat{g}$ (cf. Section 3.4.1 and A.3), the third angle γ reads,

$$\gamma = \varphi_{12} + \varphi_{34}\,. \tag{4.20}$$

It is interesting to see how the standard Hopf map fits into the different definitions of the Hopf invariant. We start by noting that all image points are regular. Thus, n_{H} solves the problem of *filling the (compactified) three-space with non-intersecting loops, such that any two loops link just once.* The pre-image of the north pole, $\beta_{\mathrm{H}} = 0$, and the south pole, $\beta_{\mathrm{H}} = \pi$, are the loops ($r_{34} = 0$, $r_{12} = \text{const.}$) and ($r_{12} = 0$, $r_{34} = \text{const.}$), respectively. When S^3 is decompactified to $\dot{\mathbb{R}}^3$ along the fourth direction, the former is

[14] since it would just give zeros on lines like for monopoles.

[15] since any two-sphere in four dimensions can be contracted to a point using the fourth dimension.

[16] All investigations of this background in the MAG suggest the same to happen there: the singular instanton, if really being the gauge fixed configuration, has a pointlike singularity at the origin.

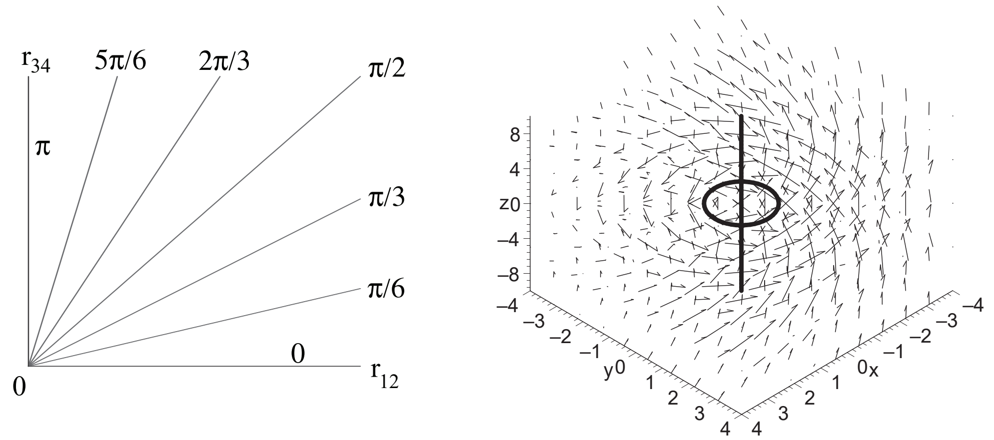

Fig. 4.2.: Left: Lines of constant β for the pointlike defect, being the standard Hopf map (cf. (4.19)). Right: A vector plot of the standard Hopf map on $\mathbb{R}^3$ as decompactified S^3, together with the preimages of the north pole (circle) and the south pole (line).

a circle in the x_1x_2-plane, while the latter turns into the x_3-axis. They are obviously linked once (Figure 4.2). For other preimages the visualisation of the linking is a bit more complicated.

The ground state soliton of the Faddeev-Niemi model is of this form, with 'north' and 'south' interchanged [50]. By definition, the position of the soliton is the pre-image of the south pole, since asymptotically n points towards the north pole and the FN energy measures the gradient of 'spins' n. The $H = 1$ soliton maps a planar circle to the south pole, therefore it is called *torus shaped un-knot soliton*.

In this context, let us investigate the symmetry of the standard Hopf map. It is inherited from the symmetry of $\hat{g}$, but partially broken by referring to τ_3 in the relation $n_{\rm H} = \hat{g}\tau_3\hat{g}^\dagger$. (From the point of view of a solution of the covariant Laplacian: the standard Hopf map inherits the symmetry of the background $A_{\rm inst}$, but we choose a particular mapping from the multiplet (A.25)). Plugging the $SO(4)$ symmetry (2.62) of $\hat{g}$ into this relation, the anti-selfdual (infinitesimal) rotations lead to the correct change of $n_{\rm H}$ given by the commutator,

$$\delta n_{\rm H} = [i\lambda, n_{\rm H}]\,, \qquad \lambda = 2\,\omega_a\tau_a \qquad \text{for } \ \omega_{\mu\nu} = \omega_a\,\bar{\eta}^a_{\mu\nu}\,. \tag{4.21}$$

The symmetry group is an $SO(3)$ subgroup of $SO(4)$. The selfdual rotations cannot be compensated unless λ *commutes with* τ_3, in which case,

$$\delta n_{\rm H} = 0 \qquad \text{for } \ \omega_{\mu\nu} = \omega\,\eta^3_{\mu\nu}\,. \tag{4.22}$$

This leaves only an $SO(2)$ symmetry group. Upon projecting the space-time S^3 stereographically onto $\mathbb{R}^3$ along the fourth direction, all symmetries containing x_4 (the 'Euclidean boosts') are lost. They correspond to the difference of selfdual and anti-selfdual rotations. Still, the sum of them is unbroken, but only as an axial symmetry due to the considerations above. This visualises the fact that non-trivial Hopf maps are at most axially symmetric.

For the standard Hopf map, the two-form $\eta_{(2)}$ (4.13) from the definition of the Hopf invariant becomes

$$\eta_{(2)} = \sin\beta \mathrm{d}\beta \wedge \mathrm{d}\alpha = 2\sin(2\theta)\mathrm{d}\theta \wedge \mathrm{d}(\varphi_{12} - \varphi_{34})\,. \tag{4.23}$$

Notice that the polar angles, which suffer from singularities, combine to an entirely smooth form. This effect also restricts the one-form $\eta_{(1)}$ to be[17]

$$\eta_{(1)} = -\cos(2\theta)\mathrm{d}(\varphi_{12} - \varphi_{34}) - \mathrm{d}(\varphi_{12} + \varphi_{34})\,. \tag{4.24}$$

The resulting three-form is simply the volume form on the preimage three-sphere,

$$H(n_{\mathrm{H}}) = \int_{S^3} 2\sin(2\theta)\mathrm{d}\theta \wedge \mathrm{d}(\varphi_{12} + \varphi_{34}) \wedge \mathrm{d}(\varphi_{12} - \varphi_{34}) = \int_{S^3} \mathrm{d}V(S^3) = 1\,. \tag{4.25}$$

The winding number of the diagonalising gauge transformation $\hat{g}$ is obviously one. In accordance with our general considerations on gauge invariance in Section 4.2.1, the gauge transformation $\hat{g}$ is singular only at the origin and does not induce further 'Dirac strings'. By fictitiously switching the perturbation of Section 3.4.1 off or on, one can view pointlike defects as *monopole loops shrunk to points*[18] or as *seeds for monopole loops*.

This picture will become important in the next section.

4.3.2. Small Monopole Loops: Twist

The relation between local and global properties for the small monopole loop is easy to explain. The loop emerges upon perturbing the pointlike defect (the standard Hopf map). Such a local perturbation does not change the global properties. In more technical terms, the two angles α and β describing these Higgs fields coincide exactly or at least asymptotically (cf. (4.7) and Figure 4.1 vs. (4.19) and Figure 4.2). As discussed in Section 4.1.2, the monopole charge (the hedgehog shape) of the Higgs field comes from the dependence of n on φ_{34} and θ, being coordinates perpendicular to the loop. For the integrand of the Hopf invariant, however, one needs a three-form. That is, the dependence of n on the monopole worldline coordinate must be used. Indeed, from the

[17] up to an exact form which does not contribute to the Hopf invariant.

[18] The Dirac sheets shrink to points as well.

relation to the pointlike defect we have $\alpha = \alpha(\varphi_{12})$. This dependence induces a *twist* [250]: *The hedgehog rotates around the third colour axis while moving along the loop.* In our case, it performes (uniformly) just one full rotation, $\alpha \sim \varphi_{12}$. The associated relation [239],

$$\nu(A) = H(n) = \text{magnetic charge} \times \text{twist}\,, \qquad 1 = 1 \times 1\,, \tag{4.26}$$

is just the simplest realisation of a formula given by Jahn [190]. Notice that on our choice of the Dirac sheet the Higgs field is constant (on the south pole, $\beta = \pi$, cf. Figure 4.1). It has been shown explicitly that, for the same monopole loop but a different choice of the Dirac sheet, a second term becomes relevant for the instanton number [190].

The physical meaning of the twist is most obvious for static monopoles, where the worldline coordinate is time [251]. A twisted gauge field is time-dependent and hence carries electric charge which is necessary to generate an instanton number, $\nu \sim \int \vec{E}\vec{B}\,\mathrm{d}V$.

4.3.3. Localisation via an Abelian Gauge Field

An important link between global and local properties of defects in terms of physical quantities is provided by an *auxiliary Abelian gauge field*,

$$a \equiv (ig\,\mathrm{d}g^\dagger)_3 = (ig\,\mathrm{d}g^\dagger, \tau_3)\,. \tag{4.27}$$

This field is the inhomogeneous part of the diagonal gauge field in the space fixed frame[19], $A^{||}_{\mathrm{AG}} = gAg^\dagger + a\,\tau_3$. Thus, under the residual Abelian gauge freedom $h = \exp(i\lambda\tau_3)$, it transforms according to (3.23),

$$g \to hg\,, \qquad a \to a + \mathrm{d}\lambda\,. \tag{4.28}$$

This gauge field is *not pure gauge*, its field strength being

$$\mathrm{d}a = f + f^{\mathrm{sg}}\,, \qquad f = (i\,\mathrm{d}g \wedge \mathrm{d}g^\dagger)_3 = (-ig\,\mathrm{d}g^\dagger \wedge g\,\mathrm{d}g^\dagger)_3\,, \qquad f^{\mathrm{sg}} = (ig\,\mathrm{d}^2 g^\dagger)_3\,, \tag{4.29}$$

where we have taken a singular part into account for later use. In terms of the angles (α, β, γ) from (4.6) these fields read,

$$a = \mathrm{d}\gamma + \cos\beta\,\mathrm{d}\alpha\,, \qquad f = -\sin\beta\,\mathrm{d}\beta \wedge \mathrm{d}\alpha\,, \qquad f^{\mathrm{sg}} = \mathrm{d}^2\gamma + \cos\beta\,\mathrm{d}^2\alpha\,. \tag{4.30}$$

We mention in passing that this so-called 'Clebsch form' is the most general form of an Abelian gauge field in three dimensions [252].

[19] As $a = (-ig^\dagger \mathrm{d}g, n)$, this field also appears in the body fixed frame, $A^{||n} = Cn + ig^\dagger \mathrm{d}g + a\,n$.

For the *global properties* we assume that the defects[20] do not extend into the transition region. Then the diagonalising gauge transfromations $g_{N,S}$ of the local Higgs fields $n_{N,S}$ are smooth there, and the instanton number is given by the difference of Hopf invariants or winding numbers on the transition three-sphere. To avoid writing these differences we assume one of the charts to be free of defects and restrict ourselves to the other chart in the following (i.e. we suppress chart indices N and S). For the same reason, we also assume that the full gauge field approaches a pure gauge at the three-sphere, $A \to ig\mathrm{d}g^\dagger$.

The crucial observation is now that *the instanton number of the full field A can be recovered from the Abelian field a.* We first stay on the three-sphere and consider the Chern-Simons numbers. For a it is defined via its embedding into the full group,

$$CS(a) \equiv CS(a\tau_3) = \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\,(a\tau_3 \wedge \mathrm{d}a\tau_3) = \frac{1}{16\pi^2}\int_{S^3} a \wedge f \tag{4.31}$$

The cubic term does not contribute due to the Abelian nature of a, while f^{sg} is absent by our assumptions. Using the relation [252, 190],

$$f = \frac{1}{2}\,\epsilon_{ab3}(g\mathrm{d}g^\dagger)_a \wedge (g\mathrm{d}g^\dagger)_b\,, \tag{4.32}$$

we find,

$$\begin{aligned} CS(a) &= \frac{i}{32\pi^2}\int_{S^3} \epsilon_{ab3}(g\mathrm{d}g^\dagger)_3 \wedge (g\mathrm{d}g^\dagger)_a \wedge (g\mathrm{d}g^\dagger)_b = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}\,(g\mathrm{d}g^\dagger)^3 \\ &= \deg(g) = CS(A)\,. \end{aligned} \tag{4.33}$$

To show that the Chern-Simons number of a non-Abelian pure gauge can be recovered from projecting this field to one of its components, involves a tricky combination of the quadratic and cubic term[21]. It explains why the instanton number of the regular instanton survives projecting this field onto its third component (see Section 3.6) and was found in the context of magnetic helicity, too [252].

The Chern-Simons number of a is related to the Hopf invariant of n by

$$f = -(n, i\mathrm{d}n \wedge \mathrm{d}n) = -\eta_{(2)}\,, \tag{4.34}$$

and $a = -\eta_{(1)}$ [190], hence $CS(a) = H(n)$.

Now we consider the 'bulk', i.e. the chart bounded by the three-sphere, where, by virtue of Stokes' theorem,

$$\nu(A) = CS(A) = CS(a) = \frac{1}{16\pi^2}\int_{S^3} a \wedge f = \frac{1}{16\pi^2}\int_{V^4} \mathrm{d}(a \wedge f)\,, \quad \partial V^4 = S^3\,. \tag{4.35}$$

[20] and possible Dirac sheets

[21] The quadratic term gives a contribution -1 for each component, while the cubic term contributes $+4$ for the full field only.

We know right from the beginning that g and hence a cannot be smooth inside the bulk V^4, else the winding number of g and the instanton number of A would vanish. In fact, one might be tempted to conclude that the integrand is the usual instanton density $f \wedge f$ for an Abelian field. But, in our case, this vanishes,

$$f \wedge f = 0 \,, \tag{4.36}$$

since f together with g is a function of three angles (α, β, γ) only and cannot constitute a four-form. Since the regular part in (4.35) vanishes, *the contribution to the instanton number of the background field reduces to the singularities of the gauge transformation g associated with the Abelian gauge, i.e. to the defects/monopole worldlines plus Dirac sheets.* For instance, the pointlike defects have $a \wedge f \sim \mathrm{d}V(S^3)$ and $\mathrm{d}(a \wedge f) \sim \delta^{(4)}(x)$, so that, the integrand is localised at the origin, the defect position. We stress that it is not the instanton density of the non-Abelian background field A that is localised, but the instanton density of an auxiliary Abelian field a representing the defects in an Abelian gauge[22].

Taking singularities into account, the instanton number splits into the sum of two terms,

$$\nu(A) = \frac{1}{16\pi^2} \int_{V^4} (f^{\mathrm{sg}} \wedge f - a \wedge \mathrm{d}f) \,, \tag{4.37}$$

where we already made use of (4.36). Both terms are absent in regions of space-time where g is regular since they are proportional to the operator d^2. In Section 2.3.1 we have shown that a non-trivial action of this operator gives rise to the Dirac sheet and the monopole current. The same happens here, as magnetic monopoles are the generic defects: f^{sg} and $\mathrm{d}f = *k$ are δ-distributions on the Dirac sheets and the monopole loops, respectively. Accordingly, in (4.37), *the contributions to the instanton number reduce to integrals of f over the Dirac sheets and to integrals of a over the monopole loops*, respectively.

At this point, a few words on singular forms are in order. δ-distributions of an n-dimensional hypersurface Σ in a d-dimensional manifold M are distribution-valued $(d-n)$-forms such that [203],

$$\int_M \omega \wedge \delta(\Sigma) = \int_\Sigma \omega \,, \qquad \Sigma \subset M \,, \tag{4.38}$$

[22]In order to detect monopoles, the gauge field a was introduced in [145, 136]. We will use it for both global and local properties.

for every n-form ω. Stokes' theorem turns into $\mathrm{d}\,\delta(\Sigma) = \delta(\partial\Sigma)$, which in our case is nothing but $\mathrm{d}f^{\mathrm{sg}} = -*k$. We locally split the coordinates $(x_1, \dots, x_d)$ into $(y_1, \dots, y_n)$ along the hypersurface Σ and $(z_{n+1}, \dots, z_d)$ normal to it, the latter chosen to vanish on Σ. Then, the relevant part of the form ω is proportional to $\mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$, while the distribution δ has the form[23],

$$\delta(\Sigma) = f(y)\, \delta^{(d-n)}(z)\, \mathrm{d}z_{k+1} \wedge \dots \wedge \mathrm{d}z_d\,. \tag{4.39}$$

Concerning the monopole term, the one-form a must depend on (the one-form generated by) the monopole worldline coordinate. Since a is intimately related to g and n, *this dependence is nothing but the twist* encountered in the previous section. The other contribution to the instanton number is proportional to the *winding number of n on the Dirac sheet*, because f is the relevant topological density (cf. (4.34)). However, subleties can arise from the fact that we are dealing with singular objects. This will become clearer below.

Another formula for the instanton number can be obtained by adding a zero to (4.35),

$$\nu(A) = \frac{1}{16\pi^2} \int_{S^3} a \wedge (f + f^{\mathrm{sg}})\,. \tag{4.40}$$

We recall that f^{sg} is assumed to vanish on the three-sphere. Stokes' theorem yields,

$$\nu(A) = \frac{1}{16\pi^2} \int_{V^4} \mathrm{d}a \wedge (f + f^{\mathrm{sg}}) = \frac{1}{16\pi^2} \int_{V^4} (2 f^{\mathrm{sg}} \wedge f + f^{\mathrm{sg}} \wedge f^{\mathrm{sg}})\,. \tag{4.41}$$

Compared to (4.37), the monopole loop term $a \wedge \mathrm{d}f$ has gone since the currents of f and f^{sg} compensate each other (cf. (2.35)). Instead, the other term, $f^{\mathrm{sg}} \wedge f$, appears twice, and a new term, $f^{\mathrm{sg}} \wedge f^{\mathrm{sg}}$, shows up. From the forms involved in f^{sg} one concludes that the support of this term reduces to *self-intersection points of the Dirac sheet* (see also Section 4.4). It will be absent for simple choices of the sheet (see below).

With this formalism at hand, we are in a position to reconsider the small monopole loops: One simply has to plug in the explicit expressions (4.7) and (4.10) into the formula for the instanton number (4.37) and take care of the singularities. The monopole loop coordinate is $y = \varphi_{12}$, while the normal coordinates are $z = \{r_{12} - R, x_3, x_4\}$ which vanish at the loop C. The monopole current is,

$$*k = \mathrm{d}f = \sin\beta\, \mathrm{d}\beta \wedge \mathrm{d}^2\alpha\,. \tag{4.42}$$

[23]The prefactor $f(y)$ can be obtained by integrating the full δ-distribution over Σ [101].

By virtue of

$$\mathrm{d}^2\alpha = 2\pi(\delta^{(2)}(x_1, x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \delta^{(2)}(x_3, x_4)\mathrm{d}x_3 \wedge \mathrm{d}x_4)\,, \tag{4.43}$$

the current is localised on two planes on which the prefactor $\sin\beta$ formally vanishes (see Figure 4.1). But β has a jump exactly at the loop C, i.e. the other 'prefactor' $\mathrm{d}\beta$ becomes infinite. Therefore, we write,

$$\sin\beta\,\mathrm{d}\beta = -\mathrm{d}(\cos\beta)\,, \qquad \mathrm{d}(\cos\beta)|_{x_3=x_4=0} = 2\,\delta(r_{12} - R)\,\mathrm{d}r_{12}\,. \tag{4.44}$$

Altogether, the monopole current,

$$*k = \delta(C) = 4\pi\delta^{(3)}(r_{12} - R, x_3, x_4)\,\mathrm{d}(r_{12} - R) \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4\,, \tag{4.45}$$

is of the desired form (4.39). According to the discussion of the twist, there is a component of the Abelian gauge field a tangential to the loop,

$$a|_C = (1 + \cos\beta)\mathrm{d}\varphi_{12}\,. \tag{4.46}$$

However, it inherits a multivaluedness from β. Its contribution to the instanton number is correctly evaluated by carefully collecting all β-terms,

$$\begin{aligned} \frac{-1}{16\pi^2}\int_{V^4} a \wedge \mathrm{d}f &= \frac{1}{16\pi^2}\int_{\mathbb{R}^4}(1 + \cos\beta)\mathrm{d}\varphi_{12} \wedge \mathrm{d}(\cos\beta) \wedge 2\pi\delta^{(2)}(x_3, x_4)\mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ &= \frac{1}{8\pi}\int_{\mathbb{R}^2_{x_1x_2}} \mathrm{d}\varphi_{12} \wedge (1 + \cos\beta)\mathrm{d}(\cos\beta) \\ &= \frac{1}{4}\left(\cos\beta + \frac{1}{2}\cos^2\beta\right)\Bigg|_{r_{12}=0}^{r_{12}\to\infty} = \frac{1}{4}(2+0) = \frac{1}{2}\,. \end{aligned} \tag{4.47}$$

For the Dirac sheet D the coordinates split as $y = \{r_{12}, \varphi_{12}\}$ and $z = \{x_3, x_4\}$, and the singular field strength is of the expected form,

$$f^{\mathrm{sg}} = \delta(D) = 4\pi\,\theta(R - r_{12})\,\delta^{(2)}(x_3, x_4)\,\mathrm{d}x_3 \wedge \mathrm{d}x_4\,. \tag{4.48}$$

The winding of n on this sheet is very peculiar: on the interior of D, n is constant (points to the south pole), while on the boundary, i.e. on the monopole loop C, it is singular and takes on all possible values once. Thus, one expects a winding number, but concentrated on the loop. Indeed, the topological density becomes localised,

$$f|_D = \mathrm{d}(\cos\beta) \wedge \mathrm{d}\varphi_{12} = 2\,\delta(r_{12} - R)\mathrm{d}r_{12} \wedge \mathrm{d}\varphi_{12}\,, \tag{4.49}$$

which, however, will not be the case for any other Dirac sheet D' of this loop. Combining (4.48) and (4.49) we again encounter an ambiguity, this time $\theta(0)$. Again, by collecting the β-terms more carefully one obtains,

$$\begin{aligned} \frac{1}{16\pi^2}\int_{V^4} f^{\mathrm{sg}}\wedge f &= \frac{1}{16\pi^2}\int_{\mathbb{R}^4} 2\pi(1-\cos\beta)\delta^{(2)}(x_3,x_4)\,\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\mathrm{d}(\cos\beta)\wedge\mathrm{d}\varphi_{12} \\ &= \frac{1}{4}(2-0)=\frac{1}{2}\,, \end{aligned} \tag{4.50}$$

in analogy to (4.47). This is consistent with formula (4.41), as the instanton number is simply twice this contribution.

Altogether, the Abelian field a contains information about both the global properties of the background (via the Chern-Simons form at the boundary) and the local properties of the defects (via its singularities in the bulk). Writing the instanton number of the background as a four-dimensional integral involving a, the integrand shows that *there must be defects* (see (4.36)). Moreover, it is localised at these singularities, their contributions being proportional to the *twist along the monopole loop* and the *winding number of n on the Dirac sheet* (plus the self-intersection of the latter, see (4.37), (4.41)).

Monopole loops which are not localised within a single chart are called *large*. They unavoidably intersect the transition region, so that the arguments presented above do not hold for these loops. The required modifications are our next topic.

4.3.4. Large Monopole Loops: Flux

The great circles induced by the instanton in the LAG (for the particular case $R=\rho$) are large monopole loops, see Section 3.4.1 and in particular Figure 3.7. They illustrate the relation between instanton number and monopole charge proposed by Tsurumaru, Tsutsui and Fujii [253]. These authors claim that *if the Higgs field n is singular on the transition three-sphere at two points, it is invariant under the transition function.* As the latter does not rotate n, it must be of the form,

$$t(x)=\exp\left(in(x)\theta(x)\right)\,. \tag{4.51}$$

The transition function is an entirely smooth function. Therefore, the singularities of n at the two points p_1 and p_2 have to be 'cured' by θ, hence θ is a multiple of 2π there.

Writing $S^3_{r=\mathrm{const}}\backslash\{p_1,p_2\}$ as the product of a two-sphere S^2 and an interval $I_{p_1p_2}$, and assuming that θ only depends on the interval variable, the authors conclude that

$$\nu(A)=\deg(t)=\deg_{S^2}(n)\,\frac{\Phi_I(\theta)}{2\pi},\quad\text{where}\quad \Phi_I(\theta)=\int_I\mathrm{d}\theta=\theta(p_2)-\theta(p_1)\in 2\pi\mathbb{Z}\,. \tag{4.52}$$

Φ is the flux of an Abelian field which is quantised due to the special behaviour of θ at the endpoints.

These results can easily be checked for the LAG solutions. For simplicity we choose the last field entries in the quadruplet (A.22) and (A.23). They vanish at the loop $C = (\vec{0}, x_4)$, and the normalised field n is just the static hegdehog field $n = x_a\tau_a/|\vec{x}|$. It is singular on any sphere $S^3_{r=\text{const}}$ at the points $p_{1,2} = (\vec{0}, x_4 = \pm r)$. Indeed, the normalised field n is the same for both charts (for singular and regular gauge), i.e. it is not rotated by the transition function $\hat{g}$. Accordingly, the latter can be written in yet another form,

$$\hat{g} = \exp\left(i\,\frac{x_a\tau_a}{|\vec{x}|}\,\theta\right)\,, \qquad \theta = 2\arccos\frac{x_4}{r}\,, \tag{4.53}$$

with $\theta \in \{0, 2\pi\}$ at the two points $p_{1,2}$.

The interval $I_{p_1p_2}$ is parametrised by $x_4 \in (-r, r)$. Indeed, θ is only a function of x_4 (r is a constant), and the flux is $\Phi_I(\theta) = 2\pi$. The two-spheres S^2 are spatial ones with fixed radius $|\vec{x}| = \sqrt{r^2 - x_4^2}$. From the hedgehog behaviour it is obvious that the winding number (magnetic charge) of n on these spheres is one. Altogether, we have the following formula for the instanton number,

$$\nu(A) = \deg(t) = \text{magnetic charge} \times \text{flux}\,, \qquad 1 = 1 \times 1\,. \tag{4.54}$$

It is the analogue of (4.26), and is valid for large monopole loops. The difference between (4.26) and (4.54) is that the twist represents the change of the hedgehog *along the worldline*, while the flux is measured on a *slice perpendicular to it*. In both cases, an additional topological quantity is required, since the magnetic charge is of lower dimension compared to the instanton number.

4.4. Outlook 2: Center Vortices

An alternative mechanism to explain colour confinement are center vortices [254, 255]. They emerge upon fixing the gauge group up to its center, and the relevant topology is encoded in $\pi_1(SU(2)/Z_2) \cong Z_2$. These vortices should give the area law for the Wilson loop – again via condensation in the QCD vacuum. There are many analogies between Abelian and center projections. For instance, the *Maximal center gauge* is defined by maximising a functional on the lattice [256]. The *center projection* replaces the gauge fixed configuration by its nearest center element at every point. *Center dominance* has been observed on the lattice [257], but it is suffering from a severe Gribov problem [258].

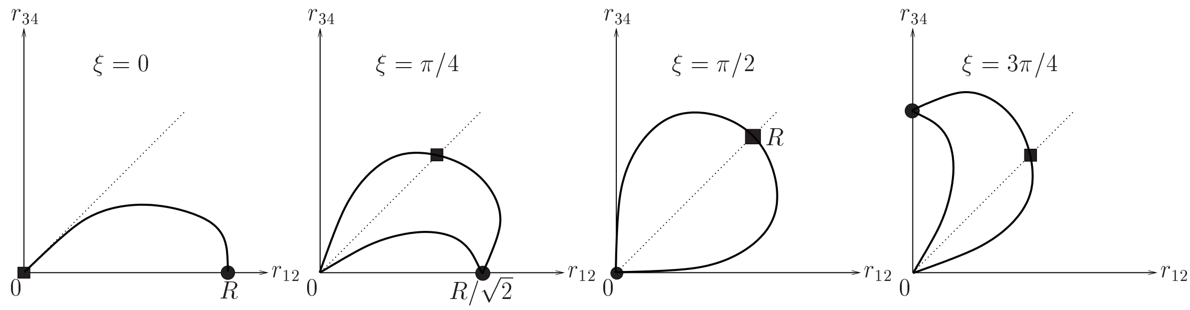

Fig. 4.3.: Vortex sheets for different values ξ of the relative perturbation strength in (4.55) together with the monopole loops for ϕ_0 (filled circles, the small monopole loops from Section 4.1.2) and ϕ_1 (filled boxes, always on the dotted line $\theta = \pi/4$).

The *Laplacian center gauge* (LCG) [227] is reached by first going to the Laplacian Abelian gauge and then fixing the residual Abelian freedom by rotating the first excited state into a particular plane in colour space. Defects occur where these two Higgs fields become collinear, including the positions where one or both vanish. They form two-dimensional *vortex sheets* (have codimension two). By definition, the magnetic monopoles of the LAG lie on these surfaces, namely where the relative orientations of the Higgs fields changes from parallel to anti-parallel [259]. A vortex sheet can be understood as half a Dirac sheet (see e.g. [170]).

The results presented in the body of this work can be used for center vortices in two ways: Repeating Schrödinger perturbation theory for *two* of the ground states in (A.25), one arrives at small vortex sheets in the LCG[24]. A selection of these defects, coming from

$$\phi_0 = \phi_{0,\text{inst}} + (0, 0, R^2 \cos\xi)^\top, \qquad \phi_1 = \phi_{1,\text{inst}} + (0, 0, R^2 \sin\xi)^\top, \tag{4.55}$$

is depicted in Figure 4.3. To the best of my knowledge, these are the first explicitly constructed vortex sheets in non-trivial backgrounds. At the origin they show an interesting behaviour: Two branches of the vortex sheet 'cross each other perpendicularly', meaning that the two tangent spaces of the intersecting branches span the whole four-dimensional tangent space. This phenomenon is called *self-intersection* (see e.g. [245]) and has been proposed to generate the instanton number [260]. In fact, one can define a Higgs field $\phi \equiv [\phi_0, \phi_1]$ and $n = \phi/|\phi|$, the defects of which exactly reflect the vortex sheets. Following the 'localisation of topology' in Section 4.3.3, the self-intersection term $f^{\text{sg}} \wedge f^{\text{sg}}$ now contributes to the instanton number (see (4.41)). This is work in progress with P. de Forcrand and M. Pepe.

[24]The vortex sheet of the unperturbed instanton in the LCG degenerates to a point.

5. Summary

This thesis has been dealing with aspects of one of the most fascinating phenomena in modern quantum field theory, the confinement of colour. An attractive scenario to explain this non-perturbative effect is the dual superconductor. In this model, magnetic monopoles – as dual Cooper pairs – condense in the QCD vacuum. As there are no monopoles in pure Yang-Mills theory, 't Hooft has proposed the Abelian projections to obtain them as defects in a partial gauge fixing. Although many lattice results support this mechanism, the results on Abelian projections in the continuum are rather limited. We have tried to fill this gap, concentrating on effects induced by instantons.

The Maximally Abelian gauge is defined by minimising the off-diagonal parts of the gauge field. We have *illustrated the MAG in terms of a toy model*, in which two vectors in three-dimensional space are rotated 'close to some axis'. The eigenvalues of the Faddeev-Popov operator have been calculated exactly. As expected, zero modes of this operator occur for reducible configurations. By looking at the eigenvalues of the FP operator, we have found *additional configurations located on the Gribov horizon*. Their Gribov copies can easily be visualised.

The Gribov problem of the 'real' MAG has been analysed by deriving the explicit expression for its FP operator. It is the difference of two positive semidefinite operators. By the same spectral flow argument as in the toy model, we conclude that *the Gribov horizon is reached by scaling up the off-diagonal components* of the gauge field. Moreover, *the single instanton is a horizon configuration of this gauge*. We have explicitly calculated the associated zero modes by solving a radial Schrödinger equation. These zero modes can be derived neither from reducibility nor from space-time symmetries. Being the first continuum result in this context, this statement can be read as a hint towards a relation between the Gribov problem and confinement. It is also interesting for lattice simulations, where the gauge fixing procedure is a highly degenerate spin glass problem.

The Polyakov gauge is special in that its defects are static. We have argued that it *cannot exactly be derived from a gauge fixing functional*. We have constructed a functional, which, however, *generically yields the PG*. A closer inspection, though, reveals that these gauges differ by additional defects, visible in their FP operators. At the moment, it is still unclear, whether there is some physics behind this formal discrepancy.

The Laplacian Abelian gauge is defined by the diagonalisation of the ground state of the covariant Laplacian (in the adjoint representation). The Higgs field of this gauge has been calculated for the first time in an instanton background. The high symmetry of this background yields a *pointlike defect* localised at the position of the instanton. The normalised Higgs field is just the standard Hopf map. Since the ground state is degenerate, the single instanton is again on the Gribov horizon.

Information about non-generic backgrounds can usually be obtained only by numerical means (which complicates topological investigations). We have used perturbation theory to show that *defects on orbits close to the instanton are small circles*. These generic defects are the *worldlines of magnetic monopoles carrying unit magnetic charge*. Viewing Hopf defects as 'seeds' for monopole loops allows for a clear topological interpretation. The *twist*, the rotation of the hedgehog field along the worldline, naturally appears from this relation.

Global properties of Higgs fields are described by the theory of fibre bundles. The instanton number of the background is translated (via the winding number of the transition function) into the *Hopf invariant* of the Higgs field. It follows that, *in non-trivial backgrounds, defects must be present*. Just like in residue calculus, the Hopf invariant is the sum of contributions of small spheres around each defect. Hence, *the instanton number of the background is determined by the local properties of the defects*, a phenomenon we have called 'localisation of the global topology'. This localisation is particularly simple for the pointlike defect, since there is only one defect and the standard Hopf map has Hopf invariant one.

In order to quantify the contributions of different kinds of defects, we have *introduced an auxiliary Abelian gauge field*. It contains information about both the global properties of the background and – via its singularities – the local properties of the defects. Again, there have to be defects to generate an instanton number. The contributions to the instanton number are proportional to the *twist along the monopole loop* and the *winding number of the Higgs field on the Dirac sheet*. We have computed these contributions explicitly for the small monopole loop. In addition, we have indicated that the self-intersection of the Dirac sheet may also contribute, namely for center vortices.

Large monopole loops, which we have shown to appear in the LAG under certain circumstances, generate the instanton number via a modified mechanism.

The method of Abelian projection always has to face the objection, whether magnetic monopoles obtained via gauge fixing can be physical. In other words, is there really a 'gauge which is most favourable for our purposes' [222]? Indeed, the actual monopole worldlines may be similar for MAG and LAG; the PG is certainly different, since it produces static monopoles. Concerning approximate results (like Abelian dominance), Abelian projections can behave differently, but after all, *physical effects should be independent of any chosen gauge*. The unified description of Abelian gauges in terms of functionals and the inevitability of defects in non-trivial backgrounds point in this direction. The latter phenomenon is quite analogous to the Gribov problem: there must be Gribov copies due to general arguments, but their particular appearance is gauge dependent. It might be that the monopoles of Abelian projections just give different emphasis to the relevant configurations [15].

The *local* correlation of monopoles with the *gauge invariant* position of instantons, i.e. with the topological density, is an even stronger argument for their physical relevance. Together with the fact that individual monopole loops can fuse to fewer large loops in instanton ensembles, this leads to a conjecture about the dynamics: instantons may indirectly induce confinement via the creation of large monopole loops in the QCD vacuum. Admittedly, this approach has not been proven so far, presumably because details of instanton interactions are not yet fully understood.

Instantons also provide the link to the physics of chiral symmetry breaking. Therefore, it is reasonable to look for zero modes in Abelian projections. We have given some first results concerning the single instanton background. In addition, it would be interesting to relate these zero modes to the ones of an Abelian Dirac operator in three dimensions which are characterised by the Hopf invariant, too [261].

We have also shown how the Abelian gauges are related to the Faddeev-Niemi action which has recently been suggested as a model for glueballs. Its derivation from the underlying Yang-Mills theory and its relation to the dual Abelian Higgs model [262] or to string theory [263] are under investigation. The influence of defects and Gribov copies in this context has to be clarified.

The presented results allow for a better understanding of non-perturbative issues in QCD. However, ‘there is still a lot of work to do to unveil the mysteries of confinement’ [264].

References

[1] R. P. Feynman, (1976), in: 'Les Houches 1976, Proceedings, Weak And Electromagnetic Interactions At High Energies', Amsterdam 1977, 121-206.

[2] M. K. Prasad, Physica **D1**, 167 (1980).

[3] G. 't Hooft, Acta Physica Austriaca Suppl. **22**, 531 (1980).

[4] T. Suzuki, Progr. Theor. Phys. **80**, 929 (1988).

[5] P. de Forcrand, O. Philipsen, Phys. Lett. **B475**, 280 (2000), hep-lat/9912050.

[6] C. Spiering, *Auf der Suche nach der Urkraft* (Teubner, Leipzig, 1989).

[7] U. G. Mitreuter, J. M. Pawlowski, A. Wipf, Nucl. Phys. **B514**, 381 (1998), hep-th/9611105.

[8] E.-M. Ilgenfritz, S. Thurner, H. Markum, M. Müller-Preußker, Phys. Rev. **D61**, 054501 (2000), hep-lat/9904010.

[9] G. S. Bali, hep-ph/9809351.

[10] A. Kovner, hep-ph/0009138.

[11] Particle Data Group, C. Caso *et al.*, Eur. Phys. J. **C3**, 1 (1998).

[12] G. B. West, hep-ph/9608258.

[13] A. Jaffe, E. Witten, http://www.claymath.org/prizeproblems/yangmills.htm.

[14] F. Karsch, Nucl. Phys. Proc. Suppl. **83**, 14 (1996), hep-lat/9909006.

[15] S. V. Shabanov, private communication.

[16] J. Kogut, L. Susskind, Phys. Rev. D **9**, 3501 (1974).

[17] P. Becher, M. Böhm, H. Joos, *Eichtheorien* (Teubner, Stuttgart, 1983).

[18] Y. A. Simonov, Physics Uspekhi **39**, 313 (1996), hep-ph/9709344.

[19] K. Wilson, Phys. Rev. **D10**, 2445 (1974).

[20] S. Mandelstam, Ann. Phys. **19**, 1 (1962).

[21] C. N. Yang, Phys. Rev. Lett. **33**, 445 (1975).

[22] F. J. Wegner, J. Math. Phys. **12**, 2259 (1971).

[23] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, 1983).

[24] H. J. Rothe, *Lattice Gauge Theories. An Introduction* (World Scientific, 1992).

[25] A. H. Guth, Phys. Rev. **21**, 2291 (1980).

[26] T. A. DeGrand, D. Toussaint, Phys. Rev. **D22**, 2478 (1980).

[27] M. Creutz, Phys. Rev. **D21**, 2308 (1980).

[28] Y. Nambu, Phys. Rev. **D10**, 4262 (1974).

[29] G. Parisi, Phys. Rev. **D11**, 970 (1975).

[30] G. 't Hooft, in: *High Energy Physics*, Proceedings of the EPS International Conference, Palermo 1975, A. Zichichi, ed., Editrice Compositori, Bologna 1976.

[31] S. Mandelstam, Phys. Rep. **C23**, 245 (1976).

[32] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, 1985).

[33] G. 't Hooft, Nucl. Phys. **B190**, 455 (1981).

[34] V. L. Ginzburg, L. D. Landau, JETP (USSR) **20**, 1064 (1950).

[35] H. B. Nielsen, P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[36] S. Maedan, T. Suzuki, Progr. Theor. Phys. **81**, 229 (1989).

[37] V. Singh, D. A. Browne, R. W. Haymaker, Phys. Lett. **B306**, 115 (1993), hep-lat/9301004.

[38] G. S. Bali, C. Schlichter, K. Schilling, Phys. Rev. **D51**, 5165 (1995), hep-lat/9409005.

[39] H. Suganuma, S. Sasaki, H. Toki, Nucl. Phys. B **435**, 207 (1995), hep-ph/9312350.

[40] H. Ichie, H. Suganuma, H. Toki, Phys. Rev. **D52**, 2944 (1995), hep-ph/9502278.

[41] Y. Koma, H. Suganuma, H. Toki, Phys. Rev. **D60**, 074024 (1999), hep-ph/9902441.

[42] F. V. Gubarev, E.-M. Ilgenfritz, M. I. Polikarpov, T. Suzuki, Phys. Lett. **B468**, 134 (1999), hep-lat/9909099.

[43] G. S. Bali, C. Schlichting, K. Schilling, Prog. Theor. Phys. Suppl. **131**, 645 (1998), hep-lat/9802005.

[44] D. Antonov, Surveys High Energy Phys. **14**, 265 (2000), hep-th/9909209.

[45] L. Faddeev, preprint IAS Print-75-QS70 (1975).

[46] L. Faddeev, in: *Einstein and contemporary tendencies in the field theory of elementary particles*, in Relativity, quanta and cosmology, vol. 1, M. Pantaleo and F. de Finis, eds., Johnson Reprint, 1979.

[47] T. H. R. Skyrme, Proc. Roy. Soc.London **A260**, 127 (1961).

[48] R. Hobart, Proc. Roy. Soc. London **82**, 201 (1963).

[49] G. H. Derrick, J. Math. Phys. **5**, 1252 (1964).

[50] L. Faddeev, A. J. Niemi, (1997), hep-th/9705176.

[51] A. F. Vakulenko, L. V. Kapitansky, Dokl. Akad. Nauk USSR **246**, 840 (1979).

[52] A. Kundu, Y. P. Rubakov, J. Phys. **A15**, 269 (1982).

[53] R. S. Ward, Nonlinearity **12**, 1 (1999), hep-th/9811176.

[54] L. Faddeev, A. J. Niemi, Nature **387**, 58 (1997), hep-th/9610193.

[55] J. Gladikowski, M. Hellmund, Phys. Rev. **D56**, 5194 (1997), hep-th/9609035.

[56] J. Hietarinta, P. Salo, Phys. Lett. **B451**, 60 (1999), hep-th/9811053.

[57] R. A. Battye, P. M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998), hep-th/9808129.

[58] J. Hietarinta, P. Salo, Phys. Rev. **D62**, 081701 (2000).

[59] M. Hirayama, J. Yamshita, (2001), hep-th/0106251.

[60] L. Faddeev, A. J. Niemi, Phys. Rev. Lett. **82**, 1624 (1998), hep-th/9807069.

[61] S. Coleman, E. Weinberg, Phys. Rev. **D7**, 1888 (1973).

[62] Y. M. Cho, H. Lee, D. G. Pak, hep-th/9905215.

[63] E. Langmann, A. J. Niemi, Phys. Lett. **B463**, 252 (1999), hep-th/9905147.

[64] H. Gies, Phys. Rev. **D63**, 125023 (2001), hep-th/0102026.

[65] Y. M. Cho, H. W. Lee, D. G. Pak, hep-th/0105198.

[66] L. Dittmann, T. Heinzl, A. Wipf, to appear.

[67] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, 1951).

[68] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* (Wiley, 1963).

[69] T. Eguchi, P. B. Gilkey, A. J. Hanson, Phys. Rep. **66**, 213 (1980).

[70] M. Nakahara, *Geometry, Topology and Physics* (Adam Hilger, 1990).

[71] S. K. Donaldson, P. B. Kronheimer, *The Geometry of Four-Manifolds* (Oxford, 1990).

[72] G. Rudolph, Lectures given at the Mitteldeutsche Physik-Combo, Halle, Leipzig and Jena, Germany, Winter Term 1996/97 (unpublished).

[73] J. Fuchs, hep-th/9506005.

[74] G. Rudolph, M. Schmidt, I. P. Volobuev, math-ph/0009018.

[75] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer, 1972).

[76] C. Ford, T. Tok, A. Wipf, Nucl. Phys. **B548**, 585 (1999), hep-th/9809209.

[77] W. Ambrose, I. M. Singer, Trans. Amer. Math. Soc. **75**, 428 (1953).

[78] Y. M. Cho, Phys. Rev. **D21**, 1080 (1980).

[79] T. Kugo, *Eichtheorie* (Springer, 1997).

[80] N. H. Christ, T. D. Lee, Phys. Rev. **D22**, 939 (1980).

[81] K. Kuchař, Phys. Rev. **D34**, 3031 (1986).

[82] R. P. Feynman, Nucl. Phys. **B188**, 479 (1981).

[83] P. van Baal, B. van den Heuvel, Nucl. Phys. **B417**, 215 (1994), hep-lat/9310005.

[84] W. Kondracki, J. Rogulski, (1986), Dissertationes Mathematicae **250**, Panstwowe Wydawnictwo Naukowe, Warszawa.

[85] T. Pause, T. Heinzl, Nucl. Phys. **B524**, 695 (1998), hep-th/9801169.

[86] V. Gribov, Nucl. Phys. **B139**, 1 (1978).

[87] I. M. Singer, Commun. Math. Phys. **60**, 7 (1978).

[88] P. van Baal, hep-th/9511119.

[89] L. D. Faddeev, V. N. Popov, Phys. Lett. **B25**, 29 (1967).

[90] P. van Baal, Nucl. Phys. Proc. Suppl. **42**, 843 (1995), hep-lat/9411047.

[91] P. de Forcrand, Nucl. Phys. Proc. Suppl. **20**, 194 (1991).

[92] P. de Forcrand, J. E. Hetrick, Nucl. Phys. Proc. Suppl. **42**, 861 (1995).

[93] L. Giusti *et al.*, hep-lat/0104012.

[94] P. A. M. Dirac, Proc. Roy. Soc. **A133**, 60 (1931).

[95] G. Wentzel, Suppl. Progr. Theor. Phys **37&38**, 163 (1966).

[96] A. Trautmann, Int. J. Theor. Phys **16**, 561 (1977).

[97] M. Minami, Prog. Theor. Phys. **62**, 1128 (1979).

[98] L. H. Ryder, J. Phys. **A13**, 437 (1980).

[99] T. T. Wu, C. N. Yang, Phys. Rev. **D12**, 3845 (1975).

[100] B. Julia, A. Zee, Phys. Rev. **D11**, 2227 (1975).

[101] H. Kleinert, *Pfadintegrale in Quantenmechanik, Statistik und Polymerphysik* (Wissenschaftsverlag, Mannheim, Leipzig, Wien, Zürich, 1993).

[102] H. Georgi, S. L. Glashow, Phys. Rev. Lett. **28**, 1494 (1972).

[103] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **23**, 435 (1976).

[104] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).

[105] J. Arafune, P. G. O. Freund, C. J. Goebel, J. Math. Phys. **16**, 433 (1975).

[106] A. M. Polyakov, Sov. Phys. JETP Lett. **20**, 194 (1974).

[107] A. A. Belavin, A. M. Polyakov, A. S. Schwarz, Y. S. Tyupkin, Phys. Lett. **B59**, 85 (1975).

[108] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976).

[109] T. Schäfer, E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998), hep-ph/9610451.

[110] K. Uhlenbeck, Bull. Amer. Math. Soc. **1**, 579 (1978).

[111] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, Y. A. Manin, Phys. Lett. **A65**, 185 (1978).

[112] K. Lee, C. Lu, Phys. Rev. **D58**, 025011 (1998), hep-th/9802108.

[113] T. C. Kraan, P. van Baal, Nucl. Phys. **B533**, 627 (1998), hep-th/9805168.

[114] T. C. Kraan, P. van Baal, Phys. Lett. **B435**, 389 (1998), hep-th/9806034.

[115] C. Ford, J. M. Pawlowski, T. Tok, A. Wipf, Nucl. Phys. **B596**, 387 (2001), hep-th/0005221.

[116] C. H. Taubes, J. Diff. Geom. **19**, 517 (1984).

[117] P. J. Braam, P. van Baal, Comm. Math. Phys **122**, 267 (1989).

[118] P. van Baal, Nucl. Phys. Proc. Suppl. **49**, 238 (1996), hep-th/9512223.

[119] W. Nahm, Phys. Lett. **B90**, 413 (1980).

[120] G. 't Hooft, Comm. Math. Phys. **81**, 455 (1981).

[121] P. van Baal, Comm. Math. Phys. **94**, 397 (1984).

[122] G. 't Hooft, unpublished (1976).

[123] F. Wilczek, in: *Quark Confinement and Field Theory*, D. Stamp and D. Weingarten, eds., Wiley, New York, 1977.

[124] E. Corrigan, D. B. Fairlie, Phys. Lett. **B67**, 69 (1977).

[125] R. Jackiw, C. Nohl, C. Rebbi, Phys. Rev. **D15**, 1642 (1977).

[126] G. 't Hooft, Phys. Rev. **D14**, 3432 (1976).

[127] R. Jackiw, *Diverse topics in theoretical and mathematical physics* (World Scientific, Singapore, 1995).

[128] H. Min, T. Lee, P. Y. Pac, Phys. Rev. **D32**, 440 (1985).

[129] M. Quandt, H. Reinhardt, Int. J. Mod. Phys. **A13**, 4049 (1998), hep-th/9707185.

[130] K.-I. Kondo, Phys. Rev. **D57**, 7467 (1998), hep-th/9709109.

[131] L. D. Faddeev, A. A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory* (Addison-Wesley, 1991).

[132] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic Publishers, 1981).

[133] N. Weiss, Phys. Rev. **D24**, 475 (1981).

[134] J. Polonyi, hep-ph/9511243.

[135] A. Di Giacomo, M. Mathur, Nucl. Phys. **B531**, 302 (1998), hep-th/9802050.

[136] C. Ford *et al.*, Ann. Phys. (N.Y.) **269**, 26 (1998), hep-th/9802191.

[137] P. Goddard, J. Nuyts, D. Olive, Nucl. Phys. **B125**, 1 (1977).

[138] J. F. Cornwell, *Group Theory in Physics,* Vol. 2 (Academic Press, London, 1984).

[139] S. V. Shabanov, Phys. Lett. **B458**, 322 (1999), hep-th/9903223.

[140] A. J. van der Sijs, Prog. Theor. Phys. Suppl. **131**, 149 (1998), hep-lat/9803001.

[141] R. C. Brower, K. N. Orginos, C.-I. Tan, Phys. Rev. **D55**, 6313 (1997), hep-th/9610101.

[142] A. V. Yung, Nucl. Phys. **B297**, 47 (1988).

[143] D. Hansen, private communication.

[144] A. S. Kronfeld, M. L. Laursen, G. Schierholz, U.-J. Wiese, Phys. Lett **B198**, 516 (1987).

[145] A. S. Kronfeld, G. Schierholz, U.-J. Wiese, Nucl. Phys. **B293**, 461 (1987).

[146] A. J. van der Sijs, Nucl. Phys. B (Proc. Suppl.) **53**, 535 (1997), hep-lat/9608041.

[147] J. C. Vink, U.-J. Wiese, Phys. Lett. **B289**, 122 (1992), hep-lat/9206006.

[148] S. Hioki *et al.*, Phys. Lett. **B271**, 201 (1991).

[149] G. S. Bali, V. Bornyakov, M. Müller-Preußker, K. Schilling, Phys. Rev. **D54**, 2863 (1996), hep-lat/9603012.

[150] A. Hart, M. Teper, Phys. Rev. **D55**, 3756 (1997), hep-lat/9606007.

[151] J. M. Arms, J. E. Marsden, V. Moncrief, Commun. Math. Phys. **78**, 455 (1981).

[152] O. Babelon, C. M. Viallet, Commun. Math. Phys. **81**, 515 (1981).

[153] M. Creutz, I. J. Muzinich, T. N. Tudron, Phys. Rev. **D19**, 531 (1979).

[154] A. Galindo, P. Pascual, *Quantum Mechanics II,* (Springer, 1991).

[155] P. van Baal, Nucl. Phys. **B369**, 259 (1992).

[156] L. Baulieu, A. Rozenberg, M. Schaden, Phys. Rev. **D54**, 7825 (1996), hep-th/9607147.

[157] F. Lenz, H. W. L. Naus, M. Thies, Annals Phys. **233**, 317 (1994).

[158] H. Reinhardt, Mod. Phys. Lett. **A11**, 2451 (1996), hep-th/9602047.

[159] C. Ford, T. Tok, A. Wipf, Phys. Lett. **B456**, 155 (1999), hep-th/9811248.

[160] H. Weyl, *The Classical Groups* (Princeton University Press, 1973).

[161] F. Palumbo, Phys. Lett. **B173**, 81 (1986).

[162] J. C. Vink, Phys. Rev. **D51**, 1292 (1995), hep-lat/9407007.

[163] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. 4, Analysis of Operators* (Academic Press, Boston, 1978).

[164] R. Jackiw, C. Rebbi, Phys. Rev. Lett. **36**, 1116 (1976).

[165] P. Hasenfratz, G. 't Hooft, Phys. Rev. Lett. **36**, 1119 (1976).

[166] A. S. Goldhaber, Phys. Rev. Lett. **36**, 1122 (1976).

[167] S. L. Adler, Phys. Rev. **D6**, 3445 (1972).

[168] P. de Forcrand, M. Pepe, Nucl. Phys. Proc. Suppl. **2001**, 498 (94), hep-lat/0010093.

[169] P. de Forcrand, private communication.

[170] H. Reinhardt, T. Tok, hep-th/0009205.

[171] C. G. Callan, R. Dashen, D. J. Gross, Phys. Lett. **B66**, 375 (1977).

[172] K. Konishi, K. Takenaga, Phys. Lett. **B508**, 392 (1999), hep-th/9911097.

[173] A. Gonzalez-Arroyo, P. Martinez, Nucl. Phys. **B459**, 337 (1996), hep-lat/9507001.

[174] J. V. Steele, hep-lat/0007030.

[175] M. Garcia Perez, T. G. Kovacs, P. van Baal, (2000), hep-ph/0006155.

[176] P. Rossi, Nucl. Phys. **B149**, 170 (1979).

[177] M. Garcia Perez, D. Hansen, work in progress.

[178] L. Faddeev, A. J. Niemi, Phys. Lett. **B464**, 90 (1999), hep-th/9907180.

[179] S. V. Shabanov, Phys. Lett. **B463**, 263 (1999), hep-th/9907182.

[180] W. S. Bae, Y. M. Cho, S. W. Kimm, hep-th/0105163.

[181] F. Gliozzi, Nucl. Phys. **B141**, 379 (1978).

[182] M. F. Atiyah, I. M. Singer, Annals Math. **93**, 119 (1971).

[183] B. Grossman, Phys. Rev. **A61**, 86 (1977).

[184] M. N. Chernodub, T. C. Kraan, P. van Baal, Nucl. Phys. Proc. Suppl. **83**, 556 (2000), hep-lat/9907001.

[185] H. Neuberger, Nucl. Phys. Proc. Suppl. **83**, 67 (2000), hep-lat/9909042.

[186] T. Banks, A. Casher, Nucl. Phys. **B169**, 103 (1980).

[187] J. Kogut *et al.*, Phys. Rev. Lett. **50**, 393 (1983).

[188] M. F. Atiyah, V. K. Patodi, I. M. Singer, Math. Proc. Cambridge Phil. Soc. **77**, 43 (1975).

[189] J. R. Schmidt, A. M. Bincer, Phys. Rev. **D35**, 3995 (1987).

[190] O. Jahn, J. Phys. **A33**, 2997 (2000), hep-th/9909004.

[191] T. Suzuki, I. Yotsuyanagi, Phys. Rev. **D42**, 4257 (1990).

[192] J. D. Stack, S. D. Neiman, R. J. Wensley, Phys. Rev. **D50**, 3399 (1994), hep-lat/9404014.

[193] P. Becher, H. Joos, Zeit. Phys. **C15**, 343 (1982).

[194] F. Brandstater, U. J. Wiese, G. Schierholz, Phys. Lett. **B272**, 319 (1991).

[195] V. G. Bornyakov, V. K. Mitrjushkin, M. Müller-Preußker, Phys. Lett. **B284**, 99 (1992).

[196] T. L. Ivanenko, A. V. Pochinsky, M. I. Polikarpov, Phys. Lett. **B302**, 458 (1993).

[197] H. Shiba, T. Suzuki, Phys. Lett. **B351**, 519 (1995), hep-lat/9408004.

[198] P. Cea, L. Cosmai, Phys. Rev. **D62**, 094510 (2000), hep-lat/0006007.

[199] J. Fröhlich, P. A. Marchetti, Commun. Math. Phys. **112**, 343 (1987).

[200] L. D. Debbio, A. D. Giacomo, G. Paffuti, Phys. Lett. **B349**, 513 (1995), hep-lat/9403013.

[201] L. D. Debbio, A. D. Giacomo, G. Paffuti, P. Pieri, Phys. Lett. **B355**, 255 (1995), hep-lat/9505014.

[202] M. N. Chernodub, M. I. Polikarpov, A. I. Veselov, Nucl. Phys. Proc. Suppl. **49**, 307 (1996), hep-lat/9512030.

[203] J. Fröhlich, P. A. Marchetti, Phys. Rev. **D64**, 014505 (2001), hep-th/0011246.

[204] M. N. Chernodub, M. I. Polikarpov, A. I. Veselov, Phys. Lett. **B399**, 267 (1997), hep-lat/9610007.

[205] A. D. Giacomo, G. Paffuti, Phys. Rev. **D56**, 6816 (1997), hep-lat/9707003.

[206] G. I. Poulis, Phys. Rev. **D54**, 6974 (1996), hep-lat/9601013.

[207] L. D. Debbio, M. Faber, J. Greensite, S. Olejnik, Nucl. Phys. Proc. Suppl. **53**, 141 (1997), hep-lat/9607053.

[208] C. Michael, (1998), hep-ph/9809211.

[209] E. M. Ilgenfritz, H. Markum, M. Müller-Preußker, S. Thurner, Nucl. Phys. Proc. Suppl. **73**, 560 (1999), hep-lat/9809154.

[210] O. Jahn, F. Lenz, Phys. Rev. **D58**, 085006 (1998), hep-th/9803177.

[211] B. J. Harrington, H. K. Shepard, Phys. Rev. **D17**, 2122 (1978).

[212] M. Garcia Perez, A. Gonzalez-Arroyo, A. Montero, P. van Baal, JHEP **06**, 001 (1999), hep-lat/9903022.

[213] M. N. Chernodub, F. V. Gubarev, JETP Lett. **62**, 100 (1995), hep-th/9506026.

[214] R. C. Brower, K. N. Orginos, C.-I. Tan, Nucl. Phys. B (Proc. Suppl.) **53**, 488 (1997), hep-lat/9608012.

[215] S. Thurner, H. Markum, E. M. Ilgenfritz, M. Müller-Preußker, hep-lat/9809153.

[216] T. G. Kovacs, Z. Schram, Nucl. Phys. Proc. Suppl. **63**, 531 (1998), hep-lat/9710062.

[217] S. Ejiri *et al.*, Nucl. Phys. Proc. Suppl. **47**, 322 (1996), hep-lat/9509013.

[218] F. Shoji, Nucl. Phys. Proc. Suppl. **94**, 525 (2001), hep-lat/0010088.

[219] H. Suganuma, K. Itakura, H. Toki, O. Miyamura, hep-ph/9512347.

[220] B. L. G. Bakker, M. N. Chernodub, M. I. Polikarpov, Phys. Rev. Lett. **80**, 30 (1998), hep-lat/9706007.

[221] H. Markum, W. Sakuler, S. Thurner, Nucl. Phys. Proc. Suppl. **47**, 254 (1996), hep-lat/9510024.

[222] S. Thurner, M. Feurstein, H. Markum, hep-lat/9606011.

[223] M. Feurstein *et al.*, Nucl. Phys. Proc. Suppl. **63**, 480 (1998), hep-lat/9709140.

[224] V. Bornyakov, G. Schierholz, Phys. Lett. **B384**, 190 (1996), hep-lat/9605019.

[225] A. Hart, M. Teper, Phys. Lett. **B371**, 261 (1996), hep-lat/9511016.

[226] M. Feurstein, H. Markum, S. Thurner, (1997), hep-lat/9702006.

[227] C. Alexandrou, M. D'Elia, P. de Forcrand, Nucl. Phys. Proc. Suppl. **83**, 437 (2000), hep-lat/9907028.

[228] A. Gonzalez-Arroyo, A. Montero, Nucl. Phys. Proc. Suppl. **47**, 294 (1996), hep-lat/9509048.

[229] C. G. Callan, R. Dashen, D. J. Gross, Phys. Rev. **D17**, 2717 (1978).

[230] E. V. Shuryak, Nucl. Phys. **B198**, 83 (1982).

[231] E. M. Ilgenfritz, M. Müller-Preußker, Z. Phys. **C16**, 339 (1983).

[232] M. I. Polikarpov, A. I. Veselov, Nucl. Phys. **B297**, 34 (1988).

[233] D. Diakonov, V. Y. Petrov, P. V. Pobylitsa, Phys. Lett. **B226**, 372 (1989).

[234] E. V. Shuryak, Nucl. Phys. **B319**, 521 (1989).

[235] D. Diakonov, V. Petrov, in: *Nonperturbative approaches to quantum chromodynamics*, Trento, 1995.

[236] M. Fukushima *et al.*, Nucl. Phys. Proc. Suppl. **53**, 494 (1997), hep-lat/9610003.

[237] T. DeGrand, A. Hasenfratz, T. G. Kovacs, Nucl. Phys. Proc. Suppl. **63**, 528 (1998), hep-lat/9709095.

[238] R. C. Brower, D. Chen, J. W. Negele, E. Shuryak, Nucl. Phys. Proc. Suppl. **73**, 512 (1999), hep-lat/9809091.

[239] P. van Baal, Nucl. Phys. Proc. Suppl. **63**, 126 (1998), hep-lat/9709066.

[240] P. de Forcrand, Lectures given at the Strong Interaction Study Days 'Exploring QCD', Kloster Banz, Germany, 2000 (unpublished).

[241] I. I. Balitsky, A. V. Yung, Phys. Lett. **B168**, 113 (1986).

[242] J. J. M. Verbaarschot, Nucl. Phys. **B362**, 33 (1991).

[243] M. Garcia Perez, T. G. Kovacs, P. van Baal, Phys. Lett. **B472**, 295 (2000), hep-ph/9911485.

[244] P. van Baal, Commun. Math. Phys. **85**, 529 (1982).

[245] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry - Methods and Applications,* Vol. 2 (Springer, 1985).

[246] N. H. Christ, R. Jackiw, Phys. Lett. **91B**, 229 (1980).

[247] H. W. Griesshammer, (1997), hep-ph/9709462.

[248] H. Reinhardt, Nucl. Phys. **B503**, 505 (1997), hep-th/9702049.

[249] H. Hopf, Math. Ann. **104**, 637 (1931).

[250] C. H. Taubes, in: *Progress in gauge field theory*, G. 't Hooft, ed., Plenum Press, New York, 1984.

[251] T. C. Kraan, P. van Baal, Nucl. Phys. **A642**, 299 (1998), hep-th/9805201.

[252] R. Jackiw, S.-Y. Pi, Phys. Rev. **D61**, 105015 (2000), hep-th/9911072.

[253] T. Tsurumaru, I. Tsutsui, A. Fujii, Nucl. Phys. **B589**, 659 (2000), hep-th/0005064.

[254] G. 't Hooft, Nucl. Phys. **B138**, 1 (1978).

[255] G. Mack, V. B. Petkova, Ann. Phys. **123**, 442 (1979).

[256] L. D. Debbio *et al.*, Phys. Rev. **D58**, 094501 (1998), hep-lat/9801027.

[257] L. Del Debbio, M. Faber, J. Greensite, Š. Olejnik, Phys. Rev. **D55**, 2298 (1997), hep-lat/9610005.

[258] V. G. Bornyakov, D. A. Komarov, M. I. Polikarpov, Phys. Lett. **B497**, 151 (2001), hep-lat/0009035.

[259] P. de Forcrand, M. Pepe, Nucl. Phys. **B598**, 557 (2001), hep-lat/0008016.

[260] M. Engelhardt, H. Reinhardt, Nucl. Phys. **B567**, 249 (2000), hep-th/9907139.

[261] C. Adam, B. Muratori, C. Nash, Phys. Rev. **D61**, 105018 (2000), hep-th/9909189.

[262] M. Mathur, hep-th/9807167.

[263] M. Miettinen, A. J. Niemi, Y. Stroganov, Phys. Lett. **B474**, 303 (2000), hep-th/9908178.

[264] M. Garcia Perez, Nucl. Phys. Proc. Suppl. **94**, 27 (2001), hep-lat/0011026.

[265] E. Merzbacher, *Quantum Mechanics* (Wiley & Sons, New York, 1998).

Publications

[FB1] F. Bruckmann, T. Heinzl, T. Tok, A. Wipf,
Instantons and Gribov copies in the maximally Abelian gauge,
Nucl. Phys. **B584**, 589-614 (2000), hep-th/0001175.

[FB2] F. Bruckmann, T. Heinzl, T. Vekua, A. Wipf,
Magnetic Monopoles vs. Hopf Defects in the Laplacian (Abelian) Gauge,
Nucl. Phys. **B593**, 545-561 (2000), hep-th/0007119.

[FB3] F. Bruckmann, G. 't Hooft,
Monopoles, Instantons and Confinement,
Lecures given by GtH at the 5th WE Heraeus Summer School, Saalburg, Germany, September 1999, notes written by FB, hep-th/0010225.

[FB4] F. Bruckmann,
Hopf defects as seeds for monopole loops,
J. High Energy Phys. **08**, 030 (2001), hep-th/0011249.

[FB5] F. Bruckmann, T. Heinzl, T. Tok, A. Wipf,
Variational Gauge Fixing,
to appear.

A. Conventions and Angular Momentum in Four Dimensions

We shall work in *Euclidean* four-space $\mathbb{R}^4$. This is natural for field theories when considering thermal effects or tunneling. In addition the path integral is a priori 'well-defined' only in Euclidean space; following most textbooks we take for granted that one can Wick rotate back to Minkowski space.

The coordinates are denoted by $x_\mu = (\vec{x}, x_4)$, sometimes also by $(x_0, \vec{x})$.

A.1. Polar Angles

We rewrite the cartesian coordinates in four dimensions by virtue of the radius and the following angles,

$$\frac{x_\mu}{r} \equiv \hat{x}_\mu \equiv (\cos\theta\cos\varphi_{12}, \cos\theta\sin\varphi_{12}, \sin\theta\cos\varphi_{34}, \sin\theta\sin\varphi_{34}) \,, \tag{A.1}$$

where

$$\varphi_{12} = \arctan\frac{x_2}{x_1} \in (0, 2\pi) \,, \qquad \varphi_{34} = \arctan\frac{x_4}{x_3} \in (0, 2\pi) \,, \tag{A.2}$$

are azimuthal angles and

$$\theta = \arctan\frac{r_{34}}{r_{12}} \in (0, \pi/2) \,, \tag{A.3}$$

is a polar angle. The two radii are $r_{12} = (x_1^2 + x_2^2)^{1/2}$, $r_{34} = (x_3^2 + x_4^2)^{1/2}$. Accordingly, the measures of the unit three-sphere and the four-space are,

$$\mathrm{d}V(S^3) = \sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi_{12}\,\mathrm{d}\varphi_{34} \,, \qquad \mathrm{d}V(\mathbb{R}^4) = r^3\mathrm{d}r\,\mathrm{d}V(S^3) \,, \tag{A.4}$$

and $\mathrm{Vol}(S^3) = \frac{1}{2}\cdot 2\pi \cdot 2\pi = 2\pi^2$.

On the three-dimensional subspace $\vec{x} = (x, y, z) \in \mathbb{R}^3$ we use,

$$\vartheta = \arctan\frac{z}{\sqrt{x^2+y^2}} \in (0, \pi) \,, \qquad \varphi = \arctan\frac{y}{x} \in (0, 2\pi) \,, \tag{A.5}$$

which leads to the following measure of the unit two-sphere,

$$\mathrm{d}V(S^2) = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \,. \tag{A.6}$$

A.2. The Lie Algebra $so(4)$

The Lie group $SO(4)$ is the symmetry group of the four-dimensional Euclidean space. It also appears in the quantum theory of the hydrogen atom, where the spatial symmetry $SO(3)$ is dynamically enhanced to $SO(4)$ (cf. [265]). Furthermore, $SO(4)$ is 'only a sign flip and its consequences' away from the group $SO(1,3)$ part of which are the well-known Lorentz transformations. It is known that $SO(4)$ is isomorphic to $SU(2) \times SU(2)/Z_2$. We only need the representation of the Lie algebra $so(4)$ as angular momenta acting on functions. The latter enters the four-dimensional Laplacian when splitted into radial and angular part just like in three dimensions.

The generators of four-dimensional rotations are,

$$L_{\mu\nu} \equiv -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad \mu, \nu = 1, \ldots, 4\,, \;\; \mu \neq \nu\,. \tag{A.7}$$

It is straightforward to check that the $L_{\mu\nu}$ indeed satisfy the commutator relations of $so(4)$. In analogy with the Lorentz group (field strength) one introduces the angular momenta (magnetic generators) and 'boosts' (electric generators),

$$L_i \equiv \frac{1}{2}\,\epsilon_{ijk} L_{jk}\,, \qquad K_i \equiv L_{i4}\,. \tag{A.8}$$

For later use we denote that their components have a very simple form in the coordinates (A.1),

$$L_3 = -i\,\frac{\partial}{\partial \varphi_{12}}\,, \qquad K_3 = -i\,\frac{\partial}{\partial \varphi_{34}}\,. \tag{A.9}$$

The splitting into two independent Lie algebras $su(2)$ is provided by the anti-selfdual and selfdual parts of $L_{\mu\nu}$, if duality is understood as the exchange of $\vec{L}$ and $\vec{K}$,

$$M_a \equiv \frac{1}{2}\,(L_a - K_a) = -\frac{i}{2}\,\bar{\eta}^a_{\mu\nu}\, x_\mu \partial_\nu\,, \qquad N_a \equiv \frac{1}{2}\,(L_a + K_a) = -\frac{i}{2}\,\eta^a_{\mu\nu}\, x_\mu \partial_\nu\,. \tag{A.10}$$

As announced these operators have the commutation relations and Casimir operators of $su(2)$,

$$[M_a, M_b] = i\epsilon_{abc} M_c\,, \qquad [N_a, N_b] = i\epsilon_{abc} N_c\,, \qquad [M_a, N_b] = 0\,, \tag{A.11}$$

$$\vec{M}^2 \to m(m+1)\,, \qquad \vec{N}^2 \to n(n+1)\,. \tag{A.12}$$

It is well known in three dimensions that only the integer representations occur as angular momenta. In four dimensions the situation is slightly different. Due to the factor 1/2 in (A.10) the total momenta m and n can still be half-integer in our representation, $m, n \in \{0, 1/2, 1, \ldots\}$, but they have to coincide, $m = n$, since,

$$\vec{M}^2 - \vec{N}^2 = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma} x_\mu \partial_\nu x_\rho \partial_\sigma = 0\,. \tag{A.13}$$

A complete set of compatible observables is provided by $\{m(=n), m_3, n_3\}$, the latter being the eigenvalues of the third components,

$$M_3\, e^{im_3(\varphi_{12}-\varphi_{34})} = -\frac{i}{2}\left(\frac{\partial}{\partial\varphi_{12}} - \frac{\partial}{\partial\varphi_{34}}\right) e^{im_3(\varphi_{12}-\varphi_{34})} = m_3\, e^{im_3(\varphi_{12}-\varphi_{34})}\,, \tag{A.14}$$

$$N_3\, e^{in_3(\varphi_{12}+\varphi_{34})} = -\frac{i}{2}\left(\frac{\partial}{\partial\varphi_{12}} + \frac{\partial}{\partial\varphi_{34}}\right) e^{in_3(\varphi_{12}+\varphi_{34})} = n_3\, e^{in_3(\varphi_{12}+\varphi_{34})}\,, \tag{A.15}$$

and $m_3 \in \{-m, -m+1, \ldots, m\}$, $n_3 \in \{-n, -n+1, \ldots, n\}$. The spherical harmonics in four dimensions are of the form,

$$\begin{aligned} Y_{m,m_3,n_3}(\theta, \varphi_{12}, \varphi_{34}) &= \Theta_{m,m_3,n_3}(\theta)\, e^{im_3(\varphi_{12}-\varphi_{34})} e^{in_3(\varphi_{12}+\varphi_{34})} \\ &= \Theta_{m,m_3,n_3}(\theta)\, e^{i(m_3+n_3)\varphi_{12}} e^{-i(m_3-n_3)\varphi_{34}}\,. \end{aligned} \tag{A.16}$$

One might worry that the exponentials (A.14) and (A.15) are non-continuous functions for half-integer m_3 or n_3. But due to $m = n$ also the third components are either both integer or both half-integer, and (A.16) shows that the product of these functions is indeed smooth.

The function $\Theta(\theta)$ satisfies the differential equation

$$\left(\frac{1}{\sin 2\theta}\frac{\partial}{\partial\theta}\sin 2\theta\frac{\partial}{\partial\theta} + 4m(m+1) - \frac{(m_3+n_3)^2}{\cos^2\theta} - \frac{(m_3-n_3)^2}{\sin^2\theta}\right)\Theta_{m,m_3,n_3}(\theta) = 0\,. \tag{A.17}$$

For the extremal states in a multiplet, $m_3 = \pm m$, one can use the usual annihilation/creation argument, in terms of abstract states,

$$M_\pm |m, \pm m, n_3\rangle = 0\,. \tag{A.18}$$

The associated differential equation is much simpler than (A.17) and straightforwardly solved in terms of the functions,

$$\Theta_{m,-m,n_3}(\theta) = \cos^{-m_3-n_3}\theta\, \sin^{-m_3+n_3}\theta\,, \quad \Theta_{m,m,n_3}(\theta) = \sin^{m_3-n_3}\theta\, \cos^{m_3+n_3}\theta\,. \tag{A.19}$$

Finally the four-dimensional Laplacian reads,

$$\Box = \partial_r^2 + \frac{3}{r}\,\partial_r + \frac{2(\vec{M}^2 + \vec{N}^2)}{r^2}\,. \tag{A.20}$$

A.3. Spherical Harmonics

In the following we list the eigenfunctions of $\vec{J}^2 = (\vec{L} + \vec{T})^2$ and $\vec{L}^2$ for the three cases of interest in Section 3.4.1 ($t = 1$). We suppress the two 'magnetic' quantum numbers labelling the vectors in each multiplet.

(i) For $(j, l) = (1, 0)$ the spherical harmonics are given by the canonical dreibein $\hat{e}^a$ of constant unit vectors,

$$Y^{\text{sg}}_{(1,0)} = Y^{\text{reg}}_{(1,0)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{A.21}$$

(ii) For $(j, l) = (1/2, 1/2)$ there are four eigenfunctions, all linear in $\hat{x}_\mu$,

$$Y^{\text{sg}}_{(1/2,1/2)} = \left\{ \begin{pmatrix} \hat{x}_4 \\ \hat{x}_3 \\ \hat{x}_2 \end{pmatrix}, \begin{pmatrix} -\hat{x}_3 \\ \hat{x}_4 \\ -\hat{x}_1 \end{pmatrix}, \begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \\ -\hat{x}_4 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \right\}, \tag{A.22}$$

$$Y^{\text{reg}}_{(1/2,1/2)} = \left\{ \begin{pmatrix} -\hat{x}_4 \\ \hat{x}_3 \\ \hat{x}_2 \end{pmatrix}, \begin{pmatrix} -\hat{x}_3 \\ -\hat{x}_4 \\ -\hat{x}_1 \end{pmatrix}, \begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \\ \hat{x}_4 \end{pmatrix}, \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} \right\}. \tag{A.23}$$

The following remarks are in order. Obviously, Y^{reg} is obtained from Y^{sg} upon exchanging $\hat{x}_4 \to -\hat{x}_4$. This is achieved via conjugation with $\hat{g}$, $Y^{\text{sg}}_{(j=1/2,m=1/2)} \sim \hat{g}^\dagger \, Y^{\text{reg}}_{(j=1/2,n=1/2)} \, \hat{g}$. Note that the 'intertwining' gauge transformation $\hat{g}$ is only defined up to rotations around the direction of the Higgs field ϕ in isospace. It is convenient to combine the members of each (1/2, 1/2) quadruplet into a 'four-vector' Y_μ. Introducing the basis matrices $\sigma_\mu \equiv (i\sigma^a, \mathbb{1})$, one finds the relation $Y^{\text{reg}}_\mu = \sigma_\mu Y^{\text{sg}}_\mu \, \sigma^\dagger_\mu$ for any $\mu = 1, \ldots, 4$. Any component $Y_\mu(\hat{x})$ vanishes, if $\hat{x}_\mu = \pm\hat{e}_\mu$, the $\hat{e}_\mu$ denoting the canonical basis of $\mathbb{R}^4$. This means that the zeros of the quadruplet eigenfunctions are given by two points located on a three-sphere with fixed radius r (see Figure 3.7).

(iii) For the case $(j, l) = (0, 1)$ one has three basic eigenfunctions, now bilinear in $\hat{x}_\mu$,

$$Y^{\text{sg}}_{(0,1)} = \left\{ \begin{pmatrix} \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_1\hat{x}_2 + \hat{x}_3\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{x}_4) \\ -\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix} \right\} \tag{A.24}$$

$$Y^{\text{reg}}_{(0,1)} = \left\{ \begin{pmatrix} \hat{x}_1^2 - \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_1\hat{x}_2 - \hat{x}_3\hat{x}_4) \\ 2(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_2 + \hat{x}_3\hat{x}_4) \\ -\hat{x}_1^2 + \hat{x}_2^2 - \hat{x}_3^2 + \hat{x}_4^2 \\ 2(\hat{x}_2\hat{x}_3 - \hat{x}_1\hat{x}_4) \end{pmatrix}, \begin{pmatrix} 2(\hat{x}_1\hat{x}_3 - \hat{x}_2\hat{x}_4) \\ 2(\hat{x}_2\hat{x}_3 + \hat{x}_1\hat{x}_4) \\ -\hat{x}_1^2 - \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 \end{pmatrix} \right\} \tag{A.25}$$

Again, the two sets of eigenfunctions are related via $\hat{x}_4 \to -\hat{x}_4$ and can most easily be obtained from case (i) by conjugation with $\hat{g}$,

$$Y^{\rm sg}_{(j=0,\,m=1)} = \hat{g}^\dagger\, Y^{\rm reg}_{(j=1,\,n=0)}\, \hat{g}\,, \qquad Y^{\rm reg}_{(j=0,\,n=1)} = \hat{g}\, Y^{\rm sg}_{(j=1,\,m=0)}\, \hat{g}^\dagger\,, \tag{A.26}$$

which, in particular, implies that they *never vanish*. The pointlike defects stem from the radial part of the wave function.

The relevant spherical harmonics for the fundamental representation $t = 1/2$ needed for the Laplacian gauge are

$$Y^{\rm sg}_{(1/2,0)} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{A.27}$$

$$Y^{\rm reg}_{(0,1/2)} = \left\{ \hat{g} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{x}_4 + i\hat{x}_3 \\ -\hat{x}_2 + i\hat{x}_1 \end{pmatrix}, \hat{g} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 + i\hat{x}_1 \\ \hat{x}_4 - i\hat{x}_3 \end{pmatrix} \right\}. \tag{A.28}$$

B. Gauge Theory

B.1. The Lie Algebra $su(2)$

For a basis of the Lie algbera $su(2)$ we take half the Pauli matrices,

$$X \in su(2): \qquad X = X_a \tau_a \,, \qquad \tau_a = \sigma_a/2\,. \tag{B.1}$$

Due to

$$\tau_a \tau_b = \frac{\delta_{ab}}{4}\,\mathbb{1}_2 + \frac{i}{2}\,\epsilon_{abc}\tau_c\,, \tag{B.2}$$

we have the following commutator (vector product) and Killing form (scalar product),

$$[X,Y] = i\epsilon_{abc}X_a Y_b \tau_c = iX \times Y\,, \qquad (X,Y) \equiv 2\,\mathrm{tr}\,XY = X_a Y_a\,, \tag{B.3}$$

and combinations thereof,

$$(X,[Y,Z]) = (Z,[X,Y]) = (Y,[Z,X]) = i\epsilon_{abc}X_a Y_b Z_c\,, \tag{B.4}$$

$$[X,[Y,Z]] = (X,Y)Z - (X,Z)Y\,. \tag{B.5}$$

For a normalised field n,

$$(n,n) = n_a n_a = 1\,, \qquad n^2 = \frac{1}{4}\,\mathbb{1}_2\,, \tag{B.6}$$

the double commutator is equivalent to the orthogonal projector,

$$[n,[n,X]] = X - (X,n)n = \mathbb{P}^{\perp n}(X) \equiv X - (X,n)n\,, \quad \mathbb{P}^{\| n}(X) \equiv (X,n)n\,. \tag{B.7}$$

An element g of the Lie group $SU(2)$ can be written as

$$g = \exp(iX_a\tau_a) = \cos\frac{|X|}{2}\,\mathbb{1}_2 + i\sin\frac{|X|}{2}\,\frac{X_a}{|X|}\,\sigma_a, \qquad |X| = \sqrt{(X,X)}\,. \tag{B.8}$$

B.2. Gauge Transformations

The (hermitean) gauge field A transforms under a gauge transformation g as,

$$A \to {}^{g}\!A = gAg^\dagger + ig\mathrm{d}g^\dagger\,. \tag{B.9}$$

Its field strength F,

$$F = \mathrm{d}A - iA \wedge A \equiv (\mathrm{d}A_c + \frac{1}{2}\,\epsilon_{abc} A_a \wedge A_b)\tau_c\,, \tag{B.10}$$

transforms homogeneously,

$$F \to {}^gF = gFg^\dagger\,, \qquad \phi \to {}^g\phi = g\phi\, g^\dagger\,. \tag{B.11}$$

The infinitesimal transformations under $g = \exp(i\lambda)$ are

$$A \to A + \delta A\,, \qquad \delta A = \mathrm{D}_A\lambda = \mathrm{d}\lambda - i[A,\lambda]\,, \tag{B.12}$$

$$F \to F + \delta F\,, \qquad \delta F = -i[F,\lambda]\,, \tag{B.13}$$

$$\phi \to \phi + \delta\phi\,, \qquad \delta\phi = -i[\phi,\lambda]\,. \tag{B.14}$$

B.3. Parametrisation, Diagonalisation and Abelian Gauge Field

We parametrise the normalised Higgs field n with two polar angles in internal space,

$$n = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix}, \qquad \alpha \in (0, 2\pi)\,,\ \ \beta \in (0,\pi)\,. \tag{B.15}$$

The topological density is

$$\mathrm{d}V(S^2) = (n, i\,\mathrm{d}n \wedge \mathrm{d}n) = \sin\beta\,\mathrm{d}\beta \wedge \mathrm{d}\alpha\,. \tag{B.16}$$

The diagonalising gauge transformation reads,

$$g \;=\; e^{i\gamma\tau_3} e^{i\beta\tau_2} e^{i\alpha\tau_3}\,, \qquad {}^gn = gng^\dagger = \tau_3\,, \tag{B.17}$$

with the residual Abelian gauge freedom expressed in γ. Another form is

$$g = \cos\frac{\beta}{2}(\cos\frac{\alpha+\gamma}{2}\mathbb{1}_2 + i\sin\frac{\alpha+\gamma}{2}\sigma_3) + i\sin\frac{\beta}{2}(\cos\frac{\alpha-\gamma}{2}\sigma_2 - i\sin\frac{\alpha-\gamma}{2}\sigma_1)\,. \tag{B.18}$$

The differential $\mathrm{d}n$ is related to the Maurer-Cartan form $g^\dagger\mathrm{d}g$ as,

$$\mathrm{d}n = [n, g^\dagger\mathrm{d}g]\,, \qquad [n, \mathrm{d}n] = \mathbb{P}^{\perp n}(g^\dagger\mathrm{d}g)\,. \tag{B.19}$$

The Abelian gauge field of Section 4.3.3 is defined by

$$a = (ig\mathrm{d}g^\dagger, \tau_3) = \mathrm{d}\gamma + \cos\beta\mathrm{d}\alpha\,. \tag{B.20}$$

The field strength may contain a singular part,

$$\mathrm{d}a \;=\; f + f^{\mathrm{sg}}\,, \tag{B.21}$$

$$f \;=\; (-ig\mathrm{d}g^\dagger \wedge g\mathrm{d}g^\dagger, \tau_3) = -\sin\beta\mathrm{d}\beta \wedge \mathrm{d}\alpha = -\eta_{(2)}\,, \tag{B.22}$$

$$f^{\mathrm{sg}} \;=\; (ig\mathrm{d}^2g^\dagger, \tau_3) = \mathrm{d}^2\gamma + \cos\beta\mathrm{d}^2\alpha\,. \tag{B.23}$$

C. Explicit Calculations

C.1. Faddeev-Popov Operator of the MAG

The Faddeev-Popov operator of the Maximally Abelian gauge follows from the general considerations about projectors as in (3.45),

$$\mathrm{FP}_{\mathrm{MAG}} = (i\,\mathrm{ad}_{A_\mu^\perp} + \mathrm{D}_{A_\mu}\mathbb{P}^\perp)\mathrm{D}_{A_\mu}\,. \tag{C.1}$$

We first show that it vanishes when acting on parallel components $\lambda^\|$,

$$\mathrm{FP}_{\mathrm{MAG}}\lambda^\| = i\,\mathrm{ad}_{A_\mu^\perp}\mathrm{D}_{A_\mu}\lambda^\| + \mathrm{D}_{A_\mu}(\mathrm{D}_{A_\mu}\lambda^\|)^\perp\,. \tag{C.2}$$

We use (3.4) to show that

$$(\mathrm{D}_{A_\mu}\lambda^\|)^\perp = -i\,\mathrm{ad}_{A_\mu^\perp}\lambda^\|\,. \tag{C.3}$$

All what is left is to show that the operators $\mathrm{ad}_{A_\mu^\perp}$ and D_{A_μ} commute on the gauge slice. Indeed the Leibniz rule gives

$$[\mathrm{D}_{A_\mu}, \mathrm{ad}_{A_\mu^\perp}] = \mathrm{ad}_{\mathrm{D}_{A_\mu}A_\mu^\perp} = 0\,. \quad \text{on } \Gamma \tag{C.4}$$

$\mathrm{FP}_{\mathrm{MAG}}$ vanishes for the parallel gauge parameters $\lambda^\|$ which reflects the residual Abelian gauge freedom.

In the perpendicular sector we write

$$\mathrm{D}_{A_\mu}\lambda^\perp = \mathrm{D}_{A_\mu^\|}\lambda^\perp - i\,\mathrm{ad}_{A_\mu^\perp}\lambda^\perp\,, \tag{C.5}$$

where for $SU(2)$ only the first term is perpendicular. Therefore,

$$i\,\mathrm{ad}_{A_\mu^\perp}\mathrm{D}_{A_\mu}\lambda^\perp = i\,\mathrm{ad}_{A_\mu^\perp}(\mathrm{D}^\|_{A_\mu} - i\,\mathrm{ad}_{A_\mu^\perp})\lambda^\perp\,, \tag{C.6}$$

$$\mathrm{D}_{A_\mu}(\mathrm{D}_{A_\mu}\lambda^\perp)^\perp = \mathrm{D}_{A_\mu}\mathrm{D}_{A_\mu^\|}\lambda^\perp = (\mathrm{D}^2_{A_\mu^\|} - i\,\mathrm{ad}_{A_\mu^\perp}\mathrm{D}_{A_\mu^\|})\lambda^\perp\,. \tag{C.7}$$

The sum of these two terms gives (3.46) without the last term (the FP operator not only *acts* on the perpendicular space $\mathcal{H}^\perp$ but also *gives* an element of this space). We finally notice that there is even another form for $\mathrm{FP}_{\mathrm{MAG}}$,

$$\mathrm{FP}_{\mathrm{MAG}} = \mathrm{ad}_{\tau_3}(\mathrm{D}^2_{A_\mu} + (A_\mu^\perp)^2)\mathrm{ad}_{\tau_3}\,. \tag{C.8}$$

It involves 'more natural' operators but does not allow for a nice scaling argument like (3.61).

For the proof that $\mathbb{P}^{\perp}\mathrm{ad}^2_{A^{\perp}}\mathbb{P}^{\perp}$ is a nonnegative operator, we define the *hermitean* matrix-valued one-form X via,

$$[A^{\perp}, \phi^{\perp}] \equiv iX\,, \tag{C.9}$$

and calculate,

$$\begin{aligned}\langle \phi, \mathbb{P}^{\perp}\mathrm{ad}^2_{A^{\perp}}\mathbb{P}^{\perp}\phi\rangle &= \langle \phi^{\perp}, \mathrm{ad}_{A^{\perp}}\mathrm{ad}_{A^{\perp}}\phi^{\perp}\rangle = -\langle \mathrm{ad}_{A^{\perp}}\phi^{\perp}, \mathrm{ad}_{A^{\perp}}\phi^{\perp}\rangle & \text{(C.10)}\\ &= -\langle iX, iX\rangle = \langle X, X\rangle \geq 0\,. & \text{(C.11)}\end{aligned}$$

C.2. Feynman-Hellmann Theorem Applied to the LAG

In order to obtain information about the Laplacian of Section 3.4.1 when $R \neq \rho$, we keep R fixed and vary ρ. We restrict ourselves to the singular gauge. The ρ-dependent part of (3.122) contains two terms,

$$V^{\mathrm{sg}}_{\rho\,(j,m)}(r) \equiv 4e^{-\alpha_R(r)}\left[\frac{\rho^2(\vec{J}^2 - \vec{M}^2)}{r^2(r^2+\rho^2)} - \frac{\vec{T}^2\rho^2}{(r^2+\rho^2)^2}\right]. \tag{C.12}$$

The ρ^2-dependence of the ground state energy is determined by the Feynman-Hellmann theorem,

$$\frac{\partial}{\partial\rho^2}E = \frac{\partial}{\partial\rho^2}\langle\phi|H|\phi\rangle = \langle\phi|\frac{\partial H}{\partial\rho^2}|\phi\rangle \equiv \langle\phi|\frac{\partial V_\rho}{\partial\rho^2}|\phi\rangle\,. \tag{C.13}$$

For the three angular momentum sectors of interest ($t = 1$) we have,

$$\begin{aligned}\frac{\partial V^{\mathrm{sg}}_{\rho\,(0,1)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4}\frac{-4r^2}{(r^2+\rho^2)^3} < 0\,,\\ \frac{\partial V^{\mathrm{sg}}_{\rho\,(1/2,1/2)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4}\frac{2(\rho^2-r^2)}{(r^2+\rho^2)^3}\,,\\ \frac{\partial V^{\mathrm{sg}}_{\rho\,(1,0)}(r)}{\partial\rho^2} &= \frac{(r^2+R^2)^2}{R^4}\frac{4\rho^2}{(r^2+\rho^2)^3} > 0\,.\end{aligned} \tag{C.14}$$

According to (C.13), these functions have to be integrated with the positive factor $|\phi|^2\sqrt{g}$. Therefore, the ground state energies in the first and the third sector are monotonic in ρ^2, their slopes satisfying

$$\frac{\partial}{\partial\rho^2}E^{\mathrm{sg}}_{(0,1)} < 0\,, \qquad \frac{\partial}{\partial\rho^2}E^{\mathrm{sg}}_{(1,0)} > 0\,. \tag{C.15}$$

As the energies meet at $R = \rho$ ('level crossing') we conclude,

$$E^{\text{sg}}_{(0,1)} < E^{\text{sg}}_{(1,0)} \quad \text{for } R < \rho\,, \qquad E^{\text{sg}}_{(0,1)} > E^{\text{sg}}_{(1,0)} \quad \text{for } R > \rho\,. \tag{C.16}$$

This explains the behaviour of the full lines in Figure 3.6.

For the sector $(1/2, 1/2)$ there is no such simple argument. Still, we can compute the slope of $E(\rho^2)$ at the point $\rho = R$ by simply inserting the known function ϕ. This amounts to ordinary perturbation theory in $\delta \equiv \rho^2 - R^2$,

$$H(\rho^2) = H(\delta = 0) + \delta \left.\frac{\partial H}{\partial \rho^2}\right|_{\delta=0} + O(\delta^2) = H_0 + H_{\text{pert}}\,. \tag{C.17}$$

In this way we find a vanishing slope for the sector $(1/2, 1/2)$,

$$\left.\frac{\partial}{\partial \rho^2} E^{\text{sg}}_{(1/2,1/2)}\right|_{\rho^2=R^2} \sim \int_0^\infty \frac{(1-r^2)}{(r^2+1)^7}\, r^5 \mathrm{d}r = 0\,. \tag{C.18}$$

The lowest-lying state of this sector is thus pinched between the other two, at least for $R \approx \rho$ (cf. Figure 3.6).

Zusammenfassung

Die vorliegende Arbeit befasst sich mit Aspekten des "Colour Confinement" (Farbeinschluss) von Quarks in der starken Wechselwirkung. Dabei handelt es sich um eines der wichtigsten und faszinierendsten Infrarot-Phänomene der Quantenchromodynamik (QCD). Zu seiner Lösung wurde das Modell des dualen Supraleiters vorgeschlagen. Demzufolge sind die chromoelektrischen Feldlinien im QCD-Vakuum zu Flussschläuchen konzentriert. Die dazu notwendigen (kondensierten) magnetischen Monopole entstehen als Defekte von abelschen Projektionen, bei denen die Eichgruppe bis auf die maximal abelsche Untergruppe fixiert wird. Das Hauptaugenmerk der Arbeit liegt auf dem Zusammenhang von Instantonen und Monopolen.

Die Arbeit beginnt mit einer Einführung in die Problemstellung und der Vorstellung der Gitter-Eichtheorie sowie zweier relevanter effektiver Theorien der QCD: Duales Abelsches Higgs-Modell und Faddeev-Niemi-Wirkung. Danach werden grundlegende Konzepte von Eichtheorien bereitgestellt, nämlich Faserbündel, Eichfixierung inklusive Gribov-Problem und solitonische Objekte wie magnetische Monopole und Instantonen.

Das folgende Kapitel befasst sich eingehend mit der Methode der abelschen Projektion. Abelsche Eichungen werden explizit bzw. mit Hilfe von Funktionalen und Higgs-Feldern definiert. Abelsche Projektionen sind eng mit dem Begriff der Reduzibilität verknüpft. Die drei populärsten abelschen Eichungen, die Maximal Abelsche Eichung (MAG), die Polyakov-Eichung (PG) und die Laplacesche Abelsche Eichung (LAG) werden ausführlich untersucht. Der Zusammenhang abelscher Eichungen mit dem Faddeev-Niemi-Modell wird erläutert.

Die auftretenden Defekte werden in einem gesonderten Kapitel besprochen. Der Tatsache, dass generische Defekte lokal magnetische Monopole sind, steht die Hopf-Invariante als globale Eigenschaft gegenüber. Der Zusammenhang von Instantonen und Defekten wird generell und anhand von Beispielen dargestellt. Alle Rechnungen werden mit analytischen Methoden durchgeführt; Gitter-Resultate werden als Vergleich herangezogen.

Als wichtigste Ergebnisse werden präsentiert:

- Die Wirkungsweise der MAG, insbesondere ihr Gribov-Problem, kann mit Hilfe eines nieder-dimensionalen Modells visualisiert werden.

- Das singuläre Insanton liegt auf dem Gribov-Horizont der MAG; die Null-Moden des Faddeev-Popov-Operators werden explizit berechnet. Die Existenz eines Gribov-Horizonts wird aus einer allgemeinen Überlegung hergeleitet.

- Für die PG gibt es ein Eichfixierungsfunktional, dass aber nur bis auf Defekte mit der expliziten Eichbedingung übereinstimmt.

- In der LAG liegt das Instanton wiederum auf dem Gribov-Horizont. Aufgrund seiner hohen Symmetrie induziert es punktförmige Defekte, die durch Standard-Hopf-Abbildungen beschrieben werden. Durch eine Störung entstehen kleine kreisförmige Weltlinien von magnetischen Monopolen mit "Twist".

- Ein abelsches Hilfsfeld reflektiert gleichzeitig die globalen Eigenschaften des Hintergrund-Feldes und die lokalen Eigenschaften der Defekte. Es beweist die Notwendigkeit von Defekten für Instantonen und quantifiziert die Beiträge zur Instanton-Zahl: für Monopole muss der Twist, für Dirac-Blätter eine Windungszahl des Higgs-Feldes auftreten.

- Die unter bestimmten Umständen ebenfalls in der LAG induzierten großen Monopol-Weltlinien generieren die Instanton-Zahl über einen anderen Mechanismus.

Ausblicke auf fermionische Null-Moden und Zentrums-Vortices werden gegeben.

Die gewonnenen Resultate, insbesondere die Korrelation zwischen Instantonen und Defekten (Monopolen), sind ein starkes Indiz für die physikalische Relevanz der Defekte abelscher Projektionen, und zwar unabhängig von der gewählten abelschen Eichung. Eine mögliche Konsequenz für die Dynamik der QCD ist, dass Instantonen indirekt über die Bildung einer langen Monopol-Weltlinie zum Confinement beitragen können.

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt meiner Dissertation stehen. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten in Anspruch genommen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die geltende Prüfungsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

Ich versichere ehrenwörtlich, dass ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, 5. Juli 2001 Falk Bruckmann

Dank

An erster Stelle möchte ich mich bei Prof. Andreas Wipf für die Betreuung und umfassende Unterstützung meiner Arbeit bedanken. Mein herzlicher Dank gilt auch Dr. Thomas Heinzl für zahlreiche Literaturanregungen sowie seine Geduld mit meinen Manuskripten. Dr. Torsten Tok danke ich besonders für seine Hilfsbereitschaft in mathematischen Fragen. Den Genannten bin ich außerdem für viele lehrreiche Diskussionen und die hervorragende Zusammenarbeit zu Dank verpflichtet.

Für die angenehme Arbeitsatmosphäre und anregende Gespräche, nicht nur über Physik, bedanke ich mich auch bei allen anderen Mitarbeiterinnen und Mitarbeitern der Arbeitsgruppe, namentlich Leander Dittmann, Dörte Hansen, Andreas Kirchberg, Ronny Koch, Dr. Thomas Strobl, Dr. Mikhail Volkov und Erik Wöhnert – ihm schulde ich besonderen Dank für seine Hilfe bei Computerproblemen – sowie bei Stefan Scheel und Temo Vekua.

Prof. Manuel Asorey, Dr. Philippe de Forcrand und Dr. Sergei Shabanov danke ich für wertvolle fachliche Diskussionen bzw. die Möglichkeit des Aufenthalts an ihren Instituten.

Nicht zuletzt möchte ich meiner Frau Astrid für ihr Verständnis und ihren Zuspruch sowie die Durchsicht des Manuskripts danken.

Lebenslauf

Name	Falk Bruckmann
geboren	am 9. Januar 1973, in Halle/Saale
Geburtsname	Ziesche
Nationalität	deutsch
Familienstand	verheiratet mit Dr. Astrid Bruckmann, Biologin; ein Kind

Schule

Sept. 1979 – Aug. 1989	Polytechnische Oberschule, Thale
Sept. 1989 – Dez. 1989	Erweiterte Oberschule, Quedlinburg
Jan. 1990 – Juni 1991	Spezialklassen für Mathematik und Physik, Martin-Luther-Universität Halle-Wittenberg
Juni 1991	Abitur

Studium

Okt. 1991 – Sept. 1994	Physik-Studium: Friedrich-Schiller-Universität Jena
Okt. 1994 – März 1996	Humboldt-Universität zu Berlin
April 1996 – Mai 1997	Friedrich-Schiller-Universität Jena
Mai 1997	Diplom (Note: mit Auszeichnung) Thema: 'Effektives Potential und Zerfall instabiler Zustände in der Quantenfeldtheorie' Betreuer: Prof. Dr. A. Wipf

Zivildienst

Juni 1997 – Juni 1998	Behindertenwerkstatt, Jena

Forschung

seit Juli 1998	Doktorand am Theoretisch-Physikalischen Institut, Friedrich-Schiller-Universität Jena